Exercises 7

1. (Segre embedding)

   (a) Show that the map $\sigma : \mathbb{RP}^1 \times \mathbb{RP}^1 \to \mathbb{RP}^3$ by $([x, y], [u, v]) \to [xu, xv, yu, yv]$ is well-defined and injective. And further show that the image of σ is $\{[z_0, z_1, z_2, z_3] \in \mathbb{RP}^3 \mid z_0 z_3 = z_1 z_2\}$.

   (b) Assume we work over field k. Let $D_+(y) \subset \mathbb{P}^1, D_+(v) \subset \mathbb{P}^1$ and $D_+(z_3) \subset \mathbb{P}^3$, we can define $\sigma_3 : D_+(y) \times_k D_+(v) \to D_+(z_3)$ which is induced by ring homomorphism:

   $$\sigma_3^\# : k[\frac{z_0}{z_3}, \frac{z_1}{z_3}, \frac{z_2}{z_3}] \to k[\frac{x}{y}] \otimes_k k[\frac{u}{v}] = k[\frac{x}{y}, \frac{u}{v}]$$

   $$\frac{z_0}{z_3} \mapsto \frac{x}{y}\frac{u}{v}, \frac{z_1}{z_3} \mapsto \frac{x}{y}, \frac{z_2}{z_3} \mapsto \frac{u}{v}$$

   Show that $\sigma_3^\#$ factor through $k[\frac{z_0}{z_3}, \frac{z_1}{z_3}, \frac{z_2}{z_3}]/(\frac{z_0}{z_3} - \frac{z_1}{z_3}\frac{z_2}{z_3})$, thus σ_3 actually is a closed embedding

   $$\sigma_3 : D_+(y) \times_k D_+(v) \xrightarrow{\cong} D_+(z_3) \cap V(z_0 z_3 - z_1 z_2) \to D_+(z_3)$$

   (c) Define $\sigma_0, \sigma_1, \sigma_2$ in the same way, then we can also define a morphism $\sigma : \mathbb{P}^1 \times_k \mathbb{P}^1 \to \mathbb{P}^3$ by gluing $\sigma_0, \sigma_1, \sigma_2, \sigma_3$. Verify that σ is well-defined and in fact, it is a closed embedding:

   $$\sigma : \mathbb{P}^1 \times_k \mathbb{P}^1 \xrightarrow{\cong} V(z_0 z_3 - z_1 z_2) \to \mathbb{P}^3$$

2. (fiber product)

   (a) Let X, Y, Z be sets with maps $f : X \to Z, g : Y \to Z$. Show that we have fiber product in Set, and

   $$X \times_Z Y = \{(x, y) \in X \times Y | f(x) = g(y) \in Z\}$$

   And let $p : X \times_Z Y \to Z, (x, y) \mapsto f(x)$ be the canonical projection. Then show that for all $z \in Z, p^{-1}(z) = f^{-1}(z) \times g^{-1}(z)$.

   (b) Let U, V be open subschemes of scheme S. Show that $U \times_S V = U \cap V$

[1] find also these exercises on https://github.com/iamcxds/AG-exercise, you can skip a question with * if it is difficult.

(c) Let k be a field, and let L be its finite separate extension. Show that $\operatorname{Spec} L \times_k \operatorname{Spec} L \cong \sqcup_{[k:L]} \operatorname{Spec} L$

(d) * Assume k is algebraically closed, let $k(x)$ be the field of fractions of $k[x]$, show that the underlying set of $\operatorname{Spec} k(x) \times_k \operatorname{Spec} k(y)$ is isomorphic to

$$\{\mathfrak{p} \in \mathbb{A}^2 | \operatorname{ht}(\mathfrak{p}) = 0 \text{ or } 1, \text{and } \forall c \in k, \mathfrak{p} \neq (x - c), \mathfrak{p} \neq (y - c)\}$$