Home Work

Using the Laplace transformation find the solution of the differential equation.

$Y'' + 2Y' + 5Y = e^{-x} \sin x$; Where $Y(0) = 0$ and $Y'(0) = 1$

Solution:

The given differential equation is

$$Y'' + 2Y' + 5Y = e^{-x} \sin x \quad \text{..................................(1)}$$

Taking the Laplace transform of both sides of (1), we get

$$\mathcal{L}\{Y''\} + 2\mathcal{L}\{Y'\} + 5\mathcal{L}\{Y\} = \mathcal{L}\{e^{-x} \sin x\}$$

$$=> \ s^2Y(s) - sy(0) - Y'(0) + 2\{s\,Y(s) - y(0)\} + 5Y(s) = \frac{1}{(s+1)^2 + 1^2}$$

$$=> s^2Y(s) - 0 - 1 + 2\,\{s\;Y\;(s) - 0\} + 5Y(s) = \frac{1}{(s+1)^2 + 1^2}$$

$$=> s^2Y(s) - 1 + 2s\,Y(s) + 5Y(s) = \frac{1}{(s+1)^2 + 1^2}$$

$$=> (s^2 + \ 2s + 5)Y(s) = \frac{1}{(s+1)^2 + 1^2} + 1$$

$$=> Y(s) = \frac{s^2 + 2s + 3}{(s^2 + 2s + 2)(s^2 + 2s + 5)}$$

$$=> Y(s) = \frac{s^2 + 2s + 2}{(s^2 + 2s + 2)(s^2 + 2s + 5)} + \frac{1}{(s^2 + 2s + 2)(s^2 + 2s + 5)}$$

$$=> Y(s) = \frac{1}{(s^2 + 2s + 5)} + \frac{1}{(s^2 + 2s + 2)(s^2 + 2s + 5)} \quad \text{........................ (2)}$$

Taking the inverse Laplace transform of both sides of (2), we get

$$\mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{(s^2+2s+5)}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{(s^2+2s+2)(s^2+2s+5)}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2+2^2}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{(s^2+2s+2)(s^2+2s+5)}\right\}$$

$$= \frac{1}{2}e^{-x}\sin 2x \ + \mathcal{L}^{-1}\left\{\frac{1}{(s^2+2s+2)(s^2+2s+5)}\right\} \quad \text{.....................} (3)$$

Now

$$\frac{1}{(s^2+2s+2)(s^2+2s+5)} = \frac{\frac{1}{3}[(s^2+2s+5)-(s^2+2s+2)]}{(s^2+2s+2)(s^2+2s+5)}$$

$$= \frac{1}{3}.\frac{1}{(s^2+2s+2)} - \frac{1}{3}.\frac{1}{(s^2+2s+5)}$$

$$= \frac{1}{3}.\frac{1}{(s+1)^2+1^2} - \frac{1}{3}.\frac{1}{(s+1)^2+2^2}$$

$$\therefore \ \mathcal{L}^{-1}\{\frac{1}{(s^2+2s+2)(s^2+2s+5)}\} = \frac{1}{3}\mathcal{L}^{-1}\{\frac{1}{(s+1)^2+1^2}\} - \frac{1}{3}\mathcal{L}^{-1}\{\frac{1}{(s+1)^2+2^2}\}$$

$$= \tfrac{1}{3}e^{-x}\sin x - \tfrac{1}{3}.\tfrac{1}{2}e^{-x}\sin 2x$$

Putting this value in (3), we get

$$y = \frac{1}{2}e^{-x}\sin 2x + \frac{1}{3}e^{-x}\sin x - \frac{1}{3}.\frac{1}{2}e^{-x}\sin 2x$$

$$= \frac{1}{3}e^{-x}\sin x + \frac{1}{3}e^{-x}\sin 2x$$

$$= \frac{1}{3}e^{-x}(\sin x + \sin 2x)$$

which is the required solution.

Formula: $\mathcal{L}\{t^n \cos at\} = (-1)^n \frac{d^n}{ds^n}\mathcal{L}\{\cos at\}$

Some Special Function:

1. **The unit impulse Function (or Dirac delta function)**

The unit impulse function is defined by

$$F_t(t) = \begin{cases} \frac{1}{\epsilon}, & 0 \le t \le \epsilon \\ 0, & t > \epsilon \end{cases}$$

Problem: Find the Laplace transform of unit impulse function.

