MATHEMATICS 2^{nd} Term Mid-Test Revision

1 $(x+\sqrt{x})^4 - (x-\sqrt{x})^2 = 159{,}600$

$$\left((x+\sqrt{x})^2\right)^2 - (x-\sqrt{x})^2 = 159{,}600$$

let $(x+\sqrt{x})^2 = t$

So, the equation becomes,

$$t^2 - t = 159{,}600$$

$$t^2 - t - 159{,}600 = 0$$

$$\Delta = b^2 - 4ac \Rightarrow (-1)^2 - 4(1)(-159{,}600)$$

$$\Rightarrow 1 + 638400$$

$$\Rightarrow 638{,}401$$

$$\sqrt{\Delta} = \sqrt{638401} = 799$$

$$t_1 = \frac{-b+\sqrt{\Delta}}{2a} = \frac{-(-1)+799}{2(1)} = \frac{800}{2} = 400$$

$$t_2 = \frac{-b-\sqrt{\Delta}}{2a} = \frac{-(-1)-799}{2(1)} = \frac{-798}{2} = -399$$

Δ t_2 is rejected because it is negative

Thus, $(x+\sqrt{x})^2 = 400$

$x + \sqrt{x} = \sqrt{400}$, $x+\sqrt{x} = 20$

$x + \sqrt{x} = 20$, note that $x = (\sqrt{x})^2$

$(\sqrt{x})^2 + \sqrt{x} = 20$, let $\sqrt{x} = y$

$= y^2 + y = 20$, $y^2 - y - 20 = 0$

$$\Delta = b^2 - 4ac$$

$$= (+1)^2 - 4(1)(-20)$$

$$= 81$$

$$\sqrt{\Delta} = \sqrt{81} = 9$$

$$y_1 = \frac{-(+1)+9}{2} = \frac{8}{2} = 4$$

$$y_2 = \frac{-(1)-9}{2} = \frac{-10}{2} = -5$$

y_2 is rejected since it is negative

Thus,

$\sqrt{x} = 4$

$x = 4^2$, $x = 16$

$$\boxed{S = \{(16)\}}$$

---

2 i) For a triangle with vertices (x_1, y_1); (x_2, y_2) and (x_3, y_3)

$$\boxed{\text{Area} = \frac{1}{2}\left|\begin{matrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{matrix}\right|}$$

$$\text{Area} = \frac{1}{2}\left|\begin{array}{ccc|cc} -3 & 1 & 1 & -3 & 1 \\ 2 & -4 & 1 & 2 & -4 \\ 5 & 1 & 1 & 5 & 1 \end{array}\right|$$

$$= \frac{1}{2}\left([12+5+2] - [-20-3+\right.$$

$$= \frac{1}{2}\left([19] - [-21]\right)$$

$$= \frac{1}{2}(40) = 20$$

The triangle has an areas of 2[

ii) points are collinear when:

$$\left|\begin{matrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{matrix}\right| = 0$$

In this case, points are "not" colline[

because $\left|\begin{matrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{matrix}\right| = 40$

(3) To determine the direction of an object we need to find its velocity:

- when velocity is negative, the object is moving to the left
- When velocity is positive, the object is moving to the right

(*) Note that velocity = derivative (displacement)

$\Rightarrow$ Velocity $= f'(t)$

Velocity $= \left[\dfrac{t}{(t+1)^2}\right]'$, $\boxed{\left(\dfrac{u}{v}\right)' = \dfrac{u'v - v'u}{v^2}}$

$\Rightarrow \dfrac{t'(t+1) - (t+1)'t}{(t+1)^2}$

Velocity$(t) = \dfrac{\cancel{t}+1-\cancel{t}}{(t+1)^2} \Rightarrow \dfrac{1}{(t+1)^2}$

