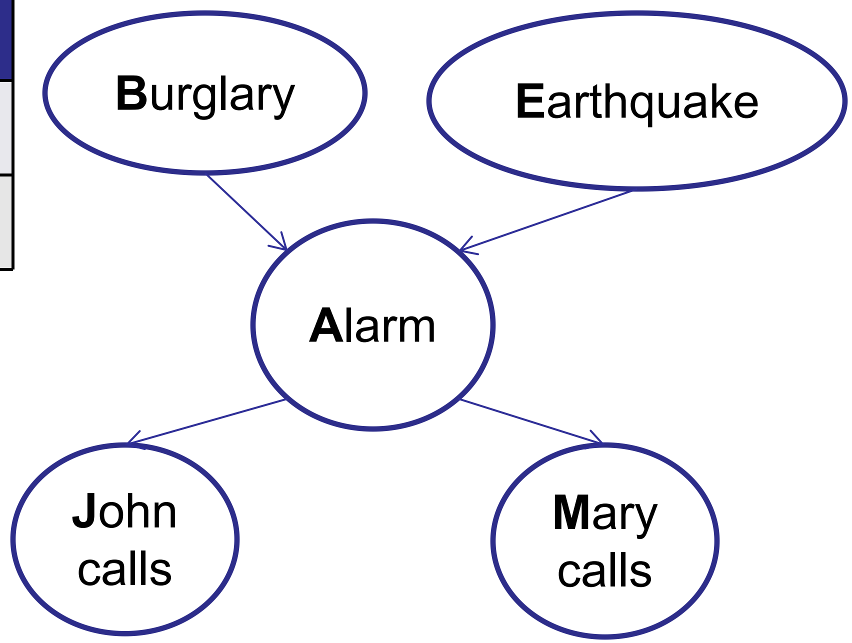
# CS 349: Artificial Intelligence

Probabilistic Inference

# Example: Alarm Network

| B | P(B) |
|---|---|
| +b | 0.001 |
| ¬b | 0.999 |

| E | P(E) |
|---|---|
| +e | 0.002 |
| ¬e | 0.998 |

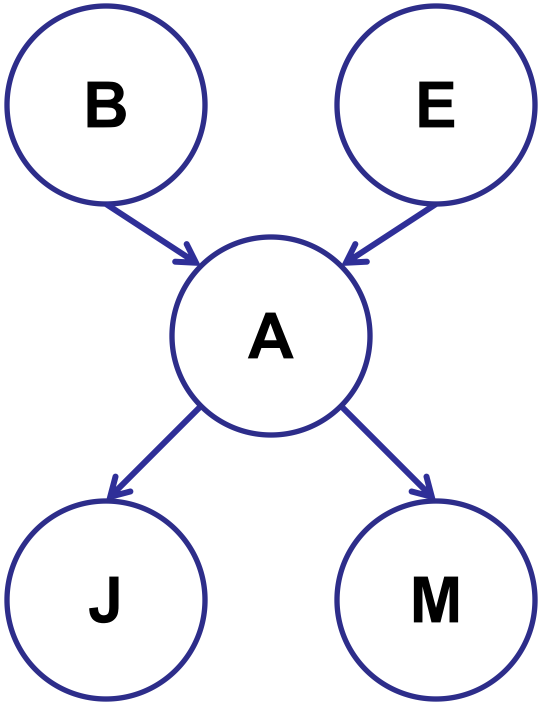


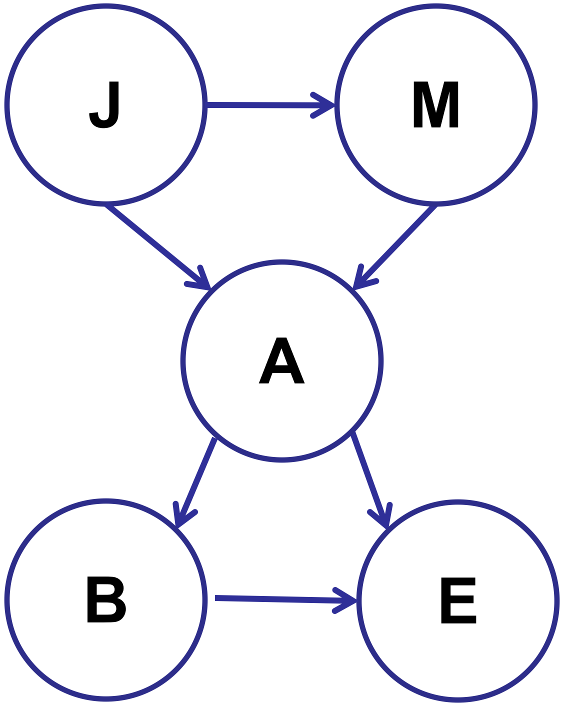

| B | E | A | P(A\|B,E) |
|---|---|---|---|
| +b | +e | +a | 0.95 |
| +b | +e | ¬a | 0.05 |
| +b | ¬e | +a | 0.94 |
| +b | ¬e | ¬a | 0.06 |
| ¬b | +e | +a | 0.29 |
| ¬b | +e | ¬a | 0.71 |
| ¬b | ¬e | +a | 0.001 |
| ¬b | ¬e | ¬a | 0.999 |

| A | J | P(J\|A) |
|---|---|---|
| +a | +j | 0.9 |
| +a | ¬j | 0.1 |
| ¬a | +j | 0.05 |
| ¬a | ¬j | 0.95 |

| A | M | P(M\|A) |
|---|---|---|
| +a | +m | 0.7 |
| +a | ¬m | 0.3 |
| ¬a | +m | 0.01 |
| ¬a | ¬m | 0.99 |

