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PH110 : Waves and Electromagnetics
Tutorial 02

Q.29 Calculate the line integral of the function $v = x^2\hat{x} + 2yz\hat{y} + y^2\hat{z}$ from the origin to the point (1,1,1) by three different routes:

(a) $(0,0,0) \rightarrow (1,0,0) \rightarrow (1,1,0) \rightarrow (1,1,1)$
(b) $(0,0,0) \rightarrow (0,0,1) \rightarrow (0,1,1) \rightarrow (1,1,1)$
(c) the direct straight line.
(d) What is the line integral around the closed loop that goes out along path (a) and back along path (b)?

(a) From (0,0,0) to (1,0,0),
$\because dx \neq 0, \; dy = dz = 0$
$\therefore dl = dx\,\hat{x}$

$\therefore$ let $I_1 = \int_0^1 v \cdot dl = \int_0^1 (x^2\hat{x} + 2yz\hat{y} + y^2\hat{z}) \cdot (dx\,\hat{x})$
$= \int_0^1 x^2\,dx = \frac{1}{3} x^3 \Big|_0^1$

$$\boxed{I_1 = \frac{1}{3}}$$

Now, from (1,0,0) to (1,1,0),
$dx = dz = 0$ and $dy \neq 0$, $x = 1$ and $z = 0$
$\therefore dl = dy\,\hat{y}$

$\therefore$ let $I_2 = \int_0^1 v \cdot dl = \int_0^1 (x^2\hat{x} + 2yz\hat{y} + y^2\hat{z}) \cdot (dy\,\hat{y})$

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$$I_2 = \int_0^1 2y(0)\,dy = 0$$

Now, from $(1,1,0)$ to $(1,1,1)$
$dx = dy = 0$, $dz \neq 0$ and $x = y = 1$, $dl = dz\,\hat{z}$

let $I_3 = \int_0^1 (x^2\hat{x} + 2yz\hat{y} + y^2\hat{z}) \cdot (dz\,\hat{z})$

$$I_3 = \int_0^1 y^2\,dz = \int_0^1 dz \qquad (\because y = 1)$$

$$I_3 = 1$$

$\therefore$ the line integral of v on the ~~path~~ given path is I_a, and

$$I_a = I_1 + I_2 + I_3 = \frac{1}{3} + 0 + 1 = \frac{4}{3}$$

Hence, line integral of v from origin to $(1,1,1)$ on the given path is $4/3$.

(b) From $(0,0,0) \longrightarrow (0,0,1)$
$dx = dy = 0$ and $dz \neq 0$
$\therefore$ $dl = dz\,\hat{z}$
$x = y = 0$.

$$\therefore I_1 = \int_0^1 (x^2\hat{x} + 2yz\hat{y} + y^2\hat{z}) \cdot (dz\,\hat{z})$$

$$= \int_0^1 y^2\hat{z} = 0 \qquad (\because y = 0)$$

$\therefore$ $I_1 = 0$

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From $(0,0,1) \rightarrow (0,1,1)$

$dx = dz = 0$ and $dy \neq 0$

$\therefore\ dl = dy\,\hat{y}$

$x = 0$ and $z = 1$

$$I_2 = \int_0^1 v\cdot(dy\,\hat{y}) = \int_0^1 2yz\,dy = \int_0^1 2y\,dy \quad (\because z=1)$$

$$\therefore\ I_2 = \frac{2}{2}\left[y^2\right]_0^1 = 1$$

$$\therefore\ I_2 = 1$$

From $(0,1,1) \rightarrow (1,1,1)$

$dy = dz = 0$ and $dx \neq 0$

$\therefore\ dl = dx\,\hat{x}$

$y = z = 1$

$$I_3 = \int_0^1 v\cdot(dx\,\hat{x}) = \int_0^1 x^2\,dx = \frac{1}{3}$$

$$\therefore\ I_3 = \frac{1}{3}$$

Hence, the line integral of v from origin to $(1,1,1)$ along the given path is I_b and I_b is equal to

$$I_b = I_1 + I_2 + I_3$$

$$I_b = 0 + 1 + \frac{1}{3} = \frac{4}{3}$$

$$\boxed{\therefore \int_{(0,0,0)}^{(1,1,1)} v\cdot dl = \frac{4}{3}}$$

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(c) Along the direct straight line from origin to (1,1,1),

$$x = y = z \quad \text{—(i)}$$

$$\therefore \; dx = dy = dz \quad \text{—(ii)}$$

$$\therefore \; d\mathbf{l} = dx\,\hat{x} + dy\,\hat{y} + dz\,\hat{z}$$

~~$I = \int v \cdot d$~~

$$v \cdot d\mathbf{l} = (x^2\hat{x} + 2yz\,\hat{y} + y^2\hat{z}) \cdot (dx\,\hat{x} + dy\,\hat{y} + dz\,\hat{z})$$

$$= (x^2\hat{x} + 2x^2\hat{y} + x^2\hat{z}) \cdot (dx\,\hat{x} + dx\,\hat{y} + dx\,\hat{z}) \quad \text{(from (i) \& (ii))}$$

$$= x^2\,dx + 2x^2\,dx + x^2\,dx$$

$$= 4x^2\,dx$$

$\therefore$ line integral of v from origin to (1,1,1) along the direct line is I,

$$I = \int_0^1 4x^2\,dx = \frac{4}{3}\,x^3\Big|_0^1$$

$$= \frac{4}{3}$$

Hence, $I = \frac{4}{3}$ is the required line integral.

(d) ~~value~~ line integral around the close loop that goes out along path in (a) and back along path (b) is

$$I = (I_a) - (I_b) = \frac{4}{3} - \frac{4}{3} = 0$$

Hence, along the given closed loop, line integral is 0.

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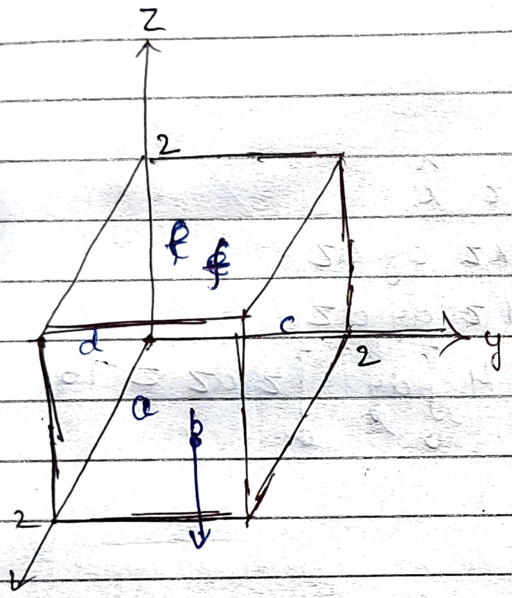
Q.30 Calculate the surface integral of the function $v = 2xz\,\hat{x} + (x+2)\,\hat{y} + y(z^2-3)\,\hat{z}$ over the bottom of the box. Does the surface integral depend only on the boundary line for this function? What is the total flux over the closed surface of the box?



