

$$0 = 1 - 1 = -1 + 1 = 0$$

# From Elementary School to Higher Algebras

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IMB

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### Question

Can you prove  $0 = 0$ , non-trivially?

## Test your math level

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$$S^3 \sim \mathbb{C}^2 \setminus \{0\} \rightarrow \mathbb{CP}^1 \sim S^2$$



$$\pi_4(S^3) \cong \mathbb{Z}/2\mathbb{Z}$$

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Hopf fibration is a map defined by

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For any point  $x \in \mathbb{S}^2$ , preimage  $H^{-1}(x)$  is a circle  $\mathbb{S}^1$ .

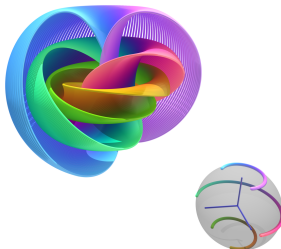
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Let us study the cases  $X = \mathbb{S}^n$ .

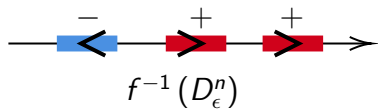
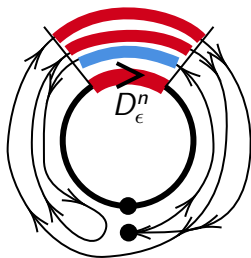
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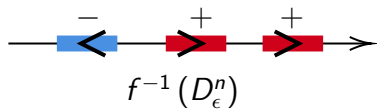
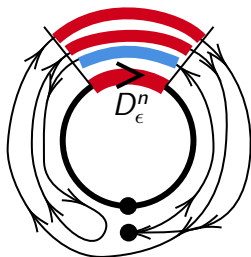


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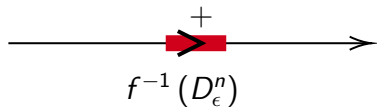
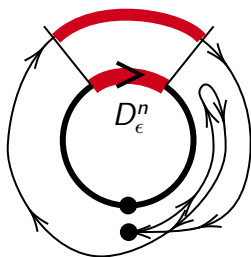
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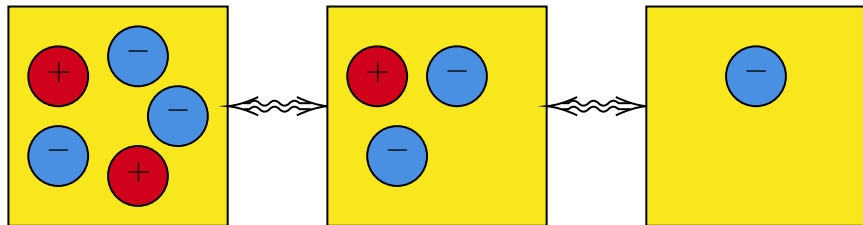

 deform



## Homotopy groups of spheres

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This gives an isomorphism to the configuration space of disks.

$$\text{End}_*(\mathbb{S}^n) \cong \text{Conf}(\mathbb{R}^n) = \bigcup_{j,k} \text{Emb}\left(\bigsqcup_j D_+^n \sqcup \bigsqcup_k D_-^n, \mathbb{R}^n\right)$$

Therefore,  $\pi_0 \text{End}_*(\mathbb{S}^n) \cong \mathbb{Z}$ ,  $(j, k) \mapsto j - k$

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## Higher homotopy groups of spheres

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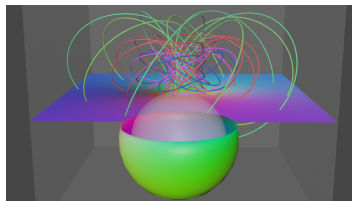
$$\text{Map}_*(\mathbb{S}^3, \mathbb{S}^2) \cong \text{Map}_*(\mathbb{S}^1, \text{End}_*(\mathbb{S}^n)) \rightarrow \text{Map}_*([0, 1], \text{End}_*(\mathbb{S}^n))$$

Then  $H$  corresponds to a path (deformation or homotopy) from  $0 \in \text{End}_*(\mathbb{S}^n)$  to 0 itself.

