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MA101 : Linear Algebra and Matrices
Tutorial : 4

1. Is the following set linearly independent and/or span R^3 ?

$u = \begin{bmatrix} 3 \\ 2 \\ -4 \end{bmatrix}$, $v = \begin{bmatrix} -6 \\ 1 \\ 7 \end{bmatrix}$, $w = \begin{bmatrix} 0 \\ -5 \\ 2 \end{bmatrix}$, and $z = \begin{bmatrix} 3 \\ 7 \\ -5 \end{bmatrix}$

a) $\{u, v, w\}$

Let the matrix formed by these vectors is A.

$$A = \begin{bmatrix} 3 & -6 & 0 \\ 2 & 1 & -5 \\ -4 & 7 & 2 \end{bmatrix}$$

If $Ax = 0$ has a non-trivial solution, then the vectors will be linearly dependent.

$R_2 \rightarrow R_2 - 2/3 R_1$, $R_3 \rightarrow R_3 + \frac{4}{3} R_1$

$$A = \begin{bmatrix} 3 & -6 & 0 \\ 0 & 4 & -5 \\ 0 & 1 & 2 \end{bmatrix}$$

$R_3 \rightarrow R_3 - \frac{1}{4} R_2$

$$A = \begin{bmatrix} 3 & -6 & 0 \\ 0 & 4 & -5 \\ 0 & 0 & 13/4 \end{bmatrix}$$

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Using row echelon form of A, we get three equations

$$3x_1 - 6x_2 = 0$$
$$4x_2 - 5x_3 = 0$$
$$\frac{13}{4}x_3 = 0$$

which, have only one solution $\{x_1\ x_2\ x_3\}^T = \{0\ 0\ 0\}^T$

Hence, the vectors u, v and w are linearly independent.
Since, in REF of A, we have three pivot entries, the vectors will also span R^3.

b) $\{u, v\}$

Two vectors are linearly independent, only if there does not exist some constant c $(c \neq 0)$ such that $b = ca$, b and a are two vectors.

Since, $u = \begin{bmatrix} 3 \\ 2 \\ -4 \end{bmatrix}$ and $v = \begin{bmatrix} -6 \\ 1 \\ 7 \end{bmatrix}$, there exist no scalar (non-zero) such that $v = cu$, hence the vectors u and v are linearly independent.

Let us call the matrix formed by u and v be A.

$$\therefore A = \begin{bmatrix} 3 & -6 \\ 2 & 1 \\ -4 & 7 \end{bmatrix}$$

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$R_2 \rightarrow R_2 - \frac{2}{3} R_1$, $R_3 \rightarrow R_3 + \frac{4}{3} R_1$

$$A = \begin{bmatrix} 3 & -6 \\ 0 & 4 \\ 0 & 1 \end{bmatrix}$$

$R_3 \rightarrow R_3 - \frac{1}{4} R_2$

$$A = \begin{bmatrix} 3 & -6 \\ 0 & 4 \\ 0 & 0 \end{bmatrix}$$

Since in REF of A, ther are only two pivot entries, hence these vectors will not span R^3.

c) $\{u, v, w, z\}$
Let us call the matrix formed by u, w, w and z be A.

$$A = \begin{bmatrix} 3 & -6 & 0 & 3 \\ 2 & 1 & -5 & 7 \\ -4 & 7 & 2 & -5 \end{bmatrix}$$

If $Ax = 0$ has only trivial solution, then the vectors $\{u, v, w, z\}$ will be linearly independent.

$\{R_2 \rightarrow R_2 - 2/3\, R_1$ and $R_3 \rightarrow R_3 + \frac{4}{3} R_1\}$

$$A = \begin{bmatrix} 3 & -6 & 0 & 3 \\ 0 & 4 & -5 & 5 \\ 0 & 1 & 2 & -1 \end{bmatrix}$$

$R_3 \rightarrow R_3 - \frac{1}{4} R_2$

$$A = \begin{bmatrix} 3 & -6 & 0 & 1 \\ 0 & 4 & -5 & 5 \\ 0 & 0 & 13/4 & -9/4 \end{bmatrix}$$

