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MA102 : Introduction to Discrete Mathematics

Tutorial 6

1. What are $A \circ B$ and $B \circ A$ when A and B are inequality relations $<$ and $>$ respectively on

(i) on $S = \{0, 1, 2 \ldots, 9\}$
(ii) on $S = \mathbb{Z}$

$A = \{(x, y) : x < y\}$
$B = \{(x, y) : x > z\}$

$\therefore A \circ B = \{(x, z) : (x, y) \in B, (y, z) \in A\}$

$= \{(x, z) : x > y, y < z\}$

$= \{(x, z) : y < x, y < z\}$

and $B \circ A = \{(x, z) : (x, y) \in A, (y, z) \in B\}$

$= \{(x, z) : x < y, y > z\}$

$= \{(x, z) : x < y, z < y\}$

$\therefore$ (i) $A \circ B = \{(x, z) : 0 < z, x \leq 9\}$

and, $B \circ A = \{(x, z) : 0 \leq x, z < 9\}$

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(ii) $A \circ B = \{(x, z) : y < x \text{ and } y < z\}$

as there is no lower bound of the $\mathbb{Z}$ set, we can say that every x and z belonging to $\mathbb{Z}$ will be in $A \circ B$.

$\therefore \quad A \circ B = \mathbb{Z} \times \mathbb{Z}$

$B \circ A = \{(x, z) : x > y \text{ and } z > y\}$

as there is no upper bound of the $\mathbb{Z}$ set, we can say that every $x \in \mathbb{Z}$ and $z \in \mathbb{Z}$ will be related.

$\therefore \quad B \circ A = \mathbb{Z} \times \mathbb{Z}$

2. Suppose that $A = \{1, 2, 3, 4\}$ and R be the relation on A defined as $(a, b) \in R$ iff $a < b$. Find the matrix graph representation of R with respect to the natural ordering

$A = \{1, 2, 3, 4\}$

$(a, b) \in R$ iff $a < b$

$\therefore R = \{(1, 2), (1, 3), (1, 4), (2, 3), (2, 4), (3, 4)\}$

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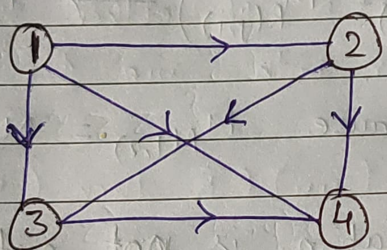
the matrix representation of R w.r.t the natural ordering is,

$$M_R = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

and the graph representation of R is



3. Suppose that the relation R on a set is represented by the matrix

$$M_R = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

Is R reflexive, symmetric, antisymmetric ?

Since, all the principal diagonal elements are 1, therefore, the relation is reflexive.

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Since, the matrix is not symmetric because

$$M_{R(2,3)} = 1 \quad \text{and} \quad M_{R(3,2)} = 0$$

~~M_R~~

$$\therefore \forall (i,j) \{ M_{R(ij)} = M_{R(ji)} \} \text{ is false.}$$

Hence, the relation is **not** symmetric.

Since, for all i, j such that $i \neq j$

$$\left(M_{R(i,j)} = M_{R(j,i)} \right) \longrightarrow i = j$$

is false, because $M_{R(1,2)} = M_{R(2,1)}$ and $2 \neq 1$.

Hence the relation is **not** anti-symmetric.

4. Determine whether the relation R on the set of real numbers is reflexive, symmetric, anti-symmetric, transitive, where $(x,y) \in R$ iff

(i) $x + y = 0$

for all $x \in$ Real $x + x = 0$ is not true.
Hence, R is **not** reflexive.

for all $x, y \in$ Real, if $x + y = 0$, then $y + x = 0$, the commutative law of addition.
$\therefore$ R is symmetric.

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As $x+y=0$ and $y+x=0$ does not necessarily imply that $x=y$, R is not antisymmetric.

Counterexample $(1,-1) \in R$ and $(-1,1) \in R$ but $1 \neq -1$.

Since, $(1,-1) \in R$ and $(-1,1) \in R$ but $(1,1)$ does not belong to R, therefore R is not transitive.

Hence R is not reflexive, symmetric, not antisymmetric and not transitive.

(ii) $xy \geq 0$
For all $x \in$ Real, $x \cdot x \geq 0$. Therefore, R is reflexive.

For all $x, y \in$ Real, if $xy \geq 0$, then $yx \geq 0$ (commutative law of multiplication $xy = yx$).
Therefore, R is symmetric.

Since, $(1,2) \in R$ and $(2,1) \in R$ but $2 \neq 1$.
$\therefore \forall (x,y) \{((x,y) \in R \wedge (y,x) \in R) \longrightarrow x=y\}$ is false.

Hence, R is not anti-symmetric.

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Since, $(2,0) \in R$ and $(0,-5) \in R$ but $(2,-5) \notin R$. Therefore R is not transitive.

Hence, R is reflexive, symmetric, not transitive and not antisymmetric.

