Name : Archit Agrawal
Student ID : 202052307
Group : A1

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MA101 : Linear Algebra and Matrices
Tutorial 7

1. Find $U + W$ for the following subspaces

i) $U = \{[x \ y]^T \in R^2 \mid 2x + y = 0\}$
$W = \{[x \ y]^T \in R^2 \mid 4x + 2y = 0\}$

Since, U is $[x \ y]^T$ such that

$$2x + y = 0$$
$$\therefore \quad y = -2x$$
Let $x = \alpha$, $\alpha \in R$
$$\therefore \quad U = \begin{bmatrix} \alpha \\ -2\alpha \end{bmatrix} \quad \text{—(I)}$$

And, W is $[x \ y]^T$ such that

$$4x + 2y = 0$$
$$\therefore \quad 2y = -4x$$
$$y = -2x$$
Let $x = \alpha$, $\alpha \in R$
$$\therefore \quad W = \begin{bmatrix} \alpha \\ -2\alpha \end{bmatrix} \quad \text{—(II)}$$

from (I) and (II),

$$U = W$$

Since, subspaces U and W coincide. Hence,

$$U + W = U \quad \text{or} \quad U + W = W$$

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(ii) $U = \{[x \;\; y]^T \in R^2 \mid 2x + y = 0\}$,
$W = \{[x \;\; y]^T \in R^2 \mid 4x + y = 0\}$

Since, U is $[x \;\; y]^T$ such that
$2x + y = 0$
$y = -2x$
Let $x = 1$
$\therefore \; y = -2$

and, W is $[x \;\; y]^T$ such that
$4x + y = 0$
$y = -4x$
Let $x = 1$
$\therefore \; y = -4$

$\because$ $U + W$ is a subspace
$\therefore$ $U + W = \text{span}\{U, W\}$ — (i)

Also, since U and W are linearly independent as a vector in U i.e. $\begin{bmatrix} 1 \\ -2 \end{bmatrix}$ is not a scalar multiple of a vector in W i.e. $[1 \;\; -4]^T$.

Hence, $\text{span}\{U, W\} = R^2$ — (ii)

from (i) & (ii),

$U + W = \text{span}\{U, W\} = R^2$

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iii) U = line passing through $[1, 1, 1]^T$ and origin.

W = line passing through $[1\ 0\ 0]^T$ and origin.

Since, U and W are two ~~non-parallel~~ intersecting lines in R^3. Therefore, U+W will be a plane in R^3, which will pass through $[1, 0\ 0]^T$, $[1\ 1\ 1]^T$ and origin.

Let the plane be $ax + by + cz = d$

As it passes through origin,

$$a(0) + b(0) + c(0) = d$$
$$\therefore d = 0$$

Also, it passes through $[1\ 0\ 0]^T$,

$$\therefore a(1) + b(0) + c(0) = 0$$
$$\therefore a = 0$$

and it passes through $[1\ 1\ 1]^T$,

$$0(1) + b(1) + c(1) = 0$$
$$\therefore b = -c$$

$\therefore$ the equation of plane,

$$by - bz = 0$$
$$y - z = 0$$

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Hence, $U+W$ is a plane in $\mathbb{R}^3$ $\{[x\ y\ z]^T\}$ given by $y - z = 0$

2. Extend the following vectors to a basis of $\mathbb{R}^n$ for suitable n.

(i) $v_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

Since v_1 is a vector in $\mathbb{R}^3$, we have to find two vectors ~~such that they span~~ v_2 and v_3 such that span$\{v_1, v_2, v_3\} = \mathbb{R}^3$.

Let $v_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ and $v_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

For v_1, v_2 and v_3 to span $\mathbb{R}^3$, the equation $Ax = 0$ must have only trivial solutions, where $A = [v_1\ v_2\ v_3]$

$Ax = 0$ has only trivial solution if it has three pivot columns.

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1$ and $R_3 \to R_3 - 3R_1$

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$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Since, there are three pivot columns in A, $Ax = 0$ has only trivial solutions and vectors v_1, v_2, v_3 span R^3.