Solution:

The unit impulse function is defined by

$$F_t(t) = \begin{cases} \frac{1}{\in}, & 0 \leq t \leq \in \\ 0, & t > \in \end{cases}$$

We know that

$$\mathcal{L}\{F(t)\} = \int_0^{\propto} e^{-st} F(t)\, dt$$

$$\therefore\ \mathcal{L}\{F_{\in}(t)\} = \int_0^{\propto} e^{-st} F_{\in}(t)\, dt$$

$$= \int_0^{\in} e^{-st} F_{\in}(t)\, dt + \int_{\in}^{\propto} e^{-st} F_{\in}(t)\, dt$$

$$= \int_0^{\in} e^{-st}.\frac{1}{\in} dt + 0$$

$$= \frac{1}{\in}\int_0^{\in} e^{-st}.\, dt$$

$$= \frac{1}{\in}[\frac{e^{-st}}{-s}]_0^{\in}$$

$$= \frac{1}{\in}\left[-\frac{e^{-\in s}}{s} + \frac{1}{s}\right]$$

$$= -\frac{e^{-\in s}}{\in s} + \frac{1}{\in s}$$

$$= \frac{1 - e^{-\in s}}{\in s}$$

$(\boldsymbol{Ans.})$

2. The Bessel Function

The Bessel Function of order n is defined by

$$J_n(t) = \frac{t^n}{2^n\Gamma(n+1)}\left[1 - \frac{t^2}{2.(2n+2)} + \frac{t^4}{2.4(2n+2)(2n+4)} - \cdots\right]$$
$$= \sum_{r=0}^{\propto} \frac{(-1)^r}{r!\ \Gamma(n+r+1)} (\frac{t}{2})^{n+2r}$$

Problem: Find the Laplace transform of Bessel Function

Prove that

$$\mathcal{L}\{J_0(t)\} = \frac{1}{\sqrt{s^2+1}}$$

Solution: By definition of Bessel Function of order n, we have

$$J_n(t) = \frac{t^n}{2^n\Gamma(n+1)}\left[1 - \frac{t^2}{2.(2n+2)} + \frac{t^4}{2.4(2n+2)(2n+4)} - \cdots\right] \text{ .... (1)}$$

Putting $n = 0$ in (1), we get

$$J_0(t) = \left[1 - \frac{t^2}{2^2} + \frac{t^4}{2^2.4^2} - \frac{t^6}{2^2.4^2.6^2} + \cdots\right] \quad \text{.................. (2)}$$

Now we taking Laplace transform in both side of eqn. (2), we get

$$\mathcal{L}\{J_0(t)\} = \mathcal{L}\{1\} - \frac{1}{2^2}\mathcal{L}\{t^2\} + \frac{1}{2^2.4^2}\mathcal{L}\{t^4\} - \frac{1}{2^2.4^2.6^2}\mathcal{L}\{t^6\} + \cdots$$
$$= \frac{1}{s} - \frac{1}{2^2}.\frac{2!}{s^3} + \frac{1}{2^2.4^2}.\frac{4!}{s^5} - \frac{1}{2^2.4^2.6^2}\ \frac{6!}{s^7} + \ ....$$
$$= \frac{1}{s}[1 - \frac{1}{2}\left(\frac{1}{s^2}\right) + \frac{1.3}{2.4}\left(\frac{1}{s^4}\right) - \frac{1.3.5}{2.4.6}\left(\frac{1}{s^6}\right) + \ ...]$$
$$= \frac{1}{s}\left(1 + \frac{1}{s^2}\right)^{-\frac{1}{2}}$$

$$= \frac{1}{s}\left(\frac{s^2+1}{s^2}\right)^{-\frac{1}{2}}$$

$$= \frac{1}{s}.\frac{s}{(s^2+1)^{\frac{1}{2}}}$$

$$= \frac{1}{\sqrt{s^2+1}}$$

- $(1+x)^{-n} = 1 - nx + \frac{n(n-1)}{2}x^2$
- $(1+x)^n = 1 + n_{C_1}x + n_{C_2}x^2 + \cdots x^n$

Hence,

$$\mathcal{L}\{J_0(t)\} = \frac{1}{\sqrt{s^2+1}}$$

$(\boldsymbol{Proved})$