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MA102: Introduction to Discrete Mathematics

Tutorial 5

1. Use rules of inference to show that if $\forall x (P(x) \to (Q(x) \land S(x)))$ and $\forall x (P(x) \land R(x))$ are true then $\forall x (R(x) \land S(x))$ is true.

Step		Rule of Inference
1.	$\forall x (P(x) \to (Q(x) \land S(x)))$	Premise
2.	$\forall x (P(x) \land R(x))$	Premise
3.	$P(c) \to (Q(c) \land S(c))$	Universal Instantiation from 1
4.	$P(c) \land R(c)$	Universal Instantiation from 2
5.	$R(c)$	Simplification from 4
6.	$P(c)$	Simplification from 4
7.	$Q(c) \land S(c)$	Modus Ponens from 3 and 6
8.	$S(c)$	Simplification from 7.
9.	$R(c) \land S(c)$	Conjunction from 5 and 8
10.	$\forall x (R(x) \land S(x))$	Universal Generalization from 9.

2. From following set of premises, what relevant conclusion or conclusions can be drawn? Explain the rules of inference used to ~~ex~~ obtain each conclusion from the premises.

(i) a) "Every student has an Internet Account."
b) "Homer does not have an Internet Account"
(c) "Maggie has an internet account"

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Let $P(x)$: "x is a student"
and $Q(x)$: " x has an internet account".

∴ Premise (a) : $\forall x \, (P(x) \rightarrow Q(x))$
Premise (b) : $\neg Q(\text{Homer})$
Premise (c) : $Q(\text{Maggie})$

Steps	Rules of Inference
1. $\forall x \, (P(x) \rightarrow Q(x))$	Premise
2. $P(\text{Homer}) \rightarrow Q(\text{Homer})$	Universal Instantiation from 1
3. $\neg Q(\text{Homer})$	Premise
4. $\neg P(\text{Homer})$	Modus Tollens from 2 and 3

Hence, we can conclude that "Homer is not a student".

(b)(a) All foods that are healthy to eat do not taste good.

(b) Tofu is healthy to eat
(c) You only eat what tastes good.
(d) You do not eat tofu.
(e) Cheeseburgers are not healthy to eat.

Let $P(x)$ = " x is healthy to eat"
$T(x)$ = " x tastes good"
$R(x)$ = " You eat x "

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Steps	Rules of Inference
1. $\forall x(P(x) \to \neg T(x))$	Premise (a)
2. $P(Tofu)$	Premise (b)
3. $\forall x(R(x) \to T(x))$	Premise (c)
4. $\neg R(Tofu)$	Premise (d)
5. $\neg P(Cheeseburgers)$	Premise (e)
6. $P(Tofu) \to \neg T(Tofu)$	Universal Instantiation from (1)
7. $\neg T(Tofu)$	Modus Ponens from 2 and 6

Hence, we can conclude that "Tofu does not taste good".

3. Use resolution to show the hypotheses "Allen is a bad boy or Hillary is a good girl" and "Allen is a good boy or David is happy" imply the conclusion "Hillary is a good girl or David is happy".

Let p = "Allen is a bad boy"
q = "Hillary is a good girl"
r = "David is happy"

$\sim p$ = "Allen is a good boy".

Steps	Rules of Inference
1. $p \vee q$	Premise
2. $\neg p \vee r$	premise
3. $q \vee r$	Resolution from 1 and 2

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Resolution is based on the tautology that
$$((p \vee q) \wedge (\neg p \vee r)) \rightarrow (q \vee r)$$

Hence, the conclusion is "Hillary is a good girl or David is happy".

4. What is wrong with this argument? Let $H(x)$ be "x is happy". Given the premise $\exists x\, H(x)$, we conclude that $H(\text{Lola})$. Therefore, Lola is happy.

$\exists x\, H(x)$ is premise

Hence, assuming premise to be true, we have the statement that "there is at least one person who is happy" but we cannot specify that person to be "Lola".
There might be some other person, say "Bunny" who is happy, then also the premise will be true. Hence, the problem in this argument is that we cannot specify a particular person to be happy.

5. Find DNF, CNF, PDNF, PCNF of the following formulae

(a) $Q \wedge (P \vee \neg Q)$
(b) $P \rightarrow (P \wedge (Q \rightarrow P))$
(c) $(Q \rightarrow P) \wedge (\neg P \wedge Q)$

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(a) $Q \wedge (P \vee \neg Q)$

P	Q	$P \vee \neg Q$	$Q \wedge (P \vee \neg Q)$	
T	T	T	T	minterm
T	F	T	F	maxterm
F	T	F	F	maxterm
F	F	T	F	maxterm

CNF :- $(Q) \wedge (P \vee \neg Q)$ is already a product of sum, hence it is CNF form.

PCNF :- $(\neg P \vee Q) \wedge (P \vee \neg Q) \wedge (P \wedge Q)$

$Q \wedge (P \vee \neg Q) \equiv (Q \wedge P) \vee (Q \wedge \neg Q)$ (distribution of ~~disj~~ conjuction over conjunction)

$\equiv (P \wedge Q)$ (as $Q \wedge \neg Q \equiv F$)

$\therefore$ $P \wedge Q$ is in product of sums form.

