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LAGRANGE MULTIPLIERS AND OPTIMALITY *

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Abstract. Lagrange multipliers used to be viewed as auxiliary variables introduced in a problem of constrained minimization in order to write first-order optimality conditions formally as a system of equations. Modern applications, with their emphasis on numerical methods and more complicated side conditions than equations, have demanded deeper understanding of the concept and how it fits into a larger theoretical picture.

A major line of research has been the nonsmooth geometry of one-sided tangent and normal vectors to the set of points satisfying the given constraints. Another has been the game-theoretic role of multiplier vectors as solutions to a dual problem. Interpretations as generalized derivatives of the optimal value with respect to problem parameters have also been explored. Lagrange multipliers are now being seen as arising from a general rule for the subdifferentiation of a nonsmooth objective function which allows black-and-white constraints to be replaced by penalty expressions. This paper traces such themes in the current theory of Lagrange multipliers, providing along the way a free-standing exposition of basic nonsmooth analysis as motivated by and applied to this subject.

Key words. Lagrange multipliers, optimization, saddle points, dual problems, augmented Lagrangian, constraint qualifications, normal cones, subgradients, nonsmooth analysis.

AMS(MOS) subject classifications. 49K99, 58C20, 90C99, 49M29

1. Optimization problems. Any problem of optimization concerns the minimization of some real-valued, or possibly extended-real-valued, function f_0 over some set C; maximization can be converted to minimization by a change of sign. For problems in finitely many "continuous" variables, which we concentrate on here, C is a subset of $\mathbb{R}^n$ and may be specified by a number of side conditions, called constraints, on $x = (x_1, \ldots, x_n)$. Its elements are called the *feasible solutions* to the problem, in contrast to the *optimal solutions* where the minimum of f_0 relative to C is actually attained in a global or local sense.

Equality constraints $f_i(x) = 0$ and inequality constraints $f_i(x) \leq 0$ are most common in describing feasible solutions, but other side conditions, like the attainability of x as a state taken on by a controlled dynamical system, are encountered too. Such further conditions can be indicated abstractly by a requirement $x \in X$ with $X \subset \mathbb{R}^n$. This notation can be convenient also in representing simple conditions for which the explicit introduction of a constraint function f_i would be cumbersome, for instance sign restrictions or upper or lower bounds on the components x_j of x. In a standard formulation of optimization from this point of view, the problem is to

$$(\mathcal{P}) \qquad \begin{array}{l} \text{minimize } f_0(x) \text{ over all } x \in X \\ \text{such that } f_i(x) \begin{cases} \leq 0 & \text{for } i = 1, \ldots, s, \\ = 0 & \text{for } i = s+1, \ldots, m, \end{cases} \end{array}$$

where $f_0, f_1, \ldots, f_m$ are real-valued functions on $\mathbb{R}^n$ and X is a certain subset of $\mathbb{R}^n$. Then C is the set of points $x \in X$ for which the listed conditions $f_i(x) \leq 0$ or $f_i(x) = 0$

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are satisfied. The presence of X can be suppressed by taking it to be all of $\mathbb{R}^n$, which we refer to as the case of $(\mathcal{P})$ where there's *no abstract constraint.* In general it's customary to suppose—and for simplicity we'll do so throughout this paper—that X is closed, and that the objective function f_0 and constraint functions f_i are smooth (i.e., at least of class $\mathcal{C}^1$).

In classical problems of optimization, only equality constraints were seriously considered. Even today, mathematics is so identified with the study of "equations" that many people find it hard at first to appreciate the importance of inequality constraints and to recognize when they may be appropriate. Yet inequalities are the hallmark of modern optimization, affecting not just the scope of applications but the very nature of the analysis that must be used. This is largely because of the computer revolution, which has opened the way to huge problems of a prescriptive kind—where the goal may be to prescribe how some device should best be designed, or how some system should best be operated. In such problems in engineering, economics and management, it's typical that actions can be taken only within certain limited ranges, and that the consequences of the actions are desired to lie within certain other ranges. Clearly, inequality constraints are essential in representing such ranges.

A set C specified as in $(\mathcal{P})$ can be very complicated. Usually there's no practical way of decomposing C into a finite number of simple pieces which can be investigated one by one. The process of minimizing f_0 over C leads inevitably to the possibility that the points of interest may lie on the boundary of C. When inequality constraints come into play, the geometry becomes one-sided and nontraditional forms of analysis are needed.

A fundamental issue despite these complications is the characterization of the locally or globally optimal solutions to $(\mathcal{P})$, if any. Not just any kind of characterization will do, however, in these days of diverse applications and exacting computational requirements. Conditions for optimality must not only be technically correct in their depiction of what's necessary or sufficient, but rich in supplying information about potential solutions and in suggesting a variety of numerical approaches. Moreover they should fit into a robust theoretical pattern which readily accommodates problem features that might be elaborated beyond the statement so far in $(\mathcal{P})$.

Lagrange multipliers have long been used in optimality conditions involving constraints, and it's interesting to see how their role has come to be understood from many different angles. This paper aims at opening up such perspectives to the reader and providing an overview not only of the properties of Lagrange multipliers that can be drawn upon in applications and numerical work, but also the new kind of analysis that has needed to be developed. We'll focus the discussion on first-order conditions for the most part, but this will already reveal differences in outlook and methodology that distinguish optimization from other mathematical disciplines.

One distinguishing idea which dominates many issues in optimization theory is convexity. A set $C \subset \mathbb{R}^n$ is said to be *convex* if it contains along with any two different points the line segment joining those points:

$$x \in C,\ x' \in C,\ 0 < t < 1 \quad \Longrightarrow \quad (1-t)x + tx' \in C.$$

(In particular, the empty set is convex, as are sets consisting of a single point.) A function f on $\mathbb{R}^n$ called *convex* if it satisfies the inequality

$$f\big((1-t)x + tx'\big) \le (1-t)f(x) + tf(x') \text{ for any } x \text{ and } x' \text{ when } 0 < t < 1.$$

It's *concave* if the opposite inequality always holds, and *affine* under equality; the affine functions $f : \mathbb{R}^n \to \mathbb{R}$ have the form $f(x) = v{\cdot}x + const.$

Convexity is a large subject which can hardly be addressed here, see [1], but much of the impetus for its growth in recent decades has come from applications in optimization. An important reason is the fact that when a convex function is minimized over a convex set every locally optimal solution is global. Also, first-order necessary conditions for optimality turn out to be sufficient. A variety of other properties conducive to computation and interpretation of solutions ride on convexity as well. In fact the great watershed in optimization isn't between linearity and nonlinearity, but convexity and nonconvexity. Even for problems that aren't themselves of convex type, convexity may enter for instance in setting up subproblems as part of an iterative numerical scheme.

By the *convex case of* problem $(\mathcal{P})$, we'll mean the case where X is a convex set and, relative to X, the objective function f_0 and the inequality constraint functions $f_1, \ldots, f_s$ are convex, and the equality constraints functions $f_{s+1}, \ldots, f_m$ are affine. The feasible set C is then convex, so that a convex function is indeed being minimized over a convex set. In such a problem the vectors of Lagrange multipliers that may be introduced to express optimality have a remarkable significance. They generally solve an auxiliary problem of optimization which is *dual* to the given problem. Moreover, as will be explained later, these two problems are the strategy problems associated with the two players in a certain zero-sum game. Such game concepts have had a great impact on optimization, especially on its applications to areas like economics.

Although we're supposing in problem $(\mathcal{P})$ that the f_i's are smooth, it's unavoidable that questions of *nonsmooth analysis* eventually be raised. In order to relate Lagrange multipliers to perturbations of $(\mathcal{P})$ with respect to certain canonical parameters, for instance, we'll need to consider the optimal value (the minimum value of f_0 over C) as a function of such parameters, but this function can't be expected to be smooth no matter how much smoothness is imposed on the f_i's.

Another source of nonsmoothness in optimization—there are many—is the frequent use of penalty expressions. Instead of solving problem $(\mathcal{P})$ as stated, we may wish to minimize a function of the form

$$f(x) = f_0(x) + \rho_1\big(f_1(x)\big) + \cdots + \rho_m\big(f_m(x)\big) \tag{1.1}$$

over the set X, where each ρ_i is a function on $\mathbb{R}^1$ that gives the value 0 when $f_i(x)$ lies in the desired range, but some positive value (a penalty) when it lies outside that range. As an extreme case, infinite penalties might be used. Indeed, in taking

$$\begin{aligned} &\text{for } i = 1, \ldots, s: && \rho_i(u_i) = \begin{cases} 0 & \text{if } u_i \le 0, \\ \infty & \text{if } u_i > 0, \end{cases} \\ &\text{for } i = s+1, \ldots, m: && \rho_i(u_i) = \begin{cases} 0 & \text{if } u_i = 0, \\ \infty & \text{if } u_i \ne 0, \end{cases} \end{aligned} \tag{1.2}$$

we get for f in (1.1) the so-called *essential* objective function in $(\mathcal{P})$, whose minimization over X is equivalent to the minimization of f_0 over C. Obviously, the essential objective function is far from smooth and even is discontinuous, but even finite penalties may be incompatible with smoothness. For example, linear penalty expressions

$$\begin{aligned} &\text{for } i = 1, \ldots, s: && \rho_i(u_i) = \begin{cases} 0 & \text{if } u_i \le 0, \\ d_i u_i & \text{if } u_i > 0, \end{cases} \\ &\text{for } i = s+1, \ldots, m: && \rho_i(u_i) = \begin{cases} 0 & \text{if } u_i = 0, \\ d_i|u_i| & \text{if } u_i \ne 0, \end{cases} \end{aligned} \tag{1.3}$$

with positive constants d_i have "kinks" at the origin which prevent f from being smooth. Linear penalties have widely been used in numerical schemes since their introduction by Pietrzykowski [2] and Zangwill [3]. Quadratic penalty expressions

$$\begin{aligned} &\text{for } i = 1,\ldots,s: && \rho_i(u_i) = \begin{cases} 0 & \text{if } u_i \le 0, \\ \frac{1}{2}d_i u_i^2 & \text{if } u_i > 0, \end{cases} \\ &\text{for } i = s+1,\ldots,m: && \rho_i(u_i) = \begin{cases} 0 & \text{if } u_i = 0, \\ \frac{1}{2}d_i u_i^2 & \text{if } u_i \ne 0, \end{cases} \end{aligned} \tag{1.4}$$

with coefficients $d_i > 0$, first proposed in the inequality case by Courant [4], are first-order smooth but discontinuous in their second derivatives. Penalty expressions with a possible mixture of linear and quadratic pieces have been suggested by Rockafellar and Wets [5], [6], [7], and Rockafellar [8] as offering advantages over the black-and-white constraints in (1.2) even in the modeling of some situations, especially large-scale problems with dynamic or stochastic structure. Similar expressions ρ_i have been introduced in connection with augmented Lagrangian theory, which will be described in §6 and 7, but differing from penalty functions in the usual sense of that notion because they take on negative as well as positive values. Such "monitoring functions" nevertheless have the purpose of facilitating problem formulations in which standard constraints are replaced by terms incorporated into a modified objective function, although at the possible expense of some nonsmoothness.

For most of this paper we'll keep to the conventional format of problem $(\mathcal{P})$, but in §10 we'll explain how the results can be extended to a more flexible problem statement which covers the minimization of penalty expressions such as in (1.1) as well as other nonsmooth objective functions that often arise in optimization modeling.

2. The classical view. Lagrange multipliers first made their appearance in problems having *equality constraints only*, which in the notation of $(\mathcal{P})$ is the case where $X = \mathbb{R}^n$ and $s = 0$. The feasible set then has the form

$$C = \{\, x \mid f_i(x) = 0 \text{ for } i = 1,\ldots,m \,\}. \tag{2.1}$$

and can be approached geometrically as a "smooth manifold," like an d-dimensional hypersurface within $\mathbb{R}^n$. This approach requires a rank assumption on the Jacobian matrix of the mapping $F : x \mapsto \big(f_1(x),\ldots,f_m(x)\big)$ from $\mathbb{R}^n$ into $\mathbb{R}^m$. Specifically, if at a given point $\bar{x} \in C$ the Jacobian matrix $\nabla F(\bar{x}) \in \mathbb{R}^{m\times n}$, whose rows are the gradient vectors $\nabla f_i(\bar{x})$, has full rank m, it's possible to coordinatize C around $\bar{x}$ so as to identify it locally with a region of $\mathbb{R}^d$ for $d = n - m$. The rank condition on $\nabla F(\bar{x})$ is equivalent of course to the linear independence of the vectors $\nabla f_i(\bar{x})$ for $i = 1,\ldots,m$ and entails having $m \le n$.

The workhorse in this mathematical setting is the standard implicit mapping theorem along with its special case, the inverse mapping theorem. The linear independence condition makes possible a local change of coordinates around $\bar{x}$ which reduces the constraints to an extremely simple form. Specifically, one can write $x = G(z)$, with $\bar{x} = G(\bar{z})$, for a smooth local mapping G having invertible Jacobian $\nabla G(\bar{z})$, in such a way that the transformed constraint functions $h_i = f_i \circ G$ are just $h_i(z_1,\ldots,z_n) \equiv z_i$ and, therefore, the constraints on $z = (z_1,\ldots,z_n)$ are just $z_i = 0$ for $i = 1,\ldots,m$. For the problem of minimizing the transformed objective function $h_0 = f_0 \circ G$ subject to such constraints, there's an elementary first-order necessary condition for the optimality of $\bar{z}$: one must have

$$\frac{\partial h_0}{\partial z_i}(\bar{z}) = 0 \text{ for } i = m+1,\ldots,n. \tag{2.2}$$

A corresponding condition in the original coordinates can be stated in terms of the values

$$\bar{y}_i = -\frac{\partial h_0}{\partial z_i}(\bar{z}) \text{ for } i = 1, \ldots, m. \tag{2.3}$$

These have the property that $\nabla\big(h_0 + \bar{y}_1 h_1 + \cdots + \bar{y}_m h_m\big)(\bar{z}) = 0$, and this equation can be written equivalently as

$$\nabla\big(f_0 + \bar{y}_1 f_1 + \cdots + \bar{y}_m f_m\big)(\bar{x}) = 0. \tag{2.4}$$

Thus, a necessary condition for the local optimality of $\bar{x}$ in the original problem is the existence of values $\bar{y}_i$ such that the latter holds.

This result can be stated elegantly in terms of the *Lagrangian* for problem $(\mathcal{P})$, which is the function L on $\mathbb{R}^n \times \mathbb{R}^n$ defined by

$$L(x, y) = f_0(x) + y_1 f_1(x) + \cdots + y_m f_m(x) \text{ for } y = (y_1, \ldots, y_m). \tag{2.5}$$

THEOREM 2.1. *In the case of problem $(\mathcal{P})$ where only equality constraints are present, if $\bar{x} = (\bar{x}_1, \ldots, \bar{x}_n)$ is a locally optimal solution at which the gradients $\nabla f_i(\bar{x})$ are linearly independent, there must be a vector $\bar{y} = (\bar{y}_1, \ldots, \bar{y}_m)$ such that*

$$\nabla_x L(\bar{x}, \bar{y}) = 0, \qquad \nabla_y L(\bar{x}, \bar{y}) = 0. \tag{2.6}$$

The supplementary values $\bar{y}_i$ in this first-order condition are called *Lagrange multipliers* for the constraint functions f_i at $\bar{x}$. An intriguing question is what they might mean in a particular application, since it seems strange perhaps that a problem entirely in $\mathbb{R}^n$ should require us to search for a vector pair in the larger space $\mathbb{R}^n \times \mathbb{R}^m$.

The two equations in (2.6) can be combined into a single equation, the vanishing of the full gradient $\nabla L(\bar{x}, \bar{y})$ relative to all the variables, but there's a conceptual pitfall in this. The false impression is often gained that since the given problem in x is one of minimization, the vanishing of $\nabla L(\bar{x}, \bar{y})$ should, at least in "nice" circumstances when $\bar{x}$ is optimal, correspond to $L(x, y)$ achieving a local minimum with respect to both x and y at $(\bar{x}, \bar{y})$. But apart from the convex case of $(\mathcal{P})$, L need not have a local minimum even with respect to x at $\bar{x}$ when y is fixed at $\bar{y}$. On the other hand, it will become clear as we go along that the equation $\nabla_y L(\bar{x}, \bar{y}) = 0$ should be interpreted on the basis of general principles as indicating a *maximum* of L with respect to y at $\bar{y}$ when x is fixed at $\bar{x}$.

The immediate appeal of (2.6) as a necessary condition for optimality resides in the fact that these vector equations constitute $n + m$ scalar equations in the $n + m$ unknowns $\bar{x}_j$ and $\bar{y}_i$. The idea comes to mind that in solving the equations for $\bar{x}$ and $\bar{y}$ jointly one may hope to determine some—or every—locally optimal solution $\bar{x}$ to $(\mathcal{P})$. While this is definitely the role in which Lagrange multipliers were seen traditionally, the viewpoint is naive from current perspectives. The equations may well be nonlinear. Solving a system of nonlinear equations numerically is no easier than solving an optimization problem directly by numerical means. In fact, nonlinear equations are now often solved by optimization techniques through conversion to a nonlinear least squares problem.

Although a direct approach to the equality-constrained case of $(\mathcal{P})$ through solving the equations in (2.6) may not be practical, Lagrange multipliers retain importance for other reasons. Before delving into these, let's look at the extent to which the classical methodology behind Theorem 2.1 is able to handle inequality constraints along with equality constraints.

An inequality constraint $f_i(x) \le 0$ is *active* at a point $\bar{x}$ of the feasible set C in (P) if $f_i(\bar{x}) = 0$, whereas it's *inactive* if $f_i(\bar{x}) < 0$. Obviously, in the theoretical study of the local optimality of $\bar{x}$ only the active inequality constraints at $\bar{x}$ need to be considered with the equality constraints, but as a practical matter it may be very hard to know without a lot of computation exactly which of the inequality constraints might turn out to be active.

As a temporary notational simplification, let's suppose that the inequality constraints $f_i(x) \le 0$ for $i = 1, \dots, r$ are inactive at $\bar{x}$, whereas the ones for $i = r+1, \dots, s$ are active. (The set X is still the whole space $\mathbb{R}^n$.) As long as all the gradients $\nabla f_i(\bar{x})$ for $i = r+1, \dots, s, s+1, \dots, m$ are linearly independent, we can follow the previous pattern of introducing a change of coordinates $x = G(z)$ such that the functions $h_i = f_i \circ G$ take the form $h_i(z_1, \dots, z_n) \equiv z_i$ for $i = r+1, \dots, m$. Then the problem is reduced locally to minimizing $h_0(z_1, \dots, z_n)$ subject to

$$z_i \begin{cases} \le 0 & \text{for } i = r+1, \dots, s, \\ = 0 & \text{for } i = s+1, \dots, m. \end{cases}$$

The former point $\bar{x}$ is transformed into a point $\bar{z}$ having coordinates $\bar{z}_i = 0$ for $i = r+1, \dots, m$. The elementary first-order necessary condition for the optimality of $\bar{z}$ in this setting is that

$$\frac{\partial h_0}{\partial z_i}(\bar{z}) \begin{cases} \le 0 & \text{for } i = r+1, \dots, s, \\ = 0 & \text{for } i = m+1, \dots, n. \end{cases}$$

Then by letting

$$\bar{y}_i = \begin{cases} 0 & \text{for } i = 1, \dots, r, \\ -\dfrac{\partial h_0}{\partial z_i}(\bar{z}) & \text{for } i = r+1, \dots, m, \end{cases}$$

we obtain $\nabla\big(h_0 + \bar{y}_1 h_1 + \cdots + \bar{y}_m h_m\big)(\bar{z}) = 0$, which translates back to

$$\nabla\big(f_0 + \bar{y}_1 f_1 + \cdots + \bar{y}_m f_m\big)(\bar{x}) = 0.$$

This result can be stated as the following generalization of Theorem 2.1 in which the notation no longer supposes advance knowledge of the active set of inequality constraints.

Theorem 2.2. *In the case of problem $(\mathcal{P})$ with both equality and inequality constraints possibly present, but no abstract constraint, if $\bar{x}$ is a locally optimal solution at which the gradients $\nabla f_i(\bar{x})$ of the equality constraint functions and the active inequality constraint functions are linearly independent, there must be a vector $\bar{y}$ in*

$$Y = \big\{\, y = (y_1, \dots, y_s, y_{s+1}, \dots, y_m) \,\big|\, y_i \ge 0 \textit{ for } i = 1, \dots, s \,\big\} \tag{2.7}$$

such that

$$\nabla_x L(\bar{x}, \bar{y}) = 0, \tag{2.8}$$

$$\frac{\partial L}{\partial y_i}(\bar{x}, \bar{y}) \begin{cases} = 0 & \textit{for } i \in [1, s] \textit{ with } \bar{y}_i > 0, \textit{ and for } i \in [s+1, m], \\ \le 0 & \textit{for } i \in [1, s] \textit{ with } \bar{y}_i = 0. \end{cases} \tag{2.9}$$

The simple rule $\nabla_y L(\bar{x}, \bar{y}) = 0$ in Theorem 2.1 has been replaced in Theorem 2.2 by requirements imposed jointly on $\nabla_y L(\bar{x}, \bar{y})$ and $\bar{y}$. The significance of these complicated requirements will emerge later along with a more compact mode of expressing them.

Certainly Theorem 1.2 dispels further any illusion that the role of Lagrange multipliers is to enable an optimization problem to be solved by solving some system of smooth nonlinear equations. Beyond the practical difficulties already mentioned, there's now the fact that a whole collection of systems might be have to be inspected. For each subset I of $\{1,\ldots,s\}$ we could contemplate solving the $n+m$ equations

$$\begin{aligned}
\frac{\partial L}{\partial x_j}(\bar{x},\bar{y}) &= 0 \ \text{ for } \ j=1,\ldots,n,\\
\frac{\partial L}{\partial y_i}(\bar{x},\bar{y}) &= 0 \ \text{ for } \ i\in I \ \text{ and } \ i=s+1,\ldots,m,\\
\bar{y}_i &= 0 \ \text{ for } \ i\in\{1,\ldots,s\}\setminus I,
\end{aligned}$$

for the $n+m$ unknowns $\bar{x}_j$ and $\bar{y}_i$ and checking then to see whether the remaining conditions in Theorem 2.2, namely

$$\begin{aligned}
\frac{\partial L}{\partial y_i}(\bar{x},\bar{y}) &\le 0 \ \text{ for } \ i\in\{1,\ldots,s\}\setminus I,\\
\bar{y}_i &\ge 0 \ \text{ for } \ i\in I \ \text{ and } \ i=s+1,\ldots,m,
\end{aligned}$$

happen to be satisfied in addition. But the number of such systems to look at could be astronomical, so that an exhaustive search would be impossible.

The first-order optimality conditions in Theorem 2.2 are commonly called the *Kuhn-Tucker conditions* on the basis of the 1951 paper of Kuhn and Tucker [9], but after many years it came to light that they had also been derived in the 1939 master's thesis of Karush [10]. This thesis was never published, but the essential portions are reproduced in Kuhn's 1976 historical account [11]. The very same theorem is proved by virtually the same approach in both cases, but with a "constraint qualification" in terms of certain tangent vectors instead of the linear independence in Theorem 2.2. This will be explained in §4. Karush's motivation came not from linear programming, an inspiring new subject when Kuhn and Tucker did their work, but from the calculus of variations. Others in the calculus of variations had earlier considered inequality constraints, for instance Valentine [12], but from a more limited outlook. Quite a different approach to inequality constraints, still arriving in effect at the same conditions, was taken before Kuhn and Tucker by John [13]. His hypothesis amounted to the generalization of the linear independence condition in Theorem 2.2 in which the coefficients of the gradients of the inequality constraint functions are restricted to nonnegativity. This too will be explained in §4.

Equality and inequality constraints are handled by Theorem 2.2, but not an abstract constraint $x\in X$. The optimality condition is therefore limited to applications where it's possible and convenient to represent all side conditions explicitly by a finite number of equations and inequalities. The insistence on linear independence of constraint gradients is a further shortcoming of Theorem 2.2. While the linear independence assumption is natural for equality constraints, it's unnecessarily restrictive for inequality constraints. It excludes many harmless situations that often arise, as for instance when the constraints are merely linear (i.e., all the constraint functions are affine) but the gradients are to some degree linearly dependent because of inherent symmetries in the problem's structure. This of course is why even the early contributors just cited felt the need for closer study of the constraint geometry in optimization problems.

3. Geometry of tangents and normals. The key to understanding Lagrange multipliers has been the development of concepts pertinent to the minimization of a function f_0 over a set $C \subset \mathbb{R}^n$ without insisting, at first, on any particular kind of representation for C. This not only furnishes insights for a variety of representations of C, possibly involving an abstract constraint $x \in X$, but also leads to a better way of writing multiplier conditions like the ones in Theorem 2.2.

Proceeding for the time being under the bare assumption that C is some subset of $\mathbb{R}^n$, we discuss the local geometry of C in terms of "tangent vectors" and "normal vectors" at a point $\bar{x}$. The introduction of such vectors in a *one-sided* sense, instead of the classical two-sided manner, has been essential to advancement in optimization theory ever since inequality constraints came to the fore. It has stimulated the growth of a new branch of analysis, which is called *nonsmooth* analysis because of its emphasis on one-sided derivative properties of functions as well as kinks and corners in set boundaries.

Many different definitions of tangent and normal vectors have been offered over the years. The systematic developments began with convex analysis in the 1960s and continued in the 70s and 80s with various extensions to nonconvex sets and functions. We take this opportunity to present current refinements which advantageously cover both convex and nonconvex situations with a minimum of effort. The concepts will be applied to Lagrange multipliers in §4.

DEFINITION 3.1. A vector w is *tangent* to C at $\bar{x}$, written $w \in T_C(\bar{x})$, if there is a sequence of vectors $w^k \to w$ along with a sequence of scalars $t_k \downarrow 0$ such that $\bar{x} + t_k w_k \in C$.

DEFINITION 3.2. A vector v is *normal* to C at $\bar{x}$, written $v \in N_C(\bar{x})$, if there is a sequence of vectors $v^k \to v$ along with a sequence of points $x^k \to \bar{x}$ in C such that, for each k,

$$\big\langle v^k,\, x - x^k \big\rangle \le o\big(|x - x^k|\big) \text{ for } x \in C \tag{3.1}$$

(where $\langle \cdot, \cdot \rangle$ is the canonical inner product in $\mathbb{R}^n$, $|\cdot|$ denotes the Euclidean norm, and o refers as usual to a term with the property that $o(t)/t \to 0$ as $t \to 0$). It is a *regular* normal vector if the sequences can be chosen constant, i.e., if actually

$$\big\langle v,\, x - \bar{x} \big\rangle \le o\big(|x - \bar{x}|\big) \text{ for } x \in C. \tag{3.2}$$

Note that $0 \in T_C(\bar{x})$, and whenever $w \in T_C(\bar{x})$ then also $\lambda w \in T_C(\bar{x})$ for all $\lambda \ge 0$. These properties also hold for $N_C(\bar{x})$ and mean that the sets $T_C(\bar{x})$ and $N_C(\bar{x})$ are *cones* in $\mathbb{R}^n$. In the special case where C is a smooth manifold they turn out to be the usual tangent and normal subspaces to C at $\bar{x}$, but in general they aren't symmetric about the origin: they may contain a vector without containing its negative. As subsets of $\mathbb{R}^n$ they're always closed but not necessarily convex, although convexity shows up very often.

The limit process in Definition 3.2 introduces the set of pairs (x, v) with v a normal to C at x as the closure in $C \times \mathbb{R}^n$ of the set of pairs (x, v) with v a regular normal to C at x. A basic consequence therefore of Definition 3.2 is the following.

PROPOSITION 3.3. *The set* $\big\{ (x, v) \,\big|\, v \in N_C(x) \big\}$ *is closed as a subset of* $C \times \mathbb{R}^n$*: if* $x^k \to \bar{x}$ *in* C *and* $v^k \to v$ *with* $v^k \in N_C(x^k)$*, then* $v \in N_C(\bar{x})$.

The symbols $T_C(\bar{x})$ and $N_C(\bar{x})$ are used to denote more than one kind of tangent cone and normal cone in the optimization literature. For the purposes here there is no need to get involved with a multiplicity of definitions and technical relationships, but some remarks may be helpful in providing orientation to other presentations of the subject.

The cone we're designating here by $T_C(\bar{x})$ was first considered by Bouligand [14], who called it the *contingent cone* to C at $\bar{x}$. It was rediscovered early on in the study of Lagrange multipliers for inequality constraints and ever since has been regarded as fundamental by everyone who has dealt with the subject, not only in mathematical programming but control theory and other areas. For instance, Hestenes relied on this cone in his 1966 book [15], which connected the then-new field of optimal control with the accomplishments of the 1930s school in the calculus of variations. Another important tangent cone is that of Clarke [16], which often agrees with $T_C(\bar{x})$ but in general is a subcone of $T_C(\bar{x})$.

The normal cone $N_C(\bar{x})$ coincides with the cone of "limiting normals" developed by Clarke [16], [17], [18] under the assumption that C is closed. (A blanket assumption of closedness would cause us trouble in §§8 and 9, so we avoid it here.) Clarke used limits of more special "proximal" normals, instead of the regular normals v^k in Definition 3.2, but the cone comes out the same because every regular normal is itself a limit of proximal normals when C is closed; cf. Kruger and Mordukhovich [19], or Ioffe [20]). Clarke's tangent cone consists of the vectors w such that $\langle v, w\rangle \le 0$ for all $v \in N_C(\bar{x})$.

Clarke was the first to take the crucial step of introducing limits to get a more robust notion of normal vectors. But the normal cone really stressed by Clarke in [16], and well known now for its many successful applications to a diversity of problems, especially in optimal control and the calculus of variations (cf. [17] and [18]) isn't this cone of limit vectors, $N_C(\bar{x})$, but its closed convex hull.

