Name: Archit Agrawal
Student ID: 202052307
Group: A1

classmate
Date
Page 1

MA101 : Linear Algebra and Matrices
Tutorial 6

1. Let $B = \{b_1, b_2\}$ and $C = \{c_1, c_2\}$ be basis of R^2 given below. Find the change of coordinate matrix from B to C. Find coordinates of $e_1 = [1\ 0]^T$ with respect to both bases.

$$b_1 = \begin{bmatrix} -1 \\ 8 \end{bmatrix}, \quad b_2 = \begin{bmatrix} 1 \\ -5 \end{bmatrix}, \quad c_1 = \begin{bmatrix} 1 \\ 4 \end{bmatrix}, \quad c_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Let x be a vector in R^2.
Then, $[x]_B$ are co-ordinates of vector with respect to bases B and $[x]_C$ are co-ordinates of vector with respect to C.
The change of coordinate matrix from B to C, that is $P_{B\to C}$ is a matrix such that

$$(P_{B\to C})\,[x]_B = [x]_C$$

Now, $P_{B\to C} = \left[\,[b_1]_C \quad [b_2]_C\,\right]$ — (1)

$[b_1]_C$ is $[p\ q]^T$ such that $pc_1 + qc_2 = b_1$

$$\therefore \quad p\begin{bmatrix} 1 \\ 4 \end{bmatrix} + q\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 8 \end{bmatrix}$$

$$p + q = -1$$
$$4p + q = 8$$

Solving the equations, we get,
$$p = 3 \text{ and } q = -4$$

$$\therefore [b_1]_C = \begin{bmatrix} 3 \\ -4 \end{bmatrix}$$

Name : Archit Agrawal
Student ID : 202052307
Group : A 1

$[b_2]_C$ is $[r \quad s]^T$ such that $rc_1 + sc_2 = b_2$

$$r\begin{bmatrix}1\\4\end{bmatrix} + s\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}1\\-5\end{bmatrix}$$

$$r + s = 1$$
$$4r + s = -5$$

Solving the equations, we get,
$r = -2$ and $s = 3$

$$\therefore [b_2]_C = \begin{bmatrix}-2\\3\end{bmatrix}$$

$\therefore$ change of co-ordinate matrix from B to C is

$$P_{B\to C} = \begin{bmatrix}3 & -2\\-4 & 3\end{bmatrix}$$

Coordinates of $e_1 = \begin{bmatrix}1\\0\end{bmatrix}$ w.r.t to bases B are m and n such that

$$mb_1 + nb_2 = e_1$$

$$m\begin{bmatrix}-1\\8\end{bmatrix} + n\begin{bmatrix}1\\-5\end{bmatrix} = \begin{bmatrix}1\\0\end{bmatrix}$$

$$-m + n = 1$$
$$8m - 5n = 0$$

Solving the equations we get
$m = \frac{5}{3}$ and $n = \frac{8}{3}$

$$\therefore [e_1]_b = \begin{bmatrix}5/3\\8/3\end{bmatrix}$$

Name : Archit Agrawal
Student ID: 202052307
Group : A1

Now, $[e_1]_C = P_{B \to C} [e_1]_B$

$$[e_1]_C = \begin{bmatrix} 3 & -2 \\ -4 & 3 \end{bmatrix} \begin{bmatrix} 5/3 \\ 8/3 \end{bmatrix}$$

$$[e_1]_C = \begin{bmatrix} -1/3 \\ 4/3 \end{bmatrix}$$

Hence, coordinates of e_1 w.r.t bases B are $\begin{bmatrix} 5/3 \\ 8/3 \end{bmatrix}$

and w.r.t basis C are $\begin{bmatrix} -1/3 \\ 4/3 \end{bmatrix}$.

2. Let V be the set of all 3×3 anti-symmetric matrices with entries from R. Is V a vector space? If yes then find its basis and dimension. What can you say if 3 is replaced by n?