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Since, column 4 has no pivot entries, the solution set $\{x_1\ x_2\ x_3\ x_4\}$ has x_4 as a free variable

let $x_4 = c$

the ref of A gives following equations from $Ax = 0$

$$3x_1 - 6x_2 + 0x_3 + x_4 = 0 \quad \text{— (I)}$$
$$0x_1 + 4x_2 - 5x_3 + 5x_4 = 0 \quad \text{— (II)}$$
$$0x_1 + 0x_2 + \frac{13}{4}x_3 - \frac{9}{4}x_4 = 0 \quad \text{— (III)}$$

Put $x_4 = c$ and back substitute the values in (II) and (I)

$$\frac{13}{4}x_3 - \frac{9}{4}c = 0$$

$$\boxed{x_3 = \frac{9c}{13}}$$

$$4x_2 - 5\left(\frac{9c}{13}\right) + 5c = 0$$

$$4x_2 = \frac{(-65+45)c}{13}$$

$$\cancel{4}x_2 = \frac{-2\cancel{0}^{5}c}{13}$$

$$\boxed{x_2 = \frac{-5c}{13}}$$

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$$3x_1 - 6\left(-\frac{5c}{13}\right) + c = 0$$

$$3x_1 = \frac{(-13-30)c}{13}$$

$$x_1 = \frac{-43c}{39}$$

Hence, non trivial solution of $Ax=0$ exists and is given by

$$x = \begin{bmatrix} -43c/39 \\ -5c/13 \\ 9c/13 \\ c \end{bmatrix}, \text{ where } c \in R$$

∴ the vectors u, v, w and z are linearly dependent.

Since in REF of A, there are three pivot entries the given vectors span R^3.

d) $\{u, v, w, z, 0\}$

Let the matrix formed by the vectors be A.

$$A = \begin{bmatrix} 3 & -6 & 0 & 3 & 0 \\ 2 & 1 & -5 & 7 & 0 \\ -4 & 7 & 2 & -5 & 0 \end{bmatrix}$$

If $Ax=0$ has only trivial solutions, the vectors will be linearly independent.

$R_2 \to R_2 - \frac{2}{3}R_1$ and $R_3 \to R_3 + \frac{1}{4}R_1$

$$A = \begin{bmatrix} 3 & -6 & 0 & 1 & 0 \\ 0 & 4 & -5 & 5 & 0 \\ 0 & 1 & 2 & -9/4 & 0 \end{bmatrix}$$

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$R_3 \to R_3 - \frac{1}{4} R_2$

$$A = \begin{bmatrix} 3 & -6 & 0 & 1 & 0 \\ 0 & 4 & -5 & 5 & 0 \\ 0 & 0 & 13/4 & -9/4 & 0 \end{bmatrix}$$

Since, column 4 and 5 have no pivot entries, they are free variables.
let $x_4 = c$ and $x_5 = d$

The equations from $Ax = 0$ are,

$$3x_1 - 6x_2 + x_4 = 0 \quad \text{— (I)}$$

$$4x_2 - 5x_3 + 5x_4 = 0 \quad \text{— (II)}$$

$$\frac{13}{4}x_3 - \frac{9}{4}x_4 = 0 \quad \text{— (III)}$$

Put $x_4 = c$ in (III)

$$\boxed{x_3 = \frac{9c}{13}}$$

Putting $x_3 = \frac{9c}{13}$ and $x_4 = c$ in (II),

$$\boxed{x_2 = \frac{-5c}{13}}$$

Putting $x_2 = -\frac{5c}{13}$ and $x_4 = c$ in (I)

$$\boxed{x_1 = \frac{-43c}{39}}$$

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Hence, non-trivial solution x of $Ax = 0$ is

$$x = \begin{bmatrix} -43c/39 \\ -5c/13 \\ 9c/13 \\ c \\ d \end{bmatrix} \quad \text{where}, \; c, d \in R$$

Hence, the vectors $\{u, v, w, z, 0\}$ are linearly dependent.
Since, there are three pivot entries in REF of A, the vectors span R^3.

2. Let A be the matrix given below and $v_p = [1 \;\; -1 \;\; 0 \;\; 1]^T$ be a particular solution to $Ax = b$. Then find its all solutions.

$$A = \begin{bmatrix} 1 & -3 & 6 & 9 \\ 0 & 0 & 4 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

the solution set of $Ax = b$, if v_p is its solution, is given by w,

$$w = v_p + x$$

where x is solution set of $Ax = 0$

Augmented matrix for $Ax = 0$ is,

$$\left[\begin{array}{cccc|c} 1 & -3 & 6 & 9 & 0 \\ 0 & 0 & 4 & 5 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

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Let $x = [x_1 \ x_2 \ x_3 \ x_4]^T$ be its solution set.