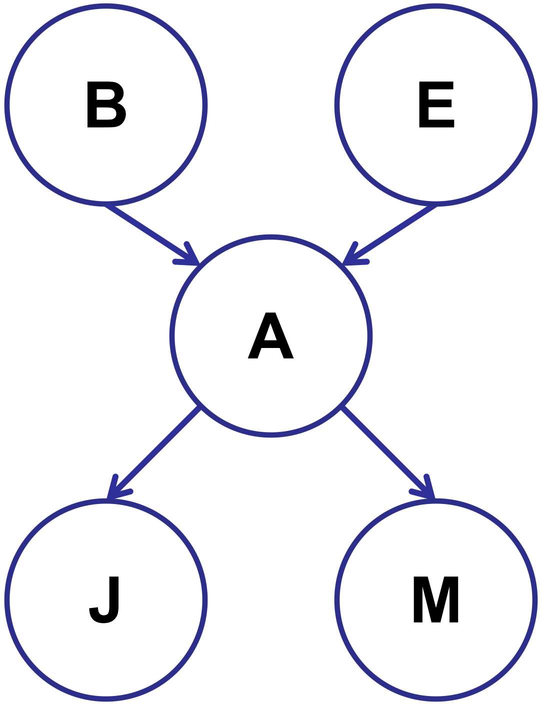
# Causation and Correlation



# Probabilistic Inference

- Probabilistic Inference: calculating some quantity from a joint probability distribution
  - Posterior probability:
    $$P(Q|E_1 = e_1, \ldots E_k = e_k)$$
  - Most likely explanation:
    $$\mathsf{argmax}_q\ P(Q = q|E_1 = e_1 \ldots)$$
  - In general, partition variables into *Query (Q or X)*, *Evidence (E)*, and *Hidden (H or Y)* variables

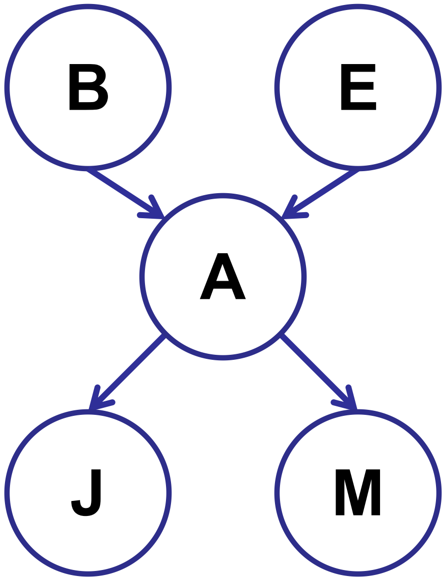


# Inference by Enumeration

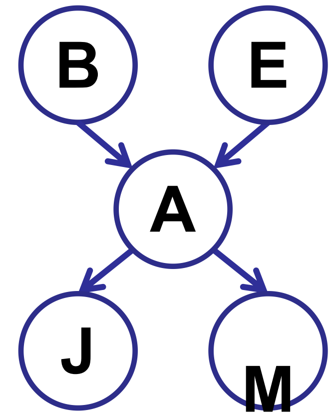
- Given unlimited time, inference in BNs is easy
- Recipe:
  - State the unconditional probabilities you need
  - Enumerate *all* the atomic probabilities you need
  - Calculate sum of products
- Example:

$$P(+b| + j, +m) =$$

$$\frac{P(+b, +j, +m)}{P(+j, +m)}$$



# Inference by Enumeration

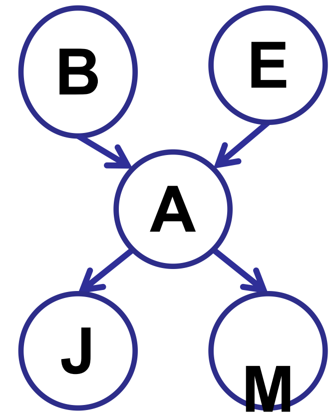



$P(+b, +j, +m)$
$= \sum_e \sum_a P(+b, +j, +m, e, a)$
$= \sum_e \sum_a P(+b)\ P(e)\ P(a|+b,e)\ P(+j|a)\ P(+m|a)$

$$= \quad P(+b)P(+e)P(+a|+b,+e)P(+j|+a)P(+m|+a)+$$
$$P(+b)P(+e)P(-a|+b,+e)P(+j|-a)P(+m|-a)+$$
$$P(+b)P(-e)P(+a|+b,-e)P(+j|+a)P(+m|+a)+$$
$$P(+b)P(-e)P(-a|+b,-e)P(+j|-a)P(+m|-a)$$

