

Random Quantum Circuits

A Glimpse into Quantum Randomness

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September 1, 2024

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2 Preliminaries

- Linear Algebra
- Measure Theory
- Group Theory

3 Quantum Mechanics

- Projective Measurements

4 Circuit Structure

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Introduction

Preliminaries

For the sake of completeness, we will briefly go through some of the basic concepts that will be used in the subsequent sections.

We will cover the following topics:

- Linear Algebra
 - ▶ Tensor Products
- Group Theory
 - ▶ Unitary Group (Special emphasis on Haar Random Unitaries)
A bit of **measure theory** on the way.
- Quantum Mechanics
 - ▶ Measurements vs Projective Measurements

Linear Algebra: Basic Concepts

Assuming that the audience is familiar with basic linear algebra concepts like

- Hermitian and Unitary matrices
- Eigenvalues and eigenvectors
- Inner product and outer product
- Diagonalization of matrices
- Matrix exponentiation

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$$H^\dagger = H$$

Hermitian

$$U^\dagger U = UU^\dagger = I$$

Unitary

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$$A|\psi\rangle = \lambda|\psi\rangle$$

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$$\langle \cdot | \cdot \rangle : V \times V \rightarrow \mathbb{C}$$

Inner Product

$$|\cdot\rangle\langle\cdot| : V \times V \rightarrow \mathcal{L}(V)$$

Outer Product

- Diagonalization of matrices
- Matrix exponentiation

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- Diagonalization of matrices

$$A = PDP^{-1} \quad \text{where} \quad D = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix}$$

- Matrix exponentiation

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$$e^A = \mathbb{I} + A + \frac{A^2}{2!} + \dots$$

Linear Algebra: Basic Concepts

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If A is diagonalizable, then

$$e^A = Pe^D P^{-1} \quad \text{where} \quad e^D = \begin{pmatrix} e^{\lambda_1} & & & \\ & e^{\lambda_2} & & \\ & & \ddots & \\ & & & e^{\lambda_n} \end{pmatrix}$$

Linear Algebra: Tensor Products

Definition (Tensor Product)

Let V and W be vector spaces over a field \mathbb{F} . The tensor product of V and W , denoted by $V \otimes W$, is a vector space over \mathbb{F} with the following properties:

- 1 For any $v \in V$ and $w \in W$, $v \otimes w \in V \otimes W$.
- 2 The tensor product is bilinear, i.e., for any $v, v' \in V$, $w, w' \in W$, and $a, b \in \mathbb{F}$,

$$(av + bv') \otimes w = a(v \otimes w) + b(v' \otimes w)$$

and

$$v \otimes (aw + bw') = a(v \otimes w) + b(v \otimes w')$$

This gives a way to combine two vector spaces to form a new vector space. In the context of quantum mechanics, the tensor product is used to represent composite systems.

Measure Theory

Definition (Measure of a Set)

Let \mathcal{A} be a σ -algebra on a set X . A function $\mu : \mathcal{A} \rightarrow [0, \infty]$ is called a measure on \mathcal{A} if it satisfies the following properties:

- 1 $\mu(\emptyset) = 0$.
- 2 Countable additivity: For any countable collection of pairwise disjoint sets $\{A_i\}_{i \in \mathbb{N}} \subset \mathcal{A}$,

$$\mu\left(\bigcup_{i \in \mathbb{N}} A_i\right) = \sum_{i \in \mathbb{N}} \mu(A_i).$$

Example (Lebesgue Measure)

For any interval $I = [a, b] \subset \mathbb{R}$, the Lebesgue measure $\mu(I) = b - a$.

Measure Theory: Translation Invariance

Definition (Translation Invariant Measure)

A measure μ on a σ -algebra \mathcal{A} is said to be translation invariant if for all $A \in \mathcal{A}$ and $x \in X$, $\mu(A) = \mu(A + x)$.

Example (Lebesgue Measure)

For any interval $I = [a, b] \subset \mathbb{R}$, and $x \in \mathbb{R}$, the Lebesgue measure $\mu(I) = \mu(I + x) = b - a$. Thus, the Lebesgue measure is translation invariant.

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We are particularly interested in translation invariant measures on a set X as they allow us to define a notion of uniformity on X .

A translation-invariant measure would ensure that every region of the space is treated equally, regardless of its position.

Group Theory

Definition (Group)

A group is a set G along with a binary operation $*$ such that the following properties are satisfied:

- 1 Closure: For all $a, b \in G$, $a * b \in G$.
- 2 Associativity: For all $a, b, c \in G$, $(a * b) * c = a * (b * c)$.
- 3 Identity: There exists an element $e \in G$ such that for all $a \in G$, $a * e = e * a = a$.
- 4 Inverse: For all $a \in G$, there exists an element $a^{-1} \in G$ such that $a * a^{-1} = a^{-1} * a = e$.

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Example (General Linear Group)

The set of all $n \times n$ invertible matrices with matrix multiplication as the binary operation forms a group called the General Linear Group, denoted by $GL(n, \mathbb{C})$.

Group Theory: Unitary Group

There is a specific subgroup of the General Linear Group that is of particular interest to us i.e., the Unitary Group.

Definition (Unitary Group)

The unitary group $U(n)$ is the group of all $n \times n$ unitary matrices.

Example (Special Unitary Group)

The set of all $n \times n$ unitary matrices with determinant 1 forms a subgroup of the unitary group called the Special Unitary Group, denoted by $SU(n)$.

Group Theory: Haar Random Unitaries

Since we are interested in constructing random quantum circuits, we need a way to generate random unitaries uniformly from the unitary group. This is where Haar measure comes into play.

Definition (Haar Measure)

Let (G, \cdot) be a **locally compact topological group**. A Haar measure μ on G is a translation-invariant measure that is non-zero on open sets.

In case of the unitary group, the Haar measure of the unitary group is finite, thus allowing us to define a **probability distribution** on the unitary group. Since the Haar measure is translation invariant, it ensures that the generated unitaries are uniformly distributed over the unitary group.

Quantum Mechanics

Using Dirac Notation, we can write Schrödinger equation as

$$i\hbar \frac{d|\Psi\rangle}{dt} = \hat{H}|\Psi\rangle \implies |\Psi; t\rangle = \mathcal{U}_t |\Psi; 0\rangle,$$

where $\mathcal{U}_t := \exp\left(-i\hat{H}t/\hbar\right)$ is the time-evolution operator. Since Hamiltonian is hermitian, then by definition time-evolution operator is unitary.

Why Unitary Operators?

This identification implies that to study the dynamics of a quantum system evolving under a random Hamiltonian, we can employ random unitaries rather than solving the Schrödinger equation.

Quantum Mechanics: Projective Measurements

Circuit Structure

Operator Spread

Anderson Chain

Thank You!