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MA102 : Introduction to Discrete Mathematics

Remote End-Semester Exam

② (i) $F_n = 2^{2^n} + 1 \quad , \quad n \geq 0$

$P : \prod_{r=0}^{r=n-1} F_r = F_n - 2$

Basis step :

~~$F_0 = 0$~~

$F_0 = 3$

$F_1 = 5$

$$\prod_{r=0}^{n-1} F_r =$$

when $n = 1$, $F_0 = 3 = F_1 - 2$

Hence, $P(1)$ is true

Inductive Step :

Let us assume that $P(k)$ is true.

$$\prod_{r=0}^{k-1} F_r = F_k - 2$$

Now, we need to prove that

$$\prod_{r=0}^{k} F_r = F_{k+1} - 2$$

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Using the hypothesis

$$F_0 \cdot F_1 \cdot \ldots \cdot F_{k-1} = F_k - 2$$

Multiplying each side by F_k

$$F_0 \cdot F_1 \cdots F_k = F_k \cdot F_k - 2F_k$$

$$= (2^{2^k}+1)(2^{2^k}+1) - 2(2^{2^k}+1)$$

$$= 2^{2^k} \cdot 2^{2^k} + \cancel{2 \cdot 2^{2^k}} + 1 - \cancel{2 \cdot 2^{2^k}} - 2$$

$$= (2^{2^{k+1}} + 1) - 2$$

$$= F_{k+1} - 2$$

Hence, proved

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3. $a_n = 2a_{n-1} + a_{n-2} - 2a_{n-3}$

the characteristic equation for the given reccurence relation is

$$x^3 - 2x^2 - x + 2 = 0$$

$$(x-2)(x^2-1) = 0$$
$$(x-2)(x-1)(x+1) = 0$$

∴ the solution of the recurrence relation is

$$a_n = \alpha_1 (1)^n + \alpha_2 (-1)^n + \alpha_3 (2)^n$$

Using initial conditions, $a_0 = 3$, $a_1 = 6$ and $a_2 = 0$.

$$3 = a_0 = \alpha_1 + \alpha_2 + \alpha_3$$
$$6 = a_1 = \alpha_1 - \alpha_2 + 2\alpha_3$$
$$0 = a_2 = \alpha_1 + \alpha_2 + 4\alpha_3$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 1 & -1 & 2 & 6 \\ 1 & 1 & 4 & 0 \end{array}\right]$$

$$R_2 \to R_2 - R_1$$
$$R_3 \to R_3 - R_1$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & -2 & 1 & 3 \\ 0 & 0 & 3 & -3 \end{array}\right]$$

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$$\boxed{\alpha_3 = -1}$$

$$-2\alpha_2 + \alpha_3 = 3$$

$$\therefore \boxed{\alpha_2 = \frac{4}{-2} = -2}$$

and, $\alpha_1 + \alpha_2 + \alpha_3 = 3$

$$\boxed{\alpha_1 = 6}$$

$$\boxed{\therefore a_n = 6 + (-2)(-1)^n + (-1)(2^n)}$$

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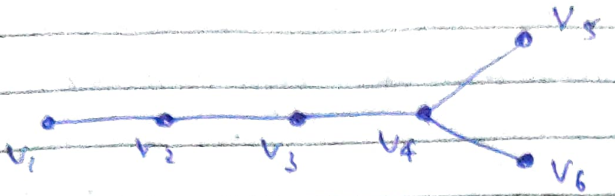
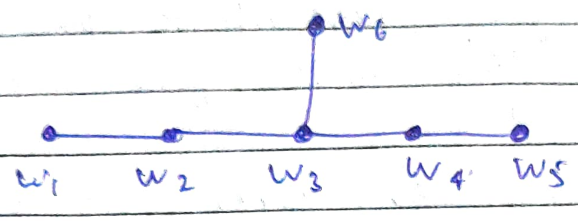
4.



Graph A



Graph B

Both graphs A and B have 6 vertices and 5 edges.
Also, graph A has 2 vertices of degree 2 ($\{v_2, v_3\}$), 3 vertices of degree 1 ($\{v_1, v_5, v_6\}$) and one vertex of degree 3 ($\{v_4\}$). Graph B also has one vertex of degree 3 ($\{w_3\}$), two vertices of degree 2 ($\{w_2, w_4\}$), and three vertices of degree 1 ($\{w_1, w_6, w_5\}$)

Since, v_4 in Graph A is adjacent ~~connected~~ to 2 one degree vertices (v_5, v_6) and one two degree vertex $\{v_3\}$, but in graph B w_3 is ~~connected~~ ~~to~~ adjacent to 2 two degree vertices $\{w_2, w_4\}$ and 1 one degree vertex $\{v_6\}$.
The graphs A and B are not isomorphic.

Q.E.D.

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5. Dividing n by 15 gives 15 possible values of remainder from $\{0, 1, \ldots, 14\}$. Since we have 16 distinct integers, therefore, by Pigeonhole principle at least $\left\lceil \frac{16}{15} \right\rceil$ will give the same remainder when divided by 15.

$$\left\lceil \frac{16}{15} \right\rceil = 2$$

Let a and b be these two numbers and x be the remainder obtained when a and b are divided by 15.

$$\therefore \quad a = 15k + x$$
$$b = 15l + x$$

where k and l are some integers.

$$\therefore \quad a - b = 15(k - l)$$

i.e. $15 \mid (a - b)$.

Hence, in a collection of 16 integers, there exists distinct integers x and y such that 15 divides $(x - y)$.