Clarke's convexified normal cone and his tangent cone are polar to each other. For a long time such duality was felt to be essential in guiding the development of nonsmooth analysis because of the experience that had been gained in convex analysis [1]. Although the cone of limiting normals was assigned a prominent role in Clarke's framework, results in the calculus of normal vectors were typically stated in terms of the convexified normal cone, cf. Clarke [17] and Rockafellar [21]. This seemed a good expedient because (1) it promoted the desired duality, (2) in most of the examples deemed important at the time the cones turned out anyway to agree with the ones in Definitions 3.1 and 3.2, and (3) convexification was ultimately needed anyway in certain infinite-dimensional applications involving weak convergence. But gradually it has become clear that convexification is an obstacle in some key areas, especially the treatment of graphs of nonsmooth mappings.

The move away from the convexifying of the cone of limiting normals has been championed by Mordukhovich, who furnished the missing results needed to fill the calculus gaps that had been feared in the absence of convexity [22], [23], [24]. Mordukhovich, like Clarke, emphasized "proximal" normals as the starting point for defining general normals through limits. The "regular" normals used here have not previously been featured in that expositional role, but they have long been familiar in optimization under an alternative definition in terms of polarity with tangent vectors (the property in Proposition 3.5(b) below); cf. Bazaraa, Gould, and Nashed [25], Hestenes [26], Penot [27].

Thanks to the efforts of many researchers, a streamlined theory is now in the offing. Its outline will be presented here. We'll be able to proceed on the basis of only the one tangent cone $T_C(\bar{x})$ in Definition 3.1 and the one normal cone $N_C(\bar{x})$ in Definition 3.2 and yet go directly to the heart of the issues about Lagrange multipliers.

We begin by demonstrating that when C is convex, $T_C(\bar{x})$ and $N_C(\bar{x})$ coincide with the tangent and normal cones originally introduced in convex analysis.

PROPOSITION 3.4. *If the set C is convex, the tangent cone $T_C(\bar{x})$ is the closure* *of the set of all vectors w such that $\bar{x}+\varepsilon w \in C$ for some $\varepsilon > 0$, whereas the normal cone $N_C(\bar{x})$ is the set of vectors v such that*

$$\langle v,\, x-\bar{x}\rangle \le 0 \text{ for all } x \in C, \tag{3.3}$$

or in other words, such that the linear function $x \mapsto \langle v, x\rangle$ achieves its maximum over C at $\bar{x}$. Every normal to a convex set is therefore a regular normal.

Proof. The assertion about tangents stems from the observation that when $\bar{x}+\varepsilon w \in C$ for some $\varepsilon > 0$, all the points $\bar{x}+tw$ with $0 < t < \varepsilon$ likewise belong to C, since these lie on the line segment joining $\bar{x}$ with $\bar{x}+\varepsilon w$.

As for normals, if v satisfies (3.3) it definitely satisfies (3.2) and thus belongs to $N_C(\bar{x})$. On the other hand, suppose v satisfies (3.2) and consider any $x \in C$. The convexity of C implies that the point $x_t = \bar{x}+t(x-\bar{x})$ belongs to C for all $t \in (0,1)$, so that $\langle v,\, x_t-\bar{x}\rangle \le o(|x_t-\bar{x}|)$ for $t \in (0,1)$, or in other words, $\langle v,\, x-\bar{x}\rangle \le o(t|x-\bar{x}|)/t$ for $t \in (0,1)$. Taking the limit on the right as $t{\downarrow}0$, we see that $\langle v,\, x-\bar{x}\rangle \le 0$. Thus, v satisfies (3.3); this demonstrates that (3.3) characterizes regular normals, at least.

Now consider a general normal v, which by Definition 3.2 is a limit of regular normals v_k at points x_k approaching $\bar{x}$ in C. For each $x \in C$ we have by the characterization already developed that $\langle v_k,\, x-x_k\rangle \le 0$, so in the limit we have $\langle v,\, x-\bar{x}\rangle \le 0$. Therefore v again has the property in (3.3). In particular, every normal is regular. □

Regular normal vectors v always have a variational interpretation, even when the set isn't convex. This previously unnoticed fact, which underscores the fundamental connection between normal vectors and optimality conditions, is brought out in property (c) of the next proposition, which also indicates how the normal cone can always be derived from the tangent cone.

PROPOSITION 3.5. *For any set $C \subset \mathbb{R}^n$, the following properties of a vector $v \in \mathbb{R}^n$ are equivalent:*

(a) *v is a regular normal to C at $\bar{x}$;*

(b) *$\langle v, w\rangle \le 0$ for every tangent vector w to C at $\bar{x}$;*

(c) *on some open neighborhood O of $\bar{x}$ there is a smooth function f_0 with $-\nabla f_0(\bar{x}) = v$, such that f_0 attains its minimum relative to $C \cap O$ at $\bar{x}$.*

Proof. Condition (a) is equivalent to the property that $\langle v, w\rangle \le 0$ whenever w is the limit of a sequence of vectors of the form $(x_k-\bar{x})/|x_k-\bar{x}|$ with $x_k \in C$, $x_k \ne \bar{x}$. Since the tangent cone $T_C(\bar{x})$ consists of all nonnegative multiples of vectors w obtainable as such limits (together with the zero vector), it's clear that this property of v holds if and only if (b) holds. Thus, (a) is equivalent to (b). On the other hand, (c) obviously implies (a) because of the expansion $f_0(x) = f_0(\bar{x}) + \langle v,\, x-\bar{x}\rangle + o(|x-\bar{x}|)$.

Only the implication from (a) to (c) is left to establish. Fix any vector v satisfying the inequalities in (3.2). Define a nondecreasing function $\theta_0 : [0,\infty) \to [0,\infty)$ by taking $\theta_0(t)$ to be the maximum of $\langle v,\, x-\bar{x}\rangle$ subject to $x \in C$, $|x-\bar{x}| \le t$, and note that (3.2) ensures that $\theta_0(t)/t \to 0$ as $t{\downarrow}0$. Next define $\theta_1(t) = (1/t)\int_t^{2t}\theta_0(s)ds$ for $t > 0$, $\theta_1(0) = 0$. Since $\theta_0(t) \le \theta_1(t) \le \theta_0(2t)$, the nondecreasing function θ_1 is continuous on $[0,\infty)$ with $\theta_0(t)/t \le \theta_1(t)/t \le 2[\theta_0(2t)/2t]$. Therefore $\theta_1(t)/t \to 0$ as $t{\downarrow}0$. Finally, define $\theta_2(t) = (1/t)\int_t^{2t}\theta_1(s)ds$ for $t > 0$, $\theta_2(0) = 0$. The same reasoning shows that θ_2 is continuous and nondecreasing on $[0,\infty)$ with $\theta_2 \ge \theta_1$ and $\theta_2(t)/t \to 0$ as $t{\downarrow}0$. But because θ_1 was itself continuous, we now have the further property that θ_2 is smooth on $(0,\infty)$; specifically, $\theta_2'(t) = [\theta_1(2t)-\theta_1(t)-\theta_2(t)]/t$ for $t > 0$, and $\theta_2'(t) \to 0$ as $t{\downarrow}0$. In particular $\theta_2 \ge \theta_1$. Now let $f_0(x) = -\langle v,\, x-\bar{x}\rangle + \theta_2(|x-\bar{x}|)$. The function f_0 is well defined and smooth on the open ball of radius 1 around $\bar{x}$, and $\nabla f_0(\bar{x}) = -v$. The construction ensures that $f_0(x) \geq 0$ for all $x \in C$ in this ball, whereas $f_0(\bar{x}) = 0$. Thus f_0 attains a local minimum relative to C at $\bar{x}$, and condition (c) has been verified. $\square$

A fundamental principle emerges from these facts.

THEOREM 3.6. *For any problem in which a smooth function f_0 is to be minimized over a closed set C, the gradient normality condition*

$$-\nabla f_0(\bar{x}) \in N_C(\bar{x}) \tag{3.4}$$

is necessary for local optimality. It is sufficient for global optimality when C is convex and, relative to C, f_0 is convex. The task of developing first-order optimality conditions comes down then in principle to determining formulas for $N_C(\bar{x})$ in various cases of C.

Proof. The necessity is given by the implication from (c) to (a) in Proposition 3.5 (which is entirely elementary, as the proof there shows). The sufficiency in the convex case follows immediately from the characterization of $N_C(\bar{x})$ in Proposition 3.3 and the inequality

$$f_0(x) \geq f_0(\bar{x}) + \langle \nabla f_0(\bar{x}), x - \bar{x} \rangle \text{ for } x \in C, \tag{3.5}$$

which holds whenever f_0 is convex relative to C. $\square$

In writing a normality condition like (3.4) we follow the convention that the symbolism implies $\bar{x} \in C$. This lifts the burden of having always to make the latter explicit. In effect we interpret $N_C(\bar{x})$ as denoting the empty set when $\bar{x} \notin C$.

Although the statement of Theorem 3.6 makes no use of it directly, $-\nabla f_0(\bar{x})$ must be a *regular* normal to C at $\bar{x}$, as the justification shows. In many applications, even when C isn't convex, every normal vector $v \in N_C(\bar{x})$ will be regular anyway. One might be tempted through this to simplify matters by dropping the limit process in the definition of $N_C(\bar{x})$ and restricting the concept of normality in the first place to the vectors v with the property in (3.2). This would work up to a point, but theoretical disadvantages would eventually become serious. In particular, the closedness in Proposition 3.3 would be lacking in general. To maintain this crucial property, often used in technical arguments, the class of sets C under consideration would have to be restricted—often in effect to the kinds of sets for which every normal in the sense of Definition 3.2 is automatically a regular normal. Such sets are common and important (all convex sets are among them by Proposition 3.4), but some of the sets of fundamental interest in optimization fall in tougher categories, for instance graphs and epigraphs of mappings that express the dependence of optimal solutions and optimal values on parameters (cf. §9).

Normality conditions in mode of (3.4) first gained a foothold in convex analysis, cf. [1], and spread from there to nonsmooth analysis in Clarke's framework [16], [17]. Such a condition in terms of the cone of regular normals was given by Hestenes [24].

The beauty of this fundamental kind of first-order optimality condition is that it covers wide territory without forcing more detail on us than we might want to cope with at a particular time. There's no way of knowing in advance of computations just where an optimal solution $\bar{x}$ might be located within C, and it would be cumbersome to have to list all possibilities explicitly every time the issue came up. For instance, from the fact that

$$N_C(\bar{x}) = \{0\} \text{ when } \bar{x} \text{ is an interior point of } C, \tag{3.6}$$

we obtain from the gradient normality condition (3.4) the classical rule that $\nabla f_0(\bar{x}) = 0$ whenever such a point gives a local minimum of f_0 relative to C. But the gradient normality condition applies equally well to situations where $\bar{x}$ is a boundary point of C at which some combination of constraints might be active.

When C is convex, (3.4) can be re-expressed through the characterization of $N_C(\bar{x})$ in Proposition 3.4. In the notation $M(x) = \nabla f_0(x)$ it takes the form

$$\big\langle M(\bar{x}),\, x - \bar{x} \big\rangle \geq 0 \text{ for all } x \in C. \tag{3.7}$$

This condition for any mapping $M : \mathbb{R}^n \to \mathbb{R}^n$ is called the *variational inequality* for M and C. The variational inequality for M and C is thus the relation

$$-M(\bar{x}) \in N_C(\bar{x}) \qquad \text{(with } C \text{ convex).} \tag{3.8}$$

Variational inequalities have taken on great importance in generalizations of partial differential equations to include "inequality type" side conditions. Because partial differential operators can often be interpreted as the gradient mappings of convex functionals on function spaces, the variational inequalities in PDE theory often constitute optimality conditions for such functionals. In such a context $\mathbb{R}^n$ would be replaced by an infinite-dimensional Hilbert space in (3.7).

To fill out the picture of the relationship between tangents and normals, we derive one more fact of nonsmooth geometry. Recall that C is *locally closed* at $\bar{x}$ if its intersection with some neighborhood of $\bar{x}$ is closed.

Definition 3.7. The set C is *Clarke regular* at $\bar{x}$, one of its points, if C is locally closed at $\bar{x}$ and every normal vector to C at $\bar{x}$ is a regular normal vector.

Clarke's original definition of regularity [16], [17], differs on the surface: he requires every tangent vector to be regular in the sense of belonging to the polar of $N_C(\bar{x})$. This means that $\langle v, w\rangle \leq 0$ for all $w \in T_C(\bar{x})$ and $v \in N_C(\bar{x})$, and it is therefore equivalent by Proposition 3.5 to the normal vector property used here.

Proposition 3.8. *Let $\bar{x} \in C$ and suppose C is Clarke regular at $\bar{x}$. Then the tangent and normal cones to C at $\bar{x}$ are convex cones polar to each other:*

$$\begin{aligned} T_C(\bar{x}) &= \big\{\, w \,\big|\, \langle v, w\rangle \leq 0 \textit{ for all } v \in N_C(\bar{x}) \,\big\}, \\ N_C(\bar{x}) &= \big\{\, v \,\big|\, \langle v, w\rangle \leq 0 \textit{ for all } w \in T_C(\bar{x}) \,\big\}. \end{aligned} \tag{3.9}$$

Proof. Because every normal is a regular normal, we know from Proposition 3.5 that the second equation in (3.9) is correct along with the "$\subset$" part of the first equation. Our job is to prove the opposite inclusion. Fixing any $\bar{w} \notin T_C(\bar{x})$, we aim at establishing the existence of a vector $\bar{v} \in N_C(\bar{x})$ with $\langle \bar{v}, \bar{w}\rangle > 0$.

Replacing C by its intersection with some closed ball around $\bar{x}$ if necessary, we can suppose that C is compact. Let B stand for some closed ball around $\bar{w}$ that doesn't meet $T_C(\bar{x})$ (this exists because $T_C(\bar{x})$ is closed). The definition of $T_C(\bar{x})$ implies the existence of a value $\varepsilon > 0$ such that the compact, convex set $S = \big\{\, \bar{x} + tw \,\big|\, w \in B,\ t \in [0, \varepsilon] \,\big\}$ meets C only at $\bar{x}$. For an arbitrary sequence of values $\varepsilon_k \in (0, \varepsilon)$ with $\varepsilon_k \downarrow 0$, consider the compact, convex sets $S^k = \big\{\, \bar{x} + tw \,\big|\, w \in B,\ t \in [\varepsilon_k, \varepsilon] \,\big\}$, which are disjoint from C.

The function $h(x, u) = \frac{1}{2}|x - u|^2$ attains its minimum over $C \times S^k$ at some (x^k, u^k). In particular, x minimizes $h(x, u^k)$ over $x \in C$, so the vector $-\nabla_x h(x^k, u^k) = u^k - x^k$ is a regular normal to C at x^k (Proposition 3.5). Likewise, the vector $-\nabla_u h(x^k, u^k) = x^k - u^k$ is a regular normal to S^k at u^k. Necessarily $x^k \neq u^k$ because $C \cap S^k = \emptyset$, but $x^k \to \bar{x}$ and $u^k \to \bar{x}$, because the sets S^k increase to D (the closure of their union), and $C \cap S = \{\bar{x}\}$.

Let $v^k = (u^k - x^k)/|u^k - x^k|$, so that v^k is a regular normal to C at x^k, while $-v^k$ is a regular normal to S^k at u^k, and $|v^k| = 1$. We can suppose that v^k converges to some $\bar{v}$ with $|\bar{v}| = 1$; then $\bar{v} \in N_C(\bar{x})$ by Definition 3.2. Because $-v_k$ is normal to S^k at u_k and S^k is convex, we have by Proposition 3.4 that $\langle v^k, u - u^k \rangle \geq 0$ for all $u \in S^k$. Since S^k increases to D while $u^k \to \bar{x}$, we obtain in the limit that $\langle \bar{v}, u - \bar{x} \rangle \geq 0$ for all $u \in D$. We can choose u in this inequality to have the form $\bar{x} + \varepsilon w$ for any $w \in B$, where w in turn can be written in terms of the radius δ of the ball B as $\bar{w} + \delta z$ for arbitrary z with $|z| \leq 1$. Hence $\big\langle \bar{v}, \varepsilon(\bar{w} + \delta z) \big\rangle \geq 0$ for all such z. This implies $\langle \bar{v}, \bar{w} \rangle \geq \delta/\varepsilon > 0$. □

Proposition 3.8 applies in particular when C is any closed, convex set, inasmuch as all normals to such sets are regular by Proposition 3.4. The polarity of the tangent and normal cones in that case is a well known fact of convex analysis [1]. We'll see in the next section that the feasible set C to $(\mathcal{P})$ exhibits the same property under a minor assumption, even though it generally isn't convex.

4. Multiplier rule with normal cones. Normal cones are useful in the development and statement of Lagrange multiplier rules for equality and inequality constraints as well as in dealing abstractly with minimization over a set C. For a starter, we show how the multiplier conditions (2.9) in Theorem 2.2 can be written in a very neat manner which ultimately gives the pattern for generalizations to problem formats beyond $(\mathcal{P})$. We focus on the closed, convex sets

$$\begin{aligned} Y &= \mathbb{R}^s_+ \times \mathbb{R}^{m-s} = \big\{\, y = (y_1, \ldots, y_m) \,\big|\, y_i \geq 0 \ \text{ for } \ i = 1, \ldots, s \,\big\}, \\ U &= \big\{\, u = (u_1, \ldots, u_m) \,\big|\, u_i \leq 0 \ \text{ for } \ i = 1, \ldots, s; \ u_i = 0 \ \text{ for } \ i = s+1, \ldots, m \,\big\}, \end{aligned} \tag{4.1}$$

the first constituting the *multiplier space* for problem $(\mathcal{P})$ as identified in Theorem 2.2, and the second allowing the feasible set in $(\mathcal{P})$ to be expressed by

$$C = \big\{\, x \in X \,\big|\, F(x) \in U \,\big\}, \ \text{ where } \ F(x) = \big(f_1(x), \ldots, f_m(x)\big). \tag{4.2}$$

Proposition 4.1. *At any $\bar{y} \in Y$ the normal cone $N_Y(\bar{y})$ consists of all vectors $u = (u_1, \ldots, u_m)$ such that*

$$u_i \begin{cases} \leq 0 & \text{for } i \in \{1, \ldots, s\} \text{ with } \bar{y}_i = 0, \\ = 0 & \text{for } i \in \{i, \ldots, s\} \text{ with } \bar{y}_i > 0 \text{ and for } i \in \{s+1, \ldots, m\}, \end{cases}$$

while at any $\bar{u} \in U$ the normal cone $N_U(\bar{u})$ consists of all $y = (y_1, \ldots, y_m)$ such that

$$y_i \begin{cases} = 0 & \text{for } i \in \{i, \ldots, s\} \text{ with } \bar{u}_i < 0, \\ \geq 0 & \text{for } i \in \{1, \ldots, s\} \text{ with } \bar{u}_i = 0, \\ \text{unrestricted} & \text{for } i \in \{s+1, \ldots, m\}. \end{cases}$$

Thus, $\bar{y} \in N_U(\bar{u})$ if and only if $\bar{u} \in N_Y(\bar{y})$, and condition (2.9) in Theorem 2.2 can be written either as $\bar{y} \in N_U\big(F(\bar{x})\big)$, where $F(\bar{x}) = \big(f_1(\bar{x}), \ldots, f_m(\bar{x})\big)$, or as

$$\nabla_y L(\bar{x}, \bar{y}) \in N_Y(\bar{y}). \tag{4.3}$$

This view of the multiplier conditions in Theorem 2.2 holds a surprise. Because Y is convex and $L(x, y)$ is affine in y, the normality relation (4.3) indicates through Proposition 3.4 that $L(\bar{x}, y) \leq L(\bar{x}, \bar{y})$ for all $y \in Y$. It refers therefore to a global *maximum* of the Lagrangian in the y argument, even though the original problem in

x was one of minimization. But this ties into notions of game theory promoted by von Neumann, as we'll see in §5.

Our discussion turns now to modern developments of multiplier rules which improve on Theorem 2.2. Ever since inequality constraints attracted major attention in optimization, the geometry of one-sided tangents and normals to a feasible set C has been studied for this purpose in one way or another. Arguments exclusively in terms of normal vectors will be offered in Theorem 4.2, but for many years the customary approach to deriving first-order optimality conditions when C is the feasible set in the standard problem $(\mathcal{P})$ has been through tangent vectors, essentially by way of the implication from (c) to (b) in Proposition 3.4. If $\bar{x}$ is locally optimal, $-\nabla f_0(\bar{x})$ must be one of the vectors v in (b), so the task has been seen as that of determining how the vectors v obtained from the condition in (b) may be represented.

It's worth tracing how this older route goes, even though we'll bypass it here. In the pioneering work of Kuhn and Tucker [9] and that of their later discovered predecessor Karush [10], the tangent vectors studied at a point $\bar{x} \in C$ were the vectors of the form $w = \dot{x}(0)$ corresponding to smooth arcs $x : (-\varepsilon, \varepsilon) \mapsto C$ such that $x(t) \in C$ for $t \geq 0$, $x(0) = \bar{x}$. Such vectors belong to the cone $T_C(\bar{x})$ as we've defined it, and often—but not always—describe it completely.

From the description of $T_C(\bar{x})$ in Definition 3.1 and the expansions

$$f_i(\bar{x} + t_k w^k) = f_i(\bar{x}) + t_k \big\langle \nabla f_i(\bar{x}), w^k x \big\rangle + o\big(t_k |w^k|\,\big) \text{ for all } x$$

it's easy to deduce that

$$w \in T_C(\bar{x}) \quad \Longrightarrow \quad \begin{cases} w \in T_X(\bar{x}), \\ \big\langle \nabla f_i(\bar{x}), w \big\rangle \leq 0 & \text{for } i \in \{1, \ldots, s\} \text{ with } f_i(\bar{x}) = 0, \\ \big\langle \nabla f_i(\bar{x}), w \big\rangle = 0 & \text{for } i \in \{s+1, \ldots, m\}. \end{cases} \tag{4.4}$$

Let $K(\bar{x})$ stand for the set of w describes on the right side of this implication; we have $T_C(\bar{x}) \subset K(\bar{x})$ in this notation. In circumstances where $\bar{x}$ is locally optimal and $T_C(\bar{x}) = K(\bar{x})$, $-\nabla f_0(\bar{x})$ must be one of the vectors v such that $\langle v, w \rangle \leq 0$ for all $w \in K(\bar{x})$. The latter condition can be translated as follows into a special representation for v. First, when $X = \mathbb{R}^n$ (so that $T_X(\bar{x}) = \mathbb{R}^n$), the Farkas Lemma [28] for linear inequalities states that the vectors v satisfying $\langle v, w \rangle \leq 0$ for every $w \in K(\bar{x})$ are precisely the ones that can be represented in the form $v = y_1 \nabla f_1(\bar{x}) + \cdots + y_m \nabla f_m(\bar{x})$ with coefficients y_i satisfying

$$y_i \begin{cases} = 0 & \text{for } i \in \{1, \ldots, s\} \text{ with } f_i(\bar{x}) < 0, \\ \geq 0 & \text{for } i \in \{1, \ldots, s\} \text{ with } f_i(\bar{x}) = 0, \\ \text{unrestricted} & \text{for } i \in \{s+1, \ldots, m\}. \end{cases} \tag{4.5}$$

This result can be extended to situations where X might not be all of $\mathbb{R}^n$ but at least is a *polyhedral* set, meaning that it's a convex set expressible as the points satisfying some finite system of linear constraints. (In particular, $\mathbb{R}^n$ is considered to be a polyhedral set.) By passing temporarily through an explicit introduction of such linear constraints, one is able to deduce that the vectors v in question can be written in the form

$$v = y_1 \nabla f_1(\bar{x}) + \cdots + y_m \nabla f_m(\bar{x}) + z$$

for some $z \in N_X(\bar{x})$, again with coefficients y_i satisfying (4.5).

It follows by this form of reasoning—in the case where the implication in (4.4) is an equivalence and X is polyhedral—that a necessary condition for the local optimality of $\bar{x}$ in problem $(\mathcal{P})$ is the existence of coefficients $\bar{y}_i$ satisfying (4.5) such that

$$\nabla f_0(\bar{x}) + \bar{y}_1 \nabla f_1(\bar{x}) + \cdots + \bar{y}_m \nabla f_m(\bar{x}) + \bar{z} = 0 \text{ for some } \bar{z} \in N_X(\bar{x}).$$

This relation reduces to $\nabla f_0(\bar{x}) + \bar{y}_1 \nabla f_1(\bar{x}) + \cdots + \bar{y}_m \nabla f_m(\bar{x}) = 0$ if $\bar{x}$ is an interior point of X, as when $X = \mathbb{R}^n$; cf. (3.6). In general it says $-\nabla_x L(\bar{x}, \bar{y}) \in N_X(\bar{x})$, while condition (4.5) on the vector $\bar{y} = (\bar{y}_1, \ldots, \bar{y}_m)$ can be expressed through Proposition 4.1 either as $\bar{y} \in N_U\big(F(\bar{x})\big)$ or as $\nabla_y L(\bar{x}, \bar{y}) \in N_Y(\bar{y})$.

The use of normal cones in expressing such necessary conditions for optimality is relatively new and hasn't yet filtered down to the textbook level. Anyway, textbooks in optimization have treated only the case of this derivation where $X = \mathbb{R}^n$, preferring to make explicit any linear constraints involved in the specification of X. Yet Kuhn and Tucker in their paper [9] allowed for abstract handling of nonnegativity constraints on certain of the components of x, in effect taking

$$X = \mathbb{R}^r_+ \times \mathbb{R}^{n-r} = \big\{\, x = (x_1, \ldots, x_n) \,\big|\, x_i \geq 0 \text{ for } i = 1, \ldots, r \,\big\}. \tag{4.6}$$

The relation $-\nabla_x L(\bar{x}, \bar{y}) \in N_X(\bar{x})$ in this case summarizes their conditions

$$\frac{\partial L}{\partial x_j}(\bar{x}, \bar{y}) \begin{cases} = 0 & \text{for } j \in [1, r] \text{ with } \bar{x}_j > 0, \text{ and for } j \in [r+1, n], \\ \geq 0 & \text{for } j \in [1, r] \text{ with } \bar{x}_j = 0, \end{cases} \tag{4.7}$$

which are pleasingly parallel to the form taken by the conditions on the components of $\nabla_y L(\bar{x}, \bar{y})$ and $\bar{y}$ in (2.9). Requirements of the form (4.7) or (2.9) are known as *complementary slackness* conditions, because they forbid two linked inequalities to be "slack" (i.e., to hold with strict inequality) at the same time. For instance, in (4.7) at least one of the two inequalities $(\partial L / \partial x_1)(\bar{x}, \bar{y}) \geq 0$ and $\bar{x}_1 \geq 0$ must hold as an equation.

As already underlined, this customary approach to deriving optimality conditions depends on equivalence reigning in the tangent cone implication (4.4). Equivalence can be established in various cases by demonstrating that the closure of some subcone of $T_C(\bar{x})$ (such as the one utilized by Kuhn and Tucker [9] and Karush [10]) contains all the vectors w on the right side of (4.4). Much effort has gone into finding convenient assumptions, called *constraint qualifications*, that guarantee such a property. As Kuhn and Tucker were well aware, no extra assumption is needed at all when every constraint is linear. For nonlinear constraints, though, a constraint qualification can't be dispensed with.

A popular constraint qualification is one stated by Mangasarian and Fromovitz [29] for the case of $(\mathcal{P})$ with no abstract constraint: the gradients $\nabla f_i(\bar{x})$ of the *equality* constraints should be linearly independent, and there should be a vector w such that $\big\langle \nabla f_i(\bar{x}), w \big\rangle = 0$ for all the equality constraints, but $\big\langle \nabla f_i(\bar{x}), w \big\rangle < 0$ for all the *active inequality* constraints. In particular this condition validates the conclusions in Theorem 2.2 without requiring the linear independence of the gradients of the active inequality constraints together with the gradients of the equality constraints. In the case of inequality constraints only, the same constraint qualification was used by Karush [10]; the Mangasarian-Fromovitz condition is what one gets when independent equality constraints are added and then linearized through a change of variables based on the implicit function theorem.

The well trod approach to optimality through tangent vectors has served adequately for many purposes, but it's unnecessarily complicated and suffers from technical and conceptual limitations which increasingly cause trouble as the theory reaches out to embrace problem structures associated with infinite-dimensional applications. The same optimality conditions, and more, can be obtained without such machinery, without assuming that the set X is polyhedral, and incidentally without relying anywhere on the implicit mapping theorem.

We'll say the *basic constraint qualification* in $(\mathcal{P})$ is satisfied at a feasible solution $\bar{x}$ when

$$(\mathcal{Q}) \qquad \begin{cases} \text{there is no vector } y \neq 0 \text{ satisfying (4.5) such that} \\ -\big[\, y_1 \nabla f_1(\bar{x}) + \cdots + y_m \nabla f_m(\bar{x}) \,\big] \in N_X(\bar{x}). \end{cases}$$

This assumption is close in spirit to the linear independence condition required by the classical methodology behind Theorem 2.2, to which it reduces when $X = \mathbb{R}^n$ and only equality constraints are involved. With inequality as well as equality constraints present (but still $X = \mathbb{R}^n$), condition $(\mathcal{Q})$ would stipulate that the only linear combination $y_1 \nabla f_1(\bar{x}) + \cdots + y_m \nabla f_m(\bar{x}) = 0$ in which the coefficients y_i for active inequality constraints are nonnegative, while those for inactive inequality constraints are 0, is the one in which *all* coefficients are 0.

