Archit Agrawal
202052307

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PH110: Waves and Electromagnetics

Tutorial 7

Q.9 A dipole p is at a distance r from a point charge q, and oriented so that p makes an angle θ with a vector r from q to p.
(a) What is the force on p?
(b) what is the force on q?

(a) $F = (p \cdot \nabla) E$

$$E = \frac{1}{4\pi\epsilon_0} \frac{q}{r^2} \hat{r} = \frac{q}{4\pi\epsilon_0} \frac{x\hat{x} + y\hat{y} + z\hat{z}}{(x^2+y^2+z^2)^{3/2}}$$

$$\therefore F_x = \left(p_x \frac{\partial}{\partial x} + p_y \frac{\partial}{\partial y} + p_z \frac{\partial}{\partial z}\right) \frac{q}{4\pi\epsilon_0} \frac{x}{(x^2+y^2+z^2)^{3/2}}$$

$$F_x = \frac{q}{4\pi\epsilon_0} \left\{ p_x \left[\frac{1}{(x^2+y^2+z^2)^{3/2}} - \frac{3}{2} x \cdot \frac{2x}{(x^2+y^2+z^2)^{5/2}} \right] \right\}$$

$$+ \frac{q}{4\pi\epsilon_0} \left\{ p_y \left[-\frac{3}{2} x \cdot \frac{2y}{(x^2+y^2+z^2)^{5/2}} \right] + p_z \left[-\frac{3x}{2} \cdot \frac{2z}{(x^2+y^2+z^2)^{5/2}} \right] \right\}$$

$$F_x = \frac{q}{4\pi\epsilon_0} \left[\frac{p_x}{r^3} - \frac{3x}{r^5} (p_x x + p_y y + p_z z) \right]$$

$$F_x = \frac{q}{4\pi\epsilon_0} \left[\frac{p}{r^3} - \frac{3r(p \cdot r)}{r^5} \right]_x$$

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$$\therefore \boxed{F = \frac{q}{4\pi\epsilon_0 r^3}\left[p - 3(p\cdot\hat{r})\hat{r}\right]}$$

(b) $$E = \frac{1}{4\pi\epsilon_0}\frac{1}{r^3}\left\{3\left(p(-\hat{r})\right)(-\hat{r}) - p\right\}$$

$$= \frac{1}{4\pi\epsilon_0}\frac{1}{r^3}\left[3(p\cdot\hat{r})\hat{r} - p\right]$$

-ve sign indicates that r points towards S.

$$F = qE$$

$$\therefore \boxed{F = \frac{q}{4\pi\epsilon_0 r^3}\left[3(p\cdot\hat{r})\hat{r} - p\right]}$$

The forces are equal and opposite, as expected from Newton's Third Law.

Q.11 A short cylinder of radius a and length L, carries a 'frozen in' uniform polarization P, parallel to its axis. Find the bound charge and sketch the electric field.

(i) for $L >> a$
(ii) for $L << a$
(iii) for $L \approx a$

$\rho_b = 0$
$\sigma_b = \pm P$ (plus sign ~~toward~~ at one end- the one P points towards: minus sign at the other end - the one P points away from).

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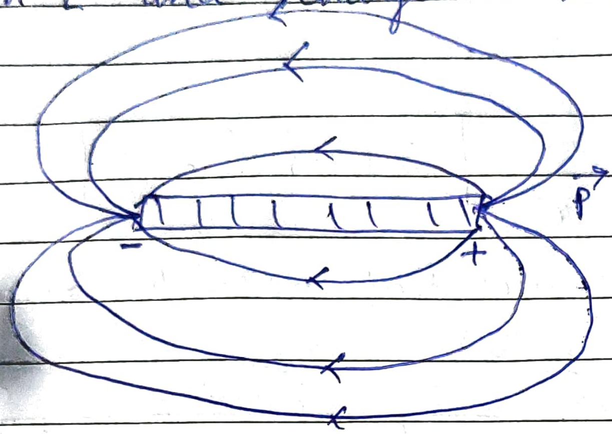
(i) $L >> a$

