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MA101: Linear Algebra and Matrices
Tutorial 2.

1. Let $A = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 0 & 1 \\ 3 & 0 & 1 \end{bmatrix}$. Show that A is row equivalent to identity matrix by finding elementary matrices $E_1 E_2 \dots E_K$ such that $E_K E_{K-1} E_{K-2} \dots E_1 A = I$.

$$A = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 0 & 1 \\ 3 & 0 & 1 \end{bmatrix}$$

$R_2 \rightarrow R_2 - 2R_1$ and $R_3 \rightarrow R_3 - 3R_1$

$$A = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & -1 \\ 0 & 3 & -2 \end{bmatrix}$$

$R_2 \rightarrow R_2 - \frac{1}{2} R_3$

$$A = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1/2 & 0 \\ 0 & 3 & -2 \end{bmatrix}$$

$R_1 \rightarrow R_1 + \frac{1}{3} R_3$

$$A = \begin{bmatrix} 1 & 0 & 1/3 \\ 0 & 1/2 & 0 \\ 0 & 3 & -2 \end{bmatrix}$$

$R_3 \rightarrow R_3 - 6R_2$

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$$A = \begin{bmatrix} 1 & 0 & 1/3 \\ 0 & 1/2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

$R_1 \rightarrow R_1 + \frac{1}{6} R_3$

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

$R_2 \rightarrow 2 R_2$ and $R_3 \rightarrow -\frac{1}{2} R_3$

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Hence, A is row equivalent to identity matrix.

Elementary matrices

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$ (By applying $R_2 \rightarrow R_2 - 2R_1$ to identity matrix)

$$E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}$$ (By applying $R_3 \rightarrow R_3 - 3R_1$ to identity matrix)

$$E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1/2 \\ 0 & 0 & 1 \end{bmatrix}$$ (By applying $R_2 \rightarrow R_2 - \frac{1}{2} R_3$ to identity matrix)

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$$E_4 = \begin{bmatrix} 1 & 0 & 1/3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$ (By applying $R_1 \to R_1 + \frac{1}{3} R_3$ to identity matrix)

$$E_5 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -6 & 1 \end{bmatrix}$$ (By applying $R_3 \to R_3 - 6R_2$ to identity matrix)

$$E_6 = \begin{bmatrix} 1 & 0 & 1/6 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$ (By applying $R_1 \to R_1 + \frac{1}{6} R_3$ to identity matrix)

$$E_7 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$ (By applying $R_2 \to 2R_2$ to identity matrix)

$$E_8 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1/2 \end{bmatrix}$$ (By applying $R_3 \to -\frac{1}{2} R_3$ to identity matrix)

Hence, $E_1, E_2 \dots, E_8$ are elementary matrices such that $E_8 E_7 \dots E_1 A = I$

2. Find the reduced row echelon form (RREF) of following matrices

a) $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 4 & 5 & 6 & 7 \\ 6 & 7 & 8 & 9 \end{bmatrix}$ b) $\begin{bmatrix} 1 & 3 & 5 & 7 \\ 3 & 5 & 7 & 9 \\ 5 & 7 & 9 & 1 \end{bmatrix}$

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a) $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 4 & 5 & 6 & 7 \\ 6 & 7 & 8 & 9 \end{bmatrix}$

Applying $R_2 \rightarrow R_2 - 4R_1$ and $R_3 \rightarrow R_3 - 6R_1$

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -3 & -6 & -9 \\ 0 & -5 & -10 & -15 \end{bmatrix}$$

Applying $R_2 \rightarrow -\frac{1}{3} R_2$

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & -5 & -10 & -15 \end{bmatrix}$$

Applying $R_3 \rightarrow R_3 + 5R_2$

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Applying $R_1 \rightarrow R_1 - 2R_2$

$$\begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Hence, RREF of A is $\begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

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b) $\begin{bmatrix} 1 & 3 & 5 & 7 \\ 3 & 5 & 7 & 9 \\ 5 & 7 & 9 & 1 \end{bmatrix}$, let A be rref of given matrix

Applying $R_2 \to R_2 - 3R_1$ and $R_3 \to R_3 - 5R_1$

$$\begin{bmatrix} 1 & 3 & 5 & 7 \\ 0 & -4 & -8 & -12 \\ 0 & -8 & -16 & -34 \end{bmatrix}$$

Applying $R_2 \to -\frac{1}{4} R_2$

$$\begin{bmatrix} 1 & 3 & 5 & 7 \\ 0 & 1 & 2 & 3 \\ 0 & -8 & -16 & -34 \end{bmatrix}$$

