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MA101: Linear Algebra and Matrices
Tutorial 10

1. Find the least squares line $y = mx + c$ and least square degree 2 polynomial that best fits the data $(2,0)$, $(1,0)$, $(0,2)$, $(1,4)$ and $(2,4)$.

Using these points in $y = mx + c$ yield five value of m and c for corresponding point.
Putting these in a 5×2 matrix with column 1 containing values of c and column 2 containing values of m.

$$X = \begin{bmatrix} 1 & 2 \\ 1 & 1 \\ 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix}$$

Matrix containing corresponding value of y

$$y = \begin{bmatrix} 0 \\ 0 \\ 2 \\ 4 \\ 4 \end{bmatrix}$$

For least square solution of $X\beta = y$, use

$$X^T X \beta = X^T y \longrightarrow (1)$$

where $\beta = [c \quad m]^T$

$$X^T X = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 2 & 1 & 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 1 & 1 \\ 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix}$$

$$X^T X = \begin{bmatrix} 5 & 6 \\ 6 & 10 \end{bmatrix}$$

and, $X^T y = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 2 & 1 & 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 2 \\ 4 \\ 4 \end{bmatrix} = \begin{bmatrix} 10 \\ 12 \end{bmatrix}$

Substituting in 1

$$\begin{bmatrix} 5 & 6 \\ 6 & 10 \end{bmatrix} \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix} = \begin{bmatrix} 10 \\ 12 \end{bmatrix}$$

$$\begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix} = \begin{bmatrix} 5 & 6 \\ 6 & 10 \end{bmatrix}^{-1} \begin{bmatrix} 10 \\ 12 \end{bmatrix}$$

$$= \frac{1}{14} \begin{bmatrix} 10 & -6 \\ -6 & 5 \end{bmatrix} \begin{bmatrix} 10 \\ 12 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

Thus, the least square line has the equation $y = 2$.

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For degree 2 polynomial

$$y = ax^2 + bx + c$$

$$\underbrace{\begin{bmatrix} x_1^2 & x_1 & 1 \\ x_2^2 & x_2 & 1 \\ x_3^2 & x_3 & 1 \\ x_4^2 & x_4 & 1 \\ x_5^2 & x_5 & 1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} a \\ b \\ c \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{bmatrix}}_{b}$$

Using $A^T A x = A^T b$

$$A = \begin{bmatrix} 4 & 2 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \\ 4 & 2 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} 0 \\ 0 \\ 2 \\ 4 \\ 4 \end{bmatrix}$$

$$A^T A = \begin{bmatrix} 4 & 1 & 0 & 1 & 4 \\ 2 & 1 & 0 & 1 & 2 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 4 & 2 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \\ 4 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 34 & 18 & 10 \\ 18 & 10 & 6 \\ 10 & 6 & 5 \end{bmatrix}$$

$\cancel{A^T b =}$

$$A^T b = \begin{bmatrix} 4 & 1 & 0 & 1 & 4 \\ 2 & 1 & 0 & 1 & 2 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 2 \\ 4 \\ 4 \end{bmatrix} = \begin{bmatrix} 20 \\ 12 \\ 10 \end{bmatrix}$$

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Substituting in $A^T A x = A^T b$

$$\begin{bmatrix} 34 & 18 & 10 \\ 18 & 10 & 6 \\ 10 & 6 & 5 \end{bmatrix} x = \begin{bmatrix} 20 \\ 12 \\ 10 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 34 & 18 & 10 & 20 \\ 18 & 10 & 6 & 12 \\ 10 & 6 & 5 & 10 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

$\therefore$ $a = 0, \; b = 0, \; c = 2$

$\therefore$ $y = 0x^2 + 0x + 2$

$y = 2$

least square of 2 degree

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2. Use the QR factorization to find the least square solution to $Ax = b$.