The face name is written at the centre of the face.

We need to find the surface integral of v over bottom i.e. base b.

For base b,

$z = 0$

$da = dx \cdot dy\,(-\hat{z})$

$\therefore\ v \cdot da = -y\,(z^2 - 3)\,dx \cdot dy$

$= 3y\,dx \cdot dy \qquad (\because z = 0)$

$\therefore \int_b v \cdot da = 3 \int_0^2 dx \cdot \int_0^2 y\,dy$

$= 3 \times 2 \times 2 = 12$

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Hence, integral of v over base b is ~~12~~ 12.

To check if the surface integral depends on boundary line only, we will need to find the total surface integral of v over the other faces of cube (other from face b)

for face a,

$x = 2$

$da = dy\,dz\,\hat{x}$

$v \cdot da = 2xz\,dy\,dz$

$= 4z\,dy\,dz$

$\therefore \int_a v \cdot da = 4 \int_0^2 dy \int_0^2 z\,dz = 16$

for face c,

$y = 2$

$da = dx\,dz\,\hat{y}$

$v \cdot da = (x+2)\,dx\,dz$

$\int_c v \cdot da = \int_0^2 (x+2)\,dx \cdot \int_0^2 dz = 12$

for face d,

$y = 0$

$da = dx\,dz\,(-\hat{y})$

$v \cdot da = -(x+2)\,dx\,dz$

$\int_d v \cdot da = -\int_0^2 (x+2)\,dx \int_0^2 dz = -12$

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for face e,

$z = 2$

$da = dx\, dy\, \hat{z}$

$v \cdot da = y(z^2 - 3)\, dx\, dy$

$= y\, dx\, dy$

$$\int_e v \cdot da = \int_0^2 dx \int_0^2 y\, dy = 4$$

for face f,

~~[illegible]~~ $x = 0$

$da = dy \cdot dz(-\hat{x})$

$v \cdot da = -2xz\, dy\, dz$

$= 0 \qquad (\because x = 0)$

$$\therefore \int_f v \cdot da = 0$$

$\therefore$ total surface integral over the boundary line

is $= \int_a v \cdot da + \int_c v \cdot da + \int_d v \cdot da + \int_e v \cdot da + \int_f v \cdot da$

$= 16 + 12 - 12 + 4 + 0 = 20$

$\because$ this is different from what we calculate earlier for face b, hence the value of surface integral does not depend on boundary line only for this function.

The total flux over the closed surface is

$$\phi = \int_a v \cdot da + \int_b v \cdot da + \int_c v \cdot da + \int_d v \cdot da + \int_e v \cdot da + \int_f v \cdot da$$

$= 16 + 12 + 12 - 12 + 4 + 0 = 32$

The total flux over the closed surface is 32.

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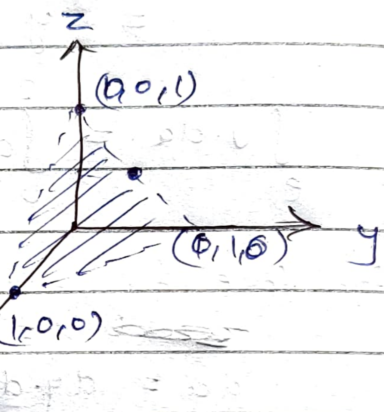
Q.3) Calculate the volume integral of the function $T = z^2$ over the tetrahedron with corners at $(0,0,0)$, $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$

$$\int T \, d\tau = \int z^2 \, dx \, dy \, dz$$



We can do the integral in any order, so let us assume

$$\int T \, d\tau = \int z^2 \left[\int \left(\int dx \right) dy \right] dz$$

The shaded plane's equation is

$$x + y + z = 1$$

$$\therefore \quad x = 1 - y - z$$

$$\therefore \int_0^{1-y-z} dx = 1 - y - z$$

Hence,

$$\int T \, d\tau = \int z^2 \left[\int (1 - z - y) \, dy \right] dz$$

Now, for a given value of z, y varies from 0 to $1-z$

$\therefore$ the y integral becomes

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$$\int_0^{(1-z)} (1-\cancel{x}-y)\,dy = \left[(1-z)y - \frac{y^2}{2}\right]_0^{1-z}$$

$$= (1-z)^2 - \frac{(1-z)^2}{2} = \frac{(1-z)^2}{2}$$

Hence, $$\int T\,d\tau = \int \frac{z^2(1-z)^2}{2}\,dz$$

and z varies from 0 to 1.

$$\therefore \int T\,d\tau = \frac{1}{2}\int_0^1 (z^4 - 2z^3 + z^2)\,dz$$

$$= \frac{1}{2}\left[\frac{z^5}{5} - \frac{z^4}{2} + \frac{z^3}{3}\right]_0^1$$

$$= \frac{1}{2}\left[\frac{1}{5} - \frac{1}{2} + \frac{1}{3}\right] = \frac{1}{60}$$

Hence, the required volume integral is $\frac{1}{60}$.

