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MA102: Introduction to Discrete Mathematics

Tutorial 11

1. Determine which of these are linear homogeneous relations with constant coefficients. Also, find the degree of those that are

a) $a_n = 3a_{n-2}$

b) $a_n = \frac{a_{n-1}}{n}$

(c) $a_n = a_{n-1} + 2$

(d) $a_n = a_{n-2}^2$

A linear homogeneous recurrence relation of degree k with constant coefficients is a recurrence relation of the form

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}$$

where $c_1, c_2 \ldots c_k$ are real numbers with $c_k \neq 0$.

(a) Here, k is 2 and the coefficient is 3.
Also, a_n is a linear relation with a_{n-2}.

Hence, it is a linear homogeneous relation with constant coefficient. Its degree is 2.

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(b) Here, a_n is a relation linear to a_{n-1}. But the constant is not constant.

Hence, it is a linear homogeneous relation without constant coefficient. Its degree is 1.

(c) As, there is a constant term (3) in the relation, it is non-homogeneous.

Hence, it is linear, non homogeneous relation with constant coefficients. Its degree is 1.

(d) Since, a_n is square of a_{n-2}, it is not a linear relation.

It is a non-linear, homogeneous relation with constant coefficient. Its degree is 2.

2. Write a recurrence relation and initial conditions for the numbers S_n of n-bit strings having no two consecutive zeroes. Compute S_6.

Let us start from initial conditions.

$S_1 \rightarrow$ no. of bit strings of length 1 having no two consecutive zeroes.

possible bit strings are 0 and 1.

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$\therefore \quad s_1 = 2$

Now, $s_2 \rightarrow$ no. of bit strings of length 2 having no two consecutive zeroes.

possible bit strings (of length 2) are 00, 01, 10, 11.
But 00 has two consecutive zeroes.

$\therefore \quad s_2 = 3$

Now, suppose a bit string of length $(n-1)$ having no two consecutive zeroes. To make this string of length n ~~such~~ there are two possibilities

$$\underbrace{* \; * \; * \; * \; * \; * \; * \; \cdots \; *}_{n-1}$$

either 1 can be added at last or 0 can be added.

i.e $\quad * \; * \; * \; \cdots \; * \; 1$
$\quad * \; * \; * \; \cdots \; * \; 0$

Now, if 1 is added, then the number of strings with no two zeroes together will be s_{n-1}.

and, if 0 is added, then,

$* \; * \; * \; * \; \cdots \; * \; * \; 0$

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The second bit from right should ~~also~~ be 1 only to prevent two ~~zer~~ consecutive zeroes. Hence, In this case the no. of bit strings of length n with no two consecutive zeroes will be S_{n-2}.

$$\therefore \quad S_n = S_{n-1} + S_{n-2}$$

Hence, the recurrence relation and initial conditions for the numbers S_n of n-bit strings having no two consecutive zeroes is

$$\boxed{\begin{array}{l} S_1 = 2, \quad S_2 = 3 \\ \text{and} \quad S_n = S_{n-1} + S_{n-2} \quad \forall n \geq 3 \end{array}}$$

$$\therefore \quad S_6 = S_5 + S_4$$

$$S_5 = S_4 + S_3$$

$$S_4 = S_3 + S_2$$

$$S_3 = S_2 + S_1 = 2 + 3 = 5$$

$$\therefore S_4 = 5 + 3 = 8$$

$$S_5 = 8 + 5 = 13$$

$$S_6 = 13 + 8 = 21$$

$$\boxed{\therefore \quad S_6 = 21}$$

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3. Find a recurrence relation for the number of ternary strings of length n that contain two consecutive zeroes. What are initial conditions?

Consider a ~~str~~ ternary string of length $(n-1)$ having ~~two~~ two consecutive 0's. The number of such strings will be t_{n-1}.

Now, to make this string of length n following operations can be done.

(i) concatenate a '2' at the end of string

$$\underbrace{* \; * \; * \; * \; \cdots \; * \; *}_{(n-1)\text{ asterisks}} \; 2$$

~~Since, 2 is added at last therefore, there will be no case where this string of length n will have two cons~~

Adding 2 at last does not increase the number of ~~str~~ strings with two consecutive zeroes and hence ternary strings of length n with two consecutive zeroes will be t_{n-1} in this case.