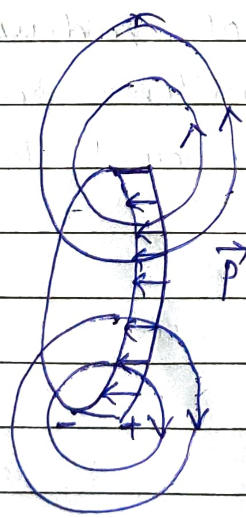
Then the end looks like point charges, and the whole thing is like a physical dipole, of length L and charge $P\pi a^2$



Like a dipole

(ii) $L << a$

Then it is like a circular plate capacitor: Field is nearly uniform inside; nonuniform "fringing" field at the edges



Like a parallel plate capacitor

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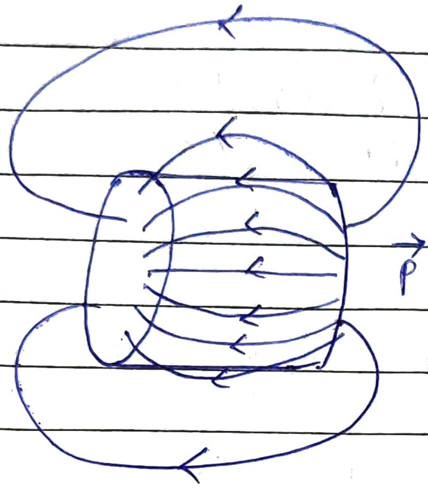
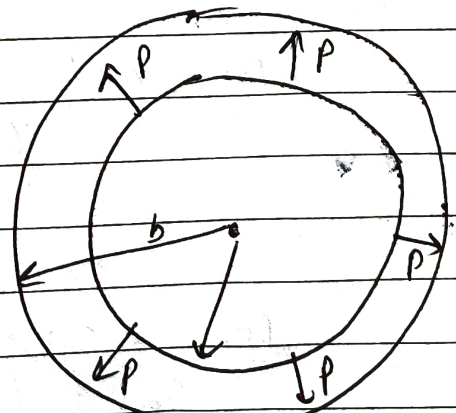
(iii) $L \approx a$



Q.15 A thick spherical shell (inner radius a, outer radius b) is made of dielectric material with a 'frozen in' polarisation

$$P(r) = \frac{k}{r}\hat{r}$$

where k is a constant and r is the distance from centre (there is no free charge in the problem). Find the electric field in all different regions by two methods:



(a) Locate all the bound charge, and use Gauss's Law to calculate the field it produces

(b) use eq 4.23 to find D, and then get E from eq. 4.21

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$$\rho_b = -\nabla \cdot P$$

$$\rho_b = -\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{k}{r}\right)$$

$$\rho_b = \frac{-k}{r^2}$$

$$\therefore \quad \sigma_b = P \cdot \hat{n}$$

$$= \begin{cases} + P\cdot\hat{r} = k/b & (\text{at } r=b) \\ - P\cdot\hat{r} = -k/a & (\text{at } r=a) \end{cases}$$

Gauss's Law $\Rightarrow$

$$E = \frac{1}{4\pi\epsilon_0}\frac{Q_{enc}}{r^2}\hat{r}$$

For $r<a$

$$Q_{enc} = 0$$

$$\therefore \quad E = 0$$

For $r>b$,

$$Q_{enc} = 0$$

$$\therefore \quad E = 0$$

For $a<r<b$

$$Q_{enc} = \left(-\frac{k}{a}\right)(4\pi a^2) + \int_a^r \left(\frac{-k}{\bar{r}^2}\right) 4\pi\bar{r}^2\, d\bar{r}$$

$$Q_{enc} = -4\pi k a - 4\pi k(r-a)$$

$$Q_{enc} = -4\pi k r$$

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$$\therefore E = -\left(\frac{k}{\epsilon_0 r}\right)\hat{r}$$

(b) $\oint D \cdot da = Q_{f\,enc} = 0$

$D = 0$ everywhere

$D = \epsilon_0 E + P = 0$

$$\therefore E = \left(\frac{-1}{\epsilon_0}\right)P$$

Q.16 Suppose the field inside a large piece of dielectric is E_0, so that the electric displacement is $D_0 = \epsilon_0 E_0 + P$

(a) Now a small spherical cavity (Fig 4.19a) is hollowed out of the material. Find the field at the centre of the cavity in terms of E_0 and P. Also, find the displacement at the centre of the cavity in terms of D_0 and P. Assume the polarisation is 'frozen in' so it doesn't change when the cavity is excavated.

