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PH110: Waves and Electromagnetics

Tutorial 4

20. One of these is an impossible electrostatic field. Which one?

(a) $E_1 = k[xy\,\hat{x} + 2yz\,\hat{y} + 3xz\,\hat{z}]$

(b) $E_2 = k[y^2\,\hat{x} + (2xy + z^2)\,\hat{y} + 2yz\,\hat{z}]$

Here k is a constant with appropriate units. For the possible one, find the potential, using the origin as your reference point. Check your answer by computing ∇V.

$$\nabla \times E_1 = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ xy & 2yz & 3xz \end{vmatrix}$$

$$= (0 - 2y)\,\hat{x} - (3z - 0)\,\hat{y} + (0 - x)\,\hat{z}$$

$$= -2y\,\hat{x} - 3z\,\hat{y} - x\,\hat{z}$$

$$\nabla \times E_1 \neq 0$$

Now, $$\nabla \times E_2 = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y^2 & (2xy + z^2) & 2yz \end{vmatrix}$$

$$= (2z - 2z)\,\hat{x} - (0 - 0)\,\hat{y} + (2y - 2y)\,\hat{z}$$

$$= 0$$

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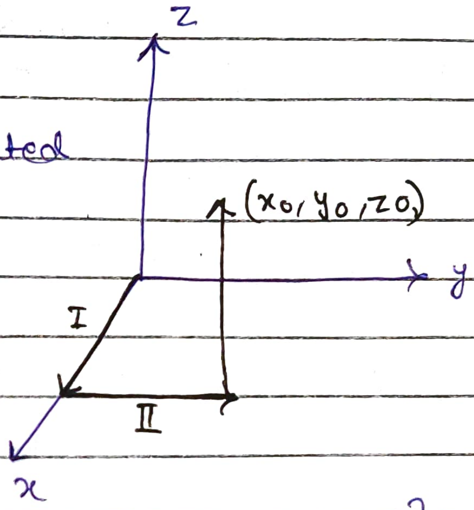
Since, electrostatic fields have $\nabla \times E = 0$.

$\therefore$ E_1 is an impossible electric field and E_2 is a possible field.

Now, lets calculate potential for E_2.
The origin is reference and let the point (x_0, y_0, z_0)

Lets go by the indicated path.



$\vec{dl} = dx\,\hat{x} + dy\,\hat{y} + dz\,\hat{z}$

$\therefore$ $\vec{E_2} \cdot \vec{dl} = k\left[dx\,y^2 + (2yx + z^2)\,dy + 2yz\,dz\right]$

for path 1
$x \neq 0$, $y = 0$, $z = 0$, $dy = dz = 0$

$E \cdot dl = ky^2\,dx$
$E \cdot dl = 0$

For path 2
$x = x_0$, $z = 0$, $dz = 0$, $dx = 0$

$E \cdot dl = k(2x_0 y)\,dy$

$$\int_0^{y_0} 2kx_0 y\,dy = \frac{2kx_0 y_0^2}{2} = kx_0 y_0^2$$

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For path 3

$x = x_0$, $y = y_0$, $dx = 0$, $dy = 0$

$$E \cdot dl = 2k y_0 z \, dz$$

$$\int_0^{z_0} E \cdot dl = k y_0 z_0^2$$

$$\therefore V(x_0, y_0, z_0) = -\int_0^{(x_0, y_0, z_0)} E \cdot dl$$

$$V(x_0, y_0, z_0) = -k\left(x_0 y_0^2 + y_0 z_0^2\right)$$

$$\boxed{\therefore V(x, y, z) = -k\left(xy^2 + yz^2\right)}$$

Checking using calculating ∇V.

$$\nabla V = -k\left(y^2 \hat{x} + \left(2xy + z^2\right)\hat{y} + 2yz\,\hat{z}\right)$$

$$\boxed{\nabla V = -E}$$

Hence, proved.