(ii) Concatenate a '1' at the end:
In this case also, the required number will be s_{n-1} using the same argument as in (i).

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(iii) Concatenate a '0' at the end. This will create three possibilities:

(a) $\underbrace{* * \cdots * *}_{(n-1)\text{ places}} 0\ 0$

In this case, we got two consecutive zeroes at the end, no matter what first $(n-2)$ places contain.
$\therefore$ for every place there are 3 possibilities $(0, 1, 2)$ therefore no. of strings will be 3^{n-2} in this case.

(b) $\underbrace{* * \cdots * 1}_{(n-1)\text{ places}} 0$

In this case, we don't have two consecutive zeroes at the end. So, there must be two consecutive zeroes in first $(n-2)$ places. Therefore, there are t_{n-2} such strings.

(c) $\underbrace{* * \cdots * 2}_{(n-1)\text{ places}} 0$

In this case also, there will be t_{n-2} strings using the same argument as in (b).

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∴ reccurence relation for the number of ternary strings of length n having two consecutive zeroes is

$$\boxed{t_n = 2t_{n-1} + 2t_{n-2} + 3^{n-2}}$$

∵ we need t_{n-1} and t_{n-2} to calculate t_n.
we need t_1 and t_2 for initial conditions

$t_1 \rightarrow$ ternary strings of length 1 with two consecutive zeroes.
possible strings of length 1 are 0, 1, 2.
None of them have two continuous 0's.

$$\boxed{\therefore \; t_1 = 0}$$

possible strings of length 2 are,
00, 01, 02, 10, 11, 12, 20, 21, 22

∴ only one string i.e. 00 have two consecutive 0's.

$$\boxed{\therefore \; t_2 = 1}$$

4. Find the solution of the recurrence relation $a_n = a_{n-1} + 6a_{n-2}$ for $n \geq 2$, $a_0 = 3$ and $a_1 = 6$.

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The characteristic equation for the given relation is

$$x^n = x^{n-1} + 6x^{n-2}$$
$$x^2 = x + 6$$
$$x^2 - x - 6 = 0$$
$$(x-3)(x+2) = 0$$

solutions of the characteristic equation are 3 and (-2).

∴ solution of recurrence relation is

$$a_n = \alpha_1 (3)^n + \alpha_2 (-2)^n$$

using initial conditions, $a_0 = 3$ and $a_1 = 6$

$$3 = \alpha_1 + \alpha_2 \quad \text{— (I)}$$
$$6 = 3\alpha_1 - 2\alpha_2 \quad \text{— (II)}$$
$$+\ 6 = 2\alpha_1 + 2\alpha_2 \quad \text{(I)} \times 2$$
$$12 = 5\alpha_1 + 0 \quad \{\text{(II)} + \text{(I)}\}$$

∴ $\alpha_1 = 12/5$

and $\alpha_2 = \frac{3}{5}$

∴ solution to the recurrence relation is

$$\boxed{a_n = \frac{12}{5}(3)^n + \frac{3}{5}(-2)^n}$$

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5. Find the solution of the recurrence relation $a_n = 5a_{n-2} - 4a_{n-4}$ with $a_0 = 3$, $a_1 = 2$, $a_2 = 6$ and $a_3 = 8$.

the characteristic equation for given recurrence relation is,

$$x^n = 5x^{n-2} - 4x^{n-4}$$

$$x^4 = 5x^2 - 4$$
$$x^4 - 5x^2 + 4 = 0$$
$$(x^2-1)(x^2-4) = 0$$
$$(x-1)(x+1)(x-2)(x+2) = 0$$

~~solution~~ roots of the characteristic equation are
1, -1, 2, -2.

∴ solution of the recurrence relation is

$$a_n = \alpha_1 (1)^n + \alpha_2 (-1)^n + \alpha_3 (2)^n + \alpha_4 (-2)^n.$$

Using initial conditions, $a_0 = 3$, $a_1 = 2$, $a_2 = 6$ and $a_3 = 8$.

$$3 = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 \quad \text{(I)}$$
$$2 = \alpha_1 - \alpha_2 + 2\alpha_3 - 2\alpha_4 \quad \text{(II)}$$
$$6 = \alpha_1 + \alpha_2 + 4\alpha_3 + 4\alpha_4 \quad \text{(III)}$$
$$8 = \alpha_1 - \alpha_2 + 8\alpha_3 - 8\alpha_4 \quad \text{(IV)}$$