(b) Do the same for a long needle-shaped cavity running parallel to P (Fig 4.19 b)

(c) Do the same for a thin wafer-shaped cavity perpendicular to P (Fig 4.19 c)

Assume that the cavities are small enough that P, E_0 and D_0 are essentially uniform.

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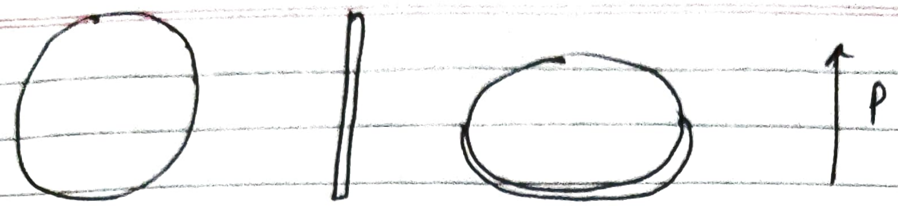

4.19 (a) 4.19(b) 4.19(c)

(a) It is same as E_0 minus the field at the centre of a sphere with uniform polarization P which is equal to $-P/3\epsilon_0$.

$$\boxed{\therefore E = E_0 + \frac{P}{3\epsilon_0}}$$

$$\therefore D = \epsilon_0 E$$

$$D = \epsilon_0 E_0 + \frac{P}{3}$$

$$\boxed{\text{So, } D = D_0 - \frac{2}{3}P}$$

(b) Same as E_0 minus the field of $\pm$ charges at the two ends of the needle - but these are small and far away, so

$$\boxed{E = E_0}$$

$$D = \epsilon_0 E = \epsilon_0 E_0$$

$$\boxed{D = D_0 - P}$$

(c) Same as E minus the field of a parallel plate capacitor with upper plate at $\sigma = P$, which is equal to $-\frac{P}{\epsilon_0}$.

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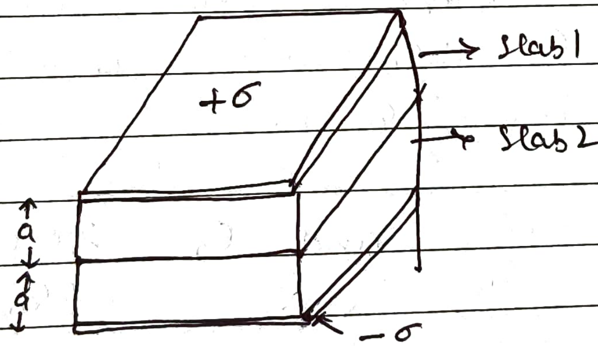
$$\therefore \boxed{E = E_0 + \frac{P}{\epsilon_0}}$$

$$\therefore D = \epsilon_0 E$$
$$= \epsilon_0 E_0 + P$$

So, $\boxed{D = D_0}$

Q.18 The space between the plates of a parallel-plate capacitor (Fig 4-24) is filled with two slabs of linear dielectric material. Each slab has thickness a, so the total distance between the plates is 2a. Slab 1 has dielectric constant of 2 and slab 2 has k = 1.5. The free charge density on the top plate is σ and on the bottom plate is $-\sigma$.



(a) Find the electric displacement D in each slab.
(b) Find the electric field E in each slab.
(c) Find the polarisation P in each slab.
(d) Find the potential difference between plates.
(e) Find the location and amount of all bound charge.
(f) Now that you know all the charge, recalculate the field in each slab, and confirm your answer to (b).

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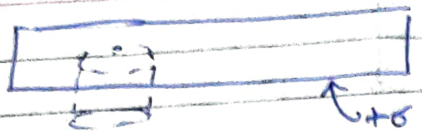
(a)



Applying $\int D \cdot da = Q_{fenc}$ to the Gaussian surface

$$DA = \sigma A$$

$$\therefore \quad D = \sigma \qquad (\text{as } D = 0 \text{ inside the metal plate})$$

This is true in both the slabs ! D points down.

