Experiment 0

Error Analysis

I Introduction

Error analysis is the study and evaluation of errors in an experiment. *Experience has shown that no measurement, howsoever carefully made can be completely free of errors.* Since the whole structure of science depends on measurements, the ability to evaluate these uncertainties and keep them to the minimum is crucially important.

II Uncertainty in measurement

II.1 Inevitability of uncertainty

It is impossible to obtain the *true* value of any physical quantity. As an example, let us set out to measure the *true* diameter of a wire.

We can choose amongst a variety of techniques and measuring instruments to perform this task. As a first crude estimate, we might select a meter scale and observe the diameter to be 2mm. The stated value of 2mm means that the diameter appears to be closer to 2mm than to any other millimeter ruling on the scale. Clearly, the measurement is subject to an uncertainty. To get a more accurate value we might use vernier callipers and find the diameter to be 2.6mm. This measurement, though closer to the true value is not free of uncertainty. If we use a screw gauge we may find that the diameter is 2.63mm which is an improvement in our estimate of the '*true*' value of the diameter. However, it is not the true value, for the true value could lie anywhere between 2.625mm and 2.635mm. We need to carry out a more accurate observation. An experienced physicist might choose an inteferometry technique (that uses light diffraction). Using this technique, let's assume that we find the diameter to be 2.6358mm. However, the diameter could be slightly larger or smaller than 2.6358mm. We have not got rid of errors. Starting from the meter-scale to the interferometry technique, we made good progress in our effort to find the *true* value of the wire's diameter. However, we were unable to remove uncertainty from our measurement.

Invention of better techniques would allow us to get closer to the '*true'* value of the diameter. However, there will always be an uncertainty, howsoever small.

II.2 Importance of knowing uncertainty

The knowledge of uncertainty in measurement is crucial for engineering tasks and scientific goals.

In basic sciences, error analysis plays a fundamental role. When any new theory is proposed it must be tested against older theories by means of one or more experiments for which the new and the old theory predict different results. In principle, a researcher can simply perform an experiment and can let the outcome decide between rival theories. In practice, however, the situation is complicated by experimental uncertainties.

A famous example of such a test of a scientific theory is the measurement of bending of light as it passes near the sun. When Einstein published his General Theory of Relativity in 1916, he pointed out that the theory predicted that light from a star would be bent through an angle of $\alpha = 1.8''$ as it passes near the sun. The simplest classical theory would predict no bending ($\alpha = 0''$) and a careful classical analysis would predict a bending through an angle of $\alpha = 0.9''$. In principle, all that was necessary was to observe a star when it was aligned with the edge of the sun and to measure the angle of bending α. If the result were $\alpha = 1.8''$ general theory would be vindicated, otherwise one of the classical theories would be correct.

In practice, measuring the bending of light by sun was extremely hard and was possible only during a solar eclipse. Nonetheless, in 1919 it was successfully measured by Dyson, Eddington and Davidson, who reported their best estimate as $\alpha = 2''$ with 95% confidence that it lay between $1.7''$ and $2.3''$. Obviously, this result was consistent with General Theory and inconsistent with classical theories.

Had the accuracy of their measurement been lower and had they reported a result like $\alpha = 1.6''$ with 95% confidence that it lay between $0.7''$ and $2.5''$, this result would have been consistent with both the theories. Clearly, it is necessary to reduce the uncertainty in the error to determine the right theory unambiguously.

II.3 Uncertainty in reading scales

In every physics experiment, values of physical quantities are measured. Measurements are always made by reading off a scale or a digital meter. This appears to be a straightforward matter. However, unless read off carefully we might risk either underestimating, overestimating or clearly misrepresenting the measurement.

Assume we are measuring the length of a wire and that the length is between 12.3cm and 12.4cm but clearly closer to 12.3cm than to 12.4cm. What value should we report as our best estimate of the length? To record the length to be 12.3cm would be an innacurate report of our observation since the length is only *close* to 12.3cm, it is *not equal* to it. Instead we will call 12.3cm as the best estimate of the length. If we do not specify the probable range of the wire's length, we can be accused of hiding (or misprepresenting) the truth. Therefore, we must state our observation as

best estimate = 12.3cm,

probable range = 12.3cm to 12.35cm.

The above observation is correct but we have overestimated the error. If we look more carefully at the scale or use a magnifying glass, we might find that the reading on the scale can actually be specified more accurately as

best estimate =12.33cm,

probable range of the length =12.32cm to 12.34cm.

