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MA101: Linear Algebra and Matrices
Tutorial 5

1. Let T be the linear transformation whose standard matrix is given below. Decide if T is one-one, onto and hence bijective.

a) $A = \begin{bmatrix} -5 & 10 & -5 & 4 \\ 8 & 3 & -4 & 7 \\ 4 & -9 & 5 & -3 \\ -3 & -2 & 5 & 4 \end{bmatrix}$

the transformation T is from R^4 to R^4.

T is one-one iff $Ax = 0$ has only trivial solutions and T is onto iff columns of A span R^4.

reducing A to REF of A

$$A = \begin{bmatrix} -5 & 10 & -5 & 4 \\ 8 & 3 & -4 & 7 \\ 4 & -9 & 5 & -3 \\ -3 & -2 & 5 & 4 \end{bmatrix}$$

$R_2 \to R_2 + \frac{8}{5} R_1$

$R_3 \to R_3 + \frac{4}{5} R_1$

$R_4 \to R_4 - \frac{3}{5} R_1$

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$$= \begin{bmatrix} -5 & 10 & -5 & 4 \\ 0 & 19 & -12 & 67/5 \\ 0 & -1 & 1 & 1/5 \\ 0 & -8 & 8 & 8/5 \end{bmatrix}$$

$R_3 \to R_3 + \frac{R_2}{19}$

$R_4 \to R_4 + \frac{8R_2}{19}$

$$= \begin{bmatrix} -5 & 10 & -5 & 4 \\ 0 & 19 & -12 & 67/5 \\ 0 & 0 & 7/19 & 86/95 \\ 0 & 0 & 56/19 & 688/95 \end{bmatrix}$$

$R_4 \to R_4 - 8R_3$

$$\text{REF of } A = \begin{bmatrix} -5 & 10 & -5 & 4 \\ 0 & 19 & -12 & 67/5 \\ 0 & 0 & 7/19 & 86/95 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\because$ there are only three pivot columns, a free variable exist, hence $Ax=0$ will have non-trivial solution. Hence, T is not one-one.

Also, there are three pivot entries in ref(A), hence it will not span $\mathbb{R}^4$. Hence, it is not onto.

$\therefore$ T is neither one-one nor onto.

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(ii) $A = \begin{bmatrix} 4 & -7 & 3 & 7 & 5 \\ 6 & -8 & 5 & 12 & -8 \\ -7 & 10 & -8 & -9 & 14 \\ 3 & -5 & 4 & 2 & -6 \\ -5 & 6 & -6 & -7 & 3 \end{bmatrix}$

the transformation T is from R^5 to R^5.
$\therefore$ T is one-one only if $Ax=0$ has only trivial solutions and T is onto only if the columns of A span R^5.

reducing A to REF of A

$$A = \begin{bmatrix} 4 & -7 & 3 & 7 & 5 \\ 6 & -8 & 5 & 12 & -8 \\ -7 & 10 & -8 & -9 & 14 \\ 3 & -5 & 4 & 2 & -6 \\ -5 & 6 & -6 & -7 & 3 \end{bmatrix}$$

$R_2 \rightarrow R_2 - \frac{3}{2} R_1$

$R_3 \rightarrow R_3 + \frac{7}{4} R_1$

$R_4 \rightarrow R_4 - \frac{3}{4} R_1$

$R_5 \rightarrow R_5 + \frac{5}{4} R_1$

$$= \begin{bmatrix} 4 & -7 & 3 & 7 & 5 \\ 0 & 5/2 & 1/2 & 3/2 & -31/2 \\ 0 & -9/4 & -11/4 & 13/4 & 91/4 \\ 0 & 1/4 & 7/4 & -13/4 & -39/4 \\ 0 & -11/4 & -9/4 & 7/4 & 37/4 \end{bmatrix}$$

