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PH110: Waves and Electromagnetics

Tutorial 3

46. (a) Show that

$$x \frac{d(\delta(x))}{dx} = -\delta(x)$$

(b) Let $\theta(x)$ be the step function:

$$\theta(x) = \begin{cases} 1 & , \text{if } x > 0 \\ 0 & , \text{if } x \leq 0 \end{cases}$$

Show that $\frac{d\theta}{dx} = \delta(x)$

(a) Two expressions involving delta functions are considered equal if

$$\int_{-\infty}^{\infty} f(x) \, D_1(x) \, dx = \int_{-\infty}^{\infty} f(x) \, D_2(x) \, dx \quad \text{--- (1)}$$

∴ Let us solve,

$$\int_{-\infty}^{\infty} f(x) \left(x \frac{d}{dx} (\delta(x)) \right) dx$$

(using integration by parts)

$$= x f(x) \delta(x) \Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \frac{d}{dx} (x f(x)) \, \delta(x) \, dx$$

$$= 0 - \int_{-\infty}^{\infty} \left(x \frac{df}{dx} + f \right) \delta(x) \, dx$$

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(first term is zero as $\delta(x) = 0$ at $\pm\infty$)

$$\therefore \int_{-\infty}^{\infty} f(x)\left(x \frac{d}{dx}(\delta(x))\right) dx = -\int_{-\infty}^{\infty}\left(x\frac{df}{dx} + f\right)\delta(x)\, dx$$

$$= -\left(\int_{-\infty}^{\infty}\left(x\frac{df}{dx}\right)\delta(x)\, dx + \int_{-\infty}^{\infty} f\,\delta(x)\, dx\right)$$

$$= -\left(0 + \int_{-\infty}^{\infty} f\,(\delta(x))\, dx\right)$$

$$= -f(0)$$

$$= -\int_{-\infty}^{\infty} f(x)\,\delta(x)\, dx \qquad \left(\because f(0) = \int_{-\infty}^{\infty} f(x)\,\delta(x)\, dx\right)$$

$$\therefore \int_{-\infty}^{\infty} f(x)\left(x\frac{d}{dx}(\delta(x))\right) dx = -\int_{-\infty}^{\infty} f(x)\,\delta(x)\, dx$$

$\therefore$ from ①, we have

$$\boxed{x\frac{d}{dx}(\delta(x)) = -\delta(x)}$$

Hence, proved.

(b) Let us assume f be a compactly supported function.

$$\int_{-\infty}^{\infty} f(x)\left(\frac{d\theta}{dx}\right) dx = f(x)\,\theta(x)\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty}\left(\frac{df}{dx}\right)\theta(x)\, dx$$

$$= f(\infty)\,\theta(\infty) - f(-\infty)\,\theta(-\infty) - \int_{-\infty}^{\infty}\frac{df}{dx}\,\theta(x)\, dx$$

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$\because$ $\theta(\infty) = 1$

and $\theta(-\infty) = 0$

$$\int_{-\infty}^{\infty} f(x)\left(\frac{d\theta}{dx}\right)dx = f(\infty) - \left[\int_{-\infty}^{0} 0\left(\frac{df}{dx}\right)dx + \int_{0}^{\infty}\left(\frac{df}{dx}\right)dx\right]$$

$$\int_{-\infty}^{\infty} f(x)\left(\frac{d\theta}{dx}\right)dx = f(\infty) - \left(f(\infty) - f(0)\right)$$

$$\int_{-\infty}^{\infty} f(x)\left(\frac{d\theta}{dx}\right)dx = f(0) = \int_{-\infty}^{\infty} f(x)\,\delta(x)\,dx \quad -(1)$$

$\because$ when,

$$\int_{-\infty}^{\infty} g(x)\,D_1(x)\,dx = \int_{-\infty}^{\infty} g(x)\,D_2(x)\,dx.$$

then, $\quad D_1(x) = D_2(x).$

$\therefore$ from (1), we have,

$$\frac{d\theta}{dx} = \delta(x)$$

Hence, proved.

47. (a) Write an expression for volume charge density $\rho(r)$ of a point charge q at r'. Make sure that the volume integral of ρ equals q.

(b) What is the volume charge density of an electric dipole, consisting of a point

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charge $-q$ at the origin and a point charge $+q$ at a?

(c) What is the volume charge density (in spherical coordinates) of a uniform, infintesimally thin spherical shell of radius R and total charge Q, centred at the origin?