(iii) $x = 1$ or $y = 1$.

- For every real x other than 1, $(x, x) \notin R$.
  Hence, R is not reflexive.

- Clearly $(x, y) \in R$ if either $x = 1$ or $y = 1$ or both are 1.

  Hence, if $(x,y) \in R$ then (y, x) will also belong to R.
  Hence, R is Symmetric.

- Let $x = 0$ and $y = 1$.
  $\therefore$ $(x,y) \in R$
  also $(y,x) \in R$
  but $x \neq y$
  Hence, R is not antisymmetric.

- Let $x = 3$, $y = 1$ and $z = 520$

  $\therefore$ $(x,y) \in R$ (as $y = 1$)
  $(y, z) \in R$ (as $y = 1$)
  but $(x, z) \notin R$

  Hence, R is not transitive.

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5. Give an example of a relation which is symmetric and anti symmetric.

Let us assume the set $A = \{a, b, c\}$ and define a relation R on A such that

$$R = \{(a,a), (b,b), (c,c)\}$$

The relation R on A is both symmetric and anti-symmetric.

6. Let A be the relation "to be wife of" and B be "to be father of" on the set of all humans. What does the relation $A \circ B$ mean in this case?

$A = \{(x,y): x \text{ is wife of } y\}$
$B = \{(x,y): x \text{ is father of } y\}$

$A \circ B = \{(x,z): (x,y) \in B \text{ and } (y,z) \in A\}$

$A \circ B = \{(x,z): x \text{ is father of } y, y \text{ is wife of } z\}$

$A \circ B = \{(x,z): x \text{ is father-in-law of } z\}$

$\therefore$ $A \circ B$ is the relation "to be father-in-law of".

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7. Find the matrix $\bar{R}$ and R^{-1} where matrix of R is given below

$$\begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}$$

the matrix representation of $\bar{R}$ is formed by replacing 1's by 0's and 0's by 1's in matrix representation of R.

$$\therefore M_{\bar{R}} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

the matrix representation of R^{-1} is the transpose of matrix representation of R.

$$\therefore M_{R^{-1}} = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

8. Show that if R_1 and R_2 are equivalence relations on S, then $R_1 \cap R_2$ is an equivalence relation on S.

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Reflexive

Let say $a \in S$.

Since, R_1 and R_2 are equivalence relations, therefore,

$(a,a) \in R_1$ and $(a,a) \in R_2$

$\Rightarrow (a,a) \in R_1 \cap R_2$

$\Rightarrow R_1 \cap R_2$ is reflexive.

Symmetric

Let $a,b \in S$ such that $(a,b) \in R_1 \cap R_2$.

$\Rightarrow (a,b) \in R_1$ and $(a,b) \in R_2$

$\Rightarrow (b,a) \in R_1$ and $(b,a) \in R_2$ (as R_1 and R_2 are equivalence relations)

$\Rightarrow (b,a) \in R_1 \cap R_2$

Hence, if $(a,b) \in R_1 \cap R_2$, $(b,a) \in R_1 \cap R_2$.

Hence, $R_1 \cap R_2$ is symmetric.

Transitive

Let $a,b,c \in S$ such that $(a,b) \in R_1 \cap R_2$ and $(b,c) \in R_1 \cap R_2$

$(a,b) \in R_1 \cap R_2 \Rightarrow (a,b) \in R_1$ and $(a,b) \in R_2$
$(b,c) \in R_1 \cap R_2 \Rightarrow (b,c) \in R_1$ and $(b,c) \in R_2$

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$(a,b) \in R_1$ and $(b,c) \in R_1 \Rightarrow (a,c) \in R_1$

$(a,b) \in R_2$ and $(b,c) \in R_2 \Rightarrow (a,c) \in R_2$

(as R_1 and R_2 are equivalence relations)

$\Rightarrow (a,c) \in R_1$ and $(a,c) \in R_2$
$\Rightarrow (a,c) \in R_1 \cap R_2$

Hence, if $(a,b) \in R_1 \cap R_2$ and $(b,c) \in R_1 \cap R_2$,
then $(a,c) \in R_1 \cap R_2$.

Hence, $R_1 \cap R_2$ is transitive relation.

Since, $R_1 \cap R_2$ is reflexive, symmetric and transitive, it is an equivalence relation.

9. Find all distinct equivalence relations on the set $\{a,b,c\}$.

The equivalence relations on set $\{a,b,c\}$ are given below,

$R_1 = \{(a,a), (b,b), (c,c)\}$

$R_2 = \{(a,a), (b,b), (c,c), (a,b), (b,a)\}$

$R_3 = R_1 \cup \{(b,c), (c,b)\}$

$R_4 = R_1 \cup \{(a,c), (c,a)\}$

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$R_5 = R_1 \cup \{(a,b), (b,a), (b,c), (c,b), (a,c), (c,a)\}$

These are the five relations that are equivalence relations on set $\{a, b, c\}$

10. Show that $\{(x,y) \mid x-y \in \mathbb{Q}\}$ is an equivalence relation on the set of real numbers. What are $\left[\frac{1}{2}\right]$, $[1]$, $[\pi]$?

