Problem 9

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1 Problem

Let $x_0 = 0$, $x_1 = 1$, and for $n \geq 1$,

$$x_{n+1} = \frac{x_n}{n+1} + (1 - \frac{1}{n+1})x_{n-1}$$

Determine $\lim_{n\to\infty} x_n$.

2 Solution

Suppose $n \in \mathbb{N}, n \geq 1$, then $x_n = \sum_{k=1}^{n} \frac{(-1)^{k+1}}{k}$

For $n = 1$:
$x_1 = \sum_{k=1}^{1} \frac{(-1)^{k+1}}{k} = \frac{1}{1} = 1$

Suppose the proposition is true up to n and $n > 1$, we need to show that the proposition is still true for $n + 1$.

$$x_{n+1} = \sum_{k=1}^{n+1} \frac{(-1)^{k+1}}{k} = \frac{(-1)^{n+2}}{n+1} + \sum_{k=1}^{n} \frac{(-1)^{k+1}}{k}$$

$$= \frac{(-1)^{n+2}}{n+1} + \frac{(-1)^{n+1}}{n} + \sum_{k=1}^{n-1} \frac{(-1)^{k+1}}{k}$$

$$= (-1)^{n+1} \left[\frac{1}{n} - \frac{1}{n+1}\right] + \sum_{k=1}^{n-1} \frac{(-1)^{k+1}}{k}$$

$$= \frac{(-1)^{n+1}}{n(n+1)} + \sum_{k=1}^{n-1} \frac{(-1)^{k+1}}{k} = \frac{1}{n+1} \left[\frac{(-1)^{n+1}}{n} + (n+1)\sum_{k=1}^{n-1} \frac{(-1)^{k+1}}{k}\right]$$

$$= \frac{1}{n+1}\left[\frac{(-1)^{n+1}}{n} + n\sum_{k=1}^{n-1}\frac{(-1)^{k+1}}{k} + \sum_{k=1}^{n-1}\frac{(-1)^{k+1}}{k}\right]$$

$$= \frac{1}{n+1}\left[n\sum_{k=1}^{n-1}\frac{(-1)^{k+1}}{k} + \sum_{k=1}^{n}\frac{(-1)^{k+1}}{k}\right] = \frac{nx_{n-1} + x_n}{n+1}$$

$$= \frac{x_n}{n+1} + \frac{nx_{n-1}}{n+1} = \frac{x_n}{n+1} + \left(1 - \frac{1}{n+1}\right)x_{n-1}$$

Which is the definition of x_{n+1} stated in the problem.

Therefore, $x_n = \sum_{k=1}^{n}\frac{(-1)^{k+1}}{k}$ and the limit

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty}\sum_{k=1}^{n}\frac{(-1)^{k+1}}{k} = ln(2)$$

Of course, a proof by induction is not very constructive. My motivation for the proposition was that the sequence $x_n - x_{n-1}$ gave the terms of the Maclaurin series of $ln(1+x)$ at $x = 1$ and the first few values of x_n seemed to confirm this, so it was a natural candidate for induction.