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & 4 & -1 & 4 \end{bmatrix}^T = \begin{bmatrix} 1/2 & -1/2 \\ 1/2 & 1/2 \\ 1/2 & -1/2 \\ 1/2 & 1/2 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 0 & 5 \end{bmatrix}$$

and, $b = \begin{bmatrix} -1 \\ 6 \\ 5 \\ 7 \end{bmatrix}$

$$Q^T b = \begin{bmatrix} 1/2 & 1/2 & 1/2 & 1/2 \\ -1/2 & 1/2 & -1/2 & 1/2 \end{bmatrix} \begin{bmatrix} -1 \\ 6 \\ 5 \\ 7 \end{bmatrix} = \begin{bmatrix} 17/2 \\ 9/2 \end{bmatrix}$$

The least square solution $\hat{x}$ satisfies $Rx = Q^T b$

$$\begin{bmatrix} 2 & 3 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 17/2 \\ 9/2 \end{bmatrix} \qquad \left\{ \hat{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right\}$$

$$\therefore \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 29/10 \\ 9/10 \end{bmatrix}$$

Hence, the least square solution to $Ax = b$ is $\hat{x} = \begin{bmatrix} 29/10 \\ 9/10 \end{bmatrix}$

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3. Find QR factorization of A where Q is also provided.

$$A = \begin{bmatrix} -2 & 3 \\ 5 & 7 \\ 2 & -2 \\ 4 & 6 \end{bmatrix}, \quad Q = \begin{bmatrix} -2/7 & 5/7 \\ 5/7 & 2/7 \\ 2/7 & -4/7 \\ 4/7 & 2/7 \end{bmatrix}$$

$\because$ $A = QR$ and $R = Q^T A$

$$\therefore R = \begin{bmatrix} -2/7 & 5/7 & 2/7 & 4/7 \\ 5/7 & 2/7 & -4/7 & 2/7 \end{bmatrix} \begin{bmatrix} -2 & 3 \\ 5 & 7 \\ 2 & -2 \\ 4 & 6 \end{bmatrix}$$

$$R = \begin{bmatrix} 7 & 7 \\ 0 & 7 \end{bmatrix}$$

Hence, R is $\begin{bmatrix} 7 & 7 \\ 0 & 7 \end{bmatrix}$.

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4. Orthogonally diagonalize following matrices

(a) $A = \begin{bmatrix} 6 & -2 \\ -2 & 9 \end{bmatrix}$

$$\det(A - \lambda I) = \begin{vmatrix} 6-\lambda & -2 \\ -2 & 9-\lambda \end{vmatrix} = (\lambda-6)(\lambda-9) - 4$$

$$\det(A - \lambda I) = \lambda^2 - 15\lambda + 50 = 0$$

$\therefore$ eigen value are $\lambda_1 = 5$ and $\lambda_2 = 10$.

Let x_0 be eigen vector corresponding to eigenvalue $\lambda = 5$.

$$\therefore (A - \lambda_1 \cancel{I}) x = 0$$

$$\begin{bmatrix} 1 & -2 \\ -2 & 4 \end{bmatrix} x = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$R_2 \to R_2 + 2R_1$$

$$\begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix} \cancel{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \left\{ \text{where } x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right\}$$

x_2 is free variable

$$x_1 - 2x_2 = 0$$

$$x_1 = 2x_2$$

$$\therefore x = c \begin{bmatrix} 2 \\ 1 \end{bmatrix} \qquad (c \text{ is real scalar})$$

Let v be eigen vector corresponding to eigenvalue $\lambda = 10$.

$$\therefore \begin{bmatrix} -4 & -2 \\ -2 & -1 \end{bmatrix} \cancel{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

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$R_2 \rightarrow R_2 - \frac{1}{2} R_1$

$$\begin{bmatrix} -4 & -2 \\ 0 & 0 \end{bmatrix} v = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \left\{ v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \right\}$$

v_2 is free variable and

$$-4v_1 - 2v_2 = 0$$
$$v_1 = \frac{-v_2}{2}$$

$$\therefore \quad v = c \begin{bmatrix} -1/2 \\ 1 \end{bmatrix} \qquad (c \text{ is real scalar})$$

