Problem 4

Evan Burton, ID: 010945129, Undergraduate

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1 The Problem

Let $X = (x_1, 0), (x_2, 0), \cdots, (x_{20}, 0)$ be a set of 20 distinct points on the positive x-axis and $Y = (0, y_1), (0, y_2), \cdots, (0, y_{16})$ be a set of 16 distinct points on the positive y-axis. Join each point in X to each point in Y by a line segment. Consider all intersection points of these line segments that are not on the x-axis or y-axis. Assuming that no three line segments meet at the same point how many different intersection points are there?

2 Finding a Pattern

Suppose there are j points on the y-axis and i points on the x-axis, where $i, j \geq 2$. We can connect x_1 to all points on the y-axis. Then, $x_2 \to y_1$ will have $j - 1$ intersections, $x_2 \to y_2$ will have $j - 2$, and so on until $x_2 \to y_j$, which will have no intersections with $x_1 \to Y$, except at the axes. This requires that no three line segments meet at the same point.

We repeat this process, for the next point on the x-axis obtaining: $x_3 \to y_1$: $(j - 1) + (j - 1)$ intersections (since the lines from $x_2 \to Y$ are intersected as well), $x_3 \to y_2$: $(j - 2) + (j - 2)$, and so on. For i points on the x-axis and j on the y-axis, we obtain:

$$1\sum_{k=1}^{j-1}(j-k) + 2\sum_{k=1}^{j-1}(j-k) + 3\sum_{k=1}^{j-1}(j-k) + \cdots + (i-1)\sum_{k=1}^{j-1}(j-k)$$

$$= \sum_{k=1}^{j-1}(j-k)\sum_{h=1}^{i-1}(h) = \sum_{k=1}^{j-1}(k)\sum_{h=1}^{i-1}(h)$$

$$= \frac{1}{2}(j-1)(j) \cdot \frac{1}{2}(i-1)(i)$$

$$= \frac{ij}{4}(ij - i - j + 1)$$

To find this pattern, I had to draw the easier cases of $6, 7, 8$ total points. There would be a cool picture here of the pattern for these cases if Geogebra didn't crash on launch.

3 Solution

For 20 points on the x-axis and 16 points on the y-axis, the number of intersections is given by:

$$\frac{16 \cdot 20}{4} \cdot (16 \cdot 20 - 16 - 20 + 1) = 80(320 - 35) = 22,800$$