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SECTION $\rightarrow$ A1

MA101 Tutrorial #8

(1) (a) $\begin{bmatrix} 5 & 0 & 0 & 0 \\ 8 & -4 & 0 & 0 \\ 0 & 7 & 1 & 0 \\ 1 & -5 & 2 & 1 \end{bmatrix}$

The characteristic polynomial is :-

$$\begin{vmatrix} 5-\lambda & 0 & 0 & 0 \\ 8 & -4-\lambda & 0 & 0 \\ 0 & 7 & 1-\lambda & 0 \\ 1 & -5 & 2 & 1-\lambda \end{vmatrix} = (5-\lambda)(-4-\lambda)(1-\lambda)(1-\lambda)$$

The roots are $\lambda_1 = 5$, $\lambda_2 = -4$, $\lambda_3 = 1$, $\lambda_4 = 1$

These are the eigen values with 5 having multiplicity 1, -4 having 1 and 1 having 2 multiplicity.

$\lambda_1 = 5$:

$$[A - \lambda I]X = 0$$

$$\Rightarrow \underbrace{\begin{bmatrix} 0 & 0 & 0 & 0 \\ 8 & -9 & 0 & 0 \\ 0 & 7 & -4 & 0 \\ 1 & -5 & 2 & -4 \end{bmatrix}}_{C} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\boxed{\text{Rank } (C) = 3}$$

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GM of $\lambda_1 = 5$ is $n - \text{rank}(A - \lambda_1 I)$

$= n - \text{rank}(C)$

$= 4 - 3 = 1$

$\therefore$ $\boxed{\text{For } \lambda_1 = 5 :- \quad A.M = 1 \quad G.M = 1}$

$\lambda_2 = -4$:-

$[A - \lambda_2 I] X = 0$

$$\underbrace{\begin{bmatrix} 9 & 0 & 0 & 0 \\ 8 & 0 & 0 & 0 \\ 0 & 7 & 5 & 0 \\ 1 & -5 & 2 & 5 \end{bmatrix}}_{C} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

rank (C) = 3

GM of $\lambda_2 = -4$ is $n - \text{rank}(C)$

$= 4 - 3 = 1$

$\therefore$ $\boxed{\text{For } \lambda_2 = -4 :- \quad A.M = 1 \quad G.M. = 1}$

$\lambda_3 = 1$

$[A - \lambda_3 I] X = 0$

$$\underbrace{\begin{bmatrix} 4 & 0 & 0 & 0 \\ 8 & -5 & 0 & 0 \\ 0 & 7 & 0 & 0 \\ 1 & -5 & 2 & 0 \end{bmatrix}}_{C} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

rank (C) = 3

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GM for $\lambda_3 = 1$ $= n - \text{rank}(C)$
$= 4 - 3 = 1$

$\therefore$ $\boxed{\text{for } \lambda_3 = 1 :- \ A.M. = 2 \quad G.M. = 1}$

$A.M. \neq G.M.$

Since A.M $\neq$ G.M. for all eigen values given matrix is not diagonalizable.

(b) $\begin{bmatrix} 6 & -2 & 0 \\ -2 & 9 & 0 \\ 5 & 8 & 3 \end{bmatrix}$

The characteristic polynomial is :-

$$|A - \lambda I| = \begin{vmatrix} 6-\lambda & -2 & 0 \\ -2 & 9-\lambda & 0 \\ 5 & 8 & 3-\lambda \end{vmatrix}$$

$$= (3-\lambda)[(6-\lambda)(9-\lambda) - 4]$$

$$= (3-\lambda)(\lambda^2 - 15\lambda + 50)$$

$$= (3-\lambda)(\lambda - 5)(\lambda - 10)$$

The roots are $\lambda_1 = 3$, $\lambda_2 = 5$, $\lambda_3 = 10$

Hence 3, 5 and 10 are the eigen values.