Clearly x and v are linearly independent.

$$x \cdot v = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} -1/2 \\ 1 \end{bmatrix} = 0$$

Let U_1 and U_2 be unit vectors in direction of x and v.

$$\therefore \quad U_1 = \frac{x}{|x|} = \begin{bmatrix} 2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix}$$

and
$$U_2 = \frac{2v}{|2v|} = \begin{bmatrix} -1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix}$$

$$\therefore \text{ matrix } P = \left[U_1 \,\vdots\, U_2 \right] = \begin{bmatrix} 2/\sqrt{5} & -1/\sqrt{5} \\ 1/\sqrt{5} & 2/\sqrt{5} \end{bmatrix}$$

and
$$D = \begin{bmatrix} 5 & 0 \\ 0 & 10 \end{bmatrix}$$

Now $A = PDP^{-1}$

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$$\therefore \quad A = \begin{bmatrix} 2/\sqrt{5} & -1/\sqrt{5} \\ 1/\sqrt{5} & 2/\sqrt{5} \end{bmatrix} \begin{bmatrix} 5 & 0 \\ 0 & 10 \end{bmatrix} \begin{bmatrix} 2/\sqrt{5} & 1/\sqrt{5} \\ -1/\sqrt{5} & 2/\sqrt{5} \end{bmatrix}$$

(b) $$A = \begin{bmatrix} 1 & -6 & 4 \\ -6 & 2 & -2 \\ 4 & -2 & -3 \end{bmatrix}$$

$\det(A - \lambda I) = 0$ will give the eigenvalues of A.

$$\begin{vmatrix} 1-\lambda & -6 & 4 \\ -6 & 2-\lambda & -2 \\ 4 & -2 & -3-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)\left[(\lambda-2)(\lambda+3) - 4\right] + 6\left[6(3+\lambda) + 8\right] + 4\left(12 + 4(\lambda-2)\right) = 0$$

$$\lambda^3 - 63\lambda - 162 = 0$$

$$\lambda_1 = 9, \quad \lambda_2 = -6 \text{ and } \lambda_3 = -3$$

Let x, v, w be corresponding eigenvectors corresponding to λ_1, λ_2 and λ_3.

$$\therefore \quad [A - \lambda_1 I]\, x = 0$$

$$\begin{bmatrix} -8 & -6 & 4 \\ -6 & -7 & 2 \\ 4 & -2 & -12 \end{bmatrix} x = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \qquad \left\{ x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \right\}$$

x_3 is free variable and

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$$x_2 + 2x_3 = 0$$
$$\therefore x_2 = -2x_3$$

and, $$x_1 - 2x_3 = 0$$
$$x_1 = 2x_3$$

$$\therefore x = c\begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$$ (c is any scalar real)

Now, $[A - \lambda_2 I] v = 0$

$$\left[\begin{array}{ccc|c} 7 & -6 & 4 & 0 \\ -6 & 8 & -2 & 0 \\ 4 & -2 & 3 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1/2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$ $(v = [v_1 \ v_2 \ v_3]^T)$

v_3 is free variable

$$\therefore v = c\begin{bmatrix} -1 \\ -1/2 \\ 1 \end{bmatrix}$$

And, $[A - \lambda_3 I] w = 0$

$$\left[\begin{array}{ccc|c} 4 & -6 & 4 & 0 \\ -6 & 5 & -2 & 0 \\ 4 & -2 & 0 & 0 \end{array}\right]$$

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$$\sim \left[\begin{array}{ccc|c} 1 & 0 & -1/2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \qquad \left(\because w = [w_1 \;\; w_2 \;\; w_3]^T\right)$$

w_3 is free variable

$$\therefore \quad w = \begin{bmatrix} 1/2 \\ 1 \\ 1 \end{bmatrix}$$

Let u_1, u_2, u_3 be orthonormal vectors in x, v, w direction.