# Inference by Enumeration

- An optimization




$$P(+b, +j, +m)$$

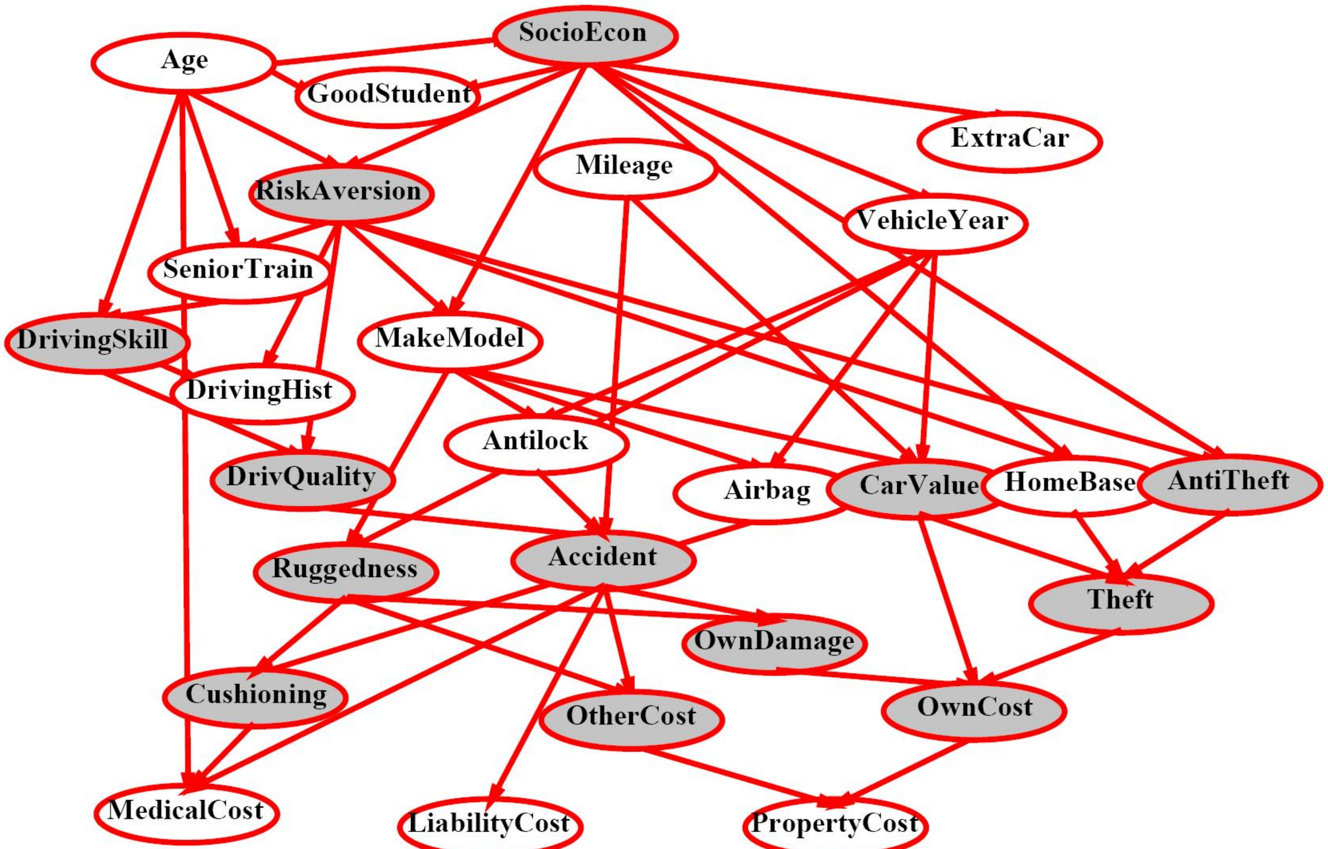
$$= \sum_e \sum_a P(+b, +j, +m, e, a)$$

$$= \sum_e \sum_a P(+b)\, P(e)\, P(a|+b,e)\, P(+j|a)\, P(+m|a)$$

$$= P(+b) \sum_e P(e) \sum_a P(a|+b,e)\, P(+j|a)\, P(+m|a)$$ **or**

$$= P(+b) \sum_a P(+j|a)\, P(+m|a) \sum_e P(e)\, P(a|+b,e)$$

# Inference by Enumeration




Problem?

Not just 4 rows; approximately $10^{16}$ rows!

# Variable Elimination

- Why is inference by enumeration so slow?
  - You join up the whole joint distribution before you sum out the hidden variables
    ( $\sum_e \sum_a P(+b)\, P(e)\, P(a|+b,e)\, P(+j|a)\, P(+m|a)$ )
  - You end up repeating a lot of work!

- Idea: interleave joining and marginalizing!
  - Called “Variable Elimination”
  - Still NP-hard, but usually much faster than inference by enumeration
  - Requires an algebra for combining “factors”

# Factor Zoo I

- Joint distribution: P(X,Y)
  - Entries P(x,y) for all x, y
  - Sums to 1

$$P(T, W)$$

| T | W | P |
|---|---|---|
| hot | sun | 0.4 |
| hot | rain | 0.1 |
| cold | sun | 0.2 |
| cold | rain | 0.3 |

- Selected joint: P(x,Y)
  - A slice of the joint distribution
  - Entries P(x,y) for fixed x, all y
  - Sums to P(x)

$$P(cold, W)$$

| T | W | P |
|---|---|---|
| cold | sun | 0.2 |
| cold | rain | 0.3 |

# Factor Zoo II

- Family of conditionals: P(X |Y)
  - Multiple conditionals
  - Entries P(x | y) for all x, y
  - Sums to |Y|

$P(W|T)$

| T | W | P |
|---|---|---|
| hot | sun | 0.8 |
| hot | rain | 0.2 |
| cold | sun | 0.4 |
| cold | rain | 0.6 |

$P(W|hot)$

$P(W|cold)$

- Single conditional: P(Y | x)
  - Entries P(y | x) for fixed x, all y
  - Sums to 1

$P(W|cold)$

| T | W | P |
|---|---|---|
| cold | sun | 0.4 |
| cold | rain | 0.6 |

# Factor Zoo III

- Specified family: P(y | X)
  - Entries P(y | x) for fixed y, but for all x
  - Sums to … who knows!

$P(rain|T)$

| T | W | P |
|---|---|---|
| hot | rain | 0.2 |
| cold | rain | 0.6 |

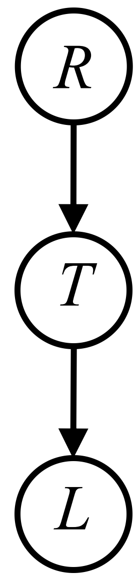
$\} P(rain|hot)$

$\} P(rain|cold)$

- In general, when we write $P(Y_1 \ldots Y_N \mid X_1 \ldots X_M)$
  - It is a "factor," a multi-dimensional array
  - Its values are all $P(y_1 \ldots y_N \mid x_1 \ldots x_M)$
  - Any assigned X or Y is a dimension missing (selected) from the array

# Example: Traffic Domain

- Random Variables
  - R: Raining
  - T: Traffic
  - L: Late for class!