Q.32 Check the fundamental theorem for gradients using $T = x^2 + 4xy + 2yz^3$, the points $a \equiv (0,0,0)$, $b \equiv (1,1,1)$ and the three paths

(a) $(0,0,0) \to (1,0,0) \to (1,1,0) \to (1,1,1)$

(b) $(0,0,0) \to (0,0,1) \to (0,1,1) \to (1,1,1)$

(c) the parabolic path $z = x^2$; $y = x$

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$$T = x^2 + 4xy + 2yz^3$$

Using the fundamental theorem for gradients the value of the line integral is, say A,

$$A = \int \nabla T \cdot dl = T(1,1,1) - T(0,0,0)$$

$$A = (1+4+2) - 0$$
$$A = 7$$

The gradient of T, ∇T is,

$$\nabla T = (2x+4y)\hat{x} + (4x+2z^3)\hat{y} + 6yz^2(\hat{z})$$

$$\therefore \nabla T \cdot dl = (2x+4y)\,dx + (4x+2z^3)\,dy + (6yz^2)\,dz$$

(a) For path (i), from (0,0,0) to (1,0,0)
$y = z = 0$
$\therefore dy = dz = 0$ and $x \in (0,1)$

$$\therefore \nabla T \cdot dl = 2x\,dx$$
$$\therefore \int_{(i)} \nabla T \cdot dl = \int_0^1 2x\,dx = 1$$

For path (ii), from (1,0,0) to (1,1,0)
$x = 1$, $z = 0$, $dx = dz = 0$
$y \in (0,1)$

$$\therefore \nabla T \cdot dl = 4\,dy$$
$$\therefore \int_{(ii)} \nabla T \cdot dl = \int_0^1 4\,dy = 4$$

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For path(iii), $(1,1,0)$ to $(1,1,1)$.
$x = y = 1$, $dx = dy = 0$
and $z \in (0,1)$

$\therefore \nabla T \cdot dl = 6z^2 dz$

$$\therefore \int_{(iii)} \nabla T \cdot dl = \int_0^1 6z^2 \, dz = 2$$

the total line integral $\int \nabla T \cdot dl$,

$$\int \nabla T \cdot dl = \int_{(i)} \nabla T \cdot dl + \int_{(ii)} \nabla T \cdot dl + \int_{(iii)} \nabla T \cdot dl$$

$$= 1 + 4 + 2$$
$$= 7$$
$$= A$$

Hence, proved

(b) For path (i), from $(0,0,0) \rightarrow (0,0,1)$
$x = y = dx = dy = 0$ and $z \in (0,1)$

$\therefore \nabla T \cdot dl = 6yz^2 \, dz = 0$

$$\int_{(i)} \nabla T \cdot dl = 0$$

For path (ii), from $(0,0,1) \rightarrow (0,1,1)$
$x = 0$, $z = 1$, $dx = dz = 0$
and $y \in (0,1)$

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$$\therefore \nabla T \cdot dl = (4x + 2z^3)\, dy$$
$$= 2\, dy$$

$$\therefore \int_{(ii)} \nabla T \cdot dl = \int_0^1 2\, dy = 2$$

for path (iii), from $(0,1,1)$ to $(1,1,1)$
$y = z = 1$, $dy = dz = 0$
and $x \in (0,1)$

$$\therefore \nabla T \cdot dl = (2x + 4y)\, dx$$
$$= (2x + 4)\, dx$$

$$\therefore \int_{(iii)} \nabla T \cdot dl = \int_0^1 (2x+4)\, dx = \left(x^2 + 4x\right)_0^1$$
$$= 5$$

the total integral of ∇T over the path given is

$$\int \nabla T \cdot dl = \int_{(i)} \nabla T \cdot dl + \int_{(ii)} \nabla T \cdot dl + \int_{(iii)} \nabla T \cdot dl$$
$$= 0 + 2 + 5$$
$$= 7$$
$$\int \nabla T \cdot dl = A$$

Hence, proved

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(c) along the path $z = x^2$ and $y = x$

$\because \quad z = x^2 \quad \text{— (I)}$

$dz = 2x\,dx \quad \text{— (II)}$

and $\because \quad y = x \quad \text{— (III)}$

$dy = dx \quad \text{— (IV)}$

$$\nabla T \cdot dl = (2x+4y)dx + (4x+2z^3)dy + (6yz^2)dz$$

$$= 6x\,dx + (4x+2x^6)dx + 12x^6\,dx$$

(using (I), (II), (III) & (IV))

$$= (10x + 14x^6)\,dx$$

$$\therefore \int \nabla T \cdot dl = \int_0^1 (10x + 14x^6)\,dx$$

$$= \left(5x^2 + 2x^7\right)\Big|_0^1$$

$$= 7$$

$$\int \nabla T \cdot dl = A$$

Hence, it is proved that $\int \nabla T \cdot dl$ is independent of the path taken and the fundamental law of gradients is proven.

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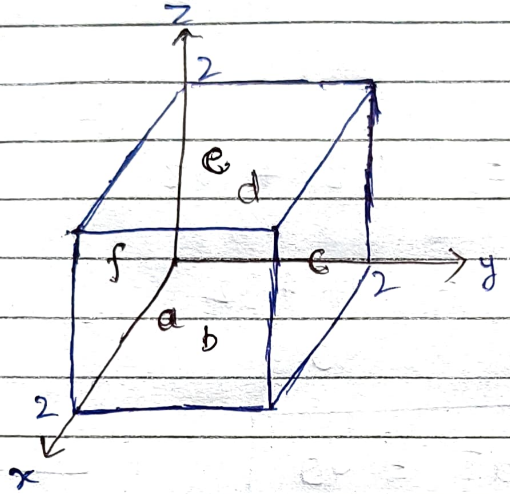
Q.33. Test the divergence theorem for the function $v = (xy)\hat{x} + (2yz)\hat{y} + (3zx)\hat{z}$.
Take as your volume the cube shown.



The divergence theorem states that

$$\int_V \nabla \cdot v \, d\tau = \oint_S v \cdot da$$

Now, let us first compute the LHS of the above equation.

$$\therefore \nabla \cdot v = y + 2z + 3x$$

$$\therefore \int_V \nabla \cdot v \, d\tau = \int_0^2 \int_0^2 \left(\int_0^2 (y + 2z + 3x) \, dx \right) dy \, dz$$