Since $x \cdot v = v \cdot w = x \cdot w = 0$, therefore x, v, w are orthogonal.

$$\therefore \quad u_1 = \frac{x}{|x|} = \begin{bmatrix} 2/3 \\ -2/3 \\ 1/3 \end{bmatrix}$$

$$u_2 = \frac{2v}{|2v|} = \begin{bmatrix} -2/3 \\ -1/3 \\ 2/3 \end{bmatrix}$$

$$u_3 = \frac{2w}{|2w|} = \begin{bmatrix} 1/3 \\ 2/3 \\ 2/3 \end{bmatrix}$$

Now, $P = [u_1 \;\; u_2 \;\; u_3]$

Since P is orthogonal, $P^T P = I$

i.e. $P^T = P^{-1}$

$$A = P D P^{-1} = P D P^T$$

and $D = \begin{bmatrix} 9 & 0 & 0 \\ 0 & -6 & 0 \\ 0 & 0 & -3 \end{bmatrix}$

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$$A = \begin{bmatrix} 2/3 & -2/3 & 1/3 \\ -2/3 & -1/3 & 2/3 \\ 1/3 & 2/3 & 2/3 \end{bmatrix} \begin{bmatrix} 9 & 0 & 0 \\ 0 & -6 & 0 \\ 0 & 0 & -3 \end{bmatrix} \begin{bmatrix} 2/3 & -2/3 & 1/3 \\ -2/3 & -1/3 & 2/3 \\ 1/3 & 2/3 & 2/3 \end{bmatrix}$$

Q.5 Show that if A and B are symmetric then A^2, $A+B$ are also symmetric. What about AB?

Since A and B are symmetric

$\therefore$ $A = A^T$ and $B = B^T$

a) $A^2 = AA = A^T A^T = (A^T)^2$

$\therefore$ A^2 is symmetric

b) $(A+B)^T = A^T + B^T = A + B$

$\therefore$ $A+B$ is symmetric

c) $\because$ $(AB)^T = B^T A^T = BA$

$\because$ AB is not always equal to BA

$\therefore$ AB is not symmetric.

6. Let B be a $n \times n$ symmetric matrix such that $B^2 = B$. Any such matrix is called a projection matrix (or an orthogonal projection matrix). Given any $y \in R^n$, let $\bar{y} = By$, $z = y - \bar{y}$

Show that z is orthogonal to $\bar{y}$. Let W be the column space of B. Show that y is the sum of a vector in W and a vector in $W^{\perp}$.

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$\because \quad B^2 = B \quad$ (given)

and B is symmetric

$\therefore \quad B^T = B$.

hence, $B \cdot B^T = B^2 = B$

$\because \quad \bar{y} = By$

and, $z = y - \bar{y}$

$$z \cdot \bar{y} = y \cdot \bar{y} - \bar{y} \cdot \bar{y}$$
$$= y^T \cdot By - (By)^T (By)$$
$$= y^T By - y^T B^T B y$$
$$= y^T By - y^T By = 0$$

$\therefore \quad z$ is orthogonal to $\bar{y}$.

Now, since $W = \text{Col}(B)$

Any vector in W has the form BU for some U.

$$(y - \bar{y}) \cdot (BU) = [B(y - \bar{y})] \cdot U$$
$$= [By - B\bar{y}] \cdot U$$
$$= (By - B^2 y) \cdot U$$
$$= (By - By) \cdot U = 0$$

$y - \bar{y}$ is in $W^{\perp}$.

$$\therefore \quad y = \bar{y} + (y - \bar{y})$$

where $\bar{y}$ is in W and $y - \bar{y}$ is in $W^{\perp}$.

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7. Find a 3x3 matrix whose minimal polynomial is x^2.

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

Clearly $A^2 = 0$ and $A \neq 0$

So, monotonic polynomial that satisfies Cayley Hamilton theorem and divides characteristic equation of lowest degree is minimal polynomial.

characteristic eq^n,

$$\det(A - \lambda I) = 0$$

$$\begin{vmatrix} -\lambda & 0 & 0 \\ 0 & -\lambda & 0 \\ 1 & 0 & -\lambda \end{vmatrix} = 0$$

$$-\lambda^3 = 0$$

$$\lambda \cdot \lambda^2 = 0$$

By Cayley Hamilton theorem,

$A^2 = 0$ and $A \neq 0$

$\therefore$ x^2 is the solution.