Velocity$(t) = \dfrac{1}{(t+1)^2}$,

At $t = 10$ hours, Velocity$(10) = \dfrac{1}{(10+1)^2} = \dfrac{1}{121}$

> Since velocity is positive, the object is moving to the right after 10 hours

(4) a) $\vec{A} = \alpha\vec{u} + \beta\vec{v}$

$$\begin{pmatrix}1\\6\end{pmatrix} = \alpha\begin{pmatrix}1\\3\end{pmatrix} + \beta\begin{pmatrix}-1\\-2\end{pmatrix}$$

⚠ solve for α and β

$$\begin{array}{l} 1 = \alpha - \beta \\ 6 = 3\alpha - 2\beta \end{array} \Bigg\| \begin{array}{l}\times 2\\ \times 1\end{array} \Rightarrow \begin{cases} 2 = 2\alpha - 2\cancel{\beta} \\ 6 = 3\alpha - 2\cancel{\beta}\end{cases}$$

$-4 = -\alpha$, $\alpha = 4$

Replace α is any equation:

$1 = 4 - \beta$, $\beta = 3$

Thus, $\boxed{\vec{A} = 4\vec{u} + 3\vec{v}}$

b) $V_1 = 2\vec{e_1} + \vec{e_2}$

$V_2 = 3\vec{e_1} + \vec{e_2}$

$$\boxed{\vec{e_1} = \begin{pmatrix}1\\0\\0\end{pmatrix}, \quad \vec{e_2} = \begin{pmatrix}0\\1\\0\end{pmatrix}}$$

$$V_1 = 2\begin{pmatrix}1\\0\\0\end{pmatrix} + \begin{pmatrix}0\\1\\0\end{pmatrix}, \quad V_1 = \begin{pmatrix}2\\0\\0\end{pmatrix} + \begin{pmatrix}0\\1\\0\end{pmatrix}$$

$$\boxed{V_1 = \begin{pmatrix}2\\1\\0\end{pmatrix}}$$

$$V_2 = 3\begin{pmatrix}1\\0\\0\end{pmatrix} + \begin{pmatrix}0\\1\\0\end{pmatrix} \Rightarrow V_1 = \begin{pmatrix}3\\0\\0\end{pmatrix} + \begin{pmatrix}0\\1\\0\end{pmatrix}$$

$$\boxed{V_2 = \begin{pmatrix}3\\1\\0\end{pmatrix}}$$

After finding V_1, V_2 let's write it as a linear combination with respect to $\vec{e_1}$

$\vec{e_1} = \alpha\vec{V_1} + \beta\vec{V_2}$

$$\begin{pmatrix}1\\0\\0\end{pmatrix} = \alpha\begin{pmatrix}2\\1\\0\end{pmatrix} + \beta\begin{pmatrix}3\\1\\0\end{pmatrix}; \begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}2\alpha\\\alpha\\0\end{pmatrix} + \begin{pmatrix}3\beta\\\beta\\0\end{pmatrix}$$

$$\begin{array}{l} 1 = 2\alpha + 3\beta \\ 0 = \alpha + \beta \end{array} \Bigg\| \begin{array}{l}\times 1\\ \times 2\end{array} \quad - \begin{array}{l} 1 = 2\cancel{\alpha} + 3\beta \\ 0 = 2\cancel{\alpha} + 2\beta \end{array}$$

$\boxed{1 = \beta}$

$\beta = 1$, Substitute the value of β into any equation,

$\alpha + \cancel{\beta} = -\cancel{1}$ $\quad \alpha + (+1) = 0$

$\boxed{\alpha = -1}$

Thus, $\boxed{\vec{e_1} = -\vec{V_1} + \vec{V_2}}$

Continuation on next page

→ continuation of 4b)

