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MA102: Introduction to Discrete Mathematics

Tutorial 9

1. Prove that

$$2 - 2.7 + 2.(7)^2 - \dots + 2.(-7)^n = \frac{1-(-7)^{n+1}}{4}$$

we can prove this by mathematical induction.

Let $P(n)$ be the ~~proposition~~ proposition:

$$2 - 2.7 + 2.(7)^2 - \dots + 2.(-7)^n = \frac{1-(-7)^{n+1}}{4}$$

BASIS STEP: $P(0)$ is true
the LHS at $n=0$ is 2.

and, the RHS is $\frac{1-(-7)^{0+1}}{4} = \frac{8}{4} = 2$

Hence, $P(0)$ is true.

INDUCTIVE STEP: we will assume $P(K)$ to be for arbitrary positive integer K and using it we will prove $P(K+1)$ is true.

$\therefore$ assumption is

$$2 - 2.7 + 2.(7)^2 - \dots + 2.(-7)^k = \frac{1-(-7)^{k+1}}{4} \quad (1)$$

is true.

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To prove

$$2 - 2.7 + 2.(7)^2 - \cdots + 2(-7)^k + 2(-7)^{k+1} = \frac{1-(-7)^{k+2}}{4}$$

Adding $2.(-7)^{k+1}$ to both sides of ①,

$$2 - 2.7 + 2.(7)^2 - \cdots + 2(-7)^k + 2(-7)^{k+1} = \frac{1-(-7)^{k+1}}{4} + 2(-7)^{k+1}$$

$$= \frac{1-(-7)^{k+1} + 8(-7)^{k+1}}{4}$$

$$= \frac{1+7(-7)^{k+1}}{4}$$

$$= \frac{1-(-7)(-7)^{k+1}}{4}$$

$$\boxed{2 - 2.7 + 2.(7)^2 - \cdots + 2(-7)^k + 2(-7)^{k+1} = \frac{1-(-7)^{k+2}}{4}}$$

Hence, proved.

2. Prove that n^2-1 is divisible by 8 whenever n is an odd positive integer.

Rewriting this proposition for all positive integers. Since, an odd integer can be

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written as $2n-1$ for $\forall n \in N$.

Therefore, the proposition is $P(n)$:

$(2n-1)^2 - 1$ is divisible by 8.

We can prove this using Principle of mathematical Induction.

BASIS STEP: for $n=1$, $(2n-1)^2 - 1 = 0$

$\because$ 0 is divisible by 8.
$\therefore$ $P(1)$ is true.

INDUCTIVE STEP -: Assuming $P(K)$ is true
i.e. $\{(2K-1)^2 - 1\}$ is divisible by 8.

To prove: $[\{2(K+1) - 1\}^2 - 1]$ is divisible by 8

Let us calculate,

$$[\{2(k+1) - 1\}^2 - 1] - [(2k-1)^2 - 1]$$

$$= 4K^2 + 4K - 4K^2 + 4K$$

$$= 8K$$

Since, $P(K)$ is true (assumption) and the difference between $[\{2(K+1) - 1\}^2 - 1]$ and $[(2K-1)^2 - 1]$ is divisible by 8 (8K).
Hence, $P(K+1)$ is true and our proof is complete

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3. Which amounts of money can be formed using notes of ₹2 and ₹5? Prove your answer using strong induction?

$2 = 2 \cdot 1 + 5 \cdot 0$
$4 = 2 \cdot 2 + 5 \cdot 0$
$5 = 2 \cdot 0 + 5 \cdot 1$
$6 = 2 \cdot 3 + 5 \cdot 0$
$7 = 2 \cdot 1 + 5 \cdot 1$
$8 = 2 \cdot 4 + 5 \cdot 0$
$9 = 2 \cdot 2 + 5 \cdot 1$
$10 = 2 \cdot 0 + 5 \cdot 2$
⋮

As we can see every amount greater than or equal to 4 can be made using ₹2 and ₹5 notes.

Let $P(n)$ be the statement that n amount of money can be ~~used~~ made using ₹2 and ₹5 notes. i.e. $n = 2x + 5y$ with x and y being non-negative integers.