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8. Find minimal polynomial of following matrix

$$A = \begin{bmatrix} 5 & -6 & -6 \\ -1 & 4 & 2 \\ 3 & -6 & -4 \end{bmatrix}$$

$$|A - \lambda I| = \begin{vmatrix} 5-\lambda & -6 & -6 \\ -1 & 4-\lambda & 2 \\ 3 & -6 & -4-\lambda \end{vmatrix}$$

$$|A - \lambda I| = (5-\lambda)\left[(\lambda-4)(\lambda+4)+12\right] + 6(\lambda+4-6) - 6\left(6+3(\lambda-4)\right)$$

$$= (5-\lambda)\left[\lambda^2 - 4\right] + 6\lambda - 12 - 18\lambda + 36$$

$$= 5\lambda^2 - 20 - \lambda^3 + 4\lambda - 12\lambda + 24$$

$$= -\lambda^3 + 5\lambda^2 - 8\lambda + 4$$

$$= (\lambda-1)(-\lambda^2 + 4\lambda - 4)$$

$$= (\lambda-1)\{-(\lambda^2 - 4\lambda + 4)\}$$

$$= (\lambda-1)(\lambda-2)^2$$

$\therefore \lambda = 1, 2, 2$

By Cayley-Hamilton Theorem, every square matrix satisfies its characteristic equation

checking, $A - I \neq 0$
and $A - 2I \neq 0$

$$(A - I)(A - 2I) = A^2 - 3A + 2I$$

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$$A^2-3A+2I=\begin{bmatrix}13 & -18 & -18\\ -3 & 10 & 6\\ 9 & -18 & -14\end{bmatrix}-\begin{bmatrix}15 & -18 & -18\\ -3 & 12 & 6\\ 9 & -18 & -12\end{bmatrix}+\begin{bmatrix}2 & 0 & 0\\ 0 & 2 & 0\\ 0 & 0 & 2\end{bmatrix}$$

$$=\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$$

$\therefore\ (A-I)(A-2I)=0$.

$\therefore\ (\lambda-2)(\lambda-1)$ is the minimal polynomial. (lowest degree polynomial).

9. Find the SVD and the pseudoinverse $V\Sigma^{+}U$ of following matrix:

$$A=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\end{bmatrix}$$

Write down A as $\sigma_1 u_1 v_1^T + \sigma_2 u_2 v_2^T$ and find AA^+, A^+A. Use matlab to verify your answers.

$$A=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\end{bmatrix},\quad A^T=\begin{bmatrix}1 & 0\\ 0 & 1\\ 0 & 0\end{bmatrix}$$

$$A^TA=\begin{bmatrix}1 & 0\\ 0 & 1\\ 0 & 0\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\end{bmatrix}=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 0\end{bmatrix}$$

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$$\left| \left(A^T A - \lambda I \right) \right| = 0$$

$$\begin{vmatrix} 1-\lambda & 0 & 0 \\ 0 & 1-\lambda & 0 \\ 0 & 0 & -\lambda \end{vmatrix} = 0$$

$$(\lambda-1)(\lambda-1)\lambda = 0$$

$$\lambda = 0 \text{ or } 1$$

eigenvector for $\lambda = 1$ (using MATLAB)

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad v_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

eigenvector for $\lambda = 0$ (using MATLAB)

$$v_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Note, $v_1 \cdot v_2 = v_2 \cdot v_3 = v_1 \cdot v_3 = 0$

and $\|v_1\| = \|v_2\| = \|v_3\| = 1$

$\{v_1, v_2, v_3\}$ form an orthonormal basis.

$$u_1 = \frac{Av_1}{\|Av_1\|} = Av_1$$

$$u_2 = \frac{Av_2}{\|Av_2\|} = Av_2$$

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$A = U \Sigma V^T$ and $A^T = V \Sigma^T U^T$

$U = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $V = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$\Sigma = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

Σ^+ can be formed by taking reciprocal of each non-zero data on the diagonal of Σ.