$P(R)$

| | |
|---|---|
| +r | 0.1 |
| -r | 0.9 |

$P(T|R)$

| | | |
|---|---|---|
| +r | +t | 0.8 |
| +r | -t | 0.2 |
| -r | +t | 0.1 |
| -r | -t | 0.9 |

$P(L|R)$

| | | |
|---|---|---|
| +t | +l | 0.3 |
| +t | -l | 0.7 |
| -t | +l | 0.1 |
| -t | -l | 0.9 |

# Variable Elimination Outline

- Track objects called factors
- Initial factors are local CPTs (one per node)

$P(R)$

| | |
|---|---|
| +r | 0.1 |
| -r | 0.9 |

$P(T|R)$

| | | |
|---|---|---|
| +r | +t | 0.8 |
| +r | -t | 0.2 |
| -r | +t | 0.1 |
| -r | -t | 0.9 |

$P(L|T)$

| | | |
|---|---|---|
| +t | +l | 0.3 |
| +t | -l | 0.7 |
| -t | +l | 0.1 |
| -t | -l | 0.9 |

- Any known values are selected
  - E.g. if we know $L = +\ell$ , the initial factors are

$P(R)$

| | |
|---|---|
| +r | 0.1 |
| -r | 0.9 |

$P(T|R)$

| | | |
|---|---|---|
| +r | +t | 0.8 |
| +r | -t | 0.2 |
| -r | +t | 0.1 |
| -r | -t | 0.9 |

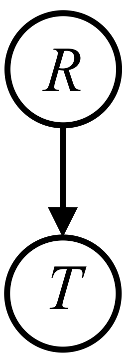
$P(+\ell|T)$

| | | |
|---|---|---|
| +t | +l | 0.3 |
| -t | +l | 0.1 |

- VE: Alternately join factors and eliminate variables

# Operation 1: Join Factors

- Combining factors:
  - Just like a database join
  - Get all factors that mention the joining variable
  - Build a new factor over the union of the variables involved
- Example: Join on R




$$P(R) \quad \times \quad P(T|R) \quad \longrightarrow \quad P(R,T)$$

$P(R)$

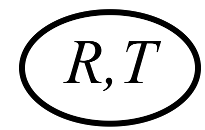
| | |
|---|---|
| +r | 0.1 |
| -r | 0.9 |

$P(T|R)$

| | | |
|---|---|---|
| +r | +t | 0.8 |
| +r | -t | 0.2 |
| -r | +t | 0.1 |
| -r | -t | 0.9 |

$P(R,T)$

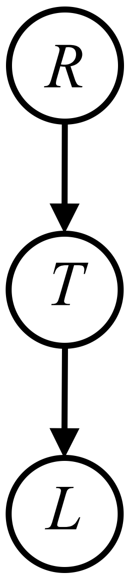
| | | |
|---|---|---|
| +r | +t | 0.08 |
| +r | -t | 0.02 |
| -r | +t | 0.09 |
| -r | -t | 0.81 |




- Computation for each entry: point wise products

$$\forall r,t: \qquad P(r,t) = P(r)\cdot P(t|r)$$

# Example: Multiple Joins




$P(R)$

| | |
|---|---|
| +r | 0.1 |
| -r | 0.9 |

$P(T|R)$

| | | |
|---|---|---|
| +r | +t | 0.8 |
| +r | -t | 0.2 |
| -r | +t | 0.1 |
| -r | -t | 0.9 |

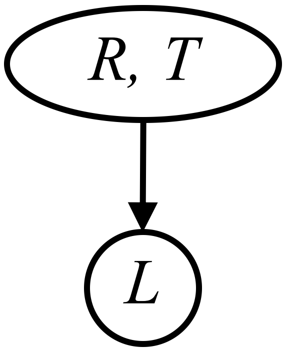
$P(L|T)$

| | | |
|---|---|---|
| +t | +l | 0.3 |
| +t | -l | 0.7 |
| -t | +l | 0.1 |
| -t | -l | 0.9 |

Join R

$P(R, T)$

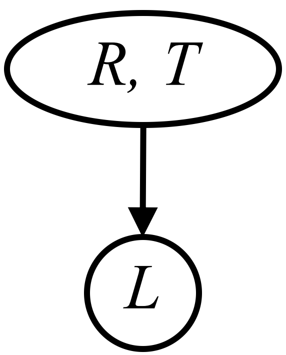
| | | |
|---|---|---|
| +r | +t | 0.08 |
| +r | -t | 0.02 |
| -r | +t | 0.09 |
| -r | -t | 0.81 |




$P(L|T)$

| | | |
|---|---|---|
| +t | +l | 0.3 |
| +t | -l | 0.7 |
| -t | +l | 0.1 |
| -t | -l | 0.9 |

# Example: Multiple Joins



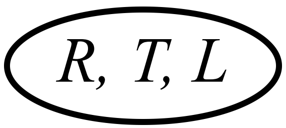
$P(R, T)$

| | | |
|---|---|---|
| +r | +t | 0.08 |
| +r | -t | 0.02 |
| -r | +t | 0.09 |
| -r | -t | 0.81 |

$P(L|T)$

| | | |
|---|---|---|
| +t | +l | 0.3 |
| +t | -l | 0.7 |
| -t | +l | 0.1 |
| -t | -l | 0.9 |

Join T



$P(R, T, L)$

| | | | |
|---|---|---|---|
| +r | +t | +l | 0.024 |
| +r | +t | -l | 0.056 |
| +r | -t | +l | 0.002 |
| +r | -t | -l | 0.018 |
| -r | +t | +l | 0.027 |
| -r | +t | -l | 0.063 |
| -r | -t | +l | 0.081 |
| -r | -t | -l | 0.729 |

