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202052307

MA102: Introduction to Discrete Mathematics

Tutorial 10

1. A coin is flipped 10 times where each flip comes up either heads or tails. How many possible outcomes -

(a) are there in total?
(b) contain exactly two heads?
(c) contain at most three tails?
(d) contain the same number of head and tails?

(a) Since, each flip can have two possibilities, the total number of possible outcomes are $2^{10} = 1024$.

(b) For exactly two heads, we need to choose two flips where head comes up. This can be done in ${}^{10}C_2$ ways. Hence, for exactly two heads out of 10 flips, there are 45 possible outcomes.

(c) For at most three tails, there can be zero tails in ${}^{10}C_0$ ways, one tail in ${}^{10}C_1$ ways, 2 tails in ${}^{10}C_2$ ways and three tails in ${}^{10}C_3$ ways.

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$\therefore$ ~~total~~ total possibilities = $^{10}C_0 + {}^{10}C_1 + {}^{10}C_2 + {}^{10}C_3$

$= 1 + 10 + 45 + 120$

$= 176.$

Hence, there will be at most three tails in 176 outcomes.

(d) Same number of heads and tails i.e. both must be present 5 times. This can be done by choosing 5 heads out of 10 flips (the other 5 will automatically be tails).

$\therefore$ possible outcomes $= {}^{10}C_5 = 252$ ways

2. How many bit strings of length 10 either begin with 3 zeroes or end with 3 ones?

No. of bit strings ~~that has~~ of length 10 and beginning with 3 zeroes,

0 0 0 _ _ _ _ _ _ _

In the 7 places, there are 2 possibilities.

$\therefore$ there are 2^7 bit strings of length 10 that begin with 3 zeroes (these include the ones that end with three 1's)

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Similarly, no. of bit string of length 10 that end with three 1's are 2^7 (these include those beginning with 3 0's)

the bit strings beginning with 3 0's and ending with three 1's are counted in both the calculation, so we need to delete them once.

0 0 0 _ _ _ _ 1 1 1

There will be 2^4 such bit strings.

Hence, no. of bit strings of length 10 that either begin with 3 0's or end with 3 1's are

$$\boxed{2^7 + 2^7 - 2^4 = 240}$$

3. How many numbers must be selected from {1, 2, 3, 4, 5, 6} to guarantee that at least one pair of these numbers add up to 7?

The possible pairs that give the sum 7 are {1, 6}, {2, 5} and {3, 4}

Selecting three numbers from the set will not guarantee that there is at least one pair that adds up to 7. If we select

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{1, 2, 3}, then no two elements add up to 7.

But, if we choose just one more element, then there will be atleast one pair that will give 7 as sum.

∴ We must choose atleast 4 numbers from the set {1, 2, 3, 4, 5, 6} so that there is at least one pair in the selection that adds up to 7.

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4. Thirteen people on softball team show up for a game.

a) How many ways are there to choose 10 players to take the field?

b) How many ways are there to assign 10 positions by selecting players from the 13 people who show up?

c) Of the 13 people who show up, three are women. How many ways are there to choose 10 players to take the field if at least one of these players must be a woman?

(a) 10 players can be selected out of 13 in ${}^{13}C_{10} = {}^{13}C_3$ ways.

$${}^{13}C_3 = \frac{13 \times \cancel{12}^{2} \times 11}{\cancel{6}}$$

$$= 286 \text{ ways}$$

Hence, there are 286 ways of choosing 10 players out of 13 who show up.

(b) This can be done by selecting 10 players out of 13 and then assigning these players 10 different positions.

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$\therefore$ no. of ways $= {}^{13}P_{10}$

$= {}^{13}C_{10} \times 10!$

$= 286 \times 3628800$

$= 1037836800$ ways

(c) Since, three are women out of 13 people who show up and there must be at least 1 women in the selected 10 people.

