$$0 = 1 - 1 = -1 + 1 = 0$$

From Elementary School to Higher Algebras

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IMB

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Question

Can you prove $0 = 0$, non-trivially?

Test your math level

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Hopf fibration

Hopf fibration is a map defined by

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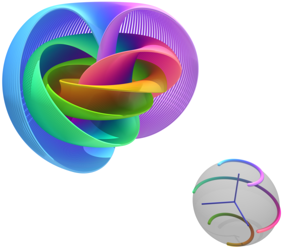
For any point $x \in \mathbb{S}^2$, preimage $H^{-1}(x)$ is a circle $\mathbb{S}^1$. We can visualize this map by considering $H : \mathbb{R}^3 \cong \mathbb{S}^3 - \{\infty\} \to \mathbb{S}^2$:



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Let us study the cases $X = \mathbb{S}^n$.

Homotopy groups of spheres

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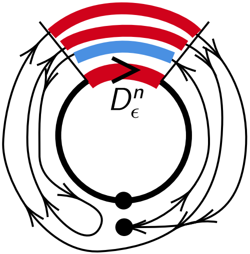
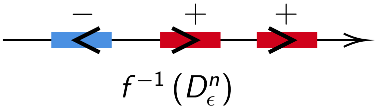
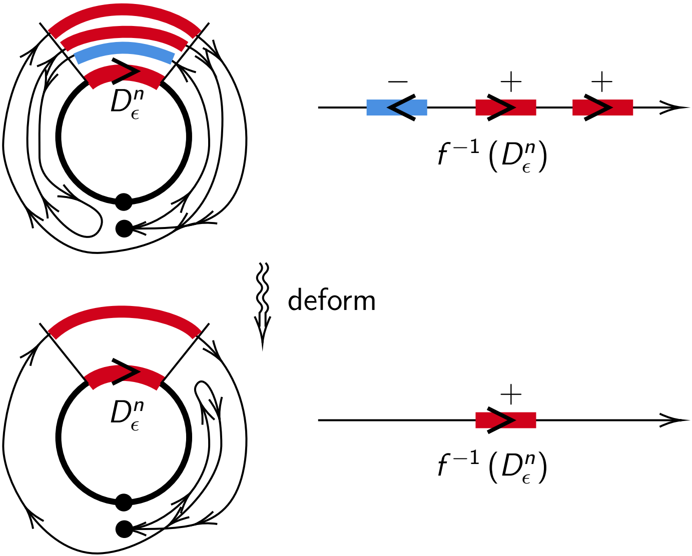

D^n_ϵ


$-$
$+$
$+$
$f^{-1}(D^n_\epsilon)$



D^n_ϵ
−
+
+
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deform
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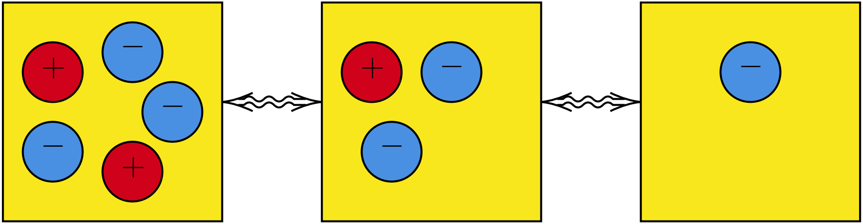

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This gives an isomorphism to the configuration space of disks.

$$\mathrm{End}_*(\mathbb{S}^n) \cong \mathrm{Conf}(\mathbb{R}^n) = \bigcup_{j,k} \mathrm{Emb}\Big(\bigsqcup_j D^n_+ \sqcup \bigsqcup_k D^n_-, \mathbb{R}^n\Big)$$

Therefore, $\pi_0 \mathrm{End}_*(\mathbb{S}^n) \cong \mathbb{Z}, (j,k) \mapsto j - k$

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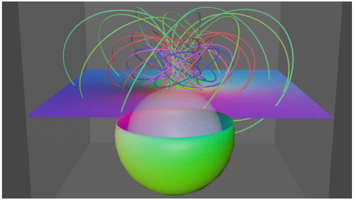
$$\mathrm{Map}_*(\mathbb{S}^3, \mathbb{S}^2) \cong \mathrm{Map}_*(\mathbb{S}^1, \mathrm{End}_*(\mathbb{S}^n)) \to \mathrm{Map}_*([0,1], \mathrm{End}_*(\mathbb{S}^n))$$

Then H corresponds to a path (deformation or homotopy) form $0 \in \mathrm{End}_*(\mathbb{S}^n)$ to 0 itself.

I made an animation to visualize this deformation.



Visualize H as deformation

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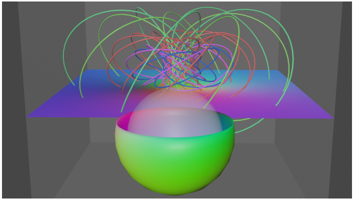

We can also visualize it as a movement in the configuration space. We will show it in the end.

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We use the **Grothendieck group** $K_0(\mathrm{FinSet})$

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We can think we formally define "$X - Y$" in this way.
But what is $K_i(\mathrm{FinSet})$ for $i > 0$?

Space of "finite sets"

Instead of thinking the set of "finite sets", we now consider the space of them $F = \mathrm{FinSet}^{\cong}$: every point $x \in F$ corresponds to a finite set, and a path between to points corresponds to an isomorphism $f : x \to y$ of two finite sets.

