# Making Simple Decisions

# Outline

- Combining Belief & Desire
- Basis of Utility Theory
- Utility Functions.
- Multi-attribute Utility Functions.
- Decision Networks
- Value of Information
- Expert Systems

# Combining Belief & Desire(1)

- Rational Decision
  - Based on Belief & Desire
  - Where uncertainty & conflicting goals exist

- Utility
  - assigned single number by some *Fn*.
  - express the desirability of a state
  - combined with outcome prob.

    → expected utility for each action

# Combining Belief & Desire(2)

- Notation
  - $U(S)$ : utility of state $S$
  - $S$ : snapshot of the world
  - $A$ : action of the agent
  - $Result_i(A)$ : outcome state by doing $A$
  - $E$ : available evidence
  - $Do(A)$ : executing A in current S

# Combining Belief & Desire(3)

- Expected Utility

$$EU(A \mid E) = \sum_i P(Result_i(A) \mid E, Do(A))U(Result_i(A))$$

- Maximum Expected Utility(MEU)
  - Choose an action which maximizes agent's expected utility

# Basis of Utility Theory(1)

- Rational preference
  - Preference of rational agent
    $\Longrightarrow$ obey constraints
  - Behavior is describable as maximization of expected utility

# Basis of Utility Theory(2)

- Notation
  - Lottery($L$): a complex decision making scenario
    - Different outcomes are determined by chance.
  - $L = [p, A;\ 1 - p, B]$
  - $A \succ B$ : $A$ is preferred to $B$
  - $A \sim B$ : indifference btw. $A$ & $B$
  - $A \succeq B$ : $B$ is not preferred to $A$

# Basis of Utility Theory(3)

- Constraints
  - Orderability

$$(A \succ B) \vee (B \succ A) \vee (A \sim B)$$

  - Transitivity

$$(A \succ B) \wedge (B \succ C) \Rightarrow (A \succ C)$$

  - Continuity

$$A \succ B \succ C \Rightarrow \exists p \ [p, A;\ 1 - p, C] \sim B$$

# Basis of Utility Theory(4)

- Constraints (cont.)
  - Substitutability

$$A \sim B \Rightarrow [p, A; 1-p, C] \sim [p, B; 1-p, C]$$

  - Monotonicity

$$A \succ B \Rightarrow (p \geq q \Leftrightarrow [p, A; 1-p, B] \succsim [q, A; 1-q, B]$$

  - Decomposability

$$[p, A; 1-p, [q, B; 1-q, C]]$$

$$\sim [p, A;\ (1-p)q, B;\ (1-p)(1-q), C]$$

# Basis of Utility Theory(5)

- Utility principle

$$U(A) > U(B) \Leftrightarrow A \succ B$$

- Maximum Expected Utility principle

$$U([p_1, S_1; \ldots; p_n, S_n]) = \sum_i p_i U(S_i)$$

$F_n$

- Utility
  - Represents that the agent's actions are **trying to achieve**.
  - Can be **constructed by observing agent's preferences**.

# Utility Functions. (1)

- Utility
  - mapping state to real numbers
  - approach
    - Compare $A$ to standard lottery $L_p$
      - $u^{\top}$: best possible prize with prob. $p$
      - $u_{\perp}$: worst possible catastrophe with prob. $1\text{-}p$
    - Adjust $p$ until

$$A \sim L_p$$

$$\text{eg)}\quad \$30 \;\sim\; L \quad \begin{cases} \xrightarrow{0.9} \text{continue} \\ \xrightarrow{0.1} \text{death} \end{cases}$$

# Utility Functions. (2)

- Utility scales
  - positive linear transform

    $$U'(x) = k_1 U(x) + k_2 \quad where \quad k_1 > 0$$

  - Normalized utility

    $$u_{\top} = 1.0,\ u^{\perp} = 0.0$$

  - Micromort（“微死亡”）

    one-millionth chance of death

    - Russian roulette, insurance
  - QALY

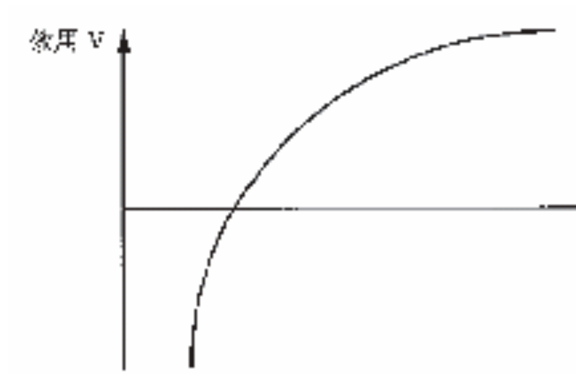
    quality-adjusted life years

# Utility Functions. (3)

- Utility of Money
  - **TV game**
    - **1million vs 3 millions by chance 0.5**

