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MA101: Linear Algebra and Matrices (Tut 1)

1. Find determinant of following matrices:

$$A = \begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix} \begin{bmatrix} 2 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 2 & -1 & 2 \\ 8 & -4 & 8 \\ 4 & -2 & 4 \end{bmatrix}$$

Taking 4 common from R_2

$$A = 4 \begin{bmatrix} 2 & -1 & 2 \\ 2 & -1 & 2 \\ 4 & -2 & 4 \end{bmatrix}$$

Since, two rows of A are same, hence $\det A = 0$

2. $$B = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Using row operations $R_1 \leftrightarrow R_5$ and $R_2 \leftrightarrow R_5$ matrix B will become identity matrix of order 5.

Since, ~~them~~ interchanging rows even number of times, leaves det B unchanged,

Hence, $\det B = \begin{vmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{vmatrix} = 1$

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$$C = \begin{bmatrix} 1 & a & a^2 & a^3 \\ 1 & b & b^2 & b^3 \\ 1 & c & c^2 & c^3 \\ 1 & d & d^2 & d^3 \end{bmatrix}$$

Using row operations $R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_2$ and $R_4 \to R_4 - R_3$

$$|C| = \begin{vmatrix} 1 & a & a^2 & a^3 \\ 0 & b-a & b^2-a^2 & b^3-a^3 \\ 0 & c-b & c^2-b^2 & c^3-b^3 \\ 0 & d-c & d^2-c^2 & d^3-c^3 \end{vmatrix}$$

$$|C| = (b-a)(c-b)(d-c) \begin{vmatrix} 1 & a & a^2 & a^3 \\ 0 & 1 & b+a & a^2+ab+b^2 \\ 0 & 1 & c+b & b^2+bc+c^2 \\ 0 & 1 & d+c & c^2+cd+d^2 \end{vmatrix}$$

Using row operations $R_3 \to R_3 - R_2$ and $R_4 \to R_4 - R_3$

$$|C| = (b-a)(c-b)(d-c) \begin{vmatrix} 1 & a & a^2 & a^3 \\ 0 & 1 & b+a & a^2+ab+b^2 \\ 0 & 0 & c-a & c^2+bc-a^2-ab \\ 0 & 0 & d-b & d^2+cd-b^2-bc \end{vmatrix}$$

$$|C| = (b-a)(c-b)(d-c)(c-a)(d-b) \begin{vmatrix} 1 & a & a^2 & a^3 \\ 0 & 1 & b+a & a^2+ab+b^2 \\ 0 & 0 & 1 & a+b+c \\ 0 & 0 & 1 & b+c+d \end{vmatrix}$$

Using $R_4 \to R_4 - R_3$

$$|C| = (b-a)(c-b)(d-c)(c-a)(d-b) \begin{vmatrix} 1 & a & a^2 & a^3 \\ 0 & 1 & b+a & a^2+ab+b^2 \\ 0 & 0 & 1 & a+b+c \\ 0 & 0 & 0 & d-a \end{vmatrix}$$

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Now the matrix is upper triangular, whose determinant is given by product of diagonal entries

$\therefore$ det $C = (b-a)(c-b)(d-c)(c-a)(d-b)(d-a)$

det $C = (a-b)(a-c)(a-d)(b-c)(b-d)(c-d)$

$$D = \begin{bmatrix} 101 & 201 & 301 \\ 102 & 202 & 302 \\ 103 & 203 & 303 \end{bmatrix}$$

Using $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_2$

$$D = \begin{bmatrix} 101 & 201 & 301 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

Since R_2 and R_3 of D are same, hence det $D = 0$

2. Show that $\det(A) = 0$, where $A = [a_{ij}]_{5\times5}$, where $a_{ij} = i + j$.

$$A = \begin{bmatrix} 2 & 3 & 4 & 5 & 6 \\ 3 & 4 & 5 & 6 & 7 \\ 4 & 5 & 6 & 7 & 8 \\ 5 & 6 & 7 & 8 & 9 \\ 6 & 7 & 8 & 9 & 10 \end{bmatrix}$$

Using $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_2$

$$A = \begin{bmatrix} 2 & 3 & 4 & 5 & 6 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 5 & 6 & 7 & 8 & 9 \\ 6 & 7 & 8 & 9 & 10 \end{bmatrix}$$

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since, R_2 and R_3 of A are same, $\det(A) = 0$

3. Suppose $CD = -DC$. Find the mistake in following sequence. Taking determinant gives $\det(C)\det(D) = -\det(D)\det(C)$, so either $\det(C) = 0$, or $\det(D) = 0$.