(a) Since, the charge density is zero ~~every~~ at every point other than r' and the integral of $\rho(r)$ over all space must be equal to q. A function with such properties is the delta function.

$$\therefore \int_{\text{all space}} \delta^3(r)\, d\tau = 1$$

$$\therefore \rho(r) = q\delta^3(r-r')$$

$$\therefore \int_{-\infty}^{\infty} \rho(r) = \int_{-\infty}^{\infty} q\, \delta^3(r-r')\, dr$$

$$= q \int_{-\infty}^{\infty} \delta^3(r-r')\, dr$$

$$\int_{-\infty}^{\infty} \rho(r) = q$$

Hence, verified.

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(b) Similarly as part (a),

the charge density at any point r due to $-q$ charge at origin is

$$\rho_1(r) = -q\,\delta^3(r)$$

and the charge density at any point due to $+q$ charge at a is

$$\rho_2(r) = q\,\delta^3(r-a)$$

$\therefore$ net charge density at any point r is

$$\boxed{\rho(r) = q\,\delta^3(r-a) - q\,\delta^3(r)}$$

(c) The charge density must be 0 everywhere except at $r=R$, and its integral over all space must be equal to total charge Q. This means the $\rho(r)$ is proportional to $\delta(r-R)$

$\therefore$ let $\rho(r) = k\cdot\delta(r-R)$

where k is some constant,

$$\because \int_0^\infty k\cdot\delta(r-R)\,dr = Q \quad \text{--- (1)}$$

The volume integration in spherical coordinates can be thought as integration over spheres $S(r)$ of radius r, and

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then summing the results over r.

Let $dA(r) = r^2 \sin\theta \, d\phi$ be the area of element on $S(r)$.

$$\therefore \int_{R^3} \rho(r)\, d^3r = \int_0^\infty \left(\int_{S(r)} \rho(r)\, dA(r) \right) dr$$

$$= k \int_0^\infty \delta(r-R) \left(\int_{S(r)} dA(r) \right) dr$$

$$= k \left(\int_{S(r)} dA(r) \right)_{r=R}$$

$$Q = k\,(4\pi R^2)$$

$$\therefore k = \frac{Q}{4\pi R^2}$$

$$\boxed{\therefore \rho(r) = \frac{Q}{4\pi R^2}\, \delta(r-R)}$$

50. (a) Let $F_1 = x^2 \hat{z}$ and $F_2 = x\hat{x} + y\hat{y} + z\hat{z}$. Calculate the divergence and curl of F_1 and F_2. Which one can be written as gradient of scalar? Find a scalar potential that does the job. Which one can be written as curl of a vector? Find a suitable vector potential.

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(b) Show that $F_3 = yz\hat{x} + zx\hat{y} + xy\hat{z}$ can be written both as gradient of a scalar and as curl of a vector, find scalar and vector potentials for the function.

(a) $\nabla \cdot F_1 = \left(\frac{\partial}{\partial x}\hat{x} + \frac{\partial}{\partial y}\hat{y} + \frac{\partial}{\partial z}\hat{z}\right) \cdot \left(x^2\hat{z}\right)$

$$\boxed{\nabla \cdot F_1 = 0}$$

and $\nabla \cdot F_2 = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z) = 3$

$$\boxed{\nabla \cdot F_2 = 3}$$

$$\nabla \times F_1 = \begin{vmatrix} x & y & z \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 0 & 0 & x^2 \end{vmatrix}$$

$$\nabla \times F_1 = \frac{\partial (x^2)}{\partial y}\hat{x} - \frac{\partial (x^2)}{\partial x}\hat{y}$$

$$\boxed{\nabla \times F_1 = -2x\,\hat{y}}$$

$$\nabla \times F_2 = \begin{vmatrix} x & y & z \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x & y & z \end{vmatrix}$$

$$\boxed{\nabla \times F_2 = 0}$$

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$\because$ F_2 is a curl-less vector, F_2 can be written as gradient of scalar.

Let $u = \frac{1}{2}(x^2+y^2+z^2)$ will do the job.

Since, F_1 is a divergence-less vector it can be written as curl of a vector. Let this vector be $A_1 = f\hat{x} + g\hat{y} + h\hat{z}$

$$\therefore \nabla \times A_1 = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ f & g & h \end{vmatrix}$$

$$F_1 = \nabla \times A_1 = \left(\frac{\partial h}{\partial y} - \frac{\partial g}{\partial z}\right)\hat{x} - \left(\frac{\partial h}{\partial x} - \frac{\partial f}{\partial z}\right)\hat{y} + \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right)\hat{z}$$

$$\therefore \frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} = x^2$$

Let $f=0$, $g = \frac{x^3}{3}$ and $h=0$, then the above equation will hold true,

$\therefore$ $A_1 = \frac{x^3}{3}\hat{y}$ will do the job.