$$E(A) = \frac{1}{2}U(S_k) + \frac{1}{2}U(S_{k+3,0\ ,0})$$

$$E(B) = U(S_{k+1,0\ ,0})$$



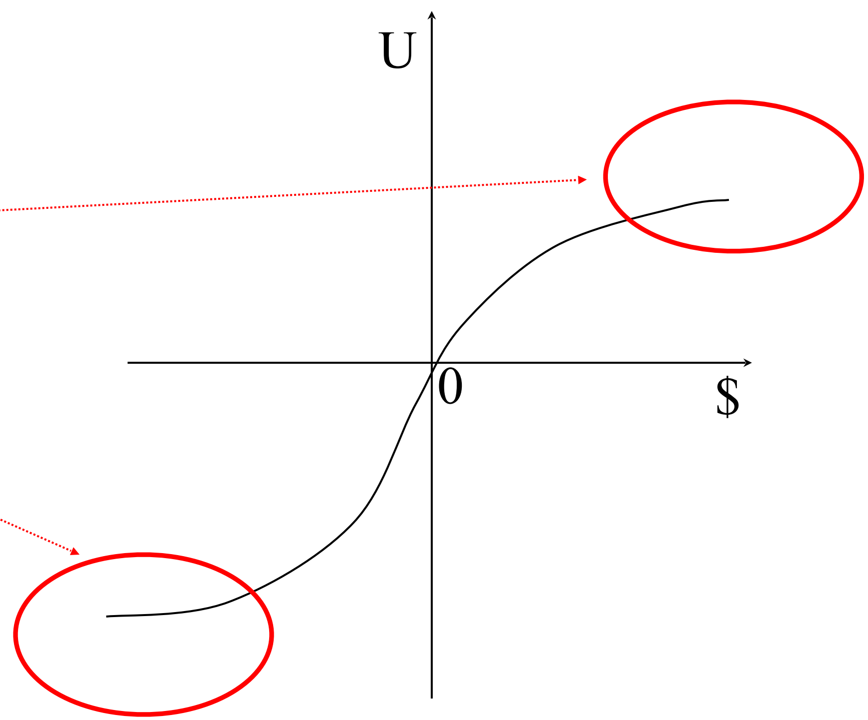

- Example (Grayson, 1960)

$$U(S_{k+n}) = -2\ .3 + 2.0\ \log(n + 1\ ,0\ )$$

for $n = -\$1\ ,0$ to $n = \$$

# Utility Functions. (4)

- Money : does NOT behave as a utility fn.
  - Given a lottery *L*
  - risk-averse（不愿承担风险）
    $$U(S_L) < U(S_{EMV(L)})$$
  - risk-seeking（风险追求）
    $$U(S_L) > U(S_{EMV(L)})$$



# Outline

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- Utility Functions
- Multi-attribute Utility Functions
- Decision Networks
- Value of Information
- Expert Systems

# Multi-attribute Utility Functions. (1)

- Multi-Attribute Utility Theory (MAUT)
  - Outcomes are characterized by 2 or more attributes.
  - eg) Site a new airport
    - disruption by construction, cost of land, noise,….
- Approach
  - Identify regularities in the preference behavior

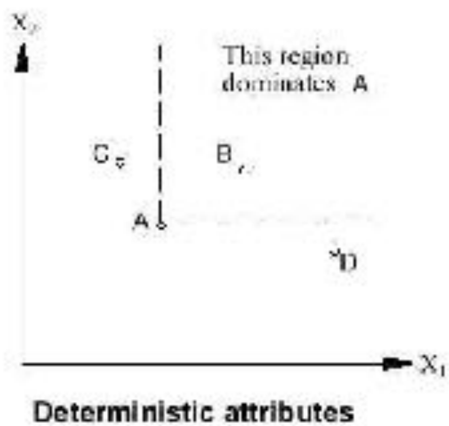
# Multi-attribute Utility Functions. (2)