# Operation 2: Eliminate

- Second basic operation: marginalization
- Take a factor and sum out a variable
  - Shrinks a factor to a smaller one
  - A projection operation
- Example:

$$P(R,T)$$

| | | |
|---|---|---|
| +r | +t | 0.08 |
| +r | -t | 0.02 |
| -r | +t | 0.09 |
| -r | -t | 0.81 |

sum $R$ $\rightarrow$

$$P(T)$$

| | |
|---|---|
| +t | 0.17 |
| -t | 0.83 |

# Multiple Elimination

$R, T, L$ $\rightarrow$ $T, L$ $\rightarrow$ $L$

$P(R, T, L)$

| | | | |
|---|---|---|---|
| +r | +t | +l | 0.024 |
| +r | +t | -l | 0.056 |
| +r | -t | +l | 0.002 |
| +r | -t | -l | 0.018 |
| -r | +t | +l | 0.027 |
| -r | +t | -l | 0.063 |
| -r | -t | +l | 0.081 |
| -r | -t | -l | 0.729 |

Sum out R

$P(T, L)$

| | | |
|---|---|---|
| +t | +l | 0.051 |
| +t | -l | 0.119 |
| -t | +l | 0.083 |
| -t | -l | 0.747 |

Sum out T

$P(L)$

| | |
|---|---|
| +l | 0.134 |
| -l | 0.886 |

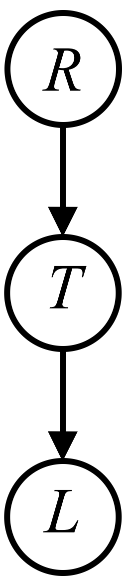
# P(L) : Marginalizing Early!

$P(R)$

| | |
|---|---|
| +r | 0.1 |
| -r | 0.9 |

$P(T|R)$

| | | |
|---|---|---|
| +r | +t | 0.8 |
| +r | -t | 0.2 |
| -r | +t | 0.1 |
| -r | -t | 0.9 |



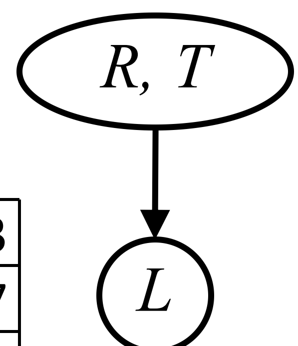

$P(L|T)$

| | | |
|---|---|---|
| +t | +l | 0.3 |
| +t | -l | 0.7 |
| -t | +l | 0.1 |
| -t | -l | 0.9 |

Join R

$P(R,T)$

| | | |
|---|---|---|
| +r | +t | 0.08 |
| +r | -t | 0.02 |
| -r | +t | 0.09 |
| -r | -t | 0.81 |



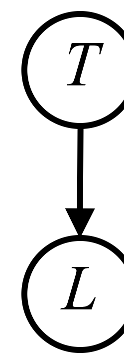

$P(L|T)$

| | | |
|---|---|---|
| +t | +l | 0.3 |
| +t | -l | 0.7 |
| -t | +l | 0.1 |
| -t | -l | 0.9 |

Sum out R

$P(T)$

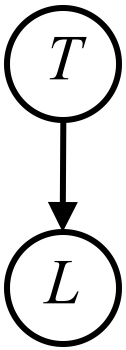
| | |
|---|---|
| +t | 0.17 |
| -t | 0.83 |



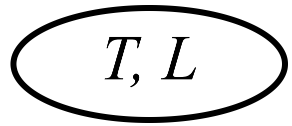

$P(L|T)$

| | | |
|---|---|---|
| +t | +l | 0.3 |
| +t | -l | 0.7 |
| -t | +l | 0.1 |
| -t | -l | 0.9 |

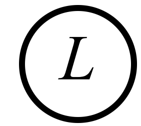
# Marginalizing Early




Join T




Sum out T




$P(T)$

| | |
|---|---|
| +t | 0.17 |
| -t | 0.83 |

$P(L|T)$

| | | |
|---|---|---|
| +t | +l | 0.3 |
| +t | -l | 0.7 |
| -t | +l | 0.1 |
| -t | -l | 0.9 |

$P(T, L)$

| | | |
|---|---|---|
| +t | +l | 0.051 |
| +t | -l | 0.119 |
| -t | +l | 0.083 |
| -t | -l | 0.747 |

$P(L)$

| | |
|---|---|
| +l | 0.134 |
| -l | 0.886 |

*Early marginalization is variable elimination*

# Evidence

- If evidence, start with factors that select that evidence
  - No evidence uses these initial factors:

$P(R)$

| | |
|---|---|
| +r | 0.1 |
| -r | 0.9 |

$P(T|R)$

| | | |
|---|---|---|
| +r | +t | 0.8 |
| +r | -t | 0.2 |
| -r | +t | 0.1 |
| -r | -t | 0.9 |

$P(L|T)$

| | | |
|---|---|---|
| +t | +l | 0.3 |
| +t | -l | 0.7 |
| -t | +l | 0.1 |
| -t | -l | 0.9 |

  - Computing $P(L|+r)$ , the initial factors become:

$P(+r)$

| | |
|---|---|
| +r | 0.1 |

$P(T|+r)$

| | | |
|---|---|---|
| +r | +t | 0.8 |
| +r | -t | 0.2 |

$P(L|T)$

| | | |
|---|---|---|
| +t | +l | 0.3 |
| +t | -l | 0.7 |
| -t | +l | 0.1 |
| -t | -l | 0.9 |

- We eliminate all vars other than query + evidence
- Compute P(+r, T)→P(+r, T)*P(L|T)→Sum on T

# Evidence II

- Result will be a selected joint of query and evidence
  - E.g. for P(L | +r), we'd end up with:

$P(+r, L)$

| | | |
|---|---|---|
| +r | +l | 0.026 |
| +r | -l | 0.074 |

Normalize

$P(L | +r)$

| | |
|---|---|
| +l | 0.26 |
| -l | 0.74 |

- To get our answer, just normalize this!

- That's it!

