# Glauber Model

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## 1 Introduction

The Glauber model, named after Nobel laureate Roy J. Glauber, is a quantitative description of high energy hadron collisions based on collision geometry. Glauber's work in the 1950s had provided the basis for theoretical understanding of scattering cross-sections of high energy nuclear collisions (see [1] for brief history). A complete description of inelastic collisions involving many-body systems is a complex quantum mechanical problem whereas the model is an approximation.

### 1.1 Assumptions

The following assumptions are taken into account,

- Collision between two nuclei is essentially a set of binary collisions between nucleons.
- Two nucleons collide while they move in straight line trajectories (*eikonal* limit).
- At relativistic speeds, nucleons undergo negligible deflection.
- Nucleons move freely within a nucleus.

### 1.2 Terminology

Following terms will be used frequently later on,

- *Beam* axis is usually taken along **z** axis in *Laboratory* frame. Colliding nuclei are highly Lorentz contracted along the beam direction, also known as the *longitudinal* direction.
- Distance between centers of the colliding nuclei is known as the *impact parameter*. By definition, the impact vector lies on **xy** plane (also referred as *transverse* plane) in the lab frame.
- Beam axis and the impact vector defines the *reaction* plane.
- An *event* corresponds to a single collision : nucleons that suffer at least one inelastic collision are known as *participants* / *wounded* nucleons and those that do not interact are referred as *spectators*.

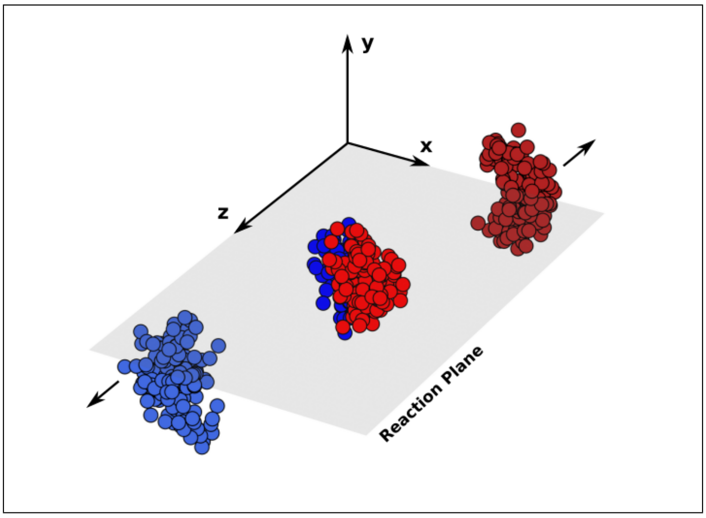



Figure 1: Reaction plane, participants and spectators

### 1.3 Input / Output

Three essential inputs to the model are nucleon density $\rho(\mathbf{r})$, impact parameter $(b)$ and inelastic (2-nucleon) scattering cross-section $(\sigma_{NN})$ that is taken to be a constant as nucleonic excitations are ignored. The outputs are number of participant nucleons $\left(N_{part}\right)$ and number of binary collisions $(N_{coll})$ that are averaged over an event ensemble generated by varying $b$.

## 2 Optical Limit

In Optical limit, all useful quantities are obtained as integral expressions [1]. Inside the nucleus, probability of finding a nucleon between $\mathbf{r}$ and $\mathbf{r} + d\mathbf{r}$ is $\rho(\mathbf{r})\, d^3\mathbf{r} \implies \int_{\text{all}} \rho(\mathbf{r})\, d^3\mathbf{r} = A$, the atomic mass number. Probability of finding the nucleon in transverse plane is referred as the *thickness* function,

$$T(\mathbf{r}_\perp) \equiv \frac{1}{A} \int_{-\infty}^{\infty} \rho(\mathbf{r}_\perp,\, z)\, dz \tag{1}$$

