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PH110: Waves and Electrodynamics

Remote End-Semester Exam

1 (a) The Maxwell's equation in differential form for free space are

(i) $\nabla \cdot E = 0$

(ii) $\nabla \cdot B = 0$

(iii) $\nabla \times E = -\frac{\partial B}{\partial t}$

(iv) $\nabla \times B = \mu_0 \varepsilon_0 \frac{\partial E}{\partial t}$

Maxwell's equation in conductors,

(i) $\nabla \cdot D = \rho_f$

(ii) $\nabla \cdot B = 0$

(iii) $\nabla \times E = -\frac{\partial B}{\partial t}$

(iv) $\nabla \times H = J_f + \frac{\partial D}{\partial t}$

$D \rightarrow$ electric displacement

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In general E, B, D and H will be discontinuous at a boundary between two media, or at a surface that carries a charge density σ or a current density K.
The explicit form of these discontinuities can be deduced from Maxwell's equations ~~written~~ in the integral form

$$\left.\begin{aligned} &\text{(i)} \quad \oint_S D \cdot da = Q_{fenc} \\ &\text{(ii)} \quad \oint_S B \cdot da = 0 \end{aligned}\right\} \text{over closed surface } S$$

$$\text{(iii)} \quad \oint_P E \cdot dl = -\frac{d}{dt}\int_S B \cdot da$$

$$\text{(iv)} \quad \oint H \cdot dl = I_{fenc} + \frac{d}{dt}\int_S D \cdot da$$

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1 (b)

(i) Since, the electric field between the plates is

$$E = \frac{\sigma}{\epsilon_0} \hat{z}$$

and, since $\sigma = \frac{Q}{\pi a^2}$

$$Q(t) = It$$

$$\boxed{\therefore \quad E(t) = \frac{It}{\pi a^2 \epsilon_0} \hat{z}}$$

Now, $B \cdot 2\pi s = \mu_0 \epsilon_0 \frac{\partial E}{\partial t} \pi s^2$

$$\therefore \quad B = \mu_0 \epsilon_0 \left(\frac{I}{\pi a^2 \epsilon_0}\right) \pi s^2 \cdot \frac{1}{2\pi s}$$

$$\boxed{B(s,t) = \frac{\mu_0 I s}{2\pi a^2} \hat{\phi}}$$

(ii) energy density is given by

$$u_{em} = \frac{1}{2}\left(\epsilon_0 E^2 + \frac{B^2}{\mu_0}\right)$$

$$= \frac{1}{2}\left\{\epsilon_0 \left(\frac{It}{\pi \epsilon_0 a^2}\right)^2 + \frac{1}{\mu_0}\left(\frac{\mu_0 I s}{2\pi a^2}\right)^2\right\}$$

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$$u_{em} = \frac{\mu_0 I^2}{2\pi^2 a^4}\left[\frac{t^2}{\mu_0\varepsilon_0} + \left(\frac{s}{2}\right)^2\right]$$

$$\boxed{u_{em} = \frac{\mu_0 I^2}{2\pi^2 a^4}\left[(ct)^2 + \left(\frac{s}{2}\right)^2\right]}$$

$c \rightarrow$ speed of light

$$c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}}$$

Poynting vector $S = \frac{1}{\mu_0}(E \times B)$

$$= \frac{1}{\mu_0}\left(\frac{It}{\pi\varepsilon_0 a^2}\right)\left(\frac{\mu_0 I s}{2\pi a^2}\right)(-\hat{s})$$

$$\boxed{S = \frac{-I^2 t s}{2\pi^2\varepsilon_0 a^4}\hat{s}}$$

~~total power flowing into the gap U_{em}~~

$$\cancel{U_{em} = \int u_{em}\, w\, 2\pi s\, ds}$$

$$\cancel{= 2\pi w \frac{\mu_0 I^2}{2\pi^2 a^4}\int_0^b \left[(ct)^2 + \left(\frac{s}{2}\right)^2\right] s\,ds}$$

$$\cancel{= \frac{\mu_0 w I^2}{\pi a^4}\left[(ct)^2\frac{s^2}{2} + \frac{s^4}{16}\right]_0^b}$$

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$$\boxed{U_{em} = \frac{\mu_0 \omega I^2 b^2}{2\pi a^4}\left[(ct)^2 + \frac{b^2}{8}\right]}$$

(ii) total power flowing by integrating Poynting vector S over a surface at radius b.

$$P_{in} = -\int S \cdot da$$

$$= \frac{I^2 t}{2\pi^2 \epsilon_0 a^4}\left[b\,\hat{s} \cdot (2\pi b \omega)\,\hat{s}\right]$$

$$\boxed{P_{in} = \frac{I^2 \omega t\, b^2}{\pi \epsilon_0 a^4}}$$

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2 (a) $J_b = \nabla \times \vec{M}$

$K_b = \vec{M} \times \hat{n}$

so, we can write

$$|J_b| = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ M_x & M_y & M_z \end{vmatrix}$$

$$J_b = \left(\frac{\partial M_z}{\partial y} - \frac{\partial}{\partial z} M_y\right)\hat{i} - \left(\frac{\partial M_z}{\partial x} - \frac{\partial M_x}{\partial z}\right)\hat{j}$$

$$+ \left(\frac{\partial}{\partial x} M_y - \frac{\partial}{\partial y} M_x\right)\hat{k}$$

so, it depends on how much magnetization changes in one component w.r.t. others.