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(b) $\nabla \cdot f_3 = 0 + 0 + 0 = 0$

and $\nabla \times f_3 = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ yz & zx & xy \end{vmatrix}$

$= (x-x)\hat{x} + (y-y)(-\hat{y}) + (z-z)\hat{z}$

$\nabla \times f_3 = 0$

Since, f_3 is divergence-less and curl-less, it can be written as both, ~~scalar~~ gradient of scalar and curl of a vector potential.

Let $\boxed{U_3 = xyz,}$ clearly

$\nabla U_3 = yz\hat{x} + xz\hat{y} + xy\hat{z} = f$

$\therefore \quad f_3 = \nabla U_3$

Now, let the vector potential be $A_3 = f\hat{x} + g\hat{y} + h\hat{z}$

$\therefore \quad F = \nabla \times A_3 = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ f & g & h \end{vmatrix}$

$\therefore \quad \frac{\partial h}{\partial y} - \frac{\partial g}{\partial z} = yz \quad \text{—} (1)$

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$$\frac{\partial f}{\partial z} - \frac{\partial g}{\partial x} = xz \quad \text{--- (II)}$$

$$\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} = xy \quad \text{--- (III)}$$

Let us set $h=0$,

From (I) we get,

$$\frac{\partial g}{\partial z} = -yz$$

$$g = -\frac{1}{2}yz^2 + g_1(x,y)$$

from (II) we get,

$$\frac{\partial f}{\partial z} = xz$$

$$f = \frac{xz^2}{2} + f_1(x,y)$$

Substituting f and g in (III), we get

$$\frac{\partial g_1}{\partial x} - \frac{\partial f_1}{\partial y} = xy$$

Let us choose $g_1 = 0$

$$f_1(x,y) = -\frac{1}{2}xy^2 + f_2(x)$$

But $\frac{\partial f}{\partial x}$ does not appear in $\nabla \times A$, so we can set $f_2 = 0$.

$\therefore$ we have

$$f = \frac{xz^2}{2} - \frac{xy^2}{2}$$

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$$g = -\frac{1}{2} y z^2$$

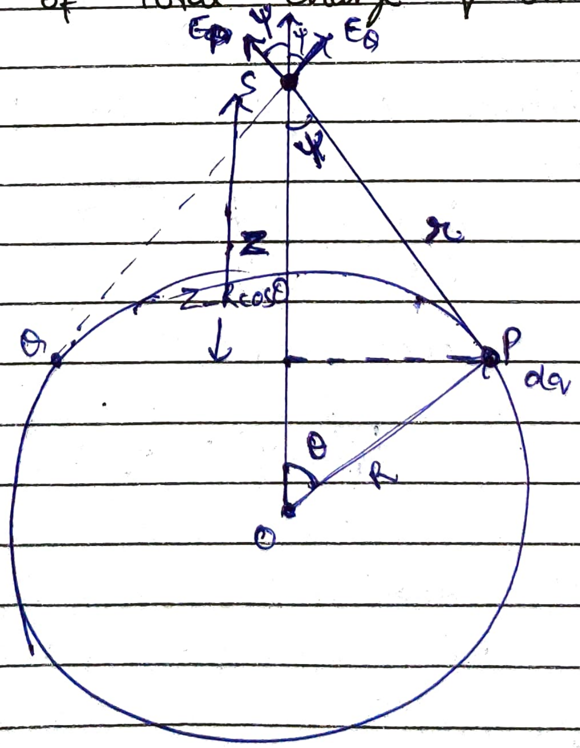
and $h = 0$

$$\therefore \boxed{A_3 = \frac{x}{2}(z^2 - y^2)\,\hat{x} - \frac{yz^2}{2}\,\hat{y} + 0\,\hat{z}}$$

$$\therefore F = \nabla U_3 = \nabla \times A_3$$

Chapter 2 Problems

7. Find the electric field at a distance z from the centre of a spherical surface of radius R that carries a uniform charge density σ. Treat the case $z < R$ as well as $z > R$. Express your answers in terms of total charge q on the sphere.