Now, $\displaystyle\int_0^2 (y + 2z + 3x) \, dx = \left(xy + 2xz + \frac{3x^2}{2} \right) \Bigg|_0^2$

$$= 2y + 4z + 6$$

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$$\int_0^2 (2y + 4z + 6)\, dy = \left(y^2 + 4yz + 6y\right)\Big|_0^2$$

$$= 16 + 8z$$

and,

$$\int_0^2 (8z + 16)\, dz = \left(4z^2 + 16z\right)\Big|_0^2$$

$$= 16 + 32$$

$$= 48$$

$$\boxed{\therefore \int_V \nabla \cdot v \, d\tau = 48} \longrightarrow (A)$$

Now, let us compute the value of RHS of divergence theorem,
the faces are numbered in the cube

for face a,

$x = 2,$

$da = dy\, dz\, \hat{x}$

$\therefore v \cdot da = xy\, dy\, dz$

$= 2y\, dy\, dz$

$$\therefore \int_a v \cdot da = 2 \int_0^2 y\, dy \int_0^2 dz$$

$$\int_a v \cdot da = 8$$

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for face b,

$z = 0$

$da = dx \cdot dy \, (-\hat{z})$

$\therefore v \cdot da = -3zx \, dx \, dy = 0 \qquad (\because z = 0)$

$\therefore \int_b v \cdot da = 0$

for face c,

$y = 2$

$da = dx \, dz \, \hat{y}$

$\therefore v \cdot da = 2yz \, dx \, dz$

$= 4z \, dx \, dz$

$\therefore \int_c v \cdot da = 4 \int_0^2 dx \int_0^2 z \, dz$

$= 16$

for face d

$x = 0$

$da = dy \cdot dz \, (-\hat{x})$

$v \cdot da = -xy \, dy \, dz = 0$

$\therefore \int_d v \cdot da = 0$

for face e,

$z = 2$

$da = dx \, dy \, \hat{z}$

$v \cdot da = 3zx \, dx \, dy$

$= 6x \, dx \, dy$

$\therefore \int_e v \cdot da = 6 \int_0^2 x \, dx \int_0^2 dy$

$\int_e v \cdot da = 24$

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for face f,

$$y = 0$$
$$da = dx\,dz\,(-\hat{y})$$
$$v \cdot da = -2yz\,dx\,dz$$
$$= 0$$

$$\therefore \int_f v \cdot da = 0$$

Now, the total value of v over the surface of cube is

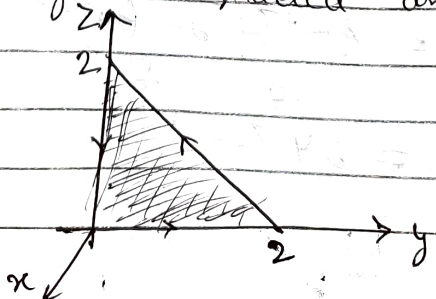
$$\oint_s v \cdot da = \int_a v \cdot da + \int_b v \cdot da + \int_c v \cdot da + \int_d v \cdot da + \int_e v \cdot da + \int_f v \cdot da$$
$$= 8 + 0 + 16 + 0 + 24 + 0$$
$$\oint_s v \cdot da = 48 \quad \text{——— (B)}$$

from (A) & (B)

$$\int_v \nabla \cdot v \, d\tau = \oint_s v \cdot da$$

Hence, checked.

Q.34 Test Stoke's theorem for the function $v = xy\,\hat{x} + 2yz\,\hat{y} + 3zx\,\hat{z}$, using the triangular shaded area in the figure



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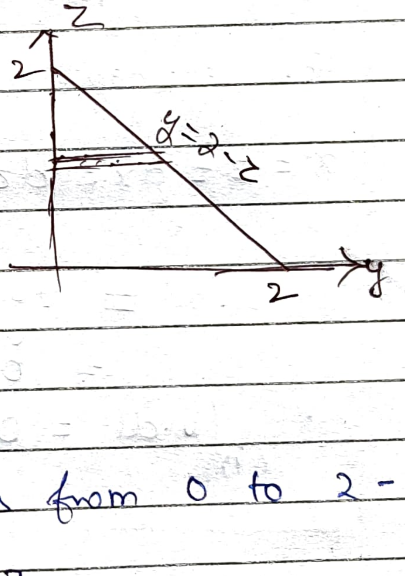
According to Stoke's theorem

$$\int_S (\nabla \times v) \cdot da = \oint_P v \cdot dl$$

Let us first compute the LHS of the above equation.

$$\therefore \nabla \times V = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ xy & 2yz & 3zx \end{vmatrix}$$

$$= (-2y)\hat{x} - 3z(\hat{y}) - x(\hat{z})$$



For the given surface
$x = 0$

$\therefore da = dy\, dz\, \hat{x}$

$\therefore (\nabla \times v) \cdot da = -2y\, dy\, dz$

$$\therefore \int (\nabla \times v) \cdot da = \int \left(\int -2y\, dy \right) dz$$

For a given z, y varies from 0 to 2 - z.

$$\therefore \int (\nabla \times v) \cdot da = \int_0^2 \left[\int_0^{2-z} -2y\, dy \right] dz$$

$$= \int_0^2 \left[(-y^2) \Big|_0^{2-z} \right] dz$$

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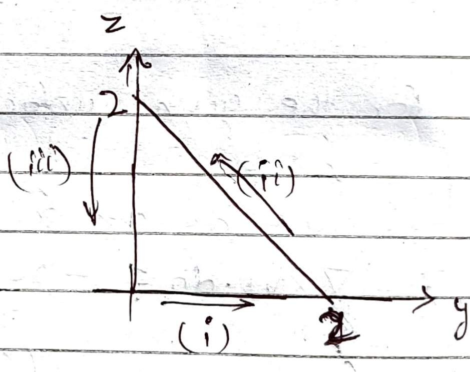
$$\int(\nabla\times v)\cdot da = -\int_0^2 \left(z^2 - 4z + 4\right) dz$$

$$= -\left[\frac{z^3}{3} - 2z^2 + 4z\right]_0^2$$

$$= -\left(\frac{8}{3} - 8 + 8\right)$$

$$\boxed{\int(\nabla\times v)\cdot da = -\frac{8}{3}} \longrightarrow (A)$$

Now, let us compute the RHS of Stoke's theorem



for segement (i)

$x = z = dx = dz = 0$

$\therefore \quad v\cdot dl = 2yz\, dy$
$= 0$

$\therefore \int_{(i)} v\cdot dl = 0$

for segment (ii)

$y = 2 - z \Rightarrow dy = -dz$
$x = dx = 0$

$$v\cdot dl = 2yz\,dy - 3\cancel{zx}^{\,0}\,dz$$
$$= 2yz\,dy$$
$$\cancel{= 2y(2-z)}$$
$$= 2y(2-y)\,dy$$

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$$\therefore \int_{(ii)} v \cdot dl = \int_2^0 (4y - 2y^2)\, dy$$

$$= \left[2y^2 - \frac{2y^3}{3}\right]_2^0$$

$$= \left[0 - \left(8 - \frac{16}{3}\right)\right]$$

$$\int_{ii} v \cdot dl = -\frac{8}{3}$$

for segment (iii)

$x = y = dx = dy = 0$

$$\therefore v \cdot dl = 3zx\, dz$$
$$= 0$$
$$\therefore \int_{(iii)} v \cdot dl = 0$$

$$\therefore \oint_P v \cdot dl = \int_{(i)} v \cdot dl + \int_{(ii)} (v \cdot dl) + \int_{(iii)} v \cdot dl$$

$$= 0 - \frac{8}{3} + 0$$

$$\oint_P v \cdot dl = -\frac{8}{3} \longrightarrow (B)$$

From (A) & (B),

$$\int_S (\nabla \times v) \cdot da = \oint_P v \cdot dl$$

Hence, checked.