$\vec{e_2} = a\vec{V_1} + b\vec{V_2}$

$\begin{pmatrix}0\\1\\0\end{pmatrix} = a\begin{pmatrix}2\\1\\0\end{pmatrix} + b\begin{pmatrix}3\\1\\0\end{pmatrix}; \begin{pmatrix}0\\1\\0\end{pmatrix} = \begin{pmatrix}2a\\a\\0\end{pmatrix} + \begin{pmatrix}3b\\b\\0\end{pmatrix}$

solve for a and b

$2a + 3b = 0 \,||\, \times 1 \quad | \quad 2a + 3b = 0$

$a + b = 1 \,||\, \times 2 \quad | \quad -2a + 2b = 2$

$\boxed{b = -2}$

b = -2, Replace the value of b into any equation

$a - 2 = 1$, $\boxed{a = 3}$

$\boxed{\text{Thus, } \vec{e_2} = 3\vec{V_1} - 2\vec{V_2}}$

(5) a)

$\boxed{P(A \cap B) = P(A) \times P(B)}$

$\boxed{P(\bar{A}) = 1 - P(A)}$

$= P(B) \Rightarrow P(A \cap B) / P(A)$

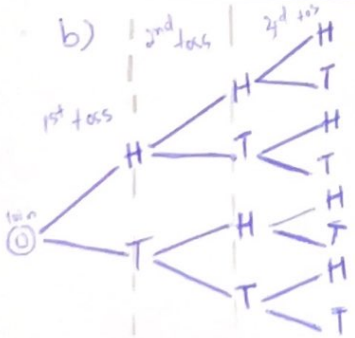
$P(B) = \frac{1}{4} \div \frac{2}{3} = \frac{1}{4} \times \frac{3}{2} = \frac{3}{8}$

$P(\bar{A}) = 1 - \frac{2}{3} = \frac{1}{3}$

$P(\bar{B}) = 1 - \frac{3}{8} = \frac{5}{8}$

$P(\overline{A \cap B}) = 1 - \frac{1}{4} = \frac{3}{4}$

b)



all possible outcomes = $2^3 = 8$

out of all 8 outcome we have three outcomes with two heads

$= \{H, T, H\}, \{H, H, T\}, \{T, H, H\}$

$\boxed{\text{Probability} = \frac{3}{8}}$

c) i)

A produces 50% = 0.5

B produces 30% = 0.3

C produces 20% = 0.2

A is defective 2% = 0.02

B is defective 3% = 0.03

C is defective 4% = 0.04

Defective parts from A = 0.02 × 0.5 = 0.01

Defective parts from B = 0.03 × 0.3 = 0.009

Defective parts from C = 0.04 × 0.2 = 0.008

Total probability of being defective

= 0.01 + 0.009 + 0.008 = 0.027

ii)

a) $\Rightarrow \frac{0.01}{0.027} \Rightarrow 0.37$

b) $\Rightarrow \frac{0.009}{0.027} \Rightarrow 0.33$

c) $\Rightarrow \frac{0.008}{0.027} \Rightarrow 0.29$

(6) $L = P + t\vec{v}$

$L = \begin{pmatrix}1\\2\\3\end{pmatrix} + t\begin{pmatrix}-1\\2\\3\end{pmatrix}$ || parametric equation

$x = 1 - 1t$, $t = \frac{1-x}{1}$

$y = 2 + 2t$, $t = \frac{y-2}{2}$

$z = 3 + 3t$, $t = \frac{z-3}{3}$

$\frac{1-x}{1} = \frac{y-2}{2} = \frac{z-3}{3}$ || symmetric equation

$x\vec{i} + y\vec{j} + z\vec{k} = \vec{i} + 2\vec{j} + 3\vec{k} + t(-\vec{i} + 2\vec{j} + 3\vec{k})$

Vector equation

6 b) $v_1 = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$ $v_2 = \begin{pmatrix} -3 \\ -2 \\ 6 \end{pmatrix}$

$$\boxed{\cos\theta = \frac{\vec{v_1} \cdot \vec{v_2}}{\|\vec{v_1}\| \cdot \|\vec{v_2}\|}}$$