- Notation
  - Attributes $X_1, X_2, X_3, \ldots$
  - Attribute value vector $X = < x_1, x_2, \ldots. >$
  - Utility Fn. $U(x_1, \ldots, x_n) = f[f_1(x_1), \ldots, f_n(x_n)]$

# Multi-attribute Utility Functions. (3)

- Dominance
  - Certain (strict dominance, Fig.1)
    - eg) airport site *S1* cost less, less noise, safer than *S2*

      : **strict dominance of *S1* over *S2***
  - Uncertain(Fig. 2)



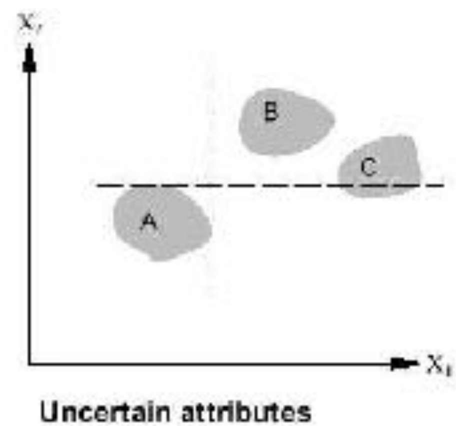
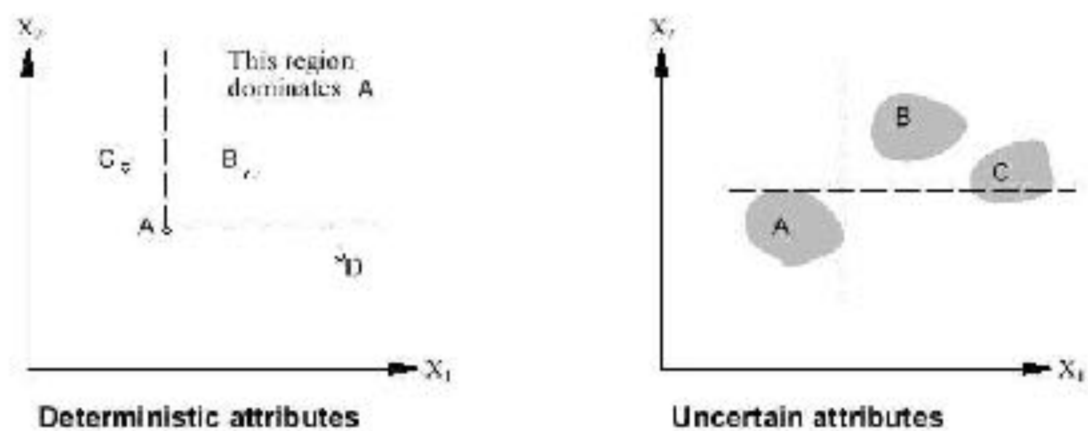




Fig. 1

Fig.2

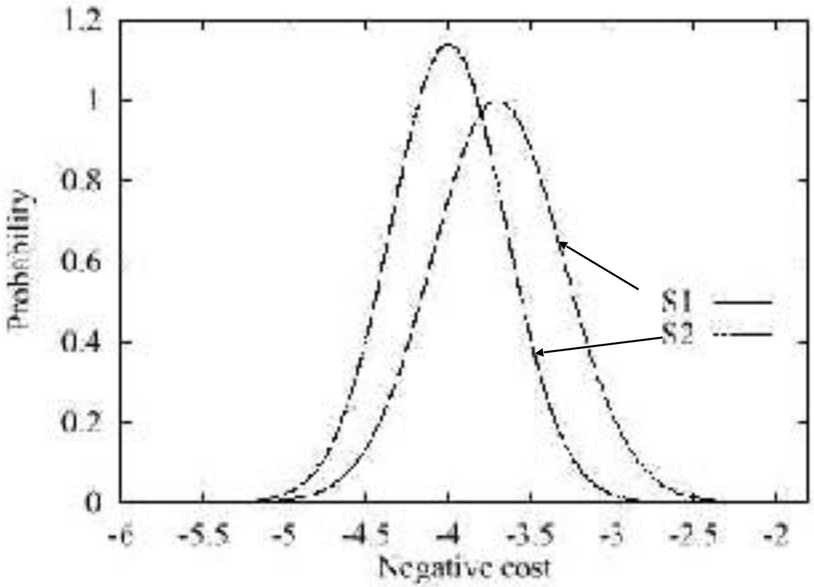
# Multi-attribute Utility Functions. (4)

