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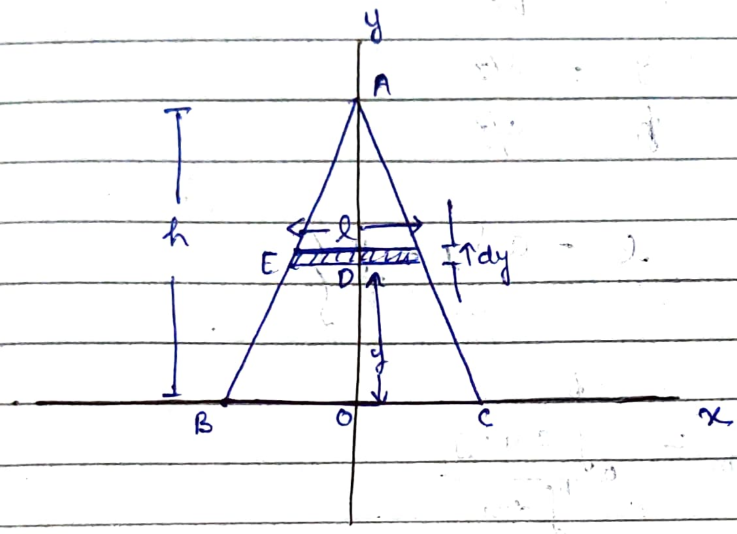
PH100: Mechanics and Thermodynamics
Tutorial # 04

① Find the centre of mass of a thin uniform plate in the shape of equilateral triangle with side a.



Let ABC be an equilateral triangle and midpoint of side BC is at the origin. Let vertical height of triangle OA = h. Let t be thickness of plate and A be area of plate. Let the mass of plate be M.

Mass per unit volume (ρ) of plate = $\dfrac{M}{\text{area} \times \text{thickness}} = \dfrac{M}{At}$

Since, the triangle is symmetrical about y-axis, the x-coordinate of centre of mass is $x = 0$.

To find the y-component, let a strip of width dy which has horizontal length l at distance y from x-axis. The mass of this strip is dm.

$\therefore \quad dm = \rho \times (\text{area of strip}) \times (\text{thickness})$

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$$dm = \frac{M}{A\cancel{t}} \times dy \times l \times \cancel{t}$$

$$dm = \frac{Ml}{A} dy$$

Since $\Delta ADE \sim \Delta AOB$,

$$\frac{AD}{AO} = \frac{DE}{BO}$$

$$\frac{h-y}{h} = \frac{l/\cancel{2}}{a/\cancel{2}}$$

$$l = a\left\{1 - \frac{y}{h}\right\}$$

Now, $$y_{com} = \frac{\int_0^h dm \cdot y}{\int_0^h dm}$$

$$= \frac{\int_0^h \frac{Ml}{A} y \, dy}{\int_0^h dm}$$

$$= \frac{\int_0^h \frac{M}{A} a\left\{1 - \frac{y}{h}\right\} y \, dy}{M}$$

$$= \int_0^h \frac{a}{A}\left(y - \frac{y^2}{h}\right) dy$$

$$= \frac{a}{A}\left[\left(\frac{y^2}{2} - \frac{y^3}{3h}\right)\right]_0^h$$

$$= \frac{a}{A}\left[\frac{h^2}{2} - \frac{h^3}{3h}\right]$$

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$$y_{com} = \frac{a h^2}{6A}$$

$\because$ area of equilateral triangle $(A) = \frac{\sqrt{3}}{4} a^2$

and height $h = \frac{\sqrt{3} a}{2}$

$$y_{com} = \frac{a \times \cancel{3} a^2 \times 4}{\cancel{8}_2 \times \cancel{4} \times \sqrt{3}\, a^2}$$

$$y_{com} = \frac{a}{2\sqrt{3}}$$

Hence, the centre of mass of triangular plate is $\left(0, \frac{a}{2\sqrt{3}}\right)$

Q. An instrument-carrying projectile accidentally explodes at top of its trajectory. The horizontal distance between the launch point and the point of explosion is L. The projectile breaks into two pieces which fly apart horizontally. The larger piece has three times the mass of the smaller piece. To the surprise of the scientist in charge, the smaller piece returns to ~~it~~ earth at launching station. How far away does the larger piece land? Neglect air resistance and effects due to earth's curvature?

