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PH110: Waves and Electrodynamics

Tutorial 10

Q.20 (a) Find the density ρ of mobile charges in a piece of copper, assuming each atom contributes one free electron

(b) Calculate the average electron velocity in a copper wire 1mm in diameter, carrying a current of 1A.

(c) What is the force of attraction between two such wires 1cm apart?

(d) If you could somehow remove the stationary positive charge what would be the electrostatic repulsion force? How many times greater than magnetic force is d?

(a) $$\rho = \frac{\text{charge}}{\text{Volume}}$$

$$= \frac{\text{charge}}{\text{atom}} \times \frac{\text{atoms}}{\text{mole}} \times \frac{\text{moles}}{\text{gram}} \times \frac{\text{grams}}{\text{Volume}}$$

$$= eN\left(\frac{1}{M}\right)(d)$$

$e = 1.6 \times 10^{-19}$ C, $N = 6 \times 10^{23}$

$M = 64$ g/mol, $d = 9$ gm/cm^3

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$$\rho = (1.6\times10^{-19})(6\times10^{23})\left(\frac{9}{64}\right)$$

$$\boxed{\rho = 1.4\times10^{4}\ C/cm^3}$$

(b) $J = \frac{I}{A} = \frac{I}{\pi s^2} = \rho v$

$$v = \frac{I}{\rho\pi s^2}$$

$$v = \frac{1}{3.14\times1.4\times10^{4}\times2.5\times10^{-3}}$$

$$\boxed{v = 9.1\times10^{-3}\ cm/s}$$

(c) $f = \frac{\mu_0 I_1 I_2}{2\pi d} = \frac{\overset{2}{4\cancel{\pi}}\times10^{-7}\times1\times1}{2\cancel{\pi}\times(10^{-2})}$

$$\boxed{f = 2\times10^{-5}\ N/m}$$

(d) $E = \frac{\lambda}{2\pi\epsilon_0 d}$

$\therefore\ f_e = \frac{\lambda_1\lambda_2}{2\pi\epsilon_0 d} = \frac{1}{v^2}\frac{1}{2\pi\epsilon_0}\left(\frac{I_1I_2}{d}\right)$

$$f_e = \frac{c^2}{v^2}\left(\frac{\mu_0}{2\pi}\right)\left(\frac{I_1I_2}{d}\right)$$

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$$f_e = \frac{c^2}{v^2} f_m$$

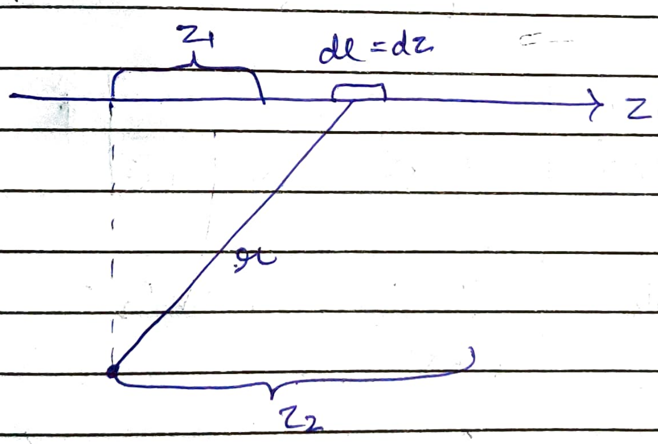
$$\therefore \frac{f_e}{f_m} = \frac{c^2}{v^2} = \left(\frac{3\times10^{8}}{9.1\times10^{-3}}\right)^2$$

$$\frac{f_e}{f_m} = 1.1\times10^{25}$$

$$\therefore f_e = (1.1\times10^{25})(2\times10^{-7})$$

$$\boxed{f_e = 2\times10^{18}\ N/cm}$$

Q.23 Find the magnetic vector potential of a finite segment of straight wire carrying current I.



$$A = \frac{\mu_0}{4\pi}\int \frac{I\hat{z}\,dz}{r} = \frac{\mu_0 I\hat{z}}{4\pi}\int_{z_1}^{z_2} \frac{dz}{\sqrt{z^2+s^2}}$$

$$A = \frac{\mu_0 I \hat{z}}{4\pi}\left[\ln\left(z+\sqrt{z^2+s^2}\right)\right]_{z_1}^{z_2}$$

