Problem 1

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1 The Problem

A game starts with 1001 numbers 1016, 1017, , 2015, 2016. During each turn two numbers are selected, say j and k. The two numbers j and k are removed and replaced by the single number jk + j + k. After 1000 turns you are left with a single number. What can you say about the final number?

2 Definitions

Definition 1. A multiset is a set with multiplicity.

Definition 2. Let $M-N$ denote that all elements of N are to be removed from M with multiplicity in mind. For example, if $M = \{1,1,1\}$ and $N = \{1,1\}$, then let $M - N = \{1\}$ and $N - M = \{\emptyset\}$.

Definition 3. Define $M \uplus N$ as the multiset of all elements of both M and N. For example, $\{1,2\} \uplus \{2,3\} = \{1,2,2,3\}$.

Definition 4. Let T be a relation that takes a multiset X with $n+1$ real elements, $n \geq 1$, and returns a multiset Y with n real elements such that

$$Y = \{x_i x_j + x_i + x_j\} \uplus X - \{x_i, x_j\}$$

where x_i and x_j are random elements in X. Let $T^n(X)$ denote repeated composition such that $T^n(X) = T \circ T \circ \cdots \circ T(X)$.

Example: Consider $V = \{a,b,c\}$.

$$T(T(V)) = T^2(V) = \begin{cases} T(\{ab+a+b,c\}) = \{abc+ab+ac+bc+a+b+c\} \\ T(\{ac+a+c,b\}) = \{abc+ab+ac+bc+a+b+c\} \\ T(\{bc+b+c,a\}) = \{abc+ab+ac+bc+a+b+c\} \end{cases}$$

Notice that the same value is returned in each case and the similarity between $T^2(V)$ and expanded form of the product

$$(1+a)(1+b)(1+c) = 1+abc+ab+ac+bc+a+b+c$$

3 Proposition

Suppose X is a multiset with $n+1$ real elements, then

$$T^n(X) = \{-1 + \prod_{\forall x \in X} (1+x)\}$$

Proof. Base Case: $n = 1$ and $X = \{x_1, x_2\}$.

$$\begin{aligned} T(X) = T(\{x_1, x_2\}) &= \{x_1x_2 + x_1 + x_2\} \\ &= \{-1 + 1 + x_1x_2 + x_1 + x_2\} \\ &= \{-1 + (1+x_1)(1+x_2)\} \\ &= \{-1 + \prod_{\forall x \in X} (1+x)\} \end{aligned}$$

Suppose the proposition is true for $n = k$ and $k \geq 1$, we need to show that it is true for $n = k+1$. For $n = k+1$, X is a multiset with $k+2$ real elements. Let $T(X) = \{x_ix_j + x_i + x_j\} \uplus X - \{x_i, x_j\} = Y$ for any two elements, x_i and x_j in X. Then,

$$Y = \{-1 + 1 + x_ix_j + x_i + x_j\} \uplus X - \{x_i, x_j\}$$

$$Y = \{-1 + (1+x_i)(1+x_j)\} \uplus X - \{x_i, x_j\}$$

$$T^{k+1}(X) = T^k(T(X)) = T^k(Y) = \{-1 + \prod_{\forall x \in Y} (1+x)\}$$

$$= \{-1 + \prod_{\forall x \in X - \{x_i, x_j\}} (1+x)\,(1 - 1 + (1+x_i)(1+x_j))\}$$

$$= \{-1 + \prod_{\forall x \in X - \{x_i, x_j\}} (1+x)(1+x_i)(1+x_j)\}$$

$$= \{-1 + \prod_{\forall x \in X} (1+x)\}$$

□

4 Solution

Let $X = \{1016, 1017, \cdots, 2015, 2016\}$. We want to find out facts about the number left over after 1000 turns. From the proposition, I found that the number after 1000 turns is the same no matter which elements are picked each turn. Using a computer I found the exact number, that it is odd, and that it is relatively prime to every number from 1016 to 2016. Using Java's BigInteger class, I calculated $T^{1000}(X)$ using the formula in the proposition. The number is printed on the following page.