Given the impact vector ($\mathbf{b}$), probability of of finding two nucleons (from different nucleus) at $\mathbf{r}_\perp$ is,

$$T_A\left(\mathbf{r}_\perp\right)\, T_B\left(\mathbf{r}_\perp - \mathbf{b}\right)\, d^2\mathbf{r}_\perp \equiv T_A\left(\mathbf{r}_\perp - \frac{\mathbf{b}}{2}\right)\, T_B\left(\mathbf{r}_\perp + \frac{\mathbf{b}}{2}\right) = T_A^-(\mathbf{r}_\perp)\, T_B^+(\mathbf{r}_\perp)$$

Define the *nuclear overlap* function,

$$T_{AB}(\mathbf{b}) \equiv \int T_A^-(\mathbf{r}_\perp)\, T_B^+(\mathbf{r}_\perp)\, d^2\mathbf{r}_\perp \Longrightarrow \int T_{AB}(\mathbf{b})\, d^2\mathbf{b} = 1 \tag{2}$$

Given an inelastic scattering cross-section, probability of a binary collision at impact parameter $b$ is $\sigma_{NN}\, T_{AB}(\mathbf{b})$. Then probability of having $n$ collisions out of $A\,B$ possible interactions is given by the *binomial* distribution,

$$P(n,\, \mathbf{b}) = \binom{A\,B}{n} \left[\sigma_{NN}\, T_{AB}(\mathbf{b})\right]^n \left[1 - \sigma_{NN}\, T_{AB}(\mathbf{b})\right]^{A\,B-n} \tag{3}$$

$n \geq 1$ gives total the probability of all possible binary collisions,

$$\sum_{n=1}^{A\,B} P(n,\, \mathbf{b}) = \sum_{n=0}^{A\,B} P(n,\, \mathbf{b}) - P(0, \mathbf{b}) = 1 - \left[1 - \sigma_{NN}\, T_{AB}(\mathbf{b})\right]^{A\,B} \tag{4}$$

Total number of binary collisions is just the mean of this distribution,

$$N_{coll}(\mathbf{b}) = \sum_{n=0}^{A\,B} n\, P(n,\, \mathbf{b}) = \sigma_{NN}\, T_{AB}(\mathbf{b}) \tag{5}$$

$$\Longrightarrow N_{coll}\left(\mathbf{r}_\perp,\, \mathbf{b}\right) \equiv \sigma_{NN}\, T_A^-(\mathbf{r}_\perp)\, T_B^+(\mathbf{r}_\perp) \tag{6}$$

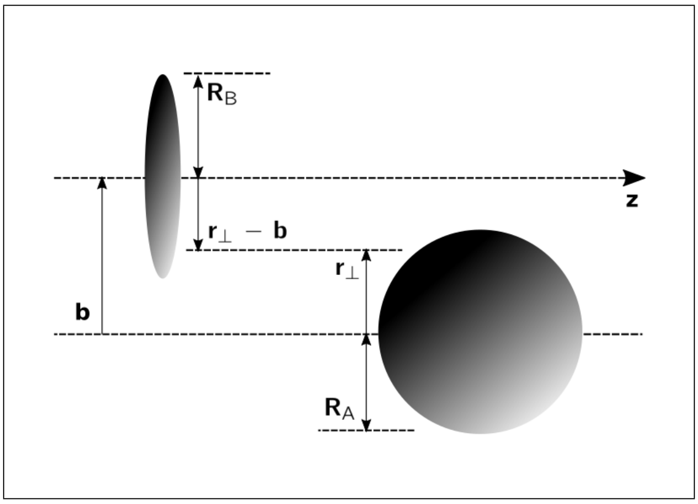



Figure 2: In the rest frame of target nucleus (here nucleus A)

To determine number of participant nucleons, consider nucleon-nucleus interaction : at $\mathbf{r}_\perp$, probability of binary collision between nucleus A and a nucleon from nucleus B is,

$$\sum_{n=1}^{A} P(n,\, \mathbf{r}_\perp, \mathbf{b}) = 1 - \left[1 - \sigma_{NN}\, T_A^-(\mathbf{r}_\perp)\right]^{A} \tag{7}$$

