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PH110 : Waves and Electrodynamics

Tutorial 6

13. Find the potential in the infinite slot of Ex 3.3 if the boundary at $x=0$ consists of two metal stripes : one from $y=0$ to $y=a/2$ and held at constant potential V_0 and the other from $y=a/2$ to $y=a$ at potential $-V_0$.

$$\frac{\partial^2 V}{\partial^2 x} + \frac{\partial^2 V}{\partial^2 y} = 0$$

boundary conditions

$V = 0$ at $y = 0, a$

$V \to 0$ as $x \to \infty$

$$V = \begin{cases} V_0 & \text{at } x=0, 0<y<a/2 \\ -V_0 & \text{at } x=0, a/2 \leq y < a \end{cases}$$

Let $V = \left(Ae^{kx} + Be^{-kx}\right)\left(C\sin ky + D\cos ky\right)$

$V=0$ at $y=0$

$$0 = \left(Ae^{kx} + Be^{-kx}\right)(0 + D)$$

$\therefore\ D = 0$

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$V = 0$ at $x \to \infty$

$$0 = \left(Ae^{kx} + Be^{-kx}\right)\left(C \sin ky\right)$$

$e^{kx} \to \infty$ as $x \to \infty$

$\therefore A = 0$

$$\therefore V(x,y) = C_n e^{-kx} \sin ky$$

$V = V_0(y)$ at $x = 0$

$$V_0(y) = C \sin(ky)$$

$V = 0$ at $y = a$

$$\therefore \sin(ka) = 0$$

$$ka = n\pi$$

$$k = \frac{n\pi}{a}$$

$$V(x,y) = C e^{-\frac{n\pi x}{a}} \sin\left(\frac{n\pi y}{a}\right)$$

$$V(x,y) = \sum_{n=1}^{\infty} C_n e^{-\frac{n\pi x}{a}} \sin\left(\frac{n\pi y}{a}\right)$$

$$V(0,y) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi y}{a}\right) = V_0(y)$$

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$$C_n = \frac{2V_0}{a}\left\{\int_0^{a/2} \sin(n\pi y/a)\, dy - \int_{a/2}^{a} \sin\left(\frac{n\pi y}{a}\right) dy\right\}$$

$$C_n = \frac{2V_0}{a}\left\{ -\frac{\cos(n\pi y/a)}{(n\pi/a)}\Bigg|_0^{a/2} + \frac{\cos(n\pi y/a)}{(n\pi/a)}\Bigg|_{a/2}^{a}\right\}$$

$$C_n = \frac{2V_0}{\pi n}\left\{-\cos\left(\frac{n\pi}{2}\right) + 1 + \cos(n\pi) - \cos\left(\frac{n\pi}{2}\right)\right\}$$

$$C_n = \frac{2V_0}{\pi n}\left\{1 + (-1)^n - 2\cos\left(\frac{n\pi}{2}\right)\right\}$$

$$1 + (-1)^n - 2\cos\left(\frac{n\pi}{2}\right) = \begin{cases} 0 , & (n-2)\% 4 \mathrel{!}= 0 \\ 4 , & (n-2)\% 4 = 0 \end{cases}$$

Therefore

$$C_n = \begin{cases} \frac{8V_0}{n\pi} & \text{if } n = 2, 6, 10, 14 \ldots \\ 0 & \text{otherwise} \end{cases}$$

$$\therefore \quad V(x,y) = \sum_{n=2,6,10,14\ldots} \frac{8V_0}{\pi} \frac{e^{-n\pi x/a}\sin(n\pi y/a)}{n}$$

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3-27. A sphere of radius R, centered at the origin, carries charge density

$$\rho(r,\theta) = k\frac{R}{r^2}(R-2r)\sin\theta,$$

where k is a constant, and r, θ are usual spherical coordinates. Find approximate potential for points on z-axis far from sphere.