$\therefore$ basis of R^n = $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

(ii) $v_1 = \begin{bmatrix} -9 \\ -7 \\ 8 \\ -5 \\ 7 \end{bmatrix}, v_2 = \begin{bmatrix} 9 \\ 4 \\ 1 \\ 6 \\ -7 \end{bmatrix}, v_3 = \begin{bmatrix} 6 \\ 7 \\ -8 \\ 5 \\ -7 \end{bmatrix}$

Since v_1, v_2 and v_3 are vectors in R^5. We have to find basis for R^5. Hence, we need two vectors v_4 and v_5 in R^5 such that span$\{v_1, v_2, v_3, v_4, v_5\}$ is R^5.

Let $v_4 = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \end{bmatrix}^T$ and $v_5 = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \end{bmatrix}^T$

Let A be 5×5 matrix such that $A = [v_1 \ v_2 \ v_3 \ v_4 \ v_5]$

$$\therefore A = \begin{bmatrix} -9 & 9 & 6 & 0 & 0 \\ -7 & 4 & 7 & 0 & 0 \\ 8 & 1 & -8 & 0 & 0 \\ -5 & 6 & 5 & 1 & 0 \\ 7 & -7 & -7 & 0 & 1 \end{bmatrix}$$

Name : Archit Agrawal
Student ID : 2020X2307
Group : A1

$R_1 \rightarrow -\frac{1}{9} R_1$

$$\begin{bmatrix} 1 & -1 & -2/3 & 0 & 0 \\ -7 & 4 & 7 & 0 & 0 \\ 8 & 1 & -8 & 0 & 0 \\ -5 & 6 & +5 & 1 & 0 \\ 7 & -7 & -7 & 0 & 1 \end{bmatrix}$$

$R_2 \rightarrow R_2 + 7 R_1$
$R_3 \rightarrow R_3 - 8 R_1$
$R_4 \rightarrow R_4 + 5 R_1$
$R_5 \rightarrow R_5 - 7 R_1$

$$\begin{bmatrix} 1 & -1 & -2/3 & 0 & 0 \\ 0 & -3 & 7/3 & 0 & 0 \\ 0 & 9 & -8/3 & 0 & 0 \\ 0 & 1 & 5/3 & 1 & 0 \\ 0 & 0 & -7/3 & 0 & 1 \end{bmatrix}$$

$R_2 \rightarrow -\frac{1}{3} R_2$

$$\begin{bmatrix} 1 & -1 & -2/3 & 0 & 0 \\ 0 & 1 & -7/9 & 0 & 0 \\ 0 & 9 & -8/3 & 0 & 0 \\ 0 & 1 & 5/3 & 1 & 0 \\ 0 & 0 & -7/3 & 0 & 1 \end{bmatrix}$$

$R_3 \rightarrow R_3 - 9 R_2$
$R_4 \rightarrow R_4 - R_2$

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$$\begin{bmatrix} 1 & -1 & -2/3 & 0 & 0 \\ 0 & 1 & -7/9 & 0 & 0 \\ 0 & 0 & 13/3 & 0 & 0 \\ 0 & 0 & 22/9 & 1 & 0 \\ 0 & 0 & -7/3 & 0 & 1 \end{bmatrix}$$

$$R_3 \rightarrow \frac{3}{13} R_3$$

$$\begin{bmatrix} 1 & -1 & -2/3 & 0 & 0 \\ 0 & 1 & -7/9 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 22/9 & 1 & 0 \\ 0 & 0 & -7/3 & 0 & 1 \end{bmatrix}$$

$$R_4 \rightarrow R_4 - \frac{22}{9} R_1$$

$$R_5 \rightarrow R_5 + \frac{7}{3} R_1$$

$$\begin{bmatrix} 1 & -1 & -2/3 & 0 & 0 \\ 0 & 1 & -7/9 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Since, there are five pivot column in A.
$Ax = 0$ has only trivial solutions.
Hence, v_1, v_2, v_3, v_4 and v_5 are linearly independent and span $\mathbb{R}^5$.

Hence, basis is $\{v_1, v_2, v_3, v_4, v_5\}$.

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3. Let $B = \{1, \cos t, \cos(2t), \dots, \cos(6t)\}$, $C = \{1, \cos t, \cos^2 t, \dots, \cos^6(t)\}$ and $H = \text{span } C$ be 7 dimensional subspace of real valued functions of $\mathbb{R}$. Show that both B, C are basis of H and evaluate

$$\int \{5\cos^3 t - 6\cos^4 t + 5\cos^5 t - 12\cos^6 t\}\, dt$$

Since, $\cos 2t = 2\cos^2 t - 1$

and $\cos 3t = 4\cos^3 t - 3\cos t$

Similarly $\cos 4t = 1 - 8\cos^2 t + 8\cos^4 t$

$\cos 5t = 5\cos t - 20\cos^3 t + 16\cos^5 t$

$\cos(6t) = 1 + 18\cos^2 t - 48\cos^4 t + 32\cos^6 t$

As all seven vectors in C are linearly independent because they have different powers.