21. Find the potential inside and outside a uniformly charged sphere (solid) whose radius is R and total charge is q. Use infinity as your reference point. Compute the gradient of V in each region, and check that it yields the correct field. Sketch V(r).

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The electric field inside and outside a solid sphere is given by,

$$E(r) = \begin{cases} \dfrac{q}{4\pi\epsilon_0 r^2}\,\hat{r} & r \geq R \\[2ex] \dfrac{qr}{4\pi\epsilon_0 R^3}\,\hat{r} & r < R \end{cases}$$

∴ potential when $r \geq R$

$$V(r) = -\int_{\infty}^{r} E \cdot dr$$

$$= -\int_{\infty}^{r} \frac{q}{4\pi\epsilon_0 r^2}\, dr$$

$$= -\frac{q}{4\pi\epsilon_0}\left(-\frac{1}{r}\Bigg|_{\infty}^{r}\right)$$

$$= \frac{q}{4\pi\epsilon_0}\left(\frac{1}{r} - \frac{1}{\infty}\right)$$

$$V(r) = \frac{q}{4\pi\epsilon_0 r}$$

when $r < R$

$$V(r) = -\int_{\infty}^{R} \frac{q}{4\pi\epsilon_0 r^2}\, dr - \int_{R}^{r} \frac{qr}{4\pi\epsilon_0 R^3}\, dr$$

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$$V(r) = \frac{q}{4\pi\epsilon_0}\left(\frac{1}{r}\right)\Bigg|_{\infty}^{R} - \frac{q}{4\pi\epsilon_0 R^3}\left(\frac{r^2}{2}\right)\Bigg|_{R}^{r}$$

$$V(r) = \frac{q}{4\pi\epsilon_0}\left(\frac{1}{R} - \frac{1}{R^3}\left(\frac{r^2}{2} - \frac{R^2}{2}\right)\right)$$

$$V(r) = \frac{q}{4\pi\epsilon_0}\left(\frac{1}{R} + \frac{1}{2R} - \frac{r^2}{2R^3}\right)$$

$$V(r) = \frac{q}{8\pi\epsilon_0 R}\left(3 - \frac{r^2}{R^2}\right)$$

$$\therefore V(r) = \begin{cases} \dfrac{q}{4\pi\epsilon_0 r} & r \geq R \\ \dfrac{q}{8\pi\epsilon_0 R}\left(3 - \dfrac{r^2}{R^2}\right) & r \leq R \end{cases}$$

$$\nabla V = \begin{cases} \dfrac{-q}{4\pi\epsilon_0 r^2} & r \geq R \\ \dfrac{q}{8\pi\epsilon_0 R}\left(\dfrac{-2r}{R^2}\right) & r < R \end{cases}$$

$$-\nabla V = \begin{cases} q/4\pi\epsilon_0 r^2 & r \geq R \\ \dfrac{q r}{4\pi\epsilon_0 R^3} & r < R \end{cases}$$

$$-\nabla V = E$$

Hence, checked

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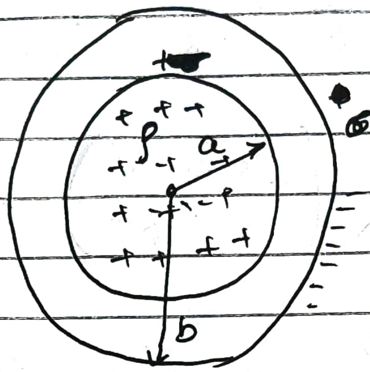
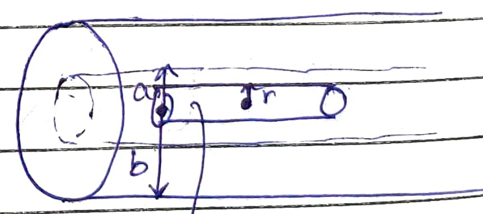
24. For the given configuration, find the potential difference between a point on the axis and a point on the outer cylinder. Note that it is not necessary to commit yourself to a particular reference point.