$\vec{v_1} \cdot \vec{v_2} = (2 \times -3) + (1 \times -2) + (2 \times 6)$
$-6 - 2 + 12 = 4$

$\|\vec{v_1}\| = \sqrt{2^2 + 1^2 + 2^2} = \sqrt{9} = 3$

$\|\vec{v_2}\| = \sqrt{(-3)^2 + (-2)^2 + (6)^2} = \sqrt{49} = 7$

$\cos\theta = \frac{4}{3 \cdot 7} = \frac{4}{21}$

$\theta = \cos^{-1}\left(\frac{4}{21}\right)$, $\boxed{\theta = 79.019°}$

7 In 2040, $t = 2040 - 2000$, $\boxed{t = 40}$

$P_1 = 10{,}000 e^{40k}$
$P_2 = 20{,}000 e^{0.01 \times 40} \Rightarrow 20{,}000 e^{0.4}$

$P_1 = P_2 \Rightarrow \frac{10{,}000}{10{,}000} e^{40k} = \frac{20{,}000}{10{,}000} e^{0.4}$

$e^{40k} = 2 e^{0.4}$ // Apply ln

~~$40k \ln e = 0.8 \ln e$ // $\ln e = 1$~~
~~$40k =$~~

$\ln e^{40k} = \ln(2 e^{0.4})$

$40k \ln e = \ln 2 + \ln e^{0.4}$

$40k = \ln 2 + 0.4$

$k = \frac{\ln 2 + 0.4}{40}$

$\boxed{k = 0.027}$

8

x	y	xy	x^2	y^2
5	4	20	25	16
5	8	40	25	64
7	10	70	49	100
7	7	49	49	49
9	10	90	81	100
11	10	110	121	100
13	12	156	169	144
15	13	195	225	169
14	15	210	196	225
13	16	208	169	256
16	17	272	256	289
17	17	289	289	289
$\Sigma x = 132$	$\Sigma y = 139$	$\Sigma xy = 1709$	$\Sigma x^2 = 1654$	

$\text{slope} = \frac{n \Sigma xy - \Sigma x \Sigma y}{n \Sigma x^2 - (\Sigma x)^2}$

$\text{Slope} = \frac{12(1709) - 132 \cdot 139}{12(1654) - (132)^2}$

$= \frac{20508 - 18348}{19848 - 17424} = \frac{2160}{2424} = 0.89$

$\boxed{\text{slope} = 0.89}$

$\text{y-intercept} = \frac{\Sigma y - \text{slope} \cdot \Sigma x}{n} =$ ~~$139 - 0.89$~~

$= \frac{139 - 0.89 \times 132}{12} = 1.79$

$\boxed{\text{y-intercept} = 1.79}$

$\boxed{y = 0.89x + 1.79}$

MATHEMATICS 2nd Term Mid-Test Revision

9)

$$z = \left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)^{2010}$$

$$r = \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} = \sqrt{\frac{1}{4} + \frac{3}{4}} = 1$$

$$z = [r(\cos\theta + i\sin\theta)]^{2010}$$

$$z = r^{2010}[\cos(2010\theta) + i\sin(\theta \cdot 2010)]$$

$$z = 1^{2010}(\cos(2010\cdot\theta) + i\sin(2010\cdot\theta))$$

$$\left.\begin{array}{l}\cos\theta = \frac{a}{r} = 1/2 \\ \sin\theta = \frac{b}{r} = \sqrt{3}/2\end{array}\right\} \theta = 60°$$

$$z = 1(\cos(60\times2010) + i\sin(60\times2010))$$

$$\boxed{z = 1}$$

⚠ $\boxed{[r(\cos\theta + i\sin\theta)]^n = r^n(\cos n\theta + i\sin n\theta)}$

10)