The latter property is known to be equivalent to the Mangasarian-Fromovitz constraint qualification by way of the Farkas Lemma. It's the constraint qualification employed in 1948 by John [13], although John adopted a different form of presentation, which will be described after the proof of the theorem we're leading up to. The basic constraint qualification $(\mathcal{Q})$ can be regarded therefore as the natural extension of the Mangasarian-Fromovitz condition and the John condition to cover a general abstract constraint $x \in X$.

Like the gradient condition in $(\mathcal{Q})$, the restriction (4.5) on y in $(\mathcal{Q})$ can be written in normal cone form, namely $y \in N_U\big(F(\bar{x})\big)$ for the closed, convex set U in (4.1), cf. Proposition 4.1. In the case where X is Clarke regular at $\bar{x}$, as when X is a closed, convex set, it would be possible to use the polarity relationship in Proposition 3.8 to translate $(\mathcal{Q})$ into an equivalent condition on the tangent cones $T_X(\bar{x})$ and $T_U\big(F(\bar{x})\big)$. But for general X that approach fails, and *only* a normal vector condition is effective.

Notice that if $(\mathcal{Q})$ is satisfied at $\bar{x}$, it must be satisfied at every $x \in C$ in some neighborhood of $\bar{x}$. Otherwise there would be a sequence of points $x^k \to \bar{x}$ in C and nonzero vectors $y^k \in N_U\big(F(x^k)\big)$ with $-\sum_{i=1}^m y_i^k \nabla f_i(x^k) \in N_X(x^k)$. These vectors y^k could be normalized to length 1 without affecting the property in question, and because of the smoothness of the f_i's any cluster point $\bar{y}$ of the sequence $\{y^k\}_{k=1}^\infty$ would then present a violation of $(\mathcal{Q})$ (because of the closedness of normality relations in Proposition 3.3). The basic constraint qualification is thus a "stable" kind of condition.

Theorem 4.2. *If $\bar{x} \in X$ is a locally optimal solution to $(\mathcal{P})$ at which the basic constraint qualification $(\mathcal{Q})$ is satisfied, there must exist a vector $\bar{y} \in Y$ such that*

$$(\mathcal{L}) \qquad -\nabla_x L(\bar{x}, \bar{y}) \in N_X(\bar{x}), \qquad \nabla_y L(\bar{x}, \bar{y}) \in N_Y(\bar{y}).$$

Proof. Local optimality means that for some compact neighborhood V of $\bar{x}$, we have $f_0(x) \geq f_0(\bar{x})$ for all $x \in C \cap V$. Replacing X by $\widetilde{X} = X \cap V$ if necessary, we can suppose without loss of generality that $\bar{x}$ is globally optimal in $(\mathcal{P})$, and X is compact. Then by replacing $f_0(x)$ by $\widetilde{f}_0(x) = f_0(x) + \varepsilon |x - \bar{x}|^2$ if necessary, where

$\nabla\widetilde{f}_0(\bar{x}) = \nabla f_0(\bar{x})$, we can ensure further without loss of generality that $\bar{x}$ is the *only* optimal solution to $(\mathcal{P})$.

With U continuing to denote the set in (4.1) and $F(x) = \big(f_1(x), \dots, f_m(x)\big)$, observe that the optimization problem

$$(\widehat{\mathcal{P}}) \qquad \text{minimize } \widehat{f}(x,u) = f_0(x) \text{ over all } (x,u) \in X \times U \text{ such that } F(x) - u = 0$$

then has $(\bar{x}, \bar{u}) = \big(\bar{x}, F(\bar{x})\big)$ as its unique optimal solution. For a sequence of values $\varepsilon_k \downarrow 0$, consider the penalty approximations

$$(\widehat{\mathcal{P}}^k) \qquad \text{minimize } \widehat{f}^k(x,u) = f_0(x) + \frac{1}{2\varepsilon_k}\big|F(x) - u\big|^2 \text{ over all } (x,u) \in X \times U.$$

Because $(\bar{x}, \bar{u})$ is a feasible solution to $(\widehat{\mathcal{P}}^k)$, the closed set

$$S^k = \big\{\, (x,u) \in X \times U \;\big|\; \widehat{f}^k(x,u) \leq \widehat{f}^k(\bar{x},\bar{u}) = f_0(\bar{x}) \,\big\}$$

is nonempty, and $(\widehat{\mathcal{P}}^k)$ is equivalent to minimizing $\widehat{f}^k$ over S^k. Let μ be the minimum value of f_0 over the compact set X. Then

$$S^k \subset \Big\{\, (x,u) \in X \times U \;\Big|\; |F(x) - u| \leq 2\varepsilon_k\big[f_0(\bar{x}) - \mu\big] \,\Big\}. \tag{4.8}$$

The boundedness of X implies through this that S^k is bounded, hence compact, so the minimum of $\widehat{f}^k$ over S^k is attained. Thus, $(\widehat{\mathcal{P}}^k)$ has an optimal solution.

Denoting an optimal solution to $(\widehat{\mathcal{P}}^k)$, not necessarily unique, by (x^k, u^k) for each k, we have from (4.8) that the sequence $\big\{(x^k,u^k)\big\}_{k=1}^{\infty}$ in $X \times U$ is bounded with

$$|F(x^k) - u^k| \leq 2\varepsilon_k\big[f_0(\bar{x}) - \mu\big], \qquad f_0(x^k) \leq \widehat{f}^k(x^k,u^k) \leq f_0(\bar{x}).$$

Any cluster point of this sequence is therefore a point $(\tilde{x}, \tilde{u}) \in X \times U$ satisfying $|F(\tilde{x}) - u| = 0$ and $f_0(\tilde{x}) \leq f_0(\bar{x})$. But the only such point is $(\bar{x}, \bar{u})$, the unique optimal solution to $(\widehat{\mathcal{P}})$. Therefore, $x^k \to \bar{x}$ and $u^k \to \bar{u}$.

The optimality of (x^k, u^k) in $(\widehat{\mathcal{P}}^k)$ implies that x^k minimizes $\widehat{f}^k(x, u^k)$ relative to $x \in X$, and u^k minimizes $\widehat{f}^k(x^k, u)$ relative to $u \in U$. Therefore by Theorem 3.6,

$$-\nabla_x \widehat{f}^k(x^k,u^k) \in N_X(x^k), \qquad -\nabla_u \widehat{f}^k(x^k,u^k) \in N_U(u^k). \tag{4.9}$$

Set $y^k = -\nabla_u \widehat{f}^k(x^k,u^k)$, so that $y_i^k = \big[f_i(x^k) - u_i^k\big]/\varepsilon_k$ and $\nabla_x \widehat{f}^k(x^k,u^k) = \nabla f_0(x^k) + y_1^k \nabla f_1(x^k) + \cdots + y_m^k \nabla f_m(x^k) = \nabla_x L(x^k, y^k)$. The normality conditions (4.9) then take the form

$$-\nabla_x L(x^k, y^k) \in N_X(x^k), \qquad y^k \in N_U(u^k). \tag{4.10}$$

We distinguish now between two cases: the sequence of vectors y^k is bounded or it's unbounded. If the sequence of vectors y^k is bounded, we can suppose it converges to some $\bar{y}$. Then in the limit in (4.10) we obtain through the closedness property in Proposition 3.3 that $-\nabla_x L(\bar{x}, \bar{y}) \in N_X(\bar{x})$ and $\bar{y} \in N_U(\bar{u}) = N_U\big(F(\bar{x})\big)$. The second of these normality conditions can also be written as $\nabla_y L(\bar{x}, \bar{y}) \in N_Y(\bar{y})$ according to Proposition 4.1, so we've arrived at condition $(\mathcal{L})$.

If the sequence of vectors y^k is unbounded, we can suppose (by passing to a subsequence if necessary) that $0 < |y^k| \to \infty$ and that the vectors $\bar{y}^k = y^k/|y^k|$

converge to some $\bar{y} \neq 0$. Dividing the normals in (4.10) by $|y^k|$ (they're still normals after rescaling), we get $-\big(1/|y^k|\big)\nabla f_0(x^k) - \sum_{i=1}^m \bar{y}_i^k \nabla f_i(x^k) \in N_X(x^k)$ with $\bar{y}^k \in N_U(u^k)$. In the limit this yields $-\sum_{i=1}^m \bar{y}_i \nabla f_i(\bar{x}) \in N_X(\bar{x})$ with $\bar{y} \in N_U\big(F(\bar{x})\big)$, so assumption $(\mathcal{Q})$ is violated. The unbounded case is therefore impossible. □

This theorem could have been formulated in the pattern of John [13], who was the first to prove the version in which no abstract constraint is involved. That would have meant modifying the definition of $L(x, y)$ to have a coefficient $y_0 \geq 0$ for f_0. Then $Y = \mathbb{R}_+^{s+1} \times \mathbb{R}^{m-s}$; the assertion would be the existence of a *nonzero* multiplier vector in this higher-dimensional set Y, with no mention of a constraint qualification. Of course, the conclusion in this case can be broken down into two cases, where $\bar{y}_0 > 0$ or $\bar{y}_0 = 0$, where in the first case a rescaling can make $\bar{y}_0 = 1$. That case corresponds to the conclusion in Theorem 4.2, while the other corresponds to a violation of $(\mathcal{Q})$.

John's way of formulating Lagrange multiplier rules is popular in some branches of optimization, especially optimal control, but it has distinct disadvantages. It interferes with comparisons with other approaches where the conclusion may be the same but the constraint qualification is different. Most seriously, it clashes with saddle point expressions of optimality such as will be seen in §§5 and 6.

The proof technique for Theorem 4.2, relying on a sequence of penalty approximations, resembles that of McShane [30] (see also Beltrami [31] for a similar idea). In McShane's approach, directed only to the case of $X = \mathbb{R}^n$, the approximate problem corresponding to $(\widehat{\mathcal{P}}^k)$ consists instead of minimizing $f_0(x) + (1/2\varepsilon_k) d_U\big(F(x)\big)^2$ in x, where $d_U\big(F(x)\big)$ is the distance of $F(x)$ from U. The approach here, where minimization is set up in two vectors x and u, promotes the normal cone viewpoint, thereby effecting a generalization to an arbitrary closed set X and, as already suggested, promoting connections with saddle point properties as well as extensions to problem formulations more flexible than $(\mathcal{P})$.

Although we've spoken of developing first-order optimality conditions through formulas for the normal cones $N_C(\bar{x})$ to various sets C, the proof of Theorem 4.2 appears to have taken a shortcut. But in fact, Theorem 4.2 provides such a formula for the special case of the feasible set C in problem $(\mathcal{P})$, which we state next. Theorem 4.2 follows in turn from this formula by Theorem 3.6.

THEOREM 4.3. *Let C be the feasible set in $(\mathcal{P})$, and let $\bar{x}$ be a point of C where the basic constraint qualification $(\mathcal{Q})$ is satisfied. Then*

$$N_C(\bar{x}) \subset \big\{\, v = y_1 \nabla f_1(\bar{x}) + \cdots + y_m \nabla f_m(\bar{x}) + z \;\big|\; y \in N_U\big(F(\bar{x})\big),\ z \in N_X(\bar{x}) \,\big\}. \quad (4.11)$$

If X is Clarke regular at $\bar{x}$ (as when X is convex), then C is Clarke regular at $\bar{x}$ and the inclusion is an equation.

Proof. First consider a regular normal vector $v \in N_C(\bar{x})$. By Proposition 3.5 there is a smooth function f_0 on a neighborhood of $\bar{x}$ such that f_0 has a local minimum relative to C at $\bar{x}$, and $-\nabla f_0(\bar{x}) = v$. Applying Theorem 4.2 we get a vector $\bar{y} \in Y$ with $-\nabla_x L(\bar{x}, \bar{y}) \in N_X(\bar{x})$ and $\nabla_y L(\bar{x}, \bar{y}) \in N_Y(\bar{y})$, and these conditions can be written as $v - \sum_{i=1}^m \nabla f_i(\bar{x}) \in N_X(\bar{x})$ and $F(\bar{x}) \in N_Y(\bar{y})$, the latter being the same as $\bar{y} \in N_U\big(F(\bar{x})\big)$ by Proposition 4.1. Thus, v belongs to the set on the right in (4.11).

Next consider a general normal $v \in N_C(\bar{x})$. By definition there exist sequences $x^k \to \bar{x}$ in C and $v^k \to v$ with v^k a regular normal in $N_C(x^k)$. Since the basic constraint qualification $(\mathcal{Q})$ is satisfied at $\bar{x}$, it's also satisfied at x^k once x^k is in a certain neighborhood of $\bar{x}$ (recall the comment preceding Theorem 4.2). The argument already given then provides vectors $y^k \in N_U\big(F(x^k)\big)$ and $z^k \in N_X(x^k)$ with

$$v^k = y_1^k \nabla f_1(x^k) + \cdots + y_m^k \nabla f_m(x^k) + z^k.$$

If the sequence of vectors y^k has a cluster point $\bar{y}$, then the sequence of vectors z^k has a corresponding cluster point $\bar{z}$ with

$$v = \bar{y}_1 \nabla f_1(\bar{x}) + \cdots + \bar{y}_m \nabla f_m(\bar{x}) + \bar{z},$$

and we have $N_U\big(F(\bar{x})\big)$ and $\bar{z} \in N_X(\bar{x})$ because of the closedness property in Proposition 3.3. In this case, therefore, v again belongs to the set on the right in (4.11).

On the other hand, if the sequence of vectors y^k has no cluster point, we have $|y^k| \to \infty$. Consider then a cluster point $\bar{y}$ of the normalized vectors $\bar{y}^k = y^k/|y^k|$, which still lie in $N_U\big(F(x^k)\big)$; we have $\bar{y} \in N_U\big(F(\bar{x})$ and $|\bar{y}| = 1$. Since

$$\big(1/|y^k|\big) v^k = \bar{y}_1^k \nabla f_1(x^k) + \cdots + \bar{y}_m^k \nabla f_m(x^k) + \big(1/|y^k|\big) z^k,$$

the vectors $\bar{z}^k = \big(1/|y^k|\big) z^k \in N_X(x^k)$ have a cluster point $\bar{z} \in N_X(\bar{x})$ with

$$0 = \bar{y}_1 \nabla f_1(\bar{x}) + \cdots + \bar{y}_m \nabla f_m(\bar{x}) + \bar{z}.$$

But this is impossible under assumption $(\mathcal{Q})$. The general inclusion in (4.11) is thereby established.

Suppose finally that X is Clarke regular at $\bar{x}$, i.e., that every normal $z \in N_X(\bar{x})$ is a regular normal. Let v be a vector represented as on the right in (4.11), and for the coefficient vector $y \in N_U\big(F(\bar{x})\big)$ in question let $h(x) = \sum_{i=1}^m y_i \nabla f_i(x)$, so that (from the sign conditions represented by $y \in N_U\big(F(\bar{x})\big)$, cf. Proposition 4.1) $h(x) \le 0 = h(\bar{x})$ for all $x \in C$, and the vector $v - \nabla h(\bar{x}) = z$ is a regular normal to X at $\bar{x}$. Then $\big\langle v - \nabla h(\bar{x}),\, x - \bar{x} \big\rangle \le o\big(|x - \bar{x}|\big)$ for $x \in X$, so that

$$\begin{aligned} \big\langle v,\, x - \bar{x} \big\rangle &\le \big\langle \nabla h(\bar{x}),\, x - \bar{x} \big\rangle + o\big(|x - \bar{x}|\big) \le h(x) - h(\bar{x}) + o\big(|x - \bar{x}|\big) \\ &\le o\big(|x - \bar{x}|\big) \ \text{ for all } \ x \in X. \end{aligned}$$

This tells us that v is a regular normal to C at $\bar{x}$. The inclusion in (4.11) thus turns into an equation in which all the vectors v are regular normals, and consequently C is Clarke regular at $\bar{x}$. □

Because of the importance of linear constraints in many applications, the following variant of Theorems 4.2 and 4.3 deserves attention.

THEOREM 4.4. *The conclusions of Theorem 4.2 remain valid when the basic constraint qualification* $(\mathcal{Q})$ *at* $\bar{x}$ *is replaced by*

$$(\mathcal{Q}') \qquad \begin{cases} X \ \textit{is polyhedral, and the only vectors } y \neq 0 \ \textit{satisfying (4.5) and} \\ \qquad - \big[\, y_1 \nabla f_1(\bar{x}) + \cdots + y_m \nabla f_m(\bar{x}) \,\big] \in N_X(\bar{x}), \\ \textit{if any, have } y_i = 0 \ \textit{for each index } i \ \textit{such that } f_i \ \textit{is not affine.} \end{cases}$$

Likewise, Theorem 4.3 remains valid with $(\mathcal{Q}')$ *substituted for* $(\mathcal{Q})$*, and in this case* C *is Clarke regular at* $\bar{x}$ *and the inclusion holds as an equation.*

Proof. It's enough to concern ourselves with Theorem 4.3, since the extension of this result implies the extension of Theorem 4.2 through the normal cone principle in Theorem 3.6. We'll use a bootstrap approach, reducing the general case by stages to the one already covered. We begin with the case where $X = \mathbb{R}^n$ and every f_i is affine. Because the gradient $\nabla f_i(x)$ is the same for all x, we write it for now as a_i. As a harmless notational simplification in this context we suppose that *all* the inequality constraints are active at $\bar{x}$, and that the vectors $a_1, \ldots, a_m$ span $\mathbb{R}^n$. (Otherwise we could translate the argument to the subspace that they span, identifying it with $\mathbb{R}^{n'}$ for some $n' < n$.) The task then is to demonstrate that $N_C(\bar{x})$ coincides with the cone

$$K = \{\, y_1 a_1 + \cdots + y_m a_m \,\big|\, y \in Y \,\}, \tag{4.12}$$

where Y continues to be the cone in (4.1).

Because C is polyhedral in this initial context, hence convex, we know from Proposition 3.4 that the vectors $v \in N_C(\bar{x})$ are the ones satisfying $\langle v,\, x - \bar{x}\rangle \leq 0$ for all $x \in C$. It's elementary that every vector $v \in K$ has this property, since

$$\big\langle a_i,\, x - \bar{x}\big\rangle = f_i(x) - f_i(\bar{x}) = f_i(x) \begin{cases} \leq 0 & \text{for } i = 1, \ldots, s, \\ = 0 & \text{for } i = s+1, \ldots, m, \end{cases}$$

when $x \in C$. Thus, $N_C(\bar{k}) \supset K$.

To obtain the opposite inclusion, we must first establish that K is closed. For each index set $I \subset \{1, \ldots, m\}$ such that the vectors a_i for $i \in I$ form a basis for $\mathbb{R}^n$, let K_I denote the subcone of K defined as in (4.12) but with $y_i = 0$ for all $i \notin I$. Obviously K_I is closed, since it corresponds in the coordinate system with basis $\{a_i\}_{i \in I}$ merely to a nonnegativity requirement on certain coordinates (the coordinates are the y_i values entering into the representations that define K_I). By proving that every $v \in K$ belongs to some K_I, we'll be able to conclude that K is closed, because there are only finitely many cones K_I, and the union of finitely many closed sets is closed.

Actually, we need only show that every $v \in K$ can be represented as in (4.12) with the set $A_y = \{\, a_i \,|\, y_i \neq 0 \,\}$ linearly independent, since other a_i's with coefficients $y_i = 0$ can always be thrown in from the spanning set $\{a_1, \ldots, a_m\}$ to form a basis. Suppose v has a representation as in (4.12) with the set A_y not linearly independent: there are coefficients η_i for $i \in A_v$, not all zero, such that $\sum_{i \in A_v} \eta_i a_i = 0$. For any value of $t \in \mathbb{R}$ we'll have another representation

$$v = y_1' a_1 + \cdots + y_m' a_m \ \text{ with } \ y_i' = y_i - t\eta_i.$$

It's possible to choose t in such a manner that $t\eta_i = y_i$ for at least one i with $y_i \neq 0$, but $t\eta_i \leq y_i$ for all $i \in \{1, \ldots, s\}$. Then $y' \in Y$ and $A_{y'} \subset A_y$, $A_{y'} \neq A_y$. If the vectors in $A_{y'}$ aren't linearly independent, the procedure can be continued a step further to get a representation in terms of a coefficient vector $y'' \in Y$ with $A_{y''} \subset A_{y'}$, $A_{y''} \neq A_{y'}$, and so on. Eventually a representation with linear independence will be achieved.

Having proved that K is closed, we now consider a vector $\hat{v} \notin K$ and demonstrate that $\hat{v} \notin N_C(\bar{x})$, thereby obtaining the desired relation $N_C(\bar{x}) = K$. Because K is closed, the problem of minimizing $\varphi(v) = \frac{1}{2}|v - \hat{v}|^2$ over $v \in K$ has an optimal solution $\bar{v}$. Theorem 3.6 characterizes this by the condition $-\nabla\varphi(\bar{v}) \in N_K(\bar{v})$. The cone K is convex, so this means by Proposition 3.4 that the vector $w = -\nabla\varphi(\bar{v}) = \hat{v} - \bar{v} \neq 0$ satisfies $\langle w,\, v - \hat{v}\rangle \leq 0$ for all $v \in K$. Hence $\langle w, \bar{v}\rangle \geq 0$, inasmuch as $0 \in K$. Since the vector $v = \bar{v} + z$ belongs to K for every $z \in K$, we must actually have $\langle w, z\rangle \leq 0$ for all $z \in K$, hence in particular

$$\langle w, a_i\rangle \begin{cases} \leq 0 & \text{for } i = 1, \ldots, s, \\ = 0 & \text{for } i = s+1, \ldots, m, \end{cases}$$
$$\langle w, \bar{v}\rangle \leq 0, \ \text{ so that } \ \langle w, \bar{v}\rangle = 0.$$

It follows that

$$f_i(\bar{x}+w)-f_i(\bar{x})\begin{cases}\leq 0 & \text{for } i=1,\dots,s,\\ =0 & \text{for } i=s+1,\dots,m,\end{cases}$$

so the point $\hat{x}=\bar{x}+w$ belongs to C, and yet $\langle \hat{x}-\bar{x},\hat{v}\rangle=\langle w,\hat{v}\rangle=\langle w,\,\hat{v}-\bar{v}\rangle=|w|^2>0$. Then $\hat{v}\notin N_C(\bar{x})$, because the vectors $v\in N_C(\bar{x})$ have to satisfy $\langle v,\,x-\bar{x}\rangle\leq 0$ for all $x\in C$ by Proposition 3.4.

So far we've proved that the variant of Theorem 4.3 with $(\mathcal{Q})$ replaced by $(\mathcal{Q}')$ is correct when $X=\mathbb{R}^n$ and every constraint function f_i is affine. Next we allow X to be any polyhedral set, keeping the f_i's affine. The polyhedral property ensures a representation of X as the set of points satisfying a system of constraints

$$f'_j(x)\begin{cases}\leq 0 & \text{for } j=1,\dots,s',\\ =0 & \text{for } j=s'+1,\dots,m',\end{cases}$$

where every function f'_j is affine. From what we've already established, as applied to X instead of C, the normal cone $N_X(\bar{x})$ consists of the vectors of the form

$$\begin{aligned} &z=y'_1\nabla f'_1(\bar{x})+\cdots+y'_{m'}\nabla f'_{m'}(\bar{x}) \text{ where}\\ &y'_j\begin{cases}=0 & \text{for } j\in\{1,\dots,s'\} \text{ with } f'_j(\bar{x})<0,\\ \geq 0 & \text{for } j\in\{1,\dots,s'\} \text{ with } f'_j(\bar{x})=0.\end{cases}\end{aligned} \tag{4.13}$$

On the other hand, we can think of C as specified by the f_i and f'_j constraint systems combined and thereby deduce from the same preliminary result that the normal cone $N_C(\bar{x})$ consists of all vectors of the form

$$v=y_1\nabla f_1(\bar{x})+\cdots+y_m\nabla f_m(\bar{x})+y'_1\nabla f'_1(\bar{x})+\cdots+y'_{m'}\nabla f'_{m'}(\bar{x})$$

in which the coefficients y'_j satisfy the requirements in (4.13) while the coefficients y_i satisfy the earlier ones in (4.5). Obviously in this way we confirm that the vectors in $N_C(\bar{x})$ are in this case characterized by the representation in Theorem 4.3.

We're ready now for the general case, where X is polyhedral but the functions f_i aren't necessarily affine. For simplicity we can take the notation to be such that the nonaffine inequality constraint functions are $f_1,\dots,f_{s^*}$, whereas the nonaffine equality constraint functions are $f_{s+1},\dots,f_{m^*}$. We suppress the affine constraint functions temporarily by introducing in place of X the polyhedral set X' comprised of the points $x\in X$ satisfying the linear constraints $f_i(x)\leq 0$ for $i=s^*+1,\dots,s$ and $f_i(x)=0$ for $i=m^*+1,\dots,m$. This allows us to think of C as the set of points $x\in X'$ satisfying the nonlinear constraints among the ones originally given. Condition $(\mathcal{Q}')$ is precisely the basic constraint qualification for this description of C, because the normal cone $N_{X'}(\bar{x})$ is now known to consist of the vectors of the form

$$z'=y_{s^*+1}\nabla f_{s^*+1}(\bar{x})+\cdots+y_s f_s(\bar{x})+y_{m^*+1}\nabla f_{m^*+1}(\bar{x})+\cdots+y_m f_m(\bar{x})+z$$

with $z\in N_X(\bar{x})$ and coefficients $y_i=0$ for $i\in\{s^*+1,\dots,s\}$ with $f_i(\bar{x})<0$, but $y_i\geq 0$ for $i\in\{s^*+1,\dots,s\}$ with $f_i(\bar{x})=0$. On the strength of $(\mathcal{Q})$ we may conclude from the original version of Theorem 4.3 that the vectors in $N_C(\bar{x})$ are the ones of the form

$$v=y_1\nabla f_1(\bar{x})+\cdots+y_{s^*}f_{s^*}(\bar{x})+y_{s+1}\nabla f_{s+1}(\bar{x})+\cdots+y_{m^*}f_{m^*}(\bar{x})+z'$$

with $z' \in N_{X'}(\bar{x})$ and coefficients $y_i = 0$ for $i \in \{1, \ldots, s^*\}$ having $f_i(\bar{x}) < 0$, but $y_i \geq 0$ for $i \in \{1, \ldots, s^*\}$ having $f_i(\bar{x}) = 0$. This representation along with the one for $N_{X'}(\bar{x})$ yields the full result we wanted to obtain. □

The proof of Theorem 4.4, although lengthy, uses no more than elementary linear algebra in conjunction with the results previously obtained. In particular, the Farkas Lemma wasn't invoked as such. But the first part of the proof establishes, in effect as an extension of the Farkas Lemma, the fact that the normal cone representation in Theorem 4.3 is always valid in the case where the constraints are all linear and $X = \mathbb{R}^n$. On the other hand, the Farkas Lemma is a consequence of the final result. It corresponds to the special case of Theorem 4.3 (as sharpened through Theorem 4.4) where C is defined by homogeneous linear constraints only: $f_i(x) = \langle a_i, x \rangle$ for $i = 1, \ldots, m$, $X = \mathbb{R}^n$, and $\bar{x} = 0$.

The kind of multiplier rule statement used by John [13] (as described after Theorem 4.2) wouldn't work for Theorem 4.4. This result demonstrates nonetheless that a constraint qualification in normal cone formulation is fully capable of handling the special features like linear constraints for which tangent cone formulations have traditionally been pushed.

5. Games and duality. For optimization problems of convex type, Lagrange multipliers take on a game-theoretic role that could hardly even have been imagined before the creative insights of von Neumann [32], [33], in applying mathematics to models of social and economic conflict.

THEOREM 5.1. *In the convex case of* $(\mathcal{P})$*, the Lagrangian* $L(x, y)$ *is convex in* $x \in X$ *for each* $y \in Y$*, and concave (actually affine) in* $y \in Y$ *for each* $x \in X$*. The Lagrangian normality condition* $(\mathcal{L})$ *in Theorem 4.2 is equivalent then to the saddle point condition:*

$$\begin{cases} \textit{the minimum of } L(x, \bar{y}) \textit{ in } x \in X \textit{ is attained at } \bar{x}, \\ \textit{the maximum of } L(\bar{x}, y) \textit{ in } y \in Y \textit{ is attained at } \bar{y}. \end{cases} \tag{5.1}$$

Proof. The expression $f_0(x) + y_1 f_1(x) + \cdots + y_m f_m(x)$ is always affine in $y = (y_1, \ldots, y_m)$ for fixed x. When f_i is convex for $i = 0, 1, \ldots, s$ and affine for $i = s+1, \ldots, m$, as stipulated in the convex case of $(\mathcal{P})$, this expression is convex in x as long as the coefficients y_i are nonnegative for $i = 1, \ldots, s$. Since the set X is convex in the convex case of $(\mathcal{P})$ (and Y is convex by choice), the normal cone conditions are by Theorem 3.6 not only necessary but sufficient for the minimum and maximum in question. □

A pair of elements $\bar{x}$ and $\bar{y}$ is said to give a *saddle point* of L on $X \times Y$ when (5.1) holds; this can also be written as

$$L(x, \bar{y}) \geq L(\bar{x}, \bar{y}) \geq L(\bar{x}, y) \text{ for all } x \in X,\ y \in Y \quad (\text{where } \bar{x} \in X,\ \bar{y} \in Y). \tag{5.2}$$

This relation has a life of its own as an equilibrium condition for certain "games," and it leads to further properties of Lagrange multipliers which are of prime importance for many applications.

It will be helpful to free ourselves temporarily of the specifics of the Lagrangian L for problem (P) and think of an arbitrary real-valued function L on a product of any nonempty sets X and Y, not necessarily even in $\mathbb{R}^n$ and $\mathbb{R}^m$. The triple (X, Y, L) determines a certain *two-person zero-sum game*, described as follows. There are two "agents," Player 1 and Player 2; X is the "strategy set" for Player 1, Y is the "strategy set" for Player 2, and L is the "payoff function."

