Problem 7

Evan Burton, ID:010945129, Undergraduate

October 6, 2016

1 Problem Statement

Show that the determinant of A is an integer, where

$$A = \begin{bmatrix} cos(1) & cos(11) & \cdots & cos(91) \\ cos(2) & cos(12) & \cdots & cos(92) \\ \vdots & \vdots & \ddots & \vdots \\ cos(10) & cos(20) & \cdots & cos(100) \end{bmatrix}$$

and the angles are in radians.

2 Solution

Consider $e^{i\theta} = cos(\theta) + isin(\theta)$. Since cosine is even, $cos(-\theta) = cos(\theta)$ and since sine is odd, $sin(-\theta) = -sin(\theta)$.

Then, $e^{-i\theta} = e^{i(-\theta)} = cos(-\theta) + isin(-\theta) = cos(\theta) - isin(\theta)$.

$$cos(\theta) = \frac{1}{2}(e^{i\theta} + e^{-i\theta}) = cosh(i\theta)$$

We will need a property of hyperbolic cosine:

$$4cosh(i\theta)cosh(i) = (e^{i\theta} + e^{-i\theta})(e^{i} + e^{-i})$$

$$= e^{i\theta+i} + e^{i\theta-i} + e^{-i\theta+i} + e^{-i\theta-i}$$

$$= e^{i(\theta+1)} + e^{i(\theta-1)} + e^{-i(\theta-1)} + e^{-i(\theta+1)}$$

$$= 2cosh(i(\theta+1)) + 2cosh(i(\theta-1))$$

$$2cosh(i\theta)cosh(i) = cosh(i(\theta+1)) + cosh(i(\theta-1))$$

Since $cos(\theta) = cosh(i\theta)$,

$$cos(\theta+1) = 2cos(1)cos(\theta) + (-1)cos(\theta-1)$$

Equivalently,

$$cos(u) = 2cos(1)cos(u-1) + (-1)cos(u-2)$$

We can write the angle of any entry in A as $u = r + 10(j-1)$, where r is the row and j is the column. Therefore, we can subtract 1 from u to get the row above it as long as $r > 1$.

Let $u = 3 + 10(j-1)$ where $j = 1, 2, \ldots, 10$. Then

$$A(3, j) = cos(u)$$

We can get all of Row 2 by

$$A(2, j) = cos(u-1)$$

and all of Row 1 by

$$A(1, j) = cos(u-2)$$

.

We can write $cos(u)$ as a linear combination of $cos(u-1)$ and $cos(u-2)$, where those are entries of Row 2 and Row 1, respectively. More generally, let $A(r)$ be the r^{th} row of A, $r \geq 3$,

$$A(r) = (2cos(1))A(r-1) + (-1)A(r-2)$$

Therefore, Row 3 is a linear combination of Rows 1 and 2 and $\det A = 0$, which is an integer.