Clearly, we have reduced the margin of error in our observation. However, how do we know that we haven't reduced it so much that we end up being wrong about its probable range? To ensure the correctness of your observation, you must ask others to make the same observation. Only when you are sure that you are neither underestimating nor overestimating the probable range, should you record that observation. However, the margin of error should be kept as small as possible and no smaller. The aim of any observation should be to estimate the truth (best

estimate & probable range of length) as closely as possible without being wrong.

Suppose the best estimate of a quantity is x and its values range $\pm\delta x$ on either side of it. Then, the uncertainty in the best estimate is δx. All measurements should be reported as

$$x \pm \delta x.$$

Therefore, the length of the wire should be reported as (12.33 ± 0.02)cm.

III Significant figures

As explained in the last section, every measurement contains an uncertainty. *The number of significant figures, say 5.382in a measurement, is equal to the number of figures that are known with some degree of reliability (digits 5,3,8) plus the last digit (that is, 2) which is an estimate or approximation.* As we improve the sensitivity of our instrument, the number of significant figures in the result increases.

Several basic rules for stating uncertainty are worth emphasizing. Since δx is an estimate of uncertainty, it should not be stated with too much precision. If we state the value of acceleration due to gravity $g = 9.82 \pm 0.02385$ m/sec^2, it is absurd. If the value of g is doubtful from the third decimal place, how can we know the uncertainty all the way to the fifth decimal place? The result should be stated as $g = 9.82 \pm 0.02$ m/sec^2.

Rule 1: *Experimental uncertainties should always be rounded off to the last significant figure.*

Once the uncertainty in a measurement has been estimated, the significant figures in the measured value must be considered. A statement such as Speed= 6051.78 ± 30 m/sec is ridiculuous. The uncertainty of 30 means that the digit 5 might really be as small as 2 or as large as 8. Clearly the trailing digits 1,7, and 8 have no significance at all and should be rounded off. That is, the speed would be correctly stated as 6050 ± 30m/sec.

Rule 2: *The last significant figure in any stated answer should usually be of the same order of magnitude (in the same decimal position) as uncertainty.*

For example, the answer 92.81 with an uncertainty of 0.3 should be rounded off as 92.8 ± 0.3. If its uncertainty is 3, then the same answer should be rounded off as 93 ± 3, and if the uncertainty is 30, then the answer should be 90 ± 30.

IV Sources of error

Random error

In section II.1, an observer made a single measurement and found the best estimate to be 12.31cm. Let us repeat our observation after removing the ruler and bringing it back. Since there is no change in external environment (temperature, pressure, etc.) between the two observations, the *true* length of the wire must stay unchanged. However, our 2^{nd} observation might not give the same value as our 1^{st} observation. Instead, we might get 12.33cm as the best estimate. A 3^{rd} measurement might read 12.30cm. Successive measurements might read 12.31cm, 12.36cm, 12.30cm, 12.30cm, 12.34cm, 12.35cm,... Contrary to expectations, every measurement of length will give a different value. However, we are measuring the length of the

same wire every time. Its *true length is unchanged* from one measurement to the next. Why should we obtain a different length in every measurement?

Perhaps, all our measurements, except for one, are wrong. If we have not aligned the zero mark of the ruler with the beginning of the wire, or not made the ruler parallel to the wire, our measurement will be wrong. However, it is impossible to align the zero mark with the beginning of the wire perfectly. An observer's own judgement on the alignment might vary from one measurement to the next. As a result successive measurements will be different. From the above observations how do we obtain this estimate? A discussion on this will be given in the next section on random errors.

As another example, let's consider a simple pendulum. If we perform several measurements of the time taken for a certain number of oscillations (say, 10) we might get a series like 16.3sec, 16.8sec, 15.5sec, 16.0sec, 16.4sec, 15.7sec, 15.3sec, 16.0sec,..... Every measurement yields a different value. To measure time, we must start the stop clock the instant we leave the pendulum and must stop the stop clock, the instant 10 oscillations are completed. In reality, there is always a time lag between the release of the pendulum and start of the clock. Likewise, there is also a time lag between completion of exactly 10 oscillations and stopping the clock. As a result, we get a different value of time in every measurement. There is an error in all our measurements.

A careful analysis of a large number of measurements in the above two experiments yields a random series of values. Therefore, such measurements are said to have a random error.

In a later section, we will describe how to (a) extract the best estimate to the true value from a random series of values, (b) find the error in our best estimate.

IV.1 Systematic errors

Many different factors can cause errors in an experiment. Many of these errors can be removed. We carry out a brief description of such types of errors.