Since there are two pivot entries and column 2 and 4 has no pivot entry, x_2 and x_4 are free variables.

Let $x_2 = a$ and $x_4 = b$

The equations from $Ax = 0$ are,

$$4x_3 + 5x_4 = 0$$
$$x_3 = \frac{-5x_4}{4}$$
$$\boxed{x_3 = \frac{-5b}{4}}$$

$$x_1 - 3x_2 + 6x_3 + 9x_4 = 0$$

$$x_1 = 3x_2 - 6x_3 - 9x_4$$
$$x_1 = 3a - 6\left(\frac{-5b}{4}\right) - 9b$$
$$x_1 = 3a - \frac{3b}{2}$$
$$\boxed{x_1 = \frac{6a - 3b}{2}}$$

Hence $x = \begin{bmatrix} (6a-3b)/2 \\ a \\ -5b/4 \\ b \end{bmatrix}$, where $a, b \in \mathbb{R}$

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$\therefore$ solution set of $Ax = b$ is,

$$w = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} (6a-3b)/2 \\ a \\ -5b/4 \\ b \end{bmatrix}$$

$$w = \begin{bmatrix} (6a-3b+2)/2 \\ a-1 \\ -5b/4 \\ b+1 \end{bmatrix} \quad \text{where } a, b \in R$$

3. Determine whether following is a linear transformation or not. If yes, then find its standard matrix.

a) $\underset{\mathbb{R}^4 \to \mathbb{R}^5}{T}\left(\begin{bmatrix} x_1 & x_2 & x_3 & x_4 \end{bmatrix}^T\right) = \begin{bmatrix} 0 & x_1+x_2 & x_2+2x_4 & x_2 & x_3 \end{bmatrix}^T$

If $T(x)$ is a linear transformation, it should satisfy following properties

i) $T(x+y) = T(x) + T(y)$
ii) $T(cx) = c\,T(x)$

Now, $T(x) = \begin{bmatrix} 0 \\ x_1+x_2 \\ x_2+2x_4 \\ x_2 \\ x_3 \end{bmatrix}$ and $T(y) = \begin{bmatrix} 0 \\ y_1+y_2 \\ y_2+2y_4 \\ y_2 \\ y_3 \end{bmatrix}$

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$$\therefore \quad T(x) + T(y) = \begin{bmatrix} 0 \\ (x_1+y_1)+(x_2+y_2) \\ (x_2+y_2)+2(x_4+y_4) \\ x_2+y_2 \\ x_3+y_3 \end{bmatrix}$$

and, $$T(x+y) = \begin{bmatrix} 0 \\ (x_1+y_1)+(x_2+y_2) \\ (x_2+y_2)+2(x_4+y_4) \\ x_2+y_2 \\ x_3+y_3 \end{bmatrix}$$

Hence, $T(x+y) = T(x) + T(y)$

Now, $$T(cx) = \begin{bmatrix} 0 \\ cx_1+cx_2 \\ cx_2+2cx_4 \\ cx_2 \\ cx_3 \end{bmatrix} = c\begin{bmatrix} 0 \\ x_1+x_2 \\ x_2+2x_4 \\ x_2 \\ x_3 \end{bmatrix} = c\,T(x)$$

Hence, the given transformation is a linear transformation.

The standard matrix representation for $T(x)$ is A.

Since A is mapping from $\mathbb{R}^4 \rightarrow \mathbb{R}^5$, A will be a 5×4 matrix given by

$$A = \begin{bmatrix} T(e_1) & T(e_2) & T(e_3) & T(e_4) \end{bmatrix}$$

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where e_1, e_2, e_3 and e_4 are respective columns of identity matrix of order 4.