$$T^{1000}(X) = \{-1 + \prod_{x \in X} (1 + x)\} = \{$$

101836028017116255559070133159539196728343700204310424919380543934247
005873019012970151902069601711052900771547100214253450303716869940151
257613379781852381357615833683403821094193846628993795389378047552326
431516390821002445429798442590346073038315129150503190571901328727971
193780563370624643452108751102389124136417984637622250103123713101240
465006865559339202328032056578463376045402499291012809316995749564565
434401384123782640696491465816466074884747094298260144512507208060484
919194265852610425357451307025986553715248344867899865005378207228412
624209899838720808469641516274131727228168643195543715333858894419845
828498535027080627723689625830438286376923230401524791532567923189350
049005040036101184227432616734564844370901475278725562996459268638062
223029782315254644291439675834598408072730452242676386546953764757264
566958008527879075943275757132780421561503973222190325222672570559598
015542793693641446834932336451116596991002055145924726551487247439363
238733359182268712218700874015381066669580576538191712792782843020474
750669049186205704663261644768183656064781196545909360306930997980245 4
212511032735523428889588152516598682856632667791447238509375832520749
537314452783184550441747256826598226376769095862786372909536158518766
171823724173249971091023564309670241982883242794243857155348646712555
862907827484183418929581220614796896794222288915586939798936423683452
071082189584505826061091344335441904051886564484201092480505413846773
548583352259873335995696006576383176513875332397715213826072424020886
771539178747507900590643358261898158794233630255789019605748797364916
426751610577103278229667696331100787547529256857151833234223650362345
105363093280182184797935833892962148766919768601472276870347655431383
911119146542788836016877886915082530254800159089679359762189002857266
834913469280047136813848381127449522145346071324275844047248218371977
735455877646655793541054820514477843777762418431192390248178829205048
604857424405238103799619605927866409543380718793950414508398260701911
075307101248204616566140164228171869234004890569758335556102115241357
905247108350486335025475354100426857258542220000773998666724573755254
159970826624433340650166261780798834959552777662582075744128286748145
498269715101407663297846199593542093174940791124521654937308120313848
914707005421736775574845505051649815399085473961497801626600566553810
695772370392962398642791834594748074396656661528617054572702030934439
985579329022383331197395415856033885682804281275451248525336234139437
965065089082954999903528529047554055139627772231340261127845136541257
205606387764725003970803341479285487803540972374945219518875969281959
112150238503545481397420440432021575982034029360193354472611342949605
920879591056716669908065749336566324737529325004255805394357991549311
752374325817179436727928906977446171647812045856253066329159029539923
816970580532753956249781114910660850778438078171352858370739085671276
637197524621058519522481274879999999999999999999999999999999999999999
999999999999999999999999999999999999999999999999999999999999999999999
999999999999999999999999999999999999999999999999999999999999999999999
999999999999999999999999999999999999999999999999999999999999999999999}

5 Code

```
import java.math.BigInteger;

public class Problem1 {

        public static BigInteger play(BigInteger[] list){

                BigInteger m = list[0].add(BigInteger.ONE);

                for(int i = 1; i < list.length; i++){
                        m = (BigInteger.ONE.add(list[i])).multiply(m);

                }

                return m.subtract(BigInteger.ONE);
        }

        public static void main(String[] args) {

                BigInteger [] list = new BigInteger[1001];
                BigInteger result = BigInteger.ZERO;

                for(int i = 1016; i <= 2016; i++){
                        list[i-1016] = BigInteger.valueOf(i);
                }

                result = play(list);

                BigInteger gcd;

                for(int i = 0; i <= 1000; i++){

                        gcd = result.gcd(BigInteger.valueOf(i+1016));

                        if(!gcd.equals(BigInteger.ONE)){
                                System.out.println(gcd);
                        }
                }

                System.out.println(result.toString());
        }
}
```