HW5

①

$$A = \begin{bmatrix} 2 & -1 \\ 0 & 3 \end{bmatrix} \rightarrow |A-\lambda I| = \begin{vmatrix} 2-\lambda & -1 \\ 0 & 3-\lambda \end{vmatrix} = 0 \rightarrow (2-\lambda)(3-\lambda) - 0 = 0 \rightarrow \begin{cases} \lambda_1 = 2 \\ \lambda_2 = 3 \end{cases}$$

$$\lambda_1 = 2 \rightarrow \begin{bmatrix} 0 & -1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0 \rightarrow \begin{cases} -x_2 = 0 \\ x_2 = 0 \end{cases} \rightarrow x_2 = 0 \rightarrow \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\lambda_2 = 3 \rightarrow \begin{bmatrix} -1 & -1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0 \rightarrow \begin{cases} -x_1 - x_2 = 0 \\ 0 \end{cases} \rightarrow x_1 = -x_2 \rightarrow \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$B = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \rightarrow |B-\lambda I| = \begin{vmatrix} \cos\theta-\lambda & -\sin\theta \\ \sin\theta & \cos\theta-\lambda \end{vmatrix} = (\cos\theta-\lambda)^2 + \sin^2\theta = 0$$

$$\rightarrow \lambda^2 - 2\cos\theta\,\lambda + 1 = 0 \rightarrow \begin{cases} \lambda_1 = \cos\theta + j\sin\theta \\ \lambda_2 = \cos\theta - j\sin\theta \end{cases}$$

$$\lambda_1 = \cos\theta + j\sin\theta \rightarrow \begin{bmatrix} -j\sin\theta & -\sin\theta \\ \sin\theta & -j\sin\theta \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0 \rightarrow \begin{cases} -j\sin\theta\, x_1 - \cancel{j}\sin\theta\, x_2 = 0 \\ \sin\theta\, x_1 - j\sin\theta\, x_2 = 0 \end{cases} \rightarrow x_2 = -jx_1 \rightarrow \begin{bmatrix} 1 \\ -j \end{bmatrix}$$

$$\lambda_2 = \cos\theta - j\sin\theta \rightarrow \begin{bmatrix} j\sin\theta & -\sin\theta \\ \sin\theta & j\sin\theta \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0 \rightarrow \begin{cases} j\sin\theta\, x_1 - \sin\theta\, x_2 = 0 \\ \sin\theta\, x_1 + j\sin\theta\, x_2 = 0 \end{cases} \rightarrow x_2 = jx_1 \rightarrow \begin{bmatrix} 1 \\ j \end{bmatrix}$$

$$C = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 2 & 0 \\ -2 & 0 & 0 \end{bmatrix} \rightarrow |C-\lambda I| = 0 \rightarrow \begin{vmatrix} -\lambda & 0 & 2 \\ 0 & 2-\lambda & 0 \\ -2 & 0 & -\lambda \end{vmatrix} = 0 \rightarrow -\lambda(2-\lambda)(-\lambda) + 2(2)(2-\lambda) = 0$$

$$\rightarrow (2-\lambda)(\lambda^2+4) = 0 \rightarrow \begin{cases} \lambda_1 = 2 \\ \lambda_2 = j2 \\ \lambda_3 = -j2 \end{cases}$$

$$\lambda_1 = 2 \rightarrow \begin{bmatrix} -2 & 0 & 2 \\ 0 & 0 & 0 \\ -2 & 0 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0 \rightarrow \begin{cases} -2x_1 + 2x_3 = 0 \\ -2x_1 + 2x_3 = 0 \end{cases} \rightarrow x_1 = x_3 \rightarrow \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}, \ldots$$

$$\lambda_2 = 2j \rightarrow \begin{bmatrix} -j2 & 0 & 2 \\ 0 & 2-2j & 0 \\ -2 & 0 & -2j \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0 \rightarrow \begin{cases} -2jx_1 + 2x_3 = 0 \\ (2-2j)x_2 = 0 \\ -2x_1 + 2jx_3 = 0 \end{cases} \rightarrow \begin{cases} x_3 = jx_1 \\ x_2 = 0 \\ x_3 = jx_1 \end{cases} \rightarrow \begin{bmatrix} 1 \\ 0 \\ j \end{bmatrix}$$