$$T(e_1) = T\begin{bmatrix}1\\0\\0\\0\end{bmatrix} = \begin{bmatrix}0\\1\\0\\0\\0\end{bmatrix} ; \quad T(e_2) = T\begin{bmatrix}0\\1\\0\\0\end{bmatrix} = \begin{bmatrix}0\\1\\1\\1\\0\end{bmatrix}$$

$$T(e_3) = T\begin{bmatrix}0\\0\\1\\0\end{bmatrix} = \begin{bmatrix}0\\0\\0\\0\\1\end{bmatrix} ; \quad T(e_4) = T\begin{bmatrix}0\\0\\0\\1\end{bmatrix} = \begin{bmatrix}0\\0\\2\\0\\0\end{bmatrix}$$

$$\therefore \quad A = \begin{bmatrix}0 & 0 & 0 & 0\\ 1 & 1 & 0 & 0\\ 0 & 1 & 0 & 2\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\end{bmatrix}.$$

The standard matrix representation for $T(x)$ is given as A.

3 b) $T\left(\begin{bmatrix}x_1 & x_2 & x_3 & x_4\end{bmatrix}^T\right) = \begin{bmatrix}2x_1 + 4x_4\end{bmatrix}^T$

$$T(x) = \begin{bmatrix}2x_1 + 4x_4\end{bmatrix}$$
$$T(y) = \begin{bmatrix}2y_1 + 4y_4\end{bmatrix}$$

$$T(x) + T(y) = \begin{bmatrix}2(x_1+y_1) + 4(x_4+y_4)\end{bmatrix}$$

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and, $T(x+y) = \left[2(x_1+y_1) + 4(x_4+y_4) \right]$

$\therefore \quad T(x+y) = T(x) + T(y)$

Now, $T(cx) = \left[2cx_1 + 4cx_4 \right]$
$= c\left[2x_1 + 4x_4 \right]$
$= c\, T(x)$

Hence, the given transformation is a linear transformation.
The standard matrix representation for $T(x)$ is A.

Since T is mapping $R^4 \to R^1$, the order of matrix A will be 1×4, such that,

$$A = \left[T(e_1) \quad T(e_2) \quad T(e_3) \quad T(e_4) \right]$$

where e_1, e_2, e_3 and e_4 are columns of order 4 identity matrix.

$T(e_1) = T\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = [2]$; $T(e_2) = T\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = [0]$

$T(e_3) = T\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} = [0]$; $T(e_4) = T\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = [4]$

$$\therefore \quad A = \begin{bmatrix} 2 & 0 & 0 & 4 \end{bmatrix}$$

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the standard matrix representation for T is
$A = [2 \quad 0 \quad 0 \quad 4]$.

4. How many rows and columns must a matrix A have in order to define a linear transformation from R^4 to R^6 by the rule $T(x) = A(x)$

Since, $T : R^4 \to R^6$, therefore the matrix x is a 4×1 matrix and matrix Ax is a 6×1 matrix. Let $Ax = b$

$$A_{m \times n} \, x_{4 \times 1} = b_{6 \times 1}$$

~~To~~ For matrix A and x are to be multiplied
$\therefore$ n must be equal to 4.
and since b is 6×1 matrix and order of Ax is $m \times 1$,
$\therefore$ $m = 6$.

Hence, **A** will 6 rows and 4 columns.

In general if $T : R^m \to R^n$, then the standard matrix representation of T has order $n \times m$.

5. Let $T : R^2 \to R^2$ be a linear transformation such that $T\left(\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $T\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$.

Find $T\begin{bmatrix} 5 \\ 6 \end{bmatrix}$.

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Let $T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax + by \\ px + qy \end{bmatrix}$

Since, $T\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$a - b = 1$ — (I)
$p - q = 2$ — (II)

and, $T\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$

$a + b = 3$ — (III)
$p + q = 4$ — (IV)

Adding ~~Solving~~ (I) & (III),

$2a = 4$

$\therefore \boxed{\begin{matrix} a = 2 \\ b = 1 \end{matrix}}$

Adding (II) & (IV);

$2p = 6$

$\therefore \boxed{\begin{matrix} p = 3 \\ q = 1 \end{matrix}}$

$\therefore T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2x + y \\ 3x + y \end{bmatrix}$

$\therefore T\begin{bmatrix} 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 16 \\ 21 \end{bmatrix}$

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6. Find standard matrix representation of T given in previous exercise.

