Name: Archit Agrawal
Student ID: 202052307
Group : A1

Date
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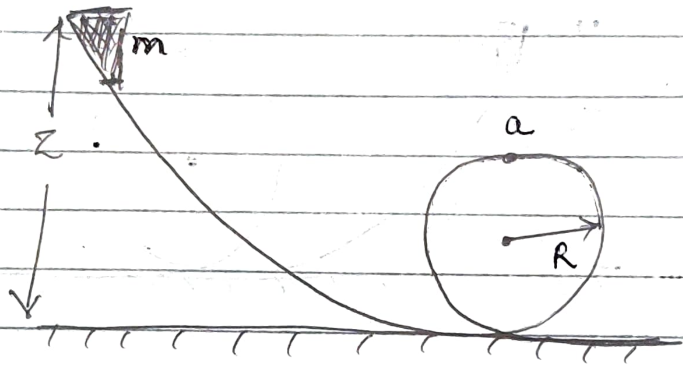
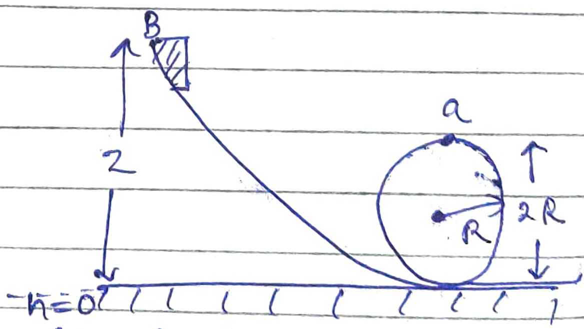
PH100: Mechanics and Thermodynamics
Tutorial #05

1. A small block of mass m starts from rest and slides along a frictionless loop-the-loop as shown in figure. What should be the initial height z, so that m pushes against the top of track (at a) with force equal to its weight?



Since, the block is pushing the top of track with a force equal to its weight, the velocity at a cannot be zero.
Let the velocity of mass m at a be v m/s.



When the mass is at B, the energies related to it are,

$$E_i = (KE) + (GPE)$$
$$= 0 + mgz$$

When the mass reaches a, the energies related to it are,

$$E_f = (KE) + (GPE) = \frac{1}{2}mv^2 + mg(2R)$$

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Student ID: 202052307
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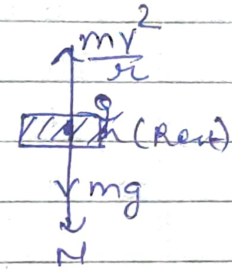
Since, no energy is lost during the motion

$$E_i = E_f$$

$$mgz = 2mgR + \frac{1}{2}mv^2 \quad \text{—— (I)}$$

Consider, the FBD of mass at a with respect to frame of block,

$$\therefore \frac{mv^2}{R} = mg + N \quad \text{—— (II)}$$



Since, it is given that block pushes the top with a force equal to its weight,

$$\therefore N = mg$$

Putting $N = mg$ in (II),

$$v^2 = 2gR$$

Putting $v^2 = 2gR$ in (I),

$$mgz = 2mgR + \frac{1}{\cancel{2}}m(\cancel{2}gR)$$

$$\cancel{mg}z = 3\cancel{mg}R$$

$$z = 3R$$

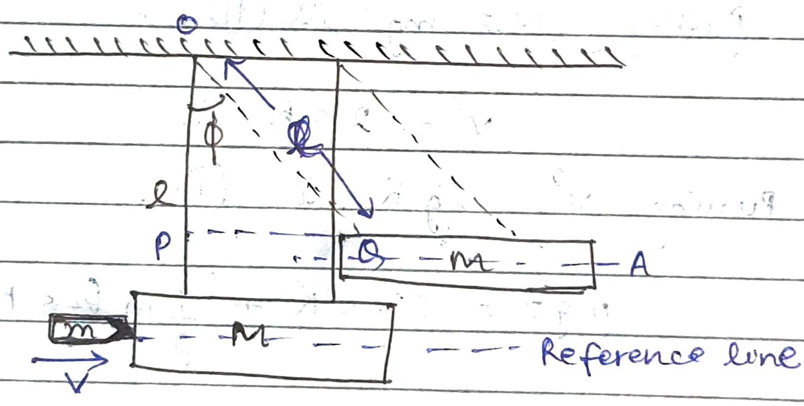
Therefore, $z = 3R$ should be the initial height such that mass pushes the top of track with a force equal to its weight.

