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PH110: Waves and Electromagnetics

Tutorial 13

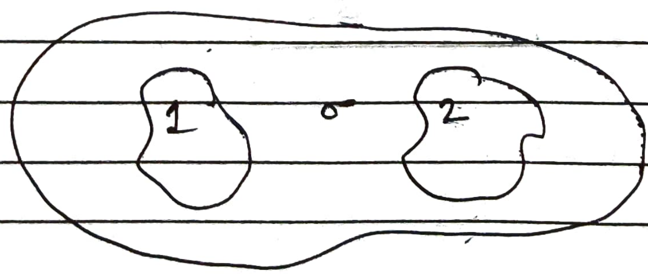
Problem 7.3 (a) Two metal objects are embedded in weakly conducting material of conductivity σ. Show that the resistance between them is related to the capacitance of the arrangement by

$$R = \frac{\varepsilon_0}{\sigma C}$$

(b) Suppose you connected a battery between 1 and 2, and charged them upto a potential difference V_0. If you disconnect the battery, the charge will gradually leak off. Show that

$$V(t) = V_0 e^{-t/\tau}$$

and find the time constant, τ, in terms of ε_0 and σ.



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(a) $\because \quad I = \int J \cdot da$

This integral is taken over a surface enclosing the positively charged conductor.

But, $\quad J = \sigma E \quad$ —(I)

and Gauss's law states

$$\int E \cdot da = \frac{Q}{\epsilon_0} \quad \text{—(II)}$$

$$\therefore \quad I = \int (\sigma E) \cdot da \quad (\text{from } (I))$$

$$= \sigma \int E \cdot da$$

$$= \frac{\sigma Q}{\epsilon_0} \quad (\text{— from } (II))$$

$\because \quad Q = CV$
and $V = IR$

$$\therefore \quad \cancel{I} = \frac{\sigma}{\epsilon_0} (C \cancel{I} R)$$

or, $\boxed{R = \frac{\epsilon_0}{\sigma C}}$

Hence, proved.

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(b) $\because$ $Q = CV$

$Q = CIR$

$I = \frac{Q}{RC}$

$-I = \frac{-Q}{RC}$

$\therefore \frac{dQ}{dt} = \frac{-Q}{RC}$

Integrating both sides from 0 to t

$$\int_{Q_0}^{Q} \frac{dQ}{Q} = -\int_{0}^{t} \frac{dt}{RC}$$

$$\ln \frac{Q}{Q_0} = \frac{-t}{RC}$$

$$Q = Q_0 e^{-t/RC}$$

$\because$ $V = \frac{Q}{C}$

$$\boxed{\therefore \quad V(t) = V_0 e^{-t/RC}}$$

The time constant is $\tau = RC = \frac{\varepsilon_0}{\sigma}$

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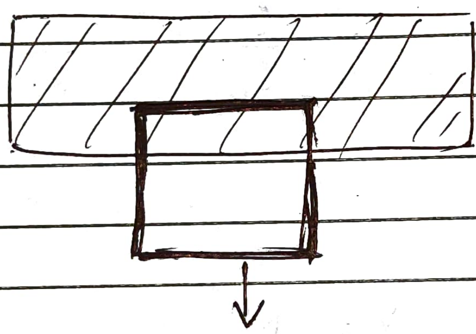
7.11 A square loop is cut out of a thick sheet of aluminium. It is then placed so that the top portion is in a uniform magnetic field B, and is allowed to fall under gravity. If the magnetic field is 1T, find the terminal velocity of the loop. Find the velocity of loop as a function of time. How long does it take to reach, say 90%, of the terminal velocity? What would happen if you cut a tiny slit in the ring, breaking the circuit?