If an experiment has not been set-up correctly our measurement will not be consistent with theoretical expectations. For instance, we might want to measure the stopping potential in photoelectric effect. Theoretically, the stopping potential $V_s = (h/e)\nu - W/e$. In the theory leading to this equation, it is assumed that the electrons move in vacuum. Therefore, it is necessary to evacuate the tube in our experimental set-up. If we do not take this step our measurement of stopping potential will be inconsistent with what we expect theoretically. In practice it is not possible to create a perfect vacuum. We must ensure that the density of air in the tube is low enough to have no measurable effect on the stopping potential. A failure to ensure this will result in a wrong comparison of experiment with theory. We would say that there is an error in our experiment.

In teaching laboratories, we often use measuring instruments that do not work properly. For example, a clock might run faster, or the divisions on a scale might be slightly more closely spaced. Measurements made using these instruments will result in an error. We might not have the time or the expertise to either set the instrument right, or to re-calibrate these instruments with a greater accuracy. Hence, it is wise to assume a fractional error of $\pm 1\%$ in all our readings.

Before starting an experiment, we might often find that the needle in a meter does not point to the 0 of the scale. It might point to x_0. This is known as a *zero error.* The value shown

by a meter subject to a zero error will be either less or more than what it should be. If the value shown by the meter is x, then the correct value $x_{\text{correct}} = x_1 - x_0$. If we fail to notice and account for the zero error, we will be reporting the wrong value.

There are many external factors that might influence our experiment. Consider an experiment to measure the magnetic field of a bar magnet using a magnetometer. In this technique the bar magnet is oriented perpendicular to the earth's magnetic field. Next, the angular position of the needle relative to its initial position (in absence of the bar magnet) is measured. Finally, a small calculation gives the magnetic field of the bar. In the technique just described, all magnetic fields except that of the bar magnet and the earth are to be regarded as external to the experimental system and must be removed from its vicinity. Failure to do so will yield an inaccurate result of the field of the magnet. External factors are often not easily apparent. They have to be imaginatively and painstakingly hunted down and removed. For instance, in the above experiment we might ensure that there are no magnets visibly lying anywhere close to the magnetometer. However, there could be hidden sources of magnetic fields. Power supplies (which contain electromagnets) are quite commonly one such source of magnetic field that can be overlooked by those not so experienced.

Systematic errors should always be reduced below the required precision. Suppose we want the value of stopping potential from a photoelectric effect experiment to have a precision of less than 2%. We must ensure that systematic errors cause an uncertainty of much less than 2%

IV.2 Problem of definition

Whenever we measure a physical quantity, we assume that we can define the points of measurement in our experiment exactly.

For instance, in an optics experiment we want the distance of an image from the center of a convex lens. We can carry out this experiment by forming an image on a movable screen. By moving the screen we obtain the position where a sharpest image is formed. Having found this position, we only need to measure the distance between this point and the center of the lens. In principle, it appears simple. In practice, the position of the sharpest image is not so clearly defined. While doing the experiment we will find that the image remains equally sharp over a range of distance. We cannot know the *exact* position. There is an uncertainty in our measurement of image distance.

In this laboratory, you will encounter such a 'problem of definition' situation in many experiments. The error cannot be removed. By using more sophisticated sensors (compared to our eyes in the above experiment), we can only reduce the uncertainty.

Suppose the image remains sharp over a distance of 7mm. Should the uncertainty in our measurement be taken as 0.5mm (the reading error) or 7mm? The uncertainty in the 'image definition' is greater than the reading error. Clearly, we should take the larger of the two (i.e., 7mm) as the uncertainty in a single measurement. In a later section on a quantitative discussion on random errors, we will show that the uncertainty in the image distance will be related to the uncertainty in 'image definition' but should not be taken to be equal to it.

V More about random errors

V.1 Best estimate and uncertainty

We explained above that repetitive measurement of a quantity, under *identical* experimental conditions, do not give the same value. To reiterate the example used earlier, in successive measuremements of time for 10 oscillations of a simple pendulum of a fixed length and mass, we obtained the a random series of values—12.31cm, 12.36cm, 12.30cm, 12.30cm,.... None of the measurements is equal to the *true* value. Each measurement is subject to a random error arising from various sources[1]

We will address the following question in this section: From the above measurements how do we extract the *true* value of a physical quantity? This question is a hard one and has no satisfactory and simple answer, since no measurement can exactly determine the true value of any continuous variable (time, position, current, etc.). However, we can obtain a *best estimate* to the true value. Additionally, we can also determing the uncertainty in the best estimate.[2]

We will *assume* that the best estimate of a physical quantity x is the mean $\bar{x}$ of our measurements. Meaning, if a quantity x takes on values $x_i, i = 1, 2, 3, ..., N$ in N measurements, then the best estimate of the quantity will be

$$x_{\text{best}} = \bar{x} = \frac{1}{N}\sum_{i=1}^{N} x_i \tag{1}$$

The above assumption appears reasonable. (A rigorous justification for it will be found in references.) As the number of measurements $N \to \infty$, the the best estimate approaches the true value $x_{best} \to x_{true}$. We will justify this in the next section.