(1) Player 1 selects an element $x \in X$, while Player 2 selects an element $y \in Y$.
(2) The two choices are revealed simultaneously.
(3) In consequence, Player 1 pays $L(x, y)$ (dollars, say) to Player 2.

The simplicity of this model is deceptive, because it can be demonstrated that a vast array of games—even chess and poker—can be dressed in such clothing. (In the case of chess, for instance, each element x can be taken to be an encyclopedic prescription of just what Player 1 would do in response to every possible board configuration and the history that led to it, and similarly for each y.) A crucial property, however, is that the amount won by either player is the amount lost by the other player (this is what's meant by "zero sum"). The amounts $L(x, y)$ are allowed to be 0, and they can be negative; then Player 1 really gets the money.

A saddle point $(\bar{x}, \bar{y})$ of L over $X \times Y$ (if one exists) gives a kind of equilibrium in the game. In choosing $\bar{x}$, Player 1 is assured that no matter what element y might be selected by Player 2, the payment can't exceed the amount $L(\bar{x}, \bar{y})$. Likewise, in choosing $\bar{y}$ Player 2 can be sure of receiving at least this same amount, regardless of the actions of Player 1. The concept of a saddle point seems associated therefore with an approach to the game in which each player tries to exercise as much control as possible over the outcome, relative to a worst-case analysis of what might happen.

This notion can be made rigorous by introducing an optimization problem for each player, to be used in determining the element to be selected. The optimization problem for Player 1 is

$$(\mathcal{P}_1) \qquad \text{minimize } f(x) \text{ over } x \in X, \text{ where } f(x) = \sup_{y \in Y} L(x, y),$$

while the optimization problem for Player 2 is

$$(\mathcal{P}_2) \qquad \text{maximize } g(y) \text{ over } y \in Y, \text{ where } g(y) = \inf_{x \in X} L(x, y).$$

These are called the *minimax strategy problems* for the two players. In $(\mathcal{P}_1)$, Player 1 distinguishes the various choices x only according to the least upper bound $f(x)$ to what might have to be paid out if that choice is made. An optimal solution $\bar{x}$ minimizes this bound. The interpretation of problem $(\mathcal{P}_2)$ is similar—from the viewpoint of Player 2. Let's write

$$\inf(\mathcal{P}_1) = \inf_{x \in X} f(x), \qquad \sup(\mathcal{P}_2) = \sup_{y \in Y} g(y). \tag{5.3}$$

Then $\inf(\mathcal{P}_1)$ is the level that Player 1 can hold payments down to, whereas $\sup(\mathcal{P}_2)$ is the level that Player 2 can force payments up to.

PROPOSITION 5.2. *In the game represented by a general choice of X, Y, and L, it is always true that* $\inf(\mathcal{P}_1) \geq \sup(\mathcal{P}_2)$. *A pair $(\bar{x}, \bar{y})$ furnishes a saddle point of $L(x, y)$ on $X \times Y$ if and only if*

$$\begin{cases} \bar{x} \text{ is an optimal solution to } (\mathcal{P}_1), \\ \bar{y} \text{ is an optimal solution to } (\mathcal{P}_2), \\ \inf(\mathcal{P}_1) = \sup(\mathcal{P}_2). \end{cases} \tag{5.4}$$

Proof. The definition of the functions f in $(\mathcal{P}_1)$ and g in $(\mathcal{P}_2)$ implies that

$$f(x) \geq L(x, y) \geq g(y) \text{ for all } x \in X \text{ and } y \in Y.$$

Each value $g(y)$ is therefore a lower bound for the minimization of f, so that $\inf(\mathcal{P}_1) \geq g(y)$ for all $y \in Y$. Similarly, $\sup(\mathcal{P}_2) \leq f(x)$, so we get

$$f(x) \geq \inf(\mathcal{P}_1) \geq \sup(\mathcal{P}_2) \geq g(y) \text{ for all } x \in X \text{ and } y \in Y.$$

Condition (5.4) is tantamount therefore to $f(\bar{x}) = L(\bar{x}, \bar{y}) = g(\bar{y})$, i.e., to the equation $\sup_{y \in Y} L(\bar{x}, y) = L(\bar{x}, \bar{y}) = \inf_{x \in X} L(x, \bar{y})$. But the latter is precisely the saddle point condition (5.1)–(5.2). □

Equipped with these concepts, let's return to the Lagrangian L in the convex case of the standard optimization problem $(\mathcal{P})$. According to Theorem 5.1, the basic necessary condition $(\mathcal{L})$ for $\bar{x}$ to be an optimal solution to $(\mathcal{P})$ can be interpreted in this case as requiring the existence of a vector $\bar{y}$ such that L has a saddle point over $X \times Y$ at $(\bar{x}, \bar{y})$. But we know now that such a saddle point corresponds to an equilibrium in the game associated with (X, Y, L). What is this game in relation to (P), and who are its "players"?

The minimax strategy problem $(\mathcal{P}_1)$ in this situation can readily be determined from the special nature of L and Y. The function f to be minimized is

$$f(x) = \sup_{y \in Y} \Big\{ f_0(x) + y_1 f_1(x) + \cdots + y_m f_m(x) \Big\} \text{ for } x \in X, \tag{5.5}$$

where the restriction of y to Y in taking the "sup" means that the coefficients y_i can be chosen arbitrarily for the terms indexed by $i = s+1, \ldots, m$, but must be nonnegative for $i = 1, \ldots, s$. It's apparent that

$$f(x) = \begin{cases} f_0(x) & \text{when } x \in C, \\ \infty & \text{when } x \notin C, \end{cases} \tag{5.6}$$

where C is the set of feasible solutions to $(\mathcal{P})$ as earlier. But the minimization of this function f over X is no different than the minimization of f_0 over the subset C of X. In other words, $(\mathcal{P}_1)$ is the same as $(\mathcal{P})$.

A mysterious piece of information has been uncovered. In our innocence we thought we were engaged straightforwardly in solving a single problem $(\mathcal{P})$. But we find we've assumed the role of Player 1 in a certain game in which we have an adversary, Player 2, whose interests are diametrically opposed to ours! Our adversary's strategy problem is

$$(\mathcal{D}) \qquad \text{maximize } g(y) = \inf_{x \in X} \Big\{ f_0(x) + y_1 f_1(x) + \cdots + y_m f_m(x) \Big\} \text{ over } y \in Y.$$

This is the optimization problem *dual* to problem $(\mathcal{P})$ in the Lagrangian sense.

In general, the essential objective function g in the dual problem $(\mathcal{D})$ might, like f in (5.6), be extended-real-valued. To learn more about the nature of $(\mathcal{D})$ in a given case with particular structure assigned to X and the f_i's, we'd have to identify the set of points y where $g(y) > -\infty$ and regard that as the feasible set in $(\mathcal{D})$. Examples will be considered below, but we first need to draw the main conclusion from these developments.

THEOREM 5.3. *In the convex case of $(\mathcal{P})$, the Lagrangian normality condition $(\mathcal{L})$ holds for $\bar{x}$ and $\bar{y}$ if and only if*

$$\begin{cases} \bar{x} \text{ is an optimal solution to } (\mathcal{P}), \\ \bar{y} \text{ is an optimal solution to } (\mathcal{D}), \\ \inf(\mathcal{P}) = \sup(\mathcal{D}). \end{cases} \tag{5.7}$$

Thus in particular, if $\bar{x}$ is an optimal solution to $(\mathcal{P})$ *at which the basic constraint qualification* $(\mathcal{Q})$ *or its variant* (Q') *is satisfied, then there exists at least one optimal solution to* $(\mathcal{D})$*, and* $\inf(\mathcal{P}) = \sup(\mathcal{D})$*.*

Proof. This is immediate from the combination of Theorem 5.1 with Proposition 5.2, in view of the existence of multiplier vectors assured by Theorems 4.2 and 4.4. □

An early and very pretty example of duality in optimization was spotlighted by Gale, Kuhn and Tucker [34] in *linear programming*, which is the case of problem $(\mathcal{P})$ where X is the special polyhedral cone in (4.6), the objective function is linear, and all the constraints are linear. Adopting the notation

$$
\begin{aligned}
f_0(x) &= c_1 x_1 + \cdots + c_n x_n, \\
f_i(x) &= b_i - a_{i1} x_1 - \cdots - a_{in} x_n \ \text{ for } \ i = 1, \ldots m,
\end{aligned}
$$

we can express the problem in this special case as

$$
(\mathcal{P}_{lin}) \qquad
\begin{aligned}
&\text{minimize } c_1 x_1 + \cdots + c_n x_n \ \text{ subject to } \ x_j \geq 0 \ \text{ for } \ j = 1, \ldots, r, \\
&\qquad a_{i1} x_1 + \cdots + a_{in} x_n \begin{cases} \geq b_i & \text{for } i = 1, \ldots s, \\ = b_i & \text{for } i = s+1, \ldots, m. \end{cases}
\end{aligned}
$$

The Lagrangian function is

$$
L(x, y) = \sum_{j=1}^{n} c_j x_j + \sum_{i=1}^{m} y_i b_i - \sum_{i=1, j=1}^{m,n} y_i a_{ij} x_j, \tag{5.8}
$$

which exhibits the same kind of symmetry between the x and y arguments as appears in the choice of X and Y. To obtain the problem dual to this, we must determine the function g defined in $(\mathcal{D})$ for this Lagrangian and see where it's finite or infinite. Elementary calculations show that $g(y) = \sum_{i=1}^{m} y_i b_i$ if $c_j - \sum_{i=1}^{m} y_i a_{ij} \geq 0$ for $j = 1, \ldots, r$ and $c_j - \sum_{i=1}^{m} y_i a_{ij} = 0$ for $j = r+1, \ldots, n$, whereas $g(y) = -\infty$ if y does not satisfy these constraints. The dual problem therefore comes out as

$$
(\mathcal{D}_{lin}) \qquad
\begin{aligned}
&\text{maximize } y_1 b_1 + \cdots + y_m b_m \ \text{ subject to } \ y_i \geq 0 \ \text{ for } \ i = 1, \ldots, s, \\
&\qquad y_1 a_{1j} + \cdots + y_m a_{mj} \begin{cases} \leq c_j & \text{for } j = 1, \ldots r, \\ = c_j & \text{for } j = r+1, \ldots, n. \end{cases}
\end{aligned}
$$

From all this symmetry it emerges that not only do the Lagrange multiplier vectors associated with an optimal solution to $(\mathcal{P}_{lin})$ have an interpretation as optimal solutions $\bar{y}$ to $(\mathcal{D}_{lin})$, but by the same token, the Lagrange multiplier vectors associated with an optimal solution to $(\mathcal{D}_{lin})$ have an interpretation as optimal solutions $\bar{x}$ to $(\mathcal{P}_{lin})$. Each of these problems furnishes the multipliers for the other.

COROLLARY 5.4. *If either of the linear programming problems* $(\mathcal{P}_{lin})$ *or* $(\mathcal{D}_{lin})$ *has an optimal solution, then so does the other, and*

$$
\inf(\mathcal{P}_{lin}) = \sup(\mathcal{D}_{lin}).
$$

The pairs $(\bar{x}, \bar{y})$ *such that* $\bar{x}$ *solves* $(\mathcal{P}_{lin})$ *and* $\bar{y}$ *solves* $(\mathcal{D}_{lin})$ *are precisely the ones that, for the choice of* L*,* X *and* Y *corresponding to these problems, satisfy the Lagrangian normality condition* $(\mathcal{L})$*, or equivalently, give a saddle point of* L *on* $X \times Y$*.*

Proof. We need only observe that the constraint qualification $(\mathcal{Q}')$ is trivially satisfied in both of these problems, because they only involve linear constraints. □

According to Kuhn [11], it was von Neumann [35] who proposed this result to Gale, Kuhn and Tucker, who then proved it in [34]. The exciting duality moved Kuhn and Tucker to give the main emphasis to saddle point conditions for optimality in their paper [9] (which was written later than [34], although it came into print earlier).

An interpretation of the Lagrangian game associated with $(\mathcal{P})$ is called for, but there's no universal story to fit all applications. An example will nonetheless convey the basic idea and indicate some relationships between this kind of mathematics and theoretical economics. Let's think of $f_0(x)$ as the cost in dollars associated with a decision vector $x = (x_1, \ldots, x_n)$ selected from $X \subset \mathbb{R}^n$. (Perhaps X is a box—a product of closed intervals, not necessarily bounded—which specifies ranges for the variables x_j.) Each conceivable decision x demands inputs of certain resources, which are available only in limited supply (space, labor, fuel, etc.). The resources are indexed by $i = 1, \ldots, m$, and $f_i(x)$ stands for the *excess* of resource i that x would require relative to the amount available. The optimization problem is to find a decision $\bar{x} \in X$ that minimizes cost subject to not demanding more in resources than is available (the excesses must be nonpositive). This is problem $(\mathcal{P})$ in a case where no equality constraints are present: $s = m$. (Note well that inequalities rather than equalities are the proper model here. Nothing requires the decision maker to use up all the supplies available. That additional restriction might lead to "busy work" inefficiencies, which could force costs higher.)

In this model, where the set Y consists of all vectors $y = (y_1, \ldots, y_m)$ with nonnegative components, the meaning of the Lagrangian $L(x, y) = f_0(x) + y_1 f_1(x) + \cdots + y_m f_m(x)$ can be gleaned from dimensional analysis. Since the term $f_0(x)$ is measured in dollars, $L(x, y)$ must be in such units as well, and therefore y_i must be measured in dollars per unit of resource i. In other words, y_i has to be a kind of *price*. The vector y thus specifies prices which act to convert excesses of resource usage into cost terms to be added to the basic costs already present. If $f_i(x) < 0$, the added cost associated with the input of resource i is negative (or at least nonpositive, since $y_i \geq 0$); a credit is given for the unused surplus of this resource.

The following game interpretation can now be made. The decision problem is enlarged to a setting where supplementary resource amounts can be purchased if needed, and surplus amounts can be sold off. Player 1 selects a decision x from this perspective, not insisting on the constraints $f_i(x) \leq 0$. Player 2, the "market," selects a price vector y. The quantity $L(x, y)$ is then the net cost to Player 1 of the decision x in accordance with any associated buying and selling of the various resources. This cost is regarded as paid by the decision maker to the "market." Player 2, as the recipient, tries to set the prices so as to gain as much income as possible out of the situation, while Player 1 tries to hold costs down.

Under the assumption that X and all the functions f_i are convex, we can apply Theorems 5.1 and 5.3. The multiplier vectors $\bar{y}$ for an optimal decision $\bar{x}$ furnish a market equilibrium: the prices have the special property that the decision maker is content with a decision that keeps within existing supplies even when offered the option of buying and selling resources at the prices $\bar{y}_i$. The clear implication is that these prices must tightly reflect the marginal value of the resources. The dual problem is therefore one of imputing value to the resources relative to what the decision maker can get out of them.

The saddle point characterization of optimality in $(\mathcal{P})$, when it's valid, as under the assumptions in Theorem 5.1, affords more than interpretations of optimality. It provides alternative methods of solving $(\mathcal{P})$. For instance, instead of applying an optimization algorithm right to $(\mathcal{P})$, one can in some cases apply it to the dual problem $(\mathcal{D})$ to get $\bar{y}$ and then get $\bar{x}$ from the minimization of $L(x, \bar{y})$ over $x \in X$. Or, one can devise "primal-dual" algorithms which search directly for a saddle point of L on $X \times Y$. In practice such "pure" numerical approaches are often blended together.

For more on duality in convex optimization, see Rockafellar [36]. Special cases beyond linear programming are developed in detail in Rockafellar [7], [8], [37], [38], [39].

When problem $(\mathcal{P})$ isn't of convex type, the assertions in Theorem 5.3 generally fail. Nonetheless, the Lagrangian dual problem $(\mathcal{D})$ can still be useful. The function g maximized over Y in $(\mathcal{D})$ is always concave (although possibly extended-real-valued), because it's defined as the pointwise infimum of the collection affine functions $x \mapsto L(x, y)$ indexed by $x \in X$. From Proposition 5.2 we know that

$$\inf(\mathcal{P}) \geq \sup(\mathcal{D}) \text{ always.} \tag{5.9}$$

By solving $(\mathcal{D})$, one can at least obtain a lower bound to the optimal value in $(\mathcal{P})$. this may provide some guideline to whether, having already calculated a certain feasible point $\hat{x}$ in $(\mathcal{P})$ with objective value $f_0(\hat{x})$, it's worth expending more effort on improving $\hat{x}$. This approach is popular in areas of optimization where the problems $(\mathcal{P})$ are extremely difficult, as for instance when the specification of the set X includes restricting some of the variables x_j to take on integer values only.

6. Canonical perturbations. Lagrange multipliers hold fundamental significance for the parametric study of problems of optimization. Through this, answers can be found as to whether saddle point conditions and duality are limited only to convex optimization or have some influence beyond.

An optimization problem typically involves not only modeling parameters but data elements that might be subject to error or fluctuation and therefore have the character of parameters as well. Much often depends on understanding how variations in such parameters may affect the optimal value and optimal solutions.

In most of applied mathematics the opinion prevails that if a problem is well formulated, and parameters enter its statement smoothly, then solutions should depend smoothly on these parameters. But in optimization this rosy hope is lost already in simple cases like linear programming. Broadly speaking, the trouble is that the operations of minimization and maximization, unlike integration, composition, and other operations at the core of classical analysis, don't always preserve differentiability and may even induce discontinuities. The mathematical prospects aren't as bleak as this may sound, however, because powerful methods of handling such features of minimization and maximization have been devised in nonsmooth analysis.

The subject of nonsmoothness in parametric optimization is far too big to lay out for general view here. But we can look at a key topic, the connection between Lagrange multipliers and "generalized derivatives" of the optimal value in a problem with respect to certain "canonical" perturbations, and in that way see the outlines of what's involved.

Let's regard our problem $(\mathcal{P})$ as embedded in an entire family of optimization problems parameterized in a special way by $u = (u_1, \ldots, u_m) \in \mathbb{R}^m$:

$$(\mathcal{P}(u)) \qquad \begin{aligned} &\text{minimize } f_0(x) \text{ over all } x \in X \\ &\text{such that } f_i(x) + u_i \begin{cases} \leq 0 & \text{for } i = 1, \ldots, s, \\ = 0 & \text{for } i = s+1, \ldots, m. \end{cases} \end{aligned}$$

Clearly $(\mathcal{P})$ is $\big(\mathcal{P}(0)\big)$; it's fruitful to think of u as a perturbation vector shifting the given problem from $\big(\mathcal{P}(0)\big)$ to $\big(\mathcal{P}(u)\big)$. Define the function p on the parameter space $\mathbb{R}^m$ by

$$p(u) = \inf\big(\mathcal{P}(u)\big) = \big[\text{the optimal value corresponding to } u = (u_1,\ldots,u_m)\big]. \quad (6.1)$$

Observe that p may be extended-real-valued. If the objective function f_0 isn't bounded below on the feasible set for problem $\big(\mathcal{P}(u)\big)$, we have $p(u) = -\infty$. If this feasible set happens to be empty (because the constraints are inconsistent), we have $p(u) = \infty$ by the convention that the greatest lower bound to the empty set of real numbers is ∞.

We are dealing here with the *canonical parameterization* of problem $(\mathcal{P})$ and its *canonical value function* p. This terminology comes from the fact that all other forms of parameterization and associated perturbation can, in principle, be cast in the same mold. For instance, if we wish to think of the functions f_i as depending on some further vector $b = (b_1,\ldots,b_d)$ in a set $B \subset \mathbb{R}^d$, the arguments being written $f_i(x,b)$, and the issue concerns what happens when b varies close to a reference value $\bar{b} \in B$, we can handle this by making the b_k's into new *variables* and introducing $x' = (x,b) \in X \times B$ as the new decision vector under the additional constraints $f_{m+k}(x,b) = 0$ for $k = 1,\ldots,d$, where $f_{m+k}(x,b) = \bar{b}_k - b_k$. Shifts of b_k away from $\bar{b}_k$ correspond then to shifts of the constraint $f_{m+k}(x,b) = 0$ to $f_{m+k}(x,b) + u_{m+k} = 0$.

THEOREM 6.1. *A pair of vectors $\bar{x} \in X$ and $\bar{y} \in Y$ furnishes a saddle point of the Lagrangian L for $(\mathcal{P})$ on $X \times Y$ if and only if*

$$\begin{cases} \bar{x} \text{ is a (globally) optimal solution to } (\mathcal{P}), \\ p(u) \geq p(0) + \big\langle \bar{y}, u\big\rangle \text{ for all } u. \end{cases} \quad (6.2)$$

Proof. The general interchange principle in minimization, applicable to any function $\Phi(x,u)$ on $\mathbb{R}^n \times \mathbb{R}^n$, says for a pair $(\bar{x},\bar{u})$ that

$$\begin{aligned} &(\bar{x},\bar{u}) \text{ minimizes } \Phi(x,u) \text{ on } \mathbb{R}^n \times \mathbb{R}^m \\ &\iff \bar{x} \text{ minimizes } \Phi(x,\bar{u}) \text{ on } \mathbb{R}^n, \text{ while } \bar{u} \text{ minimizes } \inf_x \Phi(x,u) \text{ on } \mathbb{R}^m, \\ &\iff \bar{u} \text{ minimizes } \Phi(\bar{x},u) \text{ on } \mathbb{R}^m, \text{ while } \bar{x} \text{ minimizes } \inf_u \Phi(x,u) \text{ on } \mathbb{R}^n. \end{aligned} \quad (6.3)$$

We take $\bar{u} = 0$ and $\Phi(x,u) = \varphi(x,u) - \langle \bar{y}, u\rangle$, where

$$\varphi(x,u) = \begin{cases} f_0(x) & \text{if } x \text{ is feasible for } \big(\mathcal{P}(u)\big), \\ \infty & \text{if } x \text{ isn't feasible for } \big(\mathcal{P}(u)\big). \end{cases} \quad (6.4)$$

The second condition in (6.3) can then be identified with (6.2), because $\inf_x \Phi(x,u) = p(u) - \langle \bar{y}, u\rangle$. When this condition holds, the value $\varphi(\bar{x},0)$ must be finite by virtue of equivalence with the first condition in (6.3) (since $\Phi > -\infty$ everywhere, but $\Phi \not\equiv \infty$). We argue that the saddle point condition corresponds to the third condition in (6.3). For each $x \in X$ let U_x denote the set of $u \in \mathbb{R}^m$ such that $u_i \leq -f_i(x)$ for $i = 1,\ldots,s$, but $u_i = -f_i(x)$ for $i = s+1,\ldots,m$. For each $x \notin X$ let $U_x = \emptyset$. In this notation we have

$$\begin{aligned} \inf_{u\in\mathbb{R}^m}\Big\{\varphi(x,u) - \langle y,u\rangle\Big\} &= \inf_{u\in U_x}\Big\{f_0(x) - \langle y,u\rangle\Big\} \\ &= \begin{cases} L(x,y) & \text{if } x \in X \text{ and } y \in Y, \\ -\infty & \text{if } x \in X \text{ but } y \notin Y, \\ \infty & \text{if } x \notin X, \end{cases} \end{aligned} \quad (6.5)$$

where the case $y = \bar{y}$ gives $\inf_u \Phi(x,u)$. Therefore, to assert that the third condition in (6.3) holds with the minimum values finite is to say that $\bar{y} \in Y$ and $L(x,\bar{y})$ achieves its minimum relative to $x \in X$ at $\bar{x}$, while at the same time the minimum in (6.5) for $x = \bar{x}$ and $y = \bar{y}$ is attained at $u = 0$, the minimum value being $L(\bar{x},\bar{y})$ and equaling $f_0(\bar{x})$. But from (6.5) we know that $L(\bar{x},y) \le f_0(\bar{x})$ for all $y \in Y$, because $u = 0$ is always a candidate in the minimization. Hence in these circumstances we have $L(\bar{x},y) \le L(\bar{x},\bar{y})$ for all $y \in Y$, with $L(\bar{x},\bar{y})$ finite. Conversely, if this inequality holds then necessarily $L(\bar{x},\bar{y}) = f_0(\bar{x})$ with $\bar{x}$ feasible in $(\mathcal{P})$, due to (5.5)–(5.6), and then in (6.5) for $x = \bar{x}$ and $y = \bar{y}$ we get the minimum achieved at $u = 0$. The two properties in the third condition in (6.3) are thus equivalent to the two parts of the saddle point condition for L on $X \times Y$. $\square$

As an interesting sidelight, this theorem informs us that in the convex case the multiplier vectors (if any) that satisfy condition $(\mathcal{L})$ for a particular optimal solution $\bar{x}$ are the same as the ones satisfying it for any other optimal solution. In the nonconvex case of $(\mathcal{P})$ the associated multiplier vectors could well be different.

The relationship between $\bar{y}$ and p in (6.2) can be understood geometrically in terms of the *epigraph* of p, which is the set

$$\mathit{epi}\, p = \big\{\, (u,\alpha) \in \mathbb{R}^m \times \mathbb{R} \,\big|\, \alpha \ge p(u) \,\big\}. \tag{6.6}$$

The global inequality $p(u) \ge p(0) + \langle \bar{y}, u \rangle$, with $p(0)$ finite, has the interpretation that the epigraph of the affine function of $u \mapsto p(0) + \langle \bar{y}, u \rangle$ (this epigraph being a certain "upper" closed half-space in $\mathbb{R}^m \times \mathbb{R}^n$) is a *supporting half-space* to the epigraph of p at $\big(0, p(0)\big)$. For such a half-space to exist, the set $\mathit{epi}\, p$ must in particular not be "dented in" at $\big(0, p(0)\big)$. The role of convexity in the saddle point results in §5 becomes much clearer through this.

PROPOSITION 6.2. *In the convex case of problem $(\mathcal{P})$, the canonical value function p is convex, and its epigraph is thus a convex set.*

Proof. Suppose $p(u') \le \alpha' \in \mathbb{R}$ and $p(u'') \le \alpha'' \in \mathbb{R}$, and consider any $t \in (0,1)$. Fix any $\varepsilon > 0$. From the definition of p, there's a feasible point x' for problem $\big(\mathcal{P}(u')\big)$ with $f_0(x') \le \alpha' + \varepsilon$ and also a feasible point x'' for problem $\big(\mathcal{P}(u'')\big)$ with $f_0(x'') \le \alpha'' + \varepsilon$. Let $x = (1-t)x' + tx''$. Again $x \in X$ (because X is convex), and we have

$$f_i(x) \begin{cases} \le (1-t)f_i(x') + tf_i(x'') & \text{for } i = 0,1,\ldots,s, \\ = (1-t)f_i(x') + tf_i(x'') & \text{for } i = s+1,\ldots,m \end{cases}$$

(because f_i is convex on X for $i = 0,1,\ldots,s$ and affine for $i = s+1,\ldots,m$). Therefore

$$f_0(x) \le (1-t)[\alpha' + \varepsilon] + t[\alpha'' + \varepsilon] = \alpha + \varepsilon \ \text{ for } \ \alpha = (1-t)\alpha' + +t\alpha'',$$

and for $u = (1-t)u' + tu''$ we have

$$f_i(x) + u_i \begin{cases} \le (1-t)\big[f_i(x') + u_i'\big] + t\big[f_i(x'') + u_i''\big] & \text{for } i = 1,\ldots,s, \\ = (1-t)\big[f_i(x') + u_i'\big] + t\big[f_i(x'') + u_i''\big] & \text{for } i = s+1,\ldots,m, \end{cases}$$

which implies that x is feasible for $\big(P(u)\big)$. Hence $p(u) \le \alpha + \varepsilon$. Since this is true for arbitrary $\varepsilon > 0$, we actually have $p(u) \le \alpha$. Thus, p is convex. Indeed, we've demonstrated that for any pairs (u',α') and (u'',α'') in $\mathit{epi}\, p$ and any $t \in (0,1)$, the pair $(u,\alpha) = (1-t)(u',\alpha') + t(u'',\alpha'')$ will again belong to $\mathit{epi}\, p$, which is the condition for the convexity of $\mathit{epi}\, p$. $\square$

While many applications of optimization do lie in the realm of convexity, nonconvex problems certainly come up, too. Unfortunately, without convexity and assumptions dependent on it, one can't very well ensure the existence of some $\bar{y}$ for which the p inequality in (6.2) holds, although there might be such a multiplier vector through good luck. Therefore, the saddle point interpretation of the normality condition $(\mathcal{L})$ seems more or less inevitably restricted to convex optimization along with the kind of Lagrangian duality in Theorem 5.3.

But all isn't lost in the quest for saddle point expressions of optimality in nonconvex optimization. We simply must search for other expressions of $(\mathcal{P})$ as the strategy problem for Player 1 in some game. In other words, we must be prepared to alter L and Y (even the space in which Y lies), while retaining X, with the goal of somehow still getting the objective function f in problem $(\mathcal{P}_1)$ to come out as (5.6) with C the feasible set in $(\mathcal{P})$.

This leads to the theory of *modified Lagrangians* for problem $(\mathcal{P})$, which has occupied many researchers. We'll concentrate here on a single such Lagrangian having both numerical and theoretical uses, based in part on ties to second-order conditions for optimality. This modified Lagrangian relates in a natural way to the value function p and sheds more light on perturbations of $(\mathcal{P})$ and what the Lagrange multipliers already at our disposal have to say about them.