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Solving (I), (II), (III) and (IV)

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 3 \\ 1 & -1 & 2 & -2 & 2 \\ 1 & 1 & 4 & 4 & 6 \\ 1 & -1 & 8 & -8 & 8 \end{array}\right]$$

$R_2 \rightarrow R_2 - R_1$
$R_3 \rightarrow R_3 - R_1$
$R_4 \rightarrow R_4 - R_1$

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 3 \\ 0 & -2 & 1 & -3 & -1 \\ 0 & 0 & 3 & 3 & 3 \\ 0 & -2 & 7 & -9 & 5 \end{array}\right]$$

$R_4 \rightarrow R_4 - R_2$

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 3 \\ 0 & -2 & 1 & -3 & -1 \\ 0 & 0 & 3 & 3 & 3 \\ 0 & 0 & 6 & -6 & 6 \end{array}\right]$$

$R_4 \rightarrow R_4 - 2R_3$

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 3 \\ 0 & -2 & 1 & -3 & -1 \\ 0 & 0 & 3 & 3 & 3 \\ 0 & 0 & 0 & -12 & 0 \end{array}\right]$$

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Using back substitution,

$$\alpha_4 = 0$$
$$\alpha_3 = 1$$

$$-2\alpha_2 + \alpha_3 - 3\alpha_4 = -1$$
$$-2\alpha_2 = -2$$
$$\alpha_2 = 1$$

$$\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = 3$$
$$\alpha_1 = 1$$

$\therefore$ solution to the recurrence relation is

$$\boxed{a_n = 1 + (-1)^n + 2^n}$$

6. Find explicit formula for Fibonacci sequence f_n. The Lucas numbers satisfy the recurrence relation $L_n = L_{n-1} + L_{n-2}$ and the initial conditions are $L_0 = 2$ and $L_1 = 1$. Show that $L_{n+1} = f_n + f_{n+2}$ and find explicit formula for Lucas numbers.

The fibonacci sequence is given by the recurrence relation,

$$f_n = f_{n-1} + f_{n-2}$$

and initial conditions are $f_0 = 0$ and $f_1 = 1$

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The characteristic equation for fibonacci sequence recurrence relation is,

$$x^n = x^{n-1} + x^{n-2}$$

$$x^2 - x - 1 = 0$$

$$x = \frac{1 \pm \sqrt{5}}{2}$$

$\therefore$ solution to recurrence relation is

$$f_n = \alpha_1 \left(\frac{1+\sqrt{5}}{2}\right)^n + \alpha_2 \left(\frac{1-\sqrt{5}}{2}\right)^n$$

using initial conditions $f_0 = 0$ and $f_1 = 1$,

$$0 = \alpha_1 + \alpha_2 \qquad \text{--- (1)}$$

$$1 = \alpha_1 \left(\frac{1+\sqrt{5}}{2}\right) + \alpha_2 \left(\frac{1-\sqrt{5}}{2}\right)$$

$$1 = \alpha_1 \left(\frac{1+\sqrt{5}}{2}\right) - \alpha_1 \left(\frac{1-\sqrt{5}}{2}\right) \qquad (\text{from } 1)$$

$$1 = \alpha_1 \left(0 + \sqrt{5}\right)$$

$$\therefore \quad \alpha_1 = \frac{1}{\sqrt{5}} \quad \text{and} \quad \alpha_2 = \frac{-1}{\sqrt{5}}$$

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$$\therefore \quad f_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n$$

$$\boxed{\therefore \quad f_n = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right)}$$

Proving $L_{n+1} = f_n + f_{n+2}$ using strong induction

BASIS STEP: Checking for $n = 0$

$$L_{0+1} = L_1 = 1$$

$$f_0 + f_2 = 0 + 1 = 1$$

Hence, it is true for $n = 0$.

INDUCTIVE STEP: Assuming $L_{i+1} = f_i + f_{i+2}$ for all $i \in N$, $0 \le i \le k$.

