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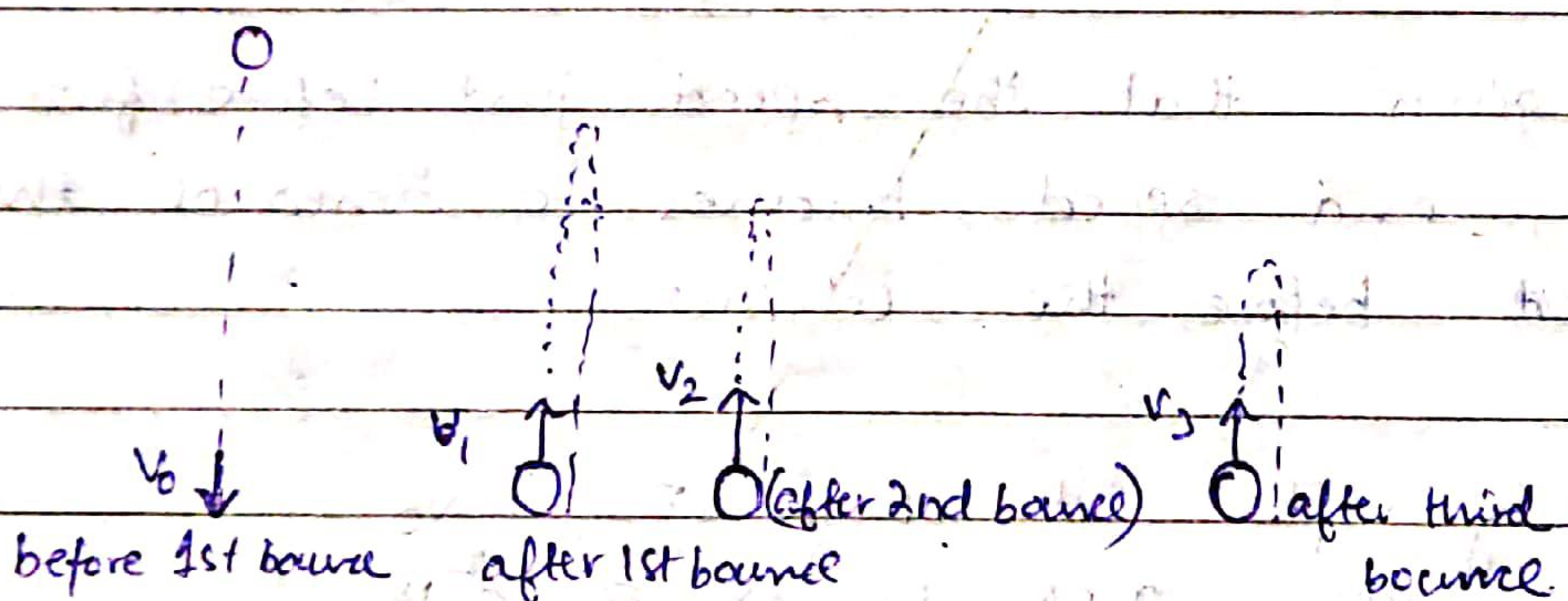
PH100: Mechanics and Thermodynamics
Tutorial 06

1. A ball drops to the floor and bounces, eventually coming to rest. Collisions between the ball and floor are inelastic; speed after each collision is e times the speed before collision where $e<1$ (e is called the coefficient of restitution). If the speed just before the first bounce is v_0, find the time to come to rest.



the maximum height of ball will keep on decreasing and eventually the ball will come to rest after infinite bounces.

$\because$ velocity after each bounce becomes e times the speed before collision.

$\therefore$ Velocity after first collision, $v_1 = ev_0$

and, velocity after second collision; $v_2 = ev_1 = e(ev_0) = e^2 v_0$

Similarly, velocity after third collision, $v_3 = ev_2 = e^3 v_0$

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$\because$ time of flight of a ball in vertical motion is, $t = \frac{2v}{g}$.