The charge in inner sphere is equal to charge on surface of outer sphere. Therefore the cable is neutral.

Consider a length l of this cable,

when $r < a$



Volume of gaussian surface $= \pi r^2 l$

$\therefore q_{enc} = \rho \pi r^2 l$

Using Gauss's Law

$$\oint E \cdot da = \frac{\rho \pi r^2 l}{\epsilon_0}$$

$$E(2\pi r l) = \frac{\rho \pi r^2 l}{\epsilon_0} \Rightarrow E = \frac{\rho r}{2\epsilon_0} \hat{r}$$

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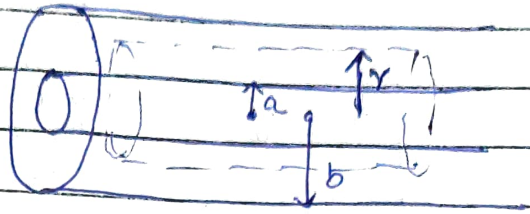
When $r \geq a$ and $r < b$



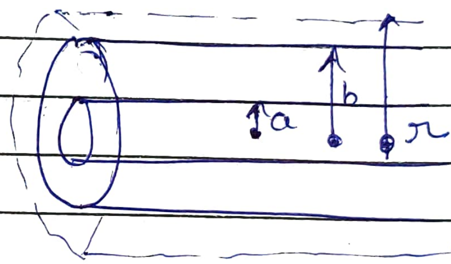
$q_{enc} = \rho \pi a^2 l$

$$\oint E \cdot da = \frac{\rho \pi a^2 l}{\epsilon_0}$$

$$E = \frac{\rho \cancel{\pi} a^2 \cancel{l}}{\epsilon_0 \cdot 2\cancel{\pi} r \cancel{l}}$$

$$E = \frac{\rho a^2}{2\epsilon_0 r}\hat{r}$$

When $r \geq b$



$q_{enc} = 0$

$\therefore$ $E = 0$

We have to calculate $V_b - V_0$

$$V_b - V_0 = -\int_0^b E \cdot dl$$

$$= -\int_0^a E \cdot dl - \int_a^b E \cdot dl$$

$$= -\int_0^a \frac{\rho r}{2\epsilon_0} dr - \int_a^b \frac{\rho a^2}{2\epsilon_0 r} dr$$

$$= -\frac{\rho}{2\epsilon_0}\left[\left(\frac{r^2}{2}\right)\Big|_0^a + a^2 \ln(r)\Big|_a^b\right]$$

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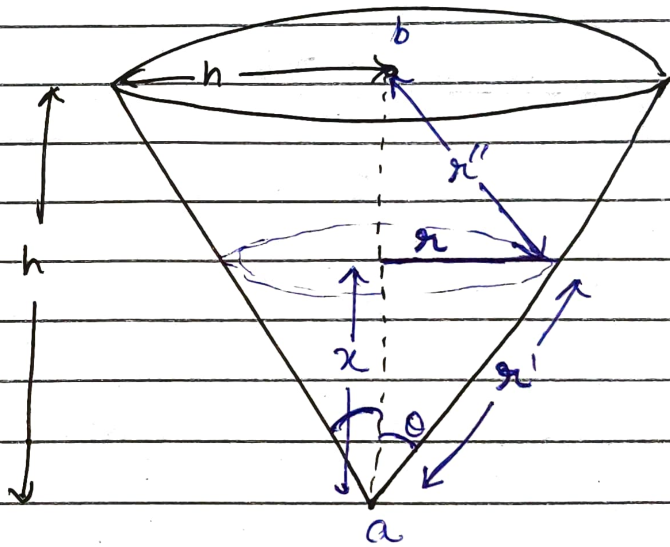
$$V_b - V_0 = \frac{-\rho}{2\epsilon_0}\left[\frac{a^2}{2} + a^2 \ln\left(\frac{b}{a}\right)\right]$$

$$\boxed{V_b - V_0 = \frac{-\rho a^2}{4\epsilon_0}\left[1 + 2\ln\left(\frac{b}{a}\right)\right]}$$

26 A conical surface carries a uniform surface charge σ. The height of cone is h, as is the radius of the top. Find the potential difference between the points a (the vertex) and b (the centre of top).