Now, we have to prove $L_{k+2} = f_{k+1} + f_{k+3}$.

$$\because \quad L_{k+2} = L_{k+1} + L_k$$

$$= f_k + f_{k+2} + f_{k-1} + f_{k+1}$$

(using inductive hypothesis)

$$\therefore \quad L_{k+2} = \underbrace{f_{k-1} + f_k} + \underbrace{f_{k+1} + f_{k+2}}$$

$$L_{k+2} = f_{k+1} + f_{k+3}$$ (using fibonacci relation)

Hence, our prove by induction is complete.

Finding explicit relation for Lucas Numbers

Since, the recurrence relation for Lucas numbers is same as recurrence relation for fibonacci numbers. The explicit solution for Lucas number will be similar to as of Fibonacci numbers. and only the initial conditions will change.

$$\therefore \quad L_n = \alpha_1 \left(\frac{1+\sqrt{5}}{2}\right)^n + \alpha_2 \left(\frac{1-\sqrt{5}}{2}\right)^n$$

Using initial conditions $L_0 = 2$ and $L_1 = 1$.

$$2 = \alpha_1 + \alpha_2 \quad \text{--- (1)}$$

$$1 = \alpha_1 \left(\frac{1+\sqrt{5}}{2}\right) + \alpha_2 \left(\frac{1-\sqrt{5}}{2}\right)$$

$$1 = \alpha_1 \left(\frac{1+\sqrt{5}}{2}\right) + (2-\alpha_1)\left(\frac{1-\sqrt{5}}{2}\right) \quad \{\text{from (1)}\}$$

$$\cancel{1} = \alpha_1 \left(\frac{1+\sqrt{5}}{2}\right) + (\cancel{1}-\sqrt{5}) - \alpha_1 \left(\frac{1-\sqrt{5}}{2}\right)$$

$$\sqrt{5} = \sqrt{5}\,\alpha_1$$

$$\alpha_1 = 1$$

$$\therefore \quad \alpha_2 = 2 - \alpha_1 = 1$$

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$$\therefore \boxed{L_n = \left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{1-\sqrt{5}}{2}\right)^n}$$

7. What is the general form of the particular solution guaranteed to exist of the linear non-homogeneous recurrence relation $a_n = 8a_{n-2} - 16a_{n-4} + f(n)$ if

(a) $f(n) = n^3$
(b) $f(n) = (-2)^n$
(c) $f(n) = (n^2-n)(-2)^n$
(d) $f(n) = n^4 2^n$.

(a) the associated linear homogeneous relation is

$$a_n = 8a_{n-2} - 16a_{n-4}$$

the characteristic equation for the above relation is

$$x^4 = 8x^2 - 16$$
$$x^4 - 8x^2 + 16 = 0$$
$$(x^2-4)^2 = 0$$
$$(x-2)^2(x+2)^2 = 0$$

roots of the characteristic equation of the associated homogeneous recurrence relation are 2 and -2 each with a multiplicity of 2.

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(a) Since 1 is not a root of the characteristic equation of the associated homogeneous recurrence relation, hence, the particular solution will be of the form

$$\boxed{P(n) = a_1 n^3 + a_2 n^2 + a_3 n + a_4}$$

(b) Since, (−2) is a root of the characteristic equation of the associated homogeneous recurrence relation with multiplicity 2, Hence, the particular solution will be of the form

$$\boxed{P(n) = n^2 (-2)^n}$$

(c) Since −2 is a root of the characteristic equation of the associated homogeneous recurrence relation with multiplicity 2. Hence, the particular solution will be of the form

$$P(n) = n^2 \left(a_1 n^2 + a_2 n + a_3 \right) (-2)^n$$

(d) Since 2 is a root of the characteristic equation of the associated homogeneous recurrence relation with multiplicity 2. Hence, the particular solution will be of the form

$$\boxed{P(n) = (n^2)\left(a_1 n^4 + a_2 n^3 + a_3 n^2 + a_4 n + a_5\right) (2^n)}$$

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8. Find the solution of the recurrence relation $a_n = 2a_{n-1} + 2n^2$ with initial condition $a_1 = 4$.

Since, it is a non-homogeneous relation, therefore the associated homogeneous relation is

$$a_n = 2a_{n-1}$$

the characteristic equation for the above relation is

$$x - 2 = 0$$

∴ the solution to associated homogeneous relation has the form $\alpha(2)^n$.

