# Random Quantum Circuits

## A Glimpse into Quantum Randomness

Piyush Kumar Singh[1] Dr. Sambuddha Sanyal[2]

[1]Department of Physical Sciences
IISER Kolkata

[2]Department of Physical Sciences
IISER Tirupati

NIUS Physics (Camp 20)

September 1, 2024

# Outline

# Introduction

# Preliminaries

For the sake of completeness, we will briefly go through some of the basic concepts that will be used in the subsequent sections.
We will cover the following topics:

- Linear Algebra
  - Tensor Products
- Group Theory
  - Unitary Group (Special emphasis on Haar Random Unitaries)
    A bit of measure theory on the way.
- Quantum Mechanics
  - Measurements vs Projective Measurements

# Linear Algebra: Basic Concepts

Assuming that the audience is familiar with basic linear algebra concepts like

- Hermitian and Unitary matrices
- Eigenvalues and eigenvectors
- Inner product and outer product
- Diagonalization of matrices
- Matrix exponentiation

# Linear Algebra: Basic Concepts

Assuming that the audience is familiar with basic linear algebra concepts like

- Hermitian and Unitary matrices

$$
\begin{aligned}
H^\dagger &= H && \text{Hermitian} \\
U^\dagger U = UU^\dagger &= I && \text{Unitary}
\end{aligned}
$$

- Eigenvalues and eigenvectors
- Inner product and outer product
- Diagonalization of matrices
- Matrix exponentiation

# Linear Algebra: Basic Concepts

Assuming that the audience is familiar with basic linear algebra concepts like

- Hermitian and Unitary matrices
- Eigenvalues and eigenvectors

$$A\left|\psi\right\rangle = \lambda\left|\psi\right\rangle$$

- Inner product and outer product
- Diagonalization of matrices
- Matrix exponentiation

# Linear Algebra: Basic Concepts

Assuming that the audience is familiar with basic linear algebra concepts like

- Hermitian and Unitary matrices
- Eigenvalues and eigenvectors
- Inner product and outer product

$$\langle\cdot|\cdot\rangle : V \times V \to \mathbb{C} \qquad \text{Inner Product}$$
$$|\cdot\rangle\langle\cdot| : V \times V \to \mathcal{L}(V) \qquad \text{Outer Product}$$

- Diagonalization of matrices
- Matrix exponentiation

# Linear Algebra: Basic Concepts

Assuming that the audience is familiar with basic linear algebra concepts like

- Hermitian and Unitary matrices
- Eigenvalues and eigenvectors
- Inner product and outer product
- Diagonalization of matrices

$$A = PDP^{-1} \quad \text{where} \quad D = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix}$$

- Matrix exponentiation

# Linear Algebra: Basic Concepts

Assuming that the audience is familiar with basic linear algebra concepts like

- Hermitian and Unitary matrices
- Eigenvalues and eigenvectors
- Inner product and outer product
- Diagonalization of matrices
- Matrix exponentiation

$$e^A = \mathbb{I} + A + \frac{A^2}{2!} + \cdots$$

# Linear Algebra: Basic Concepts

Assuming that the audience is familiar with basic linear algebra concepts like

- Hermitian and Unitary matrices
- Eigenvalues and eigenvectors
- Inner product and outer product
- Diagonalization of matrices
- Matrix exponentiation

If $A$ is diagonalizable, then

$$e^A = Pe^D P^{-1} \quad \text{where} \quad e^D = \begin{pmatrix} e^{\lambda_1} & & & \\ & e^{\lambda_2} & & \\ & & \ddots & \\ & & & e^{\lambda_n} \end{pmatrix}$$

# Linear Algebra: Tensor Products

**Definition (Tensor Product)**

Let $V$ and $W$ be vector spaces over a field $\mathbb{F}$. The tensor product of $V$ and $W$, denoted by $V \otimes W$, is a vector space over $\mathbb{F}$ with the following properties:

1. For any $v \in V$ and $w \in W$, $v \otimes w \in V \otimes W$.
2. The tensor product is bilinear, i.e., for any $v, v' \in V$, $w, w' \in W$, and $a, b \in \mathbb{F}$,
$$(av + bv') \otimes w = a(v \otimes w) + b(v' \otimes w)$$
and
$$v \otimes (aw + bw') = a(v \otimes w) + b(v \otimes w')$$

This gives a way to combine two vector spaces to form a new vector space. In the context of quantum mechanics, the tensor product is used to represent composite systems.

