Exercises 6

[1]

1. (Gluing 3 schemes) Let $X_i, i = 1, 2, 3$ be three schemes. And for pairs i, j, there are open subsets U_{ij} and isomorphisms $\varphi_{ij} : U_{ij} \xrightarrow{\sim} U_{ji}$. Now, if the isomorphisms are compatible in the sense: for each i, j, k,

   (1) $\varphi_{ij} = \varphi_{ji}^{-1}$,

   (2) $\varphi_{ij}(U_{ij} \cap U_{ik}) = U_{ji} \cap U_{jk}$,

   (3) $\varphi_{jk} \circ \varphi_{ij} = \varphi_{ik}$ on $U_{ij} \cap U_{ik}$,

   then there exists a scheme X, as gluing of X_i.

2. Let $\mathbb{RP}^n$ be the topological space constructed by quotient $\mathbb{R}^n \backslash 0$ with $\mathbb{R}^*$. And let $\mathbb{P}^n$ be the scheme of projective space. Recall a section of $p : A \to B$ is a morphism $f : B \to A$ $s.t$ $p \circ f = \mathrm{id}$.

   (a) Let $s : \mathbb{R}^n \backslash 0 \to \mathbb{RP}^n$ be the obvious projection. Show that s has no section.

   (b) Let $D_+(x_i)_n$ be the open subscheme of $\mathbb{P}^n$. Define morphisms $s : D_{x_i} \subset \mathbb{A}^n \backslash 0 \to D_+(x_i)_n$ induced by

   $$s^\# : \mathbb{Z}[\frac{x_0}{x_i}, \dots \frac{x_n}{x_i}] \to \mathbb{Z}[x_0, \dots, x_n][x_i^{-1}], \frac{x_j}{x_i} \mapsto x_j x_i^{-1}$$

   Show that we can glue s into morphism $s : \mathbb{A}^n \backslash 0 \to \mathbb{P}^n$. Then show that s has no section.

   (c) Let $i : \mathbb{RP}^1 \to \mathbb{RP}^2 \backslash [0, 0, 1]$ by $i([x, y]) = [x, y, 0]$ and $p : \mathbb{RP}^2 \backslash [0, 0, 1] \to \mathbb{RP}^1$ by $p([x, y, z]) = [x, y]$. Check there are well-defined map, i is a section of p and $p^{-1}(t \in \mathbb{RP}^1) \cong \mathbb{R}^1$. Then show that $\mathbb{RP}^2 \backslash [0, 0, 1]$ is homeomorphic to a Mobius band.

   (d) For $n \leq m$, we define morphisms $i : D_+(x_i)_n \to D_+(x_i)_m, p : D_+(x_i)_n \to D_+(x_i)_m$ induced by

   $$i^\# : \mathbb{Z}[\frac{x_0}{x_i}, \dots \frac{x_m}{x_i}] \to \mathbb{Z}[\frac{x_0}{x_i}, \dots \frac{x_n}{x_i}], \frac{x_j}{x_i} \mapsto \frac{x_j}{x_i}, j \leq n; 0, j > n$$

   $$p^\# : \mathbb{Z}[\frac{x_0}{x_i}, \dots \frac{x_n}{x_i}] \to \mathbb{Z}[\frac{x_0}{x_i}, \dots \frac{x_m}{x_i}], \frac{x_j}{x_i} \mapsto \frac{x_j}{x_i}$$

[1] find also these exercises on https://github.com/iamcxds/AG-exercise, you can skip a question with * if it is difficult.

Show that we can glue i, p into morphisms

$$i : \mathbb{P} = D_+(x_0)_n \cup \ldots \cup D_+(x_n)_n \to U_{n,m} = D_+(x_0)_m \cup \ldots \cup D_+(x_n)_m \subset \mathbb{P}^m$$

and $p : U_{n,m} \to \mathbb{P}$.

(e) * Show that sections of $p : U_{1,2} \to \mathbb{P}^1$ correspond to linear polynomials $\{ax + b\}$