- For $\lambda_1 = 3$:-

$[A - \lambda_1 I]X = 0$

$$\Rightarrow \underbrace{\begin{bmatrix} 3 & -2 & 0 \\ -2 & 6 & 0 \\ 5 & 8 & 0 \end{bmatrix}}_{C} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

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Rank (c) = 2

GM for λ_1 is n − Rank(c)

$\hookrightarrow = 3-2 = 1$

$\therefore$ $\boxed{\text{for } \lambda_1 = 3 :- \text{ A.M.} = 1, \text{ G.M.} = 1}$

- for $\lambda_2 = 5$:-

$[A - \lambda_2 I]X = 0$

$$\Rightarrow \underbrace{\begin{bmatrix} 1 & -2 & 0 \\ -2 & 4 & 0 \\ 5 & 8 & -2 \end{bmatrix}}_{C} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

Rank (c) = 2

G.M. for λ_2 is n − Rank (c)

$\hookrightarrow = 3-2 = 1$

$\therefore$ $\boxed{\text{For } \lambda_2 = 5 :- \text{ A.M.} = 1, \text{ G.M} = 1}$

- for $\lambda_3 = 10$:-

$[A - \lambda_3 I]X = 0$

$$\Rightarrow \underbrace{\begin{bmatrix} -4 & -2 & 0 \\ -2 & -1 & 0 \\ 5 & 8 & -7 \end{bmatrix}}_{C} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Rank(C) = 2

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G.M. for λ_3 is $n - \text{Rank}(C)$
$= 3 - 2 = 1$

$\therefore$ $\boxed{\text{for } \lambda_3 = 10 :- \text{A.M.} = 1 \quad \text{G.M.} = 1}$

Since, A.M. = G.M. ($\forall$) for all given eigen values (λ_1, λ_2, and λ_3). Hence given matrix is diagonalizable.

(2) Given Q is invertible. So, we can write

$A = QR$

$\Rightarrow A = QRQQ^{-1} = Q(RQ)Q^{-1}$

$\Rightarrow \boxed{A = QA_1Q^{1}}$

By the definition of similar matrices. We say that A is similar to B if there is an invertible (non-singular) $n \times n$ matrix P such that $P = P^{-1}AP$. Hence, A is similar to A_1

(3) $A = \begin{bmatrix} 0.6 & 0.3 \\ 0.4 & 0.7 \end{bmatrix}$, $v_1 = \begin{bmatrix} 3/7 \\ 4/7 \end{bmatrix}$, $x_0 = \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix}$

(a) $Av_1 = \begin{bmatrix} 0.6 & 0.3 \\ 0.4 & 0.7 \end{bmatrix} \begin{bmatrix} 3/7 \\ 4/7 \end{bmatrix} = \begin{bmatrix} 1.8/7 + 1.2/7 \\ 1.2/7 + 2.8/7 \end{bmatrix} = \begin{bmatrix} 3/7 \\ 4/7 \end{bmatrix}$

$\Rightarrow \boxed{Av_1 = 1 \cdot v_1}$

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$\Rightarrow$ 1 is the eigen value / eigen vector, find ~~another eigen value~~ the root of characteristic polynomial

$$\begin{vmatrix} 0.6-\lambda & 0.3 \\ 0.4 & 0.7-\lambda \end{vmatrix} = 0$$

$\Rightarrow (0.6-\lambda)(0.7-\lambda) - 0.12 = 0$

$\Rightarrow \lambda^2 - 1.3\lambda + 0.3 = 0$

$\Rightarrow (\lambda-1)(\lambda-3) = 0$

The eigen value are 1 and 0.3

To find the eigen vector for $\lambda = 0.3$, solve $(A - 0.3I)x = 0$

$$\Rightarrow \left[\begin{array}{cc|c} 0.3 & 0.3 & 0 \\ 0.4 & 0.4 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 1 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$x_1 + x_2 = 0$

$\hookrightarrow$ free variable

$$x = \begin{bmatrix} -c \\ c \end{bmatrix} = c\begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

eigen vector, $v_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$, $v_1 = \begin{bmatrix} 3/7 \\ 4/7 \end{bmatrix}$ $\Big\}$ linearly independent (Since, they are eigen vectors)

A basis for $\mathbb{R}^2$ is $\{v_1, v_2\}$, where v_2 is an eigen value for $\lambda = 0.3$.

