gcdSum(n) and lcmSum(n)

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1 GCD SUM FUNCTION

Given a positive integer n, $gcdSum(n)$ function is like:

$$gcdSum(n) = \sum_{i=1}^{n} gcd(i, n)$$

Example: n = 6

$$\begin{array}{c} \text{gcd(1, 6)} = 1 \\ \text{gcd(2, 6)} = 2 \\ \text{gcd(3, 6)} = 3 \\ \text{gcd(4, 6)} = 2 \\ \text{gcd(5, 6)} = 1 \\ \text{gcd(6, 6)} = 6 \\ \hline \text{gcdSum(6)} = 15 \end{array}$$

Notice that gcd of each pair (i, n) must be one of the divisors of n.

For $n = 6$, divisors are 1, 2, 3, 6. So we can think in reverse way, for each divisors how many pairs (i, n) exist. For example, when n = 6, the divisor 2 has 2 pairs (2, 6) and (4, 6). So lets calculate $gcdSum(n)$ in another way. We will denote each divisor by d.

Example: n = 6

$$\begin{array}{c} \text{Divisor d = 1 has 2 pairs. So, 1 * 2 = 2} \\ \text{Divisor d = 2 has 2 pairs. So, 2 * 2 = 4} \\ \text{Divisor d = 3 has 1 pairs. So, 3 * 1 = 3} \\ \text{Divisor d = 6 has 1 pairs. So, 6 * 1 = 6} \\ \hline \text{gcdSum(6)} = 15 \end{array}$$

Now the problem turns into counting pairs of (i, n) for a specific divisor d. There is a nice theorem using ETF (Euler Totient Function) to count this.

Theorem 1 *Given a positive integer n and let d is a divisor of n, then number of pairs (i, n) where $1 <= i <= n$ and $gcd(i, n) = d$ is:*

$$\phi(\frac{n}{d})$$

So, again lets check for n = 6

$$d = 1 : \phi(\frac{6}{1}) * 1 = 2 * 1 = 2$$

$$d = 2 : \phi(\frac{6}{2}) * 2 = 2 * 2 = 4$$

$$d = 3 : \phi(\frac{6}{3}) * 3 = 1 * 3 = 3$$

$$d = 6 : \phi(\frac{6}{6}) * 6 = 1 * 6 = 6$$

$$gcdSum(6) = 15$$

Finally we can write,

$$gcdSum(n) = \sum_{i=1}^{n} gcd(i, n) = \sum_{d|n} \phi(\frac{n}{d}) * d$$

In above formula, we have to find all divisors of n to get the result of $gcdSum(n)$. Theoretically, we have to traverse from 1 to $\sqrt{n}$ to get all divisors. We can find more efficient formula which doesn't need to generate divisors but only prime factors.

Lets check $gcdSum(n)$ for only positive prime integer like $gcdSum(p)$: When n is prime, all pairs except the pair (p, p) has gcd value 1. For example, n = 5,

gcd(1, 5) = 1
gcd(2, 5) = 1
gcd(3, 5) = 1
gcd(4, 5) = 1
gcd(5, 5) = 5

So, gcdSum(p) = (p-1) + p = 2p - 1

Now, lets check $gcdSum(n)$ for power of a prime like $gcdSum(p^a)$:
At first an example for $n = 3^3$,

gcd(1, 3^3) = 1
gcd(2, 3^3) = 1
gcd(3, 3^3) = 3
...
gcd(4, 3^3) = 1
gcd(5, 3^3) = 1
gcd(6, 3^3) = 3
...
gcd(3^2, 3^3) = 3^2
...
gcd(12, 3^3) = 3
gcd(13, 3^3) = 3
...
gcd(3^3, 3^3) = 3^2

Notice that in case of $n = p^a$, divisors of n will be $1, p, p^2, ...p^a$. So,

$$d = 1 : \phi(\frac{p^a}{1}) * 1 = \phi(p^a)$$

$$d = p : \phi(\frac{p^a}{p}) * p = \phi(p^{a-1}) * p$$

$$d = p^2 : \phi(\frac{p^a}{p^2}) * p^2 = \phi(p^{a-2}) * p^2$$

$$...$$

$$d = p^a : \phi(\frac{p^a}{p^a}) * p^a = \phi(1) * p^a$$

$$gcdSum(p^a) = \phi(p^a) + \phi(p^{a-1}) * p + \phi(p^{a-2}) * p^2 + \phi(1) * p^a$$

$$\text{We know, } \phi(p^a) = p^a - p^{a-1}$$

$$gcdSum(p^a) = (p^a - p^{a-1}) + (p^a - p^{a-1}) + (p^a - p^{a-1}) + ... + p^a$$

$$gcdSum(p^a) = a * (p^a - p^{a-1}) + p^a$$

$$gcdSum(p^a) = (a+1) * p^a - p^{a-1}$$

When n is composite integer and n is prime factorized as following:

$$n = p_1^{a_1} * p_2^{a_2} * ... * p_m^{a_m}$$

$$gcdSum(n) = gcdSum(p_1^{a_1}) * gcdSum(p_2^{a_2}) ... gcdSum(p_m^{a_m})$$

$$gcdSum(n) = \sum_{i=1}^{n} gcd(i,n) = \prod (a_i + 1) * p_i^{a_i} - p_i^{a_i - 1}$$

*It can be proved that gcdSum(N) is multiplicative like euler function.