If there is exactly one women, the team can be chosen by selecting 9 men from 10 men and 1 women from 3 women.

exactly one women in team $= ({}^{10}C_9) \cdot {}^3C_1$

$= 10 \times 3$

$= 30$

Similarly, exactly two women in team $= ({}^{10}C_8) \cdot {}^3C_2$

$= (45)(3)$

$= 135$

and, exactly 3 women in team $= ({}^{10}C_3)({}^3C_3)$

$= 120 \times 1$

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∴ no. of ways of choosing 10 players out of 13 (3 of which are women) such that there is at least one women in the selection are
30 + 135 + 120 = 285 ways.

5. How many ways are there to select 12 countries in the United Nations to serve on a security council if 3 are selected from a block of 45, 4 are selected from a block of 57, and the others are selected from remaining 69 countries?

Ways of selecting 3 countries from a block of 45 are ${}^{45}C_3$.

Ways of selecting 4 countries from a block of 57 are ${}^{57}C_4$.

Ways of selecting remaining i.e 5 countries from remaining 69 countries are ${}^{69}C_5$.

∴ ways of selecting 12 countries according to the question are

$$\boxed{{}^{45}C_3 \times {}^{57}C_4 \times {}^{69}C_5}$$

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6. Prove that if $n, k \in \mathbb{N}$ with $1 \leq k \leq n$, then $k({}^nC_k) = n({}^{n-1}C_{k-1})$ using the two sides of the equality count the number of ways to select a subset with k elements from a set with n elements and then an element of this subset.

We have to select a subset of k elements out of n and then select an element of this subset.

This can be done in two ~~ways~~ different approaches:

(i) Selecting a subset of k element from n elements can be done in ${}^nC_k$ ways and then selecting an element from this subset can be done in ${}^kC_1$ ways.

$\therefore$ no. of ways $= k \cdot {}^nC_k$ = LHS

(ii) Selecting an element from n elements in ${}^nC_1$ ways. Then, for making sure that this element belongs to the subset that we form, selecting $(k-1)$ elements from remaining $(n-1)$ elements of the set in ${}^{n-1}C_{k-1}$ ways.

$\therefore$ no. of ways $= n \cdot {}^{n-1}C_{k-1}$ = RHS

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Hence, proved.

Algebraic Proof :

To prove : $K \cdot {}^{n}C_{K} = n \cdot {}^{n-1}C_{K-1}$

$$\cancel{K} \cdot \frac{n!}{K!\,(n-K)!} = n \cdot \frac{(n-1)!}{}$$

$$n \cdot {}^{n-1}C_{k-1} = n \cdot \frac{(n-1)!}{(k-1)!\,(n-1-k+1)!}$$

$$= \frac{n!}{(k-1)!\,(n-k)!}$$

$$= \frac{k \cdot n!}{k\,(k-1)!\,(n-k)!}$$

$$= k \cdot \frac{n!}{k!\,(n-k)!}$$

$$= k \cdot \left({}^{n}C_{k}\right)$$

Hence, proved.

7. What is the minimum number of students, each of whom comes from one of the 50 states, who must be enrolled in university to guarantee that there are at least 100 students who come from same state?

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Let n be the minimum number of students to guarantee that there are at least 100 students who come from the same state.

using general form of Pigeonhole Principle

i.e. "If N objects are to be placed in K boxes $(K < N)$, there is atleast one box containing $\left\lceil \frac{N}{K} \right\rceil$ objects.

Let the objects be students and boxes be states.

$$\therefore \left\lceil \frac{n}{50} \right\rceil \geq 100$$

$$\therefore \frac{n}{50} > (100-1)$$

$$\frac{n}{50} > 99$$

$$n > 4950$$

$\therefore$ There must be atleast 4951 students to guarantee that there are at least 100 students who come from same state.

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8. A bowl contains 10 red balls and 10 white balls. How many balls must a woman select to be sure of having at least three balls of same colour.

Let there be n minimum balls to be sure of having 3 balls of same colour. Using general form of Pigeonhole principle considering objects as balls and boxes as ~~colours~~. Since, red and white colours are ~~there~~ only, there are 2 boxes.