(b) $D = \sigma E$

$$E_1 = \frac{\sigma}{\epsilon_1} \quad (\text{in slab } 1)$$

$$E_2 = \frac{\sigma}{\epsilon_2} \quad (\text{in slab } 2)$$

But, $\epsilon = \epsilon_0 \epsilon_r$

So, $\epsilon_1 = 2\epsilon_0$

$$\epsilon_2 = \frac{3}{2}\epsilon_0$$

So, $\boxed{E_1 = \frac{\sigma}{2\epsilon_0} \quad \text{and} \quad E_2 = \frac{2\sigma}{3\epsilon_0}}$

(c) $P = \epsilon_0 \chi_e E$, so

$$P = \frac{\epsilon_0 \chi d}{\epsilon_0 \epsilon_r} = \left(\frac{\chi_e}{\epsilon_r}\right)\sigma$$

$$\because \chi_e = \epsilon_r - 1$$

$$\therefore P = \left(1 - \frac{1}{\epsilon_r}\right)\sigma$$

$\therefore \quad \boxed{P_1 = \frac{\sigma}{2} \quad \text{and} \quad P_2 = \frac{\sigma}{3}}$

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(d) $V = E_1 a + E_2 a$

$$V = \frac{\sigma a}{2\epsilon_0} + \frac{2\sigma a}{3\epsilon_0}$$

$$\boxed{V = \frac{7\sigma a}{6\epsilon_0}}$$

(e) $\rho_b = 0$

$$\sigma_b = \begin{cases} +P_1 & \text{at bottom of slab 1} = \sigma/2 \\ -P_1 & \text{at top of slab 1} = -\sigma/2 \end{cases}$$

$$\sigma_b = \begin{cases} +P_2 & \text{at bottom of slab 2} = \sigma/3 \\ -P_2 & \text{at top of slab 2} = -\sigma/3 \end{cases}$$

(f) In slab 1,

total surface charge above $= \sigma - \frac{\sigma}{2} = \frac{\sigma}{2}$

total surface charge below $= \frac{\sigma}{2} - \frac{\sigma}{3} + \frac{\sigma}{3} - \sigma$

$= -\frac{\sigma}{2}$

$$\therefore \boxed{E_1 = \frac{\sigma}{2\epsilon_0}}$$

In slab 2,

total surface charge above $= \sigma - \frac{\sigma}{2} + \frac{\sigma}{2} - \frac{\sigma}{2} = \frac{2\sigma}{3}$

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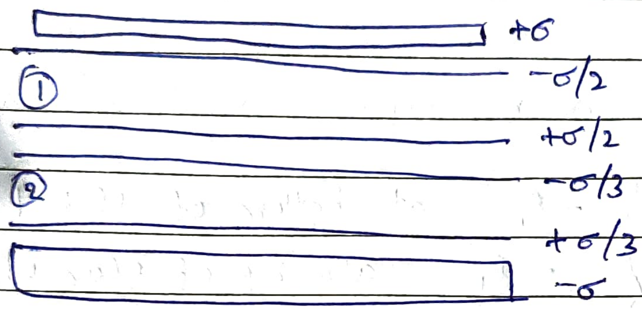
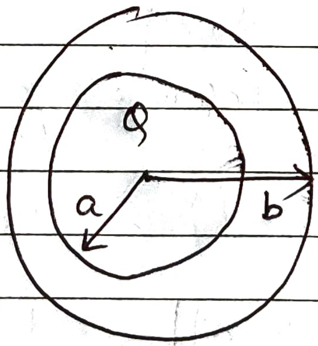
total surface charge below $= \frac{\sigma}{3} - \sigma = \frac{-2\sigma}{3}$

$\therefore \boxed{E_2 = \frac{2\sigma}{3\epsilon_0}}$

E_1 and E_2 match with the result of (b)



Q.26 A spherical conductor, of radius a, carries a charge Q. It is surrounded by linear dielectric material of susceptibility χ_e, out of the radius b. Find the energy of this configuration.



$$D = \begin{cases} 0 & , r < a \\ \frac{Q}{4\pi r^2}\hat{r} & , r \geq a \end{cases}$$

