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PH110: Waves & Electromagnetics
Tutorial 01

Q.11. Find the gradients of the following functions

(i) $f(x,y,z) = x^2 + y^3 + z^4$

(ii) $f(x,y,z) = x^2 y^3 z^4$

(iii) $f(x,y,z) = e^x (\sin y) \ln(z)$

(i) $\nabla f = \hat{x}\frac{\partial f}{\partial x} + \hat{y}\frac{\partial f}{\partial y} + \hat{z}\frac{\partial f}{\partial z}$

$= 2x\,\hat{x} + 3y^2\,\hat{y} + 4z^3\,\hat{z}$

(ii) $\nabla f = \hat{x}\frac{\partial f}{\partial x} + \hat{y}\frac{\partial f}{\partial y} + \hat{z}\frac{\partial f}{\partial z}$

$= (2xy^3z^4)\hat{x} + (3x^2y^2z^4)\hat{y} + (4x^2y^3z^3)\hat{z}$

(iii) $\nabla f = e^x(\sin y)(\ln z)\,\hat{x} + e^x(\cos y)(\ln z)\,\hat{y}$

$+ e^x(\sin y)\left(\frac{1}{z}\right)\hat{z}$

12. The height of a certain hill (in feet) is given by

$h(x,y) = 10(2xy - 3x^2 - 4y^2 - 18x + 28y + 12)$

where y is the distance (in miles) north, x the distance east of South Hadley.

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(a) Where is the top of the hill located?
(b) How high is the hill?
(c) How steep is the slope (in feet per mile) at a point 1 mile north and one mile east of South Halley? In what direction is the slope steepest, at that point?

(a) The top of hill is located where ∇h is equal to zero

$$\nabla h = 10(2y - 6x - 18)\hat{x} + 10(2x - 8y + 28)\hat{y}$$

Putting $\nabla h = 0$ gives,

$$y - 3x - 9 = 0 \quad \text{--- (i)}$$

and, $$x - 4y + 14 = 0 \quad \text{--- (ii)}$$

Multiplying (i) by 4

$$4y - 12x - 36 = 0$$
$$x - 4y + 14 = 0 \quad \longleftarrow \text{eq (ii)}$$

$$-11x - 22 = 0$$

$$x = -2$$
and $$y = 3$$

Hence, the top of hill is located at $x = -2$ and $y = 3$.

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(b) The height of hill is $h(-2,3)$

$$h(-2,3) = 10\left(2\times(-2)\times 3 - 3\times 4 - 4\times 9 - 18\times(-2) + 28\times 3 + 12\right)$$

$$= 720 \text{ feet}$$

(c) the steepness of the slope is equal to the magnitude of ∇h at $x = y = 1$.

$$\nabla h\Big|_{\text{at } x,y=1} = 10(-22)\,\hat{x} + 10(22)\,\hat{y}$$

$$= -220\,\hat{x} + 220\,\hat{y}$$

$\therefore$ steepness of slope $= \sqrt{2\times(220)^2}$

$= 311.12$ ft/mile

the slope is steepest in the direction of ∇h at $x=1$ and $y=1$, i.e. in the north-west direction.

13 Let r be the seperation vector ~~fixed~~ from a fixed point (x', y', z') to the point (x, y, z) and let $|r|$ be its length.
Show that

(a) $\nabla(|r|^2) = 2r$

(b) $\nabla\left(\frac{1}{|r|}\right) = -\frac{\hat{r}}{r^2}$

(c) What is the general formula for $\nabla(|r|^n)$?