$\therefore$ Vectors in B can be represented in terms of C as

$$[B]_C = \begin{bmatrix} 1 & 0 & -1 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & -3 & 0 & 5 & 0 \\ 0 & 0 & 2 & 0 & -8 & 0 & 18 \\ 0 & 0 & 0 & 4 & 0 & -20 & 0 \\ 0 & 0 & 0 & 0 & 8 & 0 & -48 \\ 0 & 0 & 0 & 0 & 0 & 16 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 32 \end{bmatrix}_{7\times 7}$$

There are 7 pivot entries in above matrix.

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Since, in C, powers of entries are different, then there must be 7 pivot entries in the matrix formed by vectors of C. As span of C is a 7-dimensional subspace and, ~~consists~~ column matrix of C consists 7 pivot entries, C is a basis for 7 dimension subspace H.

From $[B]_C$, we can see that there are 7 pivot entries in it, so columns of $[B]_C$ are linearly independent. Hence $[B]_C$ is a basis of H.

$\Rightarrow$ both B and C are basis of subspace H.

$$I = \int \left\{ 5\cos^3 t - 6\cos^4 t + 5\cos^5 t - 12\cos^6 t \right\} dt$$

$$= \int 5\cos^3 t \, dt - \int 6\cos^4 t \, dt + \int 5\cos^5 t \, dt - \int 12\cos^6 t \, dt$$

$$I_1 = \int 5\cos^3 t \, dt = 5 \int \frac{\cos(3t) + 3\cos t}{4} \, dt$$

$$= \frac{5}{4}\left[\frac{\sin(3t)}{3} + 3\sin t \right] + C_1$$

$$I_2 = \int 6\cos^4 t \, dt = 6 \int \frac{(\cos(4t) + 8\cos^2 t - 1)}{8} \, dt$$

$$= 6 \int \left\{ \frac{\cos(4t) - 1 + \overset{4}{\cancel{8}}\frac{(\cos(2t)+1)}{\cancel{2}}}{8} \right\} dt$$

$$= 6 \left[\frac{\sin(4t)}{4} + 3t + 2\sin 2t \right] + C_2$$

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$$I_3 = 5\int \cos^5 t \, dt = \frac{5}{16}\int \left\{ \cos(5t) + 20\cos^3 t - 5\cos t \right\} dt$$

$$I_3 = \frac{5}{16}\left[\frac{\sin(5t)}{5} + 20 \times \frac{1}{4}\left\{ \frac{\sin(3t)}{3} + 3\sin t \right\} - 5\sin t \right]$$

$$= \frac{5}{16}\left[\frac{\sin(5t)}{5} + \frac{5}{3}\sin(3t) + 10\sin t \right] + C_3$$

$$I_4 = 12\int \cos^6 t \, dt = \frac{12}{32}\int \left\{ \cos(6t) + 48\cos^4 t - 18\cos^2 t + 1 \right\} dt$$

$$I_4 = \frac{12}{32}\left[\frac{\sin(6t)}{6} + 48\left(\frac{\sin(4t)}{4} + 3t + 2\sin 2t \right) + t - \frac{9\sin(2t)}{2} - 9t \right]$$

$$I_4 = \frac{3}{8}\left[\frac{\sin(6t)}{6} + 12\sin(4t) + \frac{183}{2}\sin(2t) - 8t \right]$$

$$\therefore \quad I_0 = I_1 + I_2 + I_3 + I_4$$

After simplifying

$$I = (1 - 2\cos t)\sin^5 t + (8\cos t - 5)\sin^3 t + (10 - 12\cos t)\sin t - 6t + C$$

4. Maximize : $P = 7X + 5Y$

Subject to : $4X + 3Y \leq 240$

$2X + Y \leq 100$

$X \geq 0, \quad Y \geq 0$

Solve using graphical and simplex method.