# General Variable Elimination

- Query: $P(Q|E_1 = e_1, \ldots E_k = e_k)$

- Start with initial factors:
  - Local CPTs (but instantiated by evidence)

- While there are still hidden variables (not Q or evidence):
  - Pick a hidden variable H
  - Join all factors mentioning H
  - Eliminate (sum out) H

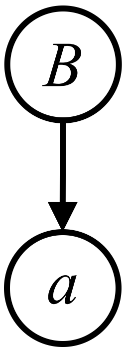
- Join all remaining factors and normalize

# Variable Elimination Bayes Rule

## Start / Select

$P(B)$

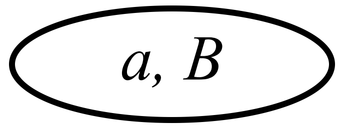
| B | P |
|---|---|
| +b | 0.1 |
| ¬b | 0.9 |




$P(A|B) \rightarrow P(a|B)$

| B | A | P |
|---|---|---|
| +b | +a | 0.8 |
| ~~b~~ | ~~¬a~~ | ~~0.2~~ |
| ¬b | +a | 0.1 |
| ~~¬b~~ | ~~¬a~~ | ~~0.9~~ |

## Join on B




$P(a, B)$

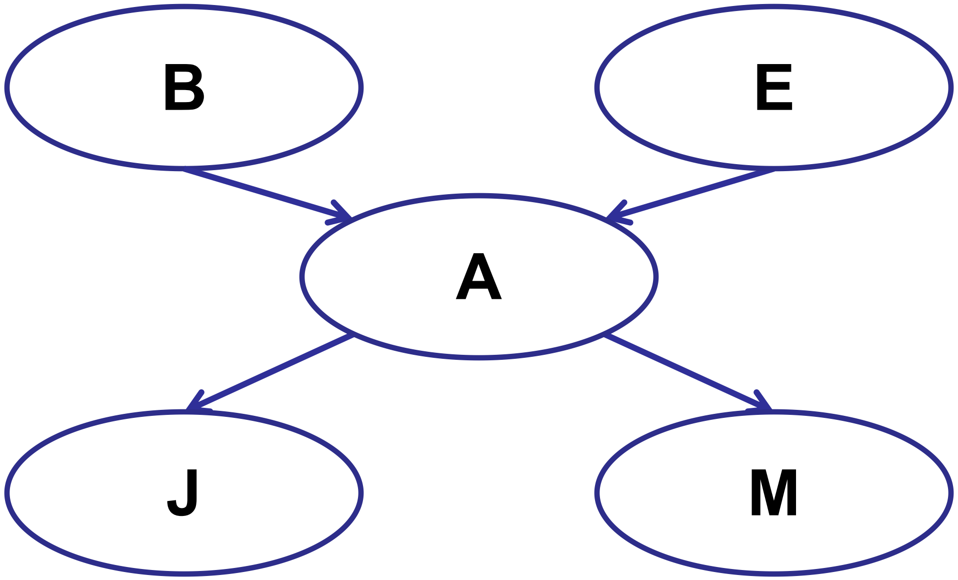
| A | B | P |
|---|---|---|
| +a | +b | 0.08 |
| +a | ¬b | 0.09 |

## Normalize

$P(B|a)$

| A | B | P |
|---|---|---|
| +a | +b | 8/17 |
| +a | ¬b | 9/17 |

# Bayes Network presentation



# Example

$$P(B|j,m) \propto P(B,j,m)$$

$$\boxed{P(B) \qquad P(E) \qquad P(A|B,E) \qquad P(j|A) \qquad P(m|A)}$$

Choose $A$

$$\left.\begin{matrix} P(A|B,E) \\ P(j|A) \\ P(m|A) \end{matrix}\right. \xrightarrow{\times} P(j,m,A|B,E) \xrightarrow{\sum} P(j,m|B,E)$$

$$\boxed{P(B) \qquad P(E) \qquad P(j,m|B,E)}$$

# Example

$$P(B) \qquad P(E) \qquad P(j, m|B, E)$$

Choose E

$$\left.\begin{matrix} P(E) \\ P(j, m|B, E) \end{matrix}\right. \xrightarrow{\times} P(j, m, E|B) \xrightarrow{\sum} P(j, m|B)$$

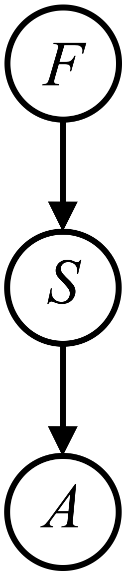
$$P(B) \qquad P(j, m|B)$$

Finish with B

$$\left.\begin{matrix} P(B) \\ P(j, m|B) \end{matrix}\right. \xrightarrow{\times} P(j, m, B) \xrightarrow{\text{Normalize}} P(B|j, m)$$

# Approximate Inference

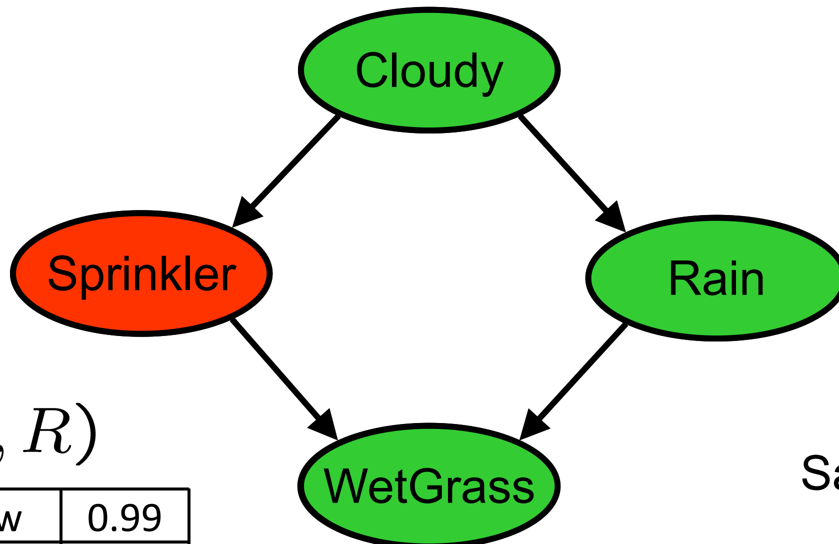
- Sampling / Simulating / Observing
- Sampling is a hot topic in machine learning, and it's really simple
- Basic idea:
  - Draw N samples from a sampling distribution S
  - Compute an approximate posterior probability
  - Show this converges to the true probability P
- Why sample?
  - Learning: get samples from a distribution you don't know
  - Inference: getting a sample is faster than computing the right answer (e.g. with variable elimination)