V is the set of all 3×3 anti-symmetric matrices,

~~Let~~ ∴ $A = \begin{bmatrix} 0 & p & q \\ -p & 0 & r \\ -q & -r & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & l & m \\ -l & 0 & n \\ -m & -n & 0 \end{bmatrix}$

~~be~~ ~~two~~

and $C = \begin{bmatrix} 0 & x & y \\ -x & 0 & z \\ -y & -z & 0 \end{bmatrix}$ belong to V.

For V to be a vector space, it must satisfy following axioms

Name : Archit Agrawal
Student ID : 202052307
Group : A1

i) $A + B \in V$

$$A+B = \begin{bmatrix} 0 & p+l & q+m \\ -(p+l) & 0 & r+n \\ -(q+m) & -(r+n) & 0 \end{bmatrix}$$

A + B is also skew-symmetric and belongs to V.

(ii) $A+B = B+A$

$$A+B = \begin{bmatrix} 0 & p+l & q+m \\ -(p+l) & 0 & r+n \\ -(q+m) & -(r+n) & 0 \end{bmatrix} = \begin{bmatrix} 0 & l+p & m+q \\ -(l+p) & 0 & n+r \\ -(m+q) & -(n+r) & 0 \end{bmatrix}$$

$$= B + A$$

Hence, $A+B = B+A$.

(iii) $(A+B)+C = A+(B+C)$

$$(A+B)+C = \begin{bmatrix} 0 & p+l & q+m \\ -(p+l) & 0 & r+n \\ -(q+m) & -(r+n) & 0 \end{bmatrix} + \begin{bmatrix} 0 & x & y \\ -x & 0 & z \\ -y & -z & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & p+l+x & q+m+y \\ -(p+l+x) & 0 & r+n+z \\ -(q+m+y) & -(r+n+z) & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & p & q \\ -p & 0 & r \\ -q & -r & 0 \end{bmatrix} + \begin{bmatrix} 0 & l+x & m+y \\ -(l+x) & 0 & n+z \\ -(m+y) & -(n+z) & 0 \end{bmatrix}$$

$$= A + (B+C)$$

Name: Archit Agrawal
Student ID: 202052307
Group: A1

(iv) $\vec{0}$ should be in V.

If $p = q = r = 0$ in A,

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

which is a skew-symmetric matrix.

(v) $A + (-A) = 0$

$$-A = \begin{bmatrix} 0 & -p & -q \\ p & 0 & -r \\ q & r & 0 \end{bmatrix}$$

$$\therefore A + (-A) = \begin{bmatrix} 0 & p & q \\ -p & 0 & r \\ -q & -r & 0 \end{bmatrix} + \begin{bmatrix} 0 & -p & -q \\ p & 0 & -r \\ q & r & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

(vi) For each matrix P in V, cP should also be in V where c is scalar

$$\because A = \begin{bmatrix} 0 & p & q \\ -p & 0 & r \\ -q & -r & 0 \end{bmatrix} \Rightarrow cA = \begin{bmatrix} 0 & cp & cq \\ -cp & 0 & cr \\ -cq & -cr & 0 \end{bmatrix}$$

cA is a skew-symmetric matrix.
Hence, $cA \in V$.

Name : Archit Agrawal
Student ID : 202052307
Group : A1

(vii) $c(A+B) = cA + cB$

$$c(A+B) = c\begin{bmatrix} 0 & p+l & q+m \\ -(p+l) & 0 & r+n \\ -(q+m) & -(r+n) & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & cp+cl & cq+cm \\ -cp-cl & 0 & cr+cn \\ -cq-cm & -cr-cn & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & cp & cq \\ -cp & 0 & cr \\ -cq & -cr & 0 \end{bmatrix} + \begin{bmatrix} 0 & cl & cm \\ -cl & 0 & cn \\ -cm & -cn & 0 \end{bmatrix}$$