$\therefore$ let ~~velocity~~ time of flight ~~during~~ after i^{th} collision and before $(i+1)^{th}$ collision be $t_{i(i+1)}$

$$\therefore \quad t_{12} = \frac{2v_1}{g} = \frac{2ev_0}{g}$$

$$\therefore \quad t_{23} = \frac{2v_2}{g} = \frac{2e^2v_0}{g}$$

and, $t_{34} = \frac{2v_3}{g} = \frac{2e^3v_0}{g}$

Now, ~~so~~ if time is taken to be 0 at the position where the ball is dropped,

time taken by the ball to come to rest will be equal to t,

$$t = \frac{v_0}{g} + t_{12} + t_{23} + t_{34} + \cdots$$

$$t = \frac{v_0}{g} + \frac{2ev_0}{g} + \frac{2e^2v_0}{g} + \frac{2e^3v_0}{g} + \cdots$$

$$t = \frac{v_0}{g} + \frac{2ev_0}{g}\left\{1 + e + e^2 + \cdots\right\}$$

$$t = \frac{v_0}{g} + \frac{2ev_0}{g(1-e)}$$

$$t = \frac{v_0}{g}\left\{\frac{1+e}{1-e}\right\}$$

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Hence, the time taken by ball to come to rest (if we assume $t=0$ where the ball is dropped) is $\frac{v_0(1+e)}{g(1-e)}$.

But, if we assume $t=0$ where the first collision took place, then,

$$t = t_{12} + t_{23} + t_{34} + \ldots$$

$$= \frac{2ev_0}{g} + \frac{2e^2v_0}{g} + \frac{2e^3v_0}{g} + \ldots$$

$$= \frac{2ev_0}{g(1-e)}$$

2. A proton makes a head-on collision with an unknown particle at rest. The proton rebounds straight back with $\frac{4}{9}$ of its initial kinetic energy. Find the ratio of mass of unknown particle to the mass of proton, assuming that the collision is elastic.

Let the mass of unknown particle be m and that of proton be m_p. Also, let initial velocity of proton be u and final velocities of proton and unknown mass m be v_1 and v_2 respectively.

From conservation of momentum, we get,

$$m_p u = m_p v_1 + m v_2 \quad \text{——— (1)}$$

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It is given that final kinetic energy of proton (KE_{pf}) is $\frac{4}{9}$ times the initial kinetic energy.

$$\therefore (KE_{pf}) = \frac{4}{9} \times \frac{1}{2} m_p u^2$$

$$\frac{1}{2} m_p v_1^2 = \frac{4}{18} m_p u^2 \quad \text{—(II)}$$

Since, the collision is elastic, no energy loss takes place, therefore

$$\frac{1}{2} m_p u^2 = \frac{1}{2} m_p v_1^2 + \frac{1}{2} m v_2^2 \quad \text{—(III)}$$

$$\frac{1}{2} m_p u^2 = \frac{4}{9} \times \frac{1}{2} m_p u^2 + \frac{1}{2} m v_2^2 \quad \{\text{from (II)}\}$$

$$\therefore \frac{1}{2} m v_2^2 = \frac{5}{9} \times \frac{1}{2} m_p u^2 \quad \text{—(IV)}$$

from (II), we get,

$$v_1^2 = \frac{4}{9} u^2$$

$$\therefore v_1 = \pm \frac{2u}{3}$$

Since, proton moves straight back,

$$\therefore v_1 = -\frac{2u}{3}$$

putting $v_1 = -\frac{2}{3}u$ in (I) gives

$$v_2 = \frac{5 m_p u}{3 m}$$

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Putting $v_2 = \frac{5}{3}\frac{m_p u}{m}$ in (iv),

$$\frac{1}{\cancel{2}}\cancel{m}\left(\frac{25}{\cancel{9}}\frac{m_p^2 \cancel{u^2}}{m^{\cancel{2}}}\right) = \frac{5}{\cancel{9}}\times\frac{1}{\cancel{2}}m_p \cancel{u^2}$$