A) $${}^{n-1}C_{n-5} = 3\,{}^{n-3}C_{n-7}$$

$\boxed{{}^{n}C_p = \frac{n!}{p!(n-p)!}}$ Thus,

$${}^{n-1}C_{n-5} = \frac{(n-1)!}{(n-5)!(n-1-(n-5))!}$$

$$= \frac{(n-1)!}{(n-5)!\,4!}$$

$${}^{n-3}C_{n-7} = \frac{(n-3)!}{(n-7)!(n-3-(n-7))!}$$

$$= \frac{(n-3)!}{(n-7)!\,4!}$$

$$\frac{(n-1)!}{(n-5)!\cancel{4!}} = \frac{(n-3)!}{(n-7)!\cancel{4!}} \cdot 3$$

continuation

$$\frac{(n-1)(n-2)\cancel{(n-3)!}}{(n-5)(n-6)\cancel{(n-7)!}} = \frac{\cancel{(n-3)!}}{\cancel{(n-7)!}} \cdot 3$$

$$(n-1)(n-2) = 3(n-5)(n-6)$$

$$n^2 - 3n + 2 = 3[n^2 - 11n + 30]$$

$$n^2 - 3n + 2 = 3n^2 - 33n + 90$$

$$2n^2 - 30n + 88 = 0$$

$$\Delta = b^2 - 4ac$$

$$= (-30)^2 - 4(2)(88)$$

$$900 - 704 = 196$$

$$\sqrt{\Delta} = \sqrt{196} = 14$$

$$n_1 = \frac{-b + \sqrt{\Delta}}{2a} = \frac{-(-30) + 14}{2(2)} = \frac{44}{4} = 11$$

$$n_2 = \frac{-b - \sqrt{\Delta}}{2a} = \frac{-(-30) - 14}{2(2)} = \frac{16}{4} = 4$$

n_2 is rejected because it is less than 7.

$$\boxed{S = \{11\}}$$

b)

A quadratic equation $az^2 + bz + c$ has real roots when $\Delta > 0$

$z^2 - 6z + c$, to have real roots,

$$b^2 - 4ac \geq 0, \quad 36 - 4(1)c \geq 0$$

$$36 - 4c \geq 0$$

$$4c \leq 36, \quad c \leq 9$$

$$\boxed{S =]-\infty, 9]}$$

c)

$$4e^{3x} - 3e^{2x} - e^x = 0$$

$$4(e^x)^3 - 3(e^x)^2 - e^x = 0, \text{ let } e^x = t$$

$$4t^3 - 3t^2 - t = 0$$

continuation on next page

~~4t~~ continuation of 10) c)

$4t^3 - 3t^2 - t = 0$

$t(4t^2 - 3t - 1) = 0$ //

$t = 0$ or $4t^2 - 3t - 1 = 0$

$4t^2 - 4t + t - 1 = 0$

$4t(t-1) + 1(t-1) = 0$

$(4t+1)(t-1) = 0$

$4t+1 = 0$, $t - 1 = 0$

$t = -\frac{1}{4}$, $t = 1$

rejected

$e^x = 1$ // Apply ln

$x \ln e = \ln 1$, $x = 0$

$$\boxed{S = \{0\}}$$

---

11) i) $z_1 = -1 + i$ $z_2 = -\sqrt{2} - \sqrt{6}$

a)

* z_1

$r_1 = \sqrt{(-1)^2 + 1^2} = \sqrt{2}$

$\cos\theta_1 = \frac{-1}{\sqrt{2}}$, $\sin\theta_1 = \frac{1}{\sqrt{2}}$

$\theta_1 = 135° = 3\pi/4$

$$\boxed{z_1 = \sqrt{2}\left(\cos\left(\frac{3\pi}{4}\right) + i\sin\left(\frac{3\pi}{4}\right)\right)}$$

* z_2

$r_2 = \sqrt{(-\sqrt{2})^2 + (\sqrt{6})^2} = \sqrt{2+6} = \sqrt{8}$
$= 2\sqrt{2}$