Since, T is mapping R^2 to R^2, the standar matrix representation of T will be 2x2 matrix. Let it be A.

$$\therefore \quad A = \begin{bmatrix} T(e_1) & T(e_2) \end{bmatrix}$$

where e_1 and e_2 are columns of order 2 identity matrix.

$$T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2x+y \\ 3x+y \end{bmatrix}$$

$$\therefore \quad T(e_1) = T\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \quad \text{and} \quad T(e_2) = T\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\therefore \quad A = \begin{bmatrix} 2 & 1 \\ 3 & 1 \end{bmatrix}$$

the standard matrix representation of T is given by $A = \begin{bmatrix} 2 & 1 \\ 3 & 1 \end{bmatrix}$.

7. Let $T: R^2 \to R^2$ be the linear transformation that reflects each point through line $X+Y=0$. Find standard matrix representation of T.

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The reflection of a point (x, y) in a line $ax + by + c = 0$ is given by (h, k), such that

$$\frac{h-x}{a} = \frac{k-y}{b} = \frac{-2(ax+by+c)}{a^2+b^2}$$

$\therefore$ reflection of (x,y) in $x + y = 0$ is given by (h,k) such that

$$h - x = k - y = \frac{-\cancel{2}(x+y)}{\cancel{2}}$$

$$\therefore \quad h - x = -(x+y)$$
$$h = -y$$

and,
$$k - y = -(x+y)$$
$$k = -x$$

$$\therefore \quad T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -y \\ -x \end{bmatrix}$$

The standard matrix representation of T will be a 2×2 matrix. Let it be A.

$$\therefore \quad A = [T(e_1) \quad T(e_2)]$$

where e_1 and e_2 are respective columns of order 2 identity matrix.

$$T(e_1) = T\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix} ; \quad T(e_2) = T\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

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$\therefore \quad A = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$

the standard matrix representation for T is

$A = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$

8. Suppose that $\{v_1, v_2, v_3\}$ is a linearly dependent set of vectors in R^n and v_4 is a vector in R^n. Show that $\{v_1, v_2, v_3, v_4\}$ is also a linearly dependent set of vectors.

Since $\{v_1, v_2, v_3\}$ is a set of linearly dependent vectors

$$\therefore \quad av_1 + bv_2 + cv_3 = 0 \qquad \text{—} (1)$$

for some a, b and $c \in R$ such that all of them (a, b, c) are not zero simultaneously.

Now if a set is $\{v_1, v_2, v_3, v_4\}$, then let a combination of the vectors be

$$av_1 + bv_2 + cv_3 + dv_4$$

Now, if we assum $d = 0$,

$$av_1 + bv_2 + cv_3 + dv_4 = 0 \qquad \{\text{from } (1)\}$$

$\therefore$ $\{v_1, v_2, v_3, v_4\}$ will be a linearly dependent vectors. because a, b, c and d are not zeroes simultaneously.

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9. Could a set of five vectors in R^4 be linearly independent? Explain. What about n vectors in R^m if $n > m$.

Let us assume we have five vectors in R^4. The matrix of these vectors will be a 4×5 matrix. ~~The~~ Let us call this matrix A. The vectors will be independent only if $Ax = 0$ has only trivial solutions.
Now, since there are five columns and four rows in A, we can have ~~on~~ a maximum of four pivot entries, i.e. at least one variable will be a free variable.
Hence, solutions other than trivial solutions will exist and the vectors will be dependent.

Hence, five vectors in R^4 will never be linearly independent.
Similarly if there are n vectors in R^m $(n > m)$, we will have a maximum of m ~~pivo~~ pivot entries, i.e. at least $(n-m)$ variables will be free, which will make the n vectors linearly dependent.

10. Given $v \neq 0$ and $p \in R^n$, the line through p in the direction of v has parametric equation $x = p + tv$. Show that a linear transformation $T: R^n \to R^n$ maps this line onto other line or onto a single point.

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$v \neq 0$ and $p \in R^n$

given, $x = p + tv$

Now, T is a linear transformation,

$\therefore \quad T(x) = T(p) + t\,T(v)$

Case-I If $T(v) \neq 0$

$\therefore \quad T(x) = T(p) + t\,T(v)$

$T(p)$ is a different point i.e. transformation of p and $T(v)$ is a line transformed from v, so, x can be transformed onto other line using linear transformation T.

Case-II If $T(v) = 0$,

~~then~~ then,

$$T(x) = T(p)$$

where $T(p)$ is just a transformation of point p. Hence, it is transformed to a single point.