$$= cA + cB$$

(viii) $(c+d)A = cA + dA$

$$(c+d)A = \begin{bmatrix} 0 & p & q \\ -p & 0 & r \\ -q & -r & 0 \end{bmatrix}(c+d)$$

$$= \begin{bmatrix} 0 & cp+dp & cq+dq \\ -cp-dp & 0 & cr+dr \\ -cq-dq & -cr-dr & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & cp & cq \\ -cp & 0 & cr \\ -cq & -cr & 0 \end{bmatrix} + \begin{bmatrix} 0 & dp & dq \\ -dp & 0 & dr \\ -dq & -dr & 0 \end{bmatrix}$$

$$= cA + dA$$

Name: Archit Agrawal
Student ID: 202052307
Group : A1

(ix) $c(dA) = (cd)(A)$

$$c(dA) = c\begin{bmatrix} 0 & dp & dq \\ -dp & 0 & dr \\ -dq & -dr & 0 \end{bmatrix} = \begin{bmatrix} 0 & (cd)p & (cd)q \\ -(cd)p & 0 & (cd)r \\ -(cd)q & -(cd)r & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & (cd)p & (cd)q \\ -p(cd) & 0 & (cd)r \\ -q(cd) & -r(cd) & 0 \end{bmatrix}$$

$$= (cd)(A)$$

(x) $1A = A$

it is clearly satisfied.

Hence, V is a vector space

Since, any matrix in V can be represented as

$$\begin{bmatrix} 0 & p & q \\ -p & 0 & r \\ -q & -r & 0 \end{bmatrix} = p\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + q\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} + r\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}$$

$\therefore$ basis of $V = \left\{ \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \right\}$

$\therefore$ dimension of V = number of vectors in basis

$\therefore$ $\dim V = 3$

Name : Archit Agrawal
Student ID : 202052307
Group : A1

Similarly a nxn skew-symmetric set will be a vector space.

For a nxn skew-symmetric matrix

dimension = no. of elements − no. of zeroes − similar elements

$$= n^2 - n - \left(\frac{n^2 - n}{2}\right)$$

$$= \frac{n(n-1)}{2}$$

3. Let V be the set of all vectors of the form

$$C = \begin{bmatrix} a - 2b + 5c \\ 2a + 5b - 8c \\ -a - 4b + 7c \\ a + b + 4c \end{bmatrix}$$. Is V a vector space? If yes find its basis.

Let A and B ∈ V.

$$\therefore A = \begin{bmatrix} p - 2q + 5r \\ 2p + 5q - 8r \\ -p - 4q + 7r \\ p + q + 4r \end{bmatrix} \text{ and } B = \begin{bmatrix} l - 2m + 5n \\ 2l + 5m - 8n \\ -l - 4m + 7n \\ l + m + 4n \end{bmatrix}$$

It can be easily verified that

- $A + B = B + A$
- $(A+B) + C = A + (B+C)$
- $A + (-A) = 0$
- $\delta(A+B) = \delta A + \delta B$

Name: Archit Agrawal
Student ID: 202052307
Group: A1

- $(\delta+\gamma)A = \delta A + \gamma A$
- $\delta(\gamma A) = (\delta\gamma)A$
- $cA \in V$
- If $p=q=r=0$, zero vector is in V.
- $1A = A$

Apart from properties mentioned above only one property remains that is necessary for V to be vector space.

that property is $(A+B) \in V$.

$$\therefore A+B = \begin{bmatrix} p-2q+5r \\ 2p+5q-8r \\ -p-4q+7r \\ p+q+4r \end{bmatrix} + \begin{bmatrix} l-2m+5n \\ 2l+5m-8n \\ -l-4m+7n \\ l+m+4n \end{bmatrix}$$

$$= \begin{bmatrix} (p+l)-2(q+m)+5(r+n) \\ 2(p+l)+5(q+m)-8(r+n) \\ -(p+l)-4(q+m)+7(r+n) \\ (p+l)+(q+m)+4(r+n) \end{bmatrix}$$

Hence, $A+B \in V$.