To calculate the uncertainty in the best estimate let's proceed by the following reasoning. Suppose one set of N measurements yields the best estimate (mean of our measurements) as $x^{(1)}_{\text{best}}$. Let us now repeat the experiment, performing another set of N measurements, denoting the best estimate by $x^{(2)}_{best}$. Likewise, we repeat N measurements a large number of times, denoting the best estimate obtained for each set i as $x^{(i)}_{best}$. One way to define uncertainty is to take it equal to the difference in the largest and the smallest value of best estimates x^i's. However, that is not conventionally done for reasons that will become clear later.

Let's calculate the mean of best estimates $\bar{x}_{\text{best}} = \frac{1}{M}\sum x^{(i)}_{best}$, using M sets of measurements. The uncertainty in the best estimate δx is then defined as the standard deviation of the mean (best estimate) of each set, σ_{SDOM}.

$$\delta x = \sigma_{SDOM} = \sqrt{\frac{1}{N}\sum (x^{(i)}_{best} - \bar{x}_{best})^2}$$

Calculation of uncertainty σ_{SDOM} involves too many sets of measurements followed by cumbersome calculations. This can be simplified.

For a *single set* of N observations $\{x_i, i = 1, N\}$, the standard deviation σ is

$$\sigma = \sqrt{\frac{1}{N}\sum (x_i - \bar{x})^2}$$

[1] Read the earlier section on random errors for a discussion on sources of random errors

[2] There is no unique definition of uncertainty. Exact value of uncertainty will depend on the way we define it.

For measurements subject to random errors, there exists a relationship between standard deviation of the mean σ_{SDOM} and the standard deviation σ. This relationship is

$$\sigma_{SDOM} = \frac{\sigma}{\sqrt{N}}$$

which gives

$$\delta x = \frac{\sigma}{\sqrt{N}}$$

Therefore, knowledge of σ from a single set of measurements also gives the uncertainty in the best estimate. The final result should be expressed as $x_{best} \pm \delta x$, where δx is given by the above equation.

We get an important conclusion from the above relationship. An increase in N is followed by a decrease in uncertainty δx. Therefore, *to reduce the uncertainty we should repeat our measurements a large number of times.* However, to decrease the δx by a factor of 10, we need to increase N by a factor of 100! The true value of a quantity is equal to $\bar{x}$ as $N \to \infty$.

V.2 Probability distribution of random errors

In a physical quantity subject to random errors, successive measurements yield a random series of values. What is the probability distribution of the random values obtained in an experiment?

Interestingly, experience has shown that *irresepective* of the source of random errors, probability distribution of quantities (time, length, current, etc.) subject to random errors is *gaussian.* That is, if we perform $N \to \infty$ measurements on any physical quantity x (with random errors), the probability of obtaining a value x_i is given by

$$P(x_i) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x_i - \bar{x})^2/(2\sigma^2)}$$

where $\bar{x}$ is the mean and σ is the standard deviation of measured values. We will not attempt to give any theoretical justification for this. Through experiments in this laboratory, you will be asked to convince yourself that random errors follow a gaussian distribution.

$P(x_i)$ is symmetrically distributed around $\bar{x}$. We argued in the previous section that $\bar{x}$ is equal to the true value (as $N \to \infty$). Therefore, $P(x_i)$ is symmetrically distributed around the *true* value. It implies that there is an equal probability for a random error to shift the result of a measurement above or below its true value. This appears to be a reasonable conclusion.

The gaussian distribution of random errors leads us to a few conclusions. In an experiment, 68% of all measurements will lie in the range $(\bar{x} \pm \sigma)$, 95% of measurements will lie in the range $(\bar{x} \pm 2\sigma)$, and 99.7% of measurements will be in the range $(\bar{x} + 3\sigma)$.[3]Expressed differently, the probability that the next measurement will lie in the range $(\bar{x} \pm \sigma)$, $(\bar{x} \pm 2\sigma)$,$(\bar{x} + 3\sigma)$ is 0.68, 0.95 and 0.997 respectively.