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$$A = \frac{\mu_0 I}{4\pi} \ln\left[\frac{z_2 + \sqrt{z_2^2 + s^2}}{z_1 + \sqrt{z_1^2 + s^2}}\right] \hat{z}$$

$$B = \nabla \times A$$
$$= -\frac{\partial A}{\partial s} \hat{\phi}$$

$$B = -\frac{\mu_0 I}{4\pi}\left[\frac{1 \quad s}{\left(z_2 + \sqrt{z_2^2+s^2}\right)\sqrt{z_2^2+s^2}} - \frac{s}{\left(z_1 + \sqrt{z_1^2+s^2}\right)\sqrt{z_1^2+s^2}}\right]\hat{\phi}$$

$$B = -\frac{\mu_0 I s}{4\pi}\left[\frac{z_2 - \sqrt{z_2^2+s^2}}{\left(z_2^2 - z_2^2 - s^2\right)\sqrt{z^2+s^2}} - \frac{z_1 - \sqrt{z_1^2+s^2}}{\left(z_1^2 - (z_1^2+s^2)\right)\sqrt{z_1^2+s^2}}\right]\hat{\phi}$$

$$B = -\frac{\mu_0 I s}{4\pi}\left(\frac{-1}{s^2}\right)\left[\frac{z_2}{\sqrt{z_2^2+s^2}} - 1 - \frac{z_1}{\sqrt{z_1^2+s^2}} + 1\right]\hat{\phi}$$

$$B = \frac{\mu_0 I}{4\pi s}\left[\frac{z_2}{\sqrt{z_2^2+s^2}} - \frac{z_1}{\sqrt{z_1^2+s^2}}\right]\hat{\phi}$$

$$\sin\theta_1 = \frac{z_1}{\sqrt{z_1^2+s^2}}, \quad \sin\theta_2 = \frac{z_2}{\sqrt{z_2^2+s^2}}$$

$$\boxed{\therefore\ B = \frac{\mu_0 I}{4\pi s}\left(\sin\theta_2 - \sin\theta_1\right)\hat{\phi}}$$

Q.24 What current density would produce the vector potential, $A = k\hat{\phi}$ (where k is a constant), in cylindrical coordinates?

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given, $A_\phi = R$

$$B = \nabla \times A = \frac{1}{s}\frac{\partial (sR)}{\partial s}\hat{z}$$

$$B = \frac{k}{s}\hat{z}$$

$$\because \quad J = \frac{1}{\mu_0}(\nabla \times B)$$

$$= \frac{1}{\mu_0}\left[\frac{-\partial}{\partial s}\left(\frac{k}{s}\right)\right]\hat{\phi}$$

$$\boxed{\therefore \quad J = \frac{k}{\mu_0 s^2}\hat{\phi}}$$

Ques 28 (a) Check that eq 5.65 is consistent with eq 5.63 by applying divergence.

(b) Check that eq 5.65 is consistent with equation 5.47 by applying curl

(c) Check that eq 5.65 is connected with 5.64 by applying laplacian.

(a) $$\nabla \cdot A = \frac{\mu_0}{4\pi}\int \nabla \cdot \left(\frac{1}{r}\right) dr'$$

$$\because \quad \nabla \cdot \left(\frac{J}{r}\right) = \frac{1}{r}(\nabla \cdot J) + (J \cdot \nabla)\left(\frac{1}{r}\right)$$

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the first term is zero, because $J(r')$ is a function of the source coordinates, not the field coordinates. And since $\mathscr{r} = r - r'$

$$\nabla\left(\frac{1}{\mathscr{r}}\right) = -\nabla'\left(\frac{1}{\mathscr{r}}\right)$$

so, $$\nabla\cdot\left(\frac{J}{\mathscr{r}}\right) = -J\cdot\nabla'\left(\frac{1}{\mathscr{r}}\right)$$

but, $$\nabla'\cdot\left(\frac{J}{\mathscr{r}}\right) = \frac{1}{\mathscr{r}}(\nabla'\cdot J) + J\cdot\nabla'\left(\frac{1}{\mathscr{r}}\right)$$

and $\nabla'\cdot J = 0 \Rightarrow$ magnetostatics

so, $$\nabla\cdot\left(\frac{J}{\mathscr{r}}\right) = -\nabla'\left(\frac{J}{\mathscr{r}}\right)$$