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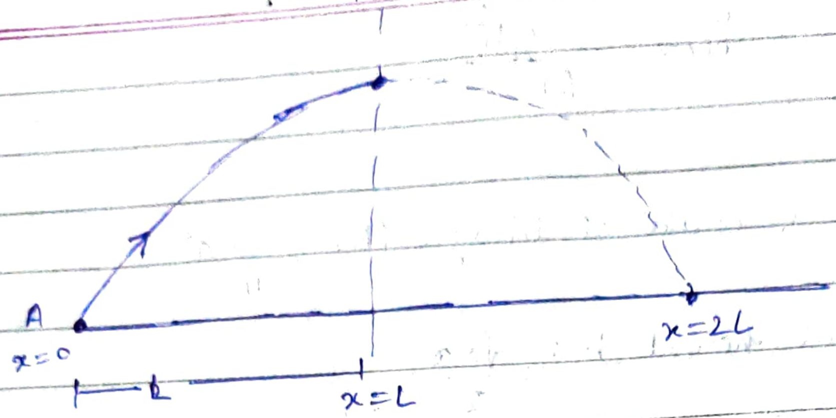

Let at $x=0$ there is launching station and at $x=L$, explosion occurs.
Since, there is no external force, the centre of mass will keep moving in original ~~of~~ motion. As time of ~~flight~~ rise equal time of fall; after reaching the ground, position of centre of mass is $x=2L$.
Let the mass of instrument be M before explosion.
After explosion, mass of smaller particle will be $\frac{M}{4}$ and that of larger particle will be $\frac{3M}{4}$.

Since, smaller particle reaches $x=0$, let x' be position of larger particle

$$\therefore M(2L) = \frac{M}{4}(0) + \frac{3M}{4}(x')$$

$$2\cancel{M}L = \frac{3\cancel{M}}{4}x'$$

$$\therefore x' = \frac{8L}{3}$$

Hence, the larger particle reaches a distance of $\frac{8L}{3}$ from launching station.

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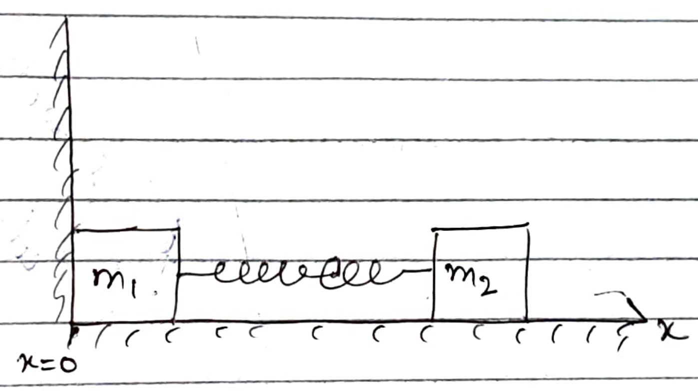
(3) A system is composed of two blocks of mass m_1 and m_2 connected by a massless spring with spring constant k. The blocks slide on frictionless plane. The unstretched length of spring is l. Initially m_2 is held so that the spring is compressed to $l/2$ and m_1 is forced against a stop, as shown. m_2 is released at $t=0$. Find the motion of centre of mass of the system as a function of time.



Let ~~center of~~ m_1 is at $x=0$.