Q.E.D.

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6. Number of men = 8
Number of women = 5

Firstly we can arrange the 8 men in a line in 8! ways.

_ M_1 _ M_2 _ M_3 _ M_4 _ M_5 _ M_6 _ M_7 _ M_8 _

Now, since no two women should be together, we can select 5 places for women from the 9 spaces seperated by men.

∴ no. of ways to arrange these 5 women in 9 spaces is $^9C_5 \times 5!$.

∴ total no. of ways to arrange 8 men and 5 women in a line so that no two women stand next to each other is

$$\boxed{\begin{aligned} & 8! \times {}^9C_5 \times 5! \\ &= 40320 \times 126 \times 120 \\ &= 609{,}638{,}400 \text{ ways} \end{aligned}}$$

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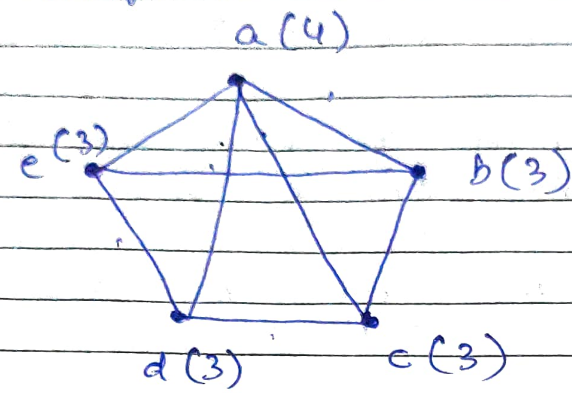
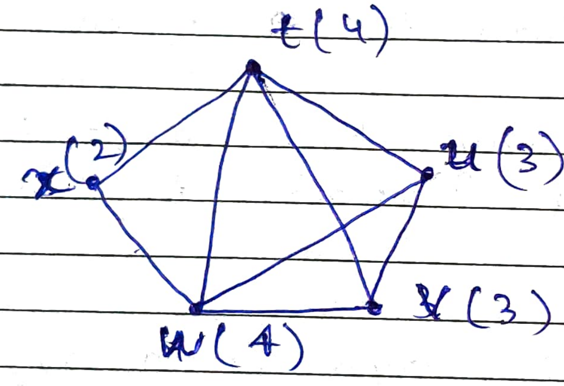
7. Graph A



Graph B



The degree of each vertex is written in parentheses along with vertex name.

Both the graphs i.e. A and B have 5 vertices and 8 edges but A and B are not isomorphic, since B has a vertex of degree 2 {x}, but A has no vertex of degree 2.

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8: given inequality

$$x_1 + x_2 + x_3 \leq 11$$

with $x_1 \geq 0$, $x_2 \geq 0$, $x_3 \geq 0$.

Let us add another variable t such that

$$x_1 + x_2 + x_3 + t = 11 \quad \text{(1)}$$

where, $t \geq 0$

the no. of solutions (non-negative integral) to the eq^n (1) is equal to the number of non-negative solutions of the given inequality.

$\therefore$ no. of solutions $= {}^{14}C_3 = 364$

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9. A number n is divisible by 22, if it an even number and satisfies the divisibility rule of 11 ; which states that the absolute difference between the sum of digits occuring in even positions and sum of digits occuring in odd position should be divisible by 11.

Proof:

Step 1: For divisibility by 2, check if the number is even, simply if it ends with $\{0, 2, 4, 6, 8\}$.

Step 2: Let $N = 10^n a_n + 10^{n-1} a_{n-1} + \cdots + a_0$

Taking mod 11 of N, we get

$$\therefore N \equiv \pm a_n \mp a_{n-1} \pm a_{n-2} \cdots \mp a_2 \pm a_1 \mp a_0 \pmod{11}$$

$$\left(\because 10^k \equiv 1 \bmod 11 \text{ if } k \text{ is even and } 10^k \equiv 10 \bmod 11 \ (10^k \equiv -1 \bmod 11) \text{ when } k \text{ is odd.} \right)$$

Suppose n is even, then

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$$N \equiv a_n - a_{n-1} + a_{n-2} - \cdots + a_2 - a_1 + a_0 \pmod{11}$$

$$\equiv \left[(a_n + a_{n-2} + a_{n-4} + \cdots + a_0) - (a_{n-1} + a_{n-3} + \cdots + a_1)\right] \pmod{11}$$

$\therefore$ $N \equiv 0 \pmod{11}$ iff

$$(a_n + a_{n-2} + a_{n-4} + \cdots + a_0) \equiv [a_{n-1} + a_{n-3} + \cdots + a_1] \pmod{11}$$

Thus, the absolute difference b/w sum of digits at even places and sum of digits at odd places must be divisible by 11 for the number to be divisible by 11.

- If the number is divisible by both 2 and 11, it is divisible by 22.

(1) My Student ID → 202052307

$$202052307 = 22(9184195) + 17$$

Hence, remainder is 17.

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10. Let ~~us~~ model a graph directed from ~~from~~ u to v iff u sent v an ~~letter~~ card.

Now, if the card by a ~~each~~ student A is sent ~~card~~ to other student B and B sent ~~A~~ card to A.

Then, there exists two edges between A and B.

If each student receives card from the same three students to whom he or she has sent card, then there must be even number of edges in the graph.

But, the no. of edges in the graph are $9.3 = 27$ and hence it is not possible that each student receives cards from the same three students to whom he or she has sent cards