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36.(a) Show that

$$\int_S f(\nabla \times A) \cdot dA = \int_S [A \times (\nabla f)] \cdot da + \oint_P fA \cdot dl$$

Using product rule

$$\nabla \times fD = f(\nabla \times D) - D \times (\nabla f)$$

$$\therefore \nabla \times fA = f(\nabla \times A) - A \times (\nabla f)$$

$$\therefore f(\nabla \times A) = \nabla \times fA + A \times (\nabla f)$$

Integrating both sides

$$\int_S f(\nabla \times A) \cdot da = \int_S [A \times (\nabla f)] \cdot da + \int_S (\nabla \times fA) \cdot da$$

Using, fundamental theorem for curl

$$\int_S (\nabla \times fA) \cdot da = \oint_P (fA) \, dl$$

$$\boxed{\int_S f(\nabla \times A) \cdot da = \int_S [A \times (\nabla f)] \cdot da + \oint_P (fA) \cdot dl}$$

Hence, proved.

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(b) Show that

$$\int_V B\cdot(\nabla\times A)\,d\tau = \int_V A\cdot(\nabla\times B)\,d\tau + \oint_S (A\times B)\cdot da$$

$$\nabla\cdot(A\times B) = B\cdot(\nabla\times A) - A\cdot(\nabla\times B)$$

(product rule)

$$B\cdot(\nabla\times A) = A\cdot(\nabla\times B) + \nabla\cdot(A\times B)$$

Integrating both sides

$$\int_V B\cdot(\nabla\times A)\,d\tau = \int_V A\cdot(\nabla\times B)\,d\tau + \int_V \nabla\cdot(A\times B)\,d\tau \quad (1)$$

Now,

$$\int_V \nabla\cdot(A\times B)\,d\tau = \oint_S (A\times B)\,da$$

(using fundamental theorem for divergence)

Now, equation (1) becomes

$$\boxed{\int_V B\cdot(\nabla\times A)\,d\tau = \int_V A\cdot(\nabla\times B)\,d\tau + \oint_S (A\times B)\cdot da}$$

Hence, proved.

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37. Find formulas for r, θ, ϕ in terms of x, y and z.

As r is the distance of point $P(x, y, z)$ from origin:

$$\therefore \quad r = \sqrt{x^2 + y^2 + z^2}$$

Now,

$$\because \quad x = r \sin\theta \cos\phi \quad \text{--- (I)}$$
$$y = r \sin\theta \sin\phi \quad \text{--- (II)}$$
$$z = r \cos\theta \quad \text{--- (III)}$$

from (III)

$$\cos\theta = \frac{z}{r}$$

$$\therefore \quad \theta = \cos^{-1}\left(\frac{z}{\sqrt{x^2+y^2+z^2}}\right)$$

from (I) & (II)

$$\frac{y}{x} = \tan\phi$$

$$\therefore \quad \phi = \tan^{-1}\left(\frac{y}{x}\right)$$

Q-38 Express the unit vectors $\hat{r}$, $\hat{\theta}$ and $\hat{\phi}$ in terms of $\hat{x}$, $\hat{y}$ and $\hat{z}$. Check your answer several ways ($\hat{r} \cdot \hat{r} = 1$, $\hat{\theta} \cdot \hat{\phi} = 0$, $\hat{r} \times \hat{\theta} = \hat{\phi}$). Also work out the inverse formulas giving $\hat{x}$, $\hat{y}$ and $\hat{z}$ in terms of $\hat{r}$, $\hat{\theta}$ and $\hat{\phi}$.

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Let $\vec{r} = x\hat{x} + y\hat{y} + z\hat{z}$

$$\vec{r} = r\sin\theta\cos\phi\,\hat{x} + r\sin\theta\sin\phi\,\hat{y} + r\cos\theta\,\hat{z}$$

let us vary r only, then $d\vec{r} = \frac{\partial(\vec{r})}{\partial r}dr$ is a short vector pointing in the direction of increase of r. To make it a unit vector, it must be divided by its length.

$$\therefore \hat{r} = \frac{\frac{\partial\vec{r}}{\partial r}}{\left|\frac{\partial\vec{r}}{\partial r}\right|}, \quad \hat{\theta} = \frac{\frac{\partial\vec{r}}{\partial\theta}}{\left|\frac{\partial\vec{r}}{\partial\theta}\right|}, \quad \hat{\phi} = \frac{\frac{\partial\vec{r}}{\partial\phi}}{\left|\frac{\partial\vec{r}}{\partial\phi}\right|}$$

Now,

$$\frac{\partial\vec{r}}{\partial r} = \sin\theta\cos\phi\,\hat{x} + \sin\theta\sin\phi\,\hat{y} + \cos\theta\,\hat{z}$$

$$\left|\frac{\partial\vec{r}}{\partial r}\right| = \sqrt{\sin^2\theta\cos^2\phi + \sin^2\theta\sin^2\phi + \cos^2\theta} = 1$$

$$\boxed{\therefore \hat{r} = \sin\theta\cos\phi\,\hat{x} + \sin\theta\sin\phi\,\hat{y} + \cos\theta\,\hat{z}}$$

$$\frac{\partial\vec{r}}{\partial\theta} = r\cos\theta\cos\phi\,\hat{x} + r\cos\theta\sin\phi\,\hat{y} - r\sin\theta\,\hat{z}$$

$$\left|\frac{\partial\vec{r}}{\partial\theta}\right| = \sqrt{r^2\cos^2\theta\cos^2\phi + r^2\cos^2\theta\sin^2\phi + r^2\sin^2\theta}$$

$$= \sqrt{r^2(\cos^2\theta + \sin^2\theta)} = r$$

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$$\boxed{\therefore \hat{\theta} = \cos\theta\cos\phi\,\hat{x} + \cos\theta\sin\phi\,\hat{y} - \sin\theta\,\hat{z}}$$

$$\frac{\partial\vec{r}}{\partial\phi} = -r\sin\theta\sin\phi\,\hat{x} + r\sin\theta\cos\phi\,\hat{y}$$

$$\left|\frac{\partial\vec{r}}{\partial\phi}\right| = \sqrt{r^2\sin^2\theta\sin^2\phi + r^2\sin^2\theta\cos^2\phi} = r\sin\theta$$

$$\boxed{\therefore \hat{\phi} = -\sin\phi\,\hat{x} + \cos\phi\,\hat{y}}$$

Checking $\hat{r}\cdot\hat{r} = 1$

$$\hat{r}\cdot\hat{r} = \sin^2\theta\cos^2\phi + \sin^2\theta\sin^2\phi + \cos^2\theta$$
$$= \sin^2\theta + \cos^2\theta$$
$$= 1$$

Q.E.D.