$\Sigma^+ = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}^T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$

$V \Sigma^+ U^T = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

$\therefore A^+ = V \Sigma U^T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$

$A A^+ = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$A^+ A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

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$$\therefore \quad A = U \Sigma V^T$$

$$= (U\Sigma) \begin{bmatrix} v_1^T \\ v_2^T \end{bmatrix}$$

$$= a_1 u_1 v_1^T + a_2 u_2 v_2^T$$

$$a_1 = 1, \quad a_2 = 1$$

$$u_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad u_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$A = a_1 u_1 v_1^T + a_2 u_2 v_2^T$$

$$= \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

CURRENT FOLDER

Name ▲

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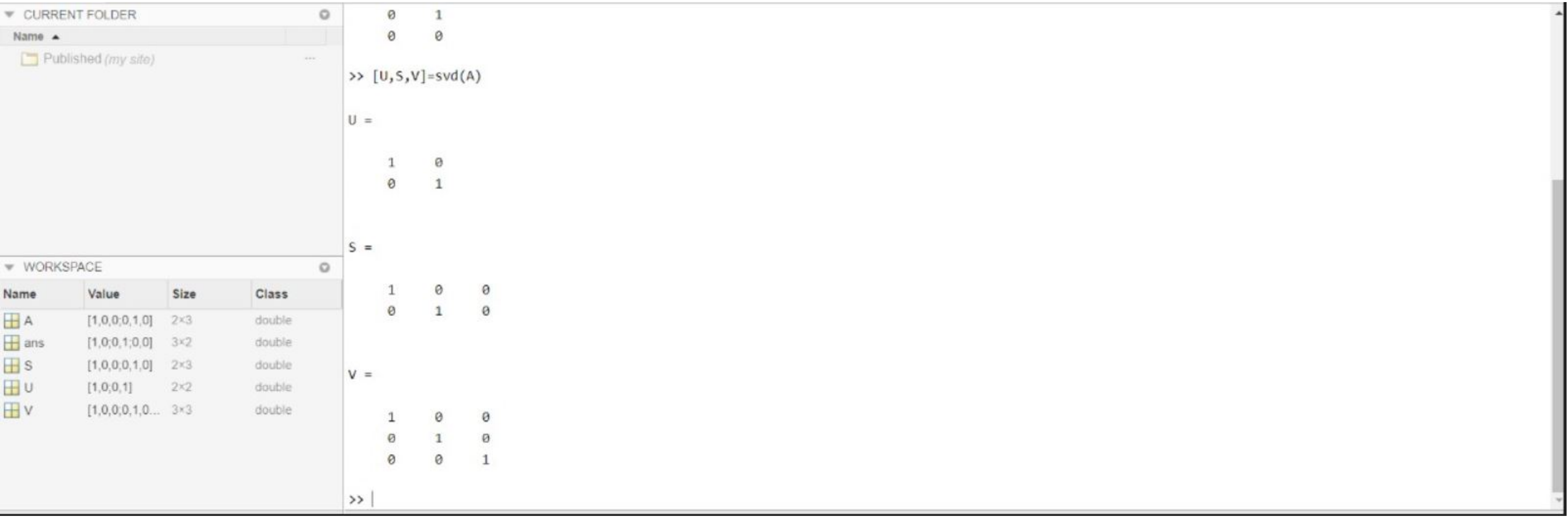
WORKSPACE

Name	Value	Size	Class
A	[1,0,0;0,1,0]	2×3	double
ans	[1,0;0,1;0,0]	3×2	double
S	[1,0,0;0,1,0]	2×3	double
U	[1,0;0,1]	2×2	double
V	[1,0,0;0,1,0...	3×3	double

```
>> A=[1,0,0;0,1,0]

A =

     1     0     0
     0     1     0

>> pinv(A)

ans =

     1     0
     0     1
     0     0

>> [U,S,V]=svd(A)

U =

     1     0
     0     1


S =
```