The *augmented Lagrangian* $\widetilde{L}$ for $(\mathcal{P})$ requires only one additional variable η. It's given by

$$\begin{aligned}\widetilde{L}(x,y,\eta) = f_0(x) &+ \sum_{i=1}^{s} \left\{ \begin{matrix} y_i f_i(x) + (\eta/2) f_i(x)^2 & \text{if } f_i(x) \geq -y_i/\eta \\ -y_i^2/2\eta & \text{if } f_i(x) \leq -y_i/\eta \end{matrix} \right\} \\ &+ \sum_{i=s+1}^{m} \Big\{ y_i f_i(x) + (\eta/2) f_i(x)^2 \Big\}, \quad \text{where } \eta > 0.\end{aligned} \tag{6.7}$$

In order to place it in the game-theoretic framework of §5, we think of dual elements $\widetilde{y} = (y,\eta)$ in the set
$$\widetilde{Y} = \mathbb{R}^m \times (0,\infty)$$
and work with the triple $(X,\widetilde{Y},\widetilde{L})$. We have the desired relation

$$\sup_{(y,\eta)\in\widetilde{Y}} \widetilde{L}(x,y,\eta) = \begin{cases} f_0(x) & \text{when } x \in C, \\ \infty & \text{when } x \notin C, \end{cases} \tag{6.8}$$

since for fixed $x \in X$ and $y \in \mathbb{R}^m$ the limit of $\widetilde{L}(x,y,\eta)$ as $\eta \to \infty$ already gives $L(x,y)$ when $x \in C$ but ∞ when $x \notin C$; cf. then (5.5)–(5.6). This observation shows that $(\mathcal{P})$ is the strategy problem for Player 1 in the game associated with $(X,\widetilde{Y},\widetilde{L})$ and suggests further that the augmented Lagrangian $\widetilde{L}$ represents a kind of mixture of the ordinary Lagrangian L and penalty expressions, with η the penalty parameter.

Note that in $\widetilde{Y}$ the multipliers y_i associated with the inequality constraint functions $f_1,\ldots,f_s$ are no longer restricted *a priori* to be nonnegative. Nevertheless they'll turn out to be nonnegative at optimality (cf. Theorem 7.3 below). Also in contrast to Y, the set $\widetilde{Y}$ isn't closed, because the variable η has been restricted to $(0,\infty)$ in order to avoid division by 0 in the formula for $\widetilde{L}$ in (6.6). This is a minor technical point we could get around, but anyway it makes little difference because $\widetilde{L}$ is monotone on the η argument (see Proposition 7.1).

The odd-looking formula for $\widetilde{L}$ receives strong motivation from the next theorem, and its proof—which indicates how the expression originated.

THEOREM 6.3. *A pair of vectors $\bar{x} \in X$ and $(\bar{y}, \bar{\eta}) \in \widetilde{Y}$ furnishes a saddle point of the augmented Lagrangian $\widetilde{L}$ on $X \times \widetilde{Y}$ if and only if*

$$\begin{cases} \bar{x} \text{ is a (globally) optimal solution to } (\mathcal{P}), \\ p(u) \geq p(0) + \langle \bar{y}, u \rangle - \dfrac{\bar{\eta}}{2} |u|^2 \text{ for all } u. \end{cases} \tag{6.9}$$

When this holds, any $\bar{\eta}' > \bar{\eta}$ will have the property that

$$\Big[\bar{x} \text{ solves } (\mathcal{P})\Big] \iff \Big[\bar{x} \text{ minimizes } \widetilde{L}(x, \bar{y}, \bar{\eta}') \text{ over } x \in X\Big]. \tag{6.10}$$

Proof. The saddle point argument is very close to the one for Theorem 6.1. Recalling the function φ in (6.5), we apply the interchange principle (6.3) to $\Phi(x, u) = \varphi(x, u) - \langle \bar{y}, u \rangle + (\bar{\eta}/2)|u|^2$ with $\bar{u} = 0$. We have $\inf_x \Phi(x, u) = \widetilde{p}(u)$, where $\widetilde{p}(u) = p(u) + (\bar{\eta}/2)|u|^2$ and $\widetilde{p}(0) = p(0)$, so the second condition in (6.3) is in this case equivalent to (6.9). To demonstrate that the third condition in (6.3) comes out as the saddle point condition for $\widetilde{L}$ on $X \times \widetilde{Y}$, we must determine $\inf_u \Phi(x, u)$ as a function of x. The first step is to observe that

$$\inf_{u_i \leq -f_i(x)} \Big\{ (\eta/2) u_i^2 - y_i u_i \Big\} = \begin{cases} y_i f_i(x) + (\eta/2) f_i(x)^2 & \text{if } f_i(x) \geq -y_i/\eta, \\ -y_i^2/2\eta & \text{if } f_i(x) \leq -y_i/\eta. \end{cases} \tag{6.11}$$

Assisted by this and the notation U_x introduced in the proof of Theorem 6.1, we're able to calculate for any x, y, and $\eta > 0$ that

$$\begin{aligned} \inf_{u \in \mathbb{R}^m} \Big\{ \varphi(x, u) - \langle y, u \rangle + (\eta/2)|u|^2 \Big\} &= \inf_{u \in U_x} \Big\{ f_0(x) - \langle y, u \rangle + (\eta/2)|u|^2 \Big\} \\ &= \begin{cases} \widetilde{L}(x, y, \eta) & \text{if } x \in X, \\ \infty & \text{if } x \notin X. \end{cases} \end{aligned} \tag{6.12}$$

(This is the way the general expression for $\widetilde{L}$ was discovered [40].) With $(y, \eta) = (\bar{y}, \bar{\eta})$ this tells us that $\inf_u \Phi(x, u)$ is $\widetilde{L}(x, \bar{y}, \bar{\eta})$ when $x \in X$, but ∞ otherwise. The third condition in (6.3) asserts therefore that the minimum of $\widetilde{L}(x, \bar{y}, \bar{\eta})$ over $x \in X$ is attained at $\bar{x}$, while at the same time the minimum in (6.12) for $(x, y, \eta) = (\bar{x}, \bar{y}, \bar{\eta})$ is attained at $u = 0$. Mimicking the argument in the proof of Theorem 6.1, one sees that the latter property means the maximum of $\widetilde{L}(\bar{x}, y, \eta)$ over $(y, \eta) \in \widetilde{Y}$ is attained at $(\bar{y}, \bar{\eta})$. The two parts of the third condition in (6.3) thus correspond to the two parts of the saddle point condition for $\widetilde{L}$ on $X \times \widetilde{Y}$.

When $(\bar{x}, \bar{y}, \bar{\eta})$ satisfies (6.9) and $\bar{\eta}' > \bar{\eta}$, not only will (6.9) still be satisfied by $(\bar{x}, \bar{y}, \bar{\eta})$, but the p inequality will be *strict* away from 0. Then, with $\bar{\eta}$ replaced by $\bar{\eta}'$ in the interchange argument, the elements $(\bar{x}, \bar{u})$ in the third condition in (6.3) necessarily have $\bar{u} = 0$; they're thus the pairs $(\bar{x}, 0)$ such that $\bar{x}$ minimizes $\widetilde{L}(x, \bar{y}, \bar{\eta}')$ over $x \in X$. But they must also be the pairs in the first condition in (6.3). This yields (6.10). □

The addition of squared constraint terms to the ordinary Lagrangian was first proposed by Hestenes [41] and Powell [42] in association with a special numerical approach—called the method of multipliers—to problems having equality constraints only. The correct extension to inequality constraints, leading to the formula in (6.7), was identified in Rockafellar [40] and first studied in detail by Buys [43] in his unpublished doctoral dissertation. A related formula was proposed by Wierzbicki [44]. The results in Theorem 4.2 were obtained in Rockafellar [45], [46] (saddle point equivalence), and [47] (the property in (6.10)). Optimal solutions to $(\mathcal{P})$ can indeed be characterized in terms of saddle point of the augmented Lagrangian when certain second-order optimality conditions are satisfied, as will be shown later in Theorem 7.4. This was first proved by Arrow, Gould and Howe [48] for a limited setting (with the saddle point just in x and y for $\bar{\eta}$ sufficiently high, and with y restricted a priori to Y) and in [46] for the general case. A deep connection between the augmented Lagrangian and monotone operator methods in the convex case was shown in [45] and [49]. Modified Lagrangians with greater degrees of smoothness have been devised by Mangasarian [50]. For an overview of modified Lagrangians and their usage in numerical optimization, see Bertsekas [51].

There's a world of difference between the inequality for $\bar{y}$ and p in (6.2) and the one in (6.9). The first can't be satisfied if p is "dented in," but in the second the graph of a concave quadratic function $u \mapsto p(0) + \langle \bar{y}, u \rangle - (\bar{\eta}/2)|u|^2$, rather than a hyperplane constituting the graph of some affine function, is pushed up against the epigraph of p. The value of $\bar{\eta} > 0$ controls the curvature of this parabolic surface, enabling it to narrow down far enough to fit into any "dent" in *epi* p. We can be confident therefore that some $\bar{y}$ and η will exist to satisfy this inequality, unless the point $\big(0, p(0)\big)$ lies in a sharp crevice of *epi* p, which isn't very likely.

The saddle point condition for the augmented Lagrangian $\widetilde{L}$ is much more powerful than the normality condition $(\mathcal{L})$ for the ordinary Lagrangian L, so the fact that it can be expected to hold in the absence of convexity assumptions on $(\mathcal{P})$ has special potential. Under condition $(\mathcal{L})$ we only know that $L(x, \bar{y})$ has a sort of stationary point at $\bar{x}$ relative to X, but the saddle point condition for $\widetilde{L}$ means that for some value of the parameter $\bar{\eta}$ the expression $\widetilde{L}(x, \bar{y}, \bar{\eta})$ achieves a definite minimum at $\bar{x}$ relative to X.

The further property in (6.1) means that the augmented Lagrangian is capable of furnishing an *exact penalty function* for $(\mathcal{P})$. Through an appropriate choice of $\bar{y}$ and $\bar{\eta}'$, we can minimize $\widetilde{L}(x, \bar{y}, \bar{\eta}')$ over $x \in X$ with the constraints $f_i(x) \le 0$ or $f_i(x) = 0$ ignored and nonetheless get the same results as by minimizing the original objective $f_0(x)$ over X subject to these constraints. This property, which has no counterpart for the ordinary Lagrangian, can be advantageous even in the convex case of $(\mathcal{P})$. Of course, we can't utilize it without first determining $(\bar{y}, \bar{\eta}')$, but this isn't as circular as may appear. For help we can turn to the *augmented dual problem*

$$(\widetilde{\mathcal{D}}) \qquad \text{maximize } \widetilde{g}(y, \eta) = \inf_{x \in X} \widetilde{L}(x, y, \eta) \text{ over all } (y, \eta) \in \widetilde{Y} = \mathbb{R}^m \times (0, \infty).$$

THEOREM 6.4. *Suppose in $(\mathcal{P})$ that X is bounded, and a feasible solution exists. Then*

$$\inf(\mathcal{P}) = \sup(\widetilde{\mathcal{D}}), \tag{6.13}$$

and $(\bar{x}, \bar{y}, \bar{\eta})$ gives a saddle point of $\widetilde{L}$ on $X \times \widetilde{Y}$ if and only if $\bar{x}$ solves $(\mathcal{P})$ and $(\bar{y}, \bar{\eta})$ solves $(\widetilde{\mathcal{D}})$. The pairs $(\bar{y}, \bar{\eta}')$ with the exact penalty property (6.10) are then the ones such that, for some $\bar{\eta} < \bar{\eta}'$, $(\bar{y}, \bar{\eta})$ is an optimal solution to $(\widetilde{\mathcal{D}})$. Furthermore in this case, the function $\widetilde{g}$ in $(\widetilde{\mathcal{D}})$ is finite everywhere on $\widetilde{Y}$, so this maximization problem is effectively unconstrained.

Proof. On the general grounds of Proposition 5.2 as applied to $(X, \widetilde{Y}, \widetilde{L})$, we know that "$\geq$" holds in (6.13). We must demonstrate that the inequality can't be strict, and then the saddle point assertion will likewise be a consequence of Proposition 5.2. Fix any $y \in \mathbb{R}^m$ and let φ be the function in (6.5). We have (6.12), and consequently

$$\begin{aligned} \inf_{x,u}\Big\{\varphi(x,u) - \langle y,u\rangle + (\eta/2)|u|^2\Big\} &= \inf_{\substack{x\in X\\ u\in U_x}}\Big\{f_0(x) - \langle y,u\rangle + (\eta/2)|u|^2\Big\} \\ &= \inf_{x\in X}\widetilde{L}(x,y,\eta) = \widetilde{g}(y,\eta). \end{aligned} \tag{6.14}$$

Because X is bounded (as well as closed) and the pairs (x, u) with $u \in U_x$ (as defined in the proof of Theorem 6.1) are simply the ones satisfying the constraints in $\big(\mathcal{P}(u)\big)$, this infimum is finite and attained. (The lower level sets of the expression being minimized are compact relative to the closed set in question.) The problem in (6.14) can be regarded as a penalty representation of the problem of minimizing the same expression subject to $u = 0$, which would yield $\inf(\mathcal{P})$ as the minimum value. Again because of the underlying closedness and boundedness, the minimum value in the penalty problem with parameter η (here y is being kept fixed) must approach this value for $u = 0$ as $\eta \to \infty$. This establishes that $\widetilde{g}(y, \eta) \to \inf(\mathcal{P})$ as $\eta \to \infty$ (for every y!) and provides us with the desired fact that the $\sup(\widetilde{\mathcal{D}})$ can't be less than $\inf(\mathcal{P})$.

The assertion about the exact penalty property is now immediate from Theorem 6.3. The finiteness of $\widetilde{g}$ results from the compactness of X and the continuity of $\widetilde{L}$ in the formula for $\widetilde{g}(y, \eta)$ in $(\widetilde{\mathcal{D}})$. □

The augmented dual problem $(\widetilde{\mathcal{D}})$ doesn't actually have to be solved fully in order to make use of the exact penalty property in (6.10). Schemes have been devised which alternate between minimizing the Lagrangian expression in (6.10) and updating the y and η values on which the expression depends. The root idea for such "multiplier methods," as they are called, came from Hestenes [41] and Powell [42] who treated equality-constrained problems in a local sense without appeal to a dual problem. The full formulation in terms of problem $(\widetilde{\mathcal{D}})$ appeared in Rockafellar [45] (convex case) and [46], [47] (nonconvex case).

7. Augmented Lagrangian properties. The augmented Lagrangian $\widetilde{L}$ has many other interesting properties which further support its numerical and theoretical roles. We go over these now, aiming in particular at the connections with second-order optimality conditions.

PROPOSITION 7.1. *The expression $\widetilde{L}(x, y, \eta)$ is nondecreasing in η and concave in (y, η) (for $\eta > 0$). In the convex case of $(\mathcal{P})$ it is convex in $x \in X$.*

Proof. For fixed x and y the terms involving η are nondecreasing functions of that variable over $(0, \infty)$, with the two expressions in the case of $i \in \{1, \dots, s\}$ agreeing at the crossover points. Therefore $\widetilde{L}(x, y, \eta)$ is nondecreasing in η. For fixed x, each of the terms with $i \in \{s + 1, \dots, m\}$ is affine in (y, η). Each of the terms with $i \in \{1, \dots, s\}$ is concave in (y, η) because it's the pointwise infimum of a collection of affine functions of y_i and η, as seen from (6.11). Since a sum of concave and affine functions is concave, we conclude that $\widetilde{L}(x, y, \eta)$ is concave in (y, η).

Suppose now that $f_0, f_1, \dots, f_s$ are convex on X, while $f_{s+1}, \dots, f_m$ are affine. In this case the terms in $\widetilde{L}(x, y, \eta)$ coming from $i = 0$ and $i \in \{s+1, \dots, m\}$ are obviously convex in $x \in X$. The term coming from each $i \in \{1, \dots, s\}$ has the form $\psi_i\big(f_i(x)\big)$ for a certain nondecreasing, convex function ψ_i on $\mathbb{R}$, so it inherits the convexity of f_i. A sum of convex functions is convex, so this gives us the convexity of $\widetilde{L}(x, y, \eta)$ in $x \in X$. □

One consequence of Proposition 7.1 is the concavity of the function $\widetilde{g}$ maximized in the augmented dual problem $(\widetilde{\mathcal{D}})$, regardless of whether $(\mathcal{P})$ is of convex type or not. This stems from the fact that $\widetilde{g}$ is by definition the "lower envelope" of the family of functions $\widetilde{L}(x,\cdot,\cdot)$ indexed by $x \in X$, and these functions are concave.

Differentiability properties of $\widetilde{L}$ are next on the agenda. The bipartite character of the formula for $\widetilde{L}$ notwithstanding, first derivatives and even one-sided second derivatives exist when the functions f_i have such derivatives. In bringing this out we'll follow the custom in optimization theory of denoting the second-derivative matrix of f_i at x by $\nabla^2 f_i(x)$, and for the Lagrangian L writing

$$\nabla^2_{xx} L(x,y) = \nabla^2 f_0(x) + y_1 \nabla^2 f_1(x) + \cdots + y_m \nabla^2 f_m(x).$$

Proposition 7.2. *The augmented Lagrangian $\widetilde{L}$ is $\mathcal{C}^1$, because the functions f_i are C^1. It is $\mathcal{C}^2$ if the functions f_i are $\mathcal{C}^2$, but only away from the points (x,y,η) satisfying the transition equation $f_i(x) = -y_i/\eta$ for some $i \in \{1,\dots,s\}$. However, the first derivatives are locally Lipschitz continuous everywhere in that case.*

At any $(\bar{x},\bar{y},\bar{\eta})$ (with $\bar{\eta} > 0$), and with $\bar{y}^+$ denoting the vector with components

$$\bar{y}_i^+ = \begin{cases} \max\{0, \bar{y}_i + \bar{\eta} f_i(\bar{x})\} & \textit{for } i = 1,\dots,s, \\ \bar{y}_i & \textit{for } i = s+1,\dots,m, \end{cases} \tag{7.1}$$

one has the first derivative formulas

$$\begin{aligned} \nabla_x \widetilde{L}(\bar{x},\bar{y},\bar{\eta}) &= \nabla L(\bar{x},\bar{y}^+), \\ \nabla_y \widetilde{L}(\bar{x},\bar{y},\bar{\eta}) &= \bar{y}^+ - \bar{y}, \\ \nabla_\eta \widetilde{L}(\bar{x},\bar{y},\bar{\eta}) &= (\bar{\eta}/2)\big|\bar{y}^+ - \bar{y}\big|^2. \end{aligned} \tag{7.2}$$

In the $\mathcal{C}^2$ case of the f_i's, even when $(\bar{x},\bar{y},\bar{\eta})$ does satisfy one or more of the transition equations, there is the second-order expansion

$$\begin{aligned} \widetilde{L}(\bar{x}+w,\bar{y},\bar{\eta}) = \widetilde{L}(\bar{x},\bar{y},\bar{\eta}) &+ \big\langle \nabla_x L(\bar{x},\bar{y}^+), w\big\rangle + \tfrac{1}{2}\big\langle w, \nabla^2_{xx} L(\bar{x},\bar{y}^+) w\big\rangle \\ &+ \frac{\bar{\eta}}{2}\Bigg[\sum_{i\in I_+} \big\langle f_i(\bar{x}), w\big\rangle^2 + \sum_{i \in I_0} \max\big\{0, \big\langle f_i(\bar{x}), w\big\rangle\big\}^2\Bigg] + o\big(|w|^2\big), \end{aligned} \tag{7.3}$$

where I_+ consists of the indices $i \in \{1,\dots,s\}$ having $f_i(\bar{x}) > -\bar{y}_i/\bar{\eta}$ along with all the indices $i \in \{s+1,\dots,m\}$, and I_0 consists of the indices $i \in \{1,\dots,s\}$ having $f_i(\bar{x}) = -\bar{y}_i/\bar{\eta}$.

Proof. The only challenge is in the terms in definition (6.7) for the inequality constraint functions. These can also be written as $(1/2\eta)\Big[\max\big\{0, y_i + \eta f_i(x)\big\}^2 - y_i^2\Big]$, and properties of the function $\theta(s) = \frac{1}{2}\max\{0,s\}^2$ on $\mathbb{R}$ then come into action. This function has continuous first derivative $\theta'(s) = \max\{0,s\} \geq 0$ as well as continuous second derivatives away from $s = 0$, where it nonetheless has right second derivative $\theta''_+(0) = 1$ and left second derivative $\theta''_-(0) = 0$. The formulas in the proposition are obtained by straightforward calculation using these properties. □

Theorem 7.3. *If $\bar{x}$ and $\bar{y}$ furnish a saddle point $(\bar{x},\bar{y},\bar{\eta})$ of the augmented Lagrangian $\widetilde{L}$ on $X \times \widetilde{Y}$ for some $\bar{\eta} > 0$, then $\bar{x}$ and $\bar{y}$ satisfy the normality condition $(\mathcal{L})$ for the ordinary Lagrangian L on $X \times Y$ (and in particular, $\bar{y}$ must belong to Y).*

In the convex case of $(\mathcal{P})$ *the two conditions are equivalent. Then in fact* $(\bar{x}, \bar{y}, \bar{\eta})$ *is a saddle point of* $\widetilde{L}$ *on* $X \times \widetilde{Y}$ *for every* $\bar{\eta} > 0$.

Proof. The saddle point condition for $\widetilde{L}$ on $X \times \widetilde{Y}$ entails

$$-\nabla_x \widetilde{L}(\bar{x}, \bar{y}, \bar{\eta}) \in N_X(\bar{x}), \qquad \nabla_y \widetilde{L}(\bar{x}, \bar{y}, \bar{\eta}) = 0, \qquad \nabla_\eta \widetilde{L}(\bar{x}, \bar{y}, \bar{\eta}) = 0,$$

where the first relation follows from Theorem 3.6 as applied to the minimization of $\widetilde{L}(x, \bar{y}, \bar{\eta})$ over $x \in X$ being attained at $\bar{x}$. The derivative conditions in the y and η arguments are equivalent in the notation of Proposition 7.2 to having $\bar{y} = \bar{y}^+$, this vector being by its definition an element of Y. Then $\nabla_x \widetilde{L}(\bar{x}, \bar{y}, \bar{\eta}) = \nabla_x L(\bar{x}, \bar{y})$ by Proposition 7.2. But also, the equation $\bar{y} = \bar{y}^+$ is another way of expressing (4.5), which is identical to $\nabla_y L(\bar{x}, \bar{y}) \in N_Y(\bar{y})$. Thus, the saddle point condition always implies condition $(\mathcal{L})$.

In the convex case of $(\mathcal{P})$ the function p is convex by Proposition 6.2, so the p inequality in (6.2) is equivalent to the one in (6.9) for any $\bar{\eta} > 0$. (This property in convex analysis will be confirmed in the subgradient discussion in §8, cf. Propositions 8.5 and 8.6.) The remaining assertions of the theorem are in this way consequences of the links forged in Theorems 6.1 and 6.3. $\square$

THEOREM 7.4. *Suppose that* X *is polyhedral (for instance* $X = \mathbb{R}^n$*) and all the functions* f_i *are* $\mathcal{C}^2$*. For any feasible solution* x *to* $(\mathcal{P})$*, let* $W(x)$ *denote the set of tangent vectors* $w \in T_X(x)$ *such that*

$$\langle \nabla f_i(x), w \rangle \begin{cases} \leq 0 & \text{for } i = 0 \text{ and for } i \in \{1, \ldots, s\} \text{ with } f_i(x) = 0, \\ = 0 & \text{for } i \in \{s+1, \ldots, m\}. \end{cases}$$

(a) *If* $\bar{x}$ *and* $\bar{y}$ *are such that, for some neighborhood* V *of* $\bar{x}$ *and some* $\bar{\eta} > 0$, $(\bar{x}, \bar{y}, \bar{\eta})$ *gives a saddle point of the augmented Lagrangian* $\widetilde{L}$ *on* $(X \cap V) \times \widetilde{Y}$*, then not only do* $\bar{x}$ *and* $\bar{y}$ *satisfy the first-order condition* $(\mathcal{L})$*, but also*

$$\Big\langle w, \nabla^2_{xx} L(\bar{x}, \bar{y}) w \Big\rangle \geq 0 \text{ for all } w \in W(\bar{x}). \tag{7.4}$$

(b) *If* $\bar{x}$ *and* $\bar{y}$ *satisfy the first-order condition* $(\mathcal{L})$ *and have the property that*

$$\Big\langle w, \nabla^2_{xx} L(\bar{x}, \bar{y}) w \Big\rangle > 0 \text{ for all nonzero } w \in W(\bar{x}), \tag{7.5}$$

then, for some neighborhood V *of* $\bar{x}$ *and some* $\bar{\eta} > 0$, $(\bar{x}, \bar{y}, \bar{\eta})$ *gives a saddle point of the augmented Lagrangian* $\widetilde{L}$ *on* $(X \cap V) \times \widetilde{Y}$*, moreover one that's strong in the sense that there exists* $\varepsilon > 0$ *with* $\widetilde{L}(x, \bar{y}, \bar{\eta}) \geq \widetilde{L}(\bar{x}, \bar{y}, \bar{\eta}) + \varepsilon |x - x|^2$ *for all* $x \in X$ *near* $\bar{x}$*. Under the additional assumptions that* X *is bounded and* $\bar{x}$ *is the unique globally optimal solution to* $(\mathcal{P})$*, the value of* $\bar{\eta}$ *can be chosen to ensure that* $(\bar{x}, \bar{y}, \bar{\eta})$ *gives a saddle point of* $\widetilde{L}$ *on the whole set* $X \times \widetilde{Y}$.

Proof. In both (a) and (b) we're in a situation where $(\mathcal{L})$ holds (as follows in the case of (a) from the preceding theorem). Since X is convex, we can use Proposition 3.4 to write the condition $\nabla_x L(\bar{x}, \bar{y}) \in N_X(\bar{x})$ in $(\mathcal{L})$ as

$$\big\langle \nabla_x L(\bar{x}, \bar{y}),\, x - \bar{x} \big\rangle \geq 0 \text{ for all } x \in X. \tag{7.6}$$

Because X isn't just convex but polyhedral, there's a value $\delta > 0$ such that

$$0 \neq w \in T_X(\bar{x}) \iff \bar{x} + \big(\delta/|w|\big) w \in X \iff \bar{x} + tw \in X \text{ for all } t \in \big[0, \delta/|w|\big], \tag{7.7}$$

so (7.6) can be expressed equally well as

$$\big\langle \nabla_x L(\bar{x},\bar{y}), w\big\rangle \geq 0 \text{ for all } w \in T_X(\bar{x}). \tag{7.8}$$

The condition $\nabla_y L(\bar{x},\bar{y}) \in N_Y(\bar{y})$ in $(\mathcal{L})$ is equivalent in the notation of Proposition 7.2 to $\bar{y}^+ = \bar{y}$, which asserts that

$$\begin{aligned} \max\big\{f_i(\bar{x}), -\bar{y}_i/\bar{\eta}\big\} = 0 \text{ for } i = 1,\dots,s,\\ f_i(\bar{x}) = 0 \text{ for } i = s+1,\dots,m. \end{aligned} \tag{7.9}$$

Hence, despite the general dependence of $\bar{y}^+$ on $\bar{\eta}$ as well as on $\bar{x}$ and $\bar{y}$, if $\bar{y}^+ = \bar{y}$ holds for some $(\bar{x},\bar{y},\bar{\eta})$ it continues to hold when $\bar{\eta}$ is shifted to any other value $\eta > 0$ (while $\bar{x}$ and $\bar{y}$ are kept fixed). Note also from (7.9) that

$$\text{for } i \in \{1,\dots,s\}: \quad \begin{cases} f_i(\bar{x}) > -\bar{y}_i/\bar{\eta} \iff \bar{y}_i > 0,\\ f_i(\bar{x}) = -\bar{y}_i/\bar{\eta} \iff \bar{y}_i = 0. \end{cases}$$

Therefore, in the context of both (a) and (b), the second-order expansion in Proposition 7.2 takes the form that, for any $\eta > 0$,

$$\begin{aligned} \widetilde{L}(\bar{x}+w,\bar{y},\eta) = \widetilde{L}(\bar{x},\bar{y},\eta) + \big\langle \nabla_x L(\bar{x},\bar{y}), w\big\rangle + \tfrac{1}{2}\big\langle w, \nabla^2_{xx} L(\bar{x},\bar{y}) w\big\rangle \\ + \frac{\eta}{2}\left[\sum_{i\in I_+} \big\langle f_i(\bar{x}), w\big\rangle^2 + \sum_{i\in I_0} \max\big\{0, \big\langle f_i(\bar{x}), w\big\rangle\big\}^2\right] + o\big(|w|^2\big) \end{aligned} \tag{7.10}$$

with the index sets given by

$$\begin{aligned} i \in I_+ &\iff i \in \{1,\dots,s\} \text{ with } \bar{y}_i > 0, \text{ or } i \in \{s+1,\dots,m\},\\ i \in I_0 &\iff i \in \{1,\dots,s\} \text{ with } \bar{y}_i = 0. \end{aligned} \tag{7.11}$$

In view of (7.8), where $\nabla_x L(\bar{x},\bar{y}) = f_0(\bar{x}) + \bar{y}_1 \nabla f_1(\bar{x}) + \cdots + \bar{y}_m \nabla f_m(\bar{x})$, these index sets can also be used to redescribe $W(\bar{x})$:

$$w \in W(\bar{x}) \iff \begin{cases} w \in T_X(\bar{x}),\\ \big\langle \nabla_x L(\bar{x},\bar{y}), w\big\rangle = 0,\\ \big\langle \nabla f_i(\bar{x}), w\big\rangle = 0 \text{ for } i \in I_+,\\ \big\langle \nabla f_i(\bar{x}), w\big\rangle \leq 0 \text{ for } i \in I_0. \end{cases} \tag{7.12}$$

Going now to the specifics of (a), let's suppose $(\bar{x},\bar{y},\bar{\eta})$ gives a saddle point of $\widetilde{L}$ on $(X \cap V) \times \widetilde{Y}$ and consider any nonzero vector $w \in W(\bar{x})$. In particular the properties in (7.7) hold for w, and since $\widetilde{L}(x,\bar{y},\bar{\eta})$ achieves a local minimum relative to $x \in X$ at $\bar{x}$ the function $\psi(t) = \widetilde{L}(\bar{x}+tw,\bar{y},\bar{\eta})$ must therefore achieve a local minimum over the interval $\big[0, \delta/|w|\big]$ at $t = 0$. This implies $\psi'(0) \geq 0$, but from (7.10)–(7.12) we have the first derivative value $\psi'_w(0) = 0$ and the *right* second derivative value $\psi''_+(0) = \big\langle w, \nabla^2_{xx} L(\bar{x},\bar{y}) w\big\rangle$. Accordingly, the latter must be nonnegative. Hence the inequality in (7.4) is correct.

