Tutorial- 6 Submission

MA 201 Probability and Statistics (2021-22) (3-1-0-4)

B. Tech. II year CSE & IT

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Tutorial 6a

Qus - 1 Let X be a continuous random variable with PDF $f_X(x) = \begin{cases} 4x^3 & \text{for} \quad 0 \le x \le 1 \\ 0 & \text{for} \quad \text{otherwise} \end{cases}$

and let $Y = \frac{1}{X}$. Find $f_Y(y)$.

Ans: $f_Y(y) = \begin{cases} 4/y^3 & \text{for} \quad 1 \le y < \infty \\ 0 & \text{for} \quad \text{otherwise} \end{cases}$

Qus - 2 Let X be a continuous random variable with PDF $f_X(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right)$ for $-\infty < x < \infty$

and let $Y = X^2$. Find $f_Y(y)$.

Ans: $f_Y(y) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{y}{2}\right)$ for $0 \le y < \infty$

Qus - 3 The joint pdf of two random variables X and Y is $f_{X,Y}(x, y) = \begin{cases} 2\, e^{-x-y} & \text{for} \quad 0 < x < y < \infty \\ 0 & \text{for} \quad \text{otherwise} \end{cases}$

Consider the transformation $Z = 2X$ and $W = Y - X$. Find the joint density of Z and W and conclude that Z and W are independent.

Ans: Considering the transformation $X = \frac{Z}{2}$ and $Y = W + \frac{Z}{2}$

$$f_{Z,W}(z, w) = \begin{cases} 2\, e^{-z-w} & \text{for} \quad 0 < z < w < \infty \\ 0 & \text{for} \quad \text{otherwise} \end{cases}$$

Tutorial 6b

```
clear all
syms pi0 pi1 pi2
p=0.5;
P = [p p 0; 0 p p; 0 p p];
solution = solve([[pi0 pi1 pi2]*P==[pi0 pi1 pi2], [pi0 pi1 pi2]*[1;1;1]==1],[pi0 pi1 pi
```

```
solution = struct with fields:
             pi0: 0
             pi1: 1/2
             pi2: 1/2
      parameters: [1×0 sym]
      conditions: symtrue
```

6.1) A small computer lab has 2 terminals. The number of students working in this lab is recorded at the end of every hour. A computer assistant notices the following pattern: If there are 0 or 1 students in a lab, then the number of students in 1 hour has a 50-50% chance to increase by 1 or remain unchanged. If there are 2 students in a lab, then the number of students in 1 hour has a 50-50% chance to decrease by 1 or remain unchanged.

(a) Write the transition probability matrix for this Markov chain.

(b) Is this a regular Markov chain? Justify your answer.

(c) Suppose there is nobody in the lab at 7 am. What is the probability of nobody working in the lab at 10 am?

$$P = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

```
clear all
p=0.5;
P = [p p 0; 0 p p; 0 p p]
```

```
P = 3×3
    0.5000    0.5000         0
         0    0.5000    0.5000
         0    0.5000    0.5000
```

```
P3 = [1 0 0]*P^3;
P3(1)
```

```
ans = 0.1250
```

6.2) A computer system can operate in two different modes. Every hour, it remains in the same mode or switches to a different mode according to the transition probability matrix.

$$P = \begin{bmatrix} \frac{2}{5} & \frac{3}{5} \\ \frac{3}{5} & \frac{2}{5} \end{bmatrix}$$

(a) Compute the 2-step transition probability matrix.

(b) If the system is in Mode I at 5:30 pm, what is the probability that it will be in Mode I at 8:30 pm on the same day?

```
clear all
P = [2/5  3/5; 3/5  2/5];
```

```
P^2
```

```
ans = 2×2
    0.5200    0.4800
    0.4800    0.5200
```

```
P3 = [1 0]*P^3
```

```
P3 = 1×2
    0.4960    0.5040
```

```
P3(1)
```

```
ans = 0.4960
```

6.3) Markov chains find direct applications in genetics. Here is an example. An offspring of a black dog is black with probability 0.6 and brown with probability 0.4. An offspring of a brown dog is black with probability 0.2 and brown with probability 0.8.

(a) Write the transition probability matrix of this Markov chain.