Checking $\hat{\theta}\cdot\hat{\phi} = 0$

$$\hat{\theta} - \hat{\phi} = -\cos\theta\cos\phi\sin\phi + \cos\theta\sin\phi\cos\phi$$
$$= 0$$

Q.E.D.

Checking $\hat{r}\times\hat{\theta} = \hat{\phi}$

$$\hat{r}\times\hat{\theta} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \end{vmatrix}$$

$$= \left(-\sin^2\theta\sin\phi - \cos^2\theta\sin\phi\right)\hat{x} - \left(-\sin^2\theta\cos\phi - \cos^2\theta\cos\phi\right)\hat{y} + 0\,\hat{z}$$

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$$\hat{r} \times \hat{\theta} = -\sin\phi\,\hat{x} + \cos\phi\,\hat{y}$$
$$= \hat{\phi}$$

Q.E.D.

Now, we have

$$\hat{r} = \sin\theta\cos\phi\,\hat{x} + \sin\theta\sin\phi\,\hat{y} + \cos\theta\,\hat{z}$$
$$\hat{\theta} = \cos\theta\cos\phi\,\hat{x} + \cos\theta\sin\phi\,\hat{y} - \sin\theta\,\hat{z}$$
$$\hat{\phi} = -\sin\phi\,\hat{x} + \cos\phi\,\hat{y}$$

and we have to find $\hat{x}$, $\hat{y}$ and $\hat{z}$ in terms of $\hat{r}$, $\hat{\theta}$ and $\hat{\phi}$.

$$\sin\theta\,\hat{r} = \sin^2\theta\cos\phi\,\hat{x} + \sin^2\theta\sin\phi\,\hat{y} + \sin\theta\cos\theta\,\hat{z}$$
$$\cos\theta\,\hat{\theta} = \cos^2\theta\cos\phi\,\hat{x} + \cos^2\theta\sin\phi\,\hat{y} - \sin\theta\cos\theta\,\hat{z}$$

$$\sin\theta\,\hat{r} + \cos\theta\,\hat{\theta} = \cos\phi\,\hat{x} + \sin\phi\,\hat{y} \quad \text{— (I)}$$
$$\hat{\phi} = -\sin\phi\,\hat{x} + \cos\phi\,\hat{y} \quad \text{— (II)}$$

Multiplying (I) by $\cos\phi$ and (II) by $\sin\phi$

$$\cos\phi\sin\theta\,\hat{r} + \cos\phi\cos\theta\,\hat{\theta} = \cos^2\phi\,\hat{x} + \cos\phi\sin\phi\,\hat{y} \quad \text{— (III)}$$
$$\sin\phi\,\hat{\phi} = -\sin^2\phi\,\hat{x} + \sin\phi\cos\phi\,\hat{y} \quad \text{— (IV)}$$

(III) − (IV)

$$\boxed{\hat{x} = \cos\phi\sin\theta\,\hat{r} + \cos\phi\cos\theta\,\hat{\theta} - \sin\phi\,\hat{\phi}}$$

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Multiplying (I) by $\sin\phi$ and (II) by $\cos\phi$

$$\sin\phi \sin\theta\, \hat{r} + \sin\phi \cos\theta\, \hat{\theta} = \sin\phi \cos\phi\, \hat{x} + \sin^2\phi\, \hat{y} \quad \text{—(V)}$$
$$\cos\phi\, \hat{\phi} = -\sin\phi \cos\phi\, \hat{x} + \cos^2\phi\, \hat{y} \quad \text{—(VI)}$$

(V) + (VI)

$$\boxed{\hat{y} = \sin\phi \sin\theta\, \hat{r} + \sin\phi \cos\theta\, \hat{\theta} + \cos\phi\, \hat{\phi}}$$

Now,

$$\cos\theta\, \hat{r} = \sin\theta \cos\theta \cos\phi\, \hat{x} + \sin\theta \cos\theta \sin\phi\, \hat{y} + \cos^2\theta\, \hat{z}$$
$$-\quad \sin\theta\, \hat{\theta} = \underset{-}{\sin\theta \cos\theta \cos\phi}\, \hat{x} \underset{-}{+} \sin\theta \cos\theta \sin\phi\, \hat{y} \underset{+}{-} \sin^2\theta\, \hat{z}$$

$$\cos\theta\, \hat{r} - \sin\theta\, \hat{\theta} = 0 + 0 + \hat{z}$$

$$\therefore \quad \boxed{\hat{z} = \cos\theta\, \hat{r} - \sin\theta\, \hat{\theta}}$$

Q.39 (a) Check the divergence theorem for the function $v_1 = r^2 \hat{r}$, using as your volume the sphere of radius R, centered at the origin.

(b) Do the same for $v_2 = \left(\frac{1}{r^2}\right) \hat{r}$.