# Prior Sampling

$P(C)$

| | |
|---|---|
| +c | 0.5 |
| -c | 0.5 |




$P(S|C)$

| | | |
|---|---|---|
| +c | +s | 0.1 |
| | -s | 0.9 |
| -c | +s | 0.5 |
| | -s | 0.5 |

$P(R|C)$

| | | |
|---|---|---|
| +c | +r | 0.8 |
| | -r | 0.2 |
| -c | +r | 0.2 |
| | -r | 0.8 |

$P(W|S,R)$

| | | | |
|---|---|---|---|
| +s | +r | +w | 0.99 |
| | | -w | 0.01 |
| | -r | +w | 0.90 |
| | | -w | 0.10 |
| -s | +r | +w | 0.90 |
| | | -w | 0.10 |
| | -r | +w | 0.01 |
| | | -w | 0.99 |

Samples:

+c, -s, +r, +w

-c, +s, -r, +w

…

# Prior Sampling

- This process generates samples with probability:

$$S_{PS}(x_1 \ldots x_n) = \prod_{i=1}^{n} P(x_i|\text{Parents}(X_i)) = P(x_1 \ldots x_n)$$

…i.e. the BN's joint probability

- Let the number of samples of an event be $N_{PS}(x_1 \ldots x_n)$

- Then

$$\begin{aligned} \lim_{N\to\infty} \widehat{P}(x_1, \ldots, x_n) &= \lim_{N\to\infty} N_{PS}(x_1, \ldots, x_n)/N \\ &= S_{PS}(x_1, \ldots, x_n) \\ &= P(x_1 \ldots x_n) \end{aligned}$$

- i.e., the sampling procedure is consistent

# Example

- We’ll get a bunch of samples from the BN:

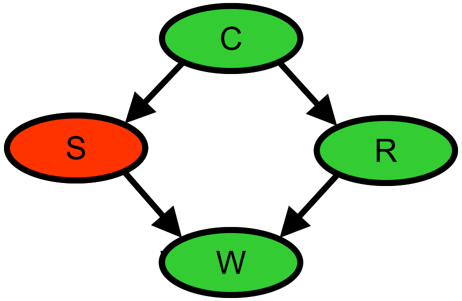
  +c, -s, +r, +w

  +c, +s, +r, +w

  -c, +s, +r, -w

  +c, -s, +r, +w

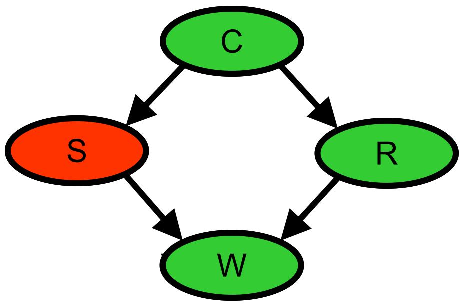
  -c, -s, -r, +w




- If we want to know P(W)
  - We have counts <+w:4, -w:1>
  - Normalize to get P(W) = <+w:0.8, -w:0.2>
  - This will get closer to the true distribution with more samples
  - Can estimate anything else, too
  - What about P(C| +w)? P(C| +r, +w)? P(C| -r, -w)?
  - Fast: can use fewer samples if less time (what’s the drawback?)

# Rejection Sampling

- Let's say we want P(C)
  - No point keeping all samples around
  - Just tally counts of C as we go

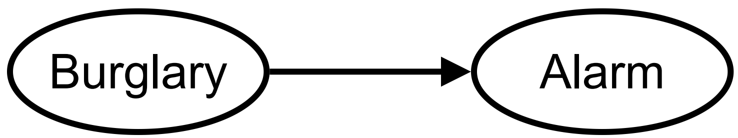



- Let's say we want P(C| +s)
  - Same thing: tally C outcomes, but ignore (reject) samples which don't have S=+s
  - This is called rejection sampling
  - It is also consistent for conditional probabilities (i.e., correct in the limit)

+c, -s, +r, +w
+c, +s, +r, +w
-c, +s, +r, -w
+c, -s, +r, +w
-c, -s, -r, +w

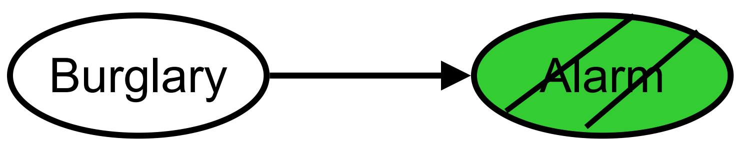
# Likelihood Weighting

- Problem with rejection sampling:
  - If evidence is unlikely, you reject a lot of samples
  - You don't exploit your evidence as you sample
  - Consider P(B|+a)




-b, -a
-b, -a
-b, -a
-b, -a
+b, +a

- Idea: fix evidence variables and sample the rest




-b +a
-b, +a
-b, +a
-b, +a
+b, +a

- Problem: sample distribution not consistent!
- Solution: weight by probability of evidence given parents