Applying $R_3 \to R_3 + 8R_2$

$$\begin{bmatrix} 1 & 3 & 5 & 7 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & -10 \end{bmatrix}$$

Applying $R_1 \to R_1 - 3R_2$

$$\begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & -10 \end{bmatrix}$$

Applying $R_3 \to -\frac{1}{10} R_3$

$$A = \begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The matrix A is RREF of given matrix

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3. Find all solutions of following linear system:

a)
$$x_1 - 3x_3 = 8$$
$$2x_1 + 2x_2 + 9x_3 = 7$$
$$x_2 + 5x_3 = -2$$

The augmented matrix for the given system is

$$\left[\begin{array}{ccc|c} 1 & 0 & -3 & 8 \\ 2 & 2 & 9 & 7 \\ 0 & 1 & 5 & -2 \end{array}\right]$$

Applying $R_2 \rightarrow R_2 - 2R_1$

$$\left[\begin{array}{ccc|c} 1 & 0 & -3 & 8 \\ 0 & 2 & 15 & -9 \\ 0 & 1 & 5 & -2 \end{array}\right]$$

Applying $R_2 \rightarrow \frac{1}{2} R_2$

$$\left[\begin{array}{ccc|c} 1 & 0 & -3 & 8 \\ 0 & 1 & 15/2 & -9/2 \\ 0 & 1 & 5 & -2 \end{array}\right]$$

Applying $R_3 \rightarrow R_3 - R_2$

$$\left[\begin{array}{ccc|c} 1 & 0 & -3 & 8 \\ 0 & 1 & 15/2 & -9/2 \\ 0 & 0 & -5/2 & 5/2 \end{array}\right]$$

Applying $R_2 \rightarrow R_2 + 3R_3$

$$\left[\begin{array}{ccc|c} 1 & 0 & -3 & 8 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & -5/2 & 5/2 \end{array}\right]$$

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Applying $R_3 \rightarrow -\frac{2}{5} R_3$

$$\left[\begin{array}{ccc|c} 1 & 0 & -3 & 8 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

Applying $R_1 \rightarrow R_1 + 3R_3$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

Hence, solution to given linear system is
$x_1 = 5$, $x_2 = 3$ and $x_3 = -1$.

b)
$$\begin{aligned} x_1 - 3x_2 \quad\quad &= 5 \\ -x_1 + x_2 + 5x_3 &= 2 \\ x_2 + x_3 &= 0 \end{aligned}$$

The augmented matrix for given system is

$$\left[\begin{array}{ccc|c} 1 & -3 & 0 & 5 \\ -1 & 1 & 5 & 2 \\ 0 & 1 & 1 & 0 \end{array}\right]$$

Applying $R_2 \rightarrow R_2 + R_1$

$$\left[\begin{array}{ccc|c} 1 & -3 & 0 & 5 \\ 0 & -2 & 5 & 7 \\ 0 & 1 & 1 & 0 \end{array}\right]$$

Applying $R_3 \rightarrow R_3 + \frac{1}{2} R_1$

$$\left[\begin{array}{ccc|c} 1 & -3 & 0 & 5 \\ 0 & -2 & 5 & 7 \\ 0 & 0 & 7/2 & 7/2 \end{array}\right]$$

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~~the~~ ~~above~~ system of equations is reduced to

$$x_1 - 3x_2 = 5 \quad \text{—(I)}$$
$$-2x_2 + 5x_3 = 7 \quad \text{—(II)}$$
$$\frac{7}{2}x_3 = \frac{7}{2} \quad \text{—(III)}$$

Eq^n (III) gives $x_3 = 1$

Substituting $x_3 = 1$ in eq^n (II)

$$-2x_2 + 5 = 7$$
$$x_2 = -1$$

Substituting $x_2 = -1$ in eq^n (I).

$$x_1 - 3(-1) = 5$$
$$x_1 = 2$$

Hence, solution to given system of equations is $x_1 = 2$, $x_2 = -1$ and $x_3 = 1$

4. Determine which matrices are in Reduced Row Echelon Form (RREF) and which others are only in Row Echelon Form (REF).

(a) $A = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

a_{ij} is element of i^{th} row and j^{th} column

Since a_{11} is 1, and all other elements in column 1 are zero. We leave row 1 and column 1 and find next non-zero column,

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which is column 3. Since a_{23} is 1 and all entries below it in column 3 are zero, leave row 2 and column 3. and move to next row. Since next row is zero (a_{34} only), the given matrix is in Reduced Row Echelon Form.

b. $\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$

a_{ij} is element of i^{th} row and j^{th} column.