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Suppose, the small charge dq at point P.
The electric field due to this charge is in E_P direction. But, since it is a sphere, there will be another charge dq at Q which provides electric field in direction E_Q at S. The resultant of these two will be in z direction.

$\because$ $$dq = \sigma\, da$$
$$= \sigma R^2 \sin\theta \, d\theta \, d\phi$$

and using cosine rule in ΔOPS,

$$r^2 = R^2 + z^2 - 2Rz\cos\theta$$
and $$\cos\psi = \frac{z - R\cos\theta}{r}$$

electric field due to charge dq at point S,

$$E_{dq} = \frac{1}{4\pi\epsilon_0}\frac{dq}{r^2}$$

and the z-component of this field is

$$(E_{dq})_z = \frac{1}{4\pi\epsilon_0}\frac{dq}{r^2}\cos\psi$$

$$(E_{dq})_z = \frac{1}{4\pi\epsilon_0} \frac{\sigma R^2 \sin\theta\, d\theta\, d\phi\, (z - R\cos\theta)}{(R^2 + z^2 - 2Rz\cos\theta)^{3/2}}$$

As ϕ goes from 0 to 2π
$\therefore$ $$\int d\phi = 2\pi$$

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The net field in z direction is

$$E_z = \frac{2\pi R^2 \sigma}{4\pi\epsilon_0} \int_0^{\pi} \frac{(z - R\cos\theta)\sin\theta \, d\theta}{(R^2+z^2-2Rz\cos\theta)^{3/2}}$$

Let $u = \cos\theta$

$\therefore \quad du = -\sin\theta \, d\theta$

when $\theta = 0, \quad u = 1$

and $\theta = \pi, \quad u = -1$

$$E_z = \frac{(2\pi R^2 \sigma)}{4\pi\epsilon_0} \int_0^{\pi} \frac{-(z - R\cos\theta)(-\sin\theta)\, d\theta}{(R^2+z^2-2Rz\cos\theta)^{3/2}}$$

$$E_z = \frac{2\pi R^2 \sigma}{4\pi\epsilon_0} \int_{-1}^{1} \frac{(z - Ru)\, du}{(R^2+z^2-2Rzu)^{3/2}}$$

$$E_z = \frac{2\pi R^2 \sigma}{(4\pi\epsilon_0)} \left[\frac{1}{z^2} \frac{zu - R}{\sqrt{R^2+z^2-2Rzu}} \right]_{-1}^{1}$$

$$\boxed{E_z = \frac{2\pi R^2 \sigma}{4\pi\epsilon_0 z^2} \left\{ \frac{z-R}{|z-R|} - \frac{(-z-R)}{|z+R|} \right\}}$$

total charge $q = 4\pi R^2 \sigma$

$$\boxed{E_z = \frac{q}{8\pi\epsilon_0 z^2} \left\{ \frac{z-R}{|z-R|} + \frac{(z+R)}{|z+R|} \right\}}$$

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For $z < R$,

$$E_z = \frac{q}{8\pi\epsilon_0 z^2}(-1+1) = 0$$

For $z > R$,

$$E_z = \frac{q}{8\pi\epsilon_0 z^2}\left(1+1\right)$$

$$\boxed{E_z = \frac{q}{4\pi\epsilon_0 z^2}\hat{z}}$$

9. Suppose the electric field in some region is found to be $E = kr^3\hat{r}$, in spherical coordinates (k is some constant)

(a) Find the charge density ρ.
(b) Find the total charge contained in the sphere of radius R, centred at the origin.

(a) According to Gauss law in differential form

$$\nabla \cdot E = \frac{\rho}{\epsilon_0}$$

$$\therefore \quad \rho = \epsilon_0 (\nabla \cdot E)$$

$$\rho = \epsilon_0 \left(\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 \cdot kr^3)\right)$$

$$\rho = \frac{5\epsilon_0 k r^4}{r^2}$$

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$\therefore \rho = 5\epsilon_0 k r^2$

(b) total charge q can be found using integration of ρ over the surface,

let the thickness of infinitesimal shell at a radius r be dr.