Name: Archit Agrawal
Student ID: 202052307
Group: A1

2. A simple way to measure the speed of a bullet is with a ballistic pendulum. As illustrated, this consists of a wooden block of ~~block~~ mass M into which the bullet is shot. The block is suspended from cables of length l, and the impact of bullet causes it to swing through a maximum angle ϕ, as shown. The initial speed of the bullet is v, and its mass is m.

a) How fast is the block moving immediately after the bullet comes to rest? (Assume that this happens quickly)

b) Show how to find velocity of bullet by measuring m, M, l and ϕ.



a) The initial momentum of system is

$$P_i = mv$$

just after the bullet collides with block it comes to rest inside it, let the velocity of mass and bullet at this instant be v'.

$\therefore$ momentum just after the bullet stops

$$P_f = (M+m)v'$$

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Student ID: 202052307
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Since, there is no external force, net momentum will be conserved.

$$\therefore \quad P_i = P_f$$

$$mv = (m+M)v'$$

$$v' = \frac{mv}{m+M}$$

Hence, the velocity immediately after the bullet comes to rest is $\frac{mv}{m+M}$.

b) Energy of the system just after the bullet stops is,

$$E_i = (KE) + (GPE)$$

$$= \frac{1}{2}(m+M)v'^2 + (0)$$

$$= \frac{1}{2}(m+M)\left(\frac{mv}{m+M}\right)^2$$

$$E_i = \frac{m^2v^2}{2(m+M)}$$

Due to impact of the bullet, the bullet-block system swings through maximum angle ϕ and stops at that instant. Let this position be A. velocity at A = 0 m/s.

Height ~~Distance~~ of position A w.r.t reference line is $(l - OP)$.

From $\triangle OPQ$,

$$\cos\phi = \frac{OP}{l} \Rightarrow OP = l\cos\phi$$

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Height of position A w.r.t. reference line is $l(1-\cos\phi)$.

Energy of the system at A,

$$E_f = (KE) + (GPE)$$
$$= 0 + (m+M)gl(1-\cos\phi)$$

~~Since~~, Using conservation of energy,

$$E_i = E_f$$
$$\frac{m^2v^2}{2(m+M)} = (m+M)gl(1-\cos\phi)$$

$$v^2 = \frac{2(m+M)^2}{m^2}gl(1-\cos\phi)$$
$$\therefore v = \left(\frac{m+M}{m}\right)\sqrt{2gl(1-\cos\phi)}$$

Hence, the velocity v of bullet is given in terms of m, M, ϕ and l as $v = \left(\frac{m+M}{m}\right)\sqrt{2gl(1-\cos\phi)}$

3. Mass m whirls on a frictionless table, held to circular motion by a string which passes through a hole in the table. The string is slowly pulled through the hole so that the radius of circle changes from l_1 to l_2. Show that the work done in pulling the string equals increase in kinetic energy of the mass.

Name : Archit Agrawal
Student ID : 20205 2307
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For circular motion in circle of radius l_1, radial force F_r is

$$F_r = \frac{m v_1^2}{l_1} = m_1 l_1 \omega_1^2 \qquad \left(\because \omega_1 = \frac{v_1}{l_1}\right)$$

Now if F_r is increased, m will move to smaller r

$$F_r(r) = \frac{m\, v^2(r)}{r} = m r \omega^2(r)$$

acceleration in circular motion is given by

$$\vec{a} = (\ddot{r} - r\dot{\theta}^2)\,\hat{r} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\,\hat{\theta}$$