Hence, by divergence theorem,

$$\nabla\cdot A = \frac{-\mu_0}{4\pi}\int \nabla'\cdot\left(\frac{J}{\mathscr{r}'}\right) d\tau'$$

$$= \frac{-\mu_0}{4\pi}\oint \frac{J}{\mathscr{r}}\cdot da'$$

But, $J = 0$ on the surface, so $\nabla\cdot A = 0$

(b) $$\nabla\times A = \frac{\mu_0}{4\pi}\int \nabla\times\left(\frac{J}{\mathscr{r}}\right) d\tau'$$

$$= \frac{\mu_0}{4\pi}\int\left[\frac{1}{\mathscr{r}}(\nabla\times J) - J\times\nabla\left(\frac{1}{\mathscr{r}}\right)\right] d\tau'$$

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but $\nabla \times J = 0$ as J is not a function of r

and $\nabla\left(\frac{1}{r}\right) = -\frac{\hat{r}}{r^2}$

so, $\nabla \times A = \frac{\mu_0}{4\pi}\int \frac{J \times \hat{r}}{r^2}\, dr'$

$\therefore \quad \nabla \times A = B$

Hence, checked.

(c) $\nabla^2 A = \frac{\mu_0}{4\pi}\int \nabla^2 \left(\frac{J}{r}\right) dr'$

But, $\nabla^2\left(\frac{J}{r}\right) = J\nabla^2\left(\frac{1}{r}\right)$

(Now as J is not a function of r)

and $\nabla^2\left(\frac{1}{r}\right) = -4\pi\delta^3(r)$

so, $\nabla^2 A = \frac{\mu_0}{4\pi}\int J(r')\left[-4\pi\delta^3(r)\right] dr'$

$\nabla^2 A = -\mu_0 J(r)$

Hence, checked.

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Q.31. (a) Complete the proof of theorem 2, section 1.6.2. That is, show that any divergence-less vector field F can be written as curl of a vector potential A. What you have to do is find A_x, A_y and A_z such that,

(i) $$\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} = F_x$$

(ii) $$\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} = F_y$$

(iii) $$\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} = F_z$$

Here's one way to do it. Pick $A_x = 0$ and solve (ii) and (iii) for A_y and A_z. Note that "the constants of integration" are themselves functions of y and z - they are constant only w.r.t. x. Now, put these in (i) and use the fact $\nabla \cdot F = 0$ to obtain

$$A_y = \int_0^x F_z(x', y, z)\, dx'$$

$$A_z = \int_0^y F_x(0, y', z)\, dy' - \int_0^x F_y(x', y, z)\, dx'$$

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(b) by direct differentiation, check that the A you obtained in (a) satisfies $\nabla \times A = F$. Is A divergenceless? [This was a very asymmetrical construction and it would be surprising if it were - although we know that there exists a vector whose curl is F and divergence is zero]

(c) As an example, let $F = y\hat{x} + z\hat{y} + x\hat{z}$. Calculate A, and confirm $\nabla \times A = F$.

(a) $$-\frac{\partial W_z}{\partial x} = F_y$$

$$W_z(x,y,z) = -\int_0^x F_y(x',y,z)\,dx' + C_1(y,z)$$

$$\frac{\partial W_y}{\partial x} = F_z$$

$$W_y(x,y,z) = \int_0^x F_z(x',y,z)\,dx' + C_2(y,z)$$

These satisfy (ii) and (iii) for any C_1 and C_2; it remains to choose C_1 and C_2 so as to satisfy (i).

$$F(x,y,z) = -\int_0^x \frac{\partial F_y(x',y,z)}{\partial y}\,\partial x' + \frac{\partial C_1}{\partial y} - \int_0^x \frac{\partial F_z(x',y,z)}{\partial z}\,\partial x' - \frac{\partial C_2}{\partial z}$$

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but, $\frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z} = 0$, so

$$\int_0^x \frac{\partial F_x(x', y, z)}{\partial x'} \, \partial x' + \frac{\partial C_1}{\partial y} - \frac{\partial C_2}{\partial z} = F_x(x, y, z)$$