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$$E = \begin{cases} 0 & , r < a \\ \dfrac{Q}{4\pi\epsilon_0 r^2}\hat{r} & , a \leq r < b \\ \dfrac{Q}{4\pi\epsilon_0 r^2}\hat{r} & , r > b \end{cases}$$

$$W = \frac{1}{2}\int D \cdot E \, d\tau$$

$$W = \frac{1}{2}\frac{Q^2 \cdot 4\pi}{(4\pi)^2}\left\{\frac{1}{\epsilon}\int_a^b \frac{1}{r^2}\cdot\frac{1}{r^2} r^2 \, dr + \frac{1}{\epsilon_0}\int_b^\infty \frac{1}{r^2} dr\right\}$$

$$= \frac{Q^2}{8\pi}\left\{\frac{1}{\epsilon}\left(\frac{-1}{r}\right)\Big|_a^b + \frac{1}{\epsilon_0}\left(\frac{-1}{r}\right)\Big|_b^\infty\right\}$$

$$= \frac{Q^2}{8\pi\epsilon_0}\left\{\frac{1}{(1+\chi_e)}\left(\frac{1}{a} - \frac{1}{b}\right) + \frac{1}{b}\right\}$$

$$\boxed{\therefore W = \frac{Q^2}{8\pi\epsilon_0 (1+\chi_e)}\left(\frac{1}{a} + \frac{\chi_e}{b}\right)}$$

Q.28. Two long co-axial cylindrical metal tubes (inner radius a, outer radius b) stand vertically in a tank of dielectric oil (susceptibility χ_e, mass density ρ). The inner one is maintained at potential V, and the outer one is grounded. To what height h does the oil rise, in the space between the tubes?

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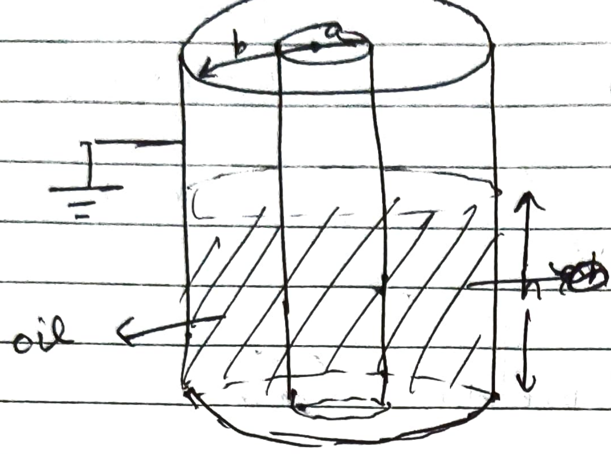

Let us find the capacitance as a function of h.

air part: $E = \frac{2\lambda}{4\pi\epsilon_0 s}$

$V = \frac{2\lambda}{4\pi\epsilon_0} \ln\left(\frac{b}{a}\right)$

$\frac{\lambda}{\epsilon_0} = \frac{\lambda'}{\epsilon}$

Oil Part: $D = \frac{2\lambda'}{4\pi s}$

$E = \frac{2\lambda'}{4\pi\epsilon_0 s}$

$V = \frac{2\lambda'}{4\pi\epsilon} \ln\left(\frac{b}{a}\right)$

$\lambda' = \frac{\epsilon}{\epsilon_0}\lambda$

$\lambda' = \epsilon_r \lambda$

$\therefore Q = \lambda' h + \lambda(l - h)$

$= \epsilon_r \lambda h - \lambda h + \lambda l$

$Q = \lambda\left[(\epsilon_r - 1)h + l\right]$

$Q = \lambda\left[\chi_e h + l\right]$ (l is total height)

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$$\therefore \quad C = \frac{Q}{V} = \frac{\lambda(\chi_e h + l)(4\pi\epsilon_0)}{2\pi \ln(b/a)}$$

$$\therefore \quad C = \frac{2\pi\epsilon_0 (\chi_e h + l)}{\ln(b/a)}$$

Net upward force,

$$f = \frac{1}{2} V^2 \frac{dC}{dh} = \frac{1}{2} V^2 \frac{2\pi\epsilon_0 \chi_e}{\ln(b/a)}$$

The gravitational force,

$$f = mg$$

$$f = \rho\pi(b^2 - a^2) g h$$

$$\therefore \quad F_{upward} = F_{downward}$$

$$\frac{V^2 \cancel{\pi}\epsilon_0 \chi_e}{\ln(b/a)} = \rho\cancel{\pi}(b^2 - a^2) g h$$

$$\boxed{h = \frac{V^2 \epsilon_0 \chi_e}{\rho(b^2 - a^2) g \ln(b/a)}}$$