## Visualize $H$ as deformation

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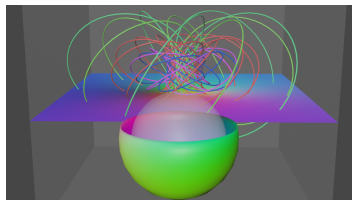
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We can also visualize it as a movement in the configuration space.  
We will show it in the end.

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We use the **Grothendieck group**  $K_0(\text{FinSet})$

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But what is  $K_i(\text{FinSet})$  for  $i > 0$  ?

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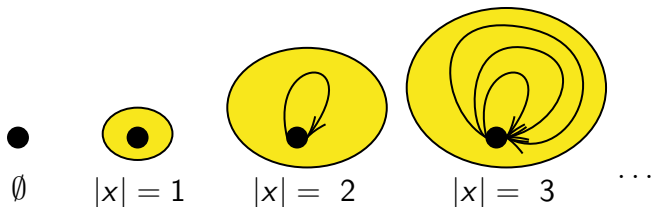
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Instead of thinking the set of "finite sets", we now consider the space of them  $F = \text{FinSet}^{\cong}$  : every point  $x \in F$  corresponds to a finite set, and a path between two points corresponds to an isomorphism  $f : x \rightarrow y$  of two finite sets.



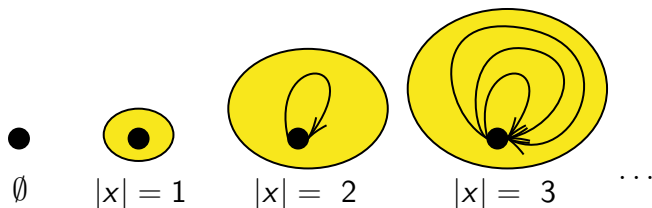
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- $\pi_0(F) = \text{FinSet}_{/\cong} \cong \mathbb{N}$ .
- $\pi_1(F, x) = S_{|x|}$  the permutation group.
- $\pi_i(F, x) = \{0\}$  for  $i > 0$  (one can never deform an isomorphism of set to another).

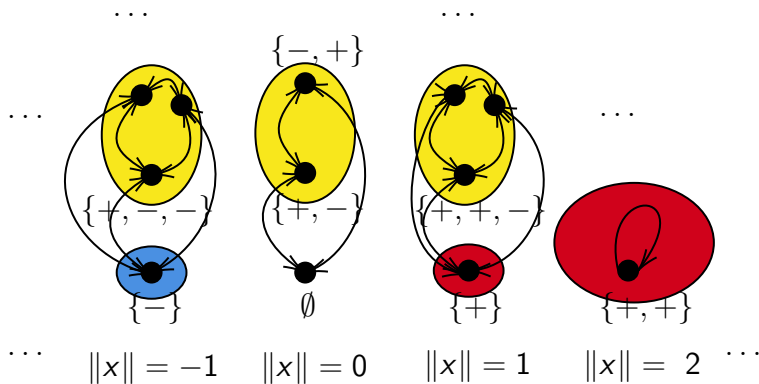
## K-space (spectrum) of "finite sets"

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We want to add "negative" to  $F$ . We can consider the space  $K(F)$  of (ordered) finite sets, whose elements are marked by  $+$  or  $-$ , and we also want to identify the sets with same "value"  $\|X\| = |X_+| - |X_-|$ , and these identifications will be the paths.

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### Answer

We can prove  $0 = 0$  non-trivially with  $h : 0 = 1 - 1 = -1 + 1 = 0!$

## What's more?

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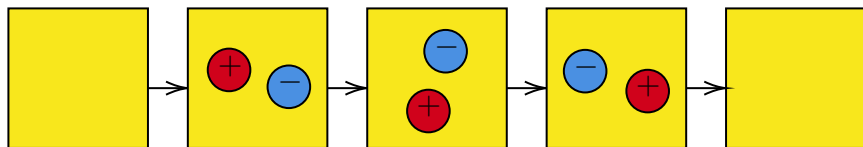
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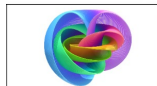
And they are the same!



# Thank you, hope you have got the brain upgrades!

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