

Exercises 6

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1. Let \mathbb{RP}^n be the topological space constructed by quotient $\mathbb{R}^{n+1} \setminus 0$ with \mathbb{R}^* . And let \mathbb{P}^n be the scheme of projective space over R . Recall a section of $p : A \rightarrow B$ is a morphism $f : B \rightarrow A$ s.t $p \circ f = \text{id}$.

- (a) Let $q : \mathbb{R}^{n+1} \setminus 0 \rightarrow \mathbb{RP}^n$ be the obvious projection. Show that q has no section.
- (b) Let $D_+(x_i)_n$ be the affine open subscheme of \mathbb{P}^n . Define morphisms $q : D_{x_i} \subset \mathbb{A}^{n+1} \setminus 0 \rightarrow D_+(x_i)_n$ induced by

$$q^\# : R[\frac{x_0}{x_i}, \dots, \frac{x_n}{x_i}] \rightarrow R[x_0, \dots, x_n][x_i^{-1}], \frac{x_j}{x_i} \mapsto x_j x_i^{-1}$$

Show that we can glue q into morphism $q : \mathbb{A}^{n+1} \setminus 0 \rightarrow \mathbb{P}^n$. Then show that q has no section.

- (c) Let $a, b \in \mathbb{R}$, $f : \mathbb{RP}^1 \rightarrow \mathbb{RP}^2 \setminus [0, 0, 1]$ by $f([x, y]) = [x, y, ax + by]$ and $p : \mathbb{RP}^2 \setminus [0, 0, 1] \rightarrow \mathbb{RP}^1$ by $p([x, y, z]) = [x, y]$. Check they are well-defined map, f is a section of p and $p^{-1}(t \in \mathbb{RP}^1) \cong \mathbb{R}^1$. Then show that $\mathbb{RP}^2 \setminus [0, 0, 1]$ is homeomorphic to a Mobius band.
- (d) For $n \leq m$, we define morphisms $i : D_+(x_i)_n \rightarrow D_+(x_i)_m$, $p : D_+(x_i)_n \rightarrow D_+(x_i)_m$ induced by

$$i^\# : R[\frac{x_0}{x_i}, \dots, \frac{x_m}{x_i}] \rightarrow R[\frac{x_0}{x_i}, \dots, \frac{x_n}{x_i}], \frac{x_j}{x_i} \mapsto \frac{x_j}{x_i}, j \leq n; 0, j > n$$

$$p^\# : R[\frac{x_0}{x_i}, \dots, \frac{x_n}{x_i}] \rightarrow R[\frac{x_0}{x_i}, \dots, \frac{x_m}{x_i}], \frac{x_j}{x_i} \mapsto \frac{x_j}{x_i}$$

Show that we can glue i, p into morphisms

$$i : \mathbb{P}^n = D_+(x_0)_n \cup \dots \cup D_+(x_n)_n \rightarrow U_{n,m} = D_+(x_0)_m \cup \dots \cup D_+(x_n)_m \subset \mathbb{P}^m$$

and $p : U_{n,m} \rightarrow \mathbb{P}^n$.

- (e) * Show that sections of $p : U_{1,2} \rightarrow \mathbb{P}^1$ correspond to linear polynomials $\{ax + by, a, b \in R\}$

¹find also these exercises on <https://github.com/iamcxds/AG-exercise>, you can skip a question with * if it is difficult.

2. (Gluing 3 schemes) Let $X_i, i = 1, 2, 3$ be three schemes. And for pairs i, j , there are open subsets U_{ij} and isomorphisms $\varphi_{ij} : U_{ij} \xrightarrow{\sim} U_{ji}$. Now, if the isomorphisms are compatible in the sense: for each i, j, k ,

(1) $\varphi_{ij} = \varphi_{ji}^{-1}$,

(2) $\varphi_{ij}(U_{ij} \cap U_{ik}) = U_{ji} \cap U_{jk}$,

(3) $\varphi_{jk} \circ \varphi_{ij} = \varphi_{ik}$ on $U_{ij} \cap U_{ik}$,

then there exists a scheme X , as gluing of X_i .