Problem 1

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1 The Problem

A game starts with 1001 numbers 1016, 1017, , 2015, 2016. During each turn two numbers are selected, say j and k. The two numbers j and k are removed and replaced by the single number jk + j + k. After 1000 turns you are left with a single number. What can you say about the final number?

2 Solution

Definition 2.1. A multiset is a set with multiplicity.

Definition 2.2. Let $M-N$ denote the usual set difference, but with multiplicity in mind. For example, if $M = \{1, 1, 1\}$ and $N = \{1, 1\}$, then $M - N = \{1\}$.

Definition 2.3. Define $M \uplus N$ as the union of M and N, adding multiplicities. Then, $\{1, 2\} \uplus \{2, 3\} = \{1, 2, 2, 3\}$.

Now, let T be a relation that takes a multiset X with $n + 1$ real elements, $n \geq 1$, and returns a multiset Y with n real elements such that

$$Y = \{x_i x_j + x_i + x_j\} \uplus X - \{x_i, x_j\}$$

for arbitrary x_i and x_j.

Let H be a function which acts on a multiset X as described above and returns the only element of $T^n(X)$. Calculations quickly show that $T(X)$ is a unique real number for $|X| = 2$, and by induction, $T^n(X) \in \mathbb{R}$ because $T^{n+1} = T \circ T^n$. In fact,

$$H(\{a, b, c\}) = (1 + a)(1 + b)(1 + c)$$

leading to the more general result:

Theorem. If X is a multiset and $|X| = n + 1$, then

$$H(X) = -1 + \prod_{x \in X} (1 + x)$$

Proof. For $n = 1$, $X = x_1, x_2$.

$$H(X) = T(X) = x_1x_2 + x_1 + x_2$$
$$= -1 + 1 + x_1x_2 + x_1 + x_2$$
$$= -1 + (1 + x_1)(1 + x_2)$$
$$= -1 + \prod_{x \in X} (1 + x)$$

Suppose the theorem is true for $n \geq 1$. Let $Y = T(X)$,

$$T^{n+1}(X) = T^n \circ T(X) = T^n(Y) = -1 + \prod_{y \in Y} (1 + y)$$

By the definition of $Y = T(X)$:

$$= -1 + \prod_{x \in X - \{x_i, x_j\}} (1 + x)(1 - 1 + (1 + x_i)(1 + x_j))$$

$$= -1 + \prod_{x \in X - \{x_i, x_j\}} (1 + x)(1 + x_i)(1 + x_j)$$

$$= -1 + \prod_{x \in X} (1 + x)$$

$\square$

Therefore, for $X = \{1016, 1017, \ldots, 2016\}$, we have the solution

$$H(X) = -1 + \prod_{x \in X} (1 + x)$$

$$= -1 + \prod_{i=1017}^{2017} i = -1 + \frac{2017!}{1016!}$$