Exercises 3

[1]

1. Let R be a ring and for any ideal J, we denote $V(J)$ for the set of prime ideals containing J. Now for any subset S of $\operatorname{Spec} R$, we define $I(S) = \cap_{\mathfrak{p}\in S}\mathfrak{p}$. Let A (resp. B) be the category of ideals of R (resp. subsets of $\operatorname{Spec} R$), whose morphisms are given by inclusion(i.e. $\operatorname{Hom}(X,Y) = \{*\}$ if $X \subset Y$, and be $\emptyset$ otherwise).

   (a) Show that $V : A^{op} \to B$ and $I : B^{op} \to A$ are well-defined functors. And furthermore, Show that $\operatorname{Hom}_A(J, I(S)) \cong \operatorname{Hom}_B(S, V(J))$

   (b) Let $\mathfrak{n}$ be the ideal of nilpotent elements, show that for any prime ideal $\mathfrak{p}$, $\mathfrak{n} \subset \mathfrak{p}$ and $\mathfrak{n} \subset I(\operatorname{Spec} R)$.

   (c) For any non-nilpotent elements $f \in R$, let R_f be the localization of R with $\{f^i\}$. Show that

   $$\operatorname{Spec} R_f \cong \{\mathfrak{p}\ prime | f \notin \mathfrak{p}\}$$

   is nonempty. Then conclude that $\mathfrak{n} = I(\operatorname{Spec} R)$

   (d) Define the radical of an ideal J to be $\sqrt{J} := \{x \in R | \exists n \in \mathbb{N}, x^n \in J\}$. Show that $V(I(S)) = \bar{S}$, the closure in $\operatorname{Spec} R$, and $I(V(J)) = \sqrt{J}$.

   (e) An ideal J is a radical ideal if $J = \sqrt{J}$. Show that there is a bijection between the closed subsets of $\operatorname{Spec} R$ and the radical ideals of R.

   (f) * Show that S is closed irreducible, iff $S = V(\mathfrak{p})$ for $\mathfrak{p}$ prime, thus there is a bijection between the irreducible closed subsets of $\operatorname{Spec} R$ and the points of $\operatorname{Spec} R$.

   (g) Show that there is a bijection between the irreducible components of $\operatorname{Spec} R$ and the minimal(with respect to inclusion) prime ideals R.

2. (a) Show that for Noetherian space X, any open subset is quasi-compact.

   (b) * Show that for Noetherian ring R, there are only finitely many minimal prime ideals. Then show that the elements of minimal prime ideals are zero divisors.

[1]find also these exercises on https://github.com/iamcxds/AG-exercise, you can skip the question with * if it is difficult.

3. We will show that every sheaf is actually a sheaf of "functions". Let X be a topological space, an etale space (E, p) over X is a topological space E with a local homeomorphism $p : E \to X$(i.e. $\forall x \in E, \exists$ neighborhood $U \ni x$, such that $p|_U$ is a homeomorphism).

   (a) For any open subset $U \subset X$, we define the sections:

   $$\Gamma_E(U) = \{f : U \to p^{-1}(U) \subset E | p \circ f = \mathrm{id}_U\}$$

   Check that Γ_E is a well-defined sheaf on X

   (b) Let $\mathcal{P}$ be a presheaf on X, we define an etale space $(Et(\mathcal{P}), p)$ as follow:

   $$Et(\mathcal{P}) = \coprod_{x \in X} \mathcal{P}_x, p : (x, f \in \mathcal{P}_x) \mapsto x \in X$$

   where $\mathcal{P}_x$ is the stalk. And the topology of $Et(\mathcal{P})$ is given by the basis $\{V_f\}_{f \in \mathcal{P}(U)}$, where

   $$V_f = \{(x \in U, f_x \in \mathcal{P}_x) \in Et(\mathcal{P})\}$$

   Check that $Et(\mathcal{P})$ is a well-defined etale space.

   (c) * Show that there is a canonical morphism $i : \mathcal{P} \to \Gamma_{Et(\mathcal{P})}$ and it is an isomorphism if $\mathcal{P}$ is a sheaf. Furthermore, $\Gamma_{Et(\mathcal{P})}$ is the sheafication of $\mathcal{P}$, i.e. for any sheaf $\mathcal{F}$, we have $\mathrm{Hom}_{PSh}(\mathcal{P}, \mathcal{F}) \cong \mathrm{Hom}_{Sh}(\Gamma_{Et(\mathcal{P})}, \mathcal{F})$.