Initially, m_1 is against the wall, hence m_2 moves according to SHM with $\omega = \sqrt{\frac{k}{m_2}}$

Let the displacement of m_2 be x_2,

$$x_2 = A\sin(\omega t) + B\cos(\omega t) + C$$

Initially, at $t=0$, $x_2 = \frac{l}{2}$

$$\frac{l}{2} = B + C \qquad \text{—— (1)}$$

$$\frac{dx_2}{dt} = v_2 = A\omega\cos(\omega t) - B\omega\sin(\omega t)$$

at $t=0$, $v_2 = 0$

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$$A\omega = 0$$

$$\therefore A = 0$$

Since, mean position of m_2 is initially at $x = l$

$$\therefore C = l$$

hence, $B = -\frac{l}{2}$

$$\therefore x_2 = \left\{1 - \frac{\cos(\omega t)}{2}\right\} l \quad \text{—— (A)}$$

Until m_1 loses contact with the wall, the x-coordinat of centre of mass is.

$$x_{com} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2} = \frac{m_1(0) + m_2 x_2}{m_1 + m_2}$$

$$= \frac{m_2 l}{m_1 + m_2}\left(1 - \frac{\cos(\omega t)}{2}\right)$$

when m_1 loses contact with wall, $x_2 = l$
$\therefore$ from equation (A),

$$l = l - \frac{l \cos(\omega t)}{2}$$

$$\therefore \omega t = \pi/2$$

after m_1 loses contact with the wall both mass will move from same velocity, i.e velocity ~~at~~ of m_2 at $x_2 = l$ i.e. $\omega t = \pi/2$

$$V_2 = \frac{dx_2}{dt} = \frac{l\omega}{2} \sin(\omega t).$$

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$\therefore$ v_2 at $\omega t = \pi/2$

$$v_2 = \frac{\omega l}{2}$$

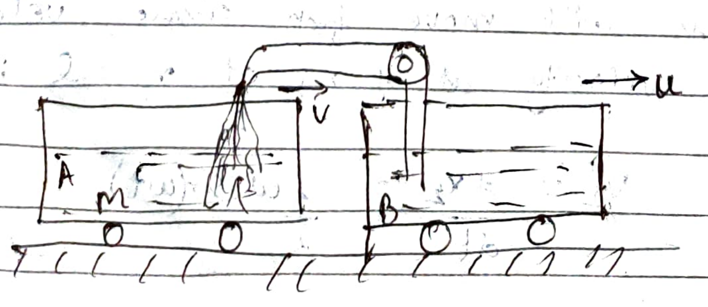
$\therefore$ velocity of centre of mass after m_1 loses contact with the wall is

$$v_{com} = \frac{m_1 v_1 + m_2 v_2}{m_1 + m_2} = \frac{m_1 \cdot \frac{\omega l}{2} + m_2 \frac{\omega l}{2}}{m_1 + m_2}$$

$$v_{com} = \frac{\omega l}{2}$$

Hence, centre of mass will be given by
$$x = \frac{m_2 l}{m_1 + m_2}\left(1 - \frac{\cos(\omega t)}{2}\right)$$ before m_1 loses contact with the wall. After m_1 loses contact with the wall the velocity of centre of mass is $\frac{\omega l}{2}$

4. Material is blown into cart A from cart B at a rate b kilograms per second. The material leaves the chute vertically downward, so that it has same horizontal velocity as cart B, u. At the moment of interest, cart A has mass M and velocity v as shown. Find $\frac{dv}{dt}$, the instantaneous acceleration of A.



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At time t, mass of cart A is M and velocity is v. Let Δm mass of sand from cart B moves to A in a time interval of Δt.

$\therefore$ the momentum at instant t, is

$$P(t) = Mv + \Delta m\, u$$

and momentum at instant $t+\Delta t$, let velocity be changed by Δv.

$$\therefore P(t+\Delta t) = (M+\Delta m)(v+\Delta v)$$

$$P(t+\Delta t) = Mv + \Delta m v + M\Delta v + \Delta m \Delta v$$

Neglecting $\Delta m \Delta v$ as it will be very small.

$$P(t+\Delta t) = Mv + M\Delta v + \Delta m v$$

$\therefore$ change in momentum, $\Delta P = P(t+\Delta t) - P(t)$

$$\therefore \Delta P = M\Delta v + \Delta m(v-u)$$

differentiating w.r.t. time gives,

$$\frac{dP}{dt} = M\frac{dv}{dt} + \frac{dm}{dt}(v-u)$$

Since, mass is blown at a rate of b kg/s.

$$\therefore \frac{dm}{dt} = b.$$

and since, net external force on system is zero, therefore $\frac{dP}{dt} = 0$.