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$$r = (x-x')\hat{x} + (y-y')\hat{y} + (z-z')\hat{z} \quad \text{--- (1)}$$

(a) $\therefore |r| = \sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}$

$\therefore |r|^2 = x^2 + y^2 + z^2 - 2xx' - 2yy' - 2zz' + x'^2 + y'^2 + z'^2$

$\therefore \nabla(|r|^2) = \hat{x}\dfrac{\partial(|r|^2)}{\partial x} + \hat{y}\dfrac{\partial(|r|^2)}{\partial y} + \hat{z}\dfrac{\partial(|r|^2)}{\partial z}$

$= (2x-2x')\hat{x} + (2y-2y')\hat{y} + (2z-2z')\hat{z}$

$= 2\{(x-x')\hat{x} + (y-y')\hat{y} + (z-z')\hat{z}\}$

$= 2r$ $\quad$ {from (1)}

Hence, proved

(b) $\dfrac{1}{|r|} = \dfrac{1}{\sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}}$

$\nabla\left(\dfrac{1}{|r|}\right) = \dfrac{-\cancel{2}(x-x')\hat{x}}{\cancel{2}|r|^2} - \dfrac{\cancel{2}(y-y')\hat{y}}{\cancel{2}|r|^2} - \dfrac{\cancel{2}(z-z')\hat{z}}{\cancel{2}|r|^2}$

$= \dfrac{-\{(x-x')\hat{x} + (y-y')\hat{y} + (z-z')\hat{z}\}}{|r|^2}$

$= \dfrac{-r}{|r|^2}$

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(c) $|r|^n = \left(\sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}\right)^n$

Now,
$$\nabla(|r|^n) = \hat{x}\frac{\partial(|r|^n)}{\partial x} + \hat{y}\frac{\partial(|r|^n)}{\partial y} + \hat{z}\frac{\partial(|r|^n)}{\partial z}$$

Now,
$$\frac{\partial(|r|^n)}{\partial x} = \frac{\partial}{\partial x}\left\{\left(\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}\right)^n\right\}$$

$$= n|r|^{n-1} \times \frac{1}{\cancel{2}|r|} \times \cancel{2}(x-x')\,\hat{x}$$

$$= n|r|^{n-2}(x-x')\hat{x}$$

Similarly
$$\frac{\partial(|r|^n)}{\partial y} = n|r|^{n-2}(y-y')\hat{y}$$

and,
$$\frac{\partial(|r|^n)}{\partial z} = n|r|^{n-2}(z-z')\hat{z}$$

$$\therefore \nabla(|r|^n) = n|r|^{n-2}\left\{(x-x')\hat{x} + (y-y')\hat{y} + (z-z')\hat{z}\right\}$$

$$\boxed{\begin{aligned}\nabla(|r|^n) &= n|r|^{n-2}\,r \\ \nabla(|r|^n) &= n|r|^{n-1}\,\hat{r}\end{aligned}}$$

Answers

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15. Calculate the divergence of the following vector functions

(a) $v_a = x^2\hat{x} + 3xz^2\hat{y} - 2xz\hat{z}$

(b) $v_b = xy\hat{x} + 2yz\hat{y} + 3zx\hat{z}$

(c) $v_c = y^2\hat{x} + (2xy + z^2)\hat{y} + 2yz\hat{z}$

(a) Divergence of v_a is given by

$$\nabla \cdot v_a = \left(\hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}\right) \cdot \left(x^2\hat{x} + 3xz^2\hat{y} - 2xz\hat{z}\right)$$

$$= \frac{\partial(x^2)}{\partial x} + \frac{\partial(3xz^2)}{\partial y} - \frac{\partial(2xz)}{\partial z}$$

$$= 2x + 0 - 2x = \underline{0} \text{ Answer}$$

(b) divergence of v_b is given by

$$\nabla \cdot v_b = \left(\hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}\right) \cdot \left(xy\hat{x} + 2yz\hat{y} + 3zx\hat{z}\right)$$

$$= \frac{\partial(xy)}{\partial x} + \frac{\partial(2yz)}{\partial y} + \frac{\partial(3zx)}{\partial z}$$

$$\boxed{\nabla \cdot v_b = y + 2z + 3x}$$

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(c) divergence of v_c

$$\nabla \cdot v_c = \left(\hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}\right) \cdot \left(y^2\hat{x} + (2xy+z^2)\hat{y} + 2yz\hat{z}\right)$$

$$\nabla \cdot v_c = \frac{\partial (y^2)}{\partial x} + \frac{\partial (2xy+z^2)}{\partial y} + \frac{\partial (2yz)}{\partial z}$$