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$$R_3 \rightarrow R_3 + \frac{9}{10} R_2$$

$$R_4 \rightarrow R_4 - \frac{1}{10} R_2$$

$$R_5 \rightarrow R_5 + \frac{11}{10} R_2$$

$$= \begin{bmatrix} 4 & -7 & 3 & 7 & 5 \\ 0 & 5/2 & 1/2 & 3/2 & -31/2 \\ 0 & 0 & -23/10 & 23/5 & 44/5 \\ 0 & 0 & 17/10 & -17/5 & -41/5 \\ 0 & 0 & -17/10 & 17/5 & -39/5 \end{bmatrix}$$

$$R_4 \rightarrow R_4 + \frac{17}{23} R_3$$

$$R_5 \rightarrow R_5 - \frac{17}{23} R_3$$

$$= \begin{bmatrix} 4 & -7 & 3 & 7 & 5 \\ 0 & 5/2 & 1/2 & 3/2 & -31/2 \\ 0 & 0 & -23/10 & 23/5 & 44/5 \\ 0 & 0 & 0 & 0 & -39/23 \\ 0 & 0 & 0 & 0 & -329/23 \end{bmatrix}$$

$$R_5 \rightarrow R_5 - \left(\frac{329}{39}\right) R_4$$

$$= \begin{bmatrix} 4 & -7 & 3 & 7 & 5 \\ 0 & 5/2 & 1/2 & 3/2 & -31/2 \\ 0 & 0 & -23/10 & 23/5 & 44/5 \\ 0 & 0 & 0 & 0 & -39/23 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

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Since, there are only four pivot columns, therefore, a free variable exists. Hence, there exists a non-trivial solution of $Ax = 0$. Therefore, the given linear transformation is not one-one.

Also, there are only four pivot columns, therefore columns of A do not span $\mathbb{R}^5$, hence the given transformation is not onto.

$\therefore$ T is neither one-one nor onto.

2. A linear transformation $T: \mathbb{R}^2 \to \mathbb{R}^2$ first reflects the points through the x_1-axis and then reflects the point through x_2-axis. Find matrix representation of T. Is T one-one or onto or both?

Let there be a point (x_1, x_2). Since the transformation reflects the points through x_1-axis, the point becomes $(x_1, -x_2)$. The transformation now reflects this point in x_2-axis. Hence, the point becomes $(-x_1, -x_2)$.

Therefore, the transformation is,

$$T : \mathbb{R}^2 \to \mathbb{R}^2, \quad T\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -x_1 \\ -x_2 \end{bmatrix}$$

Let matrix representation of T be A. Since T is transformation from $\mathbb{R}^2 \to \mathbb{R}^2$, the matrix A is a 2×2 matrix whose columns are $T(e_1)$ and $T(e_2)$, where e_1 and e_2 are columns of 2×2 identity matrix.

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$$A = \begin{bmatrix} T(e_1) & T(e_2) \end{bmatrix}$$

$$T(e_1) = T\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

$$T(e_2) = T\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

$$A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

For T to be one-one, $Ax = 0$ must have only trivial solutions. A is in its row echelon form and since A has 2 pivot columns, $Ax = 0$ has only trivial solutions, hence it is one-to-one.

For T to be onto, columns of A must span R^2. Since matrix A has two pivot entries, hence column of A span R^2. Hence, transformation is onto.

Hence, the linear transformation is both one-one and onto.

3. Is following subset of R^n a subspace? If yes, prove it else give reason?

a) $V = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \text{ such that } x_1 \geq 0 \right\}$

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For a subset to be subspace of R^n,

i) it should contain zero vector.

ii) if u and v are vectors in subset, then u+v should also be in ~~subspace~~ subset.

(iii) if u is in ~~subspace~~ subset, then cu should also be in ~~subspace~~ subset, where c is any scalar

Since, $x_1 \geq 0$ and $x_2, x_2 \in R$

$\therefore \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}^T$ lies in the given subset.

Now, suppose $u = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$ and $w = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ are in V.

$\therefore a_1 \geq 0$ and $b_1 \geq 0$ and $a_2, b_2, a_3, b_3 \in R$

$\therefore u + w = \begin{bmatrix} a_1 + b_1 \\ a_2 + b_2 \\ a_3 + b_3 \end{bmatrix}$

$\because a_1 \geq 0$ and $b_1 \geq 0$, $\therefore a_1 + b_1$ is also greater than 0.

and, $a_2 + b_2$ and $a_3 + b_3$ are also real.

