Archit Agrawal
202052307

PH110: Waves and Electrodynamics

Tutorial 14

Ch. 8. Q.1 :- Calculate the power (energy per unit time) transported down the cables of Ex 7.13 and Prob 7.62, assuming the two conductors are held at potential difference V, and carry current I (down one and back up the other).

Example 7.13

$$E = \frac{\lambda}{2\pi\epsilon_0}\frac{1}{s}\hat{s}$$

$$B = \frac{\mu_0 I}{2\pi s}\hat{\phi}$$

∴ the Poynting's vector S,

$$S = \frac{1}{\mu_0}(E \times B) = \frac{\lambda I}{4\pi^2\epsilon_0 s^2}\hat{z}$$

∴ power $P = \int S \cdot da = \int_a^b S(2\pi s)\,ds$

$$= \frac{\lambda I}{2\pi\epsilon_0}\int_a^b \frac{ds}{s}$$

$$\boxed{P = \frac{\lambda I}{2\pi\epsilon_0}\ln\left(\frac{b}{a}\right)}$$

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$\because$ potential $V = \int_a^b E \cdot dl = \frac{\lambda}{2\pi\epsilon_0} \int_a^b \frac{1}{s} ds = \frac{\lambda}{2\pi\epsilon_0} \ln\left(\frac{b}{a}\right)$

$$\therefore \boxed{P = VI}$$

Prob 7.58

$$E = \frac{\sigma}{\epsilon_0} \hat{z}$$

$$B = \mu_0 K \hat{x} = \frac{\mu_0 I}{w} \hat{x}$$

the poynting's vectors,

$$S = \frac{1}{\mu_0}(E \times B) = \frac{\sigma I}{\epsilon_0 w} \hat{y}$$

$$P = \int S \cdot da = Swh = \frac{\sigma I h}{\epsilon_0}$$

$$\boxed{\therefore P = \frac{\sigma I h}{\epsilon_0}}$$

but $V = \int E \cdot dl = \frac{\sigma h}{\epsilon_0}$

$$\therefore \boxed{P = VI}$$

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Q.2 Consider the charging capacitor in Prob. 7.34.

(a) Find the electric and magnetic fields in the gap, as functions of the distance s from the axis and the time t. (Assume the charge is zero at $t=0$).

(b) Find the energy density u_{em} and the Poynting vector S in the gap. Note especially the direction of S. Check that eq 8.12 is satisfied.

(c) Determine the total energy in the gap, as a function of time. Calculate the total power flowing into the gap, by integrating the Poynting vector over the appropriate surface. Check that the power input is equal to the rate of increase in energy in the gap.

(a) $$E = \frac{\sigma}{\epsilon_0}\hat{z}$$

and, $$\sigma = \frac{Q}{\pi a^2}$$

$\because$ $$Q(t) = It$$

$$\boxed{\therefore \quad E(t) = \frac{It}{\pi\epsilon_0 a^2}\hat{z}}$$

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Now, $B \cdot 2\pi s = \mu_0 \epsilon_0 \frac{\partial E}{\partial t} \pi s^2$

$$B = \mu_0 \epsilon_0 \frac{I}{\pi \epsilon_0 a^2} \pi s^2 \cdot \frac{1}{2\pi s}$$

$$\boxed{\therefore \; B(s,t) = \frac{\mu_0 I s}{2\pi a^2} \hat{\phi}}$$

(b) $\because$ $u_{em} = \frac{1}{2}\left(\epsilon_0 E^2 + \frac{1}{\mu_0} B^2\right)$

$$= \frac{1}{2}\left[\epsilon_0 \left(\frac{It}{\pi \epsilon_0 a^2}\right)^2 + \frac{1}{\mu_0}\left(\frac{\mu_0 I s}{2\pi a^2}\right)^2\right]$$

$$u_{em} = \frac{\mu_0 I^2}{2\pi^2 a^4}\left[\frac{t^2}{\epsilon_0 \mu_0} + \mu_0 \left(\frac{s}{2}\right)^2\right]$$

$$\boxed{u_{em} = \frac{\mu_0 I^2}{2\pi^2 a^4}\left[(ct)^2 + \left(\frac{s}{2}\right)^2\right]}$$