the Induced emf will be

$$\mathcal{E} = B l v$$

According to Ohm's law,

$$B l v = IR$$

$$\therefore I = \frac{B l v}{R}$$

$\therefore$ upward magnetic force $= I l B$

$$= \frac{B^2 l^2 v}{R}$$

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This opposes the magnetic field downward:

$$mg - \frac{B^2 l^2 v}{R} = m\frac{dv}{dt} \quad \text{--- (1)}$$

$$\frac{dv}{dt} = g - \alpha v \quad , \text{where } \alpha = \frac{B^2 l^2}{mR}$$

at terminal velocity $\frac{dv}{dt} = 0$

i.e. $g - \alpha v_t = 0$

$$\boxed{\therefore v_t = \frac{g}{\alpha} = \frac{mgR}{B^2 l^2}}$$

$\because$ $\frac{dv}{dt} = g - \alpha v$

$$\int \frac{dv}{g-\alpha v} = \int dt$$

$$-\frac{1}{\alpha} \ln(g - \alpha v) = t + c$$ (c is integration constant)

$$g - \alpha v = Ae^{-\alpha t}$$

$\because$ at $t = 0$, $v = 0$

$\therefore A = g$

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$$\therefore \quad \alpha v = g\left(1-e^{-\alpha t}\right)$$

$$\therefore \quad v = \frac{g}{\alpha}\left(1-e^{-\alpha t}\right)$$

$$\boxed{v = v_t\left(1-e^{-\alpha t}\right)}$$

At 90% of terminal velocity, $\frac{v}{v_t} = 0.9$

$$\therefore \quad 0.9 = 1 - e^{-\alpha t}$$

$$e^{-\alpha t} = 0.1$$

$$-\alpha t = \ln(0.1)$$

$$\alpha t = \ln(10)$$

$$\therefore \quad t = \frac{\ln(10)}{\alpha}$$

$$\boxed{\text{or, } \quad t_{90\%} = \frac{v_t}{g}\ln 10}$$

Now,

$$m = 4\eta A l$$

$m \to$ mass density of Al

$A \to$ cross-section area

$l \to$ length of side

$R = \frac{4l}{A\sigma}$, σ is conductivity of Al

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$$v_t = \frac{4 \eta A l g \, 4 l}{A \sigma B^2 l^2}$$

$$= \frac{16 \eta g}{\sigma B^2}$$

$$= \frac{16 g \eta \rho}{B^2}$$

$$\therefore \quad v_t = \frac{16 \times 9.8 \times 2.7 \times 10^3 \times 2.8 \times 10^{-8}}{1}$$

$$\boxed{v_t = 1.2 \text{ cm/s}}$$

$$\therefore \quad t_{90\%} = \frac{1.2 \text{ cm/s}}{9.8} \ln(10)$$

$$\boxed{t_{90\%} = 2.8 \text{ ms}}$$

If the loop will cut, there will be no current flow, hence no upward magnetic force. Hence the loop will fall freely with acceleration g.

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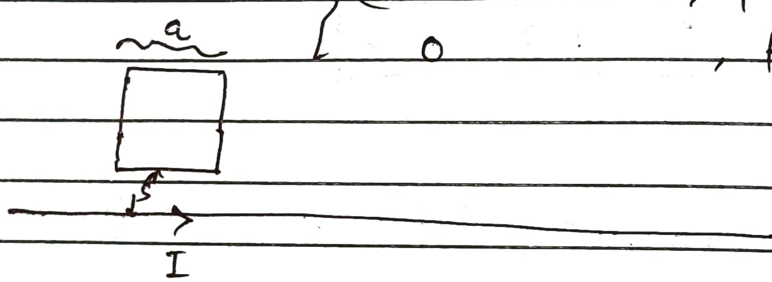
7.18 A square loop, side a, resistance R, lies a distance s from an infinite straight wire that carries current I. Now someone cuts the wire, so I drops to zero. In what direction does the induced current in the square loop flow and what total charge passes a given point in the loop during the time this current flows

$$I(t) = \begin{cases} (1-\alpha t) I & , \text{for } 0 \le t \le \frac{1}{\alpha} \\ 0 & , \text{for } t \ge \frac{1}{\alpha} \end{cases}$$



