

Exercises 3

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1. Let R be a ring and for any ideal J , we denote $V(J)$ for the set of prime ideals containing J . Now for any subset S of $\text{Spec } R$, we define $I(S) = \bigcap_{\mathfrak{p} \in S} \mathfrak{p}$. Let A (resp. B) be the category of ideals of R (resp. subsets of $\text{Spec } R$), whose morphisms are given by inclusion (i.e. $\text{Hom}(X, Y) = \{*\}$ if $X \subset Y$, and be \emptyset otherwise).
 - (a) Show that $V : A^{op} \rightarrow B$ and $I : B^{op} \rightarrow A$ are well-defined functors. And furthermore, Show that $\text{Hom}_A(J, I(S)) \cong \text{Hom}_B(S, V(J))$
 - (b) Let \mathfrak{n} be the ideal of nilpotent elements, show that for any prime ideal \mathfrak{p} , $\mathfrak{n} \subset \mathfrak{p}$ and $\mathfrak{n} \subset I(\text{Spec } R)$.
 - (c) For any non-nilpotent elements $f \in R$, let R_f be the localization of R with $\{f^i\}$. Show that

$$\text{Spec } R_f \cong \{\mathfrak{p} \text{ prime} \mid f \notin \mathfrak{p}\}$$

is nonempty. Then conclude that $\mathfrak{n} = I(\text{Spec } R)$

- (d) Define the radical of an ideal J to be $\sqrt{J} := \{x \in R \mid \exists n \in \mathbb{N}, x^n \in J\}$. Show that $V(I(S)) = \bar{S}$, the closure in $\text{Spec } R$, and $I(V(J)) = \sqrt{J}$.
 - (e) An ideal J is a radical ideal if $J = \sqrt{J}$. Show that there is a bijection between the closed subsets of $\text{Spec } R$ and the radical ideals of R .
 - (f) * Show that S is closed irreducible, iff $S = V(\mathfrak{p})$ for \mathfrak{p} prime, thus there is a bijection between the irreducible closed subsets of $\text{Spec } R$ and the points of $\text{Spec } R$.
 - (g) Show that there is a bijection between the irreducible components of $\text{Spec } R$ and the minimal (with respect to inclusion) prime ideals R .
2. (a) Show that for Noetherian space X , any open subset is quasi-compact.
 - (b) * Show that for Noetherian ring R , there are only finitely many minimal prime ideals. Then show that the elements of minimal prime ideals are zero divisors.

¹find also these exercises on <https://github.com/iamcxds/AG-exercise>, you can skip the question with * if it is difficult.

3. We will show that every sheaf is actually a sheaf of "functions". Let X be a topological space, an etale space (E, p) over X is a topological space E with a local homeomorphism $p : E \rightarrow X$ (i.e. $\forall x \in E, \exists$ neighborhood $U \ni x$, such that $p|_U$ is a homeomorphism).

- (a) For any open subset $U \subset X$, we define the sections:

$$\Gamma_E(U) = \{f : U \rightarrow p^{-1}(U) \subset E \mid p \circ f = \text{id}_U\}$$

Check that Γ_E is a well-defined sheaf on X

- (b) Let \mathcal{P} be a presheaf on X , we define an etale space $(Et(\mathcal{P}), p)$ as follow:

$$Et(\mathcal{P}) = \coprod_{x \in X} \mathcal{P}_x, p : Et(\mathcal{P}) \rightarrow X, p(x, s_x \in \mathcal{P}_x) = x \in X$$

where \mathcal{P}_x is the stalk. And the topology of $Et(\mathcal{P})$ is given by the basis $\{V_f\}_{f \in \mathcal{P}(U)}$, where

$$V_f = \{(x \in U, f_x \in \mathcal{P}_x) \in Et(\mathcal{P})\}$$

Check that $Et(\mathcal{P})$ is a well-defined etale space.

- (c) * Show that there is a canonical morphism $i : \mathcal{P} \rightarrow \Gamma_{Et(\mathcal{P})}$ and it is an isomorphism if \mathcal{P} is a sheaf. Furthermore, $\Gamma_{Et(\mathcal{P})}$ is isomorphic to the sheafification of \mathcal{P} . (First try to show that $\Gamma_{Et(\mathcal{P})}(U) \subset \prod_{x \in U} \mathcal{P}_x$ and satisfies some conditions)