$\therefore$ V is a vector space.

Since any vector in V can be represented as

$$A = \begin{bmatrix} p-2q+5r \\ 2p+5q-8r \\ -p-4q+7r \\ p+q+4r \end{bmatrix}$$

Name: Archit Agrawal
Student ID : 202052307
Group : A1

classmate
Date
Page 10

$$A = p\underbrace{\begin{bmatrix}1\\2\\-1\\1\end{bmatrix}}_{v_1} + q\underbrace{\begin{bmatrix}-2\\5\\-4\\1\end{bmatrix}}_{v_2} + r\underbrace{\begin{bmatrix}5\\-8\\7\\4\end{bmatrix}}_{v_3}$$

Hence any vector in V can be formed using the vectors v_1, v_2 and v_3.

$\therefore$ basis of $V = \{v_1, v_2, v_3\}$

$$= \left\{\begin{bmatrix}1\\2\\-1\\1\end{bmatrix}, \begin{bmatrix}-2\\5\\-4\\1\end{bmatrix}, \begin{bmatrix}5\\-8\\7\\4\end{bmatrix}\right\}$$

4. which of the following sets are linearly independent set in $R[x]$?

(a) $\{1, 2x+1, 3x^2, x\}$
(b) $\{2x+1, 3x+2\}$
(c) $\{1, x+1, x^2+x+1, x^3+x^2+x+1\}$

a) the set will be linearly independent only if

$$c_1(1) + c_2(2x+1) + c_3(3x^2) + c_4(x) = 0 \quad \text{--- (1)}$$

has only trivial solutions.

It is clear that if $c_1 = -1$, $c_4 = -2$, $c_3 = 0$ and $c_2 = 1$, then,

$$c_1(1) + c_2(2x+1) + c_3(3x^2) + c_4(x) = 0$$

Name : Archit Agrawal
Student ID : 202052307
Group : A1

Hence, a non-trivial solution of equation 1 exists.

Hence, the given set is not linearly independent.

b) for the set to be linearly independent

$c_1(2x+1) + c_2(3x+2) = 0$ must have trivial solutions only.

The above equation can be represented as

$$\begin{bmatrix} x & 1 \end{bmatrix}_{1\times 2} \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}_{2\times 2} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = 0$$

The equation will have trivial solutions only if there are 2 pivot columns in $\begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$ (say A)

$$\therefore \quad A = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$$

$$\text{ref of } A = \begin{bmatrix} 2 & 3 \\ 0 & 1/2 \end{bmatrix}$$

Since, there are two pivot entries, only trivial solution exists and hence the vectors in the set are linearly independent.

(c) for the set to be linearly independent

$$c_1(1) + c_2(x+1) + c_3(x^2+x+1) + c_4(x^3+x^2+x+1) = 0 \quad (1)$$

must have only trivial solutions.

Name: Archit Agrawal
Student ID: 202052307
Group: A1

equation 1 can be represented as

$$\begin{bmatrix} 1 & x & x^2 & x^3 \end{bmatrix} \underbrace{\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{A} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = 0$$

equation will have only trivial solution iff A has 4 pivot entries. As A is already in its row echelon form and has four pivot entries, only trivial solution exists.

Hence, the vectors in the set are linearly independent.

5. Let $T: \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation defined by

$$T\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 + 3x_2 - 2x_3 \\ 2x_1 + 3x_2 \\ x_2 - x_3 \end{bmatrix}$$

Determine whether T is an isomorphism and if so find the formula for inverse ~~of~~ linear transformation T^{-1}.