Now, Poynting Vector S,

$$S = \frac{1}{\mu_0}(E \times B) = \frac{1}{\mu_0}\left(\frac{It}{\pi \epsilon_0 a^2}\right)\left(\frac{\mu_0 I s}{2\pi a^2}\right)(-\hat{s})$$

$$\boxed{S = -\frac{I^2 t}{2\pi^2 \epsilon_0 a^4} s \, \hat{s}}$$

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$$\frac{\partial U_{em}}{\partial t} = \frac{\mu_0 I^2}{2\pi^2 a^4} 2c^2 t = \frac{I^2 t}{\pi^2 \epsilon_0 a^4}$$

and,

$$-\nabla \cdot S = \frac{I^2 t}{2\pi^2 \epsilon_0 a^4} \nabla \cdot (s\hat{s})$$

$$\boxed{-\nabla \cdot S = \frac{I^2 t}{\pi^2 \epsilon_0 a^4} = \frac{\partial U_{em}}{\partial t}}$$

Hence, eq 8.12 satisfies

(c) $U_{em} = \int u_{em}\, w\, 2\pi s\, ds$

$$= 2\pi w \frac{\mu_0 I^2}{2\pi^2 a^4} \int_0^b \left[(ct)^2 + \left(\frac{s}{2}\right)^2\right] s\, ds$$

$$= \frac{\mu_0 w I^2}{\pi a^4} \left[\frac{(ct)^2 s^2}{\cancel{4}} + \frac{s^4}{16}\right]_0^b$$

$$\boxed{U_{em} = \frac{\mu_0 w I^2 b^2}{2\pi a^4}\left[(ct)^2 + \frac{b^2}{8}\right]}$$

Over a surface at radius b:

$$P_{in} = -\int S \cdot da = \frac{I^2 t}{2\pi^2 \epsilon_0 a^4}\left[b\hat{s} \cdot (2\pi b w)\hat{s}\right]$$

$$\boxed{P_{in} = \frac{I^2 w t b^2}{\pi \epsilon_0 a^4}}$$

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Now

$$\frac{dU_{em}}{dt} = \frac{\mu_0 \omega I^2 b^2}{2\pi a^4} \cdot 2c^2 t$$

$$\boxed{\frac{dU_{em}}{dt} = \frac{I^2 \omega t b^2}{\pi \epsilon_0 a^4} = P_{in}}$$

Hence, verified.

Ch. 9 Prob 1. By explicit differentiation, check that the functions f_1, f_2 and f_3 in the text satisfy the wave equation. Show that f_4 and f_5 do not.

The wave equation is satisfied when

$$\frac{\partial^2 f}{\partial z^2} = \frac{1}{v^2}\frac{\partial^2 f}{\partial t^2} \Rightarrow \frac{\partial^2 f}{\partial t^2} = v^2 \frac{\partial^2 f}{\partial z^2}$$

Check for $f_1 = A e^{-b(z-vt)^2}$

$$\frac{\partial f_1}{\partial z} = -2Ab(z-vt)e^{-b(z-vt)^2}$$

$$\frac{\partial^2 f_1}{\partial z^2} = -2Ab\left[e^{-b(z-vt)^2} - 2b(z-vt)^2 e^{-b(z-vt)^2}\right]$$

$$\frac{\partial f_1}{\partial t} = 2Abv(z-vt)e^{-b(z-vt)^2}$$

$$\frac{\partial^2 f_1}{\partial t^2} = 2Abv\left[-ve^{-b(z-vt)^2} + 2bv(z-vt)^2 e^{-b(z-vt)^2}\right]$$

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$$\frac{\partial^2 f_1}{\partial t^2} = v^2\left\{-2Ab\left(e^{-b(z-vt)^2} - 2b(z-vt)^2 e^{-b(z-vt)^2}\right)\right\}$$

$$= v^2 \frac{\partial^2 f_1}{\partial z^2} \qquad \text{(from (1))}$$

Hence, f_1 satisfies the wave equation.

Checking for $f_2 = A\sin[b(z-vt)]$

$$\frac{\partial f_2}{\partial z} = Ab\cos[b(z-vt)]$$

$$\frac{\partial^2 f_2}{\partial z^2} = -Ab^2\sin[b(z-vt)] \qquad \text{— (II)}$$

$$\frac{\partial f_2}{\partial t} = -Abv\cos[b(z-vt)]$$

$$\frac{\partial^2 f_2}{\partial t^2} = -Ab^2v^2\sin(b(z-vt))$$

$$= v^2\frac{\partial^2 f_2}{\partial z^2} \qquad \text{(from (II))}$$

Hence, f_2 satisfies the wave equation.