the volume of this shell is therefore

$$dV = 4\pi r^2 dr$$

$\therefore$ the charge stored in this volume is

$$dq = \rho dV$$

$$dq = (5\epsilon_0 k r^2)(4\pi r^2) dr$$

$$dq = 20 k \epsilon_0 \pi r^4 dr$$

$\therefore$ charge stored in sphere of radius R centered at origin,

$$q = \int_0^R 20\pi k \epsilon_0 r^4 dr$$

$$q = 20\pi k \epsilon_0 \left. \frac{r^5}{5} \right|_0^R$$

$$\boxed{q = 4\pi k \epsilon_0 R^5}$$

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total charge can also be calculated using Gauss's Law

$$\because \oint_S E \cdot da = \frac{q}{\varepsilon_0}$$

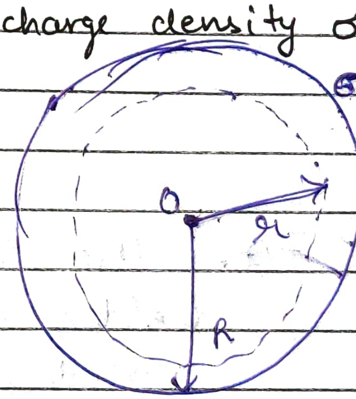
$$q = \varepsilon_0 \oint_S E \cdot da$$

Since, E is constant at particular r and $\oint_S \cdot da = 4\pi r^2$ at a given r.

$$\therefore q = \varepsilon_0 (K R^3)(4\pi R^2)$$

$$q = 4\pi \varepsilon_0 K R^5$$

11. Use Gauss's Law to find the electric field inside a spherical shell of radius R that carries uniform charge density σ.



When $r < R$,

Gaussian surface is a spherical shell centred at origin.
Since, the charge is present only at surface of shell, there is no charge enclosed inside the Gaussian surface.

$$\therefore \oint E \cdot da = 0$$

$$\therefore E = 0$$

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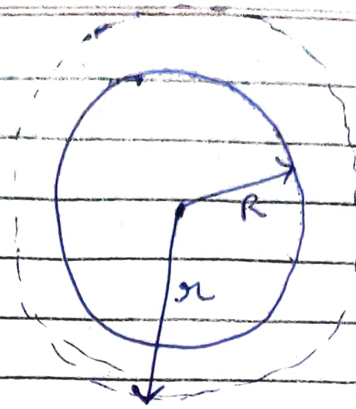
When $r \geq R$



the total charge enclosed inside the Gaussian surface is q,

$$q = (4\pi R^2)\sigma$$

$$\therefore \oint E \cdot da = \frac{4\pi R^2 \sigma}{\epsilon_0}$$

$$E(\cancel{4\pi} r^2) = \frac{\cancel{4\pi} R^2 \sigma}{\epsilon_0}$$

$$E = \frac{R^2 \sigma}{r^2 \epsilon_0} \hat{r}$$

$$\therefore \quad E = \begin{cases} 0 & r < R \\ \dfrac{R^2 \sigma}{r^2 \epsilon_0} \hat{r} & r \geq R \end{cases}$$

12. Use Gauss's Law to find the electric field inside a uniformly charged solid sphere (charge density ρ).

Clearly, the direction of electric field will be radially outwards.

Consider the sphere has radius R,

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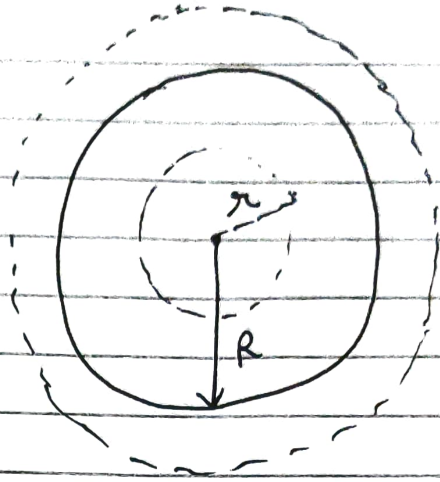

When $r < R$

total charge stored inside sphere of radius r is q

$$q = \rho\left(\frac{4}{3}\pi r^3\right)$$

From Gauss's law,

$$\oint E \cdot da = \frac{Q}{\epsilon_0}$$

$$\oint E \cdot da = \frac{4\rho\pi r^3}{3\epsilon_0}$$

$$E\,(\cancel{4\pi} r^2) = \frac{\cancel{4}\rho\cancel{\pi} r^3}{3\epsilon_0}$$

Electric field inside $\boxed{E = \frac{\rho r}{3\epsilon_0}\hat{r}}$

When $r \geq R$

$$q = \left(\frac{4}{3}\pi R^3\right)\rho$$

$$\oint E \cdot da = \frac{4\pi R^3 \rho}{3\epsilon_0}$$

Electric field Outside $\boxed{E = \frac{\rho R^3}{3\epsilon_0 r^2}\hat{r}}$