At $\mathbf{r}_\perp + \frac{\mathbf{b}}{2}$, total number of nucleons from nucleus B is $B\,T_B^+(\mathbf{r}_\perp)\,d^2\mathbf{r}_\perp$. Then, number density of participants from nucleus B is,

$$N_{part}^B(\mathbf{r}_\perp, \mathbf{b}) = B\,T_B^+(\mathbf{r}_\perp)\left(1 - \left[1 - \sigma_{NN}\,T_A^-(\mathbf{r}_\perp)\right]^A\right) \tag{8}$$

Deriving similar expression for nucleus A, we get the total number of participants,

$$\begin{aligned} N_{part}(\mathbf{r}_\perp, \mathbf{b}) = &A\,T_A^-(\mathbf{r}_\perp)\left(1 - \left[1 - \sigma_{NN}\,T_B^+(\mathbf{r}_\perp)\right]^B\right) \\ &+ B\,T_B^+(\mathbf{r}_\perp)\left(1 - \left[1 - \sigma_{NN}\,T_A^-(\mathbf{r}_\perp)\right]^A\right) \end{aligned} \tag{9}$$

$$\Longrightarrow N_{part}(\mathbf{b}) = \int N_{part}(\mathbf{r}_\perp, \mathbf{b})\,d^2\mathbf{r}_\perp \tag{10}$$

Total number of spectators is $N_{spec} \equiv A + B - N_{part}$. The integrals above can be calculated numerically. The integration limit must be set such that $\rho(\mathbf{r}) \ll 1$ holds at the boundaries.

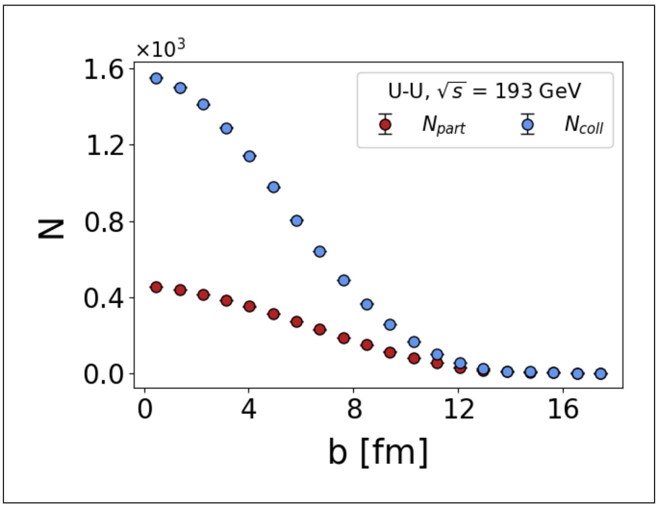



Figure 3: Impact parameter dependence of $N_{part}$ and $N_{coll}$ in Optical Glauber

## 3 Monte Carlo Approach

In Monte Carlo Glauber all useful quantities are obtained through enumeration [2],

I. Nucleon positions for each nucleus are sampled from their respective $\rho(\mathbf{r})$. Defining a hard sphere radius ensures nucleons from the same nucleus do not overlap.

II. Center of mass (CoM) for each nucleus is brought to the origin in order to shift them according to the impact vecto,

$$\begin{aligned} \mathbf{R}_{\mathrm{CM}} &= \frac{1}{A}\sum_{i=1}^{A}\mathbf{r}_i \quad \text{and} \quad \mathbf{r}_i \longrightarrow \mathbf{r}_i - \mathbf{R}_{\mathrm{CM}} \\ \mathbf{r}_i^A &\longrightarrow \mathbf{r}_i^A - \frac{\mathbf{b}}{2} \quad \text{and} \quad \mathbf{r}_i^B \longrightarrow \mathbf{r}_i^B + \frac{\mathbf{b}}{2} \end{aligned} \tag{11}$$

III. Interaction range for each nucleon is parameterized by $\sigma_{NN}$ : $D \equiv \sqrt{\frac{\sigma_{NN}}{\pi}}$. The condition for binary collision ($N_{coll} \longrightarrow N_{coll} + 1$) is then

$$r_{ij} = \left|\mathbf{r}^{\perp}_{A,i} - \mathbf{r}^{\perp}_{B,j}\right| \leq D \quad \text{where} \quad 1 \leq i \leq A,\ 1 \leq j \leq B, \tag{12}$$