Checking for $f_3 = \dfrac{A}{b(z-vt)^2+1}$

$$\frac{\partial f_3}{\partial z} = \frac{-2Ab(z-vt)}{[b(z-vt)^2+1]^2}$$

$$\frac{\partial^2 f_3}{\partial z^2} = \frac{-2Ab}{[b(z-vt)^2+1]^2} + \frac{8Ab^2(z-vt)^2}{[b(z-vt)^2+1]^3} \qquad \text{— (III)}$$

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$$\frac{\partial f_3}{\partial t} = \frac{2Abv \cdot (z-vt)}{[b(z-vt)^2+1]^2}$$

$$\frac{\partial^2 f_3}{\partial t^2} = \frac{-2Abv^2}{[b(z-vt)^2+1]^2} + \frac{8Ab^2v^2(z-vt)^2}{[b(z-vt)^2+1]^3}$$

$$= v^2 \frac{\partial^2 f_3}{\partial z^2} \qquad (\text{from (III)})$$

Hence, f_3 satisfies wave equation.

Checking for $f_4 = A e^{-b(bz^2+vt)}$

$$\frac{\partial f_4}{\partial z} = -2Ab^2 z e^{-b(bz^2+vt)}$$

$$\frac{\partial^2 f_4}{\partial z^2} = -2Ab^2\left[e^{-b(bz^2+vt)} - 2b^2z^2 e^{-b(bz^2+vt)}\right] \qquad \text{(IV)}$$

$$\frac{\partial f_4}{\partial t} = -Abv e^{-b(bz^2+vt)}$$

$$\frac{\partial^2 f_4}{\partial t^2} = Ab^2v^2 e^{-b(bz^2+vt)} \neq v^2\frac{\partial^2 f_4}{\partial z^2} \qquad (\text{from (IV)})$$

Hence, f_4 is not a wave equation.

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Checking for $f_s = A \sin(bz) \cos(bvt)^3$

$$\frac{\partial f_s}{\partial z} = Ab \cos(bz) \cos(bvt)^3$$

$$\frac{\partial^2 f_s}{\partial z^2} = -Ab^2 \sin(bz) \cos(bvt)^3 \quad \text{—— (i)}$$

$$\frac{\partial f_s}{\partial t} = -3Ab^3v^3t^2 \sin(bz) \sin(bvt)^3$$

$$\frac{\partial^2 f_s}{\partial t^2} = -6Ab^3v^3t \sin(bz) \sin(bvt)^3$$

$$- 9Ab^6v^6t^4 \sin(bz) \cos(bvt)^3$$

$$\frac{\partial^2 f_s}{\partial t^2} \neq v^2 \frac{\partial^2 f_s}{\partial z^2}$$

Hence, f_s is not a wave equation.

Ch 9. Prob 2. Show that the standing wave $f(z,t) = A \sin(kz) \cos(kvt)$ satisfies the wave equation, and express it as a sum of a wave travelling to the left and a wave travelling to the right.

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$$\frac{\partial f}{\partial z} = AK \cos(Kz) \cos(Kvt)$$

$$\frac{\partial^2 f}{\partial z^2} = -AK^2 \sin(Kz) \cos(Kvt) \quad \text{—} \; (1)$$

$$\frac{\partial f}{\partial t} = -AKv \cancel{\sin}(Kz) \sin(Kvt)$$

$$\frac{\partial^2 f}{\partial t^2} = -AK^2v^2 \sin(Kz) \cos(Kvt)$$

$$= v^2 \frac{\partial^2 f}{\partial z^2} \qquad (\text{from } (1))$$

Hence, $f(z,t)$ satisfies the wave equation.

Using the identity,

$$\sin\alpha \cos\beta = \frac{1}{2}\left[\sin(\alpha+\beta) + \sin(\alpha-\beta)\right]$$

$$\boxed{f = \frac{A}{2}\left\{\sin\left[K(z+vt)\right] + \sin\left[K(z-vt)\right]\right\}}$$

Ch.9 Prob 11: Consider a particle of charge q and mass m, free to move in the xy plane in response to an em wave propagating in the z-direction.