Now, we need to find particular solution.
We have $F(n) = 2n^2$, therefore the particular solution has the form $an^2 + bn + c$.
To find a, b, c, substitute the particular solution into the original recurrence relation.

$$a_n = 2a_{n-1} + 2n^2$$

$$an^2 + bn + c = 2\{a(n-1)^2 + b(n-1) + c\} + 2n^2$$

$$an^2 + bn + c = 2an^2 + 2n^2 + 2bn - 4an + 2a - 2b + 2c$$

$$(a+2)n^2 + (b-4a)n + (2a-2b+c) = 0(n^2) + 0(n) + 0$$

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this gives us three equations,

$$a + 2 = 0$$
$$b - 4a = 0$$
$$2a - 2b + c = 0$$

$\therefore$
$$a = -2$$
$$b = -8$$
$$c = -12$$

$\therefore$ the particular solution is $-2n^2 - 8n - 12$

$\therefore$ the general solution to the recurrence relation $a_n = 2a_{n-1} + 2n^2$ is

$$a_n = \alpha 2^n - 2n^2 - 8n - 12$$

Using initial condition $a_1 = 4$,

$$4 = 2\alpha - 2 - 8 - 12$$
$$\alpha = 13$$

$\therefore$ the solution to the recurrence relation $a_n = 2a_{n-1} + 2n^2$ with $a_1 = 4$ is

$$\boxed{a_n = 13 \cdot 2^n - 2n^2 - 8n - 12}$$

9. Find all solutions of the recurrence relation $a_n = 7a_{n-1} - 16a_{n-2} + 12a_{n-3} + n4^n$ with $a_0 = -2$, $a_1 = 0$ and $a_2 = 5$.

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This is a non-homogeneous relation, so we need to find the solution to the associated homogeneous recurrence relation (a_n^h) and a particular solution to the original relation (a_n^p).

The associated homogeneous recurrence relation is:

$$a_n = 7a_{n-1} - 16a_{n-2} + 12a_{n-3}$$

the characteristic equation for above relation is

$$x^3 - 7x^2 + 16x - 12 = 0$$

$$(x-2)(x^2-5x+6) = 0$$

$$(x-2)(x-2)(x-3) = 0$$

$$(x-2)^2(x-3) = 0$$

thus, solution to associated homogeneous recurrence relation is

$$a_n^{(h)} = \alpha 2^n + \beta n 2^n + \gamma 3^n \quad \text{— (A)}$$

Now, we need to find particular solution.
We have

$$F(n) = n4^n.$$

F(n) has polynomial part n, so degree of polynomial part is 1. F(n) has exponential part 4^n, so $s = 4$.

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$\because$ particular solution has the form

$$a_n^p = (pn+q)4^n$$

To find p and q, we substitute a_n^p in original recurrence relation.

$$a_n = 7a_{n-1} - 16a_{n-2} + 12a_{n-3} + n4^n$$

$$\Rightarrow (pn+q)4^n = 7(p(n-1)+q)4^{n-1} - 16(p(n-2)+q)4^{n-2}$$
$$+ 12(p(n-3)+q)4^{n-3} + n4^n$$

Dividing by 4^{n-3},

$$\Rightarrow 64(pn+q) = 112(pn+q-p) - 64(pn+q-2p)$$
$$+ 12(pn+q-3p) + 64n$$

$$\Rightarrow n(64p - 112p + 64p - 12p - 64)$$
$$+ 64q - 112q + 112p + 64q - 128p - 12q + 36p$$
$$= 0(n) + 0$$

$$\Rightarrow (4p-64)n + (4q+20p) = 0(n) + 0$$

$\because$ we have

$$4p - 64 = 0$$
$$q + 5p = 0$$

$$\therefore p = 16, \quad q = -80$$

$\therefore$ the particular solution is

$$a_n^p = (16n - 80)4^n \quad \text{—— (B)}$$

$\therefore$ the format of general solution is

$$a_n = a_n^h + a_n^p$$

$$a_n = \alpha 2^n + \beta n 2^n + \gamma 3^n + (16n-80)4^n$$

Using initial values, $a_0 = -2$, $a_1 = 0$ and $a_2 = 5$.

$$-2 = \alpha + \gamma - 80 \quad \text{—— (I)}$$

$$0 = 2\alpha + 2\beta + 3\gamma - 256 \quad \text{—— (II)}$$

$$5 = 4\alpha + 8\beta + 9\gamma - 768 \quad \text{—— (III)}$$

i.e.

$$\alpha + \gamma = 78$$
$$2\alpha + 2\beta + 3\gamma = 256$$
$$4\alpha + 8\beta + 9\gamma = 773$$

Solving these equation to find α, β and γ.