$$= 0 + 2x + 2y$$

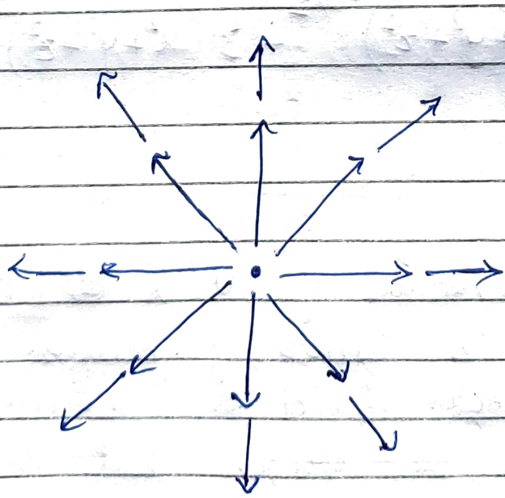
$$\boxed{\nabla \cdot v_c = 2(x+y)}$$

16. Sketch the vector function

$$v = \frac{\hat{r}}{r^2}$$

and compute its divergence. Explain the answer.

$$v = \frac{\hat{r}}{r^2} = \frac{\vec{r}}{r^3} = \frac{x\hat{x} + y\hat{y} + z\hat{z}}{r^3}$$



Plot of v is shown above.

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Now, divergence of v is

$$\nabla \cdot v = \frac{\partial}{\partial x}\left(\frac{x}{r^3}\right) + \frac{\partial}{\partial y}\left(\frac{y}{r^3}\right) + \frac{\partial}{\partial z}\left(\frac{z}{r^3}\right)$$

Now,

$$\frac{\partial}{\partial x}\left(\frac{x}{r^3}\right) = \frac{\partial}{\partial x}\left(x \cdot (x^2+y^2+z^2)^{-3/2}\right)$$

$$= (x^2+y^2+z^2)^{-3/2} + x \cdot \left(\frac{-3}{2}\right)(x^2+y^2+z^2)^{-5/2} \cdot (2x)$$

$$= (x^2+y^2+z^2)^{-3/2} - 3x^2 (x^2+y^2+z^2)^{-5/2}$$

Similarly,

$$\frac{\partial}{\partial y}\left(\frac{y}{r^3}\right) = (x^2+y^2+z^2)^{-3/2} - 3y^2 (x^2+y^2+z^2)^{-5/2}$$

and,

$$\frac{\partial}{\partial z}\left(\frac{z}{r^3}\right) = (x^2+y^2+z^2)^{-3/2} - 3z^2 (x^2+y^2+z^2)^{-5/2}$$

$$\therefore \nabla \cdot v = 3(x^2+y^2+z^2)^{-3/2} - 3(x^2+y^2+z^2)^{-5/2}(x^2+y^2+z^2)$$

$$= 0$$

Hence, divergence of v is 0.
This result is surprising because, from the plot, this vector field is obviously diverging away from the origin but still the divergence is 0. The explanation is that $\nabla \cdot v = 0$ everywhere except at the origin,

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Since, $r = 0$ at the origin the expression for v blows up. In fact $\nabla \cdot v$ is infinite at origin and zero elsewhere.

1.8. Calculate the curls of the following functions:

(a) $V_a = x^2 \hat{x} + 3xz^2 \hat{y} - 2xz \hat{z}$

(b) $V_b = xy \hat{x} + 2yz \hat{y} + 3zx \hat{z}$

(c) $V_c = y^2 \hat{x} + (2xy + z^2) \hat{y} + 2yz \hat{z}$

(a) curl of $V_a = \nabla \times V_a$

$$\nabla \times V_a = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^2 & 3xz^2 & -2xz \end{vmatrix}$$

$$= \left\{ \frac{\partial(-2xz)}{\partial y} - \frac{\partial(3xz^2)}{\partial z} \right\} \hat{x}$$

$$- \left\{ \frac{\partial(-2xz)}{\partial x} - \frac{\partial(x^2)}{\partial z} \right\} \hat{y}$$

$$+ \left\{ \frac{\partial(3xz^2)}{\partial x} - \frac{\partial(x^2)}{\partial y} \right\} \hat{z}$$

$$= (0 - 6xz)\hat{x} - (-2z - 0)\hat{y} + (3z^2 - 0)\hat{z}$$

$$\boxed{\nabla \times V_a = (-6xz)\hat{x} + (2z)\hat{y} + (3z^2)\hat{z}}$$

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(b) $\nabla \times V_b = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ xy & 2yz & 3zx \end{vmatrix}$