DNF : $P \wedge Q$

PDNF : $P \wedge Q$

(b) $P \to (P \wedge (Q \to P))$

P	Q	$Q \to P$	$P \wedge (Q \to P)$	$P \to (P \wedge (Q \to P))$
T	T	T	T	T
T	F	T	T	T
F	T	F	F	T
F	F	T	F	T

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Hence, $P \to (P \wedge (Q \to P))$ is a tautology.

$\therefore$ CNF :- $P \vee \neg P \equiv (P) \vee (\neg P)$
DNF :- $P \vee \neg P \equiv (P \vee \neg P)$

and PCNF :- tautology (or 1, or T)

PDNF :- $(P \wedge Q) \vee (P \wedge \neg Q) \vee (\neg P \wedge Q) \vee (\neg P \wedge \neg Q)$

(c) $(Q \to P) \wedge (\neg P \wedge Q)$

P	Q	$Q \to P$	$\neg P \wedge Q$	$(Q \to P) \wedge (\neg P \wedge Q)$
T	T	T	F	F
T	F	T	F	F
F	T	F	T	F
F	F	T	F	F

Hence, $(Q \to P) \wedge (\neg P \wedge Q)$ is a contradiction.

$\therefore$ CNF : $P \wedge \neg P \equiv (P) \wedge (\neg P)$
DNF : $P \wedge \neg P \equiv (P \wedge \neg P)$

and PDNF : contradiction (0 or F)

PCNF : $(\neg P \vee \neg Q) \wedge (\neg P \wedge Q) \wedge (P \wedge \neg Q) \wedge (P \wedge Q)$

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6. Prove that if x is irrational then $\frac{1}{x}$ is also irrational. Identify the method of your proof.

Let us assume that the hypothesis is true i.e. x is irrational but the conclusion to be false i.e. $\frac{1}{x}$ to be rational.

$\therefore$ There exists integer p and q such that $q \neq 0$ and $\frac{1}{x} = \frac{p}{q}$.

Since x is irrational, $x \neq 0$, as 0 is a rational number.

$$\because \frac{1}{x} = \frac{p}{q}$$

$$q = px$$

As $q \neq 0$, $x \neq 0$, we can conclude that $p \neq 0$

$$\because \frac{1}{x} = \frac{p}{q} \Rightarrow x = \frac{q}{p}$$

As x can be written in the form of q/p and $p \neq 0$, it implies that x is rational. But we assumed that x is irrational. Hence, we ~~reached~~ obtained a contradiction and then only the assumption "$\frac{1}{x}$ is a rational number" is false, thus $\frac{1}{x}$ is an

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Irrational number.

The method of proof is Proof by Contradiction.

7. Prove that if n is an integer and 3n+2 is even, then n is even using
a) proof by contraposition:
b) a proof by contradiction.

(a) p: 3n+2 is even
q: n is even
n has the domain of discourse of integers
we have to prove $p \to q$.
In contraposition, we assum $\neg q$ to be true and prove $\neg q \to \neg p$.

$\therefore$ $\neg q$ is "It is not the case that n is even".
There fore n is odd. There fore, there exists an integer k, such that

$$n = 2k+1$$

Multiplying by 3 on both sides

$$3n = 6k+3$$

adding 2 on both sides

$$3n+2 = 6k+5$$
$$3n+2 = 2(3k+2)+1$$

Since, k is integer 3k+2 is also an integer say q,

$\therefore \quad 3n+2 = 2q + 1$

Hence, 3n+2 is odd.
Because the negation of the conclusion of the conditional statement implies that the hypotheses is false, the original conditional statement is true.

Hence, we proved that if n is an integer and 3n+2 is even, then n is even.

(b) p : 3n+2 is even
q : n is even
We have to prove $p \to q$

Let us assume that 3n+2 is even and n is not even, that is n is odd.

$\because$ n is odd, there exists an integer k such that

$$n = 2k+1$$

Multiplying by 3 on both sides and then adding 2

$$3n+2 = 6k+5$$
$$3n+2 = 2(3k+2)+1$$

as k is integer, 3k+2 is also an integer (say t)

$$\therefore 3n+2 = 2t+1$$

this implies that 3n+2 is odd, but we assumed that 3n+2 is even. Hence, we obtained

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~~obtained~~ a contradiction. This completes the proof by contradiction, proving that if $3n+2$ is ~~odd~~ then n is even.

8. Prove or disprove that the product of two irrational numbers is irrational.

Let us assume two irrational numbers x and y, ~~such~~ ~~to~~

let $x = \sqrt{2}$

$y = 2\sqrt{2}$

$\therefore \quad xy = 2\sqrt{2} \times \sqrt{2}$

$= 4$, which is rational.

Hence, we found a counterexample and the theorem is disproved.

9. Let $P(n)$ be the proposition "If a and b are positive real numbers, then $(a+b)^n \geq a^n + b^n$". Prove that $P(1)$ is true. What kind of proof did you use?

