Problem 2

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1 The Problem

Two math majors Alice and Brad settle a dispute by flipping coins. Alice flips 101 coins and Brad flips 100 coins, and assume that each coin flip is equally likely to end up heads or tails. What is the probability that Alice ends up with more heads than Brad, thus wining the dispute

2 Setup and Solution?

For Brad, there are 2^{100} possible outcomes, but we are interested in the cases where he gets at most k tails. Similarly, there are 2^{101} possible outcomes for Alice's experiment and we will count the cases where she gets at most k tails. Since Alice does one more coin flip than Brad, she will get at least one more coin heads or tails up if they both get at most the same amount of tails. Likewise, we could fix the number of heads they both get and call that k, since they both cannot have the same amount of heads and tails as each other in the end. If $k = 100$, then we should find that Alice has a 50-50 shot at winning.

$$P(B) = \frac{possible}{total} = \frac{\sum_{i=0}^{k} \binom{100}{i}}{2^{100}} = 2^{-100} \sum_{i=0}^{k} \binom{100}{i}$$

$$P(A) = \frac{possible}{total} = \frac{\sum_{j=0}^{k} \binom{101}{j}}{2^{101}} = 2^{-101} \sum_{j=0}^{k} \binom{101}{j}$$

If we are to believe that Alice will have a slight edge in the competition, then we would expect $\frac{P(A)}{P(B)} > 1$.

$$\frac{P(A)}{P(B)} = \frac{2^{-101} \sum_{j=0}^{k} \binom{101}{j}}{2^{-100} \sum_{i=0}^{k} \binom{100}{i}} = \frac{1}{2} \frac{\sum_{j=0}^{k} \binom{101}{j}}{\sum_{i=0}^{k} \binom{100}{i}}$$

Note: the recursive definition of $\binom{n}{k}$, when $1 \le k \le n-1$ is

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

Now, the numerator may be written as

$$\sum_{j=0}^{k} \binom{101}{j} = \binom{101}{0} + \sum_{j=1}^{k} \binom{101}{j}$$

By the recursive formula for $\binom{n}{k}$ and restricting k to be less than 101:

$$= 1 + \sum_{j=1}^{k} \binom{100}{j-1} + \binom{100}{j} = 1 + \sum_{j=1}^{k} \binom{100}{j-1} + \sum_{j=1}^{k} \binom{100}{j}$$

Let $a = j - 1$, then

$$1 + \sum_{j=1}^{k} \binom{100}{j-1} + \sum_{j=1}^{k} \binom{100}{j} = 1 + \sum_{a=0}^{k} \binom{100}{a} + \sum_{j=1}^{k} \binom{100}{j}$$

Since $\binom{100}{0} = 1$, we have

$$= \binom{100}{0} + \sum_{a=0}^{k} \binom{100}{a} + \sum_{j=1}^{k} \binom{100}{j} = \sum_{a=0}^{k} \binom{100}{a} + \sum_{j=0}^{k} \binom{100}{j}$$

Both a and j are dummy variables, and so, the summations may be combined.

$$\sum_{a=0}^{k} \binom{100}{a} + \sum_{j=0}^{k} \binom{100}{j} = 2\sum_{j=0}^{k} \binom{100}{j}$$

Now that we have simplified the numerator, we may rewrite $\frac{P(A)}{P(B)}$ as

$$\frac{P(A)}{P(B)} = \frac{1}{2} \cdot \frac{2\sum_{j=0}^{k} \binom{100}{j}}{\sum_{i=0}^{k} \binom{100}{i}} = 1$$

Therefore, we expect Alice to win no more often than Brad at the competition, resulting in a winning chance of 50% for $0 \leq k \leq 100$ tails. So if both players get the same amount of heads or tails, the one with the extra coin will have a 50% chance of winning or losing. I am positive there is a better way of doing this problem because this approach seemed particularly difficult.

I think this problem could also be solved knowing that either Alice will get more heads or more tails than Brad and that it would be impossible for her to get both the same number of heads and tails as her competitor, since he has one less coin. Therefore, she must either win or lose half the time. But this thought just came to me now, after all of the previous work was already done. I did find it amusing, however, that the sums ended up canceling each other.