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$$M\frac{dv}{dt} + b(v-u) = 0$$

$$\frac{dv}{dt} = \frac{b(u-v)}{M}$$

Hence, the value of $\frac{dv}{dt}$ at the instant is $\frac{b(u-v)}{M}$.

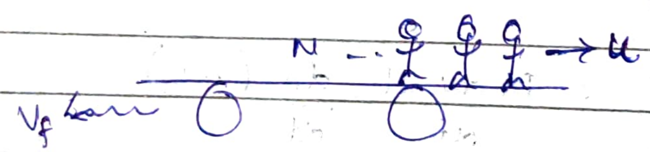
5. N men, each with mass m, stand on a railway flat car of mass M. They jump off one end of flatcar with velocity u relative to car. The car rolls in opposite direction without friction.

a) What is the final velocity of ~~car~~ a flatcar if all men jump at same time?

b) What is the final velocity of flat car if they jump off one at a time?

c) Does case a or case b yield the largest final velocity of flat car? Can you give a simple physical explanation of your answer?

a)



Since, initially the system is at rest

$\therefore$ $P_i = 0$

Let the final velocity of flatcar be V_f when all men jump together.

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$\therefore$ velocity of man w.r.t ground $= (u - v)$

The final momentum, when all men jumped together is,

$$P_f = Nm(u - v_f) - M v_f$$

Since, there is no external force,

$$P_i = P_f$$

$$\therefore Nm(u - v_f) - M v_f = 0$$

$$\therefore Nm(u - v_f) = M v_f$$

$$\therefore v_f = \frac{Nmu}{Nm + M}$$

Hence, final velocity of flat car when all men jump together is $\frac{Nmu}{Nm + M}$

b) Let us suppose the velocity of flat car be v_j after j of N men have jumped.

$\therefore$ $(N-j)$ men are on flat car.

$$\therefore P_i = ((N-j)m + M) v_j$$

Now, one more man jumps, velocity becomes v_{j+1}

$$\therefore P_f = ((N-j-1)m + M) v_{j+1} + m(v_{j+1} - u)$$

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$\therefore \Delta P = P_f - P_i$

$$\Delta P = \left[(N-j-1)m + M\right]V_{j+1} + m(V_{j+1} - u) - \left[(N-j)m+M\right]V_j$$

$$\Delta P = \left[(N-j)m + M\right](V_{j+1}) - mu - \left[(N-j)m+M\right]V_j$$

Since, net external force is zero, $\Delta P = 0$

$$\therefore \quad mu + \left[(N-j)m+M\right]V_j = \left[(N-j)m+M\right]V_{j+1}$$

$$\therefore \quad V_{j+1} = V_j + \frac{mu}{(N-j)m+M}$$

$\therefore$ Since initial velocity of railcart was zero.

$\therefore V_0 = 0$

$$V_1 = V_0 + \frac{mu}{Nm+M} = \frac{mu}{Nm+M}$$

$$V_2 = V_1 + \frac{mu}{(N-1)m+M}$$

$\vdots$

$$V_N = V_{N-1} + \frac{mu}{m+M}$$

$$\therefore V_N = \left(\frac{m}{Nm+M} + \frac{m}{(N-1)m+M} + \cdots + \frac{m}{m+M}\right)u$$

Velocity after N men jumped one at a time

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v_N is the final velocity of flat car when all men jumped one at a time.

c) velocity in part a i.e v_f can be written as

$$v_f = \underbrace{\left(\frac{m}{Nm+M} + \frac{m}{Nm+M} + \cdots + \frac{m}{Nm+M} \right)}_{N \text{ times}} u$$

and velocity in part b, v_n is

$$v_N = \left(\frac{m}{Nm+N} + \frac{m}{(N-1)m+M} + \cdots + \frac{m}{m+M} \right) u$$

Since, in each term of v_N (term ≥ 2) the denominator is lesser corresponding to the term in v_f.
i.e. each term of v_N (term ≥ 2) is greater than that of v_f.