The transformation T will be isomorphic if it is invertible. T will be invertible if the standard matrix representation of T, say A, is bijective.

$$\therefore \quad A_{3\times 3} = \begin{bmatrix} T(e_1) & T(e_2) & T(e_3) \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 3 & -2 \\ 2 & 3 & 0 \\ 0 & 1 & -1 \end{bmatrix}$$

Name: Archit Agrawal
Student ID: 202052307
Group: A1

reducing A to its Row echelon form

$R_2 \to R_2 - 2R_1$

$$\sim \begin{bmatrix} 1 & 3 & -2 \\ 0 & -3 & 4 \\ 0 & 1 & -1 \end{bmatrix}$$

$R_3 \to R_3 + \frac{1}{3} R_2$

$$\text{REF of } A = \begin{bmatrix} 1 & 3 & -2 \\ 0 & -3 & 4 \\ 0 & 0 & 1/3 \end{bmatrix}$$

Since, REF of A has three pivot entries, hence T is both one-one and onto, and hence T is isomorphic.

Let us suppose $T \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$

$$\therefore \begin{aligned} y_1 &= x_1 + 3x_2 - 2x_3 \\ y_2 &= 2x_1 + 3x_2 \\ y_3 &= x_2 - x_3 \end{aligned}$$

To find T^{-1}, we have to convert x_1, x_2 and x_3 in terms of y_1, y_2 and y_3 and then replace x by y.

the augmented matrix system for the above equations

$$\left[\begin{array}{ccc|c} 1 & 3 & -2 & y_1 \\ 2 & 3 & 0 & y_2 \\ 0 & 1 & -1 & y_3 \end{array}\right]$$

Name : Archit Agrawal
Student ID : 202052307
Group : A1

$R_2 \rightarrow R_2 - 2R_1$

$$\left[\begin{array}{ccc|c} 1 & 3 & -2 & y_1 \\ 0 & -3 & 4 & y_2 - 2y_1 \\ 0 & 1 & -1 & y_3 \end{array}\right]$$

$R_3 \rightarrow R_3 + \frac{1}{3} R_2$

$$\left[\begin{array}{ccc|c} 1 & 3 & -2 & y_1 \\ 0 & -3 & 4 & y_2 - 2y_1 \\ 0 & 0 & 1/3 & y_3 + \frac{y_2}{3} - \frac{2y_1}{3} \end{array}\right]$$

$R_3 \rightarrow 3R_3$

$$\left[\begin{array}{ccc|c} 1 & 3 & -2 & y_1 \\ 0 & -3 & 4 & y_2 - 2y_1 \\ 0 & 0 & 1 & 3y_3 + y_2 - 2y_1 \end{array}\right]$$

$R_2 \rightarrow R_2 - 4R_3$

$$\left[\begin{array}{ccc|c} 1 & 3 & -2 & y_1 \\ 0 & -3 & 0 & y_2 - 2y_1 - 12y_3 - 4y_2 + 8y_1 \\ 0 & 0 & 1 & 3y_3 + y_2 - 2y_1 \end{array}\right]$$

$R_1 \rightarrow R_1 + 2R_3$

$$\left[\begin{array}{ccc|c} 1 & 3 & 0 & y_1 + 6y_3 + 2y_2 - 4y_1 \\ 0 & -3 & 0 & 6y_1 - 3y_2 - 12y_3 \\ 0 & 0 & 1 & 3y_3 + y_2 - 2y_1 \end{array}\right]$$

$R_1 \rightarrow R_1 + R_2$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 3y_1 - y_2 - 6y_3 \\ 0 & -3 & 0 & 6y_1 - 3y_2 - 12y_3 \\ 0 & 0 & 1 & 3y_3 + y_2 - 2y_1 \end{array}\right]$$

Name: Archit Agrawal
Student ID: 202052307
Group: A1

$R_2 \to -\frac{1}{3} R_2$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 3y_1 - y_2 - 6y_3 \\ 0 & 1 & 0 & -2y_1 + y_2 + 4y_3 \\ 0 & 0 & 1 & -2y_1 + y_2 + 3y_3 \end{array}\right]$$

$\therefore$
$$x_1 = 3y_1 - y_2 - 6y_3$$
$$x_2 = -2y_1 + y_2 + 4y_3$$
$$x_3 = -2y_1 + y_2 + 3y_3$$

Replace x_1, x_2, x_3 by y_1, y_2, y_3.

