

## Problem 6

Evan Burton, ID: 010945129, Undergraduate

September 25, 2016

### 1 The Problem

Find a real-valued function  $f$  such that  $f(x) = \frac{1}{2} - \int_0^x \cos(t)[f(t)]^2 dt$ .

### 2 Solution

Consider the derivative of  $f$ :

$$\frac{d}{dx} f(x) = \frac{d}{dx} \left( \frac{1}{2} - \int_0^x \cos(t)[f(t)]^2 dt \right)$$

$$\frac{df}{dx} = -\frac{d}{dx} \int_0^x \cos(t)[f(t)]^2 dt$$

$$\frac{df}{dx} = -\cos(x)[f(x)]^2$$

Let  $f' = \frac{df}{dx}$  and  $f^2 = [f(x)]^2$  and suppose  $f^2 \neq 0$ ,

$$f' = -f^2 \cos(x)$$

$$\frac{f'}{f^2} = -\cos(x)$$

$$\int \frac{f'}{f^2} dx = \int -\cos(x) dx$$

$$\int \frac{f'}{f^2} dx = -\sin(x) + c$$

Let  $u = f(x)$ , then  $du = f'(x)dx$ :

$$\int \frac{f'}{u^2} \frac{du}{f'} = \int \frac{1}{u^2} du = -\frac{1}{u} + c_2 = -\sin(x) + c$$

$$-\frac{1}{f(x)} = -\sin(x) + d$$

$$f(x) = \frac{1}{\sin(x) - d}$$

From the problem statement, we know that  $f(0) = \frac{1}{2}$ , so

$$\begin{aligned} f(0) = \frac{1}{2} &= \frac{1}{\sin(0) - d} = \frac{1}{-d} \\ d &= -2 \end{aligned}$$

$$f(x) = \frac{1}{\sin(x) + 2}$$

### 3 Checking Solution

Checking to see that  $f$  satisfies  $f(x) = \frac{1}{\sin(x)+2} = \frac{1}{2} - \int_0^x \frac{\cos(t)}{(\sin(t) + 2)^2} dt$

By substitution,  $u = \sin(t) + 2$ ,  $du = \cos(t)dt$ :

$$\begin{aligned} f(x) &= \frac{1}{2} - \int_2^{\sin(x)+2} \frac{1}{u^2} du \\ &= \frac{1}{2} + \left[ \frac{1}{u} \right]_2^{\sin(x)+2} = \frac{1}{2} + \left( \frac{1}{\sin(x) + 2} - \frac{1}{2} \right) \\ &= \frac{1}{\sin(x) + 2} = f(x) \end{aligned}$$