(b) Rex is a brown dog. Compute the probability that his grandchild is black.

```
clear all
P = [3/5  2/5; 1/5  4/5];
P2 = [0 1]*P^2;
P2(1)
```

```
ans = 0.2800
```

6.4) Every day, Eric takes the same street from his home to the university. There are 4 street lights along his way, and Eric has noticed the following Markov dependence. If he sees a green light at an intersection, then 60% of time the next light is also green, and 40% of time the next light is red. However, if he sees a red light, then 70% of time the next light is also red, and 30% of time the next light is green.

(a) Construct the transition probability matrix for the street lights.

(b) If the first light is green, what is the probability that the third light is red?

(c) Eric's classmate Jacob has many street lights between his home and the university. If the first street light is green, what is the probability that the last street light is red? (Use the steady-state distribution.)

```
clear all
P = [6/10  4/10; 3/10  7/10];
P3 = [1 0]*P^3;
P3(2)
```

```
ans = 0.5560
```

```
syms piiG piiR
pii = [piiG piiR];
piiSolve = solve([pii*P==pii, pii*ones(size(pii,2),1)==1],pii,'ReturnConditions',true);
piiSolve
```

```
piiSolve = struct with fields:
```

```
        piiG: 3/7
        piiR: 4/7
  parameters: [1×0 sym]
  conditions: symtrue
```

6.8) A Markov chain has 3 possible states: A, B, and C. Every hour, it makes a transition to a different state. From state A, transitions to states B and C are equally likely. From state B, transitions to states A and C are equally likely. From state C, it always makes a transition to state A. Find the steady-state distribution of states.

Hint - $P = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ 1 & 0 & 0 \end{bmatrix}$

```
clear all
P = [0 1/2 1/2; 1/2 0 1/2; 1 0 0];
syms piiA piiB piiC
pii = [piiA piiB piiC];
piiSolve = solve([pii*P==pii, pii*ones(size(pii,2),1)==1],pii,'ReturnConditions',true);
piiSolve
```

```
piiSolve = struct with fields:
        piiA: 4/9
        piiB: 2/9
        piiC: 1/3
  parameters: [1×0 sym]
  conditions: symtrue
```

6.9) Tasks are sent to a supercomputer at an average rate of 6 tasks per minute. Their arrivals are modeled by a Binomial counting process with 2-second frames.

(a) Compute the probability of more than 2 tasks sent during 10 seconds.

(b) Compute the probability of more than 20 tasks sent during 100 seconds. You may use a suitable approximation.

$\lambda = 6$ tasks/min, $\Delta = 2 \text{ sec.} = 2/60 \text{ minutes}$

$$n = \frac{t}{\Delta} = \frac{10/60}{2/60} = 5$$

$$p = \lambda\Delta = 6 \times \frac{2}{60} = 0.2$$

$$\Pr\{X(n) > 2\} = 1 - \Pr\{X(n) \le 2\} = 1 - \Pr\{X(5) \le 2\} = 1 - \Pr\{X(5) = 0\} - \Pr\{X(5) = 1\} - \Pr\{X(5) = 2\}$$

```
binocdf(2,5,0.2,'upper')
```

```
ans = 0.0579
```

```
normcdf(10.5/sqrt(8),0,1,'upper')
```

```
ans = 1.0269e-04
```

6.12) On the average, 2 airplanes per minute land at a certain international airport. We would like to model the number of landings by a Binomial counting process.

(a) What frame length should one use to guarantee that the probability of a landing during any frame does not exceed 0.1?

(b) Using the chosen frames, compute the probability of no landings during the next 5 minutes.

(c) Using the chosen frames, compute the probability of more than 100 landed airplanes during the next hour.