The gaussian distribution law can be used to justify our claims that for a quantity subject to random error (a) its best estimate is the mean, and (b) the uncertainty in the best estimate is equal to standard deviation of the mean. We will not justify this claim here.

[3]For a justification consult your books on probability & statistics. You can also derive these conclusions by integrating $P(x_i)$ from $-\sigma \to +\sigma$, $-2\sigma \to +2\sigma$,$-3\sigma \to +3\sigma$.

VI Propagation of errors

Measured values are used to calculate other quantities (which we call derived quantities) through some relationships (that are often theoretically known). In the earlier sections we estimated the uncertainty in measured quantities. In the following sections, we show how to calculate the uncertainty in derived quantities from a knowledge of uncertainty in measured quantities.

VI.1 Sums & differences of quantities

Simplest rule

Suppose we add two measured quantities $x_{best} \pm \delta x$, and $y_{best} \pm \delta y$. What is the best estimate of the sum $q_{\rm best}$ and the uncertainty δq?

The best estimate will be

$$q_{\rm best} = x_{\rm best} + y_{\rm best} \tag{2}$$

and the largest probable value of the addition above will be, approximately

$$(x_{\rm best} + \delta x) + (y_{\rm best} + \delta y) \approx (x_{\rm best} + y_{\rm best}) + (\delta x + \delta y), \tag{3}$$

which gives the uncertainty in q to be

$$\delta q \approx \delta x + \delta y \tag{4}$$

The above formula is an approximation for the reason that (for random errors) uncertainty is defined as a standard deviation. The standard deviation of $q = x+y$ is not equal to $\delta q = \delta x+\delta y$. The exact formula will be given in the next section.

More refined rule—addition in quadratures

The formula obtained above for calculating uncertainty in addition of two independent quantities is likely to overestimate δq.

Let us consider how the actual value of q could equal the highest extreme $(x+\delta x)+(y+\delta y)$. Obviously, this happens if we have underestimated x by a full amount δx and overestimated y by a full amount δy. This would be a fairly unlikely event since x and y are independent. If errors are random in nature, we have a 50% chance that an overstimate of x will be accompanied by an underestimate of y and vice versa. Clearly, then, the probability we will overstimate both x and y by the full amounts δx and δy is fairly small. Therefore, the value $\delta q \approx \delta x+\delta y$ overstates our probable error.

What constitutes a better estimate for δq? The answer depends on the statistical laws governing our errors in measurement. If both x and y are independent and governed by the gaussian distribution, then the uncertainty in $q = x + y$ is given by

$$\delta q = \sqrt{(\delta x)^2 + (\delta y)^2}$$

When we combine two numbers by squaring them, adding the squares, and taking the roots, the numbers are said to be *added in quadrature*. A detailed derivation of the uncertainty δq will not be given here. However, let's gain some insight into the origin of the formula for δq above.

For random errors, we defined the uncertainties, like δx, as a standard deviation. What the above formula is telling us is that the standard deviation δq of quantity $q = x + y$ is related to the standard deviations δx and δy by the above formula. Interested readers can try to prove this statement on their own. A simple argument (that the readers can construct for themselves) will show that $\sqrt{(\delta x)^2 + (\delta y)^2} \leq \delta x + \delta y$. Therefore, as expected, the addition in quadrature gives a smaller uncertainty compared to simple addition.

The above results can be stated as follows. Suppose that $x, ..., w$ are measured with uncertainties $\delta x, ..., \delta w$ and the measured values are used to compute

$$q = x + ... + z - (u + ... + w)$$

If the uncertainties in $x, ..., w$ are independent and random, then the uncertainty in q is the quadratic sum

$$\delta q = \sqrt{(\delta x)^2 + ... + (\delta z)^2 + (\delta u)^2 + ... + (\delta w)^2}$$

of the original uncertainties. In any case, δq is never larger than the ordinary sum,

$$\delta q \leq \delta x + ... + \delta z + \delta u + ... + \delta w.$$

VI.2 Product of two quantities

Suppose a product of two measured quantities $(x_{\text{best}} + \delta x)$ and $(y_{\text{best}} + \delta y)$ occurs in a calculation, i.e., $q = xy$ What is the best estimate $q_{best}$and the uncertainty δq in the product?