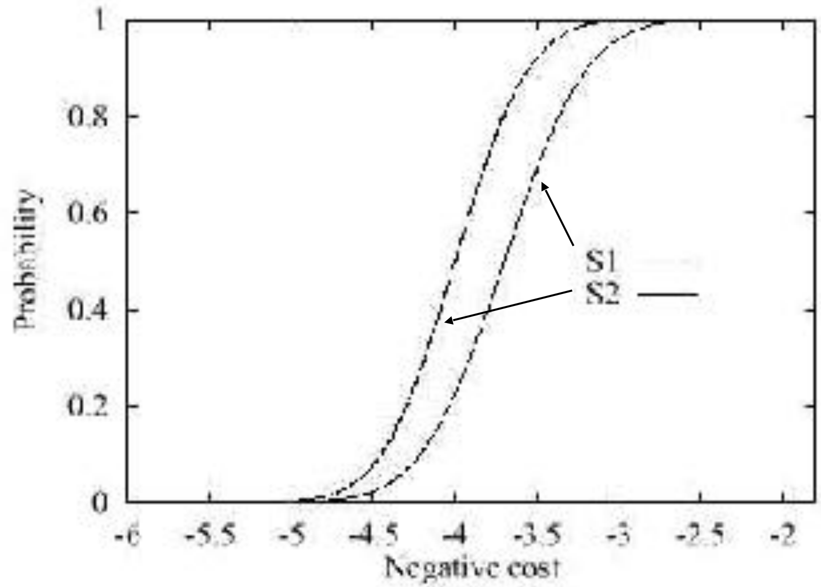
- Dominance(cont.)
  - stochastic dominance
    - In real world problem
    - eg) *S1* : avg \$3.7billion,
      standard deviation : \$0.4billion
      *S2* : avg \$4.0billion,
      standard deviation : \$0.35billion
    - *S1* **stochastic**



# Multi-attribute Utility Functions. (5)

- Dominance(cont.)




$$\forall x \quad \int_{-\infty}^{x} p_1(x)\,dx \le \int_{-\infty}^{x} p_2(x)\,dx$$

# Multi-attribute Utility Functions. (6)

- Preferences without Uncertainty
  - preferences btw. concrete outcome values.
  - Preference structure
    - *$X_1$ & $X_2$ preferentially independent* of $X_3$ iff
      Preference btw. $< x_1, x_2, x_3 >$ & $< x'_1, x'_2, x_3 >$
      Does not depend on $x_3$
    - eg) Airport site : ‹Noise,Cost,Safety›

‹20,000 suffer, $4.6billion, 0.06deaths/mpm›

vs.

‹70,000 suffer, $4.2billion, 0.06deaths/mpm›

# Multi-attribute Utility Functions. (7)

- Preferences without Uncertainty(cont.)
  - Mutual preferential independence(MPI)
    - every pair of attributes is P.I of its complements.
    - eg) Airport site : ⟨Noise, Cost, Safety⟩
      - Noise & Cost **P.I** Safety
      - Noise & Safety **P.I** Cost
      - Cost & Safety **P.I** Noise

      : ⟨Noise,Cost,Safety⟩ exhibits **MPI**

    - Agent's preference behavior

$$\max[V(x_1, x_2, \ldots, x_n) = \sum_{i=1} V_i(x_i)]$$

# Multi-attribute Utility Functions. (7)

- Preferences without Uncertainty(cont.)
  - Mutual preferential independence(MPI)
    - Airport site selection
    - ⟨Noise,Cost,Safety⟩ exhibits **MPI**