$\therefore$
$$y_1 = 3x_1 - x_2 - 6x_3$$
$$y_2 = -2x_1 + x_2 + 4x_3$$
$$y_3 = -2x_1 + x_2 + 3x_3$$

$$\therefore \quad T^{-1}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$$T^{-1}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3x_1 - x_2 - 6x_3 \\ -2x_1 + x_2 + 4x_3 \\ -2x_1 + x_2 + 3x_3 \end{bmatrix}$$

6. Let S be a finite minimal set spanning a vector space V. That is, S has the property that if a vector is removed from S, then the new set will no longer span V. Prove that S must be a basis of V.

Suppose the set S with the described property be linearly dependent. This means that there at least one vector $x \in S$ which is a

Name: Archit Agrawal
Student ID: 202052307
Group: A1

linear combination of other vectors in S. If we remove the vectors x, then also the remaining vectors $S - \{x\}$ will span same space as that of S.

This contradicts the property of S given in question. that says if you remove a vector from S it will no longer span V.
Hence, S must be a set of linearly independent vectors that spans V. Hence, S is a basis of vector space V because a set of linearly independent vectors that span the vector space is the basis of vector space.

7. Show that if A is $n \times n$ and B is $n \times p$, then rank $(AB) \le$ rank (A).

rank of a matrix is dimension of its column space.

$$\therefore \quad \text{rank}(AB) = \dim(\text{Col } AB)$$

$$\text{and,} \quad \text{rank}(A) = \dim(\text{Col } A)$$

If a vector space V is ~~subspace~~ subset of another vector space W, then

$$\dim(V) \le \dim(W) \quad \text{--- (1)}$$

Now, consider any vector $y \in \text{Col } AB$. Then there exists a vector $x \in R^p$ such that

$$y = (AB)x$$

By the definition of column space.

Name: Archit Agrawal
Student ID: 202052307
Group : A1

classmate
Date
Page 17

Let $z = Bx \in R^n$.

$\therefore$ we have

$$y = (AB)x = A(Bx)$$
$$= Az$$

hence, any vector $y \in \text{Col } AB$ also belongs to Col A.

$\therefore$ Col AB is a subset of Col A.

$$\therefore \quad \dim(\text{Col } AB) \le \dim(\text{Col } A) \qquad \{\text{from } 1\}$$

$$\text{rank}(AB) \le \text{rank } A.$$

Hence, proved.

Q. Show that if A is $n \times n$ matrix and B is $n \times p$ matrix, then $\text{rank}(AB) \le \text{rank } B$.

$$\because \quad \text{Rank}(D) = \text{Rank}(D^T) \quad \text{---} (1)$$

$$\therefore \quad \text{Rank}(AB) = \text{Rank}((AB)^T)$$
$$= \text{Rank}(B^T A^T)$$

Also from previous question,
$$\text{Rank}(CD) \le \text{Rank}(C)$$

$$\therefore \quad \text{Rank}(B^T A^T) \le \text{Rank}(B^T)$$

$$\text{Rank}\{(B^T A^T)^T\} \le \text{Rank}((B^T)^T) \qquad \{\text{from } 1\}$$

$$\text{Rank}(AB) \le \text{Rank}(B)$$

Name: Archit Agrawal
Student ID: 202052307
Group : A1

Hence, proved that $\text{rank}(AB) \leq \text{rank}(B)$.

9. Let V be a vector space over $\mathbb{R}$ of dimension n. Let V' be the set of all linear transformation from V to V. Show that V' is also a vector space over $\mathbb{R}$ and find its basis.