The best estimate is easily obtained as

$$q_{best} = x_{best} y_{best}$$

and the largest probable value of the addition approximately [4] will be

$$\approx (x_{best} + \delta x)(y_{best} + \delta y)$$

$$\approx x_{best} y_{best} + (x_{best} \delta y + y_{best} \delta x) + \delta x \delta y$$

Usually δx and δy are much smaller compared to $x_{best}$and y_{best}. Therefore, we can ignore the last term. Comparison of equations yields

$$\delta q \approx (x_{best} \delta y + y_{best} \delta x), \tag{5}$$

that can be written in terms of fractional errors as

$$\frac{\delta q}{q_{best}} \approx \frac{\delta x}{x_{best}} + \frac{\delta y}{y_{best}}$$

From the above equation, we can conclude that fractional errors add up when taking a product of two quantities. However, as indicated in the formula this is only an *approximation.* The *exact* formula will be obtained by adding in quadratures, which will give

$$\frac{\delta q}{q_{best}} = \sqrt{\left(\frac{\delta x}{x_{best}}\right)^2 + \left(\frac{\delta y}{y_{best}}\right)^2}$$

The earlier formula is the upper bound on the fractional error. These formulas can be generalized (as done for uncertainties in sums and differences) for any number of independent and random quantities.

[4] It is approximate when the uncertainties are defined as standard deviation of quantities involved.

VI.3 Quotients of two independent quantities

Suppose a quotient of two quantities $(x_{best} + \delta x)$ and $(y_{best} + \delta y)$ occurs in a calculation, i.e., $q = \frac{x}{y}$. We will assume that x,and y are independent and have random errors. What is the best estimate q_{best} and the uncertainty δq in the quotient?

The best estimate is easily obtained as

$$q_{best} = \frac{x_{best}}{y_{best}} \tag{6}$$

The two quantities x and y are independent, meaning, the value of x is independent of the value that y takes. Therefore, the highest probable value of the quotient will be obtained for a situation where x takes on the largest probable value $(x + \delta x)$ and y takes on the smallest probable value $(y - \delta y)$. The largest value of q will be *approximately*[5]

$$\approx \frac{(x_{best} + \delta x)}{(y_{best} - \delta y)} \approx y_{best}(x_{best} + \delta x)\left(1 + \frac{\delta y}{y_{best}}\right) \tag{7}$$

where we have assumed that $\delta x << x_{best}$ and $\delta y << y_{best}$. Carrying our calculation further

$$\delta q = x_{best}y_{best} + y_{best}\delta x + x_{best}\delta y + \delta x\delta y \tag{8}$$

Assuming that $\delta x << x_{best}$ and $\delta y << y_{best}$ (which should always be the case in any good experiment), we get

$$\delta q \approx y_{best}\delta x + x_{best}\delta y \tag{9}$$

and on expressing the equation in terms of fractional errors, we get

$$\frac{\delta q}{q_{best}} \approx \frac{\delta x}{x_{best}} + \frac{\delta y}{y_{best}}. \tag{10}$$

From the above equations we conclude that for a quotient of two *independent* quantities, their fractional errors add up. The *exact* result is obtained by adding in quadratures, i.e.,

$$\frac{\delta q}{q_{best}} = \sqrt{\left(\frac{\delta x}{x_{best}}\right)^2 + \left(\frac{\delta y}{y_{best}}\right)^2}.$$

The above formulas can be generalized for any number of quantities as was done for uncertainties in sums and differences.

VI.4 Errors in expression involving dependent terms

Calculation of error in a quotient (x/y) was calculated above for a case where x and y—numerator and denominator—are *independent.* We will often encounter expressions where the numerator and denominator are *not independent.* For example, consider an expression $q = (x + y)/(x + u)$. The numerator $(x + y)$ and the denominator $(x + u)$ are *not independent* of each other. Therefore, the above calculation, leading to addition of fractional uncertainties cannot be applied for the expression involving dependent quantities in the quotient. We will discuss below a method to calculate the uncertainties in such expressions.

[5]Since δx,δy are defined as a standard deviation, δq should be obtained strictly by calculating the standard deviation of x/y.

Consider a function $q = q(x, y)$. By calculus, we know that an incremental change δq in $q = q(x, y)$ due to an incremental change δx and δy is given by

$$\delta q = \left(\frac{\partial z}{\partial x}\right)\delta x + \left(\frac{\partial z}{\partial y}\right)\delta y \tag{11}$$

The above expression holds for all continuous and differentiable functions.

If physically, δx and δy are the uncertainties in x and y, then the above equation will give the uncertainty δq in q. Therefore, the uncertainty in q in the function $q = (x + y)/(x + u)$ can be calculated using the above equation. Likewise, the expression can be used wherever we encounter a quotient with terms that are not independent.

