Problem 6

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1 The Problem

Find a real-valued function f such that $f(x) = \frac{1}{2} - \displaystyle\int_0^x cos(t)[f(t)]^2 dt$.

2 Solution

Consider the derivative of f:

$$\frac{d}{dx}f(x) = \frac{d}{dx}(\frac{1}{2} - \int_0^x cos(t)[f(t)]^2 dt)$$

$$\frac{df}{dx} = -\frac{d}{dx}\int_0^x cos(t)[f(t)]^2 dt$$

$$\frac{df}{dx} = -cos(x)[f(x)]^2$$

Let $f' = \frac{df}{dx}$ and $f^2 = [f(x)]^2$ and suppose $f^2 \neq 0$,

$$f' = -f^2 cos(x)$$

$$\frac{f'}{f^2} = -cos(x)$$

$$\int \frac{f'}{f^2} dx = \int -cos(x) dx$$

$$\int \frac{f'}{f^2} dx = -sin(x) + c$$

Let $u = f(x)$, then $du = f'(x)dx$:

$$\int \frac{f'}{u^2}\frac{du}{f'} = \int \frac{1}{u^2} du = -\frac{1}{u} + c_2 = -sin(x) + c$$

$$-\frac{1}{f(x)} = -sin(x) + d$$

$$f(x) = \frac{1}{sin(x) - d}$$

From the problem statement, we know that $f(0) = \frac{1}{2}$, so

$$f(0) = \frac{1}{2} = \frac{1}{sin(0) - d} = \frac{1}{-d}$$

$$d = -2$$

$$f(x) = \frac{1}{sin(x) + 2}$$

3 Checking Solution

Checking to see that f satisfies $f(x) = \frac{1}{sin(x)+2} = \frac{1}{2} - \displaystyle\int_0^x \frac{cos(t)}{(sin(t)+2)^2} dt$

By substitution, $u = sin(t) + 2$, $du = cos(t)dt$:

$$f(x) = \frac{1}{2} - \int_2^{sin(x)+2} \frac{1}{u^2} du$$

$$= \frac{1}{2} + \left[\frac{1}{u}\right]_2^{sin(x)+2} = \frac{1}{2} + (\frac{1}{sin(x)+2} - \frac{1}{2})$$

$$= \frac{1}{sin(x)+2} = f(x)$$