Exercises 4

[1]

1. Let $f : X \to Y$ be a continuous map between topological spaces, $\mathcal{F}$ and $\mathcal{G}$ are sheaves over X and Y respectively.

   (a) Let the direct image $f_*\mathcal{F}$ be the presheaf over Y, s.t. $f_*\mathcal{F}(U) := \mathcal{F}(f^{-1}(U))$. Check that this is already a sheaf and $f_* : Sh(X) \to Sh(Y)$ defines a functor.

   (b) Let the inverse image $f^*\mathcal{G}$ be the sheafification of presheaf $\mathcal{G}(f(U)) := \operatorname{colim}_{V open, f(U)\subset V} \mathcal{G}(V)$ over X. Show that $f^*\mathcal{G}_x \cong \mathcal{G}_{f(x)}$ and $f^* : Sh(Y) \to Sh(X)$ defines a functor.

   (c) When $Y = *$ is a point, to define a sheaf $\mathcal{G}$ over $*$ is equivalent to giving the set $G = \mathcal{G}(*)$. Show that $f_*\mathcal{F} = \mathcal{F}(X)$ and $f^*G = \underline{G}$ the const sheaf.

   (d) For any $x \in X$ induces a map $x : * \to X$. Show that $x^*\mathcal{F} = \mathcal{F}_x$ and $x_*G = G_{\{x\}}$ the skyscraper sheaf supported on $\{x\}$(i.e. $G_{\{x\}}(U) = G$ if $x \in U$ and $*$ otherwise).

   (e) * Show that there is an isomorphism

   $$\operatorname{Hom}_{Sh(X)}(f^*\mathcal{G}, \mathcal{F}) \cong \operatorname{Hom}_{Sh(Y)}(\mathcal{G}, f_*\mathcal{F})$$

   (Use the fact that for sheafification functor $^+$, $\operatorname{Hom}_{PSh(X)}(\mathcal{P}, \mathcal{F}) \cong \operatorname{Hom}_{Sh(X)}(\mathcal{P}^+, \mathcal{F})$)

2. * Let $\mathcal{F}, \mathcal{G} \in Sh(X)$ and let $i : \mathcal{F} \to \mathcal{G}$ be a morphism s.t. $\forall x \in X, i_x : \mathcal{F}_x \to \mathcal{G}_x$ are isomorphisms. Show that i is an isomorphism of sheaves. (Hint: first show that i induces a homeomorphism between etale space $Et(i) : Et(\mathcal{F}) \to Et(\mathcal{G})$)

3. We define the (pre)sheaf $\mathcal{F}$ of Abelian group (i.e. Abelian (pre)sheaf) over X by changing the target to Ab, i.e. now $\mathcal{F}(U) \in Ab$. We denote $PAb(X)$ and $Ab(X)$ as the category of Abelian presheaf and sheaf. And for a sequence in $PAb(X)$ (resp. $Ab(X)$)

   $$\mathcal{F} \to \mathcal{G} \to \mathcal{H}$$

[1]find also these exercises on https://github.com/iamcxds/AG-exercise, you can skip a question with * if it is difficult.

we say it is exact if $\forall U$ open,

$$\mathcal{F}(U) \to \mathcal{G}(U) \to \mathcal{H}(U)$$

(resp. $\forall x \in X$,

$$\mathcal{F}_x \to \mathcal{G}_x \to \mathcal{H}_x$$

) are exact.

(a) Show that the exactness of presheaf implies the exactness of sheaf. (Hint: taking stalk $\operatorname{colim}_{x\in U} \mathcal{F}(U)$ is a filtered colimit)

(b) Let $X = \mathbb{C}$, show that

$$0 \to \underline{\mathbb{Z}} \xrightarrow{2\pi i} \mathcal{O} \xrightarrow{\exp} \mathcal{O}^* \to 0$$

is an exact sequence of Abelian sheaves, where $\mathcal{O}$ (resp. $\mathcal{O}^*$) is the sheaf of holomorphic functions (resp. non-vanishing holomorphic functions). But by considering the sections on $\mathbb{C}^* \subset \mathbb{C}$, show that this is not exact as presheaf.

(c) Let $\underline{\mathbb{Z}} \in Ab(X)$ be the const sheaf, and for close point $x \in X$, let $\mathbb{Z}_{\{x\}}$ be the skyscraper sheaf. and let $\mathbb{Z}_{X-\{x\}}$ be the sheaf s.t. $\mathbb{Z}_{X-\{x\}}(U) = \mathbb{Z}$ if $x \notin U$, and 0 otherwise. Show that there is an exact sequence of sheaf

$$0 \to \mathbb{Z}_{X-\{x\}} \to \underline{\mathbb{Z}} \to \mathbb{Z}_{\{x\}} \to 0$$

But then show that $\underline{\mathbb{Z}}$ is **not** isomorphic to $\mathcal{B} = \mathbb{Z}_{X-\{x\}} \oplus \mathbb{Z}_{\{x\}}$, even if there are isomorphisms $\forall x, \underline{\mathbb{Z}}_x \cong \mathbb{Z} \cong \mathcal{B}_x$.

4. Let $X = [0, 1]$ and let $C(X)$ be the ring of real-valued functions on X. Recall that the maximal spectrum $\operatorname{Spm}(R)$ is the subspace of $\operatorname{Spec}(R)$ consisting of maximal ideals. We will show that $\tilde{X} = \operatorname{Spm}(C(X))$ is homeomorphic to X.

(a) For $x \in X$, let maximal ideals $\mathfrak{m}_x = \{f \in C(X), s.t. f(x) = 0\}$. Now for any ideal $I \subsetneq C(X)$, let $V_X(I) = \{x \in X, s.t. f(x) = 0, \forall f \in I\}$. Show that $V_X(I)$ is nonempty and thus for $x \in V_X(I), I \subset \mathfrak{m}_x$. Therefore $\mathfrak{m}_- : X \to \tilde{X}$ is surjective.

(b) By construction functions separate the points of X, conclude that $\mathfrak{m}_- : X \to \tilde{X}$ is injective.

(c) Let $f \in C(X)$, and let

$$U_f = \{x \in X, s.t. f(x) \neq 0\} \subset X$$
$$\tilde{U}_f = \{m \in \tilde{X}, s.t. f \notin m\} \subset X$$

Show that $\mathfrak{m}_-(U_f) = \tilde{U}_f$ and therefore $\mathfrak{m}_-$ is a homeomorphism.

(d) * Show that any prime ideal $\mathfrak{p} \subset C(X)$ are contained in exact one $\mathfrak{m}_x$.