Hence, u + w also lie in subset V.

Now, if $u = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$, then $cu = \begin{bmatrix} ca_1 \\ ca_2 \\ ca_3 \end{bmatrix}$

If $c < 0$, ca_1 is also less than 0. Hence, ca_1 is not in the subset.

Hence, given subset of R^n is not a subspace.

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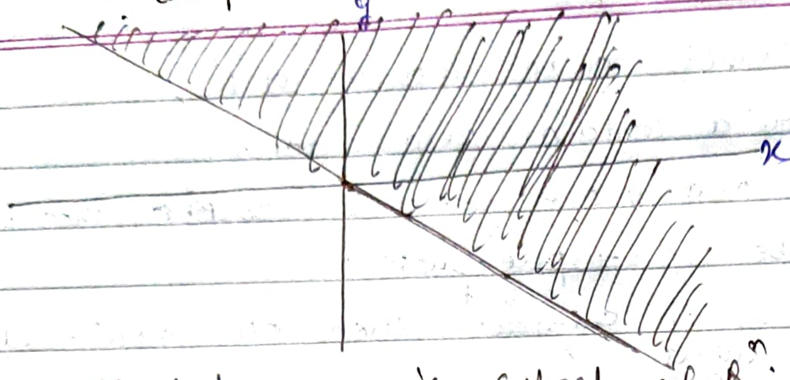
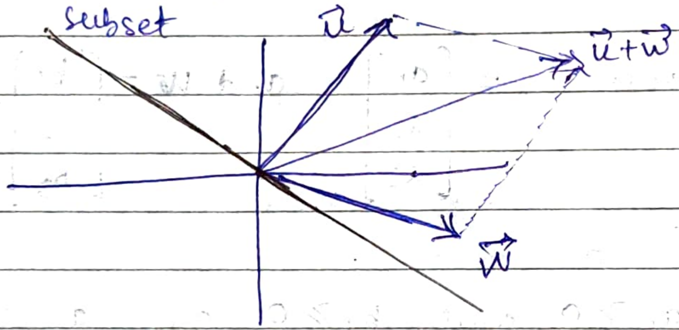
b)



Shaded area is subset of R^n.

Clearly, zero vector is in given subset.

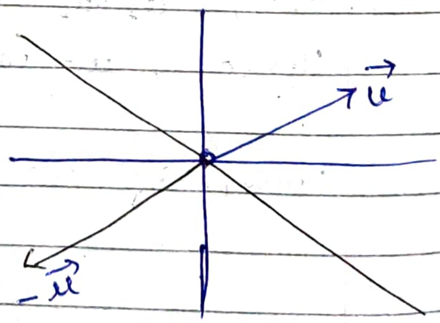
Now, suppose if there are two vectors in given subset



Since, the resultant of two vectors always lie between them. ~~Between~~ By parallelogram law an example is shown. Hence, $\vec{u}+\vec{w}$ also lies in ~~sub~~ given subset

Now, suppose a vector $\vec{u}$. ~~is~~ in given subset.

Multiplying it by a negative scalar -1, that is $-\vec{u}$ is not in given subset.



Hence, given subset is not a subspace.

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4. Determine if w is in the subspace generated by v_1, v_2 where

$$v_1 = \begin{bmatrix} 2 \\ 3 \\ -5 \end{bmatrix}, \quad v_2 = \begin{bmatrix} -4 \\ -5 \\ 8 \end{bmatrix}, \quad \text{and } w = \begin{bmatrix} 8 \\ 2 \\ -9 \end{bmatrix}$$

w will be in subspace of v_1 and v_2 only if

$$c_1 v_1 + c_2 v_2 = w$$

the ~~vector~~ matrix equation for the above equation is

$$Ax = w \text{ , has a solution.}$$

where $A = \begin{bmatrix} 2 & -4 \\ 3 & -5 \\ -5 & 8 \end{bmatrix}$ and $w = \begin{bmatrix} 8 \\ 2 \\ -9 \end{bmatrix}$