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(a) divergence theorem

$$\int_V (\nabla \cdot v)\, d\tau = \oint_S v \cdot da$$

Computing LHS of above equation:

$v = r^2 \hat{r}$

$$\therefore \nabla \cdot v = \frac{1}{r^2} \frac{\partial (r^2 \cdot r^2)}{\partial r} = \frac{4r^3}{r^2} = 4r$$

$$\therefore \int_V (\nabla \cdot v)\, d\tau = \int 4r \,(r^2 \sin\theta \, dr \, d\theta \, d\phi)$$

$$= 4 \int_0^R r^3 dr \int_0^\pi \sin\theta \, d\theta \int_0^{2\pi} d\phi$$

$$= 4 \cdot \frac{R^4}{4} \cdot 2 \times 2\pi$$

$$\int_V (\nabla \cdot v)\, d\tau = 4\pi R^4 \quad \text{—— (A)}$$

Computing RHS of divergence theorem:

Since, surface is a sphere centered at origin
$da = r^2 \sin\theta \, d\theta \, d\phi \, \hat{r}$ and r is constant $= R$

$$\therefore v \cdot da = R^4 \sin\theta \, d\theta \, d\phi$$

$$\oint_S v \cdot da = R^4 \int_0^\pi \sin\theta \, d\theta \int_0^{2\pi} d\phi$$

$$= R^4 \cdot 2 \cdot 2\pi = 4\pi R^4 \quad \text{—— (B)}$$

From (A) & (B)

$$\int_V (\nabla \cdot v)\, d\tau = \oint_S v \cdot da$$

Hence, proved.

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(b) $v = \frac{1}{r^2}\hat{r}$

$$\therefore \nabla\cdot v = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{1}{r^2}\right) = 0$$

$$\therefore \int_V (\nabla\cdot v)\, d\tau = 0 \quad \text{— (A)}$$

$$\oint_S v\cdot da = \int \left(\frac{1}{r^2}\right) r^2 \sin\theta \, d\theta \, d\phi$$

$$= \int_0^{\pi} \sin\theta \, d\theta \int_0^{2\pi} d\phi$$

$$= 4\pi \quad \text{— (B)}$$

from A and B, it is clear that divergence theorem is false for the given function. The reason for this is that the divergence is zero except at the origin, where v blows up, so our calculation of $\int \nabla\cdot v_2$ is incorrect. The right answer is 4π.

Q.40 Compute the divergence of the function

$$v = (r\cos\theta)\,\hat{r} + (r\sin\theta)\,\hat{\theta} + (r\sin\theta\cos\phi)\,\hat{\phi}$$

Check the divergence theorem for this function using the volume to be the inverted hemispherical bowl of radius R, resting on x-y plane and centred at origin.

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$$v = (r\cos\theta)\,\hat{r} + (r\sin\theta)\,\hat{\theta} + (r\sin\theta\cos\phi)\,\hat{\phi}$$

$$\nabla\cdot v = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \cdot r\cos\theta\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\cdot r\sin\theta\right)$$

$$+ \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\left(r\sin\theta\cos\phi\right)$$

$$\nabla\cdot v = \frac{3\cancel{r^2}\cos\theta}{\cancel{r^2}} + \frac{2\cancel{r\sin\theta}\cos\theta}{\cancel{r\sin\theta}} + \frac{\cancel{r\sin\theta}\,(-\sin\phi)}{\cancel{r\sin\theta}}$$

$$\boxed{\nabla\cdot v = 5\cos\theta - \sin\phi}$$

The divergence theorem states,

$$\int_V (\nabla\cdot v)\,d\tau = \oint_S v\cdot da$$

Computing LHS of divergence theorem

$$\int_V (\nabla\cdot v)\,d\tau = \int (5\cos\theta - \sin\phi)\, r^2\sin\theta\, dr\, d\theta\, d\phi$$

$$= \int_0^R r^2\,dr \int_0^{\pi/2}\left[\int_0^{2\pi}(5\cos\theta - \sin\phi)\,d\phi\right]\sin\theta\,d\theta$$

$$= \frac{R^3}{3}\int_0^{\pi/2}\left[\phi(5\cos\theta) + \cos\phi\right]_0^{2\pi}\sin\theta\,d\theta$$

$$= \frac{R^3}{3}\int_0^{\pi/2} 10\pi\cos\theta\sin\theta\,d\theta$$

$$= \frac{10\pi R^3}{6}\int_0^{\pi/2}\sin 2\theta\,d\theta$$

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$$\int_V (\nabla \cdot v)\, d\tau = \frac{5\pi R^3}{6}\left[\cos 2\theta\right]_{\pi/2}^{0}$$

$$= \frac{5\pi R^3}{6} \times 2$$

$$\therefore \int_V (\nabla \cdot v)\, d\tau = \frac{5\pi R^3}{3} \qquad \text{— (A)}$$

Computing the RHS of divergence theorem,

There are two surfaces,

(i) the hemisphere

$da = R^2 \sin\theta \, d\theta \, d\phi \, \hat{r}$

$\phi : 0 \to 2\pi$

$\theta : 0 \to \pi/2$

$r = R$

$$\int_{(i)} v \cdot da = \int (R\cos\theta) R^2 \sin\theta \, d\theta \, d\phi$$

$$= R^3 \int_0^{\pi/2} \sin\theta \cos\theta \, d\theta \int_0^{2\pi} d\phi$$

$$= R^3 \times \frac{1}{2} \times 2\pi$$

$$\int_{(i)} v \cdot da = \pi R^3$$

(ii) the flat bottom:

$da = (dr)(r\sin\theta \, d\phi)\, \hat{\theta}$

$= r \, dr \, d\phi \, \hat{\theta}$ (as θ will be $\pi/2$)

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$r: 0 \to R$ and $\phi: 0 \to 2\pi$

$$\int_{(ii)} v \cdot da = \int r^2 \sin\theta \, dr \, d\phi$$

$$= \int_0^R r^2 dr \int_0^{2\pi} d\phi$$

$$= \frac{R^3}{3} \times 2\pi = \frac{2\pi R^3}{3}$$

Now,

$$\oint_S v \cdot da = \int_{(i)} v \cdot da + \int_{(ii)} v \cdot da$$

$$= \pi R^3 + \frac{2\pi R^3}{3}$$

$$= \frac{5\pi R^3}{3} \quad \text{——— (B)}$$

from (A) & (B)

$$\int_V (\nabla \cdot v) d\tau = \oint_S v \cdot da$$

Hence, the divergence theorem is checked.

Q.41 Compute the gradient and Laplacian of the function $T = r(\cos\theta + \sin\theta \cos\phi)$. Check the Laplacian by converting T to cartesian coordinates and using equation.
Test the gradient theorem for this function, using the path shown in figure.