To argue (b) we write (7.10) as

$$\widetilde{L}(\bar{x}+w,\bar{y},\eta) = \widetilde{L}(\bar{x},\bar{y},\eta) + \Psi_1(w) + \Psi_2(w) + \eta\Psi_3(w) + o\big(|w|^2\big) \tag{7.13}$$

for the functions

$$\begin{aligned}
\Psi_1(w) &= \big\langle \nabla_x L(\bar{x},\bar{y}), w \big\rangle,\\
\Psi_2(w) &= \tfrac{1}{2}\big\langle w, \nabla^2_{xx} L(\bar{x},\bar{y}) w \big\rangle,\\
\Psi_3(w) &= \tfrac{1}{2}\left[\sum_{i\in I_+} \big\langle f_i(\bar{x}), w\big\rangle^2 + \sum_{i\in I_0} \max\big\{0, \big\langle f_i(\bar{x}), w\big\rangle\big\}^2 \right].
\end{aligned}$$

For the δ in (7.7) let $S = \big\{ w \in T_X(\bar{x}) \,\big|\, |w| = \delta \big\}$. On the compact set S the function $\Psi_1 + \Psi_3$ is positive except at the points in $S \cap W(\bar{x})$, where it vanishes. But under our assumption that (7.5) holds we have Ψ_2 positive at such points. On the compact subset of S where $\Psi_2(w) \le 0$, let λ be the minimum value of $\Psi_1 + \Psi_3$ and $-\mu$ be the minimum value of Ψ_2. Then $\lambda > 0$ and $\mu \ge 0$. Choose any value $\bar{\eta} > \mu/\lambda$. Then $\bar{\eta}(\Psi_1 + \Psi_3) + \Psi_2$ is positive on all of S; let a positive lower bound be ε_0. Consider now any nonzero $w \in T_X(\bar{x})$. We have $(\delta/|w|)w \in S$, so that

$$\varepsilon_0 \le \big[\bar{\eta}(\Psi_1+\Psi_3)+\Psi_2\big]\big((\delta/|w|)w\big) = \bar{\eta}(\delta/|w|)\Psi_1(w)+(\delta/|w|)^2\Psi_2(w)+\bar{\eta}(\delta/|w|)^2\Psi_2(w),$$

which implies that $(\varepsilon_0/\delta^2)|w|^2 \le (\bar{\eta}/\delta)|w|\Psi_1(w) + \Psi_2(w) + \bar{\eta}\Psi_3(w)$ and therefore that

$$\big(\Psi_1 + \Psi_2 + \bar{\eta}\Psi_3\big)(w) \ge (\varepsilon/\delta^2)|w|^2 \text{ when } |w| \le \delta/\bar{\eta}.$$

By virtue of the expansion (7.10) there will then exist $\delta_0 > 0$ and $\varepsilon > 0$ such that

$$\widetilde{L}(\bar{x}+w,\bar{y},\bar{\eta}) \ge \widetilde{L}(\bar{x},\bar{y},\bar{\eta}) + \varepsilon|w|^2 \text{ when } w \in T_X(\bar{x}),\ |w| \le \delta_0.$$

By taking $\delta_0 \le \delta$ and utilizing (7.7), we can transform this condition into the assertion

$$\widetilde{L}(x,\bar{y},\bar{\eta}) \ge \widetilde{L}(\bar{x},\bar{y},\bar{\eta}) + \varepsilon|x-\bar{x}|^2 \text{ when } x \in X,\ |x-\bar{x}| \le \delta_0.$$

Then $\widetilde{L}(x,\bar{y},\bar{\eta})$ has a minimum at $\bar{x}$ relative to $x \in X \cap V$ for some neighborhood V of $\bar{x}$.

At the same time we have $\nabla_y \widetilde{L}(\bar{x},\bar{y},\bar{\eta}) = 0$ and $\nabla_\eta \widetilde{L}(\bar{x},\bar{y},\bar{\eta}) = 0$, because these equations are equivalent by Proposition 7.2 to $\bar{y}^+ = \bar{y}$, which as we've seen earlier is a consequence of the condition $\nabla_y L(\bar{x},\bar{y}) \in N_Y(\bar{y})$. Because $L(\bar{x},y,\eta)$ is a smooth, concave function of (y,η) on $\widetilde{Y} = \mathbb{R}^m \times (0,\infty)$ (by Propositions 7.1 and 7.2), these equations say that $\widetilde{L}(\bar{x},y,\eta)$ achieves its maximum relative to $\widetilde{Y}$ at $(\bar{y},\bar{\eta})$. Thus, $\widetilde{L}$ has a saddle point on $X \times \widetilde{Y}$ at $(\bar{x},\bar{y},\bar{\eta})$. □

The sufficient condition for a saddle point in Theorem 7.4(b) was proved in Rockafellar [46] for the case $X = \mathbb{R}^n$; the extension to any polyhedral set X is new here. A somewhat more limited result with $X = \mathbb{R}^n$ was proved earlier by Arrow, Gould and Howe [48]. This property in combination with the earlier characterization of saddle points of the augmented Lagrangian in Theorem 6.3 leads at once to a sufficient condition for optimality in $(\mathcal{P})$ that doesn't require convexity.

THEOREM 7.5. *Suppose in $(\mathcal{P})$ that X is polyhedral and all the functions f_i are $\mathcal{C}^2$. If the vectors $\bar{x} \in X$ and $\bar{y} \in Y$ satisfy the first-order condition $(\mathcal{L})$ together with the strict second-order condition (7.5), $\bar{x}$ must be a locally optimal solution to problem $(\mathcal{P})$. In fact there must exist $\varepsilon > 0$ and $\delta > 0$ such that*

$$f_0(x) \ge f_0(\bar{x}) + \varepsilon|x-\bar{x}|^2 \textit{ for all } x \in C \textit{ with } |x-\bar{x}| \le \delta.$$

Proof. We apply Theorem 7.4(b). When $(\bar{x}, \bar{y}, \bar{\eta})$ is a saddle point of $\widetilde{L}$ on $(X \cap V) \times \tilde{Y}$, we obtain from Theorem 6.3 that $\bar{x}$ is an optimal solution to the localization of $(\mathcal{P})$ in which X is replaced by $X \cap V$. The final assertion of the theorem yields the final assertion of the corollary, because $f_0(x) \geq \widetilde{L}(x, \bar{y}, \bar{\eta})$ for all feasible x. $\square$

In the case of $X = \mathbb{R}^n$ this sufficient condition is well known, although most writers on the subject formulate the second-order inequality in terms of the set of vectors w satisfying

$$\langle \nabla f_i(x), w \rangle \begin{cases} \leq 0 & \text{for } i \in \{1, \ldots, s\} \text{ with } f_i(x) = 0 \text{ and } \bar{y}_i = 0, \\ = 0 & \text{for } i \in \{1, \ldots, s\} \text{ with } f_i(x) = 0 \text{ and } \bar{y}_i > 0 \\ & \quad \text{and for } i \in \{s+1, \ldots, m\}, \end{cases}$$

instead of $W(\bar{x})$. The two sets are identical for any multiplier vector $\bar{y}$ appearing in condition $(\mathcal{L})$, so the seeming dependence on $\bar{y}$ in this other formulation is misleading. The sufficiency for $\bar{x}$ to be locally optimal when a multiplier vector exists with all components $\bar{y}_i > 0$ (the case known as "strict complementarity") was proved already in 1939 by Karush [10]. For other early precedents, see Pennisi [52].

Second-order sufficient conditions for optimality like the ones in Theorem 7.5 are heavily exploited in the development of numerical methods, not only the "multiplier methods" mentioned in §6. The analysis provided here shows that such approaches correspond in principle to exploiting saddle point properties of $\widetilde{L}$, although this isn't well recognized by practitioners and therefore perhaps not exploited as far as it might go. The underlying saddle point implications of the sufficient condition in Theorem 7.5 partly explain the vitality of this condition, in contrast to technically more refined second-order conditions available in the literature, which typically involve more than one multiplier vector $\bar{y}$ in association with $\bar{x}$, cf. Ioffe [53], Ben-Tal [54], and Rockafellar [55].

Despite all this good news about the augmented Lagrangian, $\widetilde{L}$ has some shortcomings in comparison with L. Most notably it does not inherit from L any separability which might be present with respect to components of x. Such separability is sometimes crucial in numerical schemes for solving large-scale problems.

8. Subderivatives and subgradients. In Theorems 6.1 and 6.3 the Lagrange multiplier vector $\bar{y}$ appears as the gradient at $u = 0$ of a smooth function h such that $h \leq p$ and $h(0) = p(0)$. This relationship leads to one of the central notions in nonsmooth analysis, that of a subgradient of a possibly nonsmooth and even extended-real-valued function p. In explaining this notion we'll keep to the notation of a function p on $\mathbb{R}^m$, because that's where we initially want to make applications, namely to the potential interpretation of Lagrange multipliers as "generalized derivatives" in a sense. But for the time being p can be any function on $\mathbb{R}^m$, not necessarily the canonical value function in (6.1). In §9 the canonical value function will resurface. In §10 the same ideas will be applied instead to generalized objective functions f on $\mathbb{R}^n$.

As with the concept of normal vectors in Definition 3.2, there are different kinds of subgradients in the literature of nonsmooth analysis, and we're presenting only one of these, which nowadays appears to offer the greatest advantages. In analogy with the geometric relationship between normals and tangents, we consider along with these basic subgradients a particular kind of directional derivative, the basic subderivative.

DEFINITION 8.1. Consider any function $p : \mathbb{R}^m \to [-\infty, \infty]$ and any point $\bar{u}$ where $p(\bar{u})$ is finite. For each vector $\bar{w} \in \mathbb{R}^m$ the *subderivative* of p at $\bar{u}$ with respect to w is the value

$$dp(\bar{u})(\bar{w}) = \liminf_{\substack{w \to \bar{w} \\ t \downarrow 0}} \big[p(\bar{u} + tw) - p(\bar{u})\big]/t. \tag{8.1}$$

DEFINITION 8.2. Consider any function $p : \mathbb{R}^m \to [-\infty, \infty]$ and any point $\bar{u}$ where $p(\bar{u})$ is finite. A vector $y \in \mathbb{R}^m$ is a *subgradient* of p at $\bar{u}$, written $y \in \partial p(\bar{u})$, if there's a sequence of vectors $y^k \to y$ along with a sequence of points $u^k \to \bar{u}$ with $p(u^k) \to p(\bar{u})$ such that, for each k,

$$p(u) \geq p(u^k) + \langle y^k, u - u^k \rangle + o\big(|u - u^k|\big). \tag{8.2}$$

It is a *regular* subgradient if the sequences can be chosen constant, i.e., if actually

$$p(u) \geq p(\bar{u}) + \langle y, u - \bar{u} \rangle + o\big(|u - \bar{u}|\big). \tag{8.3}$$

If p happens to be differentiable at $\bar{u}$, the vector $\bar{y} = \nabla p(\bar{u})$ is obviously a regular subgradient at $\bar{u}$, and then $dp(\bar{u})(w) = \langle \bar{y}, w \rangle$ for all w. But subgradients can usefully be studied in the absence of differentiability, which may well have to be faced when p is the canonical value function in §6. In general, the subderivative function $dp(\bar{u})$ can be extended-real-valued like p.

If p has a local minimum at $\bar{u}$, the vector $y = 0$ is a regular subgradient there. Right from the definitions, therefore, we have a basic necessary condition for local optimality:

$$0 \in \partial p(\bar{u}).$$

A major preoccupation of nonsmooth analysis is the development of calculus rules to facilitate the application of this condition specific situations where a function is minimized and is expressible in terms of operations like addition, composition, etc. In §10 we'll see a chain rule along such lines. Anyway, out of such considerations in contexts where p may be extended-real-valued it's convenient to adopt the convention that $\partial p(\bar{u})$ denotes the empty set when $p(\bar{u}) = \infty$, but all of $\mathbb{R}^m$ if $p(\bar{u}) = -\infty$. This corresponds to the convention in §3 of interpreting the notation $N_C(\bar{x})$ as referring to the empty set when $\bar{x}$ doesn't happen to lie in C.

The limit process used in defining subgradients is parallel to the one used for normals in Definition 3.2 and has similar motivations. It ensures the following property.

PROPOSITION 8.3. *If $y^k \in \partial p(u^k)$ and $y^k \to y$, $u^k \to \bar{u}$, $p(u^k) \to p(\bar{u})$, then $y \in \partial p(\bar{u})$.*

The place of these concepts of subderivatives and subgradients in the literature of nonsmooth analysis is much like that of the tangents and normals in §3. The idea of defining one-sided substitutes for classical directional derivatives and gradients first took on fundamental importance in convex analysis (cf. [1]), where more special formulas suffice (see Proposition 8.5 below). The generalized directional derivatives at $\bar{u}$ expressed by the function $dp(\bar{u}) : \mathbb{R}^m \to [-\infty, \infty]$ as defined through (8.1) have been considered at least implicitly by most researchers on optimality conditions, although when made explicit the names have varied; see Penot [27] and Ioffe [56] for some background. The same goes for the vectors we're calling regular subgradients, which have been introduced at times through an inequality like (8.3) but also through an equivalent relationship with the set $dp(\bar{u})$ (cf. property (b) in Proposition 8.6 below); again see [27] and [56].

Clarke [16] achieved a breakthrough by introducing a limit process in the definition of subgradients; this opened up an impressive array of new applications, cf. [17], [18]. Clarke's formulation wasn't that of Definition 8.2 but involved a three-stage process, where (1) Lipschitz continuous functions were handled in terms of limits of their actual gradients where such exist (which is almost everywhere), (2) this was applied to distance functions to define normal cones to general closed sets, and (3) normals to epigraph sets were used to get subgradients of non-Lipschitz functions (much in the spirit of Proposition 8.4 below). Rockafellar [57] demonstrated how the three stages could be collapsed to one by taking limits of "proximal subgradients," essentially following the pattern in Definition 8.2 but with the "o" term replaced by a second-order term. In all this, however, Clarke took not only limits but convex hulls, and in the one-stage version of Rockafellar these were complicated to express. As explained in §3, it's now appreciated, especially through the efforts of Mordukhovich (cf. [24]), that wholesale convexification in nonsmooth analysis can be avoided, although infinite-dimensional applications such as those treated by Clarke tend to necessitate it in the end anyway (because of properties of weak convergence).

The study of subderivatives and subgradients is very closely related to that of tangent and normal vectors in the geometric setting of the epigraph of p, as defined in (6.6).

PROPOSITION 8.4. *For any function $p : \mathbb{R}^m \to [-\infty, \infty]$ and any point $\bar{u}$ where $p(\bar{u})$ is finite, one has*

$$y \in \partial p(\bar{u}) \iff (y, -1) \in N_E\big(\bar{u}, p(\bar{u})\big), \text{ where } E = epi\, p. \tag{8.4}$$

Regular subgradients y of p correspond in this manner to regular normals $(y, -1)$ to E. Furthermore, the epigraph of $dp(\bar{u})$ is the cone $T_E\big(\bar{u}, p(\bar{u})\big)$.

Proof. The tangent cone assertion is an elementary consequence of the observation that, for $t > 0$, one has $\big(\bar{u}, p(\bar{u})\big) + t(w, \beta) \in E$ if and only if $\beta \geq \big[p(\bar{u} + tw) - p(\bar{u})\big]/t$.

Next we take on the assertion that y is a regular subgradient of p at $\bar{u}$ if and only if $(y, -1)$ is a regular normal to E at $\big(\bar{u}, p(\bar{u})\big)$. It suffices to treat the case where $y = 0$, since the general case can be reduced to this by substituting the function $q(u) = p(u) - \langle y, u - \bar{u}\rangle$ for p. Applying Proposition 3.5, we see that $(0, -1)$ is a regular normal to E at $\big(\bar{u}, p(\bar{u})\big)$ if and only if $\big\langle (0, -1), (w, \beta)\big\rangle \leq 0$ for all $(w, \beta) \in \mathbb{R}^m \times \mathbb{R}$ with $\beta \geq dp(\bar{u})(w)$. This property is the same as $dp(\bar{u})(w) \geq 0$ for all w, which in view of Definition 8.1 is equivalent to $p(u) \geq p(\bar{u}) + o\big(|u - \bar{u}|\big)$, the condition for 0 to be a regular subgradient. Thus, the subgradient-normal correspondence is correct in the regular case. The general case is then immediate from the limit process in Definition 8.2 and Definition 3.2. □

For orientation with the general literature, although it won't be needed here, we remark that subgradients in the sense of Clarke are the vectors obtained in replacing the normal cone $N_E\big(\bar{u}, p(\bar{u})\big)$ in (8.4) by its closed convex hull.

Through the pipeline in Proposition 8.4 the tangent and normal vector results in §3 can be transferred quickly to the theory of subderivatives and subgradients.

PROPOSITION 8.5. *If the function p is convex, the subgradient set $\partial p(\bar{u})$ is the set of vectors y such that*

$$p(u) \geq p(\bar{u}) + \big\langle y, u - \bar{u}\big\rangle \text{ for all } u. \tag{8.5}$$

Every subgradient of a convex function is therefore a regular subgradient. On the other hand, for any $\bar{w}$ such that an $\varepsilon > 0$ exists with $p(\bar{u} + \varepsilon w) < \infty$ for all w in a neighborhood of $\bar{w}$ (this being true for every vector $\bar{w}$ when p is finite on a neighborhood of $\bar{u}$), one has

$$dp(\bar{u})(\bar{w}) = \lim_{t \downarrow 0} \big[p(\bar{u} + tw) - p(\bar{u})\big]/t. \tag{8.6}$$

Proof. Invoke Proposition 3.4 for the set $E = epi\, p$ to get the subgradient result. The subderivative result is more subtle in utilizing continuity properties of convex functions, and we won't prove it here; see [1, §23]. □

PROPOSITION 8.6. *For any function $p : \mathbb{R}^m \to [-\infty, \infty]$ and any point $\bar{u}$ where $p(\bar{u})$ is finite, the following properties of a vector y are equivalent:*

(a) *y is a regular subgradient of p at $\bar{u}$;*

(b) *$\langle y, w\rangle \le dp(\bar{u})(w)$ for all w;*

(c) *on some open neighborhood O of $\bar{u}$ there is a smooth function h with $\nabla h(\bar{u}) = y$, such that $h(u) \le p(u)$ for all $u \in O$, and $h(\bar{u}) = p(\bar{u})$.*

Proof. While this can be derived from Proposition 3.5 through Proposition 8.4, the earlier proof can also be imitated in the new context. □

We likewise want to translate Proposition 3.8 to subderivatives and subgradients, and this requires a bit of technical groundwork because closedness properties of $E = \mathit{epi}\, p$ get involved. The closedness of E as a subset of $\mathbb{R}^m \times \mathbb{R}$ corresponds to the *lower semicontinuity* of p on $\mathbb{R}^m$, which means that for every $\alpha \in \mathbb{R}$ the level set $\{\, u \,|\, p(u) \le \alpha \,\}$ is closed, or equivalently that $\liminf_{u' \to u} p(u') = p(u)$ for all $u \in \mathbb{R}^m$. Similarly, for a point $\bar{u}$ where p is finite, E is locally closed at $\big(\bar{u}, p(\bar{u})\big)$ if and only if p is *locally lower semicontinuous* at $\bar{u}$ in the sense that, for some $\varepsilon > 0$ and some neighborhood V of $\bar{u}$, the sets of the form $\{\, u \in V \,|\, p(u) \le \alpha \,\}$ with $\alpha \le p(\bar{u}) + \varepsilon$ are all closed. (The combination of p being locally lower semicontinuous and locally upper semicontinuous at $\bar{u}$ is equivalent to p being continuous on a neighborhood of $\bar{u}$. However, local lower semicontinuity of p at $\bar{u}$ doesn't require p to be lower semicontinuous relative to a neighborhood of $\bar{u}$, but only relative to the intersection of some neighborhood with a level set $\big\{\, u \,\big|\, p(u) \le p(\bar{u}) + \varepsilon \,\big\}$.)

PROPOSITION 8.7. *Let $p : \mathbb{R}^m \to [-\infty, \infty]$ be finite at $\bar{u}$, and suppose the set $E = \mathit{epi}\, p$ is Clarke regular at $\big(\bar{u}, p(\bar{u})\big)$, as is true in particular if p is locally lower semicontinuous at $\bar{u}$ and convex. Then $\partial p(\bar{u}) \ne \emptyset$ if and only if $dp(\bar{u})(w)$ is finite for some w, in which case $dp(\bar{u})(w) > -\infty$ for all w, and $dp(\bar{u})(0) = 0$. Furthermore, if $\partial p(\bar{u}) \ne \emptyset$ the function dp and set ∂p are dual to each other in the sense that*

$$\begin{aligned} dp(\bar{u})(w) &= \sup_{y \in \partial p(\bar{u})} \big\langle y, w \big\rangle \ \text{ for all } w, \\ \partial p(\bar{u}) &= \big\{\, y \,\big|\, \big\langle y, w \big\rangle \le dp(\bar{u})(w) \ \text{ for all } w \,\big\}. \end{aligned} \tag{8.7}$$

Proof. The hypothesis enables us to apply Proposition 3.8 to E and translate the results to the language of subderivatives and subgradients through the use of Proposition 8.4. □

Clarke regularity of $E = \mathit{epi}\, p$ at $\big(\bar{u}, p(\bar{u})\big)$ requires, along with the local lower semicontinuity of p at $\bar{u}$, that every normal in $N_E\big(\bar{u}, p(\bar{u})\big)$ be a regular normal. Every normal is either of the form $\lambda(y, -1)$ with $\lambda \ge 0$ or of the form $(y, 0)$ with $y \ne 0$. If there are no normals in $N_E\big(\bar{u}, p(\bar{u})\big)$ of the second kind, or more broadly, if every normal of the second kind is a limit of normals in $N_E\big(\bar{u}, p(\bar{u})\big)$ of the first kind, the Clarke regularity property is equivalent through Proposition 8.4 to having every subgradient $y \in \partial p(\bar{u})$ be a regular subgradient. (In the limit case we use here the fact that the regular normals to E at any point form a closed set, cf. property (b) in Proposition 3.5.) In general, though, there could be normals $(y, 0) \in N_E\big(\bar{u}, p(\bar{u})\big)$ that arise in more complicated ways. These can be developed as "horizon subgradients" of p at $\bar{u}$, but we won't go into that here.

Incidentally, although we've been taking the position that facts about subderivatives and subgradients are consequences of facts about tangents and normals, the opposite position is just as tenable. For any set $D \subset \mathbb{R}^m$ and any point $\bar{u} \in D$, the indicator function $p = \delta_D$ (with the value 0 on D but ∞ everywhere else) has

$$\partial \delta_D(\bar{u}) = N_D(\bar{u}).$$

The regular subgradients of δ_D at $\bar{u}$ are likewise the regular normals to D. The associated subderivative function is the indicator function for the set $T_D(\bar{u})$.

9. Multiplier vectors as subgradients. We return now to the particular function p examined in §6, the canonical value function for problem $(\mathcal{P})$. Our attention is centered on the relationship between the subderivatives and subgradients of p at $\bar{u} = 0$ on the one hand, and multiplier vectors $\bar{y}$ for $(\mathcal{P})$ on the other.

THEOREM 9.1. *When p is the canonical value function for $(\mathcal{P})$, the Lagrange multiplier vectors $\bar{y}$ in Theorems 6.1 and 6.3 are regular subgradients $\bar{y} \in \partial p(0)$, so that*

$$dp(0)(w) \geq \langle \bar{y}, w \rangle \text{ for all } w. \tag{9.1}$$

If p is happens to be differentiable at 0, this inequality implies

$$\bar{y}_i = \frac{\partial p}{\partial u_i}(0) \text{ for } i = 1, \ldots, m. \tag{9.2}$$

Similarly, if $\bar{y}$ satisfies together with some $\bar{x}$ the sufficient condition for local optimality in Theorem 7.5 (the set X being polyhedral), then for some neighborhood V of $\bar{x}$ the multiplier vector $\bar{y}$ will be a regular subgradient in $\partial p_V(0)$, where p_V is the value function obtained instead of p when X is replaced by $X \cap V$ in $(\mathcal{P})$; one will have

$$dp_V(0)(w) \geq \langle \bar{y}, w \rangle \text{ for all } w.$$

Proof. The initial assertions reinterpret the inequalities in (6.3) and (6.7) in the light of Definitions 8.1 and 8.2. The final assertion is validated by the observation that since this sufficient condition for optimality furnishes through Theorem 7.4(b) a saddle point of the augmented Lagrangian relative to $X \cap V$, it yields the inequality in (6.7) for p_V. □

The partial derivative interpretation of Lagrange multipliers in (9.2) is very appealing, but it suffers from the lack of any verifiable assumption on problem $(\mathcal{P})$ that ensures the differentiability of p at 0 (and the same for the localized value function p_V, which could be substituted for p in these considerations). It's necessary therefore to be content with looser interpretations. In the convex case, at least, there's an especially satisfying substitute which will be described in Theorem 9.3. To set the stage for it, we have to supply a good criterion for p to be locally lower semicontinuous at 0.

PROPOSITION 9.2. *A sufficient condition for the canonical value function p to be finite at 0 and locally lower semicontinuous there is the existence of a feasible solution $\hat{x}$ and an $\varepsilon > 0$ such that the set of points $x \in X$ satisfying*

$$\begin{aligned} f_0(x) &\leq f_0(\hat{x}) + \varepsilon, \\ f_i(x) &\leq \varepsilon \text{ for } i = 0, 1 \ldots, s, \\ |f_i(x)| &\leq \varepsilon \text{ for } i = s + 1 \ldots, m, \end{aligned} \tag{9.3}$$

is bounded (this being true certainly when X itself is bounded). Then for all $u \in \mathbb{R}^m$ with $|u| \leq \varepsilon$ and $p(u) < p(0)+\varepsilon$, the perturbed problem $\big(P(u)\big)$ has an optimal solution. In particular, $(\mathcal{P})$ has an optimal solution.

Proof. We have $p(0) \leq f_0(\hat{x})$, so $p(0) < \infty$. Let V denote the closed ε ball around the origin of $\mathbb{R}^m$. Our hypothesis gives the boundedness of the set

$$S_\alpha = \left\{ (x, u) \in \mathbb{R}^m \times \mathbb{R}^n \,\middle|\, u \in U,\ x \text{ feasible in } \big(P(u)\big),\ f_0(x) \leq \alpha \right\}$$

for every $\alpha \leq \bar{\alpha} = f_0(\hat{x})+\varepsilon$. Also, each such S_α is closed, hence compact. In particular, for any $u \in V$ with $p(u) < f_0(\hat{x})+\varepsilon$ the nonempty set consisting of the feasible solutions x to $\big(P(u)\big)$ with $f_0(x) \leq f_0(\hat{x}) + \varepsilon$ is the section $\{\, x \,|\, (x,u) \in S_{\bar{\alpha}} \,\}$, which is compact, so an optimal solution to $\big(P(u)\big)$ exists and $p(u) > -\infty$. In particular, $p(0) > -\infty$. It follows that for each $\alpha < \bar{\alpha}$ the level set $\{\, u \in V \,|\, p(u) \leq \alpha \,\}$ is the projection of S_α on $\mathbb{R}^m$, which is compact. Since $\bar{\alpha} > p(0)$ we conclude that p is locally lower semicontinuous at 0. □

THEOREM 9.3. *In the convex case of problem* $(\mathcal{P})$*, suppose at least one feasible solution exists and* p *is locally lower semicontinuous at* 0 *(cf. Proposition 9.2). Then*

$$\begin{aligned} \partial p(0) = & Y_{opt}, \ \ \textit{where} \\ & Y_{opt} = \big[\, \textit{set of all Lagrange multiplier vectors } \bar{y} \textit{ in } (\mathcal{P}) \,\big]. \end{aligned} \tag{9.4}$$

This set is nonempty if and only if $dp(0)(w)$ *is finite for some* w*, in which case* $\partial p(0)$ *can be identified also with the set of optimal solutions to the dual problem* $(\mathcal{D})$ *and one has*

$$dp(0)(w) \;=\; \sup_{\bar{y} \in Y_{opt}} \big\langle \bar{y}, w \big\rangle \;\; \textit{for all } w \in \mathbb{R}^m. \tag{9.5}$$

Proof. We have p convex by Proposition 6.2. By Proposition 8.5 and Theorem 6.1 the subgradients $\bar{y} \in \partial p(0)$ are the Lagrange multiplier vectors in $(\mathcal{P})$. When there is such a vector, the set of them is the optimal solution set to $(\mathcal{D})$ by Theorems 5.1 and 5.3. Since p is Clarke regular at $\bar{u}$ by Proposition 8.11, we have (9.5) by Proposition 8.7. □

The idea to digest here is that even though p may not be differentiable at 0, there's a perfect duality between the multiplier vectors $\bar{y}$ for $(\mathcal{P})$ and the subderivative values $dp(0)(w)$. From the knowledge of either it's possible to obtain the other. This, by the way, is a point that usually gets garbled in elementary books on optimization, especially linear programming texts. There Lagrange multipliers $\bar{y}_i$ are often called "shadow prices" and interpreted as if they were partial derivatives in the sense of (9.2), but this is erroneous. What's true and well known in convex analysis [1, §25] is that if p is convex and $\partial p(0)$ consists of a *unique* subgradient $\bar{y}$, then p must be differentiable at 0 with $\nabla p(0) = \bar{y}$. The partial derivative interpretation of Lagrange multipliers is correct therefore as long as the multiplier vector is unique, or equivalently, the dual problem has a unique optimal solution. But in general, increases and decreases in the parameters u_i relative to 0 may affect the optimal value at different rates, and the full set of vectors $\bar{y} \in Y_{opt}$ may need to come into play as in (9.5).