augmented matrix for the system is

$$\left[\begin{array}{cc|c} 2 & -4 & 8 \\ 3 & -5 & 2 \\ -5 & 8 & -9 \end{array}\right]$$

$$R_2 \to R_2 - \frac{3}{2} R_1$$

$$R_3 \to R_3 + \frac{5}{2} R_1$$

$$\left[\begin{array}{cc|c} 2 & -4 & 8 \\ 0 & 1 & -10 \\ 0 & -2 & 11 \end{array}\right]$$

$$R_3 \to R_3 + 2R_2$$

$$\left[\begin{array}{cc|c} 2 & -4 & 8 \\ 0 & 1 & -10 \\ 0 & 0 & -9 \end{array}\right]$$

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Since $0 \neq -9$, the given system of equation is inconsistent.
Hence w is not in subspace of v_1 and v_2.

5. Find a basis of Col(A) and a basis for Nul(A).
Here matrix and its REF is given.

$$A = \begin{bmatrix} -3 & 9 & -2 & -7 \\ 2 & -6 & 4 & 8 \\ 3 & -9 & -2 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 6 & 9 \\ 0 & 0 & 4 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Col A is the set spanned by columns of A.
Since, there are only two pivot entries in REF of A, Col A is R^2.
the non-pivot columns are ~~only~~ linear combinations of columns having pivot entries, i.e.

$$\begin{bmatrix} -3 \\ 0 \\ 0 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

and

$$\begin{bmatrix} 9 \\ 5 \\ 0 \end{bmatrix} = \frac{5}{4} \begin{bmatrix} 6 \\ 4 \\ 0 \end{bmatrix} + \frac{3}{2} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$\therefore$ basis of Col A contains pivot columns of A.

$$B(\text{Col}A) = \left\{ \begin{bmatrix} 6 \\ 4 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}$$

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Null(A) is the solution of matrix equation $AX = 0$

$\therefore$ augmented matrix for the system will be

$$\left[\begin{array}{cccc|c} -3 & 9 & -2 & -7 & 0 \\ 2 & -6 & 4 & 8 & 0 \\ 3 & -9 & -2 & 2 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{cccc|c} 1 & -3 & 6 & 9 & 0 \\ 0 & 0 & 4 & 5 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Since, column 2 and 4 are non-pivot columns, variable x_2 and x_4 are free variables.

New system of equation is,

$$4x_3 + 5x_4 = 0$$

$$x_3 = \frac{-5x_4}{4} \quad \text{—— (1)}$$

$$x_1 - 3x_2 + 6x_3 + 9x_4 = 0$$

$$x_1 = 3x_2 - 6x_3 - 9x_4$$

$$x_1 = 3x_2 - 6\left(\frac{-5x_4}{4}\right) - 9x_4$$

$$x_1 = 3x_2 - \frac{3x_4}{2}$$

Parametric solution of $AX = 0$ is,

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_2 \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -3/2 \\ 0 \\ -5/4 \\ 1 \end{bmatrix}$$

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Since x_2 and x_4 are free variables

$$\text{Nul } A = \text{Span}\left\{ \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3/2 \\ 0 \\ -5/4 \\ 1 \end{bmatrix} \right\}$$

Clearly, the vectors $\begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} -3/2 \\ 0 \\ -5/4 \\ 1 \end{bmatrix}$ are linearly independent.

Hence, basis of $\text{Nul } A = \left\{ \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3/2 \\ 0 \\ -5/4 \\ 1 \end{bmatrix} \right\}$

Q.6 What can you say about Nul (B) when B is 5×4 matrix with linearly independent columns?

Since, B is 5×4 matrix with linearly independent columns, we have 4 pivot entries in REF of B.

$\therefore$ dimension of Col B = rank of B = 4

If a matrix B, has n columns,

$$\text{rank } B + \dim(\text{Nul } B) = n$$
$$4 + \dim(\text{Nul } B) = 4$$
$$\dim(\text{Nul } B) = 0$$

$\because$ dimension of $\{0\}$ is 0.

hence, $\text{Nul } B = \{0\}$.