Since, we need to ensure that there must be 3 balls of same colour, ∴ according to Pigeonhole principle,

$$\left\lceil \frac{n}{2} \right\rceil \geq 3$$

$$\frac{n}{2} > 2$$

$$n > 4$$

Hence, there should be ~~at least~~ 5 balls the woman must draw to be sure of having at least 3 balls of same colour.

9. Give a formula for the coefficient of x^k in the expansion of $\left(x + \frac{1}{x}\right)^{100}$ where $k \in N$.

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$$\left(x+\frac{1}{x}\right)^{100} = \sum_{i=0}^{100} {}^{100}C_i \; x^i \left(\frac{1}{x}\right)^{100-i}$$

$$= \sum_{i=0}^{100} {}^{100}C_i \; x^{2i-100}$$

Let ~~200 - i~~ $2i - 100 = k$

Now, $k \in \mathbb{Z}$ iff $i = \frac{100+k}{2}$ is integer. $\frac{100+k}{2}$ will be an integer when k is an integer.

$$\boxed{\text{coefficient of } x^k = \begin{cases} 0 & \text{if } k \text{ is odd} \\ {}^{100}C_{\left(\frac{100+k}{2}\right)} & \text{if } k \text{ is even.} \end{cases}}$$

10. Show that if n is a positive integer, then ${}^{2n}C_2 = 2\left({}^{n}C_2\right) + n^2$.

Algebraic Proof:

LHS

$${}^{2n}C_2 = \frac{(\cancel{2}n)(2n-1)}{\cancel{2}} = 2n^2 - n$$

RHS

$$2\left({}^{n}C_2\right) + n^2 = \frac{\cancel{2}n(n-1)}{\cancel{2}} + n^2 = 2n^2 - n$$

Hence, LHS = RHS.

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Combinatorial Proof:

Let there be two sets A and B with n elements in each and $A \cap B = \phi$

We want to select two elements from the set $A \cup B$. This can be done by two different approaches-

(i) Since, $A \cap B = \phi$, therefore the set $A \cup B$ has $2n$ elements. Selecting 2 elements from these can be done in ${}^{2n}C_2$ ways. which is left hand side of given equation.

(ii) Selecting two elements from the 2n elements can be done by another method. There are three possibilities in this method:

a) Both the elements belong to set A. This can be done in ${}^{n}C_2$ ways.

OR

b) Both the elements belong to set B. This can be done in ${}^{n}C_2$ ways.

OR

c) One element belong to set A and other to set B. This can be done in ${}^{n}C_1 \cdot {}^{n}C_1 = n^2$ ways.

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$\therefore$ total number of ways of selecting 2 elements from the 2n elements are

$$2\left({}^{n}C_{2}\right) + n^{2}$$

which is right hand side of the equation.

Hence, LHS = RHS, proved.

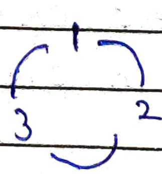
11. A circular r-permutation of n people is a seating of r of these n people around a circular table, where seatings are considered to be the same if they can be obtained from each other by rotating the table.

a) How many circular n-permutations of n people are possible where n = 3, 4 and in general?

b) How many 3-permutation of 5 people are possible?

c) How many circular r-permutation of n people are possible?

(a) In circular permutations of 3 people, the permutations



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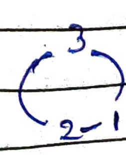

are same

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as they can be made by rotating the table only.
Hence, for circular permutation of n people, we need to first fix one position and then arrange remaining $(n-1)$ people at the table.
This can be done in $(n-1)!$ ways.

Hence,

circular n-permutations when $n=3$ are $2! = 2$.

and circular n-permutations when $n=4$ are $3! = 6$.

In general circular n-permutations are $(n-1)!$.

(b) This can be considered as selecting 3 people out of 5 and then arranging them on a circular table.

3 people can be selected from 5 in ${}^5C_3$ ways.

and 3 people can be arranged on a circular table in $2! = 2$ ways.

circular 3-permutation of 5 people are ${}^5C_3 \times 2! = 20$

(c) Using the similar argument as in (b) circular r-permutation of n are ${}^nC_r\,(r-1)!$.