A nucleon is identified as a participant ($N_{part} \longrightarrow N_{part} + 1$) if it satisfies the above criterion at least once. Collision site is taken to be the CoM of the two nucleons participating,

$$\mathbf{R}^{coll}_{ij} = \frac{A\,\mathbf{r}^{\perp}_{A,i} + B\,\mathbf{r}^{\perp}_{B,j}}{A + B} \tag{13}$$

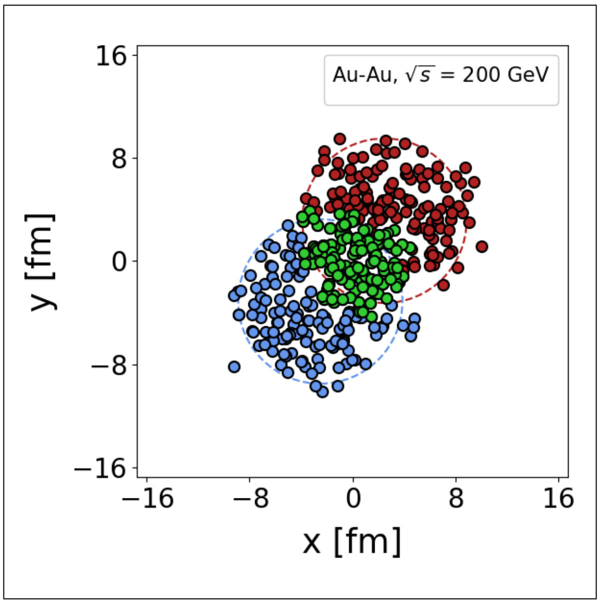



Figure 4: Beam-line view of a Monte Carlo event for $b = 8$ fm

Having obtained the nucleon positions, the thickness function can be defined assuming certain geometries [3],

$$T\left(\mathbf{r}_{\perp}\right) \sim A \times \begin{cases} \delta\left(\mathbf{r}_{\perp}\right) & \text{Point-like} \\ \Theta\left(r_{\perp} - D\right) & \text{Disk-like} \\ \exp\left(-\frac{r_{\perp}^2}{2\sigma^2}\right) & \text{Gaussian} \end{cases} \tag{14}$$

Where $[A] = L^{-2}$. The point-like assumption is trivial in this case. Considering each source to be independent, the areal densities $N_{part}(\mathbf{r}_{\perp})$ and $N_{coll}(\mathbf{r}_{\perp})$ can be defined in terms of the thickness function,

$$N_{part}\left(\mathbf{r}_{\perp}\right) = \sum_{i=1}^{N_{part}} w_i\, T\left(\mathbf{r}_{\perp} - \mathbf{R}^{part}_i\right)$$
$$N_{coll}\left(\mathbf{r}_{\perp}\right) = \sum_{i=1}^{N_{coll}} w_i\, T\left(\mathbf{r}_{\perp} - \mathbf{R}^{coll}_i\right) \tag{15}$$

Where $w_i$'s are weight factors either sampled from Gamma distribution [3] or taken to be constant for simplicity.

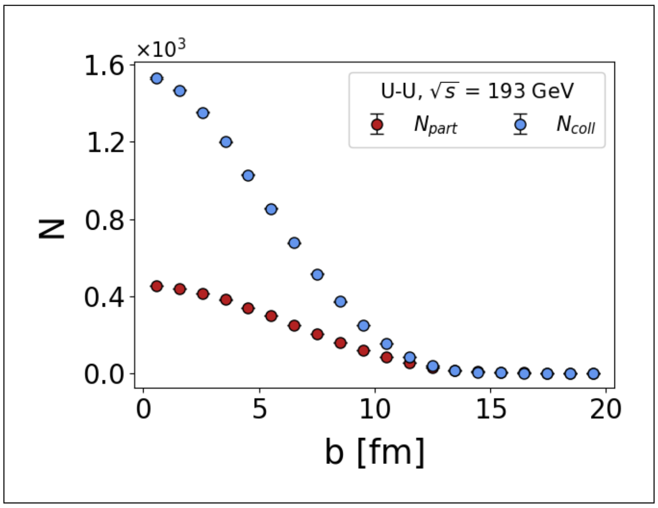



Figure 5: Impact parameter dependence of $N_{part}$ and $N_{coll}$ in Monte Carlo Glauber