Since, there is no tangential acceleration,

$$r\ddot{\theta} + 2\dot{r}\dot{\theta} = 0$$

$$r\dot{\omega} + 2\dot{r}\omega = 0$$

$$\frac{1}{\omega}\frac{d\omega}{dt} = \frac{-2}{r}\frac{dr}{dt}$$

$$\int_{\omega_1}^{\omega(r)} \frac{d\omega}{\omega} = -2\int_{l_1}^{r} \frac{dr}{r}$$

$$\ln\left(\frac{\omega(r)}{\omega_1}\right) = -2\,\ln\left(\frac{r}{l_1}\right)$$

$$\ln\left(\frac{\omega(r)}{\omega_1}\right) = \ln\left(\frac{l_1^2}{r^2}\right)$$

$$\omega(r) = \omega_1 \frac{l_1^2}{r^2} \qquad (1)$$

Now, work done can be calculated as

$$W = -\int_{l_1}^{l_2} F_r(r)\, dr = -\int_{l_1}^{l_2} m r \omega^2(r)\, dr$$

Name: Archit Agrawal
Student ID: 2020052307
Group: A1

$$W = -\int_{l_1}^{l_2} \frac{m l_1^4 \omega_1^2}{r^3}\, dr$$

$$W = \frac{1}{2} m l_1^4 \omega_1^2 \left[\frac{1}{r^2}\right]_{l_1}^{l_2}$$

$$W = \frac{1}{2} m l_1^4 \omega_1^2 \left(\frac{1}{l_2^2} - \frac{1}{l_1^2}\right)$$

Since, there is not external torque, angular momentum will remain conserved,

$$\therefore \cancel{m} l_1^2 \omega_1 = \cancel{m} l_2^2 \omega_2$$

$$l_1^2 \omega_1 = l_2^2 \omega_2 \quad \text{(11)}$$

The work done W, can be written as,

$$W = \frac{1}{2} m \left(\frac{l_1^4 \omega_1^2}{l_2^2} - l_1^2 \omega_1^2\right)$$

$$W = \frac{1}{2} m \left(\frac{l_2^4 \omega_2^2}{l_2^2} - l_1^2 \omega_1^2\right) \quad \{\text{from (11)}\}$$

$$W = \frac{1}{2} m \left(l_2^2 \omega_2^2 - l_1^2 \omega_1^2\right)$$

$$W = K_f - K_i$$

Hence, it is proved that the work done in pulling the string equals the increase in kinetic energy of mass.

4. A simple and very violent chemical reaction is

$$H + H \longrightarrow H_2 + 5eV$$

Name: Archit Agrawal
Student ID: 202052307
Group: A 1

However, when hydrogen atoms collide in free space, they simply bounce apart! The reason is that it is impossible to satisfy the laws of conservation of momentum and energy in a simple two body collision which releases energy? Can you prove this.

Let us assume that mass of hydrogen atom is m. Two H-atoms moving in similar direction with velocity $\vec{v_1}$ and $\vec{v_2}$ and a collision takes place between them and they combine to form H_2, which now moves with velocity v.

Applying law of conservation of momentum gives

$$m\vec{v_1} + m\vec{v_2} = 2m\vec{v}$$

$$\vec{v} = \frac{\vec{v_1} + \vec{v_2}}{2} \quad \text{--- (1)}$$

Applying law of conservation of energy,

$$\frac{1}{2} m v_1^2 + \frac{1}{2} m v_2^2 = \frac{1}{2}(2m) v^2 - Q$$

where $Q = 5\,eV$. Since Q energy is released it is taken as negative.

$$v_1^2 + v_2^2 = 2v^2 - \frac{2Q}{m}$$

Putting $v = \frac{v_1 + v_2}{2}$ from (1)

Name : Archit Agrawal
Student ID : 20105 2307
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$$v_1^2 + v_2^2 = 2\left(\frac{v_1+v_2}{2}\right)^2 - \frac{2Q}{m}$$

$$v_1^2 + v_2^2 = \frac{v_1^2 + v_2^2 + 2v_1v_2}{2} - \frac{2Q}{m}$$

$$\frac{2Q}{m} = \frac{-v_1^2 - v_2^2 + 2v_1v_2}{2}$$

$$\frac{Q}{m} = \frac{-(v_1-v_2)^2}{4}$$

Since, $Q = 5eV$ and m is always positive.
$\frac{Q}{m} > 0$..