The use of the above equation is not restricted to quotients with dependent terms. In general, if we have a quantity q dependent on $x, y, u, v, ...$ through a functional relationship $q = q(x, y, u, v, ...)$ then the uncertainty δq caused by uncertainties $\delta x, \delta y, \delta u, \delta v, ...$ can be calculated by generalizing the above equation , i.e.,

$$\delta q = \frac{\partial q}{\partial x}\delta x + \frac{\partial q}{\partial y}\delta y + \frac{\partial q}{\partial u}\delta u + \frac{\partial q}{\partial v}\delta v + ... \tag{12}$$

The expressions for uncertainty derived for addition, product and quotients can all be derived from the equation above. We leave this as an exercise for the reader.

VI.5 Solved example

An experiment is performed to measure the refractive index n of glass. Refractive index is related to angle of incidence i and refraction r by the relationship $n = \dfrac{\sin(i)}{\sin(r)}$. In the experiment, we measure the angle of incidence i and the angle of refraction r at the interface between glass and air.

The uncertainty in n can be found by noting that it is quotient of $\sin(i)$ and $\sin(r)$. Therefore, the fractional uncertainty in n is the quadratic sum of those in $\sin(i)$ and $\sin(r)$:

$$\frac{\delta n}{n} = \sqrt{\left(\frac{\delta sin(i)}{sin(i)}\right)^2 + \left(\frac{\delta sin(r)}{sin(r)}\right)^2}$$

The fractional uncertainty in the sine of any angle θ, we can calculate by

$$\delta \sin(\theta) = \left|\frac{\partial \sin(\theta)}{\partial \theta}\right| \delta\theta = \mid cos(\theta) \mid \delta\theta.$$

Thus, the fractional uncertainty is

$$\frac{\delta \sin(\theta)}{|\sin(\theta)|} = |\cot(\theta)|\delta\theta$$

The values of i and r obtained in the experiment are specified in the table below along with the results of calculations needed to computer the uncertainty. The uncertainty in n can be found as in the last three columns.

In this experiment each i, r measurement is done only once. The uncertainty in each due to the reading error is $\pm 1°$. If each of the two observations (in the table below) is repeated several times, our estimate of n (calculated as the mean of all values) would be more accurate and its uncertainty (σ_{SDOM}) $\frac{\delta n}{n}$ would be smaller.

i (Deg)	r (deg)	$\sin(i)$	$\sin(r)$	n	$\frac{\delta \sin(i)}{\lvert \sin(i) \rvert}$	$\frac{\delta \sin(r)}{\lvert \sin(r) \rvert}$	$\frac{\delta n}{n}$
20 ± 1	13 ± 1	0.342	0.225	1.52	0.05	0.08	0.09
40 ± 1	23.5 ± 1	0.643	0.399	1.61	0.02	0.04	0.05

VII Best fit line

VII.1 Meaning of best fit line

Consider a set of experimental data (x_1, y_1),...,(x_N, y_N) where measurement of x and y have random errors. Suppose that we expect x and y to have a linear relationship of the form

$$y = c + mx$$

where c and m are *unknown* constants that must be determined by our experiment. If our measurements were subject to no uncertainties, then each of the points (x_i, y_i) would lie *exactly* on a straight line. In practice, there are uncertainties, and we *cannot expect* all points to lie on a straight line.

The interesting problem that we address in this section, is to find the straight line $y = c+mx$ that best fits that measurements.

We will find the best estimates for the constants c and m based on the data (x_1, y_1),...,(x_N, y_N), subject to random errors. The analytical method of finding the best straight line to fit a series of experimental points is called *linear regression,* or the *least-squares fit for a line,* and is the main subject of this section.

VII.2 Slope and intercept of best fit line

Let $S_i = y_i - mx_i - c$ be the deviation of any experimental point $(x_i, y_i)$from the best fit line. Let us define a quantity

$$S = \sum (y_i - mx_i - c)^2$$

The line that best fits our data will be assumed to be that[6] for which S is minimum. We thus require

$$\frac{\partial S}{\partial m} = -2 \sum x_i (y_i - mx_i - c) = 0 \quad \text{and} \quad \frac{\partial S}{\partial c} = -2 \sum (y_i - mx_i - c) = 0$$

which give

$$m \sum x_i^2 + c \sum x_i = \sum x_i y_i \quad \text{and} \quad m \sum x_i + Nc = \sum y_i$$

The second equation can be written as $\bar{y} = m\bar{x} + c$, where $\bar{y} = \frac{1}{N} \sum y_i$and $\bar{x} = \frac{1}{N} \sum x_i$ showing that the best fit line passes through the centroid $(\bar{x}, \bar{y})$ of the points (x_i, y_i). The required values of m and c can be calculated from the above two equations to be

$$m = \frac{N \sum x_i y_i - \sum x_i \sum y_i}{\Delta}$$

and

$$c = \bar{y} - m\bar{x}$$

[6]This looks like a reasonable assumption. A more rigorous justification uses the gaussian distribution of random errors.

where

$$\Delta = N\sum x_i^2 - \left(\sum x_i\right)^2.$$

We have completed our task of finding the best fit straight line through the experimental points (x_i, y_i). The line that best fits our data is given by the m and c values in the above equations.