$$\frac{25 m_p^2}{m} = 5 m_p$$

$$\frac{m}{m_p} = \frac{25}{5}$$

$$\frac{m}{m_p} = 5$$

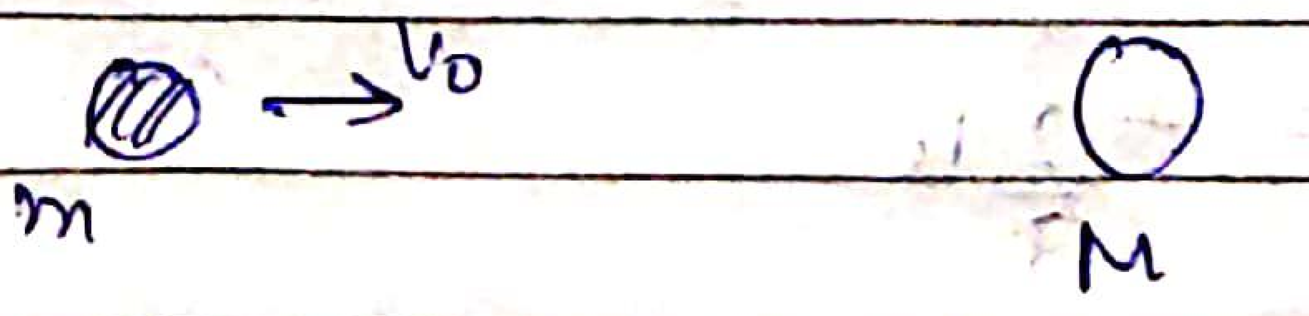
Hence, the ratio of unknown mass to that of proton is 5.

3. A particle of mass m and velocity v_0 collides elastically with a particle of mass M initially at rest and is scattered through angle θ in centre of mass system.

a. Find the final velocity of m in laboratory system.

b. Find the fraction loss of kinetic energy of m.

In the lab frame, before collision



In centre of mass frame, before collision

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(1) Applying conservation of momentum in centre of mass frame

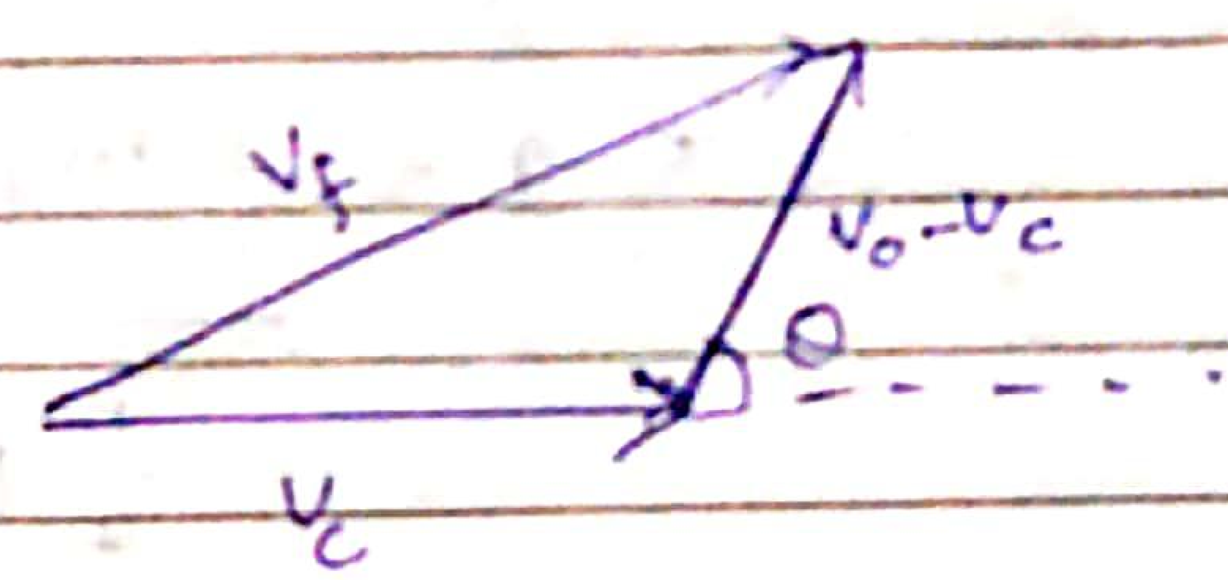
$$m(v_0 - v_c) = M v_c$$

$$\therefore \quad v_c = \frac{m v_0}{m+M}$$

$$\therefore \quad v_0 - v_c = \frac{M v_0}{m+M}$$

Since, in elastic collision velocity of centre of mass remains unchanged and to find the velocity in lab frame, we have to add v_c to all velocities after the collision.