$\cos\theta_2 = \frac{-\cancel{\sqrt{2}}}{2\cancel{\sqrt{2}}}$, $\sin\theta_2 = \frac{-\sqrt{3}\cancel{\sqrt{2}}}{2\cancel{\sqrt{2}}}$

$\cos\theta_2 = -\frac{1}{2}$, $\sin\theta_2 = -\frac{\sqrt{3}}{2}$

$\theta_2 = 240° = \frac{4\pi}{3}$

$$\boxed{z_2 = 2\sqrt{2}\left(\cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}\right)}$$

b)

$$\boxed{z_1 \cdot z_2 = r_1 \cdot r_2\left(\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)\right)}$$

$z_1 \cdot z_2 = 2\sqrt{2} \times \sqrt{2}\left(\cos(135° + 240°) + i\sin(135° + 240°)\right)$

$$\boxed{z_1 \cdot z_2 = 4\left[\cos\left(\frac{25\pi}{12}\right) + i\sin\left(\frac{25\pi}{12}\right)\right]}$$ trig form

$z_1 \cdot z_2 = (-1+i)(-\sqrt{2} - \sqrt{6}i)$

$= \sqrt{2} + \sqrt{6}i - \sqrt{2}i - \sqrt{6}i^2$

$$\boxed{\sqrt{2} + \sqrt{6} + i(\sqrt{6} - \sqrt{2})}$$ cartesian form

ii) $z = re^{\theta i}$, $z = 3e^{\pi i}$

$$\boxed{r = 3 \quad \theta = \pi}$$

$z = r(\cos\theta + i\sin\theta)$, $z = 3(\cos\pi + i\sin\pi)$

$z = 3(-1)$

$$\boxed{z = -3}$$

b) $x^3 = -8$ let $z = -8$

$x = \sqrt[3]{-8}$, $|z| = |-8| = 8$

$\cos\theta = \frac{-8}{8}$, $\sin\theta = \frac{0}{8}$

$\cos\theta = -1$, $\sin\theta = 0$

$\theta = 180°$

$$\boxed{z_k = \sqrt[n]{r}\left[\cos\left(\frac{\theta + 2k\pi}{n}\right) + i\sin\left(\frac{\theta + 2k\pi}{n}\right)\right]}$$

Where $k = 0, 1, 2, 3, \ldots$

$z_0 = \sqrt[3]{8}\left(\cos\left(\frac{180}{3}\right) + i\sin\left(\frac{180}{3}\right)\right)$

$= 2\left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)$

$= \boxed{1 + \sqrt{3}i}$

$z_1 = \sqrt[3]{8}\left[\cos\left(\frac{180+360}{3}\right) + i\sin\left(\frac{180+360}{3}\right)\right]$

$= 2(-1) = \boxed{-2}$

→ Continuation on next page

MATHEMATICS 2nd Term Mid-Test Revision

Continuation to 11) ii) b)

$$Z_2 = \sqrt[3]{8}\left[\cos\left(\frac{180+720}{3}\right) + i\sin\left(\frac{180+720}{3}\right)\right]$$

$$Z_2 = 2\left[\frac{1}{2} + i - \frac{\sqrt{3}}{2}\right]$$

$$= 1 - \sqrt{3}\,i$$

$$\boxed{S = \{1-\sqrt{3}\,i,\ 1+\sqrt{3}\,i,\ 2\}}$$

c) $\sin x + \cos x > \sqrt{2}$

$$\cancel{\sqrt{2}}\left(\frac{\sqrt{2}}{2}\sin x + \frac{\sqrt{2}}{2}\cos x\right) > \cancel{\sqrt{2}}$$

$$\frac{\sqrt{2}}{2}\sin x + \frac{\sqrt{2}}{2}\cos x > 1, \quad \frac{\sqrt{2}}{2} = \sin 45 = \cos 45$$