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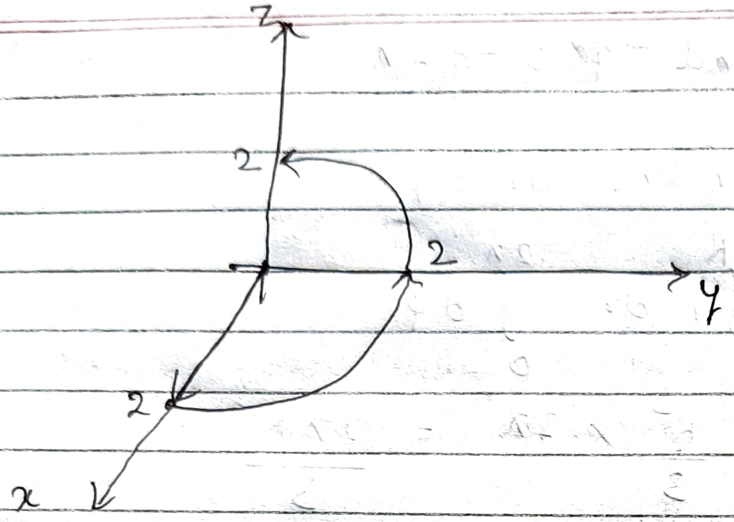

$$T = r(\cos\theta + \sin\theta\cos\phi)$$

$$\nabla T = \frac{\partial T}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial T}{\partial \theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial T}{\partial \phi}\hat{\phi}$$

$$= (\cos\theta + \sin\theta\cos\phi)\hat{r} + \frac{r(-\sin\theta + \cos\theta\cos\phi)}{r}\hat{\theta}$$

$$+ \frac{\cancel{r}}{r\sin\theta}(-\sin\theta\sin\phi)\hat{\phi}$$

$$\boxed{\nabla T = (\cos\theta + \sin\theta\cos\phi)\hat{r} + (\cos\theta\cos\phi - \sin\theta)\hat{\theta} - \sin\phi\,\hat{\phi}}$$

Laplacian of T,

$$\nabla\cdot(\nabla T) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2(\cos\theta + \sin\theta\cos\phi)\right)$$

$$+ \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta(\cos\theta\cos\phi - \sin\theta)\right)$$

$$+ \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}(-\sin\phi)$$

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$$\nabla^2 T = \frac{2}{r}(\cos\theta + \sin\theta\cos\phi) + \frac{\cos\theta(\cos\theta\cos\phi - \sin\theta) + \sin\theta(-\sin\theta\cos\phi - \cos\theta)}{r\sin\theta} + \frac{(-\cos\phi)}{r\sin\theta}$$

$$\nabla^2 T = \frac{2}{r}(\cos\theta + \sin\theta\cos\phi) + \frac{\cos^2\theta\cos\phi - \sin^2\theta\cos\phi - 2\sin\theta\cos\theta}{r\sin\theta} + \frac{(-\cos\phi)}{r\sin\theta}$$

$$= \frac{1}{r\sin\theta}\left[\cancel{2\sin\theta\cos\theta} + 2\sin^2\theta\cos\phi + \cos^2\theta\cos\phi - \cancel{\sin^2\theta\cos\phi} - \cancel{2\sin\theta\cos\theta} - \cos\phi\right]$$

$$= \frac{1}{r\sin\theta}\left[\cos\phi(\sin^2\theta + \cos^2\theta) - \cos\phi\right]$$

$$\boxed{\nabla^2 T = 0}$$

Checking Laplacian by converting in cartesian coordinates

$$x = r\sin\theta\cos\phi$$
$$y = r\sin\theta\sin\phi$$
$$z = r\cos\theta$$

$$\therefore T = (x+z) = r(\cos\phi\sin\theta + \cos\theta)$$

$$\therefore T = (x+z)$$

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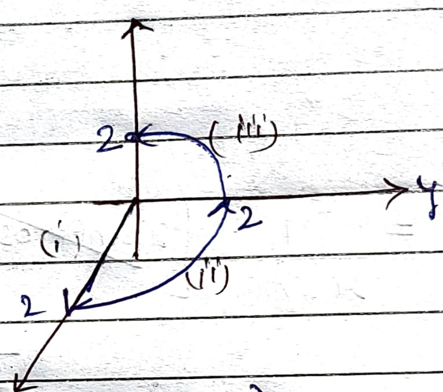
Now, $\nabla^2 T = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}$

$= 0$

Hence, we have checked Laplacian of T in cartesian coordinates

Testing the Gradient theorem

$$\int_a^b \nabla T \cdot dl = T(b) - T(a)$$



Here $a \equiv (0,0,0)$
and $b \equiv (0,0,2)$
and, $T = (x+z)$

$$\therefore \int_a^b \nabla T \cdot dl = T(0,0,2) - T(0,0,0)$$

$$\boxed{\int_a^b \nabla T \cdot dl = 2} \quad \text{--- (A)}$$

Now, computing the line integral along the path

Segment (i)

$\theta = \pi/2$, $\phi = 0$, $r: 0 \to 2$

$\therefore dl = dr\,\hat{r}$

$\therefore \nabla T \cdot dl = (\cos\theta + \sin\theta\cos\phi)\,dr$

$= dr$

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$$\int_{(i)} \nabla T \cdot dl = \int_0^2 dr = 2$$

Segment 2 :

$\theta = \pi/2$; $\phi : 0 \to \pi/2$; $r = 2$

$$dl = d\phi\, (r \sin\theta)\, \hat{\phi}$$
$$= 2\, d\phi\, \hat{\phi}$$

$$\therefore \nabla T \cdot dl = -2 \sin\phi \, d\phi$$

$$\therefore \int_{(ii)} \nabla T \cdot dl = -2 \int_0^{\pi/2} \sin\phi \, d\phi = -2$$

Segment 3 :

$r = 2$, $\phi = \pi/2$, $\theta : \pi/2 \to 0$

$$dl = r\, d\theta\, \hat{\theta} = 2\, d\theta\, \hat{\theta}$$
$$\nabla T \cdot dl = 2(\cos\theta \cos\phi - \sin\theta)\, d\theta$$
$$= -2 \sin\theta \, d\theta$$

$$\int_{(iii)} \nabla T \cdot dl = 2 \int_{\pi/2}^{0} (-\sin\theta)\, d\theta = 2$$

$$\therefore \int_a^b \nabla T \cdot dl = \int_{(i)} \nabla T \cdot dl + \int_{(ii)} \nabla T \cdot dl + \int_{(iii)} \nabla T \cdot dl$$

$$= 2 - 2 + 2$$

$$\boxed{\int_a^b \nabla T \cdot dl = 2} \quad \text{—— (B)}$$

from A & B , the gradient theorem is checked .