Let final velocity of m (after collision) be v_f



$$\therefore \quad v_f^2 = v_c^2 + (v_0 - v_c)^2 - 2 v_c (v_0 - v_c) \cos(\pi - \theta)$$

$$= \left(\frac{m v_0}{m+M}\right)^2 + \left(\frac{M v_0}{m+M}\right)^2 + 2\left(\frac{m v_0}{m+M}\right)\left(\frac{M v_0}{m+M}\right)\cos\theta$$

$$= \left(\frac{v_0}{m+M}\right)^2 \left(m^2 + M^2 + 2mM\cos\theta\right)$$

$$\therefore \quad v_f = \left(\frac{v_0}{m+M}\right)\sqrt{m + M + 2mM\cos\theta}$$

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b) initial kinetic energy $(K_i) = \frac{1}{2} m v_0^2$

final kinetic energy of m $(K_f) = \frac{1}{2} m v_f^2$

$$K_f = \frac{1}{2} m \left(\frac{v_0}{m+M}\right)^2 \left(\sqrt{m^2+M^2+2mM\cos\theta}\right)^2$$

Fractional loss in kinetic energy $= \frac{K_i - K_f}{K_i}$

fractional loss $= 1 - \frac{K_f}{K_i}$

$$= 1 - \frac{\frac{1}{2} m \left(\frac{v_0}{m+M}\right)^2 (m^2+M^2+2mM\cos\theta)}{\frac{1}{2} m v_0^2}$$

$$= 1 - \frac{(m^2+M^2+2mM\cos\theta)}{(m+M)^2}$$

fraction loss in KE $= \frac{2mM(1-\cos\theta)}{(m+M)^2}$

Hence, fraction loss in KE is equal to $\frac{2mM(1-\cos\theta)}{(m+M)^2}$.

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4. A 0.3 kg mass is attached to a spring and oscillates at 2 Hz with a Q of 60. Find the spring constant and damping constant.

Since frequency $f = 2\,Hz$:

$\therefore$ angular frequency $\omega = 2\pi f$
$= 4\pi$ rad/s

Since, spring constant k is related to ω by

$$\omega = \sqrt{\frac{k}{m}}$$

$$\therefore k = m\omega^2$$
$$k = (0.3) \times (4\pi)^2$$
$$= 47.32 \text{ N/m}$$