BASE STEP: Checking if $P(4)$ and $P(5)$ are correct -

$4 = 2 \cdot 2 + 5 \cdot 0$
$5 = 2 \cdot 0 + 5 \cdot 1$

Hence, $P(4)$ and $P(5)$ are true.

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INDUCTIVE STEP :- Let $P(j)$ be true $\forall j$ such that $4 \le j \le k$.

Now, we need to prove that $P(k+1)$ is true.

$(k+1)$ can be written as

$$k+1 = (k-1) + 2 \quad \text{— (1)}$$

Since, $P(k-1)$ is true {inductive hypothesis}
$(k-1)$ can be written as,

$$(k-1) = 2a + 5b$$

for some a, b (non-negative integers).

$$\therefore \; k+1 = 2a + 5b + 2$$
$$k+1 = 2(a+1) + 5b$$

Now, since a is integer, $(a+1)$ is also an integer and hence there exists x and y such that

$$(k+1) = 2x + 5y$$

$\therefore$ $P(k+1)$ is true.

Hence, our proof is complete.

All amounts of money greater than or equal to 4 and 2 can be made using notes of ₹2 and ₹5.

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4. Let $p_0 = 1$, $p_1 = \cos\theta$ (for θ some fixed constant) and $p_{n+1} = 2p_1 p_n - p_{n-1}$ for $n \geq 1$. Prove that $p_n = \cos(n\theta)$ for $n \geq 0$

For any $n \geq 0$ let $P(n)$ be the statement that $p_n = \cos(n\theta)$

We can prove that $P(n)$ is true $\forall n \in N$ using strong form of mathematical induction.

Base Case: P_0 is the statement that $p_0 = \cos(0 \times \theta)$, i.e. $p_0 = 1$, which is true.
The statement P_1 says that $p_1 = \cos(1 \times \theta) = \cos\theta$, which is also true.

Inductive Step: Assume $P_0, P_1, \ldots, P_K$ be true.

i.e.

$$p_0 = \cos(0 \times \theta)$$
$$p_1 = \cos(1 \times \theta)$$
$$\vdots$$
$$p_K = \cos(K\theta)$$

Now, we have to prove that $p_{K+1} = \cos\{(K+1)\theta\}$ using these assumptions.

$$\cancel{p_{K+1} = 2(\cos\theta)(\cos(K\theta)) - \cos}$$

$$p_{K+1} = 2p_1 p_K - p_{K-1}$$

$$p_{K+1} = 2(\cos\theta)\cos(K\theta) - \cos\{(K-1)\theta\}$$

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$$p_{k+1} = 2\cos\theta \cdot \cos(k\theta) - \cos(k\theta - \theta)$$

Using $\cos(a-b) = \cos a \cdot \cos b + \sin a \cdot \sin b$

$$p_{k+1} = 2\cos\theta \cdot \cos(k\theta) - (\cos(k\theta)\cos\theta + \sin(k\theta)\sin\theta)$$

$$p_{k+1} = \cos\theta \cdot \cos(k\theta) - \sin(k\theta) \cdot \sin\theta$$

Using $\cos(a+b) = \cos a \cdot \cos b - \sin a \cdot \sin b$

$$\therefore \quad p_{k+1} = \cos(k\theta + \theta)$$

$$\boxed{p_{k+1} = \cos\{(k+1)\theta\}}$$

Hence, proved.

5. Prove that $f_1 + f_3 + \cdots + f_{2n-1} = f_{2n}$ where $n \in \mathbb{N}$ and f_i is i^{th} Fibonacci number.

The Fibonacci series is given by

$$f_1 = 1$$
$$f_2 = 1$$
$$f_n = f_{n-1} + f_{n-2} \quad \forall n \geq 3$$

Let $P(n)$ be the statement that

$$f_1 + f_3 + \cdots + f_{2n-1} = f_{2n}$$

$\forall n \in \mathbb{N}$.

This can be proved using mathematical induction.

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Base Step: Proving P(1) is true.

P(1) $\Rightarrow$

$$f_1 = f_{2\times 1}$$
$$f_1 = f_2 = 1$$

Hence, P(1) is true.

Inductive Step: Let P(K) be true for some arbitrary positive integer K.

i.e. $f_1 + f_3 + f_5 + \cdots + f_{2K-1} = f_{2K}$ be true.