$\because$ $\det(-A) = (-1)^n \det(A)$, where n is order of matrix.

given,

$$CD = -DC$$

taking determinant on both sides

$$\det(C)\det(D) = \det(-D)\det(C)$$

$$\det(C)\det(D) = (-1)^n \det(D)\det(C)$$

where n is order of matrix D.

Hence, the given statement in the question is wrong.

4. If $AB = BC = I$, then show that $A = C$.

$BC = I$ (given)

Multiplying by A from left side

$$ABC = A.I$$

$$(AB)C = A$$

$$I.C = A \quad (\because AB = I)$$

$$C = A$$

Q.E.D.

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5. Expand $(A+B)^2$, where A and B are square matrices of same order.

$$(A+B)^2 = A(A+B) + B(A+B)$$
$$= A.A + A.B + B.A + B.B$$
$$= A^2 + AB + BA + B^2$$

6. Let A be a 2x2 matrix which commutes with all 2x2 matrices (i.e. AB = BA for every 2x2 matrix B). What can you say about A.

Suppose A is a fixed matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and

B is an arbitrary matrix $\begin{bmatrix} p & q \\ r & s \end{bmatrix}$.

$$\therefore AB = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} p & q \\ r & s \end{bmatrix} = \begin{bmatrix} ap+br & aq+bs \\ cp+dr & cq+ds \end{bmatrix}$$

and, $$BA = \begin{bmatrix} p & q \\ r & s \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} ap+cq & bp+dq \\ ar+cs & br+ds \end{bmatrix}$$

Equating AB = BA, we get,

$br = cq$ —① (equating row1 column1)

$\cancel{aq+bs}$

Since B is an arbitrary matrix, q and r could be anything. and for AB to be equal to BA, equation ① must be true, which is only possible when,

$$b = c = 0$$

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Putting $b = c = 0$ in AB and BA,

$$AB = \begin{bmatrix} ap & aq \\ dr & ds \end{bmatrix}, \quad BA = \begin{bmatrix} ap & dq \\ ar & ds \end{bmatrix}$$

Now, for $AB = BA$,

$aq = dq$ (equating row 1, column 2)

and, $dr = ar$ (equating row 2, column 1)

which gives us $a = d$,

. Matrix A (2×2) which commutes with every other matrix B (2×2) is given by

$$A = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}$$

i.e it is any scalar multiple of ~~mat~~ identity matrix.

7. Find the determinant of a $n\times n$ identity matrix whose j^{th} column is replaced by $[x_1 \ x_2 \ x_3 \ \cdots \ x_n]^T$.

Considering A to be 3×3 identity matrix and replacing its first column by $[x_1 \ x_2 \ x_3]^T$.

$$A = \begin{bmatrix} x_1 & 0 & 0 \\ x_2 & 1 & 0 \\ x_3 & 0 & 1 \end{bmatrix}$$

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A is a lower triangular matrix, hence its determinant is product of its diagonal entries

i.e. $\det A = x_1 \times 1 \times 1 = x_1$

Now, consider B to be a 3×3 matrix whose second column is replaced by $[x_1 \; x_2 \; x_3]^T$.

$$|B| = \begin{vmatrix} 1 & x_1 & 0 \\ 0 & x_2 & 0 \\ 0 & x_3 & 1 \end{vmatrix}$$

Using $C_2 \to C_2 - x_1 C_1$

$$B = \begin{vmatrix} 1 & 0 & 0 \\ 0 & x_2 & 0 \\ 0 & x_3 & 1 \end{vmatrix}$$

Now, B is lower triangular matrix.
$\therefore \det B = 1 \times x_2 \times 1 = x_2$

Similarly, for a $n \times n$ identity matrix whose i^{th} column is replaced by $[x_1 \; x_2 \; \cdots \; x_n]^T$. We can apply row/column operations to make it a triangular matrix. As the x_i element will be in principal diagonal, therefore determinant of such a matrix will be x_i.