The importance of this fact for applications in economics was brought out by Gale [58]. Note that since Theorem 9.3 is directed to the convex case, the subderivatives have through Proposition 8.4 a simpler expression as limits when p is finite on a neighborhood of 0, which is the situation of main economic interest:

$$dp(0)(w) \;=\; \lim_{t \downarrow 0} \big[p(0 + tw) - p(0)\big]/t.$$

Then, taking $w = e_1 = (1, 0, \ldots, 0)$ for instance, we can interpret $dp(0)(e_1)$ as the right partial derivative of p at 0 with respect to u_1, and $-dp(0)(-e_1)$ as the corresponding left partial derivative. The equation in (9.5) informs us that the right partial derivative is the highest value of the Lagrange multiplier $\bar{y}_1$ relative to the set of all multiplier vectors $\bar{y}$ associated with $(\mathcal{P})$, whereas the left partial derivative is the lowest value. But even these left and right partial derivatives for every coordinate aren't enough to determine $dp(0)(w)$ for general w, because the function $dp(0)$ won't usually be separable with respect to its different arguments w_i.

Unfortunately, in the nonconvex case of $(\mathcal{P})$ the epigraph of p won't always be Clarke regular as demanded by Proposition 8.7, so tight analogs of (9.5) can't be hoped for. At least we can sometimes ensure however that *certain* of the multiplier vectors will be subgradients, as an extension of the facts in Theorem 9.1.

THEOREM 9.4. *Suppose in $(\mathcal{P})$ that $\bar{x}$ is a strict locally optimal solution in the sense that for some compact neighborhood V of $\bar{x}$ one has $f_0(x) > f_0(\bar{x})$ for all $x \neq \bar{x}$ in $C \cap V$. If the basic constraint qualification $(\mathcal{Q})$ is satisfied at $\bar{x}$, there not only exists a Lagrange multiplier vector $\bar{y}$ for which $(\mathcal{L})$ holds, but one such that $\bar{y} \in \partial p_V(0)$, where p_V is the value function obtained instead of p when X is replaced by $X \cap V$ in $(\mathcal{P})$.*

Proof. Replacing X by $X \cap V$, we can simplify to the case where X is compact and $\bar{x}$ is the unique globally optimal solution to $(\mathcal{P})$. Define the function φ on $\mathbb{R}^n \times \mathbb{R}^m$ as in (6.4). The unique optimal solution to the problem of minimizing $\varphi(x,u)$ subject to $u = 0$ is $(\bar{x}, 0)$. For a sequence of values $0 < \eta_k \to \infty$ consider the penalty approximation of this problem in which the function $\varphi(x,u) = \varphi(x,u) + (\eta_k/2)|u|^2$ is minimized over $\mathbb{R}^n \times \mathbb{R}^m$. Because X is not only closed but bounded, φ_k achieves its minimum at some point (x^k, u^k). The assumption that $\eta_k \to \infty$ ensures in this setting that $(x^k, u^k) \to (\bar{x}, 0)$.

Since (x^k, u^k) minimizes $\varphi^k(x,u)$ over all $(x,u) \in \mathbb{R}^n \times \mathbb{R}^m$ we know that x^k minimizes $\varphi^k(x, u^k)$ over $\mathbb{R}^n$, while u^k minimizes $\inf_x \varphi^k(x,u)$ over $\mathbb{R}^m$. But $\inf_x \varphi^k(x,u) = p(u) + (\eta_k/2)|u|^2$. Therefore x^k is an optimal solution to problem $\big(\mathcal{P}(u^k)\big)$ and $p(u) + (\eta_k/2)|u|^2 \geq p(u^k) + (\eta_k/2)|u^k|^2$ for all u. The latter can be written in terms of $y^k = -\eta_k u^k$ as

$$p(u) \geq p(u^k) + \big\langle y^k,\, u - u^k \big\rangle - \frac{\eta_k}{2}|u - u^k|^2 \text{ for all } u.$$

We deduce not only that y^k is a regular subgradient of p at u^k, but by Theorem 6.3 that (x^k, y^k, η^k) gives a saddle point of the augmented Lagrangian $\widetilde{L}^k$ for problem $\big(\mathcal{P}(u^k)\big)$, this function being the same as $\widetilde{L}$ except that the constraint functions f_i are shifted to $f_i + u_i^k$. Then also by Theorem 6.3, x^k and y^k satisfy the first-order optimality condition relative to the ordinary Lagrangian L^k for $\big(\mathcal{P}(u^k)\big)$, which likewise differs from L only by such a shift: $L^k(x,y) = L(x,y) + \langle y, u^k \rangle$. Thus,

$$-\nabla_x L(x^k, y^k) \in N_X(x^k), \qquad \nabla_y L(x^k, y^k) + u^k \in N_Y(y^k). \tag{9.6}$$

If the sequence of vectors y^k is bounded, we can suppose it converges to some $\bar{y}$, and then, since $(x^k, u^k) \to (\bar{x}, 0)$, we obtain $\bar{y} \in \partial p(0)$ by Definition 8.2, and at the same time $-\nabla_x L(\bar{x}, \bar{y}) \in N_X(\bar{x})$ and $\nabla_y L(\bar{x}, \bar{y}) \in N_Y(\bar{y})$, which is condition $(\mathcal{L})$. On the other hand, if the y^k sequence isn't bounded, we can simplify to the case where $0 < |y^k| \to \infty$ and by dividing both relations in (9.6) by $|y^k|$ have

$$-\frac{1}{|y^k|}\nabla_x L(x^k, y^k) \in N_X(x^k), \qquad \frac{1}{|y^k|}\nabla_y L(x^k, y^k) + \frac{1}{\eta_k}\frac{u^k}{|u^k|} \in N_Y\Big(\frac{y^k}{|y^k|}\Big). \tag{9.7}$$

(Here we use the definition $y^k = -\eta^k u^k$ along with the fact that normal cones contain all positive multiples of their elements. We also invoke the fact that Y is itself a cone, which implies that $N_Y(y)$ coincides with $N_Y(ty)$ for any $t > 0$.) Without loss of generality we can suppose that $y^k/|y^k|$ converges to some vector $\bar{y} \neq 0$, and then by taking the limit in (9.7) we'll have in the notation of the singular Lagrangian L_0 that $-\nabla_x L_0(\bar{x},\bar{y}) \in N_X(\bar{x})$ and $\nabla_y L_0(\bar{x},\bar{y}) \in N_Y(\bar{y})$, which is impossible under assumption $(\mathcal{Q})$. Thus, the y^k sequence has to be bounded after all, and the properties we want must hold. □

Results like Theorem 9.4 are primarily of theoretical significance, inasmuch as the assumptions may be hard to verify. The next result, however, has definite practical import, because it provides estimates of the magnitude and location of Lagrange multiplier vectors. It can be regarded as a kind of extension of Theorem 6.1 in which a single multiplier vector is replaced by a set of candidates.

THEOREM 9.5. *For the canonical value function p, suppose there is a subderivative estimate of the form*

$$dp(0)(w) \;\geq\; \min_{y\in Y_{est}} \langle y, w\rangle \;\; \text{for all } w, \tag{9.8}$$

where the set $Y_{est} \subset \mathbb{R}^m$ is nonempty, compact, and convex. Then every (globally) optimal solution $\bar{x}$ to $(\mathcal{P})$ satisfies condition $(\mathcal{L})$ for some multiplier vector $\bar{y} \in Y_{est}$.

Proof. We postpone the proof of this result to the next section, right after Theorem 10.1, because it will be much simpler in the broader framework that is available there. □

An immediate consequence of Theorem 9.5, which is stated here for the first time, is an alternative criterion, not involving the basic constraint qualification $(\mathcal{Q})$, for optimal solutions to $(\mathcal{P})$ to satisfy the first-order optimality condition $(\mathcal{L})$.

COROLLARY 9.6. *If problem $(\mathcal{P})$ is calm in the sense that $dp(0)(w) > -\infty$ for all $w \neq 0$, then for every (globally) optimal solution $\bar{x}$ to $(\mathcal{P})$ there must exist at least one multiplier vector $\bar{y}$ such that $\bar{x}$ and $\bar{y}$ satisfy condition $(\mathcal{L})$.*

More specifically, if for some $r > 0$ one has $dp(0)(w) \geq -r$ for all vectors w with $|w| = 1$, then for every optimal solution $\bar{x}$ to $(\mathcal{P})$ there is a multiplier vector $\bar{y}$ with $|\bar{y}| \leq r$ such that $\bar{x}$ and $\bar{y}$ satisfy condition $(\mathcal{L})$.

Proof. This is the case of Theorem 9.5 where Y_{est} is the closed Euclidean ball of radius r around the origin. □

Calmness in the sense of Corollary 9.6 was first used as a substitute for a direct constraint qualification by Clarke [59]. Such norm estimates are important in the theory of exact penalty methods based on linear penalties, cf. Burke [60].

Theorem 9.5 also furnishes additional insights on the matter of when Lagrange multipliers can be interpreted as partial derivatives of p as in (9.2).

COROLLARY 9.7. *If the canonical value function p happens to be differentiable at 0, then every (globally) optimal solution $\bar{x}$ satisfies condition $(\mathcal{L})$ with $\bar{y} = \nabla p(0)$ as the multiplier vector. (But $\bar{x}$ could also satisfy $(\mathcal{L})$ for some multiplier vector $\bar{y} \neq \nabla p(0)$.)*

Proof. Apply Theorem 9.5 with $Y_{est} = \{\nabla p(0)\}$. □

Other results connecting generalized differentiability properties of the canonical value function p with Lagrange multipliers for $(\mathcal{P})$ are provided in Rockafellar [21], [61], [62].

Even though caution is advisable in interpreting multipliers $\bar{y}_i$ as partial derivatives $(\partial p/\partial u_i)(0)$, Corollary 9.7 furnishes some generic support for this view. The convention of regarding the parameter vector u as a perturbation relative to 0 could be relaxed, and we could think the entire family of problems $\big(\mathcal{P}(u)\big)$ on an equal footing. The multiplier vectors for any particular problem $\big(\mathcal{P}(\bar{u})\big)$ would be analyzed in the context of the properties of p around $\bar{u}$ instead of 0. In this context Corollary 9.7 tells us that whenever p is differentiable at a point $\bar{u}$, the vector $\bar{y} = \nabla p(\bar{u})$ gives

Lagrange multipliers that work for every optimal solution $\bar{x}$ to $(\mathcal{P}(\bar{u}))$. We can then ask the question: Is it true in some situations that p fails to be differentiable only for a "few" choices of $\bar{u}$? The answer is yes in the following sense. In circumstances where p is sure to be lower semicontinuous locally at 0 (cf. Proposition 9.2), the constraint qualification $(\mathcal{Q})$ for problem $(\mathcal{P})$ implies that p is Lipschitz continuous relative to a neighborhood of 0 (we won't prove this here). A Lipschitz continuous function is differentiable almost everywhere. Then, for almost every parameter vector $\bar{u}$ in some neighborhood of 0, the partial derivative interpretation of Lagrange multipliers in $(\mathcal{P}(\bar{u}))$ is valid at least for a certain choice of the multiplier vector $\bar{y}$ in $(\mathcal{P}(\bar{u}))$.

10. Extension to composite problem models. Most of the Lagrange multiplier results we've been presenting carry over to a far more flexible statement of the basic problem of optimization than $(\mathcal{P})$. Many applications, while they can be forced into the formulation of $(\mathcal{P})$, don't fit that very comfortably because their structure has to be partially disguised. Let's consider now, as a compromise between traditional statements and full generality, the following *composite* problem model:

$$(\overline{\mathcal{P}}) \qquad \begin{aligned} &\text{minimize } f(x) = f_0(x) + \rho\big(F(x)\big) \text{ over } x \in X, \\ &\quad \text{where } F(x) = \big(f_1(x), \ldots, f_m(x)\big). \end{aligned}$$

The assumptions on X and the f_i's are as before, but ρ is any convex, possibly *extended-real-valued* function on $\mathbb{R}^m$ that's lower semicontinuous and "proper" (everywhere $> -\infty$, somewhere $< \infty$). This problem is "composite" because its structure emphasizes the composition of a mapping $F : x \mapsto \big(f_1(x), \ldots, f_m(x)\big)$ with a function ρ on $\mathbb{R}^m$. Smooth assumptions are centered on the *data mapping* F (and f_0 as a tag-along), while aspects of constraints, penalties and nonsmoothness are built into the *model function* ρ.

To get a sense of what $(\overline{\mathcal{P}})$ covers, and why it might hold advantages over $(\mathcal{P})$, let's start by considering cases where ρ is separable:

$$\rho(u) = \rho(u_1, \ldots, u_m) = \rho_1(u_1) + \cdots + \rho_m(u_m) \tag{10.1}$$

with each ρ_i a convex, proper, lower semicontinuous function on $\mathbb{R}$. Then $f(x)$ has the form (1.1) mentioned in §1 in which ρ_i might be a penalty function. The particular choice (1.2) reduces $(\overline{\mathcal{P}})$ to $(\mathcal{P})$. This is the sense in which we'll view $(\mathcal{P})$ as a special case of $(\overline{\mathcal{P}})$ and be able to interpret results about $(\mathcal{P})$ as consequences of more general results for $(\overline{\mathcal{P}})$. We'll refer to it as the *traditional case* of $(\overline{\mathcal{P}})$.

The *linear penalty case* of $(\overline{\mathcal{P}})$ is the one corresponding to (10.1) and (1.3). The *quadratic penalty case* of $(\overline{\mathcal{P}})$ instead takes (10.1) and (1.4). This should be enough to give the flavor. Various mixtures and sums of the ρ_i functions in (1.2), (1.3), and (1.4) can be used along with other expressions.

It shouldn't be overlooked that the Lagrangian functions whose minimum relative $x \in X$ was considered in §§5, 6, and 7 also fit this elementary pattern. The ordinary Lagrangian L gives us expressions $f_0(x)+y_1 f_1(x)+\cdots+y_m f_m(x)$ which can be thought of as corresponding to linear functions $\rho_i(u_i) = y_i u_i$. This is an idle remark, but the augmented Lagrangian $\widetilde{L}$ involves expressions $f_0(x) + \rho_1\big(f_1(x)\big) + \cdots + \rho_m\big(f_m(x)\big)$ in which the functions ρ_i, parameterized by y_i and η, are linear-quadratic for $i = s+1, \ldots, m$ but only piecewise linear-quadratic for $i = 1, \ldots, s$.

Besides the special cases of $(\overline{\mathcal{P}})$ where ρ is separable, there are others which introduce nonsmoothness from a different angle. In the *max function case* of $(\overline{\mathcal{P}})$, we take

$$f_0 \equiv 0, \qquad \rho(u) = \rho(u_1, \ldots, u_m) = \max\{u_1, \ldots, u_m\}, \tag{10.2}$$

the latter being a convex, piecewise linear function on $\mathbb{R}^m$. The problem consists then of minimizing $f(x) = \max\{f_1(x), \ldots, f_m(x)\}$ over all $x \in X$. Again, this is just a "pure" form of example which could readily be combined with others. For instance, a function of such "max" type could be minimized over X subject to various equality or inequality constraints or penalty expressions that substitute for them.

Because ρ can in general be extended-real-valued in $(\overline{\mathcal{P}})$, there may be implicit constraints in this problem. These are brought out in terms of the set

$$D = dom\,\rho = \{\, u \in \mathbb{R}^m \,|\, \rho(u) < \infty \,\}, \tag{10.3}$$

which is convex and nonempty. In $(\overline{\mathcal{P}})$ we really minimize $f(x)$ subject to $F(x) \in D$. The set of feasible solutions isn't X but

$$C = \{\, x \in X \,|\, F(x) \in D \,\}. \tag{10.4}$$

Of course when ρ is finite everywhere on $\mathbb{R}^m$, as in the linear penalty case, the quadratic penalty case and the max function case, we do just have $C = X$, but in the traditional case C is the feasible set for $(\mathcal{P})$ as studied up to now. Other deviations of C from X arise in various mixed cases.

Our immediate goal is to establish a Lagrange multiplier rule for $(\overline{\mathcal{P}})$ that subsumes the one in Theorem 4.2 for $(\mathcal{P})$. Since traditional equality and inequality constraints have been relegated only to a special case in the problem formulation, the very idea of what a Lagrange multiplier should now be might be questioned. However a satisfying extension will come to light in which, again, there's a coefficient $\bar{y}_i$ associated with each of the functions f_i, $i = 1, \ldots, m$. In the end we'll also have a Lagrangian function and an analog $(\overline{\mathcal{L}})$ of condition $(\mathcal{L})$.

By the *basic constraint qualification* for the extended problem $(\overline{\mathcal{P}})$ at a feasible solution $\bar{x}$ we'll mean the following condition in the notation (10.3):

$$(\overline{\mathcal{Q}}) \qquad \begin{cases} \text{there is no vector } y \neq 0 \text{ such that} \\ y \in N_D\big(F(\bar{x})\big), \quad -\big[y_1 \nabla f_1(\bar{x}) + \cdots + y_m \nabla f_m(\bar{x})\big] \in N_X(\bar{x}). \end{cases}$$

To bring this to specifics, note that in the conventional case where $(\overline{\mathcal{P}})$ reduces to $(\mathcal{P})$ the set D coincides with the cone U treated in Section 4, consisting of the vectors u such that $u_i \leq 0$ for $i = 1, \ldots, s$, but $u_i = 0$ for $i = s+1, \ldots, m$. Then the requirement $y \in N_D\big(F(\bar{x})\big)$ summarizes the sign relations (4.5) in $(\mathcal{Q})$, according to Proposition 4.1, and $(\overline{\mathcal{Q}})$ reduces to $(\mathcal{Q})$. In the examples mentioned where D is all of $\mathbb{R}^m$, we have $N_D\big(F(\bar{x})\big) = \{0\}$, so condition $(\overline{\mathcal{Q}})$ is automatically fulfilled.

THEOREM 10.1. *If $\bar{x}$ is a locally optimal solution to $(\overline{\mathcal{P}})$ at which the basic constraint qualification $(\overline{\mathcal{Q}})$ is satisfied, there must exist a vector $\bar{y}$ such that*

$$\bar{y} \in \partial\rho\big(F(\bar{x})\big), \qquad -\big[\nabla f_0(\bar{x}) + \bar{y}_1 \nabla f_1(\bar{x}) + \cdots + \bar{y}_m \nabla f_m(\bar{x})\big] \in N_X(\bar{x}). \tag{10.5}$$

Proof. We'll follow the scheme used to prove Theorem 4.2 with appropriate generalizations. As in that proof, it suffices to treat the case where X is bounded and $\bar{x}$ is the unique globally optimal solution to $(\overline{\mathcal{P}})$. Likewise we can suppose that the set $D = dom\,\rho$ in (10.3) is bounded; if necessary, we could redefine $\rho(u)$ be ∞ outside some closed ball in $\mathbb{R}^m$ big enough to include $F(x)$ for every $x \in X$, which wouldn't affect the subgradients of ρ at $F(\bar{x})$ or the fact that ρ is convex, lower semicontinuous and proper. In this framework $f_0(x)+\rho(u)$ has a finite minimum value μ on $X\times\mathbb{R}^m$. The problem

$$(\widehat{\mathcal{P}})\qquad \text{minimize } \widehat{f}(x,u)=f_0(x)+\rho(u) \text{ subject to } F(x)-u=0,\ (x,u)\in X\times\mathbb{R}^m,$$

has $(\bar{x},\bar{u})=\big(\bar{x},F(\bar{x})\big)$ as its unique optimal solution.

For a sequence of values $\varepsilon_k\downarrow 0$, the approximate problems

$$(\widehat{\mathcal{P}}^k)\qquad \text{minimize } \widehat{f}^k(x,u)=f_0(x)+\rho(u)+\frac{1}{2\varepsilon_k}\big|F(x)-u\big|^2 \text{ over } (x,u)\in X\times\mathbb{R}^m$$

have solutions (x^k,u^k), since the level sets in $X\times\mathbb{R}^m$ where $\widehat{f}^k(x,u)\le\alpha$ are closed and bounded (because of the lower semicontinuity of ρ, the continuity of f_0 and F, the closedness of X, and the boundedness of X and D). We have

$$\mu+\frac{1}{2\varepsilon_k}\big|F(x^k)-u^k\big|^2\le\widehat{f}^k(x^k,u^k)\le\widehat{f}^k(\bar{x},\bar{u})=f_0(\bar{x})+\rho(\bar{u}),$$

so any cluster point $(\hat{x},\hat{u})$ of the sequence $\big\{(x^k,u^k)\big\}_{k=1}^{\infty}$, which is bounded, must satisfy $\big|F(\hat{x})-\hat{u}\big|=0$ and (because ρ is lower semicontinuous)

$$\begin{aligned} f_0(\bar{x})+\rho(\bar{u})\ge\limsup_{k\to\infty}\widehat{f}^k(x^k,u^k)&=f_0(\hat{x})+\limsup_{k\to\infty}\rho(u^k)\\ &\ge f_0(\hat{x})+\liminf_{k\to\infty}\rho(u^k)\ge f_0(\hat{x})+\rho(\hat{u}).\end{aligned}$$

Thus $(\hat{x},\hat{u})$ must be an optimal solution to $(\widehat{\mathcal{P}})$, coinciding therefore with $(\bar{x},\bar{u})$; necessarily also, $lim_{k\to\infty}\rho(u^k)=\rho(\bar{u})$. This being true for any cluster point, we deduce for the full sequence $\big\{(x^k,u^k)\big\}_{k=1}^{\infty}$ that $x^k\to\bar{x}$, $u^k\to\bar{u}$, and $\rho(u^k)\to\rho(\bar{u})$.

The optimality of (x^k,u^k) in $(\widehat{P}^k)$ implies that x^k minimizes $\widehat{f}^k(x,u^k)$ in $x\in X$, and u^k minimizes $\widehat{f}^k(x^k,u)$ in $u\in\mathbb{R}^m$. Hence by Proposition 3.5 the vector $-\nabla_x\widehat{f}^k(x^k,u^k)$ belongs to the normal cone $N_X(x^k)$. To say that the minimum of $\widehat{f}^k(x^k,u)$ is attained at u^k is to say that

$$\rho(u^k)+\frac{1}{2\varepsilon_k}\big|F(x^k)-u^k\big|^2\le\rho(u)+\frac{1}{2\varepsilon_k}\big|F(x^k)-u\big|^2 \text{ for all } u\in\mathbb{R}^m,$$

which in terms of the function

$$h^k(u)=\rho(u^k)+\frac{1}{2\varepsilon_k}\big|F(x^k)-u^k\big|^2-\frac{1}{2\varepsilon_k}\big|F(x^k)-u\big|^2$$

means that $h^k(u)\le\rho(u)$ everywhere, with $h^k(u^k)=\rho(u^k)$. Then by Proposition 8.6 the vector $\nabla h^k(u^k)$ is a subgradient of ρ at u^k. Let's denote this vector by y^k; its components are $y_i^k=\big[f_i(x^k)-u_i^k\big]/\varepsilon_k$, and in such terms the vector $\nabla_x\widehat{f}^k(x^k,u^k)$ calculates out to $\nabla f_0(x^k)+y_1^k\nabla f_1(x^k)+\cdots+y_m^k\nabla f_m(x^k)$. We thus have

$$y^k\in\partial\rho(u^k),\qquad -\big[\nabla f_0(x^k)+y_1^k\nabla f_1(x^k)+\cdots+y_m^k\nabla f_m(x^k)\big]\in N_X(x^k).\tag{10.6}$$

If the sequence of vectors y^k is bounded, we can suppose it converges to some $\bar{y}$. Then in taking the limit in (10.6) (where $x^k\to\bar{x}$, $u^k\to\bar{u}=F(\bar{x})$, and $\rho(u^k)\to\rho(\bar{u})=\rho\big(F(\bar{x})\big)$, so Proposition 8.3 is applicable), we obtain the targeted condition

(10.5). If on the other hand the sequence of vectors y^k is unbounded, we can suppose (by passing to a subsequence if necessary) that $0 < |y^k| \to \infty$ and that the vectors $\hat{y}^k = y^k/|y^k|$ converge to some $\hat{y} \neq 0$. Then from the second condition in (10.6) we have

$$-\big(1/|y^k|\big)\nabla f_0(x^k) - \big[\hat{y}_1^k \nabla f_1(x^k) + \cdots + \hat{y}_m^k \nabla f_m(x^k)\big] \in N_X(x^k)$$

and in the limit

$$-\big[\hat{y}_1 \nabla f_1(\bar{x}) + \cdots + \hat{y}_m \nabla f_m(\bar{x})\big] \in N_X(\bar{x}). \tag{10.7}$$

But at the same time, from $y^k \in \partial\rho(u^k)$ and the convexity of ρ we have for any $u \in D$ that $\rho(u) \geq \rho(u^k) + \langle y^k, u - u^k\rangle$ so that $\big(1/|\hat{y}^k|\big)\rho(u) \geq \big(1/|\hat{y}^k|\big)\rho(u^k) + \big\langle \hat{y}^k, u - u^k\big\rangle$. Then in the limit we get $0 \geq \big\langle \hat{y}, u - \bar{u}\big\rangle$. This being true for arbitrary $u \in D$, we see that $\hat{y} \in N_D(\bar{u}) = N_D\big(F(\bar{x})\big)$. A contradiction to assumption $(\overline{\mathcal{Q}})$ has been detected. Our conclusion is that the unbounded case can't arise. □

Before continuing, we tie up a loose end from earlier by applying the extended multiplier rule to deduce Theorem 9.5.

Proof of Theorem 9.5. First we prove the theorem under the assumption that the estimate is strict in the sense that

$$dp(0)(w) > \min_{y \in Y_{est}} \big\langle y, w\big\rangle \text{ for all } w \neq 0. \tag{10.9}$$

Let $\sigma(w)$ denote the value on the right obtained with maximization instead of minimization; the minimum is $-\sigma(-w)$. This switch is advantageous because σ is a finite, continuous, convex function. Our inequality can then be written $dp(0)(w)+\sigma(-w) > 0$ for all $w \neq 0$. It follows then from the limit definition of $dp(0)$ that for some $\delta > 0$ we have

$$\big[p(0 + tw) - p(0)\big]/t + \sigma(-w) > 0 \text{ when } 0 < t \leq \delta,\ |w| = 1.$$

In view of the fact that $t\sigma(-w) = \sigma(-tw)$ when $t > 0$, this can be expressed as

$$p(u) + \sigma(-u) > p(0) \text{ for all } u \neq 0 \text{ with } |u| \leq \delta.$$

Recalling the definition of $p(u)$ at the beginning of Section 6, we see that the optimal solutions to the problem

$$\begin{aligned} &\text{minimize } f_0(x) + \sigma(-u) \text{ subject to } x \in X,\ |u| \leq \delta, \\ &f_i(x) + u_i \begin{cases} \leq 0 & \text{for } i = 1, \ldots, s, \\ = 0 & \text{for } i = s+1, \ldots, m, \end{cases} \end{aligned}$$

are precisely the pairs $(\bar{x}, \bar{u})$ such that $\bar{x}$ is an optimal solution to $(\mathcal{P})$ and $\bar{u} = 0$. Let U denote the set of vectors $u = (u_1, \ldots, u_m)$ such that $u_i \leq 0$ for $i = 1, \ldots, s$ but $u_i = 0$ for $i = s+1, \ldots, m$. The inequality and equality constraints then have the form $F(x) + u = w \in U$, and this allows us to write the problem instead as

$$\text{minimize } f_0(x) + \rho\big(F(x) - w\big) \text{ subject to } x \in X,\ w \in W,$$

where $\rho(u) = \sigma(u)$ when $|u| \leq \delta$, but $\rho(u) = \infty$ when $|u| > \delta$. The solutions now are precisely the pairs $\big(\bar{x}, F(\bar{x})\big)$ such that $\bar{x}$ is an optimal solution to $(\mathcal{P})$. We cast this as a special case of problem $(\overline{\mathcal{P}})$ in the form

$$\begin{aligned} &\text{minimize } \overline{f}_0(x, w) + \rho\big(\overline{F}(x, w)\big) \text{ over all } (x, w) \in X \times U, \\ &\quad \text{where } \overline{f}_0(x, w) = f_0(x), \quad \overline{F}(x, w) = F(x) - w. \end{aligned}$$

The optimal solutions $(\bar{x},\bar{w})$ (if any) have $\bar{w}=F(\bar{x})$ and therefore $\overline{F}(\bar{x},\bar{w})=0$, this point being in the interior of the effective domain $D=dom\,\rho$, which is the closed ball of radius δ around the origin of $\mathbb{R}^m$. The constraint qualification $(\overline{\mathcal{Q}})$ is automatically satisfied in this case, so we obtain from Theorem 10.1 the existence of a multiplier vector $\bar{y}\in\partial\rho(0)$ such that

$$-\big[\nabla\overline{f}_0(\bar{x},\bar{w})+\bar{y}_1\nabla\overline{f}_1(\bar{x},\bar{w})+\cdots+\bar{y}_m\nabla\overline{f}_m(\bar{x},\bar{w})\big]\in N_{X\times U}(\bar{x},\bar{w}), \tag{10.10}$$

where $\overline{f}_i(x,w)=f_i(x)-w_i$. The condition $\bar{y}\in\partial\rho(0)$ is identical to $\bar{y}\in\partial\sigma(0)$, because σ and ρ agree in a neighborhood of 0, and from the way σ was defined it means that $\bar{y}\in Y_{ext}$. The condition in (10.10) reduces to

$$-\big[\nabla f_0(\bar{x})+\bar{y}_1\nabla f_1(\bar{x})+\cdots+\bar{y}_m\nabla f_m(\bar{x})\big]\in N_X(\bar{x}),\qquad \bar{y}\in N_U\big(\bar{w}\big)=N_U\big(F(\bar{x})\big),$$

which by Proposition 4.1 is equivalent to $(\mathcal{L})$ holding for $\bar{x}$ and $\bar{y}$.