Hence, final velocity of flat car will be greater in part b.

To understand this, assume the mass of flat car to be small. Now, if all men jump together, the flatcar goes backward slightly with speed lesser than u, and the men are moving slowly w.r.t. ground.
Consider case (b), the last jumper by himself could cause the backward speed of flatcar to be close to u, but if there are several jumpers, each jumper also contributes to increasing speed of flatcar. Hence, the final

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speed of flatcar exceeds u.

6. A raindrop of initial mass M_0 starts falling from rest under the influence of gravity. Assume that the drop gains mass from the cloud at rate proportional to product of its instantaneous mass and its instantaneous velocity: $\frac{dM}{dt} = kMV$, where k is constant. Show that the speed of drop eventually becomes effectively constant, and give an expression for terminal speed. Neglect air resistance.

Let at $t=0$, the raindrop starts falling from rest. i.e. at $t=0$, $v=0$.

At an instant t, let mass of drop be M and its velocity be V. The force acting on the drop is its weight Mg.

$$\therefore \quad F = \frac{dp}{dt}$$

$$Mg = \frac{d(MV)}{dt}$$

$$Mg = V \cdot \frac{dM}{dt} + M\frac{dV}{dt}$$

$$Mg = kV^2M + M\frac{dV}{dt} \qquad \left(\frac{dM}{dt} = kMV\right)$$

$$g = kV^2 + \frac{dV}{dt}$$

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$$\frac{dv}{dt} = g - kv^2$$

$$\frac{dv}{g - kv^2} = dt$$

$$\frac{dv}{g/k - v^2} = k\,dt$$

Integrating both sides

$$\int_0^v \frac{dv}{g/k - v^2} = \int_0^t k\,dt$$

$$\because \int \frac{dx}{a^2 - x^2} = \frac{1}{2a} \ln\left|\frac{a+x}{a-x}\right| + c$$

$$\therefore \frac{1}{2\sqrt{g/k}} \left[\ln \left| \frac{\sqrt{g/k} + v}{\sqrt{g/k} - v} \right| \right]_0^v = kt$$

$$\ln \left| \frac{\sqrt{g/k} + v}{\sqrt{g/k} - v} \right| = 2\sqrt{gk}\, t$$

$$\frac{\sqrt{g/k} + v}{\sqrt{g/k} - v} = e^{2\sqrt{gk}\,t}$$

$$\sqrt{g/k} + v = \left(\sqrt{g/k} - v\right) e^{2\sqrt{gk}\,t}$$

$$\sqrt{g/k} \left\{1 - e^{2\sqrt{gk}\,t}\right\} = -v \left\{1 + e^{2\sqrt{gk}\,t}\right\}$$

$$v = \sqrt{g/k} \left(\frac{e^{2\sqrt{gk}\,t} - 1}{e^{2\sqrt{gk}\,t} + 1} \right)$$

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$$\therefore v = \sqrt{\frac{g}{K}}\left(\frac{1 - e^{-2\sqrt{gK}t}}{1 + e^{-2\sqrt{gK}t}}\right)$$

at a long time after $t = 0$, say $t = \infty$

v becomes nearly constant, known as terminal velocity v_T.

$$v_T = \sqrt{\frac{g}{K}}\left(\frac{1-0}{1+0}\right) = \sqrt{\frac{g}{K}}$$

Hence, the speed of drop eventually becomes constant and its value is $\sqrt{\frac{g}{K}}$.