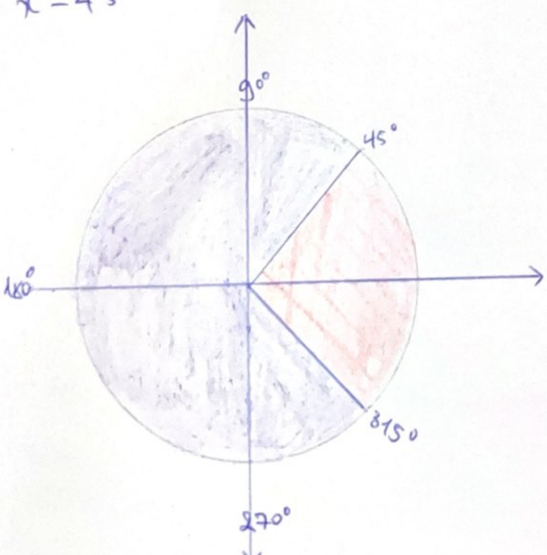
$$= \sin x \sin 45 + \cos x \cos 45 > 1$$

$$\cos(x - 45) > 1$$

$$\cos(x - 45) > \cos(0)$$

$x - 45 = 0$, $+x \cancel{+} 45 = 270$

$x = 45$, $x = 315°$



let If we choose an angle from the "red" region ex: 0°

$\cos(0 - 45) \overset{?}{>} 1$ $\quad \frac{\sqrt{2}}{2} < 1$

If choose an from the purple region

ex: 180°

$$\cos(180 - 45) \overset{?}{>} 1$$

$$-\frac{\sqrt{2}}{2} < 1$$

None of the regions satisfy our condition the inequality has no solution

---

$$F_n(x) = \frac{x^n}{1+x^2}, \quad x \in \mathbb{R}$$

$$L_n = \int_0^1 F_n(x)\,dx$$

$$L_1 = \int_0^1 F_1(x)\,dx, \quad F_1(x) = \frac{x}{1+x^2}$$

$$L_1 = \int_0^1 \frac{x}{1+x^2}\,dx$$

, let $1 + x^2 = t$

$2x\,dx = dt$

$x\,dx = \frac{dt}{2}$

$$L_1 = \int_0^1 \frac{\frac{dt}{2}}{t} = \frac{1}{2}\int_0^1 \frac{dt}{t}$$

$$= \frac{1}{2}\ln|t| = \boxed{\left[\frac{1}{2}\ln(1+x^2)\right]_0^1}$$

$$L_1 = \frac{1}{2}\ln(1+1^2) - \frac{1}{2}\ln(1+0^2)$$

$$L_1 = \frac{1}{2}\ln 2 - 0$$

$$\boxed{L_1 = \frac{\ln 2}{2} \text{ or } \ln\sqrt{2}}$$

continuation on the next page

Continuation of 12 b)

$$L_1 + L_3 = \int_0^1 F_1(x)\,dx + \int_0^1 F_3(x)\,dx$$

$$L_1 + L_3 = \int_0^1 [F_1(x) + F_3(x)]\,dx$$

$$f_1(x) = \frac{x}{1+x^2} \quad , \quad f_3(x) = \frac{x^3}{1+x^2}$$

$$L_1 + L_3 = \int_0^1 \frac{x + x^3}{1+x^2}\,dx$$

$$L_1 + L_3 = \int_0^1 \frac{x\cancel{(1+x^2)}}{\cancel{(1+x^2)}}\,dx$$

$$L_1 + L_3 = \int_0^1 x$$

$$L_1 + L_3 = \left| \frac{x^2}{2} \right|_0^1$$

$$\boxed{L_1 + L_3 = \frac{1}{2}} \quad (iv)$$

from (iv) $L_3 = \frac{1}{2} - L_1$

$$L_3 = \frac{1}{2} - \frac{\ln 2}{2}$$

$$\boxed{L_3 = \frac{1 - \ln 2}{2}}$$

c) $L_{2p} + L_{2p+2} = \int_0^1 (F_{2p} + F_{2p+2})\,dx$

$$F_{2p} = \frac{x^{2p}}{1+x^2} \quad , \quad F_{2p+2} = \frac{x^{2p+2}}{1+x^2}$$