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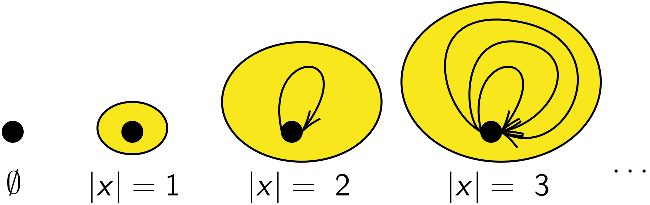

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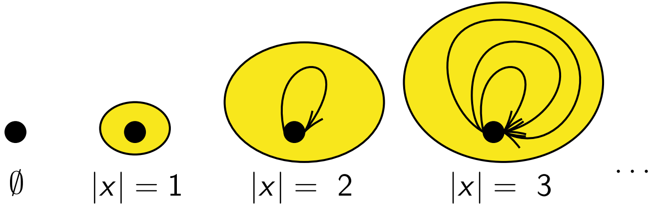

- $\pi_0(F) = \mathrm{FinSet}_{/\cong} \cong \mathbb{N}$.
- $\pi_1(F, x) = \mathrm{S}_{|x|}$ the permutation group.
- $\pi_i(F, x) = \{0\}$ for $i > 0$ (one can never deform an isomorphism of set to another).

K-space (spectrum) of "finite sets"

We want to add "negative" to F. We can consider the space $K(F)$ of (ordered) finite sets, whose elements are marked by $+$ or $-$, and we also want to identify the sets with same "value" $\|X\| = |X_+| - |X_-|$, and these identifications will be the paths.

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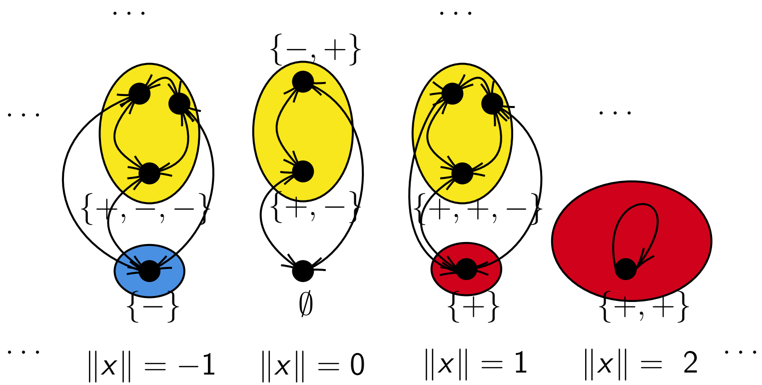

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Answer

We can prove $0 = 0$ non-trivially with $h : 0 = 1 - 1 = -1 + 1 = 0$!

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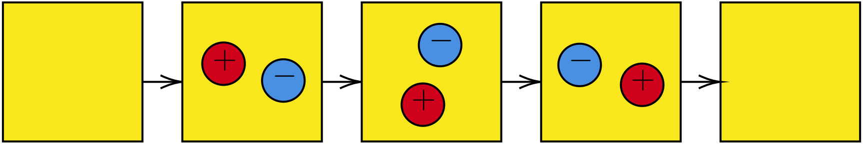
And they are the same!



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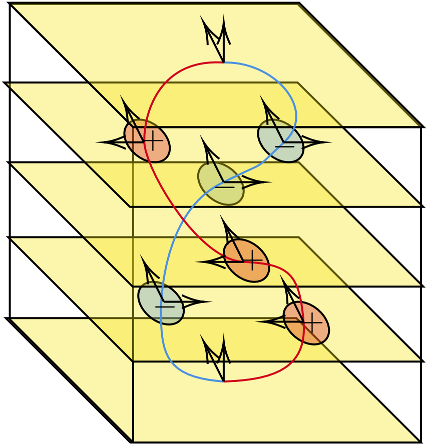

What about cobordism?

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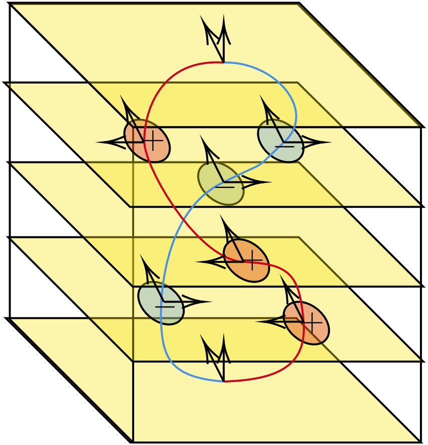

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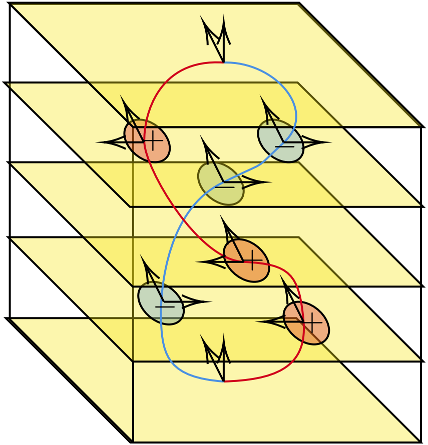

This provides a framed circle, meaning an embedding $\mathbb{S}^1 \hookrightarrow \mathbb{R}^n$ with a trivialization of the normal bundle. We denote Ω_i^{fr} as the cobordism classes of i-dimensional framed manifolds. And we have:

$$\Omega_i^{fr} \cong \pi_i^s(\mathbb{S})$$

Therefore, this framed circle is non-trivial and is the generator of Ω_1^{fr}.

Thank you, hope you have got the brain upgrades!