The best fit line can also be *approximated by visual means.* The way to draw it is the following: obtain the centroid and rotate a transparent ruler about it so that it passes through the extreme points on the top right and the bottom left. Similarly, find the line through the extreme points on the top left and bottom right. these two lines give the maximum error $\pm\Delta m$ in the slope m on either side of the best fit line. The best fit line is drawn as the bisector of the angle between these two lines.

VII.3 Uncertainties in m and c of a best fit line

If we repeat the experiment, we will get a different set of data points $(x_1^{'}, y_1^{'}), ..., (x_N^{'}, y_N^{'})$. The m and c for a line that has the best fit to this data will not be the same as the m and c for the first set of data. Likewise, if we repeat our measurement several times, we will get a different m and c each time. It follows that m and c obtained from any data set will have an uncertainty.

A rigorous analysis give the formula for *uncertainty in* m

$$\delta m = \sqrt{\frac{S}{\Delta}}.$$

VIII Comparison of your result with other results

In this laboratory you will be required to compare your results with (a)those known from theory, (b)the most precise values obtained so far by the scientific community, (c)those of your friends. Unless your experiment is consistent with (a) and (b), you cannot consider your experiment a success. You must also compare your experiment with those of your friends to see whether two results (obtained from the same setup) are consistent or not.

Lets consider another example. Student A measures the speed of sound at some temperature and pressure and gets a result (329 ± 5)m/sec. Student B carries out a measurement and gets (325 ± 3)m/sec. Student C gets the result (345 ± 2)m/sec. The standard accepted value is, say, (330 ± 1)m/sec. Which of the students gets a satisfactory result?

The accepted range (330 ± 1)m/sec lies within the error margin of A's result (329 ± 5)m/sec. Therefore, we can consider A's result to be consistent with the accepted value.

Meaning of uncertainty δx is that the correct value of x probably likes between $(x_{best} - \delta x)$ and $(x_{best} + \delta x)$; it is certainly possible that the correct value lies slightly outside this range. Therefore, a measurement can be regarded as satisfactory even if the accepted value lies slightly outside the estimated range of the measured value. For example, if student B found the value (325 ± 3)m/sec, he can certainly claim that his measurement is consistent with the accepted value (330 ± 1)m/sec.

On the other hand, if the accepted value is well outside the margins of error (the discrepancy is appreciably more than twice the uncertainty, say) there is reason to think something has gone wrong. For example, the unlucky student finds the speed to be (345 ± 2)m/sec. Compared with

the accepted speed (330 ± 1)m/sec student C's discrepancy is 14m/sec, which is seven times bigger than his stated uncertainty. He will need to check his experiment and find out what has gone wrong.

Unfortunately, the tracing of C's mistake may be a tedious business because of numerous possibilities. He may have made a mistake in the measurement or calculations that led to 345m/sec. He may have estimated his uncertainties incorrectly. (The answer (345 ± 15)m/sec would have been acceptable.) He might also be comparing his measurement with the wrong accepted value. For example, the accepted value (330 ± 1)m/sec is the speed of sound at standard temperature and pressure. Since the standard temperature is 0°C there is a good chance the measured speed was not taken at standard temperature. In fact, if the measurement was made at 20°C, the correct accepted value for the speed of sound is 343m/sec, and the measurement would be entirely acceptable.

Finally, and perhaps most likely, a discrepancy such as the one between student C's and the accepted value might indicate some undetected source of systematic error (such as a clock that runs slow). Detection of such systematic errors requires careful checking of calibration of all instruments and detailed review of all procedures.

IX Experiments

Students will do following experiments in the laboratory.

1. A pendulum will be given. Each student in the laboratory must make 2 to 3 measurements of (a)its length, (b)time for 5 oscillations, for a *fixed* length of the pendulum. A data set of roughly 70 observations will get collected in the end. Of these, each student will receive 40 randomly selected observations. Using these observations, the student must answer the questions given in the observation manual.

2. In this experiment the length of the pendulum is varied. Time for 5 oscillations is measured for 10 different lengths. Using these observations, students must answer the questions in the observation manual.

References

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