# Measure Theory

## Definition (Measure of a Set)

Let $\mathcal{A}$ be a $\sigma$-algebra on a set $X$. A function $\mu : \mathcal{A} \to [0, \infty]$ is called a measure on $\mathcal{A}$ if it satisfies the following properties:

1. $\mu(\emptyset) = 0$.
2. Countable additivity: For any countable collection of pairwise disjoint sets $\{A_i\}_{i \in \mathbb{N}} \subset \mathcal{A}$,

$$\mu\left(\bigcup_{i \in \mathbb{N}} A_i\right) = \sum_{i \in \mathbb{N}} \mu(A_i).$$

## Example (Lebesgue Measure)

For any interval $I = [a, b] \subset \mathbb{R}$, the Lebesgue measure $\mu(I) = b - a$.

# Measure Theory: Translation Invariance

**Definition (Translation Invariant Measure)**

A measure $\mu$ on a $\sigma$-algebra $\mathcal{A}$ is said to be translation invariant if for all $A \in \mathcal{A}$ and $x \in X$, $\mu(A) = \mu(A + x)$.

**Example (Lebesgue Measure)**

For any interval $I = [a, b] \subset \mathbb{R}$, and $x \in \mathbb{R}$, the Lebesgue measure $\mu(I) = \mu(I + x) = b - a$. Thus, the Lebesgue measure is translation invariant.

# Measure Theory: Translation Invariance

**Definition (Translation Invarianct Measure)**

A measure $\mu$ on a $\sigma$-algebra $\mathcal{A}$ is said to be translation invariant if for all $A \in \mathcal{A}$ and $x \in X$, $\mu(A) = \mu(A + x)$.

**Example (Lebesgue Measure)**

For any interval $I = [a, b] \subset \mathbb{R}$, and $x \in \mathbb{R}$, the Lebesgue measure $\mu(I) = \mu(I + x) = b - a$. Thus, the Lebesgue measure is translation invariant.

We are particularly interested in translation invariant measures on a set $X$ as they allow us to define a notion of uniformity on $X$.
A translation-invariant measure would ensure that every region of the space is treated equally, regardless of its position.

# Group Theory

## Definition (Group)

A group is a set $G$ along with a binary operation $*$ such that the following properties are satisfied:

1. Closure: For all $a, b \in G$, $a * b \in G$.
2. Associativity: For all $a, b, c \in G$, $(a * b) * c = a * (b * c)$.
3. Identity: There exists an element $e \in G$ such that for all $a \in G$, $a * e = e * a = a$.
4. Inverse: For all $a \in G$, there exists an element $a^{-1} \in G$ such that $a * a^{-1} = a^{-1} * a = e$.

# Group Theory

## Definition (Group)

A group is a set $G$ along with a binary operation $*$ such that the following properties are satisfied:

1. Closure: For all $a, b \in G$, $a * b \in G$.
2. Associativity: For all $a, b, c \in G$, $(a * b) * c = a * (b * c)$.
3. Identity: There exists an element $e \in G$ such that for all $a \in G$, $a * e = e * a = a$.
4. Inverse: For all $a \in G$, there exists an element $a^{-1} \in G$ such that $a * a^{-1} = a^{-1} * a = e$.

## Example (General Linear Group)

The set of all $n \times n$ invertible matrices with matrix multiplication as the binary operation forms a group called the General Linear Group, denoted by $\mathrm{GL}(n, \mathbb{C})$.

# Group Theory: Unitary Group

There is a specific subgroup of the General Linear Group that is of particular interest to us i.e., the Unitary Group.

**Definition (Unitary Group)**

The unitary group $\mathrm{U}(n)$ is the group of all $n \times n$ unitary matrices.

**Example (Special Unitary Group)**

The set of all $n \times n$ unitary matrices with determinant 1 forms a subgroup of the unitary group called the Special Unitary Group, denoted by $\mathrm{SU}(n)$.