Since a_{11} is 1 and all other elements of column 1 are zero, leave row 1 and column 1.
Next we move to a_{22}, which is also 1, and elements of column 2 below a_{22} are 0.
Leaving row 2 and column 2, a_{33} is 1 and there are no more rows below it.

Hence, given matrix is in Row Echelon form (REF)

c. $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$

a_{ij} is element of i^{th} row and j^{th} column.
a_{11} is 1, but elements in column 1 below a_{11} are not zero.
Hence, given matrix is neither RREF nor REF.

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5. If A and B are square matrices and $I-AB$ is invertible, then show that $I-BA$ is also invertible. Use Hint
$B(I-AB) = (I-BA)B$

Let $C = (I-AB)^{-1}$

Given, $(I-BA)B = B(I-AB)$
Multiplying by C from right
$(I-BA)BC = B(I-AB)C$
$B = (I-BA)BC$ $\{\because C \text{ is } (I-AB)^{-1}\}$

~~$-I\,A\,I$~~

Multiply by A from right
$BA = (I-BA)BCA$
$-I + I + BA = (I-BA)BCA$
$I = I - BA + (I-BA)BCA$
$I = (I-BA)(I+BCA)$

$\because (I-BA)(I+BCA) = I$, $(I+BCA)$ must be inverse of $(I-BA)$

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6. Find the interpolating polynomial $P(t) = a_0 + a_1 t + a_2 t^2$ whose graph will pass through (1, 12), (2, 15) and (3, 16). Does there exist a cubic polynomial which will pass through these points? What about degree n polynomial.

$$P(t) = a_2 t^2 + a_1 t + a_0$$

$$12 = a_2 + a_1 + a_0 \Rightarrow a_0 + a_1 + a_2 = 12$$

$$15 = 4a_2 + 2a_1 + a_0 \Rightarrow a_0 + 2a_1 + 4a_2 = 15$$

$$16 = 9a_2 + 3a_1 + a_0 \Rightarrow a_0 + 3a_1 + 9a_2 = 16$$

Augmented matrix for given system of equations is,

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 12 \\ 1 & 2 & 4 & 15 \\ 1 & 3 & 9 & 16 \end{array}\right]$$

$R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 12 \\ 0 & 1 & 3 & 3 \\ 0 & 2 & 8 & 4 \end{array}\right]$$

$R_3 \to R_3 - 2R_2$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 12 \\ 0 & 1 & 3 & 3 \\ 0 & 0 & 2 & -2 \end{array}\right]$$

$R_3 \to \frac{1}{2} R_3$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 12 \\ 0 & 1 & 3 & 3 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

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System of equations is reduced to

$$a_0 + a_1 + a_2 = 12 \quad \text{—(I)}$$
$$a_1 + 3a_2 = 3 \quad \text{—(II)}$$
$$a_2 = -1 \quad \text{—(III)}$$

Substituting $a_2 = -1$ in (II)

$$a_1 - 3 = 3$$
$$a_1 = 6$$

Substituting $a_1 = 6$ and $a_2 = -1$ in (I)

$$a_0 + 6 - 1 = 12$$
$$a_0 = 7$$

Hence, required polynomial is
$P(t) = 7 + 6t - t^2$

For a three degree polynomial

$$P(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3$$

$$12 = a_0 + a_1 + a_2 + a_3$$
$$15 = a_0 + 2a_1 + 4a_2 + 8a_3$$
$$16 = a_0 + 3a_1 + 9a_2 + 27a_3$$

Augmented matrix for given system of equations is

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 12 \\ 1 & 2 & 4 & 8 & 15 \\ 1 & 3 & 9 & 27 & 16 \end{array}\right]$$

$R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 12 \\ 0 & 1 & 3 & 7 & 3 \\ 0 & 2 & 8 & 26 & 4 \end{array}\right]$$

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$R_3 \rightarrow R_3 - 2R_2$

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 12 \\ 0 & 1 & 3 & 7 & 3 \\ 0 & 0 & 2 & 12 & -2 \end{array}\right]$$

the given system of equation's is reduced to

$$a_0 + a_1 + a_2 + a_3 = 12$$
$$a_1 + 3a_2 + 7a_3 = 3$$
$$2a_2 + 12a_3 = -2$$

Here, a_0, a_1 and a_2 are dependent variables and a_3 ~~ca~~ is arbitrary. we can put any value of a_3 and find a_0, a_1 and a_2.
Hence, there exist some three degree polynomial which will pass through the given points.