    - Agent's preference behavior

$$V(noise, cost, deaths) = -noise \times 10^4 - cost - deaths \times 10^{12}$$

# Multi-attribute Utility Functions. (8)

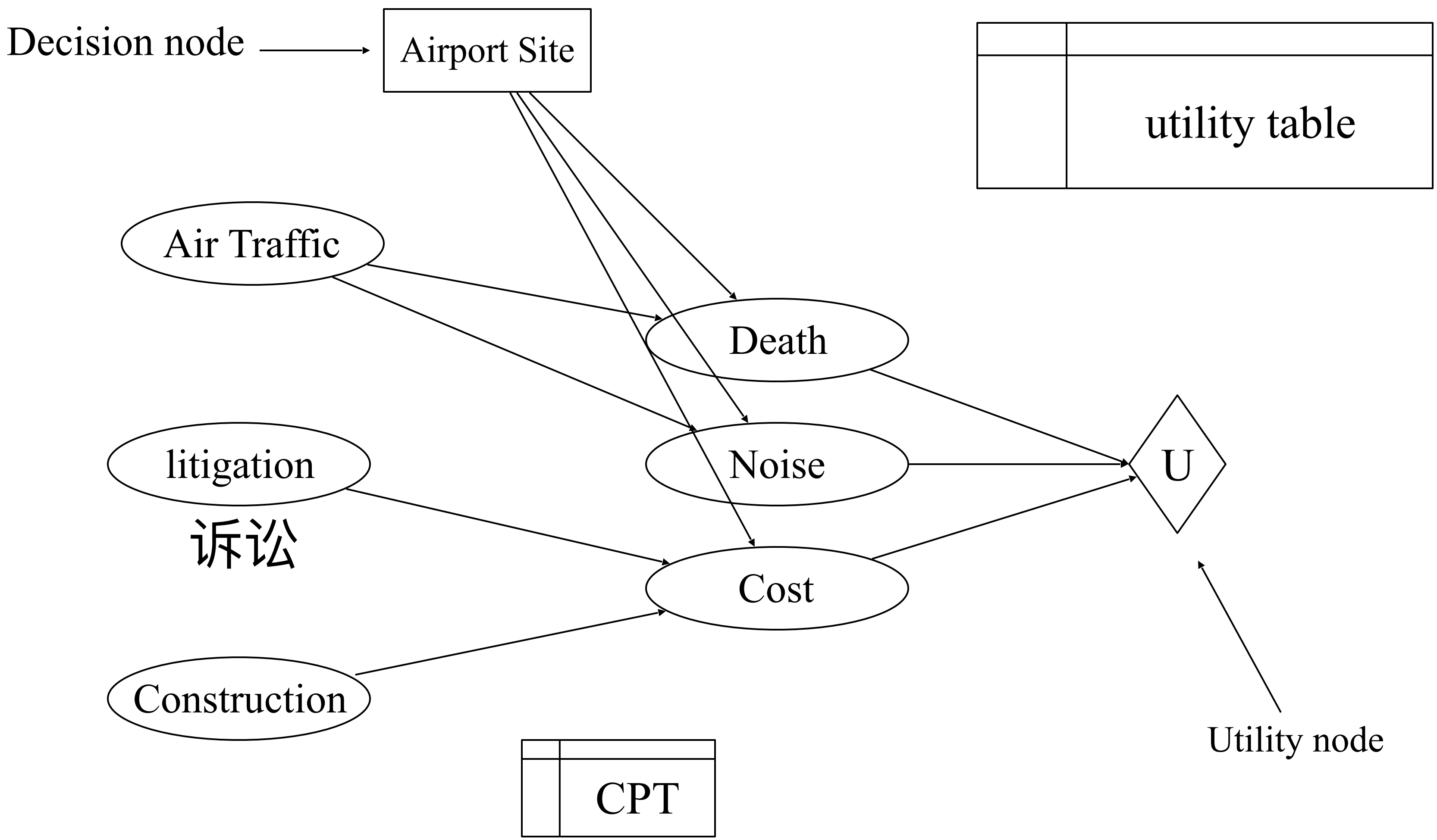
- Preferences with Uncertainty
  - Preferences btw. Lotteries' utility
  - Utility Independence(U.I)
    - **X** is *utility-independent* of **Y** iff preferences over lotteries' attribute set **X** do not depend on particular values of a set of attribute **Y.**
  - Mutual U.I. (MUI)
    - Each subset of attributes is U.I of the remaining attributes.
    - agent's behavior( for 3 attributes) :**multiplicative Utility Fn.**

$$U = k_1U_1 + k_2U_2 + k_3U_3 + k_1k_2U_1U_2 + k_2k_3U_2U_3 + k_3k_1U_3U_1 + k_1k_2k_3U_1U_2U_3$$