Now, we need to prove that P(K+1) is true.

$$f_1 + f_3 + \cdots + f_{2K-1} + f_{2(K+1)-1} = f_{2(K+1)}$$

Using our assumption

$$f_1 + f_3 + \cdots + f_{2K-1} = f_{2K}$$

Adding f_{2K+1} on both sides,

$$f_1 + f_3 + \cdots + f_{2K-1} + f_{2K+1} = f_{2K} + f_{2K+1}$$

By definition of Fibonacci sequence, RHS of above equation is f_{2K+2}

$$\therefore\ f_1 + f_3 + \cdots + f_{2K-1} + f_{2(K+1)-1} = f_{2(K+1)}$$

Hence, P(K+1) is true.

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Hence, $f_1 + f_3 + \cdots + f_{2n-1} = f_{2n}$ is true $\forall n \in \mathbb{N}$ where f_i are terms of Fibonacci sequence.

6. Show that $f_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right]$

for each $n \geq 1$, where f_n is n^{th} Fibonacci number.

The Fibonacci series is given by

$$f_1 = 1$$
$$f_2 = 1$$
$$f_n = f_{n-1} + f_{n-2} \qquad \forall n \geq 3$$

Let for $n \geq 1$, $P(n)$ be the statement that

$$f_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right]$$

We can prove the statement using strong form of mathematical induction,

Base Case :- The statement P_1 is

$$f_1 = \frac{1}{\sqrt{5}}\left[\frac{1+\sqrt{5}}{2} - \frac{1-\sqrt{5}}{2}\right]$$

$$f_1 = \frac{1}{\sqrt{5}} \cdot \frac{2\sqrt{5}}{2} = 1$$

Hence, $P(1)$ is true.

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The statement P(2) is,

$$f_2 = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^2 - \left(\frac{1-\sqrt{5}}{2}\right)^2\right]$$

$$f_2 = \frac{1}{\sqrt{5}}\left[\frac{1+5+2\sqrt{5}-1-5+2\sqrt{5}}{4}\right]$$

$$f_2 = 1$$

Hence, P(2) is also true.

Inductive Step: Assume P(1), P(2) --- P(K) holds true for some positive integer K.

Now, we have to prove that P(K+1) is true, i.e.

$$f_{K+1} = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{K+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{K+1}\right]$$

Using the definition of Fibonacci Sequence,

$$f_{K+1} = f_K + f_{K-1}$$

Since, we have assumed P(K) and P(K-1) are true.

$$\therefore f_{K+1} = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^K - \left(\frac{1-\sqrt{5}}{2}\right)^K\right] + \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{K-1} - \left(\frac{1-\sqrt{5}}{2}\right)^{K-1}\right]$$

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$$f_{k+1} = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{k} + \left(\frac{1+\sqrt{5}}{2}\right)^{k-1} - \left(\frac{1-\sqrt{5}}{2}\right)^{k} - \left(\frac{1-\sqrt{5}}{2}\right)^{k-1}\right]$$

$$= \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{k-1}\left(1+\frac{1+\sqrt{5}}{2}\right) - \left(\frac{1-\sqrt{5}}{2}\right)^{k-1}\left(1+\frac{1-\sqrt{5}}{2}\right)\right]$$

$$= \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{k-1}\left(\frac{3+\sqrt{5}}{2}\right) - \left(\frac{1-\sqrt{5}}{2}\right)^{k-1}\left(\frac{3-\sqrt{5}}{2}\right)\right]$$

$$= \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{k-1}\left(\frac{6+2\sqrt{5}}{4}\right) - \left(\frac{1-\sqrt{5}}{2}\right)^{k-1}\left(\frac{6-2\sqrt{5}}{4}\right)\right]$$

$$= \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{k-1}\left(\frac{1+2\sqrt{5}+5}{2\times 2}\right) - \left(\frac{1-\sqrt{5}}{2}\right)^{k-1}\left(\frac{1-2\sqrt{5}+5}{2\times 2}\right)\right]$$

$$= \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{k-1}\left(\frac{1+\sqrt{5}}{2}\right)^{2} - \left(\frac{1-\sqrt{5}}{2}\right)^{k-1}\left(\frac{1-\sqrt{5}}{2}\right)^{2}\right]$$

$$f_{k+1} = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{k+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{k+1}\right]$$

Hence, $P(k+1)$ is proved.