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Q. 12 How many non-negative integer solution are there to the equation $x_1 + x_2 + x_3 + x_4 + x_5 = 21$ with $x_1, x_2 \geq 1$, $x_3 = 3$, $x_4 \geq 2$ and $x_5 \geq 0$?

$\because$ $x_1 \geq 1$

$a_1 + 1 \geq 1$ (let $x_1 = a_1 + 1$)

$a_1 \geq 0$

$\because$ $x_2 \geq 1$

$a_2 + 1 \geq 1$ (let $x_2 = a_2 + 1$)

$a_2 \geq 0$

given, $x_3 = 3$

$\because$ $x_4 \geq 2$

$a_4 + 2 \geq 2$ (let $x_4 = a_4 + 2$)

$a_4 \geq 0$

and let $a_5 = x_5 \Rightarrow a_5 \geq 0$

$\therefore$ $(a_1 + 1) + (a_2 + 1) + 3 + (a_4 + 2) + a_5 = 21$

$\therefore$ $a_1 + a_2 + a_4 + a_5 = 14$

with $a_1, a_2, a_4, a_5 \geq 0$

$\therefore$ number of non-negative integral solutions to the above equation is

$$^{14+4-1}C_{4-1} = \boxed{^{17}C_3}.$$

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Q.13 A book publisher has 3000 copies of a discrete mathematics book. How many ways are there to store these books in their three warehouse if the copies of the book are indistinguishable?

Let there be x_1, x_2, x_3 books in the warehouse 1, 2 and 3 respectively.
Therefore,

$$x_1 + x_2 + x_3 = 3000$$

with $x_1 \geq 0$, $x_2 \geq 0$ and $x_3 \geq 0$.

$\because$ no. of books in a warehouse cannot be negative, we need to find the number of ~~way~~ non-negative integral solutions of the equation

$$x_1 + x_2 + x_3 = 3000$$

Hence, the number of ways to store these books in their three warehouses are

$$\boxed{= {}^{3002}C_2 = 4504501}$$

Q.14 How many ways are there to distribute six indistinguishable balls into nine distinguishable bins?

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Let there be $x_1, x_2 \cdots x_9$ balls in the bin $1, 2, \cdots, 9$ respectively.
Therefore,

$$x_1 + x_2 + \cdots + x_9 = 6$$
with $x_i \geq 0$ where $1 \leq i \leq 9$

Hence, the number of ways to distribute six indistinguishable balls into nine distinguishable bins are

$$^{6+9-1}C_{9-1} = \boxed{^{14}C_8 = 3003 \text{ ways}}$$

15. Suppose that S is a set with n elements. How many ordered pairs (A, B) are there such that A and B are subsets of S with $A \subseteq B$.

Let us say that B has k elements $(0 \leq k \leq n)$. Therefore, number of ways of selecting k elements out of n for forming the set B are ^nC_k.

Now, since $A \subseteq B$, there are 2^k possibilities for A.

$\therefore$ for a fixed B with k elements, there are $^nC_k \cdot 2^k$ ways of forming

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(A, B) such that $A \subseteq B$,

Since, B is also a subset of S, k goes from 0 to n.

$\therefore$ total ways to form ordered pair (A, B) such that A and B are subset of S with $A \subseteq B$. are

$$\boxed{\sum_{k=0}^{n} {}^{n}C_{k} \cdot 2^{k}} \longrightarrow (1)$$

The binomial expansion of $(1+x)^n$ is

$$(1+x)^n = \sum_{k=0}^{n} {}^{n}C_{k}\, x^{k}$$

i.e $$\sum_{k=0}^{n} {}^{n}C_{k}\, x^{k} = (1+x)^n$$

Putting $x = 2$

$$\boxed{\sum_{k=0}^{n} {}^{n}C_{k}\, 2^{k}} = 3^n$$

(1)

$\therefore$ no. of ways to form ordered pair (A, B) such that A and B are subsets of S (having n elements) with $A \subseteq B$ are 3^n.