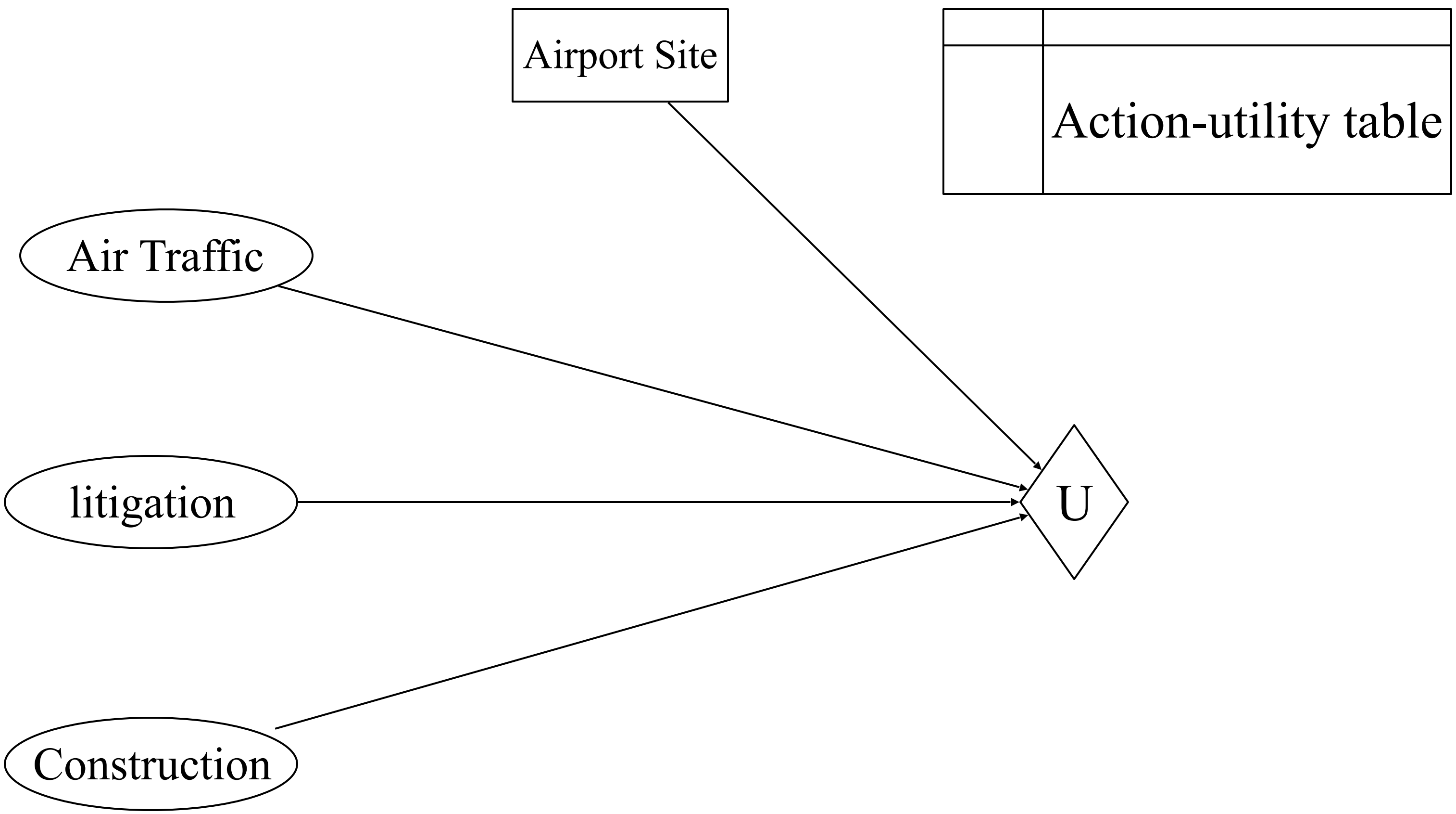
# Decision Networks

- Simple formalism for expressing & solving decision problem
- Belief networks + decision & utility nodes
- Nodes
  - Chance nodes
  - Decision nodes
  - Utility nodes

# A Simple Decision Network



# A Simplified representation



# Evaluating Decision Networks

Function DN-eval (*percept*) returns *action*

static *D*, a decision network

set evidence variables for the current state

for each possible value of decision node

set decision node to that value

calculate Posterior Prob. For parent nodes of the utility node

calculate resulting utility for the action

select the *action* with the highest utility

return *action*

# Value of Information (1)

- Idea
  - Compute value of acquiring each of evidence

- Example : Buying oil drilling rights
  - Three blocks A,B and C, exactly one has oil, worth $k$ *dollars*
  - Prior probabilities 1/3 each, mutually exclusive
  - Current price of each block is $k/3$
  - Consultant offers accurate survey of A. Fair price?