$\tan\theta = 1$

$\therefore x = r$

area of the circular ring of radius r and width dr'

$$dS = 2\pi r\, dr'$$

$$\because r' = \sqrt{r^2 + x^2} = \sqrt{2}\, r \quad (\text{as } x = r)$$

$$\therefore dS = 2\sqrt{2}\pi r\, dr$$

charge on the circular ring of radius r.

$$dq = 2\sqrt{2}\sigma\pi r\, dr$$

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$\therefore$ potential due to this ring at point a

$$dV_a = \frac{1}{\cancel{2\pi}\epsilon_0} \frac{\sigma \cdot \cancel{2}\sqrt{2}\cancel{\pi} r\, dr}{r'}$$

$$dV_a = \frac{\sigma \cdot \cancel{\sqrt{2}} \cancel{r}\, dr}{2\epsilon_0 (\cancel{\sqrt{2} r})} \qquad (\because r' = \sqrt{2} r)$$

$$dV_a = \frac{\sigma\, dr}{2\epsilon_0}$$

$\therefore$ potential due to whole cone at point a

$$\int_0^h dV_a = \frac{\sigma}{2\epsilon_0} \int_0^h dr$$

$$V_a = \frac{\sigma h}{2\epsilon_0} \quad \text{——} \; (1)$$

Potential due to ring at b,

$$r'' = \sqrt{r^2 + h^2 + r^2 - 2hr}$$

$$r'' = \sqrt{2r^2 + h^2 - 2hr}$$

$$\therefore \; dV_b = \frac{1}{4\cancel{\pi}\epsilon_0} \frac{2\sqrt{2}\sigma \cancel{\pi} r\, dr}{\sqrt{2r^2 + h^2 - 2hr}}$$

$$dV_b = \frac{\sigma}{\sqrt{2}\epsilon_0} \frac{r\, dr}{\sqrt{2r^2 + h^2 - 2hr}}$$

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$$\therefore \int_0^h dV_b = \int_0^h \frac{\sigma}{\sqrt{2}\,\epsilon_0} \frac{r\,dr}{\sqrt{2r^2+h^2-2hr}}$$

$$V_b = \frac{\sigma}{2\epsilon_0} \int_0^h \frac{r\,dr}{\sqrt{r^2-hr+\frac{h^2}{2}}}$$

$$V_b = \frac{\sigma}{4\epsilon_0} \int_0^h \frac{2r\,dr}{\sqrt{r^2-hr+h^2/2}}$$

$$V_b = \frac{\sigma}{4\epsilon_0}\left[\int_0^h \frac{2r-h}{\sqrt{r^2-hr+h^2/2}} + \int_0^h \frac{h\,dr}{\sqrt{r^2-hr+h^2/2}}\right]$$

$$V_b = \frac{\sigma}{4\epsilon_0}\left[\left(2\sqrt{r^2-hr+h^2/2}\right)\Big|_0^h + h\int_0^h \frac{dr}{\sqrt{\left(r-\frac{h}{2}\right)^2+\frac{h^2}{4}}}\right]$$

$$V_b = \frac{\sigma}{4\epsilon_0}\left[0 + h\left\{\log\left(\left(r-\frac{h}{2}\right)+\sqrt{\left(r-\frac{h}{2}\right)^2+\frac{h^2}{4}}\right)\right\}\Big|_0^h\right]$$

$$V_b = \frac{\sigma h}{4\epsilon_0}\left[\log\left(\frac{h}{2}+\frac{h}{\sqrt{2}}\right) - \log\left(\frac{h}{\sqrt{2}}-\frac{h}{2}\right)\right]$$

