

Exercises 4

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1. Let $f : X \rightarrow Y$ be a continuous map between topological spaces, \mathcal{F} and \mathcal{G} are sheaves over X and Y respectively.
 - (a) Let the direct image $f_*\mathcal{F}$ be the presheaf over Y , s.t. $f_*\mathcal{F}(U) := \mathcal{F}(f^{-1}(U))$. Check that this is already a sheaf and $f_* : Sh(X) \rightarrow Sh(Y)$ defines a functor.
 - (b) Let the inverse image $f^*\mathcal{G}$ be the sheafification of presheaf $\mathcal{G}(f(U))$ over X . Show that $f^*\mathcal{G}_x \cong \mathcal{G}_{f(x)}$ and $f^* : Sh(Y) \rightarrow Sh(X)$ defines a functor.
 - (c) When $Y = *$ is a point, to define a sheaf \mathcal{G} over $*$ is equivalent to give the set $G = \mathcal{G}(*)$. Show that $f_*\mathcal{F} = \mathcal{F}(X)$ and $f^*G = \underline{G}$ the const sheaf.
 - (d) For any $x \in X$ induces a map $x : * \rightarrow X$. Show that $x^*\mathcal{F} = \mathcal{F}_x$ and $x_*G = G_{\{x\}}$ the skyscraper sheaf supported on $\{x\}$ (i.e. $G_{\{x\}}(U) = G$ if $x \in U$ and $*$ otherwise).
 - (e) * Show that there is an isomorphism

$$\mathrm{Hom}_{Sh(X)}(f^*\mathcal{G}, \mathcal{F}) \cong \mathrm{Hom}_{Sh(Y)}(\mathcal{G}, f_*\mathcal{F})$$

2. * Let $\mathcal{F}, \mathcal{G} \in Sh(X)$ and let $i : \mathcal{F} \rightarrow \mathcal{G}$ be a morphism s.t. $\forall x \in X, i_x : \mathcal{F}_x \rightarrow \mathcal{G}_x$ are isomorphisms. Show that i is an isomorphism of sheaves. (Hint: first show that i induces a homeomorphism between etale space $Et(i) : Et(\mathcal{F}) \rightarrow Et(\mathcal{G})$)
3. We define the (pre)sheaf \mathcal{F} of Abelian group (i.e. Abelian (pre)sheaf) over X by changing the target to Ab , i.e. now $\mathcal{F}(U) \in Ab$. We denote $PAb(X)$ and $Ab(X)$ as the category of Abelian presheaf and sheaf. And for a sequence in $PAb(X)$ (resp. $Ab(X)$)

$$\mathcal{F} \rightarrow \mathcal{G} \rightarrow \mathcal{H}$$

we say it is exact if $\forall U$ open,

$$\mathcal{F}(U) \rightarrow \mathcal{G}(U) \rightarrow \mathcal{H}(U)$$

¹find also these exercises on <https://github.com/iamcxds/AG-exercise>, you can skip the question with * if it is difficult.

(resp. $\forall x \in X$,

$$\mathcal{F}_x \rightarrow \mathcal{G}_x \rightarrow \mathcal{H}_x$$

) are exact.

- (a) Show that the exactness of presheaf implies the exactness of sheaf.
(Hint: taking stalk $\text{colim}_{x \in U} \mathcal{F}(U)$ is a filtered colimit)

- (b) Let $X = \mathbb{C}$, show that

$$0 \rightarrow \underline{\mathbb{Z}} \xrightarrow{2\pi i} \mathcal{O} \xrightarrow{\exp} \mathcal{O}^* \rightarrow 0$$

is a exact sequence of Abelian sheaves, where \mathcal{O} (resp. \mathcal{O}^*) is the sheaf of holomorphic functions (resp. non-vanishing holomorphic functions). But by considering the sections on $\mathbb{C}^* \subset \mathbb{C}$, show that this is not exact as presheaf.

- (c) Let $\underline{\mathbb{Z}} \in \text{Ab}(X)$ be the const sheaf, and for close point $x \in X$, let $\mathbb{Z}_{\{x\}}$ be the skyscraper sheaf. and let $\mathbb{Z}_{X-\{x\}}$ be the sheaf s.t. $\mathbb{Z}_{X-\{x\}}(U) = \mathbb{Z}$ if $x \notin U$, and 0 otherwise. Show that there is exact sequence of sheaf

$$0 \rightarrow \mathbb{Z}_{X-\{x\}} \rightarrow \underline{\mathbb{Z}} \rightarrow \mathbb{Z}_{\{x\}} \rightarrow 0$$

But then show that $\underline{\mathbb{Z}}$ is **not** isomorphic to $\mathcal{B} = \mathbb{Z}_{X-\{x\}} \oplus \mathbb{Z}_{\{x\}}$, even if there are isomorphisms $\forall x, \underline{\mathbb{Z}}_x \cong \mathbb{Z} \cong \mathcal{B}_x$.

4. Let $X = [0, 1]$ and let $C(X)$ be the ring of real-valued functions on X . Recall that the maximal spectrum $\text{Spm}(R)$ is the subspace of $\text{Spec}(R)$ consisting of maximal ideals. We will show that $\tilde{X} = \text{Spm}(C(X))$ is homeomorphic to X .

- (a) For $x \in X$, let maximal ideals $\mathfrak{m}_x = \{f \in C(X), s.t. f(x) = 0\}$. Now for any ideal $I \subsetneq C(X)$, let $V_X(I) = \{x \in X, s.t. f(x) = 0, \forall f \in I\}$. Show that $V_X(I)$ is nonempty and thus for $x \in V_X(I), I \subset \mathfrak{m}_x$. Therefore $\mathfrak{m}_- : X \rightarrow \tilde{X}$ is surjective.
- (b) By construction functions separate the points of X , conclude that $\mathfrak{m}_- : X \rightarrow \tilde{X}$ is injective.
- (c) Let $f \in C(X)$, and let

$$U_f = \{x \in X, s.t. f(x) \neq 0\} \subset X$$

$$\tilde{U}_f = \{m \in \tilde{X}, s.t. f \notin m\} \subset \tilde{X}$$

Show that $\mathfrak{m}_-(U_f) = \tilde{U}_f$ and therefore \mathfrak{m}_- is a homeomorphism.

- (d) * Show that any prime ideal $\mathfrak{p} \subset C(X)$ are contained in exact one \mathfrak{m}_x .