Monopole Term

$$Q = \int \rho\, d\tau = kR\int \frac{1}{r^2}(R-2r)\sin\theta\, r^2 \sin\theta\, dr\, d\theta\, d\phi$$

but the r integral

$$\int_0^R (R-2r)\, dr = (Rr-r^2)\Big|_0^R = 0,$$

hence, $Q = 0$.

Dipole Term

$$\int r\cos\theta\,\rho\, d\tau = kR\int r\cos\theta \cdot \frac{1}{r^2}\sin\theta(R-2r)r^2\sin\theta\, dr\, d\theta\, d\phi$$

but the θ integral

$$\int_0^\pi \sin^2\theta\cos\theta\, d\theta = \frac{\sin^3\theta}{3}\Big|_0^\pi = 0$$

So, dipole contribution is zero.

Quadrupole Term

$$\int r^2 \left(\frac{3}{2}\cos^2\theta - \frac{1}{2}\right) \rho \, d\tau$$

$$= \frac{kR}{2} \int\int\int r^2 (3\cos^2\theta - 1) \left[\frac{1}{r^2}(R-2r)\sin\theta\right] r^2 \sin\theta \, dr \, d\theta \, d\phi$$

r integral

$$\int_0^R r^2 (R-2r)\, dr = \left.\frac{r^3 R}{3} - \frac{r^4}{2}\right|_0^R = \frac{-R^4}{6}$$

θ- integral

$$\int_0^\pi (3\cos^2\theta - 1)\sin^2\theta \, d\theta = 2\int_0^\pi \sin^2\theta \, d\theta - 3\int_0^\pi \sin^4\theta \, d\theta$$

$$= 2\left(\frac{\pi}{2}\right) - 3\left(\frac{3\pi}{8}\right) = \frac{-\pi}{8}$$

ϕ integral

$$\int_0^{2\pi} d\phi = 2\pi$$

$\therefore$ the whole integral is $= \frac{1}{2} kR \left(\frac{-R^4}{6}\right)\left(\frac{-\pi}{8}\right)(2\pi)$

$$= \frac{k\pi^2 R^5}{48}.$$

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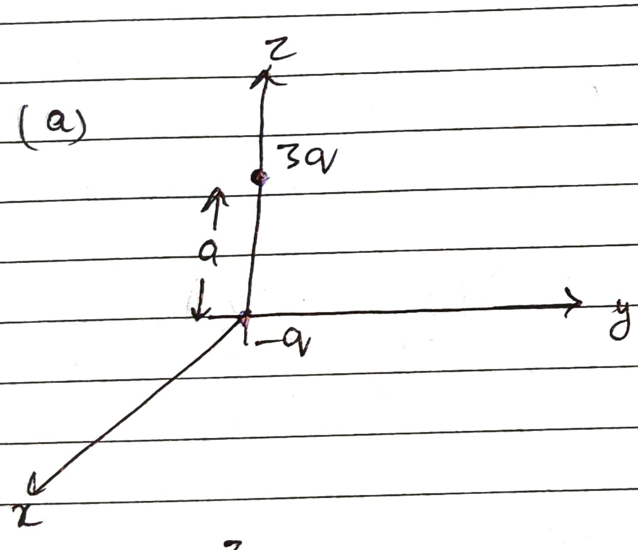
For point P on z-axis, the approximate potential is