Hint (a) $\lambda = 2$ landings/minute, $p = 0.1$

Now $p = \lambda\Delta \Rightarrow \Delta = \frac{p}{\lambda} = \frac{0.1}{2} minutes = 3sec$

$$n = \frac{t}{\Delta} = \frac{5}{0.05} = 100$$

(b) $\Pr(X(n) = 0) = \Pr(X(100) = 0) = ^{100} C_0 (0.9)^1 00 = 2.6561 \times 10^{-5}$

Or Using central limit theorem -

$$\Pr\{X(n) < 0.5\} = \Pr\{\frac{X(n) - np}{\sqrt{npq}} < \frac{0.5 - 100 \times 0.1}{\sqrt{100 \times 0.1 \times 0.9}}\} = \Pr\{Z < \frac{-9.5}{3}\} = 7.7098 \times 10^{-4}$$

```
normcdf(-9.5/3,0,1) %or
```

```
ans = 7.7098e-04
```

```
normcdf(0.5,10,3)
```

```
ans = 7.7098e-04
```

$$n = 20 \times 60 = 1200$$

Using central limit theorem -

$$\Pr\{X(n) > 100.5\} = \Pr\{\frac{X(n) - np}{\sqrt{npq}} > \frac{100.5 - 1200 \times 0.1}{\sqrt{1200 \times 0.1 \times 0.9}}\} = .9697$$

```
normcdf(100.5,120,sqrt(120*0.9),'upper')
```

```
ans = 0.9697
```

Consider a Binomial counting process with count arrival rate λ. It counts $X(t)$ until time t.

When $\Delta \downarrow 0$, $n = \frac{t}{\Delta} \uparrow \infty$. Therefore $p = \lambda\Delta \downarrow 0$ when $\Delta \downarrow 0$

$$\mathbf{E}X(t) = np = \frac{t}{\Delta} p = \lambda t$$

therefore in the limiting case, as $\Delta \downarrow 0$, $n \uparrow \infty$, and $p \downarrow 0$, it becomes a Poisson Process with parameter $np = \lambda t$,

$$X(t) = \text{Binomial}(n, p) \to \text{Poisson}(\lambda)$$

The interarrival time T becomes a random variable with cdf:

$$F_T(t) = \Pr\{T \le t\} = \Pr\{Y \le n\} = 1 - (1-p)^n = 1 - (1 - \frac{\lambda t}{n})^n \to 1 - e^{-\lambda t}$$

which is the cdf of exponential distribution with parameter λ.

Moreover, the time T_k of the k-th arrival is the sum of exponential intrarrival times that has $\text{Gamma}(k, \lambda)$ distribution. This brings us the Gamma-Poisson formula.

$$\Pr\{T_k \le t\} = \Pr\{\text{k-th arrival before time t}\} = \Pr\{X(t) \ge k\}$$

where T_k is $\text{Gamma}(k, \lambda)$ and $X(t)$ is $\text{Poisson}(\lambda t)$

Summery:

$$X(t) \equiv \text{Poisson}(\lambda t)$$

$$T \equiv \text{Exponential}(\lambda)$$

$$T_k \equiv \text{Gamma}(k, \lambda)$$

$$\Pr\{T_k \le t\} = \Pr\{X(t) \ge k\}$$

$$\Pr\{T_k > t\} = \Pr\{X(t) < k\}$$

6.17) Messages arrive at an electronic message center at random times, with an average of 9 messages per hour.

(a) What is the probability of receiving at least five messages during the next hour?

(b) What is the probability of receiving exactly five messages during the next hour?

Hint -

```
poisscdf(4,9,'upper')
```

```
ans = 0.9450
```

```
poisspdf(5,9)
```

```
ans = 0.0607
```

6.20) Power outages are unexpected rare events occurring according to a Poisson process with the average rate of 3 outages per month. Compute the probability of more than 5 power outages during three summer months.

```
poisscdf(5,3*3,"upper")
```

```
ans = 0.8843
```

6.22) Network blackouts occur at an average rate of 5 blackouts per month.

(a) Compute the probability of more than 3 blackouts during a given month.

(b) Each blackout costs $1500 for computer assistance and repair. Find the expectation and standard deviation of the monthly total cost due to blackouts.

```
poisscdf(3,5,"upper")
```

```
ans = 0.7350
```

```
5*1500 %lambda*cost
```

```
ans = 7500
```

6.23) An internet service provider offers special discounts to every third connecting customer. Its customers connect to the internet according to a Poisson process with the rate of 5 customers per minute. Compute: (a) the probability that no offer is made during the first 2 minutes (b) expectation and variance of the time of the first offer.

```
poisscdf(2,5*2)
```

```
ans = 0.0028
```

```
[3/5 3/5^2] % [E[tk]= k/lambda Var[tk]= k/lambda^2]
```

```
ans = 1×2
    0.6000    0.1200
```