So far we've concentrated on the case in (10.8) where the inequality is strict, but the general case now follows easily. For any $\varepsilon>0$ let Y^{ε}_{est} be the set of all points whose distance from Y_{est} doesn't exceed ε. This is another compact, convex set, and we have

$$\min_{y\in Y^{\varepsilon}_{est}}\big\langle y,w\big\rangle \;=\; \min_{y\in Y_{est}}\big\langle y,w\big\rangle-\varepsilon|w|.$$

The argument we've given therefore works for Y^{ε}_{est}: for any optimal solution $\bar{x}$ to $(\mathcal{P})$ there's a multiplier vector $\bar{y}^{\varepsilon}\in Y^{\varepsilon}_{est}$ for which $(\mathcal{L})$ holds. As $\varepsilon\downarrow 0$, $\bar{y}^{\varepsilon}$ remains bounded and its distance from Y_{est} dwindles to 0. Taking $\bar{y}$ to be any cluster point, we still have $(\mathcal{L})$, but also $\bar{y}\in Y_{est}$. □

The optimality result in Theorem 10.1 corresponds to the following calculus rule for subgradients of nonsmooth functions.

THEOREM 10.2. *For the function*

$$f(x)=\begin{cases} f_0(x)+\rho\big(F(x)\big) & \text{if } x\in X,\\ \infty & \text{if } x\notin X,\end{cases}$$

under the assumptions on f_0, F, X and ρ in $(\mathcal{P})$, suppose $\bar{x}$ is a point where $f(\bar{x})$ is finite and $(\overline{\mathcal{Q}})$ is satisfied. Then $\partial f(\bar{x})$ consists of the vectors of the form

$$v=\nabla f_0(\bar{x})+y_1\nabla f_1(\bar{x})+\cdots+y_m\nabla f_m(\bar{x})+z \quad \text{with} \quad y\in\partial\rho\big(F(\bar{x})\big),\ z\in N_X(\bar{x}).$$

Proof. This extends Theorem 10.1 and its proof by essentially the same argument that was used in deriving Theorem 4.3 from Theorem 4.2. □

Theorem 10.1 can be derived in turn from Theorem 10.2 through the vehicle of the elementary necessary condition

$$0\in\partial f(\bar{x}) \tag{10.10}$$

for $\bar{x}$ to give local minimum of f over $\mathbb{R}^n$—which is the same as $\bar{x}$ being a locally optimal solution to problem $(\overline{\mathcal{P}})$. To say that the vector 0 can be represented as described in Theorem 10.2 is to say that condition (10.5) holds in Theorem 10.1. As a simple illustration, if for a smooth function f_0 and a closed set X we take

$$f(x)=\begin{cases} f_0(x) & \text{when } x\in X,\\ \infty & \text{when } x\notin X,\end{cases}$$

we get $\partial f(\bar{x}) = \partial f_0(\bar{x}) + N_X(\bar{x})$ at any point $\bar{x} \in X$; there's no worry about condition $(\overline{\mathcal{Q}})$ in this case because F and ρ fall away. This special rule when invoked in the necessary condition (10.11) yields the normality condition in Theorem 3.6.

The subgradient formula in Theorem 10.2 is essentially a *chain rule* in subgradient calculus. To see this most clearly, consider the case where $f_0 \equiv 0$ and $X = \mathbb{R}^n$. Thinking of $y_1 \nabla f_i(x) + \cdots + y_m \nabla f_m(x)$ as $y \nabla F(x)$ for the Jacobian $\nabla F(x) \in \mathbb{R}^m \times \mathbb{R}^n$, we get the following specialization.

COROLLARY 10.3. *For a smooth mapping $F : \mathbb{R}^n \to \mathbb{R}^m$ and any convex, lower semicontinuous, proper function $\rho : \mathbb{R}^m \to [-\infty, \infty]$, the chain rule*

$$\partial(\rho\circ F)(\bar{x}) = \partial\rho\big(F(\bar{x})\big)\nabla F(\bar{x}) = \Big\{\, y\nabla F(\bar{x}) \,\Big|\, y \in \partial\rho\big(F(x)\big) \,\Big\}$$

is valid at any $\bar{x}$ such that ρ is finite at $F(\bar{x})$ and the only vector $y \in N_D\big(F(\bar{x})\big)$ with $y\nabla F(\bar{x}) = 0$ is $y = 0$.

It's remarkable that this chain rule covers not just the kinds of situations people are used to thinking about in terms of composition, but geometric situations involving normal vectors. For example, when ρ is the indicator of the set D consisting of the vectors u such that $u_i \leq 0$ for $i = 1, \ldots, s$ but $u_i = 0$ for $i = s+1, \ldots, m$, the composed function $\rho\circ F$ is the indicator of the set C defined by the constraints $f_i(x) \leq 0$ for $i = 1, \ldots, s$ and $f_i(x) = 0$ for $i = s+1, \ldots, m$. Then $\partial\rho\big(F(\bar{x})\big)$ is the normal cone $N_D\big(F(\bar{x})\big)$, whereas $\partial(\rho\circ F)(\bar{x})$ is the normal cone $N_C(\bar{x})$. The chain rule thus gives us a formula for $N_C(\bar{x})$; it corresponds to the formula in Theorem 4.3 for the case where $X = \mathbb{R}^n$.

Another case of the chain rule to be noted is the one where ρ is the function in (10.3), so that $\rho\circ F$ is the pointwise max of $f_1, \ldots, f_m$. Here the constraint qualification is satisfied trivially, because ρ is finite everywhere; $D = \mathbb{R}^m$. The subgradient set $\rho\big(F(\bar{x})\big)$ consists of the vectors y with $y_i \geq 0$, $y_1 + \cdots + y_m = 1$, such that $y_i = 0$ for the "inactive" functions f_i at $\bar{x}$, i.e., the ones for which $f_i(\bar{x})$ falls short of the max.

To use these results, it's necessary to be able to determine the subgradients of the convex function ρ. An extensive calculus is available in convex analysis [1], but in many applications elementary considerations suffice. For instance,

$$\rho(u) = \rho_1(u_1) + \cdots + \rho_m(u_m) \quad\Longrightarrow\quad \partial\rho(u) = \partial\rho_1(u_1) \times \cdots \times \partial\rho_m(u_m). \tag{10.11}$$

In the separable case, therefore, the condition on each multiplier $\bar{y}_i$ in Theorem 10.1 depends only on the function ρ_i on $\mathbb{R}$ and the value $f_i(\bar{x})$, and it takes the form $\bar{y}_i \in \rho_i\big(f_i(\bar{x})\big)$. This restricts $\bar{y}_i$ to lie in a certain closed interval in $\mathbb{R}$ whose bounds are determined by the left and right derivatives of ρ_i at $\bar{u}_i = f_i(\bar{x})$; see Rockafellar [39, Chapter 8] for details and examples in this one-dimensional setting.

A Lagrangian expression for the necessary condition in Theorem 10.1 can be developed through the notion of the convex function ρ^* *conjugate* to ρ, which is defined by

$$\rho^*(y) = \sup_{u\in\mathbb{R}^m} \Big\{ \langle y, u\rangle - \rho(u) \Big\} \text{ for } y \in \mathbb{R}^m. \tag{10.12}$$

Some fundamental facts in convex analysis [1] are that ρ^* is again lower semicontinuous and proper, and its conjugate function $(\rho^*)^*$ is in turn ρ: one has

$$\rho(u) = \sup_{y\in\mathbb{R}^m} \Big\{ \langle y, u\rangle - \rho^*(y) \Big\}. \tag{10.13}$$

Furthermore, there's the subgradient relationship

$$y \in \partial\rho(u) \iff u \in \partial\rho^*(y). \tag{10.14}$$

A substantial simplification often in the calculation of conjugates is the observation that

$$\rho(u) = \rho_1(u_1) + \cdots + \rho_m(u_m) \implies \rho^*(y) = \rho_1^*(y_1) + \cdots + \rho_m^*(y_m). \tag{10.15}$$

Thus, when ρ is separable, only the conjugates of convex functions of a single variable need to be calculated in order to obtain ρ^*. This case is covered thoroughly in [39, Chap. 8] with many illustrations.

Before taking up specific examples, let's look at the main connections of this duality concept with generalized optimality rules. We continue with the notation

$$L(x,y) = f_0(x) + y_1 f_1(x) + \cdots + y_m f_m(x).$$

PROPOSITION 10.4. *In terms of the function ρ^* conjugate to ρ, the first-order necessary condition in Theorem 10.1 can be expressed as*

$$-\nabla_x L(\bar{x},\bar{y}) \in N_X(\bar{x}), \qquad \nabla_y L(\bar{x},\bar{y}) \in \partial\rho^*(\bar{y}). \tag{10.17}$$

Proof. This is obvious from (10.14). □

This form of the basic first-order optimality conditions could be elaborated further. It could be symmetrized by replacing the indicator function δ_X implicit in $(\mathcal{P})$ by a convex function σ, so that the first part of (10.16) would read $-\nabla_x L(\bar{x},\bar{y}) \in \sigma(\bar{x})$. But instead of heading off in that direction we prefer here to identify the context in which a normal cone expression of optimality can be maintained.

DEFINITION 10.5. The function ρ will be said to have a *smooth dual representation* if

$$\rho(u) = \sup_{y\in Y} \Big\{ \langle y,u\rangle - k(y) \Big\} \tag{10.17}$$

with Y a nonempty, closed, convex subset of $\mathbb{R}^m$ (not necessarily $\mathbb{R}^s_+ \times \mathbb{R}^{m-s}$) and k some smooth function on $\mathbb{R}^m$ that is convex relative to Y (possibly $k \equiv 0$).

THEOREM 10.6. *Suppose in problem $(\overline{\mathcal{P}})$ that ρ has a dual smooth representation. Then in terms of the extended Lagrangian function*

$$\overline{L}(x,y) = f_0(x) + y_1 f_1(x) + \cdots + y_m f_m(x) - k(y) \tag{10.18}$$

the first-order necessary condition in Theorem 10.1 can be expressed as

$$(\overline{\mathcal{L}}) \qquad -\nabla_x \overline{L}(\bar{x},\bar{y}) \in N_X(\bar{x}), \qquad \nabla_y \overline{L}(\bar{x},\bar{y}) \in N_Y(\bar{y}).$$

Proof. The function ψ on $\mathbb{R}^m$ defined by $\psi(y) = k(y)$ when $y \in Y$, but $\psi(y) = \infty$ when $y \notin Y$, is convex, lower semicontinuous, and proper. According to (10.17) its conjugate ψ^* is ρ, and so in turn we have $\rho^* = \psi$. The vectors $u \in \partial\rho^*(y) = \partial\psi(y)$ are those of the form $\nabla k(y) + z$ with $z \in N_Y(y)$ (cf. the comments following Theorem 10.2). The condition $\nabla_y L(\bar{x},\bar{y}) \in \partial\rho^*(\bar{y})$ in Proposition 10.4 is the same therefore as $\nabla_y L(\bar{x},\bar{y}) - \nabla k(\bar{y}) \in N_Y(\bar{y})$. Since $\nabla_x \overline{L}(\bar{x},\bar{y}) = \nabla_x L(\bar{x},\bar{y})$ and $\nabla_y \overline{L}(\bar{x},\bar{y}) = \nabla_y L(\bar{x},\bar{y}) - \nabla k(\bar{y})$, we're able to write (10.16) equivalently as $(\overline{\mathcal{L}})$. □

The existence of a smooth dual representation for ρ is being emphasized because that fits with all the special examples of problem $(\overline{\mathcal{P}})$ mentioned until now. The traditional case, where $(\overline{\mathcal{P}})$ reduces to $(\mathcal{P})$, corresponds of course to $k \equiv 0$ and $Y = \mathbb{R}^s_+ \times \mathbb{R}^{m-s}$. The linear penalty case (1.3) corresponds to

$$k \equiv 0, \qquad Y = Y_1 \times \cdots \times Y_m, \text{ with } Y_i = \begin{cases} [0, d_i] & \text{for } i = 1, \ldots, s, \\ [-d_i, d_i] & \text{for } i = s+1, \ldots, m. \end{cases} \tag{10.19}$$

For the quadratic penalty case (1.4) we have

$$k(y) = \sum_{i=1}^{m} \frac{y_i^2}{2d_i}, \qquad Y = \mathbb{R}^s_+ \times \mathbb{R}^{m-s}. \tag{10.20}$$

Finally, the max function case (10.3)—which isn't separable—arises from

$$k \equiv 0, \qquad Y = \{\, y \mid y_i \geq 0,\ y_1 + \cdots + y_m = 1 \,\}. \tag{10.21}$$

The generalized Lagrangian format in Theorem 10.6 is a springboard for extending the saddle point and duality results in Section 5 to composite models. By the *convex case* of $(\overline{\mathcal{P}})$ in this setting, we'll mean the case where X is convex and, for every $y \in Y$, the function $f_0 + y_1 f_1 + \cdots + y_m f_m$ is convex relative to X. (In the specialization of $(\overline{\mathcal{P}})$ to $(\mathcal{P})$, where $Y = \mathbb{R}^s_+ \times \mathbb{R}^{m-s}$, this criterion gives the convex case of $(\mathcal{P})$.)

THEOREM 10.7. *In the convex case of $(\overline{\mathcal{P}})$ when ρ has a smooth dual representation, the extended Lagrangian $\overline{L}(x, y)$ is convex in $x \in X$ for each $y \in Y$, and concave in $y \in Y$ for each $x \in X$. The normality condition $(\overline{\mathcal{L}})$ means then that $\overline{L}$ has a saddle point on $X \times Y$ at $(\bar{x}, \bar{y})$.*

Proof. The two normality conditions in Theorem 10.6 translate through Theorem 3.5 to the max and min conditions that define a saddle point. □

As an example, in the max function case of $(\mathcal{P})$ in (10.2) we obtain the fact that the necessary and sufficient condition for the global optimality of $\bar{x}$ in the minimization of $f(x) = \max\{f_1(x), \ldots, f_m(x)\}$ over X, when the f_i's and X are convex, is the existence of $\bar{y}$ such that $(\bar{x}, \bar{y})$ is a saddle point of $y_1 f_1(x) + \cdots + y_m f_m(x)$ on $X \times Y$, where Y is the unit simplex in (10.21). Kuhn and Tucker stated this extended Lagrangian optimality condition in their early paper [9]. It's interesting to see that for them the concept of a Lagrangian was thus much more general than it came to be during the intervening years in which only cones were thought of as suitable candidates for a multiplier space Y.

The saddle point result in Theorem 10.7 leads very naturally to the introduction of the *extended dual problem*

$$(\overline{\mathcal{D}}) \qquad \text{maximize } g(y) = \inf_{x \in X} \Big\{ f_0(x) + y_1 f_1(x) + \cdots + y_m f_m(x) \Big\} - k(y) \text{ over } y \in Y$$

for the convex case of $(\overline{\mathcal{P}})$ when ρ has a smooth dual representation. Problems $(\overline{\mathcal{P}})$ and $(\overline{\mathcal{D}})$ are the strategy problems for the two players in the game corresponding to $(X, Y, \overline{L})$, and they thus enjoy the basic relationships in Theorem 5.3. The multiplier vectors $\bar{y}$ associated by condition $(\overline{\mathcal{L}})$ with optimal solutions $\bar{x}$ to $(\overline{\mathcal{P}})$ are then the optimal solutions to $(\overline{\mathcal{D}})$.

Duality in this extended sense, which in particular covers the various penalty problems that have been described, has many potential uses. The case where f_0 and k are quadratic functions (with affine as a special case), the other functions f_i are affine and X and Y are polyhedral, has been developed as *extended linear-quadratic programming* in Rockafellar and Wets [5], [6], [7], Rockafellar [8], [63], Rockafellar and Zhu [64]. Forms of *extended convex programming* have been utilized in Rockafellar [65], [66], and *extended nonlinear programming* in [67], [68]; the Lagrangian format in Theorem 10.6 for a smooth dual representation of ρ appeared first in [67].

Second-order optimality conditions for have been developed $(\overline{\mathcal{P}})$ by Poliquin and Rockafellar [69], [70] under the assumption that X is polyhedral and ρ is "piecewise linear-quadratic." For second-order theory under weaker assumptions, but also with weaker accompanying properties, see Burke and Poliquin [71] and Ioffe [72]. For other perspectives on nonsmooth analysis see Clarke [17], [18], Mordukhovich [24], Aubin and Ekeland [73], Aubin and Frankowska [74]. The last of these books offers a full theory of tangent cones and their applications.

REFERENCES

[1] R. T. Rockafellar, *Convex Analysis*, Princeton University Press, Princeton, NJ, 1970.

[2] T. Pietrzykowski, *An exact potential method for constrained maxima,* SIAM J. Numer. Anal., 6 (1969), pp. 299–304.

[3] W. I. Zangwill, *Nonlinear programming via penalty functions,* Management Science, 13 (1967), pp. 344–358.

[4] R. Courant, *Variational methods for the solution of problems of equilibrium and vibrations,* Bull. Amer. Math. Soc., 49 (1943), pp. 1–23.

[5] R. T. Rockafellar and R. J-B Wets, *A Lagrangian finite generation technique for solving linear-quadratic problems in stochastic programming,* Math. Programming Studies, 28 (1986), pp. 63–93.

[6] R. T. Rockafellar and R. J-B Wets, *Linear-quadratic problems with stochastic penalties: the finite generation algorithm,* Numerical Techniques for Stochastic Optimization Problems, Y. Ermoliev and R. J-B Wets, eds., Springer-Verlag Lecture Notes in Control and Information Sciences No. 81, 1987, pp. 545-560.

[7] R. T. Rockafellar and R. J-B Wets, *Generalized linear-quadratic problems of deterministic and stochastic optimal control in discrete time,* SIAM J. Control Opt., 320 (1990), pp. 810–822.

[8] R. T. Rockafellar, *Linear-quadratic programming and optimal control,* SIAM J. Control Opt., 25 (1987), pp. 781–814.

[9] H. W. Kuhn and A. W. Tucker, *Nonlinear programming,* Proceedings of the Second Berkeley Symposium on Mathematical Statistics and Probability, J. Neyman, ed., Univ. Calif. Press, Berkeley, 1951, pp. 481–492.

[10] W. Karush, *Minima of functions of several variables with inequalities as side conditions,* master's thesis, Dept. of Mathematics, Univ. of Chicago, 1939.

[11] H. W. Kuhn, *Nonlinear programming: a historical view,* Nonlinear Programming, R. W. Cottle and C. E. Lemke, eds., SIAM-AMS Proceedings, Vol. IX, Amer. Math. Soc. Providence R.I., 1976, pp. 1–26.

[12] F. A. Valentine, *The problem of Lagrange with differential inequalities as added side conditions,* Contributions to the Calculus of Variations 1933–37, Univ. Chicago Press, 1937, pp. 407–448.

[13] F. John, *Extremum problems with inequalities as subsidiary conditions,* Studies and Essays: Courant Anniversary Volume, K. O. Friedrichs, O. E. Neugebauer, and J. J. Stoker, eds., Wiley-Interscience, New York, 1948, pp. 187–204.

[14] G. Bouligand, *Introduction à la Géométrie Infinitésimale Directe,* Gauthier-Villars, Paris, 1932.

[15] M. R. Hestenes, *Calculus of Variations and Optimal Control Theory,* Wiley, 1966.

[16] F. H. Clarke, *Generalized gradients and applications,* Transactions Amer. Math. Math. Soc., 205 (1975), pp. 247–262.

[17] F. H. Clarke, *Optimization and Nonsmooth Analysis*, Wiley, 1983. Republished by the Centre de Recherches Mathématiques, Université de Montréal (C.P. 6128 *A,* Montréal, Québec, Canada, H3C 3J7).

[18] F. H. Clarke, *Methods of Dynamic and Nonsmooth Optimization*, CBMS-NSF Regional Conference Series in Applied Mathematics, 57, SIAM Publications, Philadelphia, PA, 1989.

[19] A. Kruger and B. S. Mordukhovich, *Extremal points and the Euler equation in nonsmooth optimization,* Dokl. Akad. Nauk BSSR, 24 (1980), pp. 684–687 (Russian).

[20] A. D. Ioffe, *Approximate subdifferentials and applications, I: the finite dimensional theory,* Trans. Amer. Math. Soc., 281, pp. 389–416.

[21] R. T. Rockafellar, *Extensions of subgradient calculus with applications to optimization,* Nonlinear Analysis, 9 (1985), pp. 665–698.

[22] B. S. Mordukhovich, *Maximum principle in the problem of time optimal response with nonsmooth constraints,* J. Appl. Math. Mech., 40 (1976), pp. 960–969.

[23] B. S. Mordukhovich, *Metric approximations and necessary optimality conditions for general classes of nonsmooth extremal problems,* Soviet Math. Dokl., 22 (1980), pp. 526–530.

[24] B. S. Mordukhovich, *Approximation Methods in Problems of Optimization and Control,* Nauka, Moscow (1988) (in Russian); English translation to appear in Wiley-Interscience.

[25] M. S. Bazaraa, J. J. Gould and M. Z. Nashed, *On the cones of tangents with application to mathematical programming,* J. Optim. Theory Appl., 13 (1974), pp. 389–426.

[26] M. R. Hestenes, *Optimization Theory*, Wiley, 1975.

[27] J.-P. Penot, *Calcul sous-différentiel et optimisation,* J. Funct. Analysis, 27 (1978), pp. 248–276.

[28] J. Farkas, *Über die Theorie der einfachen Ungleichungen,* J. Reine Angew. Math., 124 (1901), pp. 1–27.

[29] O. L. Mangasarian and S. Fromovitz, *The Fritz John conditions in the presence of equality and inequality constraints,* J. Math. Anal. Appl., 17 (1967), pp. 73–74.

[31] E. J. McShane, *The Lagrange multiplier rule,* Amer. Math. Monthly, 80 (1973), pp. 922–925.

[32] E. J. Beltrami, *A constructive proof of the Kuhn-Tucker multiplier rule,* Amer. Math. Monthly, 80 (1973), pp. 297–306.

[32] J. von Neumann, *Zur Theorie der Gesellschaftsspiele,* Math. Ann., 100 (1928), pp. 295–320.

[33] J. von Neumann and O. Morgenstern, *Theory of Games and Economic Behavior,* Princeton Univ. Press, 1944.

[34] D. Gale, H. W. Kuhn, and A. W. Tucker, *Linear programming and the theory of games,* Activity Analysis of Production and Allocation, T. C. Koopmans, ed., Wiley, 1951, pp. 317–328.

[35] J. von Neumann, *Discussion of a maximum problem,* John von Neumann: Collected Works, Vol. VI, Pergamon, 1963, pp. 89–95.

[36] R. T. Rockafellar, *Conjugate Duality and Optimization*, Regional Conference Series No. 16, SIAM Publications, 1974.

[37] R. T. Rockafellar, *Ordinary convex programs without a duality gap,* J. Opt. Theory. Appl., 7 (1971), pp. 143–148.

[38] R. T. Rockafellar, *Some convex programs whose duals are linearly constrained,* Nonlinear Programming, J. B. Rosen and O. L. Mangasarian, eds., Academic Press, 1970, pp. 293–322.

[39] R. T. Rockafellar, *Network Flows and Monotropic Optimization*, Wiley, 1984.

[40] R. T. Rockafellar, *New applications of duality in convex programming,* Proceedings of the 4th Conf. on Probability (Brasov, Rumania, 1971), pp. 37–81; written version of a talk also given at the 7th International Symposium on Math. Programming in the Hague, 1970.

[41] M. R. Hestenes, *Multiplier and gradient methods,* J. Opt. Theory Appl., 4 (1969), pp. 303–320.

[42] M. J. D. Powell, *A method for nonlinear constraints in minimization problems,* Optimization, R. Fletcher, ed., Academic Press, 1969, pp. 283–298.

[43] J. D. Buys, *Dual Algorithms for Constrained Optimization Problems*, doctoral dissertation, Univ. Leiden, 1972.

[44] A. P. Wierzbicki, *A penalty function shifting method in constrained static optimization and its convergence properties,* Archivum Automatyki i Telemechaniki, 16 (1971), pp. 395–415.

[45] R. T. Rockafellar, *The multiplier method of Hestenes and Powell applied to convex programming,* J. Opt. Theory Appl., 12 (1973), pp. 555–562.

[46] R. T. Rockafellar, *Augmented Lagrange multiplier functions and duality in nonconvex programming,* SIAM J. Control, 12 (1974), pp. 268–285.

[47] R. T. Rockafellar, *Solving a nonlinear programming problem by way of a dual problem,* Symposia Mathematica, 29 (1976), pp. 135–160.

[48] K. J. Arrow, F. J. Gould, and S. M. Howe, *A general saddle point result or constrained optimization,* Math. Programming, 5 (1973), pp. 225–134.

[49] R. T. Rockafellar, *Augmented Lagrangians and applications of the proximal point algorithm in convex programming,* Math. Oper. Research, 1 (1976), pp. 97–116.

[50] O. L. Mangasarian, *Unconstrained Lagrangians in nonlinear programming,* SIAM J. Control, 13 (1975), pp. 772–791.

[51] D. P. Bertsekas, *Constrained Optimization and Lagrange Multiplier Methods*, Academic Press, 1982.

[52] L. L. Pennisi, *An indirect sufficiency proof for the problem of Lagrange with differential inequalities as added side conditions,* Trans. Amer. Math. Soc., 74 (1953), pp. 177–198.

[53] A. D. Ioffe, *Necessary and sufficient conditions for a local extremum, 3: second order conditions and augmented duality,* SIAM J. Control Opt., 17 (1979), pp. 266–288.

[54] A. Ben-Tal, *Second-order and related extremality conditions in nonlinear programming,* J. Optim. Theory Appl., 31 (1980), pp. 141–165.

[55] R. T. Rockafellar, *Second-order optimality conditions in nonlinear programming obtained by way of epi-derivatives,* Math. Oper. Research, 14 (1989), pp. 462–484.

[56] A. D. Ioffe, *Calculus of Dini subdifferentials of functions and contingent coderivatives of set-valued maps,* Nonlinear Analysis, 8 (1984), pp. 517–539.

[57] R. T. Rockafellar, *Proximal subgradients, marginal values, and augmented Lagrangians in nonconvex optimization,* Math. Oper. Research, 6 (1981), pp. 427–437.

[58] D. Gale, *A geometric duality theorem with economic application,* Review Econ. Studies (1967).

[59] F. H. Clarke, *A new approach to Lagrange multipliers,* Math. Oper. Research, 1 (1976), pp. 165–174.

[60] J. V . Burke, *An exact penalization viewpoint of constrained optimization, SIAM Journal on Control and Optimization*, 29 (1991), pp. 968–998.

[61] R. T. Rockafellar, *Lagrange multipliers and subderivatives of optimal value functions in nonlinear programming,* Math. Programming Study 17 (1982), pp. 28–66.

[62] R. T. Rockafellar, *Directional differentiability of the optimal value function in a nonlinear programming problems,* Math. Programming Studies, 21 (1984), pp. 213–226.

[63] R. T. Rockafellar, *Computational schemes for solving large-scale problems in extended linear-quadratic programming,* Math. Programming, Ser. B, 48 (1990), pp. 447–474.

[64] C. Zhu and R. T. Rockafellar, *Primal-dual projected gradient algorithms for extended linear-quadratic programming,* SIAM J. Optimization, to appear.

[65] R. T. Rockafellar, *Multistage convex programming and discrete-time optimal control,* Control and Cybernetics, 17 (1988), pp. 225–246.

[66] R. T. Rockafellar, *Hamiltonian trajectories and duality in the optimal control of linear systems with convex costs,* SIAM J. Control Opt., 27 (1989), pp. 1007–1025.

[67] R. T. Rockafellar, *Perturbation of generalized Kuhn-Tucker points in finite dimensional optimization,* Nonsmooth Optimization and Related Topics, F. H. Clarke, V. Demyanov, and F. Gianessi, eds., Plenum Press, 1989, pp. 393–402.

[68] R. T. Rockafellar, *Nonsmooth analysis and parametric optimization,* Methods of Nonconvex Analysis (A. Cellina, ed.), Springer-Verlag Lecture Notes in Math. No. 1446 (1990), pp. 137–151.

[69] R. A. Poliquin and R. T. Rockafellar, *A calculus of epi-derivatives applicable to optimization,* Canadian J. Math., to appear.

[70] R. A. Poliquin and R. T. Rockafellar, *Amenable functions in optimization*, Nonsmooth Optimization Methods and Applications (F. Giannessi, ed.), Gordon and Breach, Philadelphia, 1992, pp. 338–353.

[71] J. V. Burke and R. A. Poliquin, *Optimality conditions for non-finite valued convex composite functions,* Math. Programming, to appear.

[72] A. D. Ioffe, *Variational analysis of a composite function: a formula for the lower second order directional derivative,* J. Math. Anal. Appl., to appear.

[73] J.-P. Aubin and I. Ekeland, *Applied Nonlinear Analysis*, Wiley, 1984.

[74] J.-P. Aubin and H. Frankowska, *Set-Valued Analysis*, Birkhäuser, 1990.