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 78 \\ 2 & 2 & 3 & 256 \\ 4 & 8 & 9 & 773 \end{array}\right]$$

$$R_2 \rightarrow R_2 - 2R_1$$
$$R_3 \rightarrow R_3 - 4R_1$$

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$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 78 \\ 0 & 2 & 1 & 100 \\ 0 & 8 & 5 & 461 \end{array}\right]$$

$R_3 \rightarrow R_3 - 4R_2$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 78 \\ 0 & 2 & 1 & 100 \\ 0 & 0 & 1 & 61 \end{array}\right]$$

$\gamma = 61$

$2\beta + \gamma = 100$

$2\beta = 39$

$\beta = \frac{39}{2}$

$\alpha + \gamma = 78$

$\alpha = 17$

$\therefore \quad \alpha = 17, \quad \beta = \frac{39}{2}, \quad \gamma = 61$

Hence, the recurrence relation $a_n = 7a_{n-1} - 16a_{n-2} + 12a_{n-3} + n4^n$ with $a_0 = -2$, $a_1 = 0$ and $a_2 = 5$ has the solution,

$$\boxed{a_n = 17 \cdot 2^n + \frac{39}{2} n \cdot 2^n + 61 \cdot 3^n + (16n - 80)4^n}$$

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10. Find all solutions of the recurrence relation $a_n = 5a_{n-1} - 6a_{n-2} + 2^n + 3n$
[Hint: Look for a particular solution of the form $q \cdot n \cdot 2^n + p_1 n + p_2$, where q, p_1 and p_2 are constants].

This is a non-homogeneous relation, so we need to find the solution of the associated homogeneous recurrence relation a_n^h and a particular solution to the original recurrence relation (a_n^p).

The associated homogeneous recurrence relation is:

$$a_n = 5a_{n-1} - 6a_{n-2}$$

the characteristic equation for the above relation is:

$$x^2 - 5x + 6 = 0$$
$$(x-2)(x-3) = 0$$

$\therefore$ Solution to associated homogeneous recurrence relation is

$$a_n^h = \alpha 2^n + \beta \cdot 3^n. \quad \text{—(A)}$$

Now, we need to find a particular solution. From the given hint, the particular solution is of the form $q \cdot n \cdot 2^n + p_1 n + p_2 = a_n^p$ where q, p_1 and p_2 are constants.

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To find q, p_1 and p_2, substitute a_n^p in original recurrence relation.

$$a_n = 5a_{n-1} - 6a_{n-2} + 2^n + 3n$$

$$\Rightarrow q \cdot n \cdot 2^n + p_1 n + p_2 = 5\left(q \cdot (n-1) \cdot 2^{n-1} + p_1(n-1) + p_2\right)$$
$$- 6\left(q \cdot (n-2) \cdot 2^{n-2} + p_1(n-2) + p_2\right)$$
$$+ 2^n + 3n$$

$$\Rightarrow n(p_1 - 5p_1 + 6p_1 - 3) + (p_2 + 5p_1 - 5p_2 - 12p_1 + 6p_2)$$
$$+ q \cdot n \cdot 2^n - 5q n \cdot 2^{n-1} + 6qn \cdot 2^{n-2} + 5q \cdot 2^{n-1}$$
$$- 12q \cdot 2^{n-2} - 2^n = 0$$

$$\Rightarrow n(2p_1 - 3) + (2p_2 - 7p_1) + q \cdot n \cdot 2^n - \frac{5}{2} q \cdot n \cdot 2^n + \frac{6}{4} q \cdot n \cdot 2^n$$
$$+ \frac{5}{2} q \cdot 2^n - \frac{12}{4} q \cdot 2^n - 2^n = 0$$

$$\Rightarrow n(2p_1 - 3) + (2p_2 - 7p_1) + 2^n\left(-\frac{q}{2} - 1\right) = 0(n) + 0(2^n) + 0$$

