Experiment 10

Resonance in LCR Circuits

Apparatus

Oscillator (1 to 1 M Hz), resistors, capacitors, inductors, AC milli-ammeter.

Purpose of Experiment

To study resonance effect in series and parallel LCR circuits.

Basic methodology

In the series LCR circuit, an inductor (L), a capacitor (C) and a resistor (R) are connected in series with a variable frequency sinusoidal emf source and the voltage across the resistance is measured. As the frequency is varied, the current in the circuit (and hence the voltage across R) changes, becoming maximum at the resonance frequency $\nu_0 = 1/(2\pi\sqrt{LC})$. In the parallel LCR circuit, the current is minimum at the resonance frequency.

I Theory

There is in general an analogy between resonating mechanical systems (like a driven spring-mass system) and electrical systems involving inductors, resistors and capacitors. In the electrical case, it is the charge $q(t)$ on the capacitor (or the current $I = dq/dt$) that satisfies a differential equation analogous to the displacement of the mass in the familiar spring mass system.

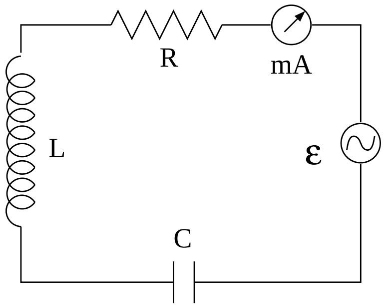

Figure 1: Circuit with L, C and R in series.

Consider the circuit Fig.1, consisting of an inductor (L), capacitor (C) and resistance (R) connected in series with a source of sinusoidally varying emf $\mathcal{E}(t) = \mathcal{E}_0 \cos(\omega t)$. Equating the voltage drops across the resistor and capacitor to the total emf, we get,

$$RI + \frac{q}{C} = V_L + \mathcal{E}_0 \cos(\omega t) = -L\frac{dI}{dt} + \mathcal{E}_0 \cos(\omega t) \quad . \tag{1}$$

Differentiating the equation with respect to time and rearranging, we get

$$L\frac{d^2 I}{dt^2} + R\frac{dI}{dt} + \frac{I}{C} = -\omega\mathcal{E}_0 \sin(\omega t) \quad , \tag{2}$$

which is analogous to the equation of motion for a forced damped oscillator.

The current $I(t)$ has the solution

$$I(t) = I_0 \cos(\omega t - \alpha) \quad . \tag{3}$$

where, I_0 exhibits resonance behaviour. The amplitude I_0 is given by

$$I_0 = \frac{\mathcal{E}_0}{\left[R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2\right]^{\frac{1}{2}}} \tag{4}$$

and

$$\tan\alpha = \frac{\omega L - \frac{1}{\omega C}}{R} \tag{5}$$

gives the phase of the current relative to applied emf. We can write $I_0 = \mathcal{E}_0/Z$, where

$$Z = \left[R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2\right]^{\frac{1}{2}} \tag{6}$$

is the impedance of the circuit. The reactance X of the circuit is

$$X = \omega L - \frac{1}{\omega C} \tag{7}$$

so that the impedance Z is given by

$$Z = (R^2 + X^2)^{\frac{1}{2}}$$

Clearly the impedance will be minimum (and I_0 will be maximum) at resonance, when the reactance vanishes. This happens at the angular frequency (known as resonance frequency)

$$\omega_0 = \frac{1}{\sqrt{LC}} \tag{8}$$

which is the natural frequency of electromagnetic oscillations in LCR circuit without an external source of emf.

Resistance, Capacitance and Inductance in AC circuits: Consider a resistor with a voltage drop $V_R = V_{R0}\cos(\omega t)$ across it (Fig. 2a). By Ohm's law the current through the resistor is

$$I_R = \frac{V_R}{R} = \frac{V_{R0}}{R}\cos(\omega t) \tag{9}$$

The current and voltage across a resistor are in phase.