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(b) $x_0 = v_1 + c v_2$

$$\begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix} = \begin{bmatrix} 3/7 \\ 4/7 \end{bmatrix} + c v_2$$

$$\Rightarrow c v_2 = \begin{bmatrix} 1/2 - 3/7 \\ 1/2 - 4/7 \end{bmatrix}$$

$$\Rightarrow c \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/2 - 3/7 \\ 1/2 - 4/7 \end{bmatrix} \Rightarrow c \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/14 \\ -1/14 \end{bmatrix}$$

$$\cancel{c = [-1/14]} \quad \boxed{c = -1/14}$$

c) $x_k = A^k x_0$

$x_1 = A(v_1 + c v_2) = A v_1 + c A v_2$ { v_1 & v_2 are eigenvector with corresponding eigen values 1 & 0.3 }

$x_2 = A^2 x_0 = A(A x_0) = A x_1$

$\Rightarrow x_2 = A(v_1 + (0.3) v_2) = v_1 + c(0.3)(0.3) v_2$

$\vdots$

$x_k = v_1 + c(0.3)^k v_2$

(4) If A is diagonalizable then

$\Rightarrow \exists$ Invertible P and diagonal D such that

$$P^{-1} A P = D$$

$$\Rightarrow D^2 = D . D = (P^{-1} A P) . (P^{-1} A P)$$

$\Rightarrow D^2 = P^{-1} A (P P^{-1}) A P = P^{-1} A^2 P$, which is also diagonalizable.

Hence, A^2 will be diagonalizable. If A is digon..

$A = PD_1P^{-1}$

and $B = QD_2Q^{-1}$

$\Rightarrow A \cdot B = PD_1P^{-1}QD_2Q^{-1}$

- If $\boxed{P=Q}$ ✓ then $AB = PD_1D_2P^{-1}$

and we know that the product of two diagonal matrix is diagonal

So, $AB = PDP^{-1} \Rightarrow AB$ is diagonalizable

- If $P \neq Q$:- we can't conclude directly thus $P^{-1}Q$ will be identity or $D_1P^{-1}QD_2$ will be a diagonal matrix.

(5) $A = \begin{bmatrix} 4 & -3 \\ 2 & -1 \end{bmatrix}$

First we need to diagonalize this matrix A :-

Step 1 :- find the eigen ~~vectors~~ values of A :-

$|A - \lambda I| = 0 \Rightarrow \begin{vmatrix} 4-\lambda & 3 \\ 2 & -1-\lambda \end{vmatrix} = 0$

$\Rightarrow (4-\lambda)(-1-\lambda) + 6 = 0$

$\Rightarrow -4 - 4\lambda + \lambda + \lambda^2 + 6 = 0$

$\Rightarrow \lambda^2 - 3\lambda + 2 = 0 \Rightarrow (\lambda - 1)(\lambda - 2) = 0$

$\Rightarrow \boxed{\lambda_1 = 1}$ and $\boxed{\lambda_2 = 2}$

Step 2 :- find eigenvectors :-

- for $\lambda_1 = 1$:-

Solve $(A - \lambda_1 I)x = 0$

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- $(P^{-1})^T = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} = Q$

- $(P)^T = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = ((P^{-1})^T)^{-1} = Q^{-1}$

- $A^T = (P^{-1})^T D (P)^T = Q D Q^{-1}$

So, $Q = \{u_1, u_2, u_3\}$ where, u_1, u_2 & u_3 are eigen vectors of Q.