Thus, by strong form of induction, the statement $P(n)$ is proved $\forall\, n \in N$.

7. Give recursive definition of following functions:

(i) $f(n) = 2n+1 \quad \forall n \in N$

(ii) $f(n) = 10^{n} \quad \forall n \in N$

(iii) $f(n) = 5 \quad \forall n \in N$

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(a) $f(1) = 3$
$f(2) = 5$
$f(3) = 7$
$\vdots$
$f(n) = 2n+1$

$\therefore$ recursive definition :

$f(1) = 3$ and $f(n) = f(n-1)+2 \quad \forall n \geq 2$

(b) $f(1) = 10$
$f(2) = 100$
$f(3) = 1000$
$\vdots$
$f(n) = 10^n$

$\therefore$ recursive definition :

$f(1) = 10$ and $f(n) = 10\, f(n-1) \quad \forall n \geq 2$

(c) $f(1) = 5$
$f(2) = 5$
$\vdots$
$f(n) = 5$

$\therefore$ recursive definition :

$f(1) = 5$ and $f(n) = f(n-1) \quad \forall n \geq 2.$

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Q. Give the recursive definition of the following sets:

(i) the set of natural numbers congruent to 2 modulo 3.

The numbers that are congruent to 2 modulo 3 are 2, 5, 8, ...

∴ recursive definition of the set is:

$2 \in S$ and if $x \in S$, then $x+3 \in S$.

(ii) the set of natural numbers not divisible by 5

The ~~fte~~ natural numbers not divisible by 5 are 1, 2, 3, 4, 6, 7, 8, 9, ...

∴ recursive definition of the set is:

$1, 2, 3, 4 \in S$ and if $x \in S$, then $x+5 \in S$.

(iii) the set of polynomials in one variable with integer coefficients

Let S be the required set.

$$S = \{a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 : a_i \in \mathbb{Z}\}$$

recursive definition of this set can be given as:

$1, -1 \in S$, if $a, b \in S$, then $(a+b \in S$ and $a \cdot x \in S)$

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(iv) $\{(a,b) \in \mathbb{N} \times \mathbb{N} \mid (a+b) \text{ is odd}\}$

Let S be the required set.
The recursive definition for the set is

$(1,2), (2,1) \in S$
if $(a,b) \in S$, then $(a+2,b) \in S$ and $(a, b+2) \in S$

9. A flagpole is n feet tall. On this pole we display flags of the following types:

red flags that are 1 feet tall
green flags that are 2 feet tall.
blue flags that are 2 feet tall.

The sum of heights of the flags is exactly n feet. Let f_n be the number of ways to display these flags on an n feet tall flagpole.

Use induction to show that,

$$f_n = \frac{2}{3} 2^n + \frac{1}{3}(-1)^n \quad \forall\, n \geq 1$$

Let us first calculate $f(1)$, $f(2)$ and $f(3)$.

Clearly, $f(1) = 1$, since only red flag is 1 foot tall.

Now, $n=2$, that is, either a blue flag or a green flag or two red flags

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can be displayed.

$\therefore f(2) = 3$

Now, for $n = 3$, the possible ways are:-

→ three red flags → 1 way
→ one green and one red → 2 ways
→ One blue and one red → 2 ways

that is, a total of 5 ways are there.

$\therefore f(3) = 5$.

Now, let us start our prove by induction.

~~BASE STEP~~ :- Let $P(n)$ be the statement that

$$f_n = \frac{2}{3} 2^n + \frac{1}{3}(-1)^n \quad \forall n \geq 1$$

BASE STEP: check if $P(1)$, $P(2)$, $P(3)$ is true

$$\therefore f_1 = \frac{2}{3}(2) + \frac{1}{3}(-1) = \frac{3}{3} = 1$$

$$f_2 = \frac{2}{3}(4) + \frac{1}{3} = \frac{9}{3} = 3$$

$$f_3 = \frac{2}{3}(8) + \frac{1}{3}(-1) = \frac{15}{3} = 5$$

Hence, $P(1)$, $P(2)$ and $P(3)$ is true.

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INDUCTIVE ~~HYP~~ STEP: Assume that $P(1), P(2) \ldots P(k)$ holds true.