CURRENT FOLDER
Name
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WORKSPACE
Name Value Size Class
A [1,0,0;0,1,0] 2×3 double
ans [1,0;0,1;0,0] 3×2 double
S [1,0,0;0,1,0] 2×3 double
U [1,0;0,1] 2×2 double
V [1,0,0;0,1,0... 3×3 double
0 1
0 0
>> [U,S,V]=svd(A)
U =
1 0
0 1
S =
1 0 0
0 1 0
V =
1 0 0
0 1 0
0 0 1
>>

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10. What is the minimum length least squares solution, $x = A^+ b$ to the following linear system?

$$Ax = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} C \\ D \\ E \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix}$$

$$A^T A = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

$$|A^T A - \lambda I| = \begin{vmatrix} 3-\lambda & 1 & 1 \\ 1 & 1-\lambda & 1 \\ 1 & 1 & 1-\lambda \end{vmatrix} = 0$$

$$\lambda = 0, 1, 4$$

For $\lambda = 4$
eigenvector $v_1 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$, $l = \sqrt{6}$

For $\lambda = 1$
eigenvector $v_2 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$, $l = \sqrt{3}$

For $\lambda = 0$,
eigenvector $v_3 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$, $l = \sqrt{2}$

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$$V_1 = \begin{bmatrix} 2/\sqrt{6} \\ 1/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix}, \quad V_2 = \begin{bmatrix} -1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}, \quad V_3 = \begin{bmatrix} 0 \\ -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

$$\Sigma = \begin{bmatrix} \sqrt{4} & 0 & 0 \\ 0 & \sqrt{1} & 0 \\ 0 & 0 & \sqrt{0} \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$AA^T = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 3 \end{bmatrix}$$

$$|AA^T - \lambda I| = \begin{vmatrix} 1-\lambda & 1 & 1 \\ 1 & 1-\lambda & 1 \\ 1 & 1 & 3-\lambda \end{vmatrix} = 0$$

$$\lambda = 0, 1, 4$$

$\lambda = 4$

$$V_1 = \begin{bmatrix} 1/2 \\ 1/2 \\ 1 \end{bmatrix}, \quad L = \sqrt{6/4} \; ; \quad u_1 = \begin{bmatrix} 1/\sqrt{6} \\ 1/\sqrt{6} \\ 2/\sqrt{6} \end{bmatrix}$$

$\lambda = 1$

$$V_2 = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}, \quad L = \sqrt{3}, \quad U_2 = \begin{bmatrix} -1/\sqrt{3} \\ -1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}$$

$\lambda = 0$

$$V_3 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \quad L = \sqrt{2}, \quad U_3 = \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix}$$

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$$U = \begin{bmatrix} 1/\sqrt{6} & -1/\sqrt{3} & -1/\sqrt{2} \\ 1/\sqrt{6} & -1/\sqrt{3} & 1/\sqrt{2} \\ 2/\sqrt{6} & 1/\sqrt{3} & 0 \end{bmatrix}$$

$$V = \begin{bmatrix} 2/\sqrt{6} & -1/\sqrt{3} & 0 \\ 1/\sqrt{6} & 1/\sqrt{3} & -1/\sqrt{2} \\ 1/\sqrt{6} & 1/\sqrt{3} & 1/\sqrt{2} \end{bmatrix}$$