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8 Multiply AB using column times rows :

$$AB = \begin{bmatrix} 1 & 0 \\ 2 & 4 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 3 & 3 & 0 \\ 1 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} \begin{bmatrix} 3 & 3 & 0 \end{bmatrix} + \_\_\_ = \_\_\_$$

$$AB = \begin{bmatrix} 1 & 0 \\ 2 & 4 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 3 & 3 & 0 \\ 1 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} \begin{bmatrix} 3 & 3 & 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 4 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 3 & 0 \\ 6 & 6 & 0 \\ 6 & 6 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 4 & 8 & 4 \\ 1 & 2 & 1 \end{bmatrix}$$

$$AB = \begin{bmatrix} 3 & 3 & 0 \\ 10 & 14 & 4 \\ 7 & 8 & 1 \end{bmatrix}$$

9. Find the inverse of following matrix using block matrix inversion discussed in class :

$$M = \begin{bmatrix} 1 & 2 & 1 & 1 \\ 0 & 3 & 1 & 1 \\ 1 & 2 & -1 & 2 \\ 1 & 1 & 2 & 2 \end{bmatrix}$$

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Partitioning M into 4 2x2 square matrices

$$M = \left[\begin{array}{cc|cc} 1 & 2 & 1 & 1 \\ 0 & 3 & 1 & 1 \\ \hline 1 & 2 & -1 & 2 \\ 1 & 1 & 2 & 2 \end{array}\right] = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

where $A = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$ and $D = \begin{bmatrix} -1 & 2 \\ 2 & 2 \end{bmatrix}$

$M^{-1} = \begin{bmatrix} E & F \\ G & H \end{bmatrix}$, where E, F, G & H are 2x2 matrices

$|A| = 3$, $|B| = 0$, $|C| = -1$ and $|D| = -6$

Checking if $D - CA^{-1}B$ is invertible.

$A^{-1} = \frac{1}{3}\begin{bmatrix} 3 & -2 \\ 0 & 1 \end{bmatrix}$, $C^{-1} = -\begin{bmatrix} 1 & -2 \\ -1 & 1 \end{bmatrix}$, $D^{-1} = \frac{-1}{6}\begin{bmatrix} 2 & -2 \\ -2 & -1 \end{bmatrix}$

$$CA^{-1} = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \frac{1}{3}\begin{bmatrix} 3 & -2 \\ 0 & 1 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 3 & 0 \\ 3 & -1 \end{bmatrix} \quad \text{—(I)}$$

$$(CA^{-1})B = \frac{1}{3}\begin{bmatrix} 3 & 0 \\ 3 & -1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 3 & 3 \\ 2 & 2 \end{bmatrix} \quad \text{—(II)}$$

$$D - CA^{-1}B = \begin{bmatrix} -1 & 2 \\ 2 & 2 \end{bmatrix} - \frac{1}{3}\begin{bmatrix} 3 & 3 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 4/3 & 4/3 \end{bmatrix} \quad \text{—(III)}$$

$$|D - CA^{-1}B| = -\frac{8}{3} - \frac{4}{3} = -4$$

$\because$ det $(D - CA^{-1}B)$ is non-zero, hence

$$M^{-1} = \begin{bmatrix} E & F \\ G & H \end{bmatrix} = \begin{bmatrix} A^{-1} + A^{-1}B(D - CA^{-1}B)^{-1}CA^{-1} & -A^{-1}B(D - CA^{-1}B)^{-1} \\ -(D - CA^{-1}B)^{-1}CA^{-1} & (D - CA^{-1}B)^{-1} \end{bmatrix}$$

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$$(D - CA^{-1}B)^{-1} = \frac{-1}{4}\begin{bmatrix} 4/3 & -1 \\ -4/3 & -2 \end{bmatrix} \longrightarrow \text{(IV)}$$

$$\therefore \text{ matrix } H = (D - CA^{-1}B)^{-1} = \frac{-1}{4}\begin{bmatrix} 4/3 & -1 \\ -4/3 & -2 \end{bmatrix} = \begin{bmatrix} -1/3 & 1/4 \\ 1/3 & 1/2 \end{bmatrix}$$

$$\text{matrix } G = -\left\{(D - CA^{-1}B)^{-1}\right\}\left\{CA^{-1}\right\} = -\left\{\frac{-1}{4}\begin{bmatrix} 4/3 & -1 \\ -4/3 & -2 \end{bmatrix}\right\}\left\{\frac{1}{3}\begin{bmatrix} 3 & 0 \\ 3 & -1 \end{bmatrix}\right\}$$

$$= \frac{1}{12}\begin{bmatrix} 1 & 1 \\ -10 & 2 \end{bmatrix} = \begin{bmatrix} 1/12 & 1/12 \\ -5/6 & 1/6 \end{bmatrix}$$