$$V(z) \cong \frac{1}{4\pi\epsilon_0} \frac{k\pi^2 R^5}{48 z^3}$$

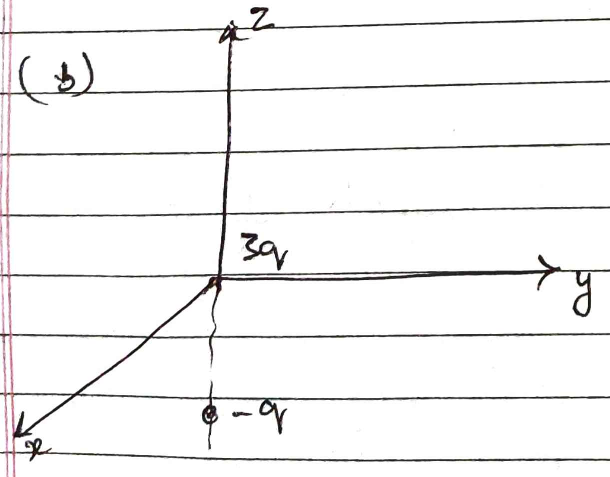
(32) Two point charges 3q and -q, are seperated by distance a. For each of the arrangements, find (i) the monopole moment, (ii) dipole moment

(iii) approximate potential at large r.

(a)



(b)



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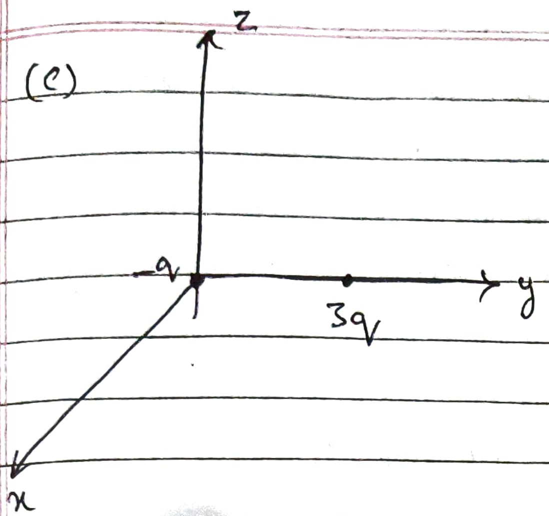
(c)



(a) monopole moment $Q = 2q$

dipole moment $p = 3qa\,\hat{z}$

$$V \cong \frac{1}{4\pi\epsilon_0}\left[\frac{Q}{r} + \frac{p\cdot\hat{r}}{r^2}\right] = \frac{1}{4\pi\epsilon_0}\left[\frac{2q}{r} + \frac{3qa\cos\theta}{r^2}\right]$$

(b) monopole moment $Q = 2q$

dipole moment $p = qa\,\hat{z}$

$$V \cong \frac{1}{4\pi\epsilon_0}\left[\frac{2q}{r} + \frac{qa\cos\theta}{r^2}\right]$$

(c) monopole moment $= 2q$

dipoler moment $= 3qa\,\hat{y}$

$$V \cong \frac{1}{4\pi\epsilon_0}\left[\frac{2q}{r} + \frac{3qa\sin\theta\sin\phi}{r^2}\right]$$

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(33) A "pure" dipole is situated at origin, pointing in z-direction.

(a) What is the force on a point charge q at $(a, 0, 0)$?

(b) What is the force on q at $(0, 0, a)$?

(c) How much work does it take to move q from $(a, 0, 0)$ to $(0, 0, a)$?

(a) This point is at $r = a$, $\theta = \pi$ and $\phi = 0$. so

$$\vec{E} = \frac{p}{4\pi\epsilon_0 a^3}\hat{\theta}, \quad = \frac{p}{4\pi\epsilon_0 a^3}(-\hat{z})$$

$$\therefore \vec{F} = q\vec{E} = -\frac{pq}{4\pi\epsilon_0 a^3}(\hat{z})$$

(b) $r = a$, $\theta = 0$

$$\vec{E} = \frac{p}{4\pi\epsilon_0 a^3}(2\hat{r}) = \frac{2p}{4\pi\epsilon_0 a^3}\hat{z}$$

$$\therefore \vec{F} = q\vec{E} = \frac{2pq}{4\pi\epsilon_0 a^3}\hat{z}$$

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(c) $W = q\left[V_{(0,0,a)} - V_{(a,0,0)} \right]$

$$= \frac{q p}{4\pi\epsilon_0 a^2}\left(\cos 0 - \cos \pi/2\right)$$

$$= \frac{pq}{4\pi\epsilon_0 a^2}$$

Chapter 4 Questions

2. According to Quantum Mechanics, the electron cloud for a hydrogen atom in ground state has a charge density

$$\rho(r) = \frac{q}{\pi a^3} e^{-2r/a}$$

where q is the charge of electron and a is Bohr's radius. Find the atomic polarizibility of such an atom.