$$\Sigma = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

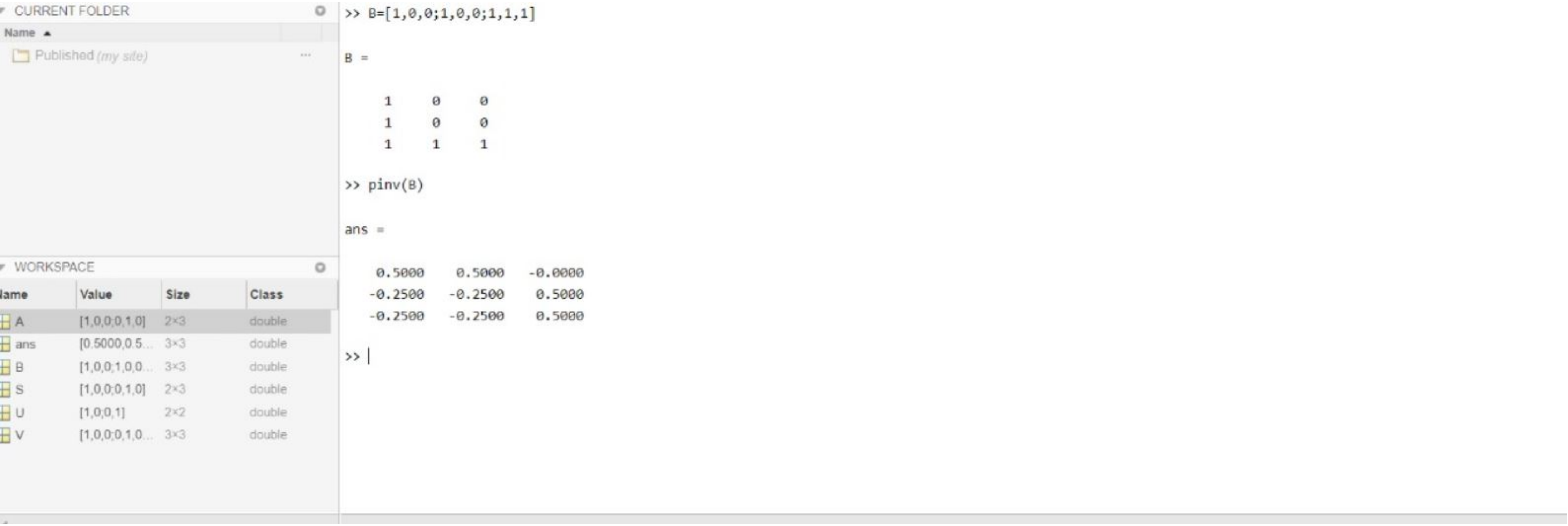
$$A^{+} = (U \Sigma V^{T})^{T} = V \Sigma^{+} U^{T}$$

$$\Sigma^{+} = \begin{bmatrix} 1/2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$U^{T} = \begin{bmatrix} 1/\sqrt{6} & 1/\sqrt{6} & 2/\sqrt{6} \\ -1/\sqrt{3} & -1/\sqrt{3} & 1/\sqrt{3} \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 \end{bmatrix}$$

$$V \Sigma^{+} = \begin{bmatrix} 1/\sqrt{6} & -1/\sqrt{3} & 0 \\ 1/2\sqrt{6} & 1/\sqrt{3} & 0 \\ 1/2\sqrt{6} & 1/\sqrt{3} & 0 \end{bmatrix}$$

$$(V \Sigma^{+}) U^{T} = \begin{bmatrix} 1/2 & 1/2 & 0 \\ -1/4 & -1/4 & 1/2 \\ -1/4 & -1/4 & 1/2 \end{bmatrix}$$


CURRENT FOLDER
Name
Published (my site)
WORKSPACE
Name
Value
Size
Class
A
[1,0,0;0,1,0]
2×3
double
ans
[0.5000,0.5...
3×3
double
B
[1,0,0;1,0,0...
3×3
double
S
[1,0,0;0,1,0]
2×3
double
U
[1,0;0,1]
2×2
double
V
[1,0,0;0,1,0...
3×3
double
>> B=[1,0,0;1,0,0;1,1,1]
B =
1 0 0
1 0 0
1 1 1
>> pinv(B)
ans =
0.5000 0.5000 -0.0000
-0.2500 -0.2500 0.5000
-0.2500 -0.2500 0.5000
>>