Now, we have to prove that $P(k+1)$ is true.

i.e. $f_{k+1} = \frac{2}{3}(2)^{k+1} + \frac{1}{3}(-1)^{k+1}$.

Let there be a flag pole of height $(k+1)$ feet.

If, let us say, the top flag is a red flag (1 foot height), then the remaining pole can be displayed in f_k ways.

Similarly, if the top flag is blue (or green) (height 2 feet), then the remaining pole can be displayed in f_{k-1} ways.

$\therefore$ total ways to arrange (or display) flags on the pole of height $(k+1)$ is,

$$f_{k+1} = f_k + 2f_{k-1}$$

$$= \frac{2}{3}2^k + \frac{1}{3}(-1)^k + 2\left(\frac{2}{3}2^{k-1} + \frac{1}{3}(-1)^{k-1}\right)$$

— By inductive hypothesis

$$f_{k+1} = 2 \cdot \frac{2}{3}(2^k) + \frac{1}{3}(-1)^{k-1}(-1+2)$$

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$$f_{k+1} = \frac{2}{3} 2^{k+1} + \frac{1}{3} (-1)^{k-1} (-1)^2$$

$$f_{k+1} = \frac{2}{3} 2^{k+1} + \frac{1}{3} (-1)^{k+1}$$

Hence, $P(k+1)$ is true and our proof is complete.

10. Find a formula for

$$\frac{1}{1\cdot2} + \frac{1}{2\cdot3} + \cdots + \frac{1}{n\cdot(n+1)}$$

by examining the values of the expression for small values of n. Prove the formula you conjectured.

Let f_n be the sum of first n terms of the series

$$\frac{1}{1\cdot2} + \frac{1}{2\cdot3} + \cdots + \frac{1}{n\cdot(n+1)}$$

$$\therefore \quad f(1) = \frac{1}{1\cdot2} = \frac{1}{2}$$

$$f(2) = \frac{1}{1\cdot2} + \frac{1}{2\cdot3} = \frac{1}{2} + \frac{1}{6} = \frac{4}{6} = \frac{2}{3}$$

$$f(3) = \frac{1}{1\cdot2} + \frac{1}{2\cdot3} + \frac{1}{3\cdot4} = \frac{1}{2} + \frac{1}{6} + \frac{1}{12} = \frac{3}{4}$$

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$$f(4) = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \frac{1}{4\cdot 5}$$

$$= \frac{30+10+5+3}{60}$$

$$= \frac{48}{60}$$

$$= \frac{4}{5}$$

observing the values of $f(1)$, $f(2)$, $f(3)$ and $f(4)$, it can be said that

$$f(n) = \frac{n}{n+1}$$

Now, let us prove this conjecture, using induction.

Let $P(n)$ be the statement that

$$f_n = \frac{n}{n+1}$$

where f_n is sum of first n terms of the given series.

BASE STEP: Check if $P(1)$ is true

$$f_1 = \frac{1}{1\cdot 2} = \frac{1}{2}$$

and for $n=1$, $\frac{n}{n+1} = \frac{1}{2}$

Hence, $P(1)$ is true.

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INDUCTIVE STEP :- Assume $P(K)$ to be true.

i.e. $$\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \cdots + \frac{1}{K\cdot(K+1)} = \frac{K}{K+1} \quad \text{--- } (1)$$

Now, we have to prove that $P(K+1)$ is true, i.e.

$$\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \cdots + \frac{1}{(K+1)(K+2)} = \frac{K+1}{K+2}$$

Adding $\frac{1}{(K+1)(K+2)}$ to both sides of (1),

$$\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \cdots + \frac{1}{K(K+1)} + \frac{1}{(K+1)(K+2)} = \frac{K}{K+1} + \frac{1}{(K+1)(K+2)}$$

$$f_{K+1} = \frac{1}{(K+1)}\left(K + \frac{1}{K+2}\right)$$

$$= \frac{K^2+2K+1}{(K+1)(K+2)}$$

$$= \frac{(K+1)^{\cancel{2}}}{\cancel{(K+1)}(K+2)}$$

$$f_{K+1} = \frac{K+1}{K+2}$$

Hence, $P(K+1)$ is true and our proof is complete.