In the case of a capacitor (Fig. 2b), the current $I_c = dq/dt$, where q is the charge on the capacitor. If the potential drop across the capacitor is $V_C = V_{C0}\cos(\omega t)$, the charge $q = CV_C = CV_{C0}\cos(\omega t)$. Then,

$$I_c = -\omega C V_{C0}\cos\left(\omega t + \frac{\pi}{2}\right) \tag{10}$$

Thus, the current through the capacitor is ahead of the voltage by phase angle $\pi/2$.

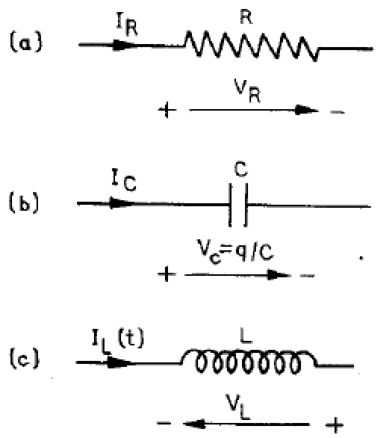

Figure 2: Voltage drop across (a) R, (b) C and (c) L.

Consider now an inductor (Fig. 2c) with current $I_L(t) = I_{L0}\cos(\omega t)$. Assume that the current flows and increases in the direction shown. The back emf induced in the inductor opposes the current and the potential drop across the inductor is

$$V_L = L\frac{dI}{dt} = -\omega L I_{L0}\sin(\omega t) = \omega L I_{L0}\cos\left(\omega t + \frac{\pi}{2}\right) \tag{11}$$

The voltage across the inductor is ahead of the current in phase by an angle $\pi/2$.

Complex Impedance: It is convenient to use complex phasors to represent the current and voltage in an AC circuit. For example, the phasor $\bar{V} = V_0 e^{j\omega t} = V_0\left(\cos(\omega t) + j\sin(\omega t)\right)$

represents a sinusoidally varying voltage $V_0 \cos(\omega t)$, which is its real part. For any component A we define its complex impedance by $\bar{V}_A = \bar{Z}_A \bar{I}_A$. We write

$$\bar{Z} = R + jX$$

where the real part of $\bar{Z}$ is the resistive impedance (R), while the imaginary part of is the reactive impedance (X).

The complex impedances of the resistor, capacitor and inductor can be obtained by generalizing eqs. (9), (10) and (11) to phasor equations:

$$\bar{I}_R = \frac{I}{R}\bar{V}_R \quad \Rightarrow \quad \bar{Z}_R = R \tag{12}$$

$$\bar{I}_C = \omega L V_{C0} e^{j\left(\omega t + \frac{\pi}{2}\right)} = j\omega C \bar{V}_C \quad \Rightarrow \quad \bar{Z}_C = \frac{1}{j\omega C} \tag{13}$$

$$\bar{V}_L = \omega L I_{L0} e^{j\left(\omega t + \frac{\pi}{2}\right)} = j\omega L I_L \quad \Rightarrow \quad \bar{Z}_L = j\omega L \tag{14}$$

Thus, the impedance of a resistor is its resistance itself, while the impedance of a capacitor and inductance are reactive with $X_C = -1/(\omega C)$ and $X_L = \omega L$.

It can be shown from Kirchhoff's rules that complex impedances in series or parallel combine just like resistors in series or parallel. Thus, for the series LCR circuit Fig 1, the net impedance of the circuit is

$$\bar{Z} = \bar{Z}_R + \bar{Z}_C + \bar{Z}_L = R + j\left(\omega L - \frac{1}{\omega C}\right) \tag{15}$$

The current in the circuit is then

$$\bar{I} = \frac{\bar{\mathcal{E}}}{\bar{Z}} = \frac{\mathcal{E}_0 e^{j\omega t}}{R + j\left(\omega L - \frac{1}{\omega C}\right)} = I_0 e^{j(\omega t - \alpha)} \tag{16}$$

From eq. (16) it can be easily seen that

$$I_0 = \frac{\mathcal{E}_0}{\left[R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2\right]^{\frac{1}{2}}} = \Re(\bar{I})$$

and

$$\alpha = \tan^{-1}\left(\frac{\omega L - \frac{1}{\omega C}}{R}\right)$$

which clearly reproduce eqs. (4) and (5). The physical current in the circuit is, of course, the real part of the phasor $\bar{I}$ in eq (16).