$$\lambda_3 = -2j \rightarrow \begin{bmatrix} 2j & 0 & 2 \\ 0 & 2+2j & 0 \\ -2 & 0 & 2j \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0 \rightarrow \begin{cases} 2jx_1 + 2x_3 = 0 \\ (2+2j)x_2 = 0 \\ -2jx_1 + 2jx_3 = 0 \end{cases} \rightarrow \begin{cases} x_3 = -jx_1 \\ x_2 = 0 \\ x_3 = -jx_1 \end{cases} \rightarrow \begin{bmatrix} 1 \\ 0 \\ -j \end{bmatrix}$$

$$D = \begin{bmatrix} 2 & 1-j & 3 \\ 1+j & 4 & j \\ 3 & -j & 2 \end{bmatrix} \rightarrow \begin{cases} \lambda_1 = -1.01 \\ \lambda_2 = 2.92 \\ \lambda_3 = 6.18 \end{cases}$$

$[V,D] = eig(D)$ با استفاده از دستور eig

$$\lambda_1 = -1.01 \rightarrow \begin{bmatrix} 0.71 - j0.05 \\ -0.15 + j0.005 \\ -0.69 \end{bmatrix} \quad \lambda_2 = 2.92 \rightarrow \begin{bmatrix} -0.36 + j0.16 \\ 0.48 + j0.63 \\ -0.48 \end{bmatrix} \quad \lambda_3 = 6.18 \rightarrow \begin{bmatrix} -0.58 - j0.08 \\ -0.23 - j0.55 \\ -0.55 \end{bmatrix}$$

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(2) (a) $\det(A^T) = (\lambda_1)(\lambda_2)(\lambda_3) = (-1)(2)(3) = \boxed{-6}$ مقادیر ویژه A و A^T یکسان است.

(b) $Trace(A^{-1})$

مقادیر ویژه ماتریس A^{-1} برابر با معکوس مقادیر ویژه A است.

$$A^{-1}: \frac{1}{\lambda_1}, \frac{1}{\lambda_2}, \frac{1}{\lambda_3} = -1, \frac{1}{2}, \frac{1}{3} \rightarrow Trace(A^{-1}) = -1 + \frac{1}{2} + \frac{1}{3} = \frac{-6+3+2}{6} = \boxed{-\frac{1}{6}}$$

(c) $\det(A - 2I)$

$$AX = \lambda X \rightarrow (A - 2I)X = AX - 2X = \lambda X - 2X = (\lambda - 2)X \Rightarrow \begin{cases} \lambda_1 = -1-2 = -3 \\ \lambda_2 = 2-2 = 0 \\ \lambda_3 = 3-2 = 1 \end{cases}$$

$$\det(A - 2I) = (\lambda_1)(\lambda_2)(\lambda_3) = (-3)(0)(1) = 0$$

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(3) $$|A - \lambda I| = 0 \rightarrow \begin{vmatrix} 1-\lambda & 1 & 1 \\ 0 & 2-\lambda & 1 \\ 0 & 0 & 3-\lambda \end{vmatrix} \xrightarrow{\text{ماتریس بالا مثلثی}} \lambda_1 = 1,\ \lambda_2 = 2,\ \lambda_3 = 3$$

$$\lambda_1 = 1 \rightarrow \begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0 \rightarrow \begin{cases} x_2 + x_3 = 0 \\ x_2 + x_3 = 0 \\ x_3 = 0,\ x_2 = 0 \end{cases} \rightarrow \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\lambda_2 = 2 \rightarrow \begin{bmatrix} -1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0 \rightarrow \begin{cases} -x_1 + x_2 + x_3 = 0 \\ x_3 = 0 \end{cases} \rightarrow x_1 = x_2 \rightarrow \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$\lambda_3 = 3 \rightarrow \begin{bmatrix} -2 & 1 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0 \rightarrow \begin{cases} -2x_1 + x_2 + x_3 = 0 \rightarrow x_1 = x_2 \\ -x_2 + x_3 = 0 \rightarrow x_2 = x_3 \\ x_1 = x_2 = x_3 \end{cases} \rightarrow \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$T = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} V_1 & V_2 & V_3 \end{bmatrix} \qquad T^{-1}AT = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

