

Exercises 5

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1. For any topological space X , we think of it as a ringed space, with the structure sheaf $C^0(U) = \{f : U \rightarrow \mathbb{R}, \text{continuous}\}$ of continuous functions.
 - (a) Show that (X, C^0) is a locally ringed space, and any continuous map $f : X \rightarrow Y$ induces a morphism of locally ringed spaces $(f, f^\#) : (X, C^0) \rightarrow (Y, C^0)$.
 - (b) Let $X = [0, 1]$ for multiplicative system $S_U = \{f \in C^0(X), s.t. \forall x \in U, f(x) \neq 0\}$. Then show that there is an isomorphism $S_U^{-1}C^0(X) \cong C^0(U)$, thus $\text{Spm } C^0(X) \cong (X, C^0)$ as locally ringed spaces.
 - (c) Let $X = \mathbb{R}^n, Y = \mathbb{R}^m$ (or more generally differential manifolds), we can define the structure to be $C^\infty(U) = \{f : U \rightarrow \mathbb{R}, \text{smooth}\}$. Show that any smooth map $f : X \rightarrow Y$ induces a morphism of locally ringed spaces $(f, f^\#) : (X, C^\infty) \rightarrow (Y, C^\infty)$.
 - (d) Let $X = \mathbb{R}^1, Y = \mathbb{R}^2$, let $f : X \rightarrow Y$ be $f(t) = (t, |t|)$. Verify that f only induces morphism between $(X, C^0) \rightarrow (Y, C^0)$ rather than $(X, C^\infty) \rightarrow (Y, C^\infty)$. (Actually, f is smooth, iff it induces $(f, f^\#) : (X, C^\infty) \rightarrow (Y, C^\infty)$.)
2. Let LRS be the category of locally ringed space. And let (X, Γ) be a locally ringed space and R be a ring.
 - (a) Show that there is a canonical map $\text{Hom}_{LRS}(X, \text{Spec } R) \rightarrow \text{Hom}_{Ring}(R, \Gamma(X))$.
 - (b) Let $f \in \Gamma(X)$, and let $D(f) = \{x \in X, s.t. f_x \in \Gamma_x \text{ is invertible}\}$. Show that $D(f)$ is open in X .
 - (c) Recall the fact that for local ring A , $A \setminus \mathfrak{m} = A^*$. Show that $V(f) = \{x \in X, s.t. f_x \in \mathfrak{m}_x \subset \Gamma_x\}$ is closed in X .
 - (d) For $x \in X$, let $I_x = \{f \in \Gamma(X), s.t. f_x \in \mathfrak{m}_x \subset \Gamma_x\}$. Show that I_x is a prime ideal of $\Gamma(X)$. Construct a canonical morphism $i \in \text{Hom}_{LRS}(X, \text{Spec } \Gamma(X))$. (Generally, i is not injective nor surjective)
 - (e) Let $h \in \text{Hom}_{Ring}(R, \Gamma(X))$, and let $\mu(h)(x \in X) = h^{-1}(I_x) \in \text{Spec } R$. Show that $\mu(h)$ induces a morphism $\in \text{Hom}_{LRS}(X, \text{Spec } R)$.
 - (f) Show that there is a bijection $\text{Hom}_{Ring}(R, \Gamma(X)) \cong \text{Hom}_{LRS}(X, \text{Spec } R)$

¹find also these exercises on <https://github.com/iamcxds/AG-exercise>, you can skip a question with * if it is difficult.