2 LCM SUM FUNCTION

Given a positive integer n, $lcmSum(n)$ function is like:

$$lcmSum(n) = \sum_{i=1}^{n} lcm(i,n)$$

Example: n = 6

lcm(1, 6) = 6
lcm(2, 6) = 6
lcm(3, 6) = 6
lcm(4, 6) = 12
lcm(5, 6) = 30
lcm(6, 6) = 6

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lcmSum(6) = 66

Notice that when i is a divisor of n lcm is n. So number of divisors of n multiplied with n is a part of the final answer. But we can't find the value of $lcm(i, n)$ when i isn't a divisor of n. Anyway as we know the relationship between gcd and lcm which is:

$$a * b = gcd(a, b) * lcm(a, b)$$

Now we can write the $lcmSum(n)$ function in a different way like following:

$$lcmSum(n) = \sum_{i=1}^{n} \frac{i.n}{gcd(i, n)}$$

Still there is i exist which we want to remove so that overall complexity to find $lcmSum()$ will be reduced. We will use above identity later.

For now, let's discard $lcm(n, n)$ from the $lcmSum(n)$ function. We have to solve following function first.

$$X = \sum_{i=1}^{n-1} lcm(i, n)$$

$$X = lcm(1, n) + lcm(2, n) + ... + lcm(n-1, n) \tag{1}$$

Reverse the term of above function:

$$X = lcm(n-1, n) + lcm(n-2, n) + ... + lcm(1, n) \tag{2}$$

Sum (1) and (2):

$$2X = lcm(1, n) + lcm(n-1, n) + lcm(2, n) + lcm(n-2, n) + ... + lcm(n-1, n) + lcm(1, n)$$

$$2X = \sum_{i=1}^{n-1} [lcm(i, n) + lcm(n-i, n)] \tag{3}$$

Lemma 1

$$lcm(a,n) + lcm(n-a,n) = \frac{an}{\gcd(a,n)} + \frac{(n-a)n}{\gcd(n-a,n)} = \frac{n \times n}{\gcd(a,n)}$$

Notice that gcd(a,n) and gcd(n-a, n) are equal.

Lemma 2

$$\sum_{i=1}^{n} \frac{n}{\gcd(i,n)} = \sum_{d|n} \frac{n}{d} \times \phi(\frac{n}{d}) = \sum_{d|n} \phi(d).d$$

Notice that we can prove this lemma using Theorem 1.

Lemma 3

$$\sum_{i=1}^{n-1} \frac{n}{\gcd(i,n)} = \sum_{d|n,d\neq n} \frac{n}{d} \times \phi(\frac{n}{d}) = \sum_{d|n,d\neq 1} \phi(d).d$$

Notice since it goes up to $n-1$, gcd(i,n) won't be n. So we shouldn't consider d = n.

Now we can rewrite eq 3:

$$2X = \sum_{i=1}^{n-1} \frac{n \times n}{\gcd(i,n)}$$

$$2X = n \times \sum_{i=1}^{n-1} \frac{n}{\gcd(i,n)}$$

$$2X = n \times \sum_{d|n,d\neq 1} \phi(d).d$$

$$2X = n \times \sum_{d|n,d\neq 1} \phi(d).d$$

When $d = 1, \phi(d) \times 1 = 1$, if we subtract 1 then we shouldn't mention $d \neq 1$ below the sum function. right?

$$2X = n \times (\sum_{d|n} \phi(d).d - 1)$$

$$X = \frac{n}{2} \times (\sum_{d|n} \phi(d).d - 1)$$

Now we can put back the discarded term $lcm(n, n) = n$ again and calculate $lcmSum(n)$ finally,

$$lcmSum(n) = X + lcm(n, n)$$

$$lcmSum(n) = \frac{n}{2} \times (\sum_{d|n} \phi(d).d - 1) + n$$

$$lcmSum(n) = \frac{n}{2} \times \sum_{d|n} \phi(d).d - \frac{n}{2} + n$$

$$lcmSum(n) = \frac{n}{2} \times \sum_{d|n} \phi(d).d + \frac{n}{2}$$

$$lcmSum(n) = \frac{n}{2} \times (\sum_{d|n} \phi(d).d + 1)$$

Ah...Cool... Isn't it?

Refs:
1. https://cs.uwaterloo.ca/journals/JIS/VOL4/BROUGHAN/gcdsum.pdf

2. https://math.stackexchange.com/questions/761670/how-to-find-this-lcm-sum-function-textlcm1-n-textlcm2-n-cdots-t

3. https://www.quora.com/How-can-I-solve-the-problem-GCD-Extreme-GCDEX-on-SPOJ