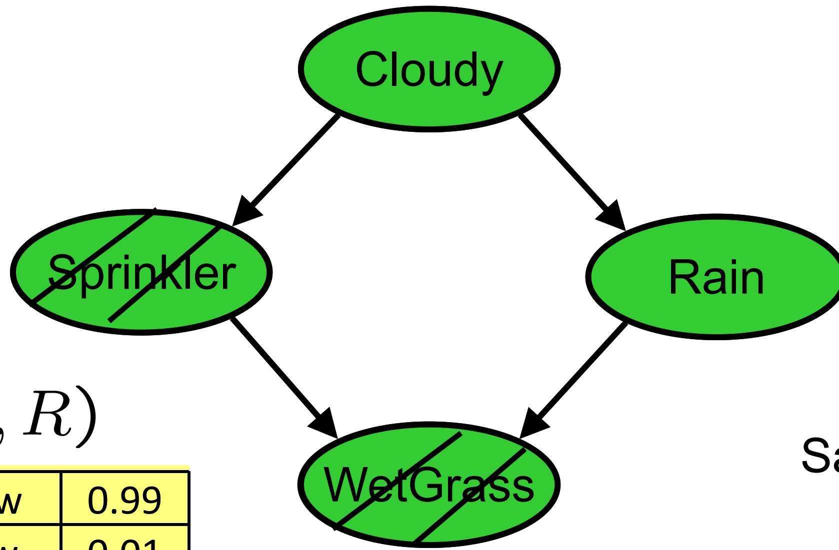
# Likelihood Weighting

$P(C)$

| | |
|---|---|
| +c | 0.5 |
| -c | 0.5 |

$P(S|C)$

| | | |
|---|---|---|
| +c | +s | 0.1 |
| | -s | 0.9 |
| -c | +s | 0.5 |
| | -s | 0.5 |




$P(R|C)$

| | | |
|---|---|---|
| +c | +r | 0.8 |
| | -r | 0.2 |
| -c | +r | 0.2 |
| | -r | 0.8 |

$P(W|S,R)$

| | | | |
|---|---|---|---|
| +s | +r | +w | 0.99 |
| | | -w | 0.01 |
| | -r | +w | 0.90 |
| | | -w | 0.10 |
| -s | +r | +w | 0.90 |
| | | -w | 0.10 |
| | -r | +w | 0.01 |
| | | -w | 0.99 |

Samples:

+c, +s, +r, +w

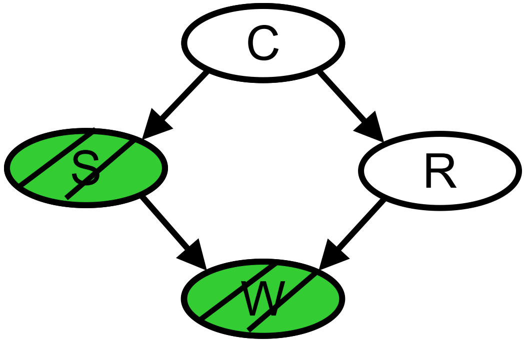
...

$$w = 1.0 \times 0.1 \times 0.99$$

# Likelihood Weighting

- Sampling distribution if z sampled and e fixed evidence

$$S_{WS}(\mathbf{z},\mathbf{e}) = \prod_{i=1}^{l} P(z_i|\text{Parents}(Z_i))$$




- Now, samples have weights

$$w(\mathbf{z},\mathbf{e}) = \prod_{i=1}^{m} P(e_i|\text{Parents}(E_i))$$

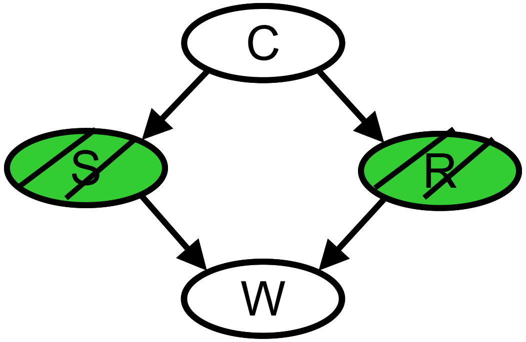
- Together, weighted sampling distribution is consistent

$$S_{\text{WS}}(z,e) \cdot w(z,e) = \prod_{i=1}^{l} P(z_i|\text{Parents}(z_i)) \prod_{i=1}^{m} P(e_i|\text{Parents}(e_i))$$

$$= P(\mathbf{z},\mathbf{e})$$

# Likelihood Weighting

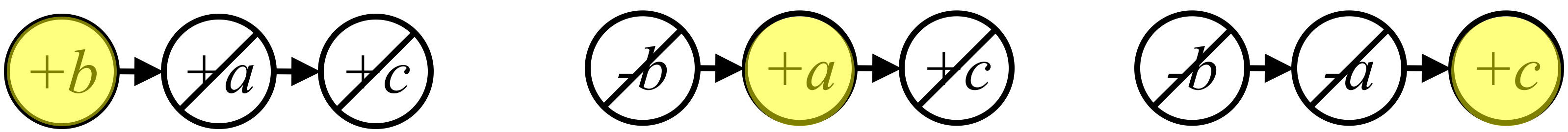
- Likelihood weighting is good
  - We have taken evidence into account as we generate the sample
  - E.g. here, W's value will get picked based on the evidence values of S, R
  - More of our samples will reflect the state of the world suggested by the evidence
- Likelihood weighting doesn't solve all our problems
  - Evidence influences the choice of downstream variables, but not upstream ones (C isn't more likely to get a value matching the evidence)
- We would like to consider evidence when we sample *every* variable



# Markov Chain Monte Carlo

- *Idea*: instead of sampling from scratch, create samples by making random change to the preceding event

- *Procedure*: resample one variable at a time, conditioned on all the rest, but keep evidence fixed. E.g., for P(b|c):




- *Properties*: Now samples are not independent (in fact they're nearly identical), but sample averages are still consistent estimators!

- *What's the point*: both upstream and downstream variables condition on evidence.