Let S be the relation

$$S = \{(x,y) \mid x-y \in \mathbb{Q}\}$$

- For every real x,

$$x - x = 0$$

Since, 0 is a rational number, relation S on real numbers is reflexive.

- For every real x and y, let $x-y$ be rational,

$$\therefore x - y = \frac{p}{q}$$

where p and q are integers and $q \neq 0$.

$$\therefore y - x = \frac{-p}{q}$$

Since, $-p$ is also an integer and $q \neq 0$.
$y - x \in \mathbb{Q}$.

Hence, S on real numbers is symmetric.

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- For every real x, y and z, let $x-y \in \mathbb{Q}$ and $y-z \in \mathbb{Q}$.

$\therefore$ $x-y = \frac{p}{q}$ and $y-z = \frac{r}{s}$

where p, q, r and s are integers and $q \neq 0$ and $s \neq 0$

$\therefore$ $(x-y) + (y-z) = \frac{p}{q} + \frac{r}{s}$

$$x - z = \frac{ps + rq}{qs}$$

as $q \neq 0$ and $s \neq 0$, $\therefore$ $qs \neq 0$.
Also $ps + rq$ is an integer.

$\therefore$ $x - z \in \mathbb{Q}$.

Hence, S on real numbers is transitive.

Since, S is reflexive, symmetric and transitive, it is an equivalence relation.

Now, $\left[\frac{1}{2}\right] = \left\{ \frac{1}{2} + \frac{p}{q} : p, q \in \mathbb{Z} \text{ and } q \neq 0 \right\}$

$\left[\frac{1}{2}\right] = \mathbb{Q}$ (as sum of two rationals is rational $\therefore \frac{1}{2} + \frac{p}{q} \in \mathbb{Q}$)

Similarly, $[1] = \mathbb{Q}$

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$$[\pi] = \left\{ \pi + \frac{p}{q} : p, q \in \mathbb{Z} \text{ and } q \neq 0 \right\}$$

Here $[\pi] \neq \mathbb{Q}$, because π is irrational, and sum of irrational and rational $\left(\pi + \frac{p}{q}\right)$ is always irrational.

11. Let R be the relation on $\mathbb{R}^2$ defined by $(x_1, y_1) R (x_2, y_2)$ iff $\lfloor y_1 \rfloor = \lfloor y_2 \rfloor$;

where $\lfloor x \rfloor$ is the largest integer less than or equal to x.

(a) Is $(7, \sqrt{3})$ related to $\left(\frac{1}{5}, \sqrt{2}\right)$ by R?

(b) Show that R is an equivalence relation on $\mathbb{R}^2$?

(c) Find the partition of $\mathbb{R}^2$ corresponding to R?

(a) $\lfloor \sqrt{3} \rfloor = \lfloor 1.73 \rfloor = 1$

$\lfloor \sqrt{2} \rfloor = \lfloor 1.41 \rfloor = 1$

as $\lfloor y_1 \rfloor = \lfloor y_2 \rfloor$

$\therefore$ $(7, \sqrt{3})$ and $\left(\frac{1}{5}, \sqrt{2}\right)$ are related by R.

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(b) Reflexive :-
For every $(x, y) \in R^2$,
$(x, y) R (x, y)$ as $\lfloor y \rfloor$ will be constant for given y.

$\therefore$ R is reflexive.

Symmetric :-
For real x_1, y_1, x_2 and y_2, let us assume that

$$(x_1, y_1) R (x_2, y_2)$$

$$\therefore \lfloor y_1 \rfloor = \lfloor y_2 \rfloor$$

$$\Rightarrow \lfloor y_2 \rfloor = \lfloor y_1 \rfloor$$
$$\Rightarrow (x_2, y_2) R (x_1, y_1)$$

Hence, R is symmetric.

Transitive :-
For (x_1, y_1), (x_2, y_2) and $(x_3, y_3) \in R^2$, let us assume that

$$(x_1, y_1) R (x_2, y_2) \text{ and } (x_2, y_2) R (x_3, y_3)$$

$$\therefore \lfloor y_1 \rfloor = \lfloor y_2 \rfloor \text{ and } \lfloor y_2 \rfloor = \lfloor y_3 \rfloor$$

$$\Rightarrow \lfloor y_1 \rfloor = \lfloor y_3 \rfloor$$

$$\Rightarrow (x_1, y_1) R (x_3, y_3)$$

Hence, R is transitive.

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Since, R is reflexive, symmetric and transitive, R is an equivalence relation.

(c) Since,

$$\lfloor i \rfloor = K \Rightarrow K \leq i < K+1 \quad \text{where } K \in \mathbb{Z}$$

$$\therefore P_i = \{(x, y_i) : x \in \mathbb{R},\ y_i \in [i, i+1) \text{ where } i \in \mathbb{Z}\}$$