Also, $Q = \dfrac{\omega}{\gamma}$, where $\gamma = \dfrac{b}{m}$

$$\therefore Q = \frac{m\omega}{b}$$

$\therefore$ damping constant $b = \dfrac{m\omega}{Q}$

$$b = \frac{(0.3)(4\pi)}{60}$$

$$b = 0.0628 \text{ N s m}^{-1}$$

and $\gamma = \dfrac{b}{m} = \dfrac{0.0628}{0.3} = 0.21\ s^{-1}$

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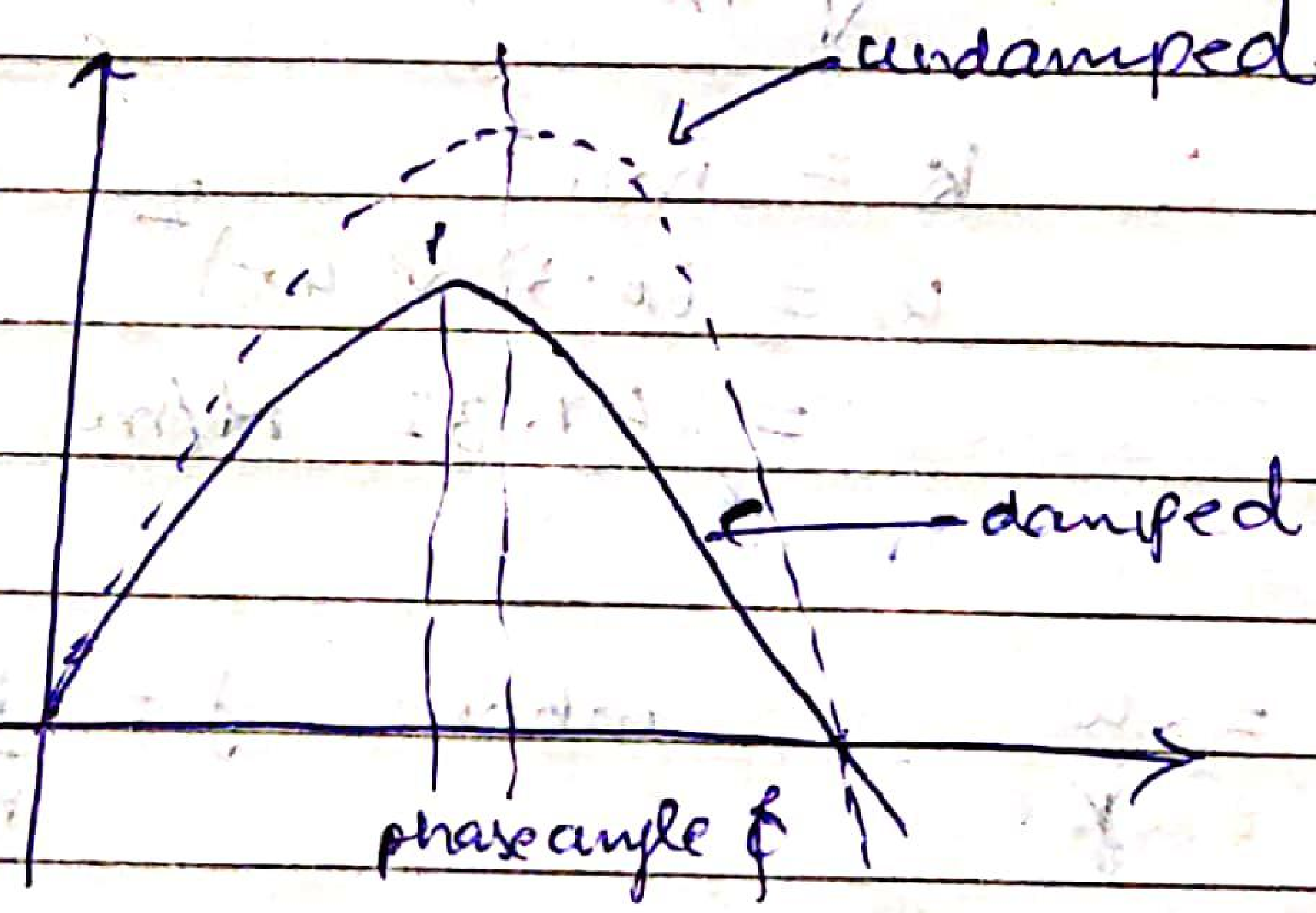
5. In an undamped free harmonic oscillator the motion is given by $x = A \sin \omega_0 t$. The displacement is maximum exactly midway between the zero crossings. In a damped oscillator the motion is no longer sinusoidal, and the maximum is advanced before the midpoint ~~by a phase angle ϕ given approximately~~ of zero crossings. Show that the maximum is advanced by a phase angle ϕ, given approximately by $\phi = \frac{1}{2Q}$, where we assume that Q is large.



For damped oscillation :

$$x = x_0 e^{-(\gamma/2)t} \sin(\omega_0 t)$$

For maxima of x; $\frac{dx}{dt} = 0$

$$\frac{dx}{dt} = x_0 e^{-\frac{\gamma t}{2}} \left[\omega_0 \cos(\omega_0 t) - \frac{\gamma}{2} \sin(\omega_0 t) \right) = 0$$

$$\omega_0 \cos(\omega_0 t) = \frac{\gamma}{2} \sin(\omega_0 t)$$

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$$\frac{\cos(\omega_0 t)}{\sin(\omega_0 t)} = \frac{\gamma}{2\omega_0} << 1 \qquad (\text{as } Q >> 1)$$

let $\omega_0 t = \alpha$

and $\phi = \frac{\pi}{2} - \alpha$

so, $\cos(\omega_0 t) = \sin\left(\frac{\pi}{2} - \alpha\right) = \sin\phi \approx \phi$

and, $\sin(\omega_0 t) = \cos\left(\frac{\pi}{2} - \alpha\right) = \cos\phi \approx \cos 0 \approx 1$