# 4 Model Inputs

## 4.1 Nucleon Density

The nucleon density profile is usually modelled after the nuclear charge density or the nuclear wave-function. Parametric form of $\rho(\mathbf{r})$ for a wide range of nucleus are given in [4]. For example, a general form valid for both spherical (e.g. Au) and deformed (e.g. U) nucleus can be set up by incorporating deformation into the 3-parameter $(R,\ a,\ w)$ Fermi model,

$$\rho(\mathbf{r}) = \rho_0 \frac{1 + w\left(\frac{r}{R}\right)^2}{1 + \exp\left[\frac{r - R_d(\theta,\phi)}{a}\right]} \tag{16}$$

Where $R$ is average nuclear *charge radius*, $a$ is *skindepth* and $w$ denotes deviation from spherical shape. Deformed charge radius $R_d(\theta,\ \phi)$, has the general form,

$$R_d(\theta,\ \phi) = R\left(1 + \sum_{l=1}^{\infty}\sum_{m=-l}^{l} \beta_{lm}\, Y_l^m(\theta,\ \phi)\right) \tag{17}$$

Where $\beta_{lm}$ are deformation parameters, $Y_l^m(\theta,\ \phi)$ are the well known spherical harmonics. In case of axial symmetry (e.g. U), $R_d$ is independent of $\phi$ i.e. $m = 0$.

## 4.2 Impact Parameter

Given an impact parameter $b$, collision cross-section in the hard sphere limit is just $\sigma = \pi b^2$. As $\sigma$ is a measure for the probability of collision, $b$ is needed to be sampled from a linear probability distribution,

$$\frac{dN}{db} \equiv \frac{d\sigma}{db} \propto b \tag{18}$$

An impact vector with arbitrary orientation in the transverse plane is then constructed by sampling the azimuth ($\phi$) from a uniform probability distribution,

$$\mathbf{b} = b\,(\cos\phi,\ \sin\phi, 0) \qquad \text{where} \qquad \phi \in [0,\ 2\pi] \tag{19}$$

### 4.3 Scattering Cross-Section

$\sigma_{NN}$ depends on the CoM energy ($\sqrt{s}$) of a 2-nucleon system. In practice, experimentally measured $\sigma_{NN}$ at specific $\sqrt{s}$ values are used. The following parametric relation [2] is useful in this context,

$$\sigma_{NN} = A + B\ \ln^n\left(\sqrt{s}\right) \tag{20}$$

### 4.4 Nucleus Orientation

$N_{part}$ and $N_{coll}$ depend on the size of the nuclear overlap region which itself will depend on the orientation of the colliding nuclei if deformed ones are present. At each event, both the nuclei are oriented randomly under *Euler* rotation : $\mathbf{r} \longrightarrow \mathbf{r}' = R_y(\theta)\, R_z(\phi)\, \mathbf{r}$ which implies,

- In Optical Glauber, evaluate $\rho(\mathbf{r}')$ for given $\mathbf{r}$.
- In Monte Carlo Glauber, rotate each nucleon after shifting their CoM to the origin.

Two sets of $\theta$ and $\phi$ must be sampled from sine and uniform probability distributions respectively. Specific orientations (e.g. *body* and *tip*) and collision configurations can be identified based on these angles [5].