$$V_b = \frac{\sigma h}{4\epsilon_0}\left[\ln\left(\frac{2h+\sqrt{2}h}{2h-\sqrt{2}h}\right)\right]$$

$$V_b = \frac{\sigma h}{4\epsilon_0} \ln\left((\sqrt{2}+1)^2\right)$$

$$V_b = \frac{\sigma h}{2\epsilon_0} \ln(1+\sqrt{2})$$

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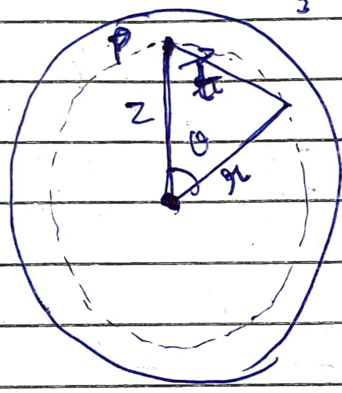
$$\therefore \boxed{V_a - V_b = \frac{\sigma h}{2\epsilon_0}\left(1 - \ln(1+\sqrt{2})\right)}$$

28. Use equation 2.29 to calculate the potential inside a uniformly charged solid sphere of radius R and total charge q.

Let ρ be the charge density of sphere.

$$\therefore \rho = \frac{q}{\frac{4}{3}\pi R^3}$$



$$\therefore V = \frac{1}{4\pi\epsilon_0}\int_V \frac{\rho(r)\,d\tau}{t}$$

and $|\vec{t}| = \sqrt{z^2 + r^2 - 2rz\cos\theta}$

and $d\tau = r^2 \sin\theta \, dr \, d\theta \, d\phi$

$$V = \frac{\rho}{4\pi\epsilon_0}\int \frac{r^2 \sin\theta \, dr \, d\theta \, d\phi}{\sqrt{z^2 + r^2 - 2rz\cos\theta}}$$

Since, ϕ will go from 0 to 2π independent of r and θ

$$\therefore \int_0^{2\pi} d\phi = 2\pi$$

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$$\therefore V = \frac{\rho}{2\epsilon_0} \int_0^R r^2 \left(\int_0^\pi \frac{\sin\theta \, d\theta}{\sqrt{r^2+z^2-2rz\cos\theta}} \right) dr$$

$$V = \frac{\rho}{2\epsilon_0} \left[\int_0^R r^2 \left(-1 \cdot \left(\frac{-1}{2rz} \right) \cdot 2 \cdot \left(\left| \sqrt{r^2+z^2-2rz\cos\theta} \right|_0^\pi \right) \right) dr \right.$$

$$V = \frac{\rho}{2\epsilon_0} \int_0^R \frac{r}{z} \left(r+z - |r-z| \right) dr$$

$$V = \frac{\rho}{2\epsilon_0 z} \int_0^R r \left(r+z - |r-z| \right) dr$$

$$\because (r+z) - |r-z| = \begin{cases} 2r & r < z \\ 2z & r \geq z \end{cases}$$

$$\therefore V = \frac{\rho}{2\epsilon_0} \left[\int_0^z \frac{r}{z}(2r)\, dr + \int_z^R \frac{r}{z}(2z)\, dr \right]$$

$$V = \frac{\rho}{2\epsilon_0} \left[\frac{2}{z} \cdot \frac{z^3}{3} + R^2 - z^2 \right]$$

$$V = \frac{\rho}{2\epsilon_0} \left[R^2 - \frac{z^2}{3} \right]$$

Since $\rho = q / \tfrac{4}{3}\pi R^3$

$$V = \frac{3q}{8\pi\epsilon_0 R^3} \left[R^2 - \frac{z^2}{3} \right] = \frac{q}{8\pi\epsilon_0 R^3} \left[3R^2 - z^2 \right]$$

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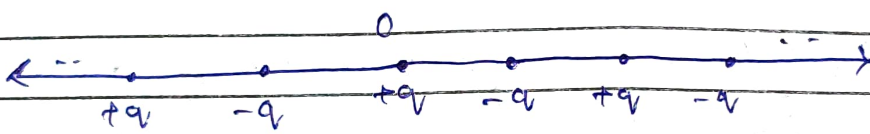
33. Consider an infinite point of charges, $\pm q$ (with alternating signs), strung out along the x-axis, each a distance a from its nearest neighbours. Find the work per particle required to assemble this system.