$$\text{matrix } F = -(A^{-1}B)(D - CA^{-1}B)^{-1}$$

$$A^{-1}B = \frac{1}{3}\begin{bmatrix} 3 & -2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

$$\text{matrix } F = -\left(\frac{1}{3}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\right)\left(\frac{-1}{4}\begin{bmatrix} 4/3 & -1 \\ -4/3 & -2 \end{bmatrix}\right)$$

$$= \frac{1}{12}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 4/3 & -1 \\ -4/3 & -2 \end{bmatrix} = \begin{bmatrix} 0 & -1/4 \\ 0 & -1/4 \end{bmatrix}$$

$$\text{matrix } E = A^{-1} + (A^{-1}B)(D - CA^{-1}B)^{-1}(CA^{-1})$$

$$E = \frac{1}{3}\begin{bmatrix} 3 & -2 \\ 0 & 1 \end{bmatrix} + \left(\frac{1}{3}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\right)\left(\frac{-1}{4}\begin{bmatrix} 4/3 & -1 \\ -4/3 & -2 \end{bmatrix}\right)\left(\frac{1}{3}\begin{bmatrix} 3 & 0 \\ 3 & -1 \end{bmatrix}\right)$$

$$E = \frac{1}{3}\begin{bmatrix} 3 & -2 \\ 0 & 1 \end{bmatrix} - \frac{1}{36}\begin{bmatrix} 0 & -3 \\ 0 & -3 \end{bmatrix}\begin{bmatrix} 3 & 0 \\ 3 & -1 \end{bmatrix}$$

$$E = \frac{1}{3}\begin{bmatrix} 3 & -2 \\ 0 & 1 \end{bmatrix} - \frac{1}{36}\begin{bmatrix} -9 & 3 \\ -9 & 3 \end{bmatrix} = \begin{bmatrix} 1 & -2/3 \\ 0 & 1/3 \end{bmatrix} - \begin{bmatrix} -1/4 & 1/12 \\ -1/4 & 1/12 \end{bmatrix}$$

$$E = \begin{bmatrix} 5/4 & -3/4 \\ 1/4 & 1/4 \end{bmatrix}$$

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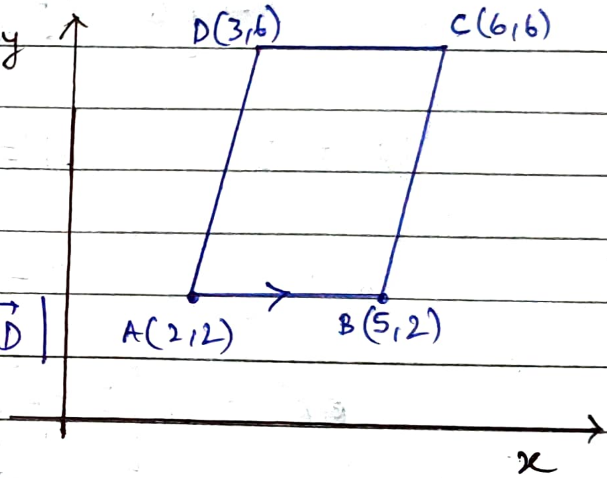
$\because$ $$M^{-1} = \begin{bmatrix} E & F \\ G & H \end{bmatrix}$$

$$M^{-1} = \begin{bmatrix} 5/4 & -3/4 & 0 & -1/4 \\ 1/4 & 1/4 & 0 & -1/4 \\ 1/12 & 1/12 & -1/3 & 1/4 \\ -5/6 & 1/6 & 1/3 & 1/2 \end{bmatrix}$$

Q.10 Find the area of the parallelogram whose corner points are (2, 2), (5, 2), (3, 6) and (6, 6).



$$\vec{AB} = 3\hat{i} + 0\hat{j} + 0\hat{k}$$
$$\vec{AD} = \hat{i} + 4\hat{j} + 0\hat{k}$$

Area of parallelogram $= |\vec{AB} \times \vec{AD}|$

$$\vec{AB} \times \vec{AD} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 0 & 0 \\ 1 & 4 & 0 \end{vmatrix}$$

$$= 0\hat{i} + 0\hat{j} + 12\hat{k}$$

$\therefore$ Area of parallelogram $= |\vec{AB} \times \vec{AD}|$
$= |12\hat{k}|$
$= 12$

Hence, area of given parallelogram is 12 units.