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Graphical Method

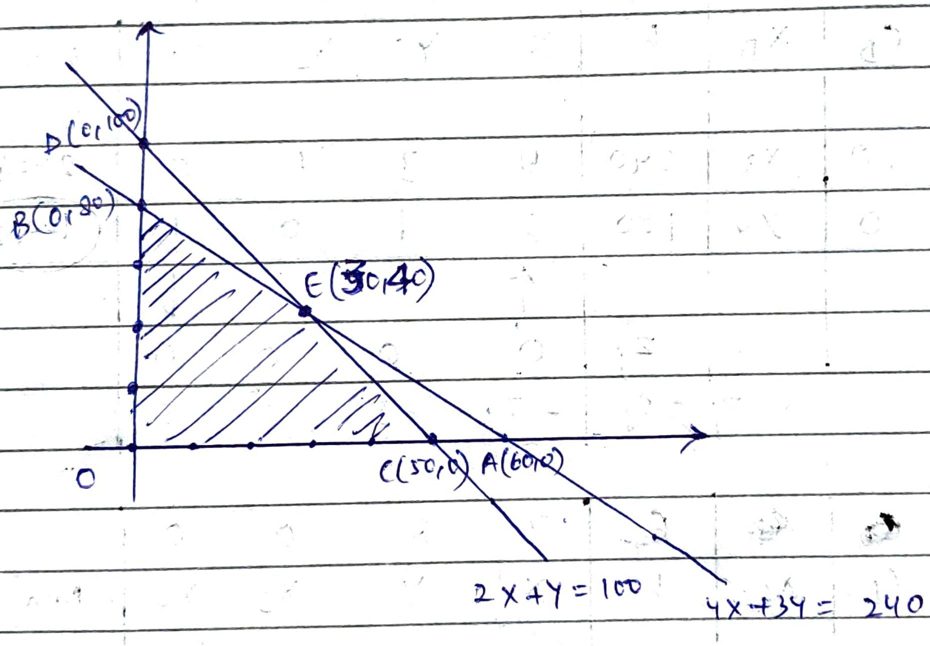
~~Let~~ $P = 7X + 5Y$

$4X + 3Y \leq 240$

	point A	point B
X	60	0
Y	0	80

$2X + Y \leq 100$

	point C	point D
X	50	0
Y	0	100



Solving $2X + Y = 100$
and $4X + 3Y = 240$

gives $X = 30$ and $Y = 40$

B (0,80) $\Rightarrow$ $P(B) = \cancel{560}\ 400$

O (0,0) $\Rightarrow$ $P(O) = 0$

C (50,0) $\Rightarrow$ $P(C) = 350$

E (30,40) $\Rightarrow$ $P(E) = 410$

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Hence optimal solution is $X = 30$, $Y = 40$
and $P_{max} = 410$

Simplex Method

constraints,

$$4X + 3Y + X_3 = 240$$
$$2X + Y + X_4 = 100$$

X_3 and X_4 are slack variables

		C_j	7	5	0	0	
C_B	X_B	b	X	Y	X_3	X_4	Min Ratio
0	X_3	240	4	3	1	0	240/4
0	X_4	100	2	1	0	1	100/2
			↑			↓	
		Z_j	0	0	0	0	
		$Z_j - C_j$	-7	-5	0	0	

		C_j	7	5	0	0	
C_B	X_B	b	X	Y	X_3	X_4	Min Ratio
0	X_3	240	4	3	1	0	
0	X_4	100	2	1	0	1	
						↓	
						0	
						30	

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Hence, optimal solution is $X = 30$ and $Y = 40$
and $P_{max} = 410$

Simplex Method

Constraints: $4X + 3Y + X_3 = 240$
$2X + Y + X_4 = 100$

X_3 and X_4 are slack variables

		C_j	7	5	0	0	
C_B	X_B	b	X	Y	X_3	X_4	Min Ratio
0	X_3	240	4	3	1	0	240/4
0	X_4	100	2	1	0	1	100/2
			↑			↓	
		Z_j	0	0	0	0	
		$Z_j - C_j$	-7	-5	0	0	

		C_j	7	5	0	0	
C_B	X_B	b	X	Y	X_3	X_4	Min Ratio
0	X_3	40	0	1	1	-2	40/1
7	X	50	1	1/2	0	1/2	50/(1/2)
		Z_j	7	7/2	0	7/2	
		$Z_j - C_j$	0	-3/2	0	7/2	

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C_B	X_B	c_j / b	X	Y	X_3	X_4	Min Ratio
5	Y	40	0	1	1	-2	
7	X	30	1	0	$-1/2$	$3/2$	
		z_j	7	5	$3/2$	$1/2$	
		$z_j - c_j$	0	0	$3/2$	$1/2$	

Now, all $z_j - c_j \geq 0$

Therefore, this is the optimal solution.