(4) $MX = \lambda X \rightarrow \sum_n e^{j\frac{2\pi}{N}(m-1)(n-1)} X_{(n)} = \lambda X_{(m)} \xrightarrow{\times e^{-j\frac{2\pi}{N}(m-1)(k-1)}}$

$\sum_m \sum_n e^{j\frac{2\pi}{N}(m-1)(n-k)} X_{(n)} = \lambda \sum_m e^{-j\frac{2\pi}{N}(m-1)(k-1)} X_m \longrightarrow$

$\sum_n e^{j\frac{2\pi}{N}(k-1)(n-1)} X_{(n)} = \lambda X_{(k)} \Rightarrow \lambda = e^{j\frac{2\pi}{N}(k-1)} \longrightarrow \lambda_k = e^{j\frac{2\pi}{N}(k-1)} \quad k = 1 \to N$

$[1, \lambda, \lambda^2, \ldots, \lambda^{N-2}, \lambda^{N-1}][1, \lambda_k, \lambda_k^2, \ldots, \lambda_k^{N-1}]^T = \lambda_k [1, \lambda_k, \lambda_k^2, \ldots, \lambda_k^{N-1}]$

$1 + \lambda\lambda_k + \lambda^2\lambda_k^2 + \cdots + \lambda^{N-1}\lambda_k^{N-1} = \lambda_k(1 + \lambda_k + \lambda_k^2 + \cdots + \lambda_k^{N-1}) \xrightarrow{\text{مجموع دنباله‌ی هندسی}}$

$\frac{1-(\lambda\lambda_k)^{n-1}}{1-(\lambda\lambda_k)} = \frac{1-\lambda_k^n}{1-\lambda_k} \longrightarrow \text{if } \lambda^n = 1 \rightarrow 1 = \lambda_k^n \rightarrow \lambda_k = e^{j\frac{2\pi}{N}k}$

$\longrightarrow x_k = [1, \lambda_k, \lambda_k^2, \ldots, \lambda_k^{N-1}]$

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(5) $|A - \lambda I| = 0 \rightarrow \begin{vmatrix} 2-\lambda & 1 & 1 \\ 1 & 3-\lambda & 0 \\ 1 & 0 & 1-\lambda \end{vmatrix} = 0 \rightarrow -\lambda^3 + 6\lambda^2 - 9\lambda + 2 = 0 \longrightarrow \begin{cases} \lambda_1 = 2 \\ \lambda_2 = 0.268 \\ \lambda_3 = 3.732 \end{cases}$

هر سه مقادیر ویژه مثبت هستند ← مثبت معین

$|B - \lambda I| = 0 \rightarrow \begin{vmatrix} -1-\lambda & -1 & -1 \\ -1 & -4-\lambda & -1 \\ -1 & -1 & -2-\lambda \end{vmatrix} = 0 \longrightarrow -\lambda^3 - 7\lambda^2 - 11\lambda - 3 = 0 \longrightarrow \begin{cases} \lambda_1 = -4.866 \\ \lambda_2 = -1.789 \\ \lambda_3 = -0.345 \end{cases}$

همه مقادیر ویژه منفی هستند ← منفی معین

$|C - \lambda I| = 0 \rightarrow \begin{vmatrix} 2-\lambda & 4 \\ 1 & 2-\lambda \end{vmatrix} = 0 \rightarrow (2-\lambda)^2 - 4 = 0 \rightarrow \lambda^2 - 4\lambda = 0 \longrightarrow \begin{cases} \lambda_1 = 4 \\ \lambda_2 = 0 \end{cases}$

مقادیر ویژه یکی صفر و دیگری مثبت است ← مثبت نیمه معین