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7. Let W be subset of R^3 defined by,

$$W = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in R^3 \;\middle|\; 5x_1 - 2x_2 + x_3 = 0 \right\}$$

Find a 1×3 matrix A such that $W = \text{Nul}(A)$, the null space of A.

Let the 1×3 matrix be $\begin{bmatrix} a_1 & a_2 & a_3 \end{bmatrix}$

Since, $W = \text{Nul}(A)$

$$\therefore \quad AW = 0$$

$$\begin{bmatrix} a_1 & a_2 & a_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$a_1x_1 + a_2x_2 + a_3x_3 = 0 \quad \text{---(I)}$$

given, $5x_1 - 2x_2 + x_3 = 0 \quad$ ---(II)

Comparing (I) and (II),

$$\frac{a_1}{5} = \frac{a_2}{-2} = \frac{a_3}{1} = d \text{ (say)}$$

$\therefore$ $a_1 = 5d$, $a_2 = -2d$ and $a_3 = d$.

$$\therefore \quad A = \begin{bmatrix} 5d & -2d & d \end{bmatrix}$$

Putting $d = 2$, gives $A = \begin{bmatrix} 10, & -4, & 2 \end{bmatrix}$

Hence, A is a matrix such that $W = \text{Nul}(A)$.

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8. Let U and V be subspaces of n-dimensional vector space R^n. Prove that the intersection $U \cap V$ is also a subspace of R^n. What can you say about union of two subspaces of R^n?

Since U and V are subspaces, they both contain zero vector, hence, $U \cap V$ also contains zero vector.

Suppose, vectors a and b are in $U \cap V$.

$\therefore$ $a \in U$ and $b \in V$
$b \in U$ and $b \in V$

Since, U and V are subspaces and $a, b \in U$ and V

$\therefore$ $a+b \in U$ and $a+b \in V$

$\Rightarrow$ $a+b \in U \cap V$.

Now, since vector $a \in U \cap V$,

$a \in U$ and $a \in V$,

since U and V are subspaces
$ca \in U$ and $ca \in V$, where c is some scalar.

$\Rightarrow$ $ca \in U \cap V$

Since, $U \cap V$ has all three properties of a subspace.
Hence, $U \cap V$ is a subspace.

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⇒ Now, $U \cup V$

Since, U and V are subspaces, they both contain $\{0\}$, hence $U \cup V$ also contains $\{0\}$.

Since in $\mathbb{R}^2$, a line through origin is a subspace.
∴ let U is x-axis and V is y-axis.

Now, ~~a vector~~ $U \cup V$ is x-axis and y-axis.

Now, a point in $U \cup V$ is $(1,0)$ and ~~a point~~ another is $(0,1)$, but their sum $(1,1)$ is not in $U \cup V$.
Hence, $U \cup V$ is not a subspace if U and V are subspaces.

9. Construct a nonzero 3x3 matrix A and a vector b such that b is not in Col(A).

Since A is 3x3 matrix and b is not in Col A, therefore, in REF of A, at most two pivot entries should be there. Because if REF of A has three pivot entries then it would span all $\mathbb{R}^3$ and there would be no vector b which is not in Col A.

$$\therefore \quad A = \begin{bmatrix} 1 & -3 & 7 \\ 0 & 5 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

Now, vector $b = \begin{bmatrix} 3 & 5 & 5 \end{bmatrix}^T$ is not in Col A.

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Q.10 Let T be a linear transformation from R^n to itself. If T is one to one show that T is onto. Is converse true?

Since, T is transformation from R^n to R^n and it is one-to one. Let A be standard matrix representation of T. Then T is one-to-one only if $Ax=0$ has only trivial solutions. $Ax=0$ will have trivial solutions only if A has n pivot entries in its REF. Now, if REF of A has n pivot columns, then columns of A will span R^n. Hence, it is onto.

Now, checking the converse i.e. if T is onto it should be one-to-one.
If T is onto, columns of A should span R^n, which only happens if A has n pivot entries. Since, A has n pivot entries, $Ax=0$ has only trivial solutions. Hence, T is one-to one.
Hence, the converse is also true.