$\therefore$ we have three equations,

$$2p_1 - 3 = 0 \Rightarrow p_1 = \frac{3}{2}$$

$$2p_2 - 7p_1 = 0 \Rightarrow p_2 = \frac{7p_1}{2} = \frac{21}{4}$$

$$-\frac{q}{2} - 1 = 0 \Rightarrow q = -2$$

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∴ the particular solution is :

$$a_n^p = -2 \cdot n \cdot 2^n + \frac{3n}{2} + \frac{21}{4}$$

∴ the general solution to the recurrence relation: $a_n = 5a_{n-1} - 6a_{n-2} + 2^n + 3n$ is

$$\boxed{a_n = \alpha \cdot 2^n + \beta \cdot 3^n - 2 \cdot n \cdot 2^n + \frac{3n}{2} + \frac{21}{4}}$$

Bonus Question Let A_n be the $n \times n$ matrix with 2's on its main diagonal, 1's in all positions next to a diagonal element, and 0's everywhere else. Find a recurrence relation for d_n, the determinant of A_n. Solve this recurrence relation to find a formula for d_n

Given $A_n = \begin{bmatrix} 2 & 1 & 0 & \cdots & & 0 \\ 1 & 2 & 1 & \cdots & & 0 \\ 0 & 1 & 2 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & 1 \; 2 & 1 \\ 0 & 0 & 0 & 0 & 0 \; 1 & 2 \end{bmatrix}_{n \times n}$

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Expanding along first row for det(A_n)

$$\det(A_n) = 2\det\left(\begin{bmatrix} 2 & 1 & \cdots & \cdots & 0 \\ 1 & 2 & 1 & 0 \cdots & 0 \\ 0 & 1 & 2 & 1 \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ & & & 2 & 1 \\ 0 & 0 & 0 \; 0 \; 0 & 1 & 2 \end{bmatrix}_{(n-1)\times(n-1)}\right) - \det\left(\begin{bmatrix} 1 & 1 & 0 & \cdots & 0 \\ 0 & 2 & 1 & \cdots & 0 \\ 0 & 1 & 2 & 1 \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots 1 & 2 \end{bmatrix}_{(n-1)\times(n-1)}\right)$$

$$\quad\quad\quad\quad\quad\quad\quad\quad = B \quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad = C$$

Clearly the matrix B is A_{n-1}.

$\therefore$ det B = det(A_{n-1}).

Now, $C = \begin{bmatrix} 1 & 1 & 0 & \cdots & 0 \\ 0 & 2 & 1 & \cdots & 0 \\ 0 & 1 & 2 & 1 \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 \; 0 & 1 & 2 \end{bmatrix}_{(n-1)\times(n-1)}$

Finding det C by expanding along 1st column

$$\det C = \det\left(\begin{bmatrix} 2 & 1 & \cdots & \cdots & 0 \\ 1 & 2 & 1 & \cdots & 0 \\ 0 & 1 & 2 & 1 \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 \; 0 \; 0 & 1 & 2 \end{bmatrix}_{(n-2)\times(n-2)}\right)$$

$$= D$$

Clearly, det C = det(A_{n-2}) as $D = A_{n-2}$

$\therefore$ $\det(A_n) = 2\det(B) - \det(C)$

$= 2\det(A_{n-1}) - \det D$

$\det(A_n) = 2\det(A_{n-1}) - \det(A_{n-2})$

$\therefore$ the recurrence relation for d_n is

$$\boxed{d_n = 2d_{n-1} - d_{n-2}}$$

the characteristic equation for the above recurrence relation is

$$x^2 - 2x + 1 = 0$$
$$(x-1)^2 = 0$$

since, characteristic equation has only one root i.e. 1 with multiplicity 2.

$\therefore$ ~~gen~~ the solution of the recurrence relation is :

$$d_n = \alpha(1)^n + \beta n(1)^n$$
$$\boxed{d_n = \alpha + \beta n}$$

to find α and β we need initial conditions d_1 and d_2.

$A_1 = \begin{bmatrix} 2 \end{bmatrix}$ and $A_2 = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$

$\therefore d_1 = 2$

$\therefore d_2 = 3$

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using initial conditions,

$$2 = \alpha + \beta$$
$$3 = \alpha + 2\beta$$
$$- \quad - \quad -$$
$$-1 = -\beta$$

$$\beta = 1, \alpha = 1$$

Hence, the solution to the recurrence relation is $n+1$.

$$\boxed{d_n = n+1}$$