Let us consider the charge at origin to be positive.



$\therefore$ The potential at origin due to all other charges is, $\left(k = \frac{1}{4\pi\epsilon_0}\right)$

$$V_0 = \frac{kq}{a}\left(-\frac{2}{a} + \frac{2}{2a} - \frac{2}{3a} + \frac{2}{4a} + \cdots\right)$$

$$V_0 = -\frac{2kq}{a}\left\{1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots\right\}$$

$$V_0 = -\frac{2kq}{a}\ln(2) \qquad \left\{\because \ln(1+x) = 1 - \frac{x}{2} + \frac{x^2}{3} + \cdots\right\}$$

$$\therefore \quad W = \frac{1}{2}qV$$

$$\therefore \quad W = \frac{1}{2} \times q \times \left(-\frac{2kq}{a}\right)\ln 2$$

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$$W = -\frac{Kq^2}{a} \ln 2$$

$$\boxed{W = -\frac{1}{4\pi\epsilon_0} \frac{q^2}{a} \ln 2}$$

$\therefore$ the Madelung constant for this system is $\ln 2$.

Q.34 Find the energy stored in a uniformly charged sphere (solid) of radius R and charge q. Do it three different ways:

(a) Use Eq. 2.43, $W = \frac{1}{2}\int \rho V \, d\tau$

(b) Use Eq. 2.45, $W = \frac{\epsilon_0}{2}\int E^2 \, d\tau$

(c) Use Eq. 2.44,
$$W = \frac{\epsilon_0}{2}\left(\int_V E^2 \, d\tau + \oint_S V E \cdot da\right)$$

Since, for a ~~po~~ solid sphere, potential is given by (at a distance r from centre).

$$V = \frac{q}{8\pi\epsilon_0 R}\left(3 - \frac{r^2}{R^2}\right)$$

$\therefore$ (a) $$W = \frac{1}{2}\int_0^R \rho \left\{\frac{q}{8\pi\epsilon_0 R}\left(3 - \frac{r^2}{R^2}\right)\right\}(4\pi r^2 dr)$$

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$$W = \frac{(\rho q)(4\pi)}{16\pi\epsilon_0 R} \int_0^R \left(3r^2 - \frac{r^4}{R^2}\right) dr$$

$$W = \frac{\rho q}{4\epsilon_0 R} \left(R^3 - \frac{1}{R^2} \cdot \frac{R^5}{5}\right)$$

$$W = \frac{\rho q}{4\epsilon_0 R} \times \frac{4R^3}{5} = \frac{\rho q R^3}{5\epsilon_0 R}$$

$$\boxed{W = \frac{q \cdot q \cancel{R^3}}{\frac{4}{3}\pi \cancel{R^3} \cdot (5\epsilon_0 R)} = \frac{1}{4\pi\epsilon_0}\left(\frac{3}{5}\frac{q^2}{R}\right)}$$

(b) $$E = \begin{cases} \frac{\rho r}{3\epsilon_0}\hat{r} & r < R \\ \frac{\rho R^3}{3\epsilon_0 r^2} & r \geq R \end{cases}$$

$$\therefore W = \frac{\epsilon_0}{2} \int_{\text{all space}} E^2 \, d\tau$$

$$\therefore W = \frac{\epsilon_0}{2}\left[\int_0^R \frac{\rho^2 r^2}{9\epsilon_0^2}(4\pi r^2 dr) + \int_R^\infty \frac{\rho^2 R^6}{9\epsilon_0^2 r^4} \cdot 4\pi r^2 dr\right]$$

$$W = \frac{\epsilon_0}{2} \cdot \left(\frac{4\pi\rho^2}{9\epsilon_0^2}\right)\left[\int_0^R r^4 dr + R^6 \int_R^\infty \frac{1}{r^2} dr\right]$$

$$W = \frac{2\pi\rho^2}{9\epsilon_0}\left[\frac{R^5}{5} + R^5\right] = \frac{12\pi\rho^2 R^5}{45\epsilon_0}$$

