Mathematical Formulation

The exact approach is based on a classical mixed-integer program (MIP) with the minimisation of total cost. This formulation, introduced by Cordeau et al. (2002) and Desaulniers et al. (2014), is commonly used when discussing the VPRTW. Generally, the vehicles are assumed homogeneous with the same capacity. The basic components of the VRPTW model are elaborated as follows.

Sets and Parameters

C: Set of customers, where the customers are denoted by 1, 2, …, n

N: Set of nodes visited, consisting of set C with node 0 (departing depot) and n+1 (returning depot)

V: Set of vehicles

q: Vehicle capacity, $q \geq 0$

d_i: Demand of customer i ($i \in C$), $d_i \geq 0$

c_{ij}: Cost of travelling from i to j ($i, j \in N,\ i \neq j$), $c_{ij} \geq 0$

t_{ij}: Travel time of edge (i, j) ($i, j \in N,\ i \neq j$), $t_{ij} > 0$

(a_i, b_i): Time window for customer i ($i \in N$). A vehicle must not reach to the customer after b_i. It can arrive before the time window begins but must wait until a_i for service. Clearly, (a_0, b_0) = (a_{n+1}, b_{n+1}). Each vehicle may does not leave the depot before a_0 and has to come back before or at b_{n+1}. Assume that a_0 = a_{n+1} = 0.

s_i: Service time at customer i ($i \in C$), $s_i \geq 0$

The model contains two sets of decision variables, as:

Variables

$$x_{ijk} = \begin{cases} 1, \text{ if vehicle } k \text{ traverses directly from } i \text{ to } j \\ 0, \text{ otherwise} \end{cases}$$

with $i, j \in N,\ k \in V,\ i \neq j,\ i \neq n+1$ (no edge starts from vertex n+1), $j \neq 0$ (no edge terminates in vertex 0)

w_{ik}: the time when vehicle k starts to serve customer i. Obviously, w_{ik} is meaningless if vehicle k does not visit customer i so its value is considered irrelevant in this case. In addition, w_{0k} = 0 for all k because a_0 = 0.

Figure 3.1 illustrates the relationship between some variables and parameters related to time factors.

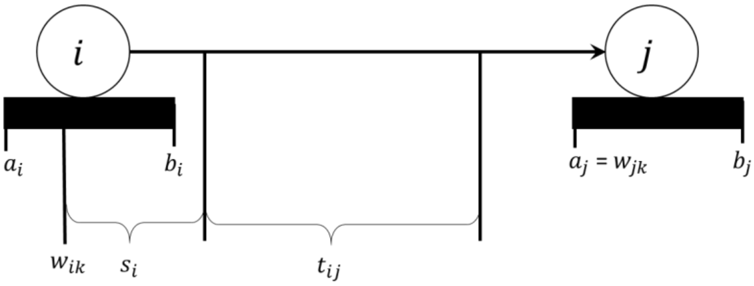

Figure. Relationship between time window, service time, travel time and service-start time

The target is to design a set of vehicle tours with the minimum total cost, satisfying (1) every customer is served only once, (2) each tour begins and ends at the depot, (3) the capacity constraints and time window are guaranteed. In that sense, the VRPTW can be formulated as a multi-commodity network flow problem as follows.

<u>Formulation</u>

Minimise

$$f_{VRPTW} = \sum_{k \in V} \sum_{i \in N} \sum_{j \in N} c_{ij} x_{ijk} \tag{3.1}$$

subject to:

$$\sum_{k \in V} \sum_{j \in N} x_{ijk} = 1 \quad \forall i \in C \tag{3.2}$$

$$\sum_{i \in C} d_i \sum_{j \in N} x_{ijk} \leq q \quad \forall k \in V \tag{3.3}$$

$$\sum_{j \in N} x_{0jk} = 1 \quad \forall k \in V \tag{3.4}$$

$$\sum_{i \in N} x_{ipk} - \sum_{j \in N} x_{pjk} = 0 \quad \forall p \in C, \forall k \in V \tag{3.5}$$

$$\sum_{i \in N} x_{i,n+1,k} = 1 \quad \forall k \in V \tag{3.6}$$

$$x_{ijk}\left(w_{ik} + s_i + t_{ij} - w_{jk}\right) \leq 0 \quad \forall i,j \in N, \forall k \in V, i \neq n+1,\ j \neq 0 \tag{3.7}$$

$$a_i \leq w_{ik} \leq b_i \quad \forall i \in N, \forall k \in V \tag{3.8}$$

$$x_{ijk} \in \{0,1\} \quad \forall i,j \in N, \forall k \in V \tag{3.9}$$

The objective function (3.1) aims at minimising the total cost of travelling. Constraint (3.2) guarantees that each customer is visited exactly once. Constraint (3.3) ensures that the vehicle capacity is not exceeded. Constraints (3.4), (3.5) and (3.6) state that each vehicle must depart from the depot 0, leave to another destination after arriving at a customer, and finally, end at the depot $n+1$. Constraint (3.7) indicates the relationship between departure time at a customer *j* with departure time, service time and travelling time from its predecessor i in a tour. Inequalities (3.8) ensures the that time window is satisfied while formulation (3.9) determines the domain of the decision variables x.

In the case that the number of vehicles is limited, one additional constraint can be added to the model, as:

$$\sum_{k \in V} \sum_{j \in N} x_{0jk} \leq |V| \quad \forall j \in N, \forall k \in V \tag{3.10}$$

Constraint (3.7) contains quadratic terms, which may result in non-convex optimisation. It can be linearized as:

$$w_{ik} + s_i + t_{ij} - w_{jk} \leq M_{ij}(1 - x_{ijk}) \quad \forall i,j \in N, \forall k \in V \tag{3.11}$$

with M_{ij} is a large scalar and can be replaced by maximum of $\{b_i + s_i + t_{ij} - a_j, 0\}$.

Moreover, it can be noted that the service-starting time variables w_{ik} impose a unique route orientation for each vehicle k, which eliminates any subtour. Therefore, the VRPTW subtour elimination constraint is unnecessary, and instead, the subtour problem can be implied in constraint (3.11).

Reference

Cordeau, J., Desaulniers, G., Desrosiers, J., Solomon, M., Soumis, F., 2002. 7. VRP with Time Windows, in: The Vehicle Routing Problem, Discrete Mathematics and Applications. Society for Industrial and Applied Mathematics, 157–193.

Desaulniers, G., Madsen, O., Ropke, S., 2014. Chapter 5: The Vehicle Routing Problem with Time Windows, in: Vehicle Routing, MOS-SIAM Series on Optimization. Society for Industrial and Applied Mathematics, 119–159.