Let us first find the field at radius r, using Gauss Law.

$$\oint E \cdot da = \frac{Q_{enc}}{\epsilon_0}$$

$$E = \frac{Q_{enc}}{4\pi\epsilon_0 r^2}$$

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$$Q_{enc} = \int_0^r \rho \, d\tau$$

$$= \frac{4\pi q}{\pi a^3} \int_0^r e^{-2\bar{r}/a} \bar{r}^2 \, d\bar{r}$$

$$= \frac{4q}{a^3} \left[-\frac{a}{2} e^{-2\bar{r}/a} \left(\bar{r}^2 + a\bar{r} + \frac{a^2}{2} \right) \right] \Bigg|_0^r$$

$$Q_{enc} = -\frac{2q}{a^2} \left[e^{-2r/a} \left(r^2 + ar + \frac{a^2}{2} \right) - \frac{a^2}{2} \right]$$

$$Q_{enc} = q \left[1 - e^{-2r/a} \left(1 + \frac{2r}{a} + \frac{2r^2}{a^2} \right) \right]$$

So the field of electron cloud is

$$E_e = \frac{1}{4\pi\epsilon_0} \frac{q}{r^2} \left[1 - e^{-2r/a} \left(1 + \frac{2r}{a} + \frac{2r^2}{a^2} \right) \right]$$

The proton will be shifted from $r=0$ to point d where $E_e = E$.

$$E = \frac{1}{4\pi\epsilon_0} \frac{q}{d^2} \left[1 - e^{-2d/a} \left(1 + \frac{2d}{a} + \frac{2d^2}{a^2} \right) \right]$$

Expanding in powers of d/a

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$$e^{-2d/a} = 1 - \left(\frac{2d}{a}\right) + \frac{1}{2}\left(\frac{2d}{a}\right)^2 - \frac{1}{3!}\left(\frac{2d}{a}\right)^3 \cdots$$

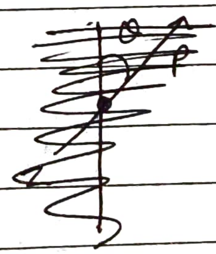
$$1 - e^{-2d/a}\left(1 + \frac{2d}{a} + \frac{2d^2}{a^2}\right) = 1 - \left(1 - \frac{2d}{a} + 2\left(\frac{d}{a}\right)^2 + \cdots\right)\left(1 + \frac{2d}{a} + \frac{2d^2}{a^2}\right)$$

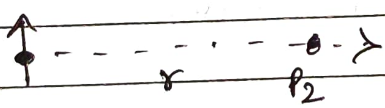
$$= \frac{4}{3}\left(\frac{d}{a}\right)^3 + \text{higher order terms}$$

$$\therefore\ E = \frac{q}{4\pi\epsilon_0 d^2}\left(\frac{4}{3}\frac{d^3}{a^3}\right)$$

$$\therefore\ \boxed{\alpha = 3\pi\epsilon_0 a^3}$$

5. In fig. p_1 and p_2 are (perfect) dipoles at distance r apart. What is the torque on p_1 due to p_2? What is the torque on p_2 due to p_1?





