

Exercises 2

¹ Here we only discuss the underlying topological space of the spectrum. And the maximal spectrum is the subspace of maximal ideals.

1. (a) Show that the maximal spectrum of $\text{Spec } \mathbb{C}[x]$ is the complex plane \mathbb{C} with cofinite topology.
- (b) Show that the maximal spectrum of $\text{Spec } \mathbb{R}[x]$ is \mathbb{C} quotiented by a $\mathbb{Z}/2\mathbb{Z}$ action with cofinite topology.
- (c) Show that the maximal spectrum of $\text{Spec } \mathbb{C}[x, y]/(x^2 + y^2 + 1)$ is

$$\{(s, t) \in \mathbb{C}^2 \mid s^2 + t^2 + 1 = 0\}$$

with cofinite topology. Then show that there is an isomorphism $i : \text{Spec } \mathbb{C}[x, x^{-1}] \rightarrow \text{Spec } \mathbb{C}[x, y]/(x^2 + y^2 + 1)$

- (d) * Show that the maximal spectrum of $\text{Spec } \mathbb{R}[x, y]/(x^2 + y^2 - 1)$ consists of

$$\{(s, t) \in \mathbb{R}^2 \mid s^2 + t^2 = 1\} \text{ and } \{l \in \text{lines}(\mathbb{R}^2) \mid l \text{ not intersect with } s^2 + t^2 = 1\}$$

with cofinite topology.

2. Let ring $A := \mathbb{Z}[x_{11}, \dots, x_{1n}, \dots, x_{nn}, \iota]/(\iota \det(x) - 1)$ where $\det(x)$ is the polynomial of determinant and let GL_n be $\text{Spec } A$. Show that for ring R

$$\text{GL}_n(R) := \text{Hom}_{\text{Ring}}(A, R) \cong \{\text{invertible matrices with coefficient } R\}$$

3. Let k be a field, find out what are the prime spectra of the following rings, and the open(or closed) sets of Zariski topology.

- (a) $\text{Spec } k[x]/(x^2)$
- (b) $\text{Spec } k[[x]]$ (where $k[[x]] := \varinjlim_n k[x]/(x^n)$ is the formal polynomial ring)
- (c) $\text{Spec } k[x]_{(x)}$
- (d) $\text{Spec } \mathbb{Z}/m\mathbb{Z}$
- (e) * $\text{Spec } \mathbb{Z}[x]$

¹find also the exercises on <https://github.com/iamcxds/AG-exercise>, you can skip the question with * if it is difficult.