$X = 30$ and $Y = 40$

$\therefore (\max)_p = 410$

6. Show that the following problem is feasible but unbounded, so it has no optimal solution.

Maximize $X + Y$
subject to $X \geq 0$, $Y \geq 0$, $-3X + 2Y \leq -1$
and $X - Y \leq 2$

Maximize $Z = X + Y$

Constraints $X \geq 0$, $Y \geq 0$

$-3X + 2Y \leq -1$

	point A	point B
X	0	$1/3$
Y	$-1/2$	0

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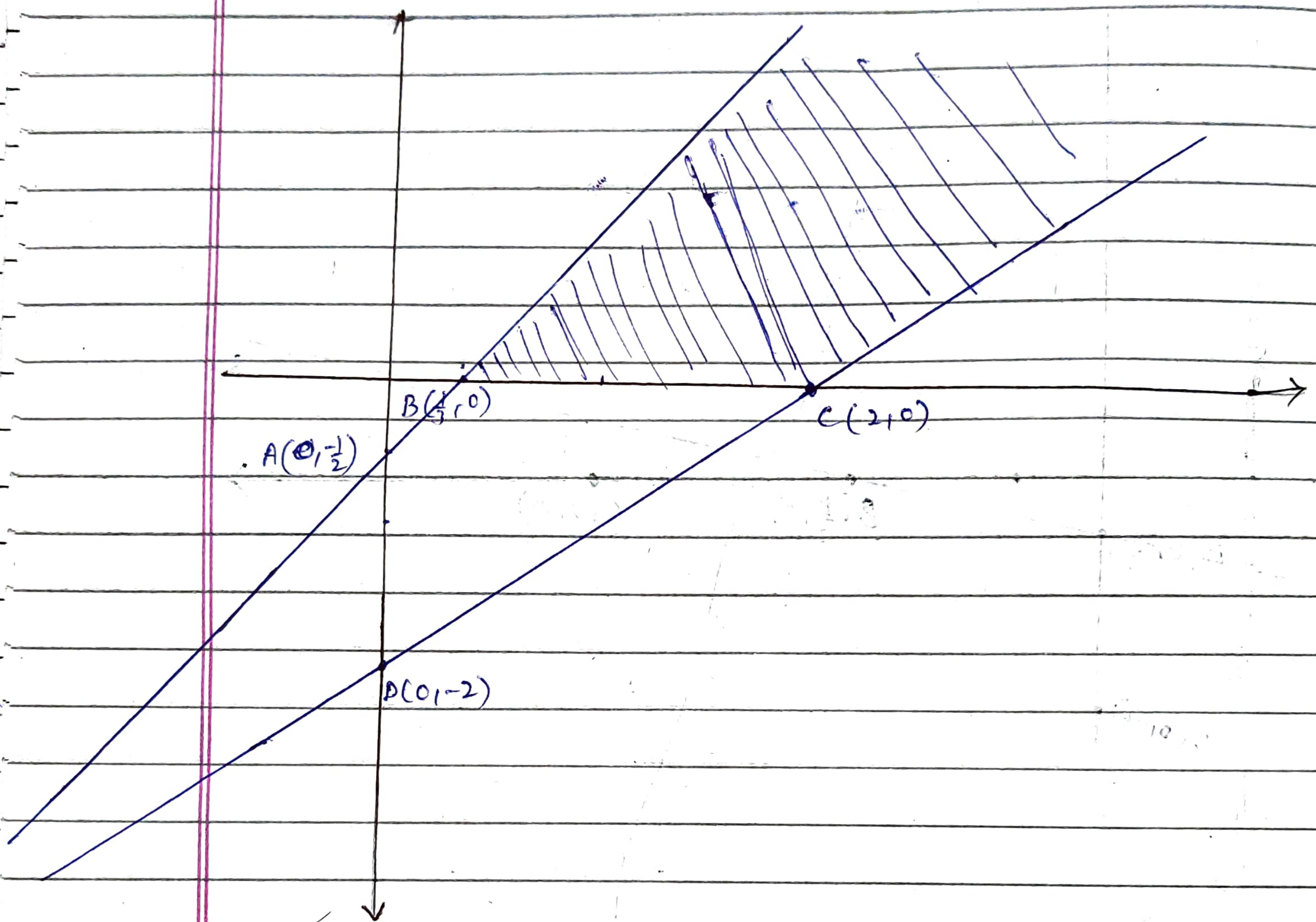
$X - Y \leq 2$

X	2	0
Y	0	-2
	point C	point D



Hence, we obtained a feasible region but the region is unbounded. Hence, the above problem has no optimal solution.