Parallel LCR circuit: Consider now the parallel LCR circuit shown in Fig. 3. The current through the resistor can be found by calculating the equivalent impedance of the circuit

$$\bar{Z} = \bar{Z}_R + \frac{1}{\frac{1}{\bar{Z}_C} + \frac{1}{\bar{Z}_L}} = R + \frac{\bar{Z}_L \bar{Z}_C}{\bar{Z}_L + \bar{Z}_C} = R - j\frac{L/C}{\omega L - \frac{1}{\omega C}} \tag{17}$$

Thus

$$\bar{I} = \frac{\bar{\mathcal{E}}}{\bar{Z}} = \frac{\mathcal{E}_0 e^{j\omega t}}{R - j\left(\frac{L/C}{\omega L - \frac{1}{\omega C}}\right)} = I_0 e^{j\omega t + \phi} \tag{18}$$

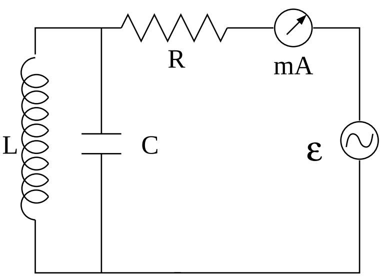

Figure 3: Parallel LCR circuit.

The magnitude of the current I_0 is given by

$$I_0 = \frac{\mathcal{E}_0}{\sqrt{R^2 + \left(\frac{L/C}{\omega L - \frac{1}{\omega C}}\right)^2}} \tag{19}$$

Viewed as a function of ω, it is clear that I_0 is now a minimum (the impedance in the denominator is maximum) when $\omega L = 1/(\omega C)$, or, where

$$\omega = \frac{1}{\sqrt{LC}} = \omega_0 \tag{20}$$

and is known as **resonance frequency** even through it corresponds to an amplitude minimum. (Note: The amplitude of the current in eq. (19) strictly falls to zero at $\omega = 1/(\sqrt{LC})$ since the denominator tends to infinity. This is because we have considered idealized (i.e. resistance-less) capacitor and inductor. A finite value of the current amplitude at resonance will be obtained if resistive impedance is included for these components).

Power Resonance: The power dissipated at the resistor is $P = IV = I^2R = V^2/R$. From eq. (3) for the series resonance circuit, the power dissipated at the resistor is

$$P = I_0^2 R \cos^2(\omega t - \alpha) \ , \tag{21}$$

where I_0 is given by eq. (4). The average power dissipated over one cycle is

$$\bar{P} = \frac{I_0^2 R}{2} = \frac{\mathcal{E}_0^2 R}{2\left[R^2 + \left(\frac{L/C}{\omega L - \frac{1}{\omega C}}\right)^2\right]} \tag{22}$$

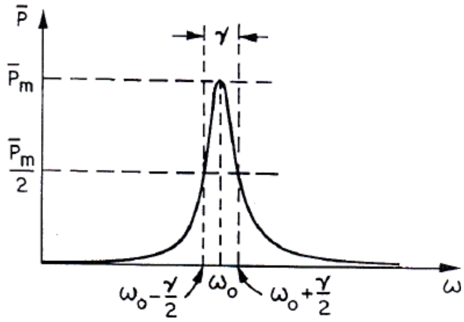

Figure 4: Average power dissipation as function of driving frequency.