Field of p_1 at p_2 $(\theta = \pi/2)$: $E_1 = \dfrac{p_1}{4\pi\epsilon_0 r^3}\ \hat{\theta}$

Torque on p_2: $N_2 = p_2 \times E_1$

$$= p_2 E_1 \sin 90^\circ = p_2 E_1$$

$$= \frac{p_1 p_2}{4\pi\epsilon_0 r^3} \quad \text{(points inside the page)}$$

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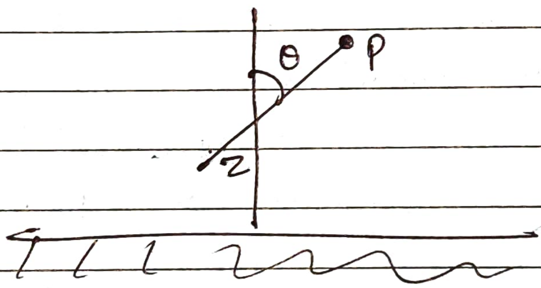
Field of p_2 at p_1 $(\theta = \pi)$: $E_2 = \dfrac{p_2}{4\pi\epsilon_0 r^3}(-2\hat{r})$

(points to the right)

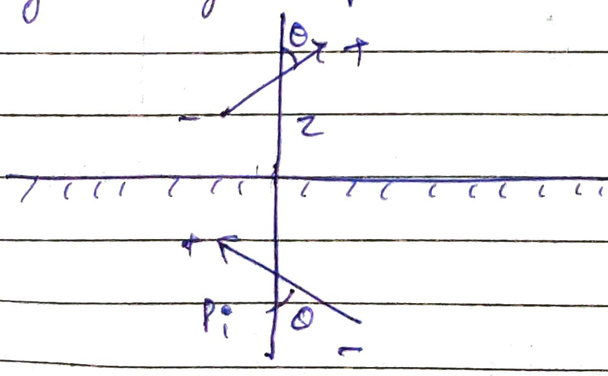
Torque on p_1: $N_1 = p_1 \times E_2 = \dfrac{2p_1p_2}{4\pi\epsilon_0 r^3}$

(points into the page)

6. A perfect dipole p is situated at a distance z above an infinite grounded conducting plane. The dipole makes an angle θ with the perpendicular to the plane. Find the torque on p. If the dipole is free to rotate. In what direction will it come to rest?



Using image dipole as shown



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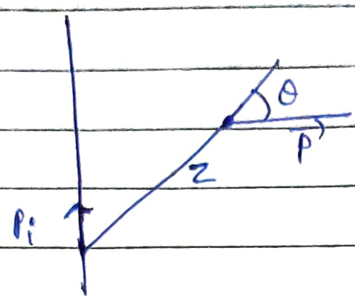
redrawing placing p_i at origin



$$E_i = \frac{p}{4\pi\epsilon_0 (2z)^3} (2\cos\theta \, \hat{r} + \sin\theta \, \hat{\theta})$$

$$p = p\cos\theta \, \hat{r} + p\sin\theta \, \hat{\theta}$$

$$N = p \times E_i$$

$$= \frac{p^2}{4\pi\epsilon_0 (2z)^3} \left[(\cos\theta \, \hat{r} + \sin\theta \, \hat{\theta}) \times (2\cos\theta \, \hat{r} + \sin\theta \, \hat{\theta}) \right]$$

$$= \frac{p^2}{4\pi\epsilon_0 (2z)^3} \left[\cos\theta \sin\theta \, \hat{\phi} + 2\sin\theta\cos\theta \, (-\hat{\phi}) \right]$$

$$N = \frac{p^2 \sin\theta \cos\theta}{4\pi\epsilon_0 (2z)^3} (-\hat{\phi}) \quad \text{(out of the page)}$$

But $\sin\theta \cos\theta = \frac{\sin 2\theta}{2}$,

so $N = \frac{p^2 \sin 2\theta}{4\pi\epsilon_0 (16 z^3)}$

For $0 < \theta < \pi/2$, N tends to rotate p counterclockwise;

for $\pi/2 < \theta < \pi$, N rotates p clockwise.

Thus the stable configuration is perpendicular to surface, either in or out.