(a) Ignoring the magnetic force, find the velocity of the particle as a function of time.

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(b) Now, calculate the resulting magnetic force on the particle.

(c) Show that the (time) average magnetic force is zero.

The problem with this naive model of light is that the velocity is 90° out of phase with the fields. For energy to be absorbed, there's got to be some resistance to the motion of the charges. Suppose we include a force of the form $-\gamma m v$, for some damping constant γ.

(d) Repeat parts a, b and c for the above scenario.

The fields are $E(z,t) = E_0 \cos(kz - \omega t)\,\hat{x}$
and $B(z,t) = \frac{1}{c} E_0 \cos(kz - \omega t)\,\hat{y}$

with $\omega = ck$.

(a) The electric force $F_e = qE$

$\therefore$ $F_e = qE_0 \cos(kz - \omega t)\,\hat{x}$

$= ma = \frac{m\,dv}{dt}$

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$$\therefore \quad v = \frac{q E_0}{m} \hat{x} \int \cos(kz - \omega t)\, dt$$

$$v = -\frac{q E_0}{m\omega} \sin(kz - \omega t)\, \hat{x} + C$$

But, since $v_{avg} = 0$.

$$\boxed{\therefore \quad v = -\frac{q E_0}{m\omega} \sin(kz - \omega t)\, \hat{x}}$$

(b) The magnetic force is

$$F_m = q\,(v \times B)$$

$$= q\left(-\frac{q E_0}{m\omega}\right)\left(\frac{E_0}{c}\right) \sin(kz - \omega t) \cdot \cos(kz - \omega t) \cdot (\hat{x} \times \hat{y})$$

$$\boxed{F_m = -\frac{q^2 E_0^2}{m\omega c} \sin(kz - \omega t) \cos(kz - \omega t)\, \hat{z}}$$

(c) The (time) average force is $(F_m)_{ave}$,

$$(F_m)_{ave} = -\frac{q^2 E_0^2}{m\omega c} \hat{z} \int_0^T \sin(kz - \omega t) \cos(kz - \omega t)\, dt$$

where $T = 2\pi/\omega$ is period.

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$$(F_m)_{ave} = \left\{ \frac{-1}{2\omega} \sin^2(kz-\omega t) \Big|_0^T \right\} \cdot \frac{-q^2 E_0^2}{m\omega c} \hat{z}$$

$$(F_m)_{ave} = \frac{q^2 E_0^2}{2m\omega^2 c} \hat{z} \left\{ \sin^2(kz-2\pi) - \sin^2(kz) \right\}$$

$$\boxed{(F_m)_{ave} = 0}$$

(d) Adding in the damping term,

$$F = qE - m\gamma v$$

$$= qE_0 \cos(kz-\omega t)\hat{x} - \gamma m v = m\frac{dv}{dt}$$

$$\Rightarrow \frac{dv}{dt} + \gamma v = \frac{qE_0}{m} \cos(kz-\omega t)\hat{x}$$

The steady state solution has the form
$v = A\cos(kz-\omega t+\theta)\hat{x}$.

$$\frac{dv}{dt} = A\omega \sin(kz-\omega t+\theta).$$

Putting this in, and using the identity

$$\cos(u) = \cos\theta \cos(u+\theta) + \sin\theta (\sin u+\theta)$$

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$A\omega \sin(kz - \omega t + \theta) + \gamma A \cos(kz - \omega t + \theta)$

$$= \frac{qE_0}{m}\left[\cos\theta \cos(kz - \omega t + \theta) + \sin\theta \sin(kz - \omega t + \theta)\right]$$

Equating like terms

$$A\omega = \frac{qE_0}{m}\sin\theta, \quad A\gamma = \frac{qE_0}{m}\cos\theta$$

$\therefore$ $$\tan\theta = \frac{\omega}{\gamma}$$

and, $$A^2(\omega^2 + \gamma^2) = \left(\frac{qE_0}{m}\right)^2$$

$$\Rightarrow \quad A = \frac{qE_0}{m\sqrt{\omega^2 + \gamma^2}}$$

$$\boxed{\therefore \; v = \frac{qE_0}{m\sqrt{\omega^2+\gamma^2}}\cos(kz - \omega t + \theta)\,\hat{x} \quad \text{where } \theta = \tan^{-1}(\omega/\gamma)}$$

$$\boxed{F_m = \frac{q^2E_0^2}{mc\sqrt{\omega^2+\gamma^2}}\cos(kz - \omega t + \theta)\cos(kz - \omega t)\,\hat{z}}$$

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To calculate the time average, write
$\cos(kz - \omega t + \theta) = \cos\theta \cos(kz - \omega t) - \sin\theta \sin(kz - \omega t)$.