$= (0 - 2y)\hat{x} - (3z - 0)\hat{y} + (0 - x)\hat{z}$

$\boxed{\nabla \times V_b = (-2y)\hat{x} - (3z)\hat{y} - (x)\hat{z}}$

(c) $\nabla \times V_c = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y^2 & (2xy + z^2) & 2yz \end{vmatrix}$

$= (2z - (0 + 2z))\hat{x} - (0 - 0)\hat{y} + ((2y + 0) - 2y)\hat{z}$

$\boxed{\nabla \times V_c = 0}$

20. Construct a vector function that has zero divergence and zero curl everywhere.

Let the function be

$V = yz\,\hat{x} + xz\,\hat{y} + xy\,\hat{z}$

Now, divergence of V

$\nabla \cdot V = \frac{\partial}{\partial x}(yz) + \frac{\partial}{\partial y}(xz) + \frac{\partial}{\partial z}(xy)$

$= 0$

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and curl of v,

$$\nabla \times v = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ yz & xz & xy \end{vmatrix}$$

$$= (x - x)\hat{x} + (y - y)(-\hat{y}) + (z - z)\hat{z}$$
$$= 0$$

Hence, v is vector function that has zero divergence and zero curl.

21. Prove the following product rules

(i) $\nabla(fg) = f(\nabla g) + g\nabla f$

(ii) $\nabla \cdot (A \times B) = B \cdot (\nabla \times A) - A \cdot (\nabla \times B)$

(iii) $\nabla \times (fA) = f(\nabla \times A) - A \times (\nabla f)$

(i) $$\nabla(fg) = \frac{\partial}{\partial x}(fg) + \frac{\partial}{\partial y}(fg) + \frac{\partial}{\partial z}(fg)$$

$$= f\frac{\partial g}{\partial x} + g\frac{\partial f}{\partial x} + f\frac{\partial g}{\partial y} + g\frac{\partial f}{\partial y} + f\frac{\partial g}{\partial z} + g\frac{\partial f}{\partial z}$$

$$= f\left(\frac{\partial g}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial g}{\partial z}\right) + g\left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} + \frac{\partial f}{\partial z}\right)$$

$$= f\nabla g + g\nabla f$$

Hence, proved

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(ii) $\nabla \cdot (A \times B) = \left\{ \left(\frac{\partial}{\partial x}\right)\hat{x} + \left(\frac{\partial}{\partial y}\right)\hat{y} + \left(\frac{\partial}{\partial z}\right)\hat{z} \right\} \cdot \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$

$$\nabla \cdot (A \times B) = \frac{\partial}{\partial x}(A_y B_z - A_z B_y) - \frac{\partial}{\partial y}(A_x B_z - A_z B_x) + \frac{\partial}{\partial z}(A_x B_y - A_y B_x)$$

$$= A_y \frac{\partial B_z}{\partial x} + B_z \frac{\partial A_y}{\partial x} - A_z \frac{\partial B_y}{\partial x} - B_y \frac{\partial A_z}{\partial x}$$

$$- A_x \frac{\partial B_z}{\partial y} - B_z \frac{\partial A_x}{\partial y} + A_z \frac{\partial B_x}{\partial y} + B_x \frac{\partial A_z}{\partial y}$$

$$+ A_x \frac{\partial B_y}{\partial z} + B_y \frac{\partial A_x}{\partial z} - A_y \frac{\partial B_x}{\partial z} - B_x \frac{\partial A_y}{\partial z}$$

$$= B_x\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) - B_y\left(\frac{\partial A_z}{\partial x} - \frac{\partial A_x}{\partial z}\right) + B_z\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)$$

$$- \left(A_x\left(\frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z}\right) - A_y\left(\frac{\partial B_z}{\partial x} - \frac{\partial B_x}{\partial z}\right) + A_z\left(\frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y}\right)\right)$$

$$= B \cdot \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ A_x & A_y & A_z \end{vmatrix} - A \cdot \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ B_x & B_y & B_z \end{vmatrix}$$

$$\boxed{\nabla \cdot (A \times B) = B \cdot (\nabla \times A) - A \cdot (\nabla \times B)}$$

