

# Exercises 7

## 1. (Segre embedding)

- (a) Show that the map  $\sigma : \mathbb{RP}^1 \times \mathbb{RP}^1 \rightarrow \mathbb{RP}^3$  by  $([x, y], [u, v]) \rightarrow [xu, xv, yu, yv]$  is well-defined and injective. And further show that the image of  $\sigma$  is  $\{[z_0, z_1, z_2, z_3] \in \mathbb{RP}^3 \mid z_0 z_3 = z_1 z_2\}$ .
- (b) Assume we work over field  $k$ . Let  $D_+(y) \subset \mathbb{P}^1, D_+(v) \subset \mathbb{P}^1$  and  $D_+(z_3) \subset \mathbb{P}^3$ , we can define  $\sigma_3 : D_+(y) \times_k D_+(v) \rightarrow D_+(z_3)$  which is induced by ring homomorphism:

$$\sigma_3^\# : k\left[\frac{z_0}{z_3}, \frac{z_1}{z_3}, \frac{z_2}{z_3}\right] \rightarrow k\left[\frac{x}{y}\right] \otimes_k k\left[\frac{u}{v}\right] = k\left[\frac{x}{y}, \frac{u}{v}\right]$$

$$\frac{z_0}{z_3} \mapsto \frac{x}{y} \frac{u}{v}, \frac{z_1}{z_3} \mapsto \frac{x}{y}, \frac{z_2}{z_3} \mapsto \frac{u}{v}$$

Show that  $\sigma_3^\#$  factor through  $k\left[\frac{z_0}{z_3}, \frac{z_1}{z_3}, \frac{z_2}{z_3}\right]/\left(\frac{z_0}{z_3} - \frac{z_1}{z_3} \frac{z_2}{z_3}\right)$ , thus  $\sigma_3$  actually is a closed embedding

$$\sigma_3 : D_+(y) \times_k D_+(v) \xrightarrow{\cong} D_+(z_3) \cap V(z_0 z_3 - z_1 z_2) \rightarrow D_+(z_3)$$

- (c) Define  $\sigma_0, \sigma_1, \sigma_2$  in the same way, then we can also define a morphism  $\sigma : \mathbb{P}^1 \times_k \mathbb{P}^1 \rightarrow \mathbb{P}^3$  by gluing  $\sigma_0, \sigma_1, \sigma_2, \sigma_3$ . Verify that  $\sigma$  is well-defined and in fact, it is a closed embedding:

$$\sigma : \mathbb{P}^1 \times_k \mathbb{P}^1 \xrightarrow{\cong} V(z_0 z_3 - z_1 z_2) \rightarrow \mathbb{P}^3$$

## 2. (fiber product)

- (a) Let  $X, Y, Z$  be sets with maps  $f : X \rightarrow Z, g : Y \rightarrow Z$ . Show that we have fiber product in Set, and

$$X \times_Z Y = \{(x, y) \in X \times Y \mid f(x) = g(y) \in Z\}$$

And let  $p : X \times_Z Y \rightarrow Z, (x, y) \mapsto f(x)$  be the canonical projection. Then show that for all  $z \in Z, p^{-1}(z) = f^{-1}(z) \times g^{-1}(z)$ .

- (b) Let  $U, V$  be open subschemes of scheme  $S$ . Show that  $U \times_S V = U \cap V$

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<sup>1</sup>find also these exercises on <https://github.com/iamcxdx/AG-exercise>, you can skip a question with \* if it is difficult.

- (c) Let  $k$  be a field, and let  $L$  be finite Galois extension of  $k$ . Show that  $\operatorname{Spec} L \times_k \operatorname{Spec} L \cong \sqcup_{[k:L]} \operatorname{Spec} L$
- (d) \* Assume  $k$  is algebraically closed, let  $k(x)$  be the field of fractions of  $k[x]$ , show that the underlying set of  $\operatorname{Spec} k(x) \times_k \operatorname{Spec} k(y)$  is isomorphic to

$$\{\mathfrak{p} \in \mathbb{A}^2 \mid \operatorname{ht}(\mathfrak{p}) = 0 \text{ or } 1, \text{ and } \forall c \in k, \mathfrak{p} \neq (x - c), \mathfrak{p} \neq (y - c)\}$$