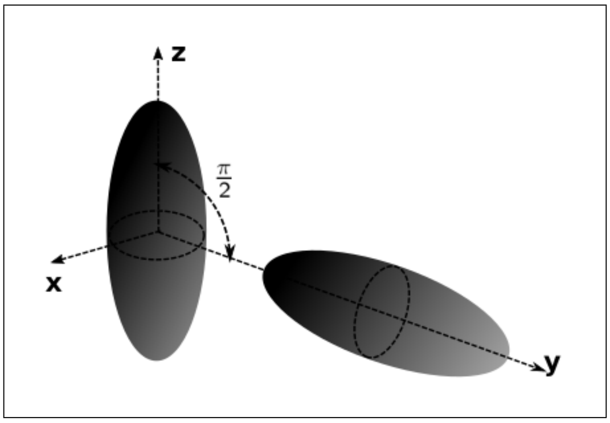



Figure 6: Body-Tip collision configuration

## 5 Collision Centrality

$N_{part}$, $N_{coll}$ as well as $b$ can not be measured in collider experiments. Since more binary collisions occur for smaller values of $b$ (fig. 3 and 5), these *central* events are expected to produce more charged particles. In practice, produced charged particle *multiplicity* at at mid-rapidity $\left(N_{ch} \equiv \frac{dN_{ch}}{d\eta}\Big|_{\eta\approx 0}\right)$ is used to determine *centrality* classes [1],

I. The distribution $\frac{dN}{dN_{ch}}$ has boundaries $\left[N_{ch}^{\text{min}},\, N_{ch}^{\text{max}}\right]$. Hence, total number of charged particles is $N^{\text{total}} = \int_{N_{ch}^{\text{min}}}^{N_{ch}^{\text{max}}} \frac{dN}{dN_{ch}}\, dN_{ch}$.

II. Centrality classes of this distribution are like percentile divisions e.g. boundaries of the $P - Q\%$ central class, $\left[N_{ch}^{\text{Q}},\, N_{ch}^{\text{P}}\right]$ is defined to be,

$$\frac{1}{N^{\text{total}}} \int_{N_{ch}^{\text{Q}}}^{N_{ch}^{\text{max}}} \frac{dN}{dN_{ch}}\, dN_{ch} \equiv Q\% \quad \text{and} \quad \frac{1}{N^{\text{total}}} \int_{N_{ch}^{\text{P}}}^{N_{ch}^{\text{max}}} \frac{dN}{dN_{ch}}\, dN_{ch} \equiv P\% \tag{21}$$

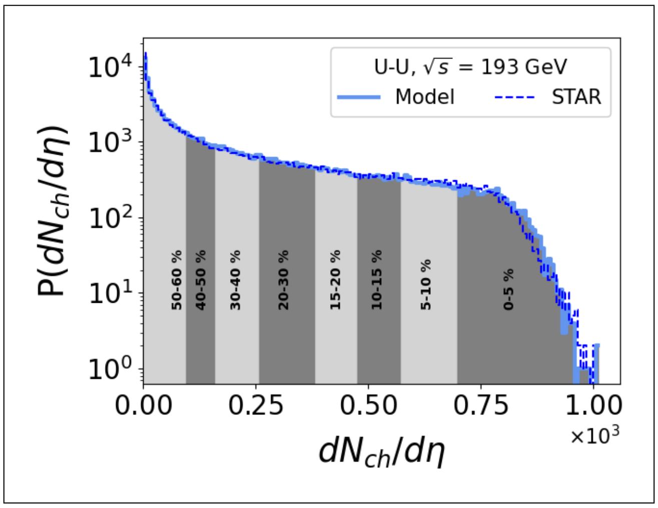



Figure 7: Centrality Division : estimate of charged particle multiplicity distribution ($n_{pp} = 2.1$, $f = 0.135$) compared against experimental data [6]

### 5.1 2-Component Formula

The *2-Component* formula is a phenomenological expression that relates the Glauber model outputs $N_{part}$ and $N_{coll}$ to the distribution $\frac{dN}{dN_{ch}}$ measured experimentally. For nucleus-nucleus collision, the expression is given by

$$N_{ch} = \frac{1-f}{2} \sum_{i=1}^{N_{part}} w_i + f \sum_{i=1}^{N_{coll}} w_i \tag{22}$$

Where $f$ is referred as the *hardness* factor and $w_i$'s are the source weight factors that are either taken equal to $n_{pp}$ i.e. average charged particle multiplicity of p-p collision or sampled from Negative Binomial distribution with mean $n_{pp}$ [7]. $N_{ch}$ is essentially a measure of total entropy deposited on transverse plane,

$$\begin{aligned} S \equiv s_0\, N_{ch} &= s_0 \int N_{ch}(\mathbf{r}_\perp)\, d^2\mathbf{r}_\perp \\ \Longrightarrow s(\mathbf{r}_\perp) &= s_0 \left[\frac{1-f}{2} N_{part}(\mathbf{r}_\perp) + f\, N_{coll}(\mathbf{r}_\perp)\right] \end{aligned} \tag{23}$$