Hence, proved.

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(iii) $\nabla \times (fA) = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ fA_x & fA_y & fA_z \end{vmatrix}$

$$\nabla \times (fA) = \left\{ \frac{\partial (fA_z)}{\partial y} - \frac{\partial (fA_y)}{\partial z} \right\} \hat{x} - \left\{ \frac{\partial (fA_z)}{\partial x} - \frac{\partial (fA_x)}{\partial z} \right\} \hat{y}$$

$$+ \left\{ \frac{\partial (fA_y)}{\partial x} - \frac{\partial (fA_x)}{\partial y} \right\} \hat{z}$$

$$= \left\{ A_z \frac{\partial f}{\partial y} + f \frac{\partial A_z}{\partial y} - A_y \frac{\partial f}{\partial z} - f \frac{\partial A_y}{\partial z} \right\} \hat{x}$$

$$- \left\{ A_z \frac{\partial f}{\partial x} + f \frac{\partial A_z}{\partial x} - A_x \frac{\partial f}{\partial z} - f \frac{\partial A_x}{\partial z} \right\} \hat{y}$$

$$+ \left\{ A_y \frac{\partial f}{\partial x} + f \frac{\partial A_y}{\partial x} - A_x \frac{\partial f}{\partial y} - f \frac{\partial A_x}{\partial y} \right\} \hat{z}$$

$$= f \left\{ \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \right) \hat{x} - \left(\frac{\partial A_z}{\partial x} - \frac{\partial A_x}{\partial z} \right) \hat{y} + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right) \hat{z} \right\}$$

$$- \left\{ \left(-A_z \frac{\partial f}{\partial y} + A_y \frac{\partial f}{\partial z} \right) \hat{x} - \left(A_z \frac{\partial f}{\partial x} - A_x \frac{\partial f}{\partial z} \right) \hat{y} \right.$$

$$\left. + \left(-A_y \frac{\partial f}{\partial x} + A_x \frac{\partial f}{\partial y} \right) \hat{z} \right\}$$

$$= f \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ A_x & A_y & A_z \end{vmatrix} - \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ A_x & A_y & A_z \\ \partial f/\partial x & \partial f/\partial y & \partial f/\partial z \end{vmatrix}$$

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$$\boxed{\nabla \times (fA) = f(\nabla \times A) - (A \times (\nabla f))}$$ Hence, proved

26. Calculate the Laplacian of the following functions:

(a) $T_a = x^2 + 2xy + 3z + 4$

(b) $T_b = \sin x \cdot \sin y \cdot \sin z$

(c) $T_c = e^{-5x} \sin 4y \cdot \cos 3z$

(d) $v = x^2 \hat{x} + 3xz^2 \hat{y} - 2xz \hat{z}$

(a) $\nabla T_a = \hat{x} \dfrac{\partial T_a}{\partial x} + \hat{y} \dfrac{\partial T_a}{\partial y} + \hat{z} \dfrac{\partial T_a}{\partial z}$

$= (2x + 2y) \hat{x} + (2x) \hat{y} + 3 \hat{z}$

Laplacian of T_a is given by

Laplacian of $T_a = \nabla \cdot (\nabla T_a)$

$= \dfrac{\partial}{\partial x}(2x + 2y) + \dfrac{\partial}{\partial y}(2x) + \dfrac{\partial}{\partial z}(3)$

$$\boxed{\nabla^2 (x^2 + 2xy + 3z + 4) = 2}$$ Answer

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(b) $\nabla T_b = \hat{x}\dfrac{\partial T_b}{\partial x} + \hat{y}\dfrac{\partial T_b}{\partial y} + \hat{z}\dfrac{\partial T_b}{\partial z}$