We know that average of $\cos(kz - \omega t)$. $\sin(kz - \omega t)$ is zero, so

$$(F_m)_{ave} = \frac{q^2 E_0^2}{mc\sqrt{\omega^2 + \gamma^2}} \hat{z} \cos\theta \int_0^T \cos^2(kz - \omega t)\, dT$$

$$\boxed{(F_m)_{ave} = \frac{\pi \gamma q^2 E_0^2}{m\omega c(\omega^2 + \gamma^2)} \hat{z}}$$

Ch.9 Prob 12. In the complex notation there is a clever device for finding the time average of a product. Suppose $f(r,t) = A\cos(kr - \omega t + \delta_a)$ and $g(r,t) = B\cos(kr - \omega t + \delta_B)$. Show that

$\langle fg \rangle = \frac{1}{2} Re(\tilde{f}\tilde{g}^*)$, where the star denotes complex conjugation. For example

$$\langle u \rangle = \frac{1}{4} Re\left[\varepsilon_0 \tilde{E} \cdot \tilde{E}^* + \frac{1}{\mu_0} \tilde{B} \cdot \tilde{B}^*\right]$$

and $\langle S \rangle = \frac{1}{2\mu_0} Re(\tilde{E} \times \tilde{B}^*)$

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$$\langle fg \rangle = \frac{1}{T}\int_0^T a\cos(kr-\omega t+\delta_a)\, b\cos(kr-\omega t+\delta_b)\, dt$$

$$= \frac{ab}{2T}\int_0^T \left[\cos(2kr-2\omega t+\delta_a+\delta_b)+\cos(\delta_a-\delta_b)\right] dt$$

$$= \frac{ab}{2T}\cos(\delta_a-\delta_b)T = \frac{ab\cos(\delta_a-\delta_b)}{2} \longrightarrow (1)$$

Meanwhile in the complex notation:

$$\tilde{f} = \tilde{a}\, e^{i(kr-\omega t)}$$

$$\tilde{g} = \tilde{b}\, e^{i(kr-\omega t)}$$

where $\tilde{a} = a\, e^{i\delta_a}$

and $\tilde{b} = b\, e^{i\delta_b}$.

So, $\frac{1}{2}\tilde{f}\tilde{g}^* = \frac{1}{2}\tilde{a}\, e^{i(kr-\omega t)}\, \tilde{b}^*\, e^{-i(kr-\omega t)}$

$$= \frac{1}{2}\tilde{a}\tilde{b}^* = \frac{1}{2}ab\, e^{i(\delta_a-\delta_b)}$$

$\therefore\ \mathrm{Re}\left(\frac{1}{2}\tilde{f}\tilde{g}^*\right) = \frac{1}{2}ab\cos(\delta_a-\delta_b) = \langle fg \rangle$

from (1)

Hence, shown.

Arlut Agrawal
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Ch.9 Prob.10 The intensity of sunlight hitting the earth is about 1300 W/m². If sunlight strikes a perfect absorber, what pressure does it exert? How about a perfect reflector? What fraction of atmospheric pressure does this amount to?

$$\text{Pressure} = \frac{\text{Intensity}}{\text{Speed of light}}$$

$$P = \frac{1.3 \times 10^{3}}{3 \times 10^{8}} = 4.3 \times 10^{-6}\ N/m^2$$

Hence, for a perfect absorber the pressure is $4.3 \times 10^{-6}\ N/m^2$.

For a perfect reflector, the pressure is twice as great: $8.6 \times 10^{-6}\ N/m^2$.

∵ Atmospheric pressure is $1.03 \times 10^{5}\ N/m^2$, so the pressure of light on a reflector is

$$\frac{8.6 \times 10^{-6}}{1.03 \times 10^{5}} = 8.3 \times 10^{-11}\ \text{atm.}$$