Now,

$$\int_0^x \frac{\partial F_x(x', y, z)}{\partial x'} \, \partial x' = F_x(x, y, z) - F_x(0, y, z)$$

so,

$$\frac{\partial C_1}{\partial y} - \frac{\partial C_2}{\partial z} = F_x(0, y, z)$$

let us choose $C_2 = 0$, $C_1(y, z) = \int_0^y F_x(0, y', z) \, dy'$

so,

$$W_x = 0, \quad W_y = \int_0^x F_z(x', y, z) \, dx'$$

$$\therefore \quad W_z = \int_0^y F_x(0, y', z) \, dy' - \int_0^x F_y(x', y, z) \, dx'$$

(b) $$\nabla \times W = \left(\frac{\partial W_z}{\partial y} - \frac{\partial W_y}{\partial z} \right) \hat{x} + \left(\frac{\partial W_x}{\partial z} - \frac{\partial W_z}{\partial x} \right) \hat{y} + \left(\frac{\partial W_y}{\partial x} - \frac{\partial W_x}{\partial y} \right) \hat{z}$$

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$$\nabla \times W = \left[F_x(0,y,z) - \int_0^x \frac{\partial F_y(x',y,z)}{\partial y} \partial x' - \int_0^x \frac{\partial F_z(x',y,z)}{\partial z} \partial x' \right] \hat{x}$$

$$+ \left[0 + F_y(x,y,z) \right] \hat{y} + \left[F_z(x,y,z) - 0 \right] \hat{z}$$

But $\nabla \cdot F = 0$, so x-term is

$$\left[F_x(0,y,z) + \int_0^x \frac{\partial F_x(x',y,z)}{\partial x'} \partial x' \right] = F_x(0,y,z) + F_x(x,y,z) - F_x(0,y,z)$$

So, $\nabla \times W = F$

$$\nabla \cdot W = \frac{\partial W_x}{\partial x} + \frac{\partial W_y}{\partial y} + \frac{\partial W_z}{\partial z}$$

$$= 0 + \int_0^x \frac{\partial F_z(x',y,z)}{\partial y} \partial x' + \int_0^y \frac{\partial F_x(0,y',z)}{\partial z} \partial y'$$

$$- \int_0^x \frac{\partial F_y(x',y,z)}{\partial z} \partial x' \neq 0$$

in general.

(c) $$W_y = \int_0^x x' \, dx' = \frac{x^2}{2}$$

$$W_z = \int_0^y y' \, dy' - \int_0^x z \, dx' = \frac{y^2}{2} - xz$$

$$\boxed{\therefore W = \frac{x^2}{2} \hat{y} + \left(\frac{y^2}{2} - xz \right) \hat{z}}$$

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$$\nabla \times W = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 0 & x^2/2 & (y^2/2 - xz) \end{vmatrix}$$

$$\nabla \times W = y\hat{x} + z\hat{y} + x\hat{z} = F$$

Hence, checked.

Q.33 Prove eq 5.78 using eq 5.63, 5.76 and 5.77.

Because $A_{above} = A_{below}$ at every point on the surface, it follows that $\frac{\partial A}{\partial x}$ and $\frac{\partial A}{\partial y}$ are same above and below: any discontinuity is confined to the normal derivative.

$$B_{above} - B_{below} = \left(-\frac{\partial A_{y\,above}}{\partial z} + \frac{\partial A_{y\,below}}{\partial z}\right)\hat{x}$$

$$+ \left(\frac{\partial A_{x\,above}}{\partial z} - \frac{\partial A_{x\,below}}{\partial z}\right)\hat{y}$$

But, according to eq 5.76, this equals $\mu_0 K(-\hat{y})$

So, $$\frac{\partial A_{y\,above}}{\partial z} = \frac{\partial A_{y\,below}}{\partial z}$$

and,

$$\frac{\partial A_{x\,above}}{\partial z} - \frac{\partial A_{x\,below}}{\partial z} = -\mu_0 K$$

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Thus normal derivative of component of A parallel to K suffers discontinuity $-\mu_0 K$, or,

$$\boxed{\frac{\partial A_{above}}{\partial n} - \frac{\partial A_{below}}{\partial n} = -\mu_0 K}$$