$= \cos x \sin y \sin z\,\hat{x} + \sin x \cos y \sin z\,\hat{y} + \sin x \sin y \cos z\,\hat{z}$

Laplacian of T_b is given by

$\nabla^2 T_b = \nabla\cdot(\nabla T_b)$

$= \dfrac{\partial}{\partial x}(\cos x \sin y \sin z) + \dfrac{\partial}{\partial y}(\sin x \cos y \sin z) + \dfrac{\partial}{\partial z}(\sin x \sin y \cos z)$

$= -\sin x \sin y \sin z - \sin x \sin y \sin z - \sin x \sin y \sin z$

$\boxed{\nabla^2 T_b = -3\sin x \sin y \sin z}$ Answer

(c) $\nabla T_c = \hat{x}\dfrac{\partial T_c}{\partial x} + \hat{y}\dfrac{\partial T_c}{\partial y} + \hat{z}\dfrac{\partial T_c}{\partial z}$

$= -5e^{-5x}\sin 4y \cos 3z\,\hat{x} + 4e^{-5x}\cos 4y \cos 3z\,\hat{y} - 3e^{-5x}\sin 4y \sin 3z\,\hat{z}$

Laplacian of T_c is given by

$\nabla^2 T_c = \nabla\cdot(\nabla T_c)$

$= \dfrac{\partial}{\partial x}(-5e^{-5x}\sin 4y \cos 3z) + \dfrac{\partial}{\partial y}(4e^{-5x}\cos 4y \cos 3z) + \dfrac{\partial}{\partial z}(-3e^{-5x}\sin 4y \sin 3z)$

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~~$\nabla^2 T$~~

$$\nabla^2 T_c = 25 e^{-5x} \sin 4y \cos 3z - 16 e^{-5x} \sin 4y \cos 3z - 9 e^{-5x} \sin 4y \cos 3z$$

$$\boxed{\nabla^2 T_c = 0}$$ Answer

(d) $v = x^2 \hat{x} + 3xz^2 \hat{y} - 2xz \hat{z}$

$$\nabla^2 v = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\left(x^2 \hat{x} + 3xz^2 \hat{y} - 2xz \hat{z}\right)$$

$$= \frac{\partial^2 (x^2)}{\partial x^2} \hat{x} + \cancel{\frac{\partial^2 (x^2)}{\partial y^2} \hat{x}}^{\,0} + \cancel{\frac{\partial^2 (x^2)}{\partial z^2} \hat{x}}^{\,0}$$

$$+ \cancel{\frac{\partial^2 (3xz^2)}{\partial x^2}}^{\,0} \hat{y} + \cancel{\frac{\partial^2 (3xz^2)}{\partial y^2}}^{\,0} \hat{y} + \frac{\partial^2 (3xz^2)}{\partial z^2} \hat{y}$$

$$- \cancel{\frac{\partial^2 (2xz)}{\partial x^2}}^{\,0} \hat{z} - \cancel{\frac{\partial^2 (2xz)}{\partial y^2}}^{\,0} \hat{z} - \cancel{\frac{\partial^2 (2xz)}{\partial z^2}}^{\,0} \hat{z}$$

$$\boxed{\nabla^2 v = 2\hat{x} + 6x\hat{y}}$$ Answer

27. Prove that the divergence of a curl is always zero. Check its for the function
(a) $x^2 \hat{x} + 3xz^2 \hat{y} - 2xz \hat{z}$

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Let $V = V_x \hat{x} + V_y \hat{y} + V_z \hat{z}$ be a vector function

$\therefore$ curl of $V \Rightarrow$

$$\nabla \times V = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ V_x & V_y & V_z \end{vmatrix}$$

$$= \left(\frac{\partial V_z}{\partial y} - \frac{\partial V_y}{\partial z}\right)\hat{x} - \left(\frac{\partial V_z}{\partial x} - \frac{\partial V_x}{\partial z}\right)\hat{y} + \left(\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y}\right)\hat{z}$$

