Exercises 8

1. (Functor of points) Recall the Yoneda lemma: An object X is decided by the functor $\mathrm{Hom}(-, X)$. Conversely, from a functor F, we can try to find the object that represents it. (e.g. we have seen that the line bundles with $n+1$ section over X is represented by $\mathbb{P}^n$)

   (a) For a set S, let $P(S)$ be the set of subsets. Can you find the set that represents P? (i.e. find such set B s.t. $P(S) \cong \mathrm{Hom}_{\mathrm{Set}}(S, B)$)

   (b) For a polynomial $f \in \mathbb{Z}[x, y]$. Let $h_f(\operatorname{Spec} R) = \{(x, y) \in R^2, f(x, y) = 0\}$. Can you find the scheme that represents h_f?

   (c) For any scheme X, let $X(R)$ denote the $\mathrm{Hom}_{\mathrm{Sch}}(\operatorname{Spec} R, X)$. which is the **functor of points**. If X is a variety over an algebraically closed field k, then show that $X(k)$ is isomorphic to the set of closed points.

   (d) If X is a variety over $\mathbb{C}$, then show that there is a $\mathbb{Z}/2\mathbb{Z}$ action on $X(\mathbb{C})$, and that the fixed points are corresponds to $X(\mathbb{R})$

   (e) Show that $(X \times_Z Y)(R) \cong X(R) \times_{Z(R)} Y(R)$. So we can define the fiber product $X \times_Z Y$ as the scheme representing $X(R) \times_{Z(R)} Y(R)$. But how can we find such a scheme?[2]

2. * For a topological space X, show that X is proper (i.e. the projection $X \times Y \to Y$ is closed for any Y) iff X is compact. (More generally, for proper map $f : X \to Y$, for $y \in Y$, we have the fibre $f^{-1}(y)$ is compact)

3. * Recall the fact that holomorphic functions on compact Riemann surfaces are constant. Let k be an algebraically closed field, and let X be a connected reduced proper k-scheme. Then $\mathcal{O}_X(X) = k$.[3]

[1] find also these exercises on https://github.com/iamcxds/AG-exercise, you can skip a question with * if it is difficult.

[2] see more on https://stacks.math.columbia.edu/tag/01JF

[3] see also in *The Rising Sea*, 10.3.7