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$$W = \frac{\cancel{12}^{3}\cancel{\pi}R^{\cancel{5}}}{\cancel{45}_{5}\epsilon_0}\left(\frac{\cancel{\pi}q^2}{\cancel{16}_{4}\pi^{\cancel{2}}R^{\cancel{6}}}\right)$$

$$\boxed{W = \frac{1}{4\pi\epsilon_0}\left(\frac{3}{5}\frac{q^2}{R}\right)}$$

(c) Let us consider a sphere of radius 'a' Such that $a > R$.

$$\therefore \quad W = \frac{\epsilon_0}{2}\left(\int_V E^2 d\tau + \oint_S VE\cdot da\right)$$

potential at a distance $r > R$ is $\frac{q}{4\pi\epsilon_0 r}$

$$\therefore \quad V_a = \frac{q}{4\pi\epsilon_0 a}$$

and $$E_a = \frac{q}{4\pi\epsilon_0 a^2}$$

$$\therefore \quad \oint_S VE\cdot da = \oint_S \frac{q}{4\pi\epsilon_0 a}\cdot\frac{q}{4\pi\epsilon_0 a^2}\cdot a^2 \sin\theta\, d\theta\, d\phi$$

$$= \frac{q^2}{(4\pi\epsilon_0)^2 a}\left(\int_0^{\pi}\sin\theta\, d\theta\right)\left(\int_0^{2\pi} d\phi\right)$$

$$\oint VE\cdot da = \frac{q^2}{(4\pi\epsilon_0)^2}\left(\frac{4\pi}{a}\right)$$

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$$\int_V E^2 d\tau = \int_0^R \frac{\rho^2 r^2}{9\epsilon_0^2}(4\pi r^2 dr) + \int_R^a \frac{\rho^2 R^6}{9\epsilon_0^2 r^4} 4\pi r^2 dr$$

$$\int_V E^2 d\tau = \frac{4\pi\rho^2}{9\epsilon_0^2} \cdot \frac{R^5}{5} + \frac{4\pi\rho^2 R^6}{9\epsilon_0^2}\left(\frac{1}{R} - \frac{1}{a}\right).$$

$$\int_V E^2 d\tau = \frac{4\pi\rho^2 R^5}{9\epsilon_0^2}\left(\frac{1}{5} + 1 - \frac{R}{a}\right)$$

$$= \frac{4\pi R^5}{9\epsilon_0^2}\left(\frac{9q^2}{16\pi^2 R^6}\right)\left(\frac{6}{5} - \frac{R}{a}\right)$$

$$= \frac{q^2}{(4\pi\epsilon_0)^2}\left(\frac{4\pi}{R}\right)\left(\frac{6}{5} - \frac{R}{a}\right)$$

$$\therefore W = \frac{\epsilon_0}{2}\left[\frac{q^2}{(4\pi\epsilon_0)^2}\left(\frac{4\pi}{a}\right) + \frac{q^2}{(4\pi\epsilon_0)^2}\left(\frac{4\pi}{R}\right)\left(\frac{6}{5} - \frac{R}{a}\right)\right]$$

$$W = \frac{\cancel{4\pi\epsilon_0}}{2} \cdot \frac{q^2}{(4\pi\epsilon_0)^{\cancel{2}}}\left[\frac{1}{a} + \frac{6}{5R} - \frac{1}{a}\right]$$

$$\boxed{W = \frac{1}{4\pi\epsilon_0}\left(\frac{3}{5}\frac{q^2}{R}\right)}$$