It also means that it depends on materials as there are 2 mechanism of magnetic polarization.

(i) Paramagnetisation
(ii) Diamagnetisation

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2(b) The electric charge will move parallel to the direction of local electric field, so we just need to find the electric field induced by the changing magnetic field (because the field is switched off). We know the magnetic field, and maxwell equations give:

$$\nabla \cdot \vec{E} = 0 \quad \text{and} \quad \nabla \times E = -\frac{\partial \vec{B}}{\partial t}$$

This, by itself, is not enough to determine the field uniquely. Thus we cannot know in which direction the charge will move.

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3(a) The equation of continuity,

$$\nabla \cdot J = -\frac{\partial \rho}{\partial t}$$

J is the current density and ρ is the charge density.

If the divergence of J is positive, then more charge is exiting than entering the specified volume. If charge is exiting faster, the charge within the volume must be decreasing. Hence, it states that if there is a net electric current flowing ~~tha~~ ~~that~~ out of a region, then charge in that region must be decreasing with time. If there is more current flowing into a given volume than exiting, then the ~~electric~~ amount of charge in that volume must be increasing.

In Electrodynamics, the continuity equation is as follows

$$\frac{\partial U}{\partial t} = -\nabla \cdot S$$

It is precisely the continuity equation for energy.

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$U \rightarrow$ energy density
energy density plays the role of ρ and
S (Poynting vector) takes the part of J.
It expresses the local conservation
of electromagnetic energy.

In general, electromagnetic energy is
not conserved by itself. The fields
do ~~so~~ work on charges and charges
create field i.e. energy is tossed
back and forth between them.
Thus it is necessary to consider
contribution of both matter and
fields. This is so because fields
themselves carry momentum and
combinely, the em energy is conserved.

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3(b) According to maxwell's equation $\nabla \cdot B = 0$. This invites us to the conclusion that a vector potential A exists in magnetostatics such that

$$B = \nabla \times A$$

~~Thus~~ Any function whose curl vanishes can be added to A, with no effect on B.

The dipole magnetic field is

$$\vec{B} = \frac{\mu_0 m}{4\pi r^3}\left[2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}\right]$$

Now, if the dipole is oriented along z-axis, we can write it in the form

$$\vec{m} = m\hat{z} = m\left(\cos\theta\,\hat{r} - \sin\theta\,\hat{\theta}\right)$$

$$m\cos\theta\,\hat{r} = (\vec{m}\cdot\hat{r})\,\hat{r}$$

$$m\sin\theta\,\hat{\theta} = m\cos\theta\,\hat{r} - \vec{m}$$

$$= (\vec{m}\cdot\hat{r})\,\hat{r} - \vec{m}$$

$$\therefore\ B = \frac{\mu_0}{4\pi r^3}\left(2(\vec{m}\cdot\hat{r})\hat{r} + (\vec{m}\cdot\hat{r})\hat{r} - \vec{m}\right)$$

$$\boxed{B = \frac{\mu_0}{4\pi r^3}\left[3(\vec{m}\cdot\hat{r})\hat{r} - \vec{m}\right]}$$

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4(a) Bound and free charges are not same.

Bound charges are a result of polarization. However, free charges ~~are~~ can be anything but not the result of polarisation.

$$\rho = \rho_b + \rho_f$$

the new equation for Gauss law becomes

$$\varepsilon(\nabla \cdot E) = \rho$$

$$\varepsilon(\nabla \cdot E) = \rho_f + \rho_b \qquad (\rho_b = -\nabla \cdot P)$$

$$\varepsilon(\nabla \cdot E) = -\nabla \cdot P + \rho_f$$

$$\varepsilon(\nabla \cdot E) + \nabla \cdot P = \rho_f$$

$$\nabla(\varepsilon E + P) = \rho_f \qquad (\text{if } J_b = 0)$$

$$\nabla \cdot (D) = \rho_f$$

D is known as electric dispacement.

∴ Normal Gauss's law

$$\nabla \cdot (\varepsilon E) = \rho_f$$

other wise we get

$$\nabla(D) = \rho_f$$