# Group Theory: Haar Random Unitaries

Since we are interested in constructing random quantum circuits, we need a way to generate random unitaries uniformly from the unitary group. This is where Haar measure comes into play.

**Definition (Haar Measure)**

Let $(G, \cdot)$ be a locally compact topological group. A Haar measure $\mu$ on $G$ is a translation-invariant measure that is non-zero on open sets.

In case of the unitary group, the Haar measure of the unitary group is finite, thus allowing us to define a probability distribution on the unitary group. Since the Haar measure is translation invariant, it ensures that the generated unitaries are uniformly distributed over the unitary group.

# Quantum Mechanics

Using Dirac Notation, we can write Schrödinger equation as

$$i\hbar \frac{\mathrm{d}\left|\Psi\right\rangle}{\mathrm{d}t} = \hat{H}\left|\Psi\right\rangle \implies \left|\Psi; t\right\rangle = \mathcal{U}_t \left|\Psi; 0\right\rangle ,$$

where $\mathcal{U}_t := \exp\left(-i\hat{H}t \middle/ \hbar\right)$ is the time-evolution operator. Since Hamiltonian is hermitian, then by definition time-evolution operator is unitary.

**Why Unitary Operators?**

This identification implies that to study the dynamics of a quantum system evolving under a random Hamiltonian, we can employ random unitaries rather than solving the Schrödinger equation.

# Quantum Mechanics: Projective Measurements

**Definition (Projective Measurement)**

# Circuit Structure

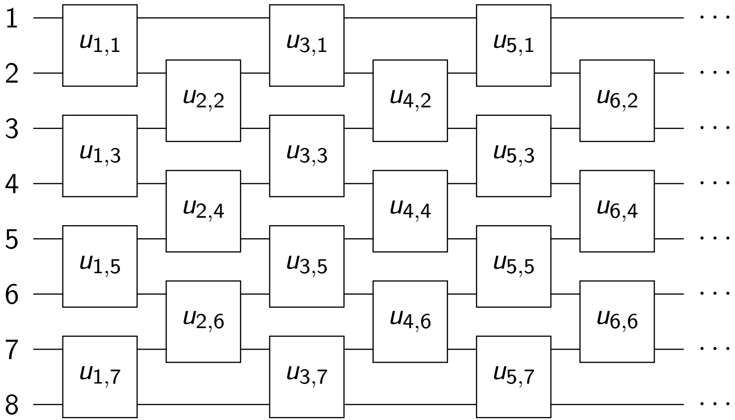


**Figure:** A spacetime diagram of 8 qubits RQC. This arrangement of two-site unitary gates will be called a brickwork structure.

# State Trajectory

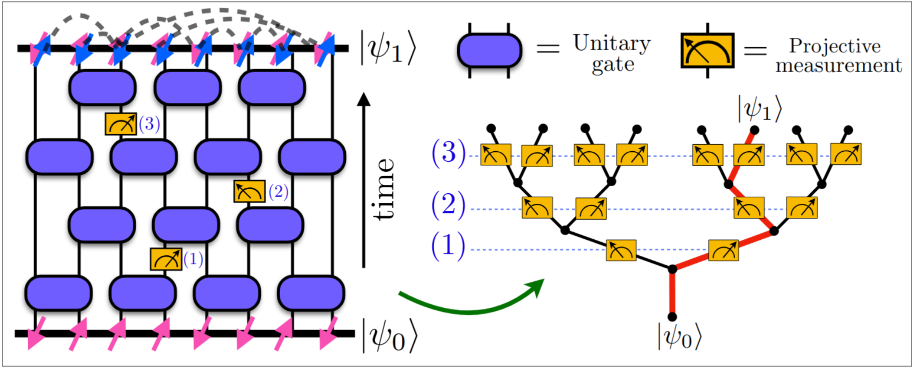


**Figure:** Hello

# Fidelity

**Definition (Fidelity)**

Fidelity measures "overlap" between two states from a given Hilbert space $\mathcal{H}$. Consider two states $|\Psi\rangle$ and $|\Phi\rangle$, the fidelity between them is defined as

$$F(|\Psi\rangle, |\Phi\rangle) = |\langle\Psi|\Phi\rangle|^2.$$

# Operator Spread

# Anderson Chain

Thank You!