for n degree polynomial)

$$P(t) = a_0 + a_1 t + \cdots + a_n t^n$$

$$a_0 + a_1 + \cdots + a_n = 12$$
$$a_0 + 2a_1 + \cdots + 2^n a_n = 15$$
$$a_0 + 3a_1 + \cdots + 3^n a_n = 16$$

Augmented matrix for these equations is

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 \cdots\cdots & 1 & 12 \\ 1 & 2 & 4 \cdots\cdots & 2^n & 15 \\ 1 & 3 & 9 \cdots\cdots & 3^n & 16 \end{array}\right]$$

$R_2 \rightarrow R_2 - R_1$ and $R_3 \rightarrow R_3 - R_1$

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$$\left[\begin{array}{cccccc|c} 1 & 1 & 1 & \cdots & \cdots & 1 & 12 \\ 0 & 1 & 3 & \cdots & \cdots & 2^n-1 & 3 \\ 0 & 2 & 8 & \cdots & \cdots & 3^n-1 & 4 \end{array}\right]$$

$R_3 \rightarrow R_3 - 2R_2$

$$\left[\begin{array}{cccccc|c} 1 & 1 & 1 & \cdots & \cdots & 1 & 12 \\ 0 & 1 & 3 & \cdots & \cdots & 2^n-1 & 3 \\ 0 & 0 & 2 & \cdots & \cdots & (3^n-1)-2(2^n-1) & -2 \end{array}\right]$$

Now in this system of equations variable $a_3, a_4 \cdots a_n$ are arbitrary and a_0, a_1, a_2 can be found using back substitution. Hence, there exist n degree polynomial which will pass through given points $(n \geq 2)$.

7. Choose h and k such that the system has (a) no solution, (b) unique solution and (c) infinitely many solutions.

$$x_1 + hx_2 = 2$$
$$4x_1 + 8x_2 = k$$

a) Augmented matrix for given system is

$$\left[\begin{array}{cc|c} 1 & h & 2 \\ 4 & 8 & k \end{array}\right]$$

$R_2 \longrightarrow R_2 - 4R_1$

$$\left[\begin{array}{cc|c} 1 & h & 2 \\ 0 & 8-4h & k-8 \end{array}\right]$$

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for no solution, pivot entry $8-4h$ must be zero and ~~any~~ $k-8$ should not be equal to zero.

i.e. $8-4h = 0$
$h = 2$

and, $k-8 \neq 0$
$k \neq 8$

Hence, for no solution $h=2$ and $k \neq 8$.

b) $\left[\begin{array}{cc|c} 1 & h & 2 \\ 4 & 8 & k \end{array}\right]$

$R_2 \to R_2 - 4R_1$

$\left[\begin{array}{cc|c} 1 & h & 2 \\ 0 & 8-4h & k-8 \end{array}\right]$

~~$R_1 \to R_1$~~

for unique solution, pivot ~~column~~ entry should be non-zero

$\therefore$ $8-4h \neq 0$
$h \neq 2$

Hence, for unique solution to exist $h \neq 2$ and k can be anything.

c) $\left[\begin{array}{cc|c} 1 & h & 2 \\ 4 & 8 & k \end{array}\right] \xrightarrow{R_2 \to R_2 - 4R_1} \left[\begin{array}{cc|c} 1 & h & 2 \\ 0 & 8-4h & k-8 \end{array}\right]$

for infinitely many solutions one of the pivot entry should be zero and corresponding row element of b should also be 0.

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i.e. $8-4h=0$ and $k-8=0$

Hence, for infinitely many solutions to exist $h=2$ and $k=8$.

8. Suppose a 3×5 matrix (coefficient matrix) for a system has three pivot columns. Is the system consistent? Why or why not?

Since, the coefficient matrix is 3×5 and there are three pivot columns.
Since, there is one ~~entry~~ pivot entry in each row and it is given that they are non-zero.
So we cannot obtain a row of the form $[0, 0 \ 0 \ 0 \ 0 \mid b]$ where b is part of augmented matrix.
As we can not obtain a row of the form $[0 \ 0 \ 0 \ 0 \ 0 \mid b]$, the given system of equation is consistent.

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9. Suppose a system of linear equations has a 3x5 augmented matrix whose fifth column is the pivot column. Is the system consistent? Why or why not?

Here it is given that the 3x5 matrix is augmented i.e. first four columns ~~are~~ represent variables.
Since, fifth column is pivot column, we ~~may~~ obtain a row of the form of
[0 0 0 0 | b]
As a system with a row of augmented matrix [0 0 0 0 | b] will not have any solution, Hence, such a system would be inconsistent.