$\because$ $\phi = \int B \cdot da$

$$B = \frac{\mu_0 I}{2\pi s} \hat{\phi}$$

$\therefore$ $\phi = \frac{\mu_0 I a}{2\pi} \int_{s}^{s+a} \frac{ds}{s}$

$$\phi = \frac{\mu_0 I a}{2\pi} \ln\left(\frac{s+a}{a}\right)$$

$\because$ $\mathcal{E} = IR = \frac{dQ}{dt} R$

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$$\because \quad \varepsilon = -\frac{d\phi}{dt}$$

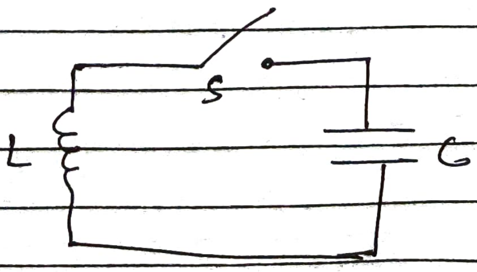
$$\therefore \quad \frac{dQ}{dt} \cdot R = -\frac{\mu_0 a}{2\pi} \ln\left(\frac{s+a}{a}\right)\frac{dI}{dt}$$

$$dQ = -\frac{\mu_0 a}{2\pi R} \ln\left(\frac{s+a}{a}\right) dI$$

$$\Rightarrow \quad \boxed{Q = -\frac{\mu_0 a}{2\pi R} \ln\left(\frac{s+a}{a}\right) I}$$

The field of the wire, at the square loop, is out of the page, and decreasing, so the field of the induced current must point out of the page, within the loop, and hence induced current flows counterclockwise.

7.27 A capacitor C is charged up to a voltage V and connected to an inductor L, as shown. At time t = 0, the switch S is closed. Find the current in the circuit as a function of time. How does your answer change if a resistor R is included in series with C and L?



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With I positive clockwise,

$$\mathcal{E} = -L\frac{dI}{dt} = \frac{Q}{C}$$

where Q is charge on capacitor.

$$\because \quad I = \frac{dQ}{dt},$$

so, $$\frac{d^2Q}{dt^2} = -\frac{Q}{LC} = -\omega^2 Q, \text{ where } \omega = \frac{1}{\sqrt{LC}}$$

The general solution to the above differential equation is

$$Q(t) = A\cos\omega t + B\sin\omega t$$

at $t=0$, $Q = CV$

$$\therefore \quad CV = A$$

$$\frac{dQ}{dt} = I(t) = -A\omega\sin\omega t + B\omega\cos\omega t$$

At $t=0$, $I=0$

$$\therefore \quad B = 0$$

$$\therefore \quad I(t) = -CV\omega\sin\omega t$$

$$\boxed{I(t) = -V\sqrt{\frac{C}{L}}\sin\left(\frac{t}{\sqrt{LC}}\right)}$$

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If you put a resistor, the oscillation is damped. This time

$$-L\frac{dI}{dt} = \frac{Q}{C} + \frac{I}{R}$$

So, $$L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{Q}{C} = 0.$$

7.33. An infinite cylinder of radius R carries a uniform surface charge σ. We propose to set it spinning about its axis, at an angular velocity ω_f. How much work will this take, per unit length? Do it two ways and compare your answers.

(a) Find the magnetic field and induced electric field, inside and outside the cylinder in terms of ω, $\dot{\omega}$ and s. Calculate the torque you must exert, and from that obtain the work done per unit length ($W = \int N\, d\phi$)

(b) Use Eq 7.31 to determine the energy stored in the resulting magnetic field.