Let $T, S \in V'$. Then, S and T are linear transformations from V to V,

then $T+S : V \to V$
such that $(T+S)(u) = T(u) + S(u)$

Let us prove that $T+S$ is a linear map :

$$\therefore \quad (T+S)(u+v) = T(u+v) + S(u+v)$$
$$= T(u) + T(v) + S(u) + S(v)$$
$$= (T+S)u + (T+S)v$$

and, $$(T+S)(cu) = T(cu) + S(cu)$$
$$= cT(u) + cS(u) \quad (\because T \,\&\, S \text{ are LT})$$
$$= c(T+S)(u)$$

Hence, $T+S$ is a linear transformation, hence it belongs to vector space V'.

Now, we have to prove V' is a vector space.

(i) for $T, S \in V'$, $T+S \in V'$ (proved above)
(ii) for $T \in V'$, cT also belongs to V'
(iii) $T = 0$ is also a linear transformation.

Name: Archit Agrawal
Student ID: 202052307
Group : A1

(iv) $(c+d)T = cT + dT \quad \forall T \in V'$

(v) $c(dT) = (cd)T \quad \forall T \in V'$

(vi) for every T in V', there exists $-T$ such that
$T + (-T) = 0$

(vii) For every $T, U, V \in V'$
$T + (U+V) = (T+U) + V$

Hence, the ~~vector~~ space V' satisfies all properties of a vector ~~sub~~space, hence it is a vector space.

Now, suppose $\{v_1, v_2, \dots v_n\}$ is a basis for V.
for $i = 1 \dots n$ define f_i belongs to V' by,

$$f_i(a_1v_1 + a_2v_2 + \dots + a_nv_n) = a_i$$

for constants a_i.

Note that f is uniquely varied for each v_j in V.
Also, $f_i(v_j) = 1$ for $i=j$ and 0 otherwise.

Consider the linear combination $a_1f_1 + \dots + a_nf_n = 0$.

Taking the value at every v_j, we get $a_j = 0$ showing that $\{f_1, f_2, \dots f_n\}$ is linearly independent.

Let g be a vector in V'. Assume g cannot be written as $a_1f_1 + \dots + a_nf_n$. Then the value of g is zero. This shows that every member of V^* is either 0 or linear combination of $\{f_1, f_2, \dots f_n\}$. Hence, $\{f_1, f_2, \dots, f_n\}$ spans V'.

Name: Archit Agrawal
Student ID: 202052307
Group : A1

$\therefore \{f_1, f_2, \ldots, f_n\}$ is a basis of V'.

10. Suppose that U and W are finite dimensional subspaces of a vector space V. Then show that $U+W$ is also finite dimensional subspace and $\dim(U+W) = \dim(U) + \dim(W) - \dim(U \cap W)$ where $U+W = \{x \mid x = y+z \text{ for some } y \in U, z \in W\}$

(Note: $U+W$ is smallest subspace containing $U \cup W$)

Let $v \in U+W$ and $v' \in U+W$. Suppose $v = u+w$ and $v' = u'+w'$ where $u, u' \in U$ and $w, w' \in W$.

Then $v+v' = (u+u') + (w+w')$. This is in $U+W$. Since U and W are closed under addition, thus $u+u' \in U$ and $w+w' \in W$.

Theorem:- If U, W are two subspaces of a vector space V, then

$$\dim(U+W) = \dim U + \dim W - \dim(U \cap W)$$

• Proof: Let $\dim(U \cap W) = k$

And the set $S = \{\lambda_1, \lambda_2 \ldots \lambda_k\}$ be a basis of $U \cap W$.

Then $S \subseteq U$ and $S \subseteq W$.

Since, S is linearly independent and $S \subseteq U$
Therefore, S can be extended to form a basis of U.

Name : Archit Agrawal
Student ID : 202052307
Group : A1

Let $\{\lambda_1, \lambda_2, \dots \lambda_k, \alpha_1, \alpha_2, \dots \alpha_m\}$ be a basis of U.