$s(\mathbf{r}_\perp)$ is the transverse entropy density and $N_{part}(\mathbf{r}_\perp)$, $N_{coll}(\mathbf{r}_\perp)$ follows from eqn. 15. CoM of this density distribution is

$$\mathbf{R}_{CM} = \frac{\int \mathbf{r}_\perp \, s\left(\mathbf{r}_\perp\right) \, d^2\mathbf{r}_\perp}{\int \, s\left(\mathbf{r}_\perp\right) \, d^2\mathbf{r}_\perp} \tag{24}$$

For point-like assumption i.e. $N(\mathbf{r}_\perp) \sim \sum_i w_i \, \delta(\mathbf{r}_\perp - \mathbf{R}_i)$, the above integral reduces to summation. For any calculation involving $s(\mathbf{r}_\perp)$, the origin must be brought to $\mathbf{R}_{CM}$ i.e. apply the coordinate transformation : $\mathbf{r}_\perp \longrightarrow \mathbf{r}_\perp - \mathbf{R}_{CM}$.

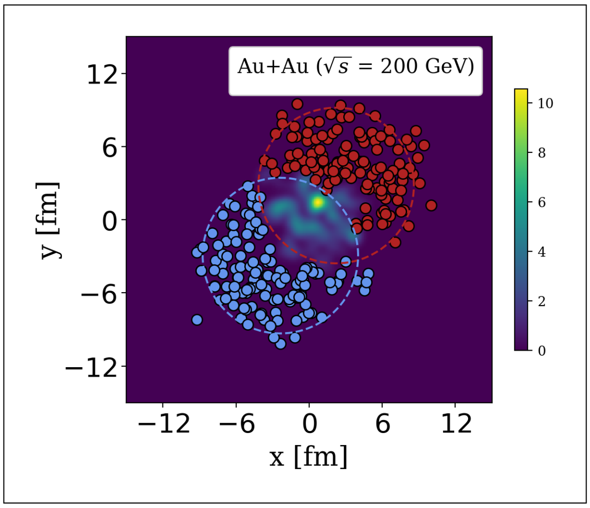



Figure 8: Transverse entropy density of a Monte Carlo event at $b = 8$ fm

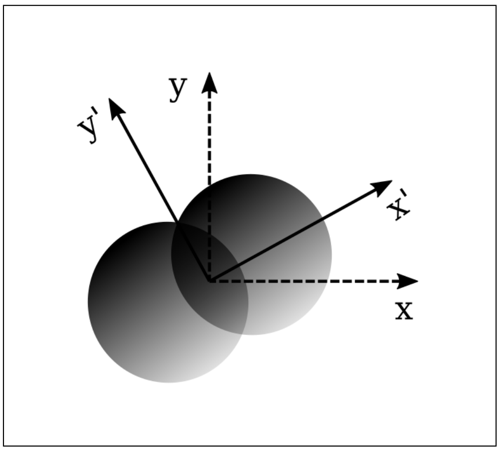



Figure 9: Tilted Reaction Plane : **x-y** are lab frame axes whereas **x'-y'** are principal axes of inertia of the nuclear overlap region