But $-\frac{(v_1-v_2)^2}{4}$ is always negative

Hence, LHS $\neq$ RHS

Hence, the law of conservation of energy and momentum do not hold simultaneously in a simple two body collision where energy is released.

5. A commonly used potential energy function to describe the interaction between two atoms is the Lennard-Jones potential :

$$U = \epsilon\left[\left(\frac{r_0}{r}\right)^{12} - 2\left(\frac{r_0}{r}\right)^{6}\right]$$

a) Show that the radius at the potential minimum is r_0, and that the depth of potential well is ϵ.

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b) Find the frequency of small oscillations about equilibrium for 2 identical atoms of mass m bound to each other by Lennard-Jones interaction.

a) The potential will be minimum at r where $\frac{dU}{dr} = 0$ and $\frac{d^2U}{dr^2} > 0$.

$$\therefore \quad \frac{dU}{dr} = \epsilon \left[-12 \frac{r_0^{12}}{r^{13}} + 12 \frac{r_0^6}{r^7} \right]$$

Putting $\frac{dU}{dr} = 0$

$$\epsilon \left[-12 \frac{r_0^{12}}{r^{13}} + 12 \frac{r_0^6}{r^7} \right] = 0$$

$$\frac{r_0^{12}}{r^{13}} = \frac{r_0^6}{r^7}$$

$$\boxed{r = r_0}$$

Now, $$\frac{d^2U}{dr^2} = \epsilon \left[168 \frac{r_0^{12}}{r^{14}} - 96 \frac{r_0^6}{r^8} \right]$$

at $r = r_0$

$$\frac{d^2U}{dr^2} = \frac{\epsilon}{r_0^2}(72) = \frac{72\epsilon}{r_0^2} > 0$$

Hence, the radius at potential minimum is r_0.

The depth of potential well at $r = r_0$ will be the potential energy at $r = r_0$.

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$$\therefore U_{r_0} = \epsilon\left[\left(\frac{r_0}{r_0}\right)^{12} - 2\left(\frac{r_0}{r_0}\right)^{6}\right]$$

$$U_{r_0} = -\epsilon$$

Hence, the depth of potential well will be ϵ joules.

b) the frequency of small oscillations about equilibrium is given by

$$\omega = \sqrt{\frac{K}{\mu}}$$

where $K = \left.\frac{d^2U}{dr^2}\right|_{r=r_0}$ and μ is reduced mass of two atoms.

$\therefore$ reduced mass $\mu = \frac{m_1 m_2}{m_1 + m_2} = \frac{m \times m}{m+m}$

$$= \frac{m}{2}$$

and $K = \left.\frac{d^2U}{dr^2}\right|_{at\ r=r_0}$

$$= \frac{72\epsilon}{r_0^2}$$ $\left\{\text{calculated in part a}\right\}$

$\therefore$ frequency of small oscillations, ω,

$$\omega = \sqrt{\frac{\frac{72\epsilon}{r_0^2}}{\frac{m}{2}}} = \frac{12}{r_0}\sqrt{\frac{\epsilon}{m}}$$

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Hence frequency of small oscillations about equilibrium is $\frac{12}{r_0}\sqrt{\frac{C}{m}}$.

6. A particle of mass m moves in one dimension along positive x-axis. It is acted on by a constant force directed toward the origin with magnitude B, and an inverse square law repulsive force with magnitude $\frac{A}{x^2}$.

a) Find the potential energy function $U(x)$.
b) Sketch the energy diagram for the system when the maximum kinetic energy is $K_0 = \frac{1}{2}mV_0^2$.
c) Find the equilibrium position x_0.
d) what is frequency of small oscillations about x_0?

