1. Define the functions “less than” and “greater than” of two numerical arguments.

1.1 less than:

m is less than n if (sub m n) is less than zero. Also, less than ≡ not(greater or equal to)

LT := λaλb. NOT (GEQ b a)

Which

GEQ := λab. LEQ b a

LEQ := λmn. ISZERO (SUB m n)

ISZERO := λn. n (λx. FALSE) TRUE

SUB := λmn. n PRED m

PRED := λnfx. n (λgh. h (g f)) (λu. x) (λu. u)

1.2 greater than:

m is greater than n if (sub m n) is greater than zero. Also, greater than ≡ not(less than or equal to)

GT := λaλb. NOT (LEQ a b)

Which LEQ is defined in 1.1

2. Define the positive and negative integers using pairs of natural numbers.

we define natural number as (a, b) = -1^a \* b.

Positive integer = +n = (0, n)

Negative integer = -n = (1, n)

We define pair as

pair => λx. λy. λf. ((f x) y)

first => λp. (p (λx. λy. x))

second => λp. (p (λx. λy. y))

Thus, +n = λx. pair 0 x

-n = λx. pair 1 x

3. Define addition and subtraction of integers

addition

m + n means add one to m, n times. For example, 5 + 3 means add one to five, three times.

The step of adding one to the previous number is called successor which is defined as

SUCC := λn.λf.λx.f (n f x)

Thus, addition can be defined as ADD := λmλn. n succ m

subtraction

m – n means take one off m, n times. For example, 5 -3 means take one off five, three times.

The step of taking one of is called predecessor ( PRED n = n − 1 ) which is defined in 1.1 as

λn.λf.λx.n (λg.λh.h (g f)) (λu.x) (λu.u)

Thus, subtraction can be defined as SUB := λmλn. n PRED m

4. Define the division of positive integers recursively

we define division as a pair (m,n) = m/n

DIV := Y (λg.λq. λa. λb. LT a b (PAIR q a) (g (SUCC q) (SUB a b) b)) 0

Which

Y := λg. (λx. g (x x)) (λx. g (x x)) ≡ S (K (S I I)) (S (S (K S) K) (K (S I I)))

LT, PAIR, SUCC and SUB are defined in the previous questions

5. Define the function n! = n. (n - 1) … 1 recursively

(λx . (λy. x (y y)) (λy. x (y y))) (λ f. λ n.(ISZERO n) 1 (MULT n (f (PRED n))))

6. Define the rational numbers as pairs of integers

We define the rational numbers as Q = pair (m,n) = m/n ; m ∈ Z, n ∈ Z

Q = λm. λn. pair m n

7. Define functions for the addition, subtraction, multiplication and division of rationals.

Let and are rational numbers

We define pair m n = (m,n) = m / n and define x and y as rational numbers which

x := λa. λb. Pair a b and y := λc. λd. Pair c d

Addition is so we can define the addition as a pair of ad+cb and bd

ADD := λx. λy. (pair ad+cb bd ) x y

Subtraction is so we can define the addition as a pair of ad-cb and bd

SUB := λx. λy. (pair ad-cb bd ) x y

Multiplication is so we can define the addition as a pair of ac and bd

MUL := λx. λy. (pair ac bd ) x y

Division is so we can define the addition as a pair of ad and bc

DIV := λx. λy. (pair ad bc ) x y

8. Define a data structure to represent a list of numbers.

A list can be defined as a pair of head and the rest

empty list: NIL := λx. TRUE

LIST := λx. λy. pair x y

which x is head and y is the rest of the list

HEAD := λp. p TRUE

REST := λp. p FALSE

9. Define a function which extracts the first element from a list.

HEAD := λp. p TRUE

10. Define a recursive function which counts the number of elements in a list.

LENGTH := Y (λg. λc. λx. NULL x c (g (SUCC c) (REST x))) 0