Problem 13

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1 The Problem

(1) A point (x,y) in $\mathbb{R}^2$ is said to be a lattice point if x and y, are both integers. Show that for any set S of 13 lattice points in $\mathbb{R}^2$, there exist four points in S whose centroid is also a lattice point, where the centroid of 4 lattice points (x_i, y_i), $i = 1, 2, 3, 4$ is $\left(\frac{x_1+x_2+x_3+x_4}{4}, \frac{y_1+y_2+y_3+y_4}{4}\right)$.

(2) What is the smallest number of lattice points that S must have to guarantee that there exist five points in S whose centroid is also a lattice point.

2 Solution to (1)

First, we will solve an easier problem: how many lattice points are required for a midpoint to be guaranteed? This will be used to show that if we can get a guarantee for n points and we have n midpoints, then the midpoint of two midpoints will be the desired result.

For a midpoint to be guaranteed, we need (x_1, y_1) and (x_2, y_2) such that their sum is divisible by 2. Consider the function $f(x, y) = (x \bmod 2, y \bmod 2)$.

The range of f is: $\{(0, 0), (0, 1), (1, 0), (1, 1)\}$. Now, we need a domain on which f is not one-to-one, then we know that some value is mapped to more than once, and we can take those two values, add them, and get a point which is $(0, 0)$ mod 2. Since the range of f has only 4 elements, a domain of 5 elements will guarantee one point in the range of f will be mapped to more than once by the pigeonhole principle. We now have a criterion which guarantees a midpoint which is a lattice point.

Now we are ready to solve problem (1). Suppose we have a set of 13 lattice points $\{p_1, p_2, \ldots, p_{13}\} \subset \mathbb{Z}^2$. Taking groups of 5 lattice points, we are guaranteed that there are some p_i and p_j, $i \neq j$, such that $p_i + p_j$ has components divisible by 2.

Thus, let p_1 and p_2 in the set $\{p_1, p_2, \ldots, p_5\}$ be those points and their midpoint be M_1. We take the next 5 points, $\{p_3, p_4, \ldots, p_7\}$, getting another midpoint from p_3 and p_4, M_2. We can continue this process to get 5 midpoints: $\{M_1, M_2, \ldots, M_5\}$. But a set of 5 lattice points determines a midpoint, so the midpoint of 2 of the midpoints is a lattice point. Therefore, we finally get:

$$\frac{M_i + M_j}{2} = \frac{1}{2}\left(\frac{p_{i1} + p_{i2}}{2} + \frac{p_{j1} + p_{j2}}{2}\right) = \frac{p_{i1} + p_{i2} + p_{j1} + p_{j2}}{4}$$

Which is a lattice point satisfying the form of the centroid of four points.

3 Example for (2)

Consider the 16 points:

$$\begin{array}{cccc}
(0,0) & (0,1) & (1,0) & (1,1) \\
(0,0) & (0,1) & (1,0) & (1,1) \\
(0,0) & (0,1) & (1,0) & (1,1) \\
(0,0) & (0,1) & (1,0) & (1,1)
\end{array}$$

No matter which 5 we pick, the sum will not have both components divisible by 5 since the maximum values are (4,0), (0,4), and (4,4) and the minimum values are (1,0), (0,1), and (1,1). Therefore, the number of lattice points which guarantees a centroid of 5 points has a lower bound of 17.

Suppose we have a function $f : \{x_1, x_2, \ldots, x_{17}\} \to \{0, 1, 2, 3, 4\}$. Then applying f to each of the coordinates of our 17 points $p_1, \ldots, p_{17}$ we know some image is mapped to more than once. Moreover, suppose each image is mapped to at most 4 times, otherwise we would have a trivial combination. Check this