a) attractive force $F_a = -B\ \hat{i}$

$\therefore$ potential energy due to this force, $U_a(x)$, is

$$U_a(x) - U_a(0) = -\int_0^x F_a\, dx = \int_0^x B\, dx$$

$$= Bx$$

and, repulsive force, $F_r = \frac{A}{x^2}\ \hat{i}$

$\therefore$ potential energy due to this force, $U_r(x)$, is

$$U_r(x) - U_r(\infty) = -\int_\infty^x F_r\, dx = -\int_\infty^x \frac{A\, dx}{x^2}$$

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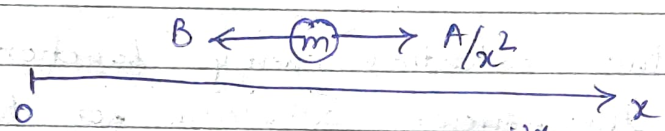
$$U_r(x) = \left[\frac{A}{x}\right]_{\infty}^{x}$$

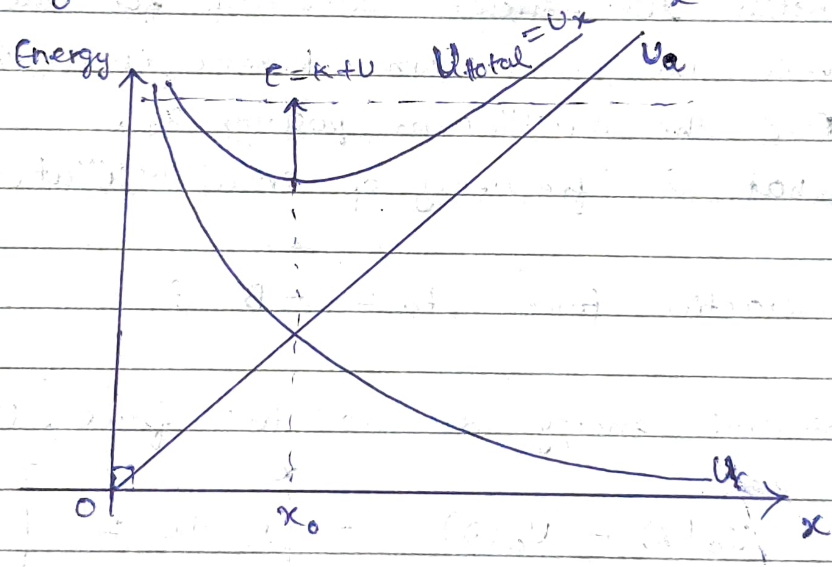
$$= \frac{A}{x}$$

$\therefore$ potential energy with respect to x, U_x is

$$U_x = Bx + \frac{A}{x}$$

b) The forces on m are represented below





c) At equilibrium position, net force on the mass is 0, let this be at $x = x_0$.

$\therefore$ $$F_{net} = B - \frac{A}{x^2}$$

$$B - \frac{A}{x_0^2} = 0$$

$$x_0 = \sqrt{\frac{A}{B}}$$

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Hence at $x_0 = \sqrt{\frac{A}{B}}$, the mass will be in equilibrium.

d) frequency of small oscillations about x_0, $\omega = \sqrt{\frac{k}{\mu}}$

where $k = \left.\frac{d^2U}{dx^2}\right|_{x=x_0}$ and μ is reduced mass.

Since, there is only one particle,

$\mu = m$

and, $U_x = Bx + \frac{A}{x}$

$\therefore \frac{dU_x}{dx} = B - \frac{A}{x^2}$

and, $\frac{d^2U_x}{dx^2} = \frac{2A}{x^3}$

$\therefore k = \left.\frac{d^2U_x}{dx^2}\right|_{at\ x=x_0} = \frac{2A}{x_0^3}$

$k = 2A\left(\frac{B}{A}\right)^{3/2} = \frac{2B^{3/2}}{A^{1/2}}$

$\therefore \omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{2B^{3/2}}{A^{1/2} \times m}}$

$= \sqrt{\frac{2}{m}}\left(\frac{B^3}{A}\right)^{1/4}$

Hence, frequency of small oscillation about x_0 is $\sqrt{\frac{2}{m}}\left(\frac{B^3}{A}\right)^{1/4}$.