$\therefore$ $\dim U = k+m$

Similarly let $\{\lambda_1, \lambda_2, \dots \lambda_k, \beta_1, \beta_2, \dots, \beta_n\}$ be a basis of W.

$\therefore$ $\dim W = k+m+n$

Now, to prove that $\dim(U+W) = k+m+n$, we claim set S_1 to be

$$S_1 = \{\lambda_1, \lambda_2, \dots \lambda_k, \alpha_1, \alpha_2, \dots \alpha_m, \beta_1, \beta_2, \dots, \beta_n\}$$

is a basis of $U+W$.

First we have to show S_1 is linearly independent.

Let $c_1\lambda_1 + c_2\lambda_2 + \dots c_k\lambda_k + a_1\alpha_1 + \dots + a_m\alpha_m + b_1\beta_1 + \dots + b_n\beta_n = 0$ —(1)

$\therefore$ $-(c_1\lambda_1 + \dots c_k\lambda_k + a_1\alpha_1 + \dots + a_m\alpha_m) \in U$

and $b_1\beta_1 + b_2\beta_2 + \dots + b_n\beta_n \in W$

$\Rightarrow$ $b_1\beta_1 + \dots + b_n\beta_n = -(c_1\lambda_1 + \dots + c_k\lambda_k + a_1\alpha_1 + \dots + a_m\alpha_m)$

Hence, S_1 is linearly independent.

$\Rightarrow$ $b_1\beta_1 + b_2\beta_2 + \dots + b_n\beta_n \in U$
and
$b_1\beta_1 + b_2\beta_2 + \dots + b_n\beta_n \in W$

$\Rightarrow$ $b_1\beta_1 + b_2\beta_2 + \dots + b_n\beta_n \in U \cap W$

Name: Archit Agrawal
Student ID: 202052307
Group : A1

$\therefore$ It can be expressed as linear combination of basis of $U \cap W$.

Thus -:

$b_1\beta_1 + \cdots + b_n\beta_n = d_1\lambda_1 + d_2\lambda_2 + \cdots + d_k\lambda_k$

$\Rightarrow b_1\beta_1 + \cdots + b_n\beta_n - d_1\lambda_1 - d_2\lambda_2 \cdots - d_k\lambda_k = 0$

But $\beta_1, \beta_2 \cdots \beta_k, \lambda_1, \lambda_2 \cdots \lambda_k$ are linearly independent

$\Rightarrow b_1, b_2, b_3 \cdots, b_n = 0$

from (1) we get

$c_1\lambda_1 + c_2\lambda_2 + \cdots + c_k\lambda_k + a_1\alpha_1 + \cdots + a_m\alpha_m = 0$

$\Rightarrow c_1 = 0, c_2 = 0, \cdots c_k = 0$
$a_1 = 0, a_2 = 0 \cdots a_m = 0$ [$\because \lambda_1, \lambda_2, \cdots \lambda_k, a_1, a_2 \cdots a_m$ are linearly independent]

$\therefore S_1$ is a linearly independent set.

Now, prove $L(S_1) = U + W$

$\because U + W$ is subspace of V, and each element of S_1 belongs to $U + W$.

$\therefore L(S_1) \subseteq U + W$ ——— (C)

Again, Let α be any element of $U + W$.
$\alpha \in U + W$.

i.e. α = some element of U + some element of W

Name : Archit Agrawal
Student ID : 202052307
Group : A1

$\alpha =$ a linear combination of elements of basis U
$+$ a linear combination of elements of basis of W

$\alpha =$ a linear combination of S_1.

$\therefore \quad \alpha \in L(S_1) \Rightarrow U + W \subseteq L(S_1)$

Hence, $L(S_1) = U + W$

$\therefore$ S_1 is a basis of $U+W$

$\therefore$ $\dim(U+W) = k + m + n$