$$L_{2p} + L_{2p+2} = \int_0^1 \frac{x^{2p}}{1+x^2} + \frac{x^{2p+2}}{1+x^2}\,dx$$

$$L_{2p} + L_{2p+2} = \int_0^1 \frac{x^{2p} + x^{2p+2}}{1+x^2}\,dx$$

$$\cancel{L_{2p} + L_{2p+2} = \int_0^1 x^{2p}(1}$$

$$L_{2p} + L_{2p+2} = \int_0^1 \frac{x^{2p} + x^{2p} \cdot x^2}{1+x^2}\,dx$$

$$L_{2p} + L_{2p+2} = \int_0^1 \frac{x^{2p}\cancel{(1+x^2)}}{\cancel{1+x^2}}\,dx$$

$$L_{2p} + L_{2p+2} = \int_0^1 x^{2p} = \left| \frac{x^{2p+1}}{2p+1} \right|_0^1$$

$$= \frac{1^{2p+1}}{2p+1} - \frac{0^{2p+1}}{2p+1}$$

$$\boxed{= \frac{1}{2p+1} \quad \text{as required}}$$

d)

$$L_2 = \int F_2(x)\,dx \quad , \quad F_2(x) = \frac{x^2}{1+x^2}$$

$L_2 = \int \frac{x^2}{1+x^2}\,dx$, using long division

$$L_2 = \int 1 - \frac{1}{1+x^2}\,dx$$

$$L_2 = \int_0^1 dx - \int \frac{1}{1+x^2}\,dx \quad (i)$$

$$L_2 = \left| x - \arctan(x) \right|_0^1$$

$$\boxed{L_2 = 1 - \arctan(1) = 1 - \frac{\pi}{4}} \quad (a)$$

$$L_4 = \int f_4(x) \quad , \quad f_4(x) = \frac{x^4}{1+x^2}$$

$L_4 = \int \frac{x^4}{1+x^2}\,dx$, using long division,

$$L_4 = \int_0^1 x^2 - \frac{x^2}{x^2+1}\,dx$$

↳ Continuation on next page

$L_4 = \int_0^1 x^2\,dx - \int \frac{x^2}{x^2+1}$

Look at equation (i) in chocolate color:

$L_4 = \frac{x^3}{3} - (x - \arctan x)$

$= \left| \frac{x^3}{3} - x + \arctan x \right|_0^1$

$= \frac{1}{3} - 1 + \arctan(1)$

$\boxed{L_4 = -\frac{2}{3} + \frac{\pi}{4}}$

d)

$L_6 = \int_0^1 F_6(x)$, $f(6)x = \frac{x^6}{1+x^2}$

$L_6 = \int_0^1 \frac{x^6}{1+x^2}$, using long division

$L_6 = \int x^4 - \frac{x^4}{x^2+1}\,dx$.

~~$L_6 = \int \frac{x^5}{5}$~~

$L_6 = \frac{x^5}{5} - \int \frac{x^4}{x^2+1}\,dx$,

Note that $\int \frac{x^4}{x^2+1}\,dx = L_4$

$L_6 = \frac{x^5}{5} - L_4$

$L_6 = \left|\frac{x^5}{5}\right|_0^1 - \left(-\frac{2}{3} + \frac{\pi}{4}\right)$

$= \frac{1}{5} + \frac{2}{3} - \frac{\pi}{4}$

$\frac{3+10}{15} - \frac{\pi}{4}$

$\boxed{\Rightarrow \frac{13}{15} - \frac{\pi}{4} = L_6}$