Now, divergence of curl :

$$\nabla \cdot (\nabla \times V) = \frac{\partial}{\partial x}\left(\frac{\partial V_z}{\partial y} - \frac{\partial V_y}{\partial z}\right) - \frac{\partial}{\partial y}\left(\frac{\partial V_z}{\partial x} - \frac{\partial V_x}{\partial z}\right) + \frac{\partial}{\partial z}\left(\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y}\right)$$

$$= \left(\frac{\partial^2 V_z}{\partial x \cdot \partial y} - \frac{\partial^2 V_z}{\partial y \cdot \partial x}\right) + \left(\frac{\partial^2 V_y}{\partial z \cdot \partial x} - \frac{\partial^2 V_y}{\partial x \cdot \partial z}\right) + \left(\frac{\partial^2 V_x}{\partial y \cdot \partial z} - \frac{\partial^2 V_x}{\partial z \cdot \partial y}\right)$$

$= 0$ (by equality of cross derivative i.e. $\frac{\partial^2 A}{\partial x \cdot \partial y} = \frac{\partial^2 A}{\partial y \cdot \partial x}$)

Hence, proved.

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Now, for the function, $V_a = x^2\hat{x} + 3xz^2\hat{y} - 2xz\hat{z}$

curl $= \nabla \times V_a = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^2 & 3xz^2 & -2xz \end{vmatrix}$

$$= (0 - 6xz)\hat{x} - (-2z - 0)\hat{y} + (3z^2 - 0)\hat{z}$$

$$\nabla \times V_a = (-6xz)\hat{x} + 2z\hat{y} + 3z^2\hat{z}$$

Now, divergence of curl of V_a,

$$\nabla \cdot (\nabla \times V_a) = \frac{\partial}{\partial x}(-6xz) + \frac{\partial}{\partial y}(2z) + \frac{\partial}{\partial z}(3z^2)$$

$$= -6z + 0 + 6z$$

$$= 0 \quad \text{Ans}$$

28 Prove that curl of a gradient is always 0. Check it for the function $f(x, y, z) = x^2y^3z^4$.

Let T be a scalar function.

$\therefore$ gradient of $T = \nabla T = \frac{\partial T}{\partial x}\hat{x} + \frac{\partial T}{\partial y}\hat{y} + \frac{\partial T}{\partial z}\hat{z}$

Now, curl of gradient of T is given by

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$$\nabla \times (\nabla T) = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \partial T/\partial x & \partial T/\partial y & \partial T/\partial z \end{vmatrix}$$

$$= \left(\frac{\partial^2 T}{\partial y \cdot \partial z} - \frac{\partial^2 T}{\partial z \cdot \partial y}\right)\hat{x} - \left(\frac{\partial^2 T}{\partial x \cdot \partial z} - \frac{\partial^2 T}{\partial z \cdot \partial x}\right)\hat{y}$$

$$+ \left(\frac{\partial^2 T}{\partial x \cdot \partial y} - \frac{\partial^2 T}{\partial y \cdot \partial x}\right)\hat{z}$$

$$= 0 \qquad \left(\text{as } \frac{\partial^2 f}{\partial x \cdot \partial y} = \frac{\partial^2 f}{\partial y \cdot \partial x}\right)$$

Hence, proved.

Now, for the function $f = x^2 y^3 z^4$.

$$\nabla f = 2xy^3z^4\,\hat{x} + 3x^2y^2z^4\,\hat{y} + 4x^2y^3z^3\,\hat{z}$$

Now,

$$\nabla \times (\nabla f) = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 2xy^3z^4 & 3x^2y^2z^4 & 4x^2y^3z^3 \end{vmatrix}$$

$$= (12x^2y^2z^3 - 12x^2y^2z^3)\hat{x}$$

$$- (8xy^3z^3 - 8xy^3z^3)\hat{y}$$

$$+ (6xy^2z^4 - 6xy^2z^4)\hat{z}$$

$$= 0$$ Answer