# Value of Information (2)

- Solution: Compute expected value of Information

  = Expected value of best action given information

  - Expected value of best action without information

- Survey may say "oil in A" with pdf 1/3or 'no oil in A"( 2/3)
  - With pdf 1/3, A has oil
  - profit: k-k/3=2k/3
  - With pdf 2/3, A has no oil
  - Profit: k/2-k/3=k/6

$$EV = \frac{1}{3} \times \frac{2k}{3} + \frac{2}{3} \times \frac{k}{6} = \frac{k}{3}$$

# General Formula

- Notation
  - Current evidence E, Current best action $\alpha$
  - Possible action outcomes $\text{Result}_i(\text{A})=S_i$
  - Potential new evidence $E_j$

- Value of perfect information (VPI)

$$EU(\alpha \mid E) = \max_A \sum_i U(S_i)P(S_i \mid E, Do(A))$$

$$EU(\alpha_{E_j} \mid E, E_j) = \max_A \sum_i U(S_i)P(S_i \mid E, Do(A), E_j)$$

$$VPI_E(E_j) = \left( \sum_k P(E_j = e_{jk} \mid E) EU(\alpha_{e_{jk}} \mid E, E_j = e_{jk}) \right) - EU(\alpha \mid E)$$

# Properties of VPI

- Nonnegative

$$\forall j, E \quad VPI_E(E_j) \geq 0$$

- Nonadditive

$$VPI_E(E_j, E_k) \neq VPI_E(E_j) + VPI_{E_j}(E_k)$$

- Order-Independent

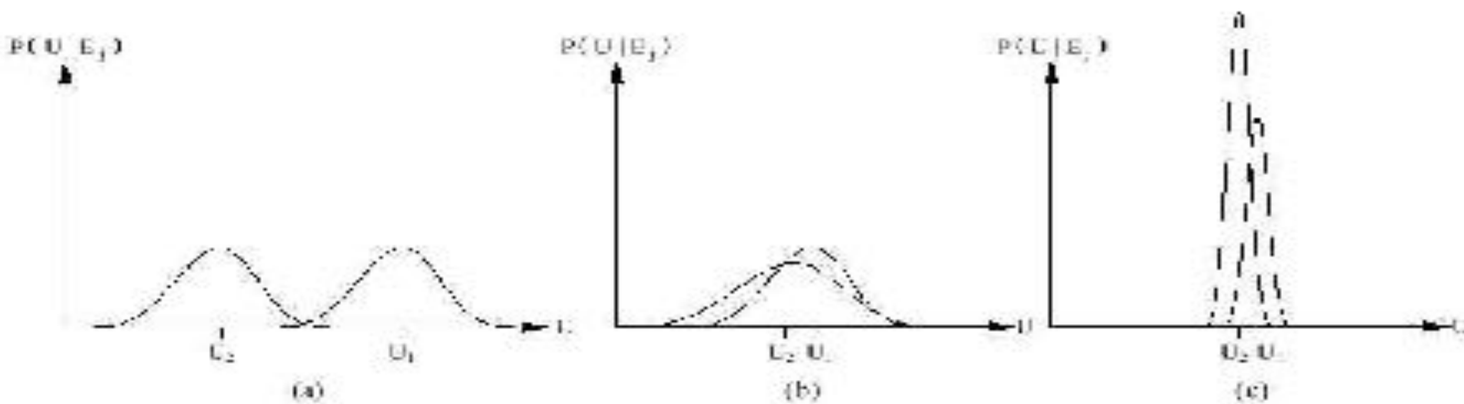
$$VPI_E(E_j, E_k) = VPI_E(E_j) + VPI_{E,E_j}(E_k) = VPI_E(E_k) + VPI_{E,E_k}(E_j)$$

# Three generic cases for VoI

a) Choice is obvious, information worth little

b) Choice is nonobvious, information worth a lot

c) Choice is nonobvious, information worth little



# Decision-Theoretic Expert Systems (1)

- Decision analysis
  - Decision maker
    - States Preferences between outcomes
  - Decision analyst
    - Enumerate possible actions & outcomes
    - Elicit preferences from decision maker to determine the best course of action
- Advantage
  - Reflect preferences of the user, not system
  - Avoid confusing likelihood and importance

# Decision-Theoretic Expert Systems (2)

- Determine the scope of the problem
- Lay out the topology
- Assign probabilities
- Assign utilities
- Enter available evidence
- Evaluate the diagram
- Obtain new evidence
- **Perform sensitivity Analysis**

# Summary

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