$\therefore$ $\boxed{\left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \right\} \text{ are the eigen vectors of } Q.}$

(7) $T : \mathbb{R}^3 \to V$

$\{v_1, v_2, v_3\}$ is the basis of V

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = (x_3 - x_2) b_1 - (x_1 + x_3) b_2 + (x_1 - x_2) b_3$$

As $T(x_1, x_2, x_3) = x_1 T(e_1) + x_2 T(e_2) + x_3 T(e_3)$ — (1)

$T(e_1) = T\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = -b_2 + b_3$

$T(e_2) = T\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) = -b_1 - b_3$

$T(e_3) = T\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = b_1 - b_2$

Put these values in eq^n (1):.

$$T(x_1, x_2, x_3) = (-b_2+b_3)x_1 + (-b_1-b_3)x_2 + (b_1-b_2)x_3$$
$$= b_1(-x_2+x_3) + b_2(-x_1-x_3) + b_3(x_1-x_2)$$

$$\Rightarrow \quad T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} -x_2+x_3 \\ -x_1-x_3 \\ x_1-x_2 \end{bmatrix}$$

Now, we have to find if the matrix A diagonalizable or not.

M-1

- Find the characteristic polynomial -

$$\begin{bmatrix} -\lambda & -1 & 1 \\ -1 & -\lambda & -1 \\ 1 & -1 & -\lambda \end{bmatrix} = -\lambda(\lambda^2-1) + (\lambda+1) + (1+\lambda)$$
$$= -\lambda^3 + \lambda + \lambda + 1 + \lambda + 1 = -\lambda^3 + 3\lambda + 2$$

$$\lambda_1 = -1, \; \lambda_2 = 2$$

- for $\lambda_1 = -1$:- $[A - \lambda_1 I]x = 0$

$$\Rightarrow \begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix} x = 0$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ -1 & 1 & -1 & 0 \\ 1 & -1 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\left.\begin{aligned} A.M. &= 2 \\ G.M &= 3-1 = 2 \end{aligned}\right\} A.M = G.M.$$

. $\therefore$

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- For $\lambda_2 = 2$:- $[A - \lambda_2 - I]x = 0$

$$\Rightarrow \left[\begin{array}{ccc|c} -2 & -1 & 1 & 0 \\ -1 & -2 & -1 & 0 \\ 1 & -1 & -2 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1/2 & -1/2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\left.\begin{array}{l} A.M. = 1 \\ G.M = 3-2 = 1 \end{array}\right\} A.M = G.M.$$

Since A.M = G.M. for all eigen values $\Rightarrow$

A is diagonalizable

Hence, $T : \mathbb{R}^3 \to V$ is diagonalizable

M-2

Oh! I just show that A is a symmetric matrix $(A = A^T)$

we could have used the fact that Real symmetric matrices not only have real eigen values, they are always diagonalizable

(8) $A = \begin{bmatrix} 5 & -5 \\ 1 & 1 \end{bmatrix}$

Find eigen values:-

$$|A - \lambda I| = 0 \Rightarrow \begin{vmatrix} 5-\lambda & -5 \\ 1 & 1-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (5-\lambda)(1-\lambda) + 5 = 0 \Rightarrow 5 - 5\lambda - \lambda + \lambda^2 + 5 = 0$$

$$\Rightarrow \lambda^2 - 6\lambda + 10 = 0 \Rightarrow \lambda = \frac{6 \pm \sqrt{-4}}{2}$$

$$\lambda_1 = 3 + i \quad \text{and} \quad \lambda_2 = 3 - i$$

Eigen vector corresponding to $\lambda_1 = 3+i$:-

$$[A - \lambda_1 I] X = 0$$

$$\begin{bmatrix} 2-i & -5 \\ 1 & -2-i \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$(2-i)x_1 = 5x_2$$

$$\Rightarrow x_1 = \frac{5x_1}{2-i}$$

$$\Rightarrow x_1 = \frac{5x_2 (2+i)}{2-i\ (2+i)}$$

$$\Rightarrow x_1 = \frac{5(2+i)x_2}{5} = (2+i)x_2$$

Eigen vector $v_1 = \begin{bmatrix} 2+i \\ 1 \end{bmatrix}$ too $\lambda_1 = 3+i$