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(a) The magnetic field is

$$B = \begin{cases} \mu_0 K \hat{z} = \mu_0 \sigma \omega R \hat{z} & (s<R) \\ 0 & (s>R) \end{cases}$$

the electric field is,

$$E = \begin{cases} -\frac{s}{2} \frac{dB}{dt} \hat{\phi} = -\frac{sR \mu_0 \sigma \dot{\omega}}{2} \hat{\phi} & (s<R) \\ -\frac{R^2}{2s} \frac{dB}{dt} \hat{\phi} = -\frac{R^3}{2s} \mu_0 \sigma \dot{\omega} \hat{\phi} & (s>R) \end{cases}$$

At the surface $(s = R)$ $E = -\frac{1}{2} \mu_0 R^2 \sigma \dot{\omega} \hat{\phi}$, so the torque on a length l of the cylinder is

$$N = -R(\sigma 2\pi R l)\left(\frac{1}{2} \mu_0 R^2 \sigma \dot{\omega}\right) \hat{z}$$

$$= -\pi \mu_0 \sigma^2 R^4 \dot{\omega} l \hat{z},$$

and the work done per unit length is,

$$\frac{W}{l} = -\pi \mu_0 \sigma^2 R^4 \int \frac{d\omega}{dt} d\phi.$$

But $d\phi = -\omega dt$, and the integral becomes

$$\int_0^{\omega_f} \omega \, d\omega = \frac{1}{2} \omega_f^2$$

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$$\therefore \boxed{\frac{W}{l} = -\frac{\mu_0 \pi}{2}(\sigma \omega_f R^2)^2}.$$

This is the work done by the field, the work you must do does not include the minus sign.

(b) Because $B = \mu_0 K \hat{z} = \mu_0 \sigma \omega_f R \hat{z}$ is uniform inside the solenoid (and zero outside).

$$W = \frac{1}{2\mu_0} B^2 \pi R^2 l.$$

$$\therefore \frac{W}{l} = \frac{1}{2\mu_0}(\mu_0 \sigma \omega_f R)^2 \pi R^2$$

$$\boxed{\frac{W}{l} = \frac{\mu_0 \pi}{2}(\sigma \omega_f R^2)^2}.$$

7.35 The preceding problem was an artificial model for the charging capacitor, designed to avoid complications associated with the current spreading out over the surface of the plates. For a more realistic model, imagine thin wires that connect to the centres of the plates. Again, the current I is constant, the radius of the capacitor is a, and the seperation of the plates is $w << a$. Assume that the current flows out over the plates in such a way that the surface charge is uniform, at any given time, and is zero at $t=0$.

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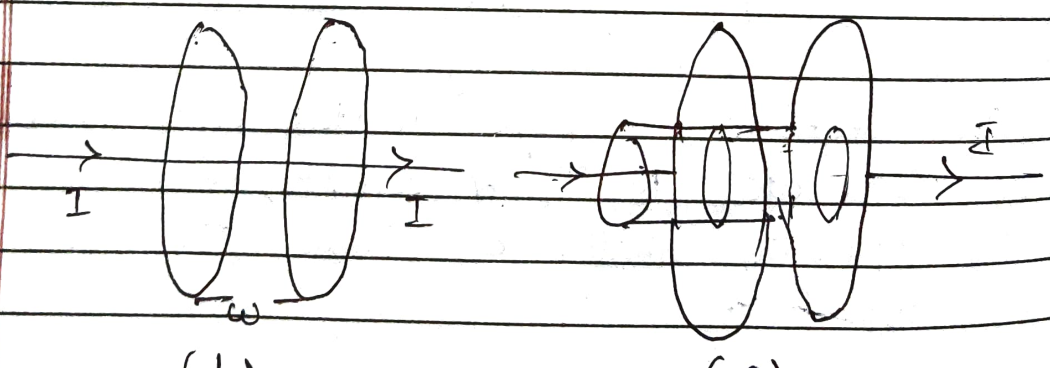
(a) Find the electric field between the plates as a function of t?

(b) Find the displacement current through the circle of radius s in the plane midway between the plates. Using this circle as your "Amperian Loop", and the flat surface that spans it, find the magnetic field at a distance s from the axis.

(c) Repeat part (b), but this time use the cylindrical surface given, which is open at right end and extends to the left through the plate and terminates outside the capacitor. Note that the displacement current through this surface is zero, and there are two contributions to I_{enc}.