The hypotheses for the $P(1)$ proposition is

"If a and b are positive real numbers" and the ~~hy~~ conclusion is "$(a+b)^1 \geq a^1 + b^1$"

$\therefore$ For all real numbers,

$$(a+b)^1 \geq a^1 + b^1$$

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the conclusion is always true, regardless of the hypotheses.

Hence, we proved that P(1) is true. The method of proof used here is a trivial proof as we only proved the conclusion of P(1) to be true.

10. Prove that if x and y are real numbers, then $\max(x+y) + \min(x+y) = x+y$. Use a proof by cases, with two cases corresponding to $x \geq y$ and $x < y$, respectively.

Consider the case when $x \geq y$

$\because x \geq y$

$\therefore \max(x, y) = x$

and $\min(x, y) = y$

$\therefore \max(x, y) + \min(x, y) = x + y$

Hence, we reached the desired conclusion in this case.

Now, consider the case when $x < y$,

$\therefore \max(x, y) = y$

and $\min(x, y) = x$

$\therefore \max(x, y) + \min(x, y) = y + x = x + y$

Hence, in this case also, we reached the

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desired conclusion.
As these two cases cover all the possible examples, and since the conclusion is true for these cases, the theorem "if x and y are real numbers, then $\max(x+y) + \min(x+y) = x+y$" is true.

11. Give a constructive proof to show that there is a positive integer that equals the sum of positive integers not exceeding it.

Let us assume that n is a positive integer that is equal to the sum of positive integers not exceeding it.

$\therefore \; n = n + (n-1) + (n-2) + \cdots + 1$

$$n = \frac{n(n+1)}{2}$$

$$2n = n^2 + n$$

$$n^2 - n = 0$$

$$n(n-1) = 0$$

$\therefore \; n = 0$ or $n = 1$

Since, n is non-zero, therefore n must be 1.

We proved that there exists a positive integer n (and it is $n=1$) which is sum of positive integers not exceeding it. As we found a witness, our proof is constructive.

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12. Prove that either $2 * 10^{500} + 15$ or $2 * 10^{500} + 16$ is not a perfect square.

We will prove this by contradiction.

Let $n = 2 * 10^{500} + 15$ and $m = 2 * 10^{500} + 16$

$\therefore m = n + 1$.

Let m and n be perfect squares. Therefore, there exists x and y such that

$$n = x^2$$
$$m = y^2$$

or,

$$n = x^2 \quad \text{— (I)}$$
$$n + 1 = y^2 \quad \text{(II)}$$

Subtracting (I) from (II),

$$y^2 - x^2 = 1$$
$$(y - x)(y + x) = 1 \quad \text{— (III)}$$

As x and y are integers, $(y - x)$ and $(y + x)$ are divisors of 1. Since, 1 has only two divisors, 1 and -1, and for the eqn (III) to hold, either both $(y - x)$ and $(y + x)$ are equal to or both are equal to -1.

Assume both are equal to 1.

$$y - x = 1$$
$$y + x = 1$$

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$\therefore$ $y = 1$
and, $x = 0$

$\therefore$ $n = 0$ and $m = 1$ (from ① & ⑪),

but ~~we~~ we assumed $n = 2 * 10^{500} + 15$ and $m = 2 * 10^{500} + 16$. Hence, we have a contradiction and both m and n cannot be perfect square.

Hence we proved that either $2 * 10^{500} + 15$ or $2 * 10^{500} + 16$ is not a perfect square.

13. Prove or disprove that there is a rational number x and an irrational number y such that x^y is irrational.

Let $x = 2$ and $y = \sqrt{2}$, that is, x is rational and y is irrational.

$$\therefore x^y = 2^{\sqrt{2}}$$

Now, if $x^y = 2^{\sqrt{2}}$ is irrational, we proved the ~~required~~ statement.

and if $2^{\sqrt{2}}$ is not irrational, then $2^{\sqrt{2}}$ is rational.

$\therefore$ let $x = 2^{\sqrt{2}}$ (rational) and $y = \dfrac{\sqrt{2}}{4}$ (irrational)

$$\therefore x^y = \left(2^{\sqrt{2}}\right)^{\frac{\sqrt{2}}{4}}$$

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$$x^y = 2^{\frac{\sqrt{2}\cdot\sqrt{2}}{4}}$$

$$x^y = 2^{1/2} = \sqrt{2} \text{ (irrational)}$$

Hence, we proved that there exists a rational number x and irrational number y such that x^y is irrational.

14. Prove that there is no positive integer n such that $n^2 + n^3 = 100$. Which method did you use?

As n is a positive integer, and since $n^3 > 100$ $\forall\, n \geq 4$. Therefore, we just need to verify if $n^2 + n^3 = 100$ for $n = 1, 2, 3, 4$

for $n = 1$,

$$n^2 + n^3 = 2$$

for $n = 2$,

$$n^2 + n^3 = 12$$

for $n = 3$,

$$n^2 + n^3 = 36$$

for $n = 4$

$$n^2 + n^3 = 80$$

Hence, there is no positive integer n such that $n^2 + n^3 = 100$. The method of proof used is Exhaustive Proof.