Using theorem 9 as described in David lay :-

Let A be a real 2X2 matrix with a complex eigen value $\lambda = a - bi$ ($b \neq 0$) and an associated eigen vector V in C^2. Then

$A = PCP^{-1}$, where $P = [\mathrm{Re}\,V.\ \mathrm{Im}\,V]$ and $C = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$

$$P = [\mathrm{Re}\,V\ \ \mathrm{Im}\,V] = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}$$

$$C = P^{-1} A P$$

$$\Rightarrow C = \begin{bmatrix} 0 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 5 & -5 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}$$

$$= -\begin{bmatrix} 0 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 5 & 5 \\ 3 & 1 \end{bmatrix} = -\begin{bmatrix} -3 & -1 \\ 1 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 1 \\ -1 & 3 \end{bmatrix}$$

(9) $A = \begin{bmatrix} 5 & -2 \\ 1 & 3 \end{bmatrix}$

given: eigen values $= 4-i$ & $4+i$

For eigen value $\lambda = 4-i$, find eigen vector :-

$[A - \lambda I]x = 0$

$$\left[\begin{array}{cc|c} 5-(4-i) & -2 & 0 \\ 1 & 3-(4-i) & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{cc|c} 1+i & -2 & 0 \\ 1 & -1+i & 0 \end{array}\right]$$

$x_1(1+i) = 2x_2$

$\Rightarrow x_1 = \frac{2x_2}{1+i} \frac{(1-i)}{(1-i)} = \frac{2(1-i)}{2} x_2$

$\Rightarrow \boxed{x_1 = (1-i)x_2}$

$v_1 = \begin{bmatrix} 1-i \\ 1 \end{bmatrix}$

$u = [Re(v)] = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$v = [Im(v)] = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$

$Au = \begin{bmatrix} 5 & -2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$

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and, $4u+v = 4\begin{pmatrix}1\\1\end{pmatrix} + \begin{pmatrix}-1\\0\end{pmatrix} = \begin{pmatrix}3\\4\end{pmatrix}$

Or $\underbrace{\begin{bmatrix}5 & -2\\1 & 3\end{bmatrix}}_{A}\underbrace{\begin{bmatrix}-1\\0\end{bmatrix}}_{v} = \begin{bmatrix}-5\\-1\end{bmatrix}$

$-u+4v = -\begin{bmatrix}1\\1\end{bmatrix} + 4\begin{bmatrix}-1\\0\end{bmatrix} = \begin{bmatrix}-5\\-1\end{bmatrix}$

$\boxed{u = \begin{pmatrix}1\\1\end{pmatrix},\ v = \begin{pmatrix}-1\\0\end{pmatrix}}$

$A(u+iv)$

$= A\left[\begin{pmatrix}1\\1\end{pmatrix} + \begin{pmatrix}-i\\0\end{pmatrix}\right] = A\begin{bmatrix}1-i\\1\end{bmatrix}$

$= \begin{bmatrix}5 & -2\\1 & 3\end{bmatrix}\begin{bmatrix}1-i\\1\end{bmatrix}$

$= \begin{bmatrix}3-5i\\4-i\end{bmatrix}$

(10) Given: A be an $n \times n$ matrix with the property that $A^T = A$

$\bar{x}^T A x = \bar{x}^T(\lambda x) = \lambda \bar{x}^T x$

$\bar{z}\, z = |z|^2 \geq 0$ for a complex number, z

$P = \bar{x}^T A x = (x^T A \bar{x})^T \ \{A = A^T\}$

$\Rightarrow \bar{P} = \overline{\bar{x}^T A x} = x^T \overline{A x} = x^T A \bar{x}$

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λ is real because $\bar{x}^T A x$ is real

$x = u + iv$, u, v, are real vectors

$Ax = A(u + iv) = Au + iAv$

Also , $\lambda x = \lambda u + i\lambda v$

Real part of Ax is Au, because entries in A, u, v are all real. ~~part of~~

$Ax = \lambda x \iff$ their real parts are also equal

$Au = \lambda u$

Real part of x is eigen vector of A.