7. Minimize $Z = 4X_1 + X_2$ subject to

$$3X_1 + X_2 = 3$$
$$4X_1 + 3X_2 = 6$$
$$X_1 + 2X_2 \leq 4$$
$$X_1, X_2 \geq 0$$

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$3X_1 + X_2 + A_1 = 3$

$4X_1 + 3X_2 + A_2 = 6$

$X_1, X_2 \geq 0$

$X_1 + 2X_2 + X_3 = 4$

A_1, A_2 are artificial variables

$X_3 \rightarrow$ slack variable

Maximize $Z^* = -4X_1 - X_2 - MA_1 - MA_2$

(I)

		c_j	4	1	0	M	M	
c_B	X_B	b	X_1	X_2	X_3	A_1	A_2	Min Ratio
M	A_1	3	3	1	0	1	0	3/3 ✓
M	A_2	6	4	3	0	0	1	6/4
0	X_3	4	1	2	1	0	0	4/1
			↑			↓		
		$z_j - c_j$	7M−4	4M−1	0	0	0	

(II)

		c_j	4	1	0	M	M	
c_B	X_B	b	X_1	X_2	X_3	A_1	A_2	Min Ratio
4	X_1	1	1	1/3	0		0	3
M	A_2	2	0	1.67	0		1	1.2 ✓
0	X_3	3	0	1.67	1		0	1.8
				↑			↓	
		$z_j - c_j$	0	1.67M + 0.33	0		0	

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C_B	X_B	b	X_1	X_2	X_3	A_1	A_2	Min Ratio
4	X_1	0.6	1	0	0			
1	X_2	1.2	0	1	0			
0	X_3	1	0	0	1			
		$Z_j - C_j$	0	0	0			

$X_1 = 0.6$

$X_2 = 1.2$

$$Z^*_{max} = -4X_1 - X_2$$
$$= -[4(0.6) + (1.2)]$$
$$= -3.6$$

$$Z_{min} = -Z^*_{max}$$

$$\boxed{Z_{min} = 3.6}$$

8. Maximize $Z = 2X_1 + 2X_2 + 4X_3$ subject to

$$2X_1 + X_2 + X_3 \leq 2$$
$$3X_1 + 4X_2 + 2X_3 \geq 8$$
$$X_1, X_2, X_3 \geq 0$$

Constraints

$$2X_1 + X_2 + X_3 + X_4 = 2$$
$$3X_1 + 4X_2 + 2X_3 - X_5 + A_1 = 8$$

$A_1 \rightarrow$ artificial variable

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$x_4 \to$ slack variable
$x_5 \to$ surplus variable

Maximize $z^* = 2x_1 + 2x_2 + 4x_3 - MA_1$

(I)

		c_j	2	2	$\boxed{4}$	0	0	M	
C_B	X_B	b	x_1	x_2	x_3	x_4	x_5	A_1	Min Ratio
0	x_4	2	2	1	1	1	0	0	2 ✓
-M	A_1	8	3	4	2	0	-1	1	2
		$z_j - c_j$	$(-3M-2)$	$(-4M-2)$	$(-2M-4)$	0	M	0	

(II)

C_B	X_B	b	x_1	x_2	x_3	x_4	x_5	A_1	Min Ratio
2	x_2	2	2	1	1	1	0	0	
-M	A_1	0	-5	0	-2	-4	-1	1	
		$z_j - c_j$	5M+2	0	2M-2	4M+2	M	0	

Now all $z_j - c_j \geq 0$

$\therefore$ we stop our iterations

$x_1 = 0$
$x_2 = 2$ $\qquad A_1 = 0$
$x_3 = 0$

$\therefore$ $z_{max} = z^*_{max}$
$z_{max} = 2 \times 0 + 2 \times 2 + 4 \times 0 = 4$