Figure 4 shows a graph of $\bar{P}$ as a function of the driving frequency ω. Maximum power value, $\bar{P}_m$, occurs at the resonance frequency $\omega_0 = 1/\left(\sqrt{LC}\right)$. It can be shown that to a good approximation, that the power falls to half the maximum value, $\bar{P}_m/2$ at $\omega = \omega_0 \pm \gamma/2$. Here γ is related to damping in the electrical circuit and is given by $\gamma = R/L$.

The width or range of ω over which the value of $\bar{P}$ falls to half the maximum value at resonance is called the Full Width at Half Maximum (FWHM). The FWHM is a characteristic of the power resonance curve and is related to the amount of damping in the system. Clearly FWHM $= \gamma = R/L$. One also define the quality factor Q as $Q = \omega_0/\gamma = 1/R\left(\sqrt{L/C}\right)$ which is also a measure of damping. Large Q (small R) implies small damping while small Q (large R) implies large damping. Clearly we have

$$\text{FWHM} = \gamma = \frac{\omega_0}{Q} \tag{23}$$

Thus, the quality factor Q can be determined from the FWHM of the power resonance graph.

II Setup and Procedure

1. Connect the series LCR circuits, as shown Fig.1. Choose an inductance L and a capacitance C so that the resonance frequency $\nu_0 = 1/\left(2\pi\sqrt{LC}\right)$ is of the order of a few kHz.

2. Vary the frequency of the oscillator in steps and record the voltage V_R across the resistor.

3. Repeat (for both series and parallel LCR circuits) for three different values of the resistor.

III Precautions

1. Calculate the expected resonance frequancy before beginnning to take readings, and make sure to take sufficinet number of readings in the vicinity ofthis frequency to be able to draw the shape of the resonance curve.

2. Make sure the range of readings you take is sufficient to go beyond the half-power points on both sides of resonance.

IV Exercies and Viva Questions.

1. Write down the Newton's law for a forced damped harmonic oscillator and map the electrical quantities appearing in eq. (2) with the corresponding mechanical quantities.

2. Verify that the solution, eq. (3) satisfies the differential equation (2).

3. Distinguish between resistive impedance and reactive impedance. What is the effect of a reactive impedance on the current and voltage in an AC circuit? In a DC circuit?

4. Calculate the (resistive or reactive) impedance of the components L, C, and R at resonance for series and parallel circuits, for your experiment.

5. Why does the series circuit give a power maximum at resonance while the parallel circuit lead to a power minimum?

6. For the circuit shown in Fig 5, with emf $\mathcal{E}(t) = \mathcal{E}_0 \cos(\omega t)$, determine the current $I(t) = I_0 \cos(\omega t - \alpha)$. (i.e. determine the amplitude I_0 and phase α.)

7. The AC millivoltmeter gives the 'rms' value of the voltage across the resistor, i.e Vrms. If $V = V_0 \cos(\omega t)$, what is V_{rms}? Show that the average power $\bar{P} = V_{rms}^2/R$..

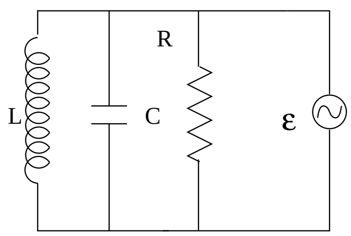

Figure 5

8. Show that eq. (22) can be rewritten as

$$\bar{P} = \frac{\mathcal{E}_0^2 RL}{2C} \frac{1}{\left[Q^2 + \left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}\right)^2\right]}$$

9. Qualitatively plot the power resonance curve for increasing values of Q. Show that the FWHM of the power resonance curve is approximately given by $\gamma = \omega_0/Q$.

10. Argue why the power maximum (minimum) for the series (parallel) LCR circuit increases (decreases) with increasing R.

References

1. *Physics*, M. Alonso and E.J. Finn, Addisson-Wesley, 1992.

2. *Linear Circuits*, M. E. Van Valkenburg and B.K. Kinariwala, Printice Hall, Englewood Cliffs, NJ, 1982.