# 6 Collision Geometry

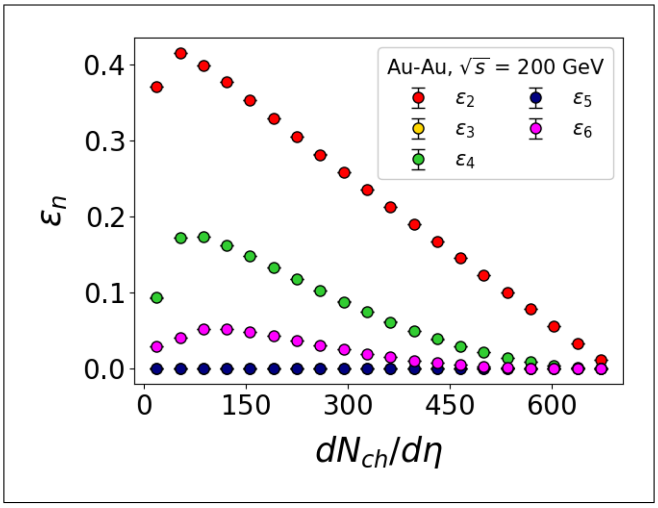
Spatial anisotropy within the nuclear overlap region results anisotropic pressure gradients that facilitate hydrodynamic expansion of strongly interacting nuclear matter. Spatial anisotropy of the initial state imprints an azimuthal anisotropy in the momentum distribution of final state particles that is measured in collider experiments [8]. Spatial anisotropy is characterized by moments referred as *eccentricity* ($\varepsilon_n$) [3],

$$\mathrm{I}_n^x = \int r_\perp^n \, s\left(\mathbf{r}_\perp\right) \, \cos n\,\phi \; d^2\mathbf{r}_\perp \quad \text{and} \quad \mathrm{I}_n^y = \int r_\perp^n \, s\left(\mathbf{r}_\perp\right) \, \sin n\,\phi \; d^2\mathbf{r}_\perp \tag{25}$$

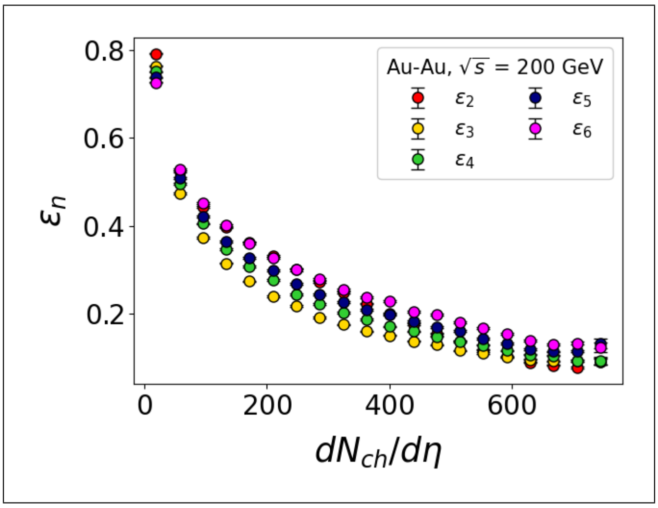
$$\varepsilon_n \, e^{i\,\Phi_n^{\mathrm{PP}}} = -\frac{\mathrm{I}_n^x + i\,\mathrm{I}_n^y}{\int r_\perp^n \, s\left(\mathbf{r}_\perp\right) \, d^2\mathbf{r}_\perp} \tag{26}$$

$$\Phi_n^{\mathrm{PP}} = \frac{1}{n} \tan^{-1}\left(\frac{\mathrm{I}_n^y}{\mathrm{I}_n^x}\right) \in \left[-\frac{\pi}{n}, \frac{\pi}{n}\right] \tag{27}$$

$\Phi_n^{\mathrm{PP}}$ is known as the $n^{\mathrm{th}}$ order *participant plane* angle. Non-vanishing $\Phi^{\mathrm{PP}}$ denotes tilt of the nuclear overlap region in the transverse plane (lab frame). The coefficients $\varepsilon_n$ are said to be associated with certain geometric deformations e.g. $\varepsilon_2$ denotes elliptic deformation, $\varepsilon_3$ denotes triangular deformation and so on. Shape of the overlap region is quite regular (almond shaped) in Optical Glauber but fluctuates from event to event in Monte Carlo Glauber. This is also evident in odd harmonics ($\varepsilon_3$, $\varepsilon_5$, ...) that vanish for spherical nucleus (e.g. Au, Pb) in Optical Glauber but are non-zero in Monte Carlo Glauber.




(a)




(b)

Figure 10: Multiplicity dependence of $\varepsilon_n$ in (a) Optical and (b) Monte Carlo Glauber

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