$\left\{ \text{as } \left(\pi/2 - \alpha\right) \to 0 \right\}$

so, $$\frac{\cos(\omega_0 t)}{\sin(\omega_0 t)} = \frac{\sin\phi}{\cos\phi} = \frac{\phi}{1} = \frac{\gamma}{2\omega_0}$$

$$\phi = \frac{\gamma}{2\omega_0}$$

$$\phi = \frac{1}{2Q} \qquad \left(\text{as } Q = \frac{\omega_0}{\gamma}\right)$$

$$\therefore \boxed{\phi = \frac{1}{2Q}}$$

6. The logarithmic decrement δ is defined to be natural logarithm of the ratio of successive maximum displacements (in same direction) of a free damped oscillator. Show that $\delta = \frac{\pi}{Q}$. Find spring constant k and damping constant b of a damped oscillator having a mass of 5 kg, frequency of

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oscillation 0.5 Hz and logarithmic decrement 0.02.

For a freely damped oscillator,

$$x = X_0 e^{-(\gamma/2)t} \sin \omega_0 t$$

Let t_1 and t_2 be the time where the successive maximum displacement takes place.

$$\therefore \omega_0 t_2 = \omega_0 t_1 + 2\pi$$

Now, let R be the ratio of successive displacements x_1 and x_2

$$\therefore R = \frac{x_1}{x_2} = \frac{x_0 e^{-\frac{\gamma t_1}{2}} \sin \omega_0 t_1}{x_0 e^{-\gamma t_2/2} \sin \omega_0 t_2}$$

$$= e^{\frac{\gamma}{2}(t_2 - t_1)} \frac{\sin(\omega_0 t_1)}{\sin(\omega_0 t_1 + 2\pi)}$$

$$= e^{\frac{\gamma}{2}(t_2 - t_1)} \quad \{\text{as } \sin(2\pi + \phi) = \sin\phi$$

$$\therefore R = e^{\frac{\gamma}{2}(t_2 - t_1)}$$

Since, t_1 and t_2 are two successive maximum instant, therefore $t_2 - t_1$ is equal to time period of oscillation

i.e. $t_2 - t_1 = \frac{2\pi}{\omega_0}$

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$$\therefore R = e^{\frac{\gamma}{2}\left(\frac{2\pi}{\omega_0}\right)}$$

$$R = e^{\left(\frac{\pi\gamma}{\omega_0}\right)}$$

$\therefore$ logarithmic decrement $\delta = \ln R = \dfrac{\pi\gamma}{\omega_0}$

$$\therefore \delta = \ln R = \frac{\pi}{Q} \qquad \left(\text{as } Q = \frac{\omega_0}{\gamma}\right)$$

Given, $\delta = 0.02$, $f = 0.5$ Hz and $m = 5$ kg

$$\therefore \omega = 2\pi f$$
$$= \pi \text{ rad/s}$$

$$\therefore \omega = \sqrt{\frac{k}{m}}$$

$$\therefore k = m\omega^2 = 5 \times \pi^2 \ Nm^{-1}$$
$$k = 49.298 \ Nm^{-1}$$

and, $\delta = \dfrac{\pi}{Q} = \dfrac{\pi\gamma}{\omega_0} = \dfrac{\pi b}{\omega_0 m}$ $\left(\text{as } \gamma = \dfrac{b}{m}\right)$

$\therefore$ damping constant b,

$$b = \frac{m\omega_0\delta}{\pi} = \frac{5 \times \pi \times 0.02}{\pi}$$
$$= 0.1 \ Nsm^{-1}$$

Hence, $k = 49.298 \ Nm^{-1}$ and $b = 0.1 \ Nsm^{-1}$