(b) (c)

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(a) $$E = \frac{\sigma(t)}{\epsilon_0}\hat{z}$$

$$\sigma(t) = \frac{Q(t)}{\pi a^2} = \frac{It}{\pi a^2}$$

$$\boxed{\therefore \quad E = \frac{It}{\pi \epsilon_0 a^2}\hat{z}}$$

(b) $$I_{denc} = J_d \pi s^2$$

$$= \epsilon_0 \frac{dE}{dt}\pi s^2$$

$$= \frac{I s^2}{a^2}$$

Now, $$\oint B \cdot dl = \mu_0 I_{denc}$$

$$B(2\pi s) = \mu_0 I \frac{s^2}{a^2}$$

$$\boxed{B = \frac{\mu_0 I s}{2\pi a^2}\hat{\phi}}$$

(c) A surface current flows radially outwards over the left ~~surface~~ plate; let $I(s)$ be the total current crossing a circle of radius s.

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The charge density (at time t) is

$$\sigma(t) = \frac{[I - I(s)]\,t}{\pi s^2}$$

Since, we are told this is independent of s, it must be that $I - I(s) = \beta s^2$, for some constant β. But $I(a) = 0$, so

$$\beta a^2 = I$$

$$\therefore \beta = \frac{I}{a^2}$$

Therefore, $I(s) = I\left(1 - \frac{s^2}{a^2}\right)$

$$B \cdot 2\pi s = \mu_0 I_{enc}$$

$$= \mu_0 (I - I(s))$$

$$= \mu_0 I \frac{s^2}{a^2}$$

$$\boxed{\therefore B = \frac{\mu_0 I s}{2\pi a^2}\,\hat{\phi}}$$

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7.37 Suppose

$$E(r,t) = \frac{1}{4\pi\epsilon_0} \frac{q}{r^2} \theta(vt - r)\hat{r} \, ; \quad B(r,t) = 0$$

(The theta function is defined in Prob. 1.46 b). Show that these fields satisfy all of Maxwell's equations, and determine ρ and J. Describe the physical situation that gives rise to these fields.

Physically, this is the field of a point charge $-q$ at the origin, out to an expanding spherical shell of radius vt; outside this shell the field is zero. Evidently the shell carries the opposite charge, $+q$. Mathematically, using product rule and Eq 1.99,

$$\nabla \cdot E = \theta(vt - r)\, \nabla \cdot \left(\frac{-1}{4\pi\epsilon_0} \frac{q}{r^2} \hat{r} \right) - \frac{q}{4\pi\epsilon_0 r^2} \hat{r} \cdot \nabla [\theta(vt-$$

$$= \frac{-q}{\epsilon_0} \delta^3(r)\, \theta(vt - r) - \frac{q}{4\pi\epsilon_0 r^2} (\hat{r} \cdot \hat{r}) \frac{\partial}{\partial r} \theta(vt-$$

But $\delta^3(r)\,\theta(vt - r) = \delta^3(r)\,\theta(t)$, and $\frac{\partial}{\partial r}\theta(vt - r) = -\delta(vt - r)$. So,

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$$\rho = \epsilon_0 \nabla \cdot E$$

$$\rho = -q\,\delta^3(r)\,\theta(t) + \frac{q}{4\pi r^2}\,\delta(vt - r)$$

(for $t < 0$, the field and the charge density are zero everywhere).

Clearly $\nabla \cdot B = 0$, and $\nabla \times E = 0$ (since E has only an r component, and it is independent of θ and ϕ). There remains only the Ampere/Maxwell law, $\nabla \times B = 0 = \mu_0 J + \mu_0 \epsilon_0 \frac{\partial E}{\partial t}$.

Evidently,

$$J = -\epsilon_0 \frac{\partial E}{\partial t} = -\epsilon_0 \left\{ -\frac{q}{4\pi\epsilon_0 r^2} \frac{\partial}{\partial t}\left[\theta(vt - r)\right]\right\} \hat{r}$$

$$\boxed{J = \frac{q}{4\pi r^2}\, v\,\delta(vt - r)\,\hat{r}.}$$

The stationary charge at the origin does not contribute to J, of course; for the expanding shall we have $J = \rho v$, as expected.