Exercises 1

1. (filtered (co)limits)
   A category I is called filtered if it satisfies the conditions below.

   (i) I is non-empty,

   (ii) for any i and j in I, there exists $k \in I$ and morphisms $i \to k, j \to k$,

   (iii) for any parallel morphisms $f, g : i \rightrightarrows j$, there exists a morphism $h : j \to k$ such that $h \circ f = h \circ g$.

   A functor from the filtered category I to C is called a filtered system of C, and the colimit of a filtered system is called the filtered colimit.

   (a) Let $M(i) = M_i$ be a filtered system of RMod(or Set) indexed by I, Show that $\operatorname{colim}_I M_i \cong \sqcup M_i / \sim$, where $M_i \ni x \sim y \in M_j$ if there exists $k \in I, s : i \to k$ and $t : j \to k$ with $M(s)(x) = M(t)(y)$. (Also check $\sim$ that is indeed an equivalence relation).

   (b) Let I be a filtered category and let J be a finite category. Given a functor $F : I \times J \to R\mathrm{Mod}$, show that $\operatorname{colim}_I \lim_J F(i,j) \cong \lim_J \operatorname{colim}_I F(i,j)$.

2. (a) Show that every poset $(P, \leq)$ can be turned into a category. Figure out what the objects and morphisms are.

   (b) Let $\mathcal{N}_{\leq}$ be the category defined by the poset $(\mathbb{N}, \leq)$ and let p be a prime number. Consider the following two functors:
   $I_p : \mathcal{N}_{\leq} \to \mathrm{Ab}$ where $I_p(n) = \mathbb{Z}/p^n\mathbb{Z}$ and

   $$I_p(n \leq m) : x \in \mathbb{Z}/p^n\mathbb{Z} \mapsto p^{m-n}x \in \mathbb{Z}/p^m\mathbb{Z};$$

   $R_p : \mathcal{N}_{\leq}^{\mathrm{op}} \to \mathrm{Ab}$ where $R_p(n) = \mathbb{Z}/p^n\mathbb{Z}$ and

   $$R_p(n \leq m) : x \in \mathbb{Z}/p^m\mathbb{Z} \mapsto x \mod p^n \in \mathbb{Z}/p^n\mathbb{Z}.$$

   Check that these two functors are well-defined.

   (c) Show that $\operatorname{colim}_{\mathcal{N}_{\leq}} I_p(n) \cong \mathbb{Z}[1/p]/\mathbb{Z}$ where $\mathbb{Z}[1/p]$ is the localization of $\mathbb{Z}$ at the multiplicative system $\{p^n | n \in \mathbb{N}\}$.

   (d) Let $\lim_{\mathcal{N}_{\leq}} R_p(n) := \mathbb{Z}_p$, the ring of p-adic integers. Show that

   $$\mathrm{Hom}_{\mathrm{Ab}}(\mathbb{Z}_{(p)}/\mathbb{Z}, \mathbb{Z}_{(p)}/\mathbb{Z}) \cong \mathbb{Z}_p.$$

(e) Let $\mathcal{N}_{\div}$ be the category defined by the poset $(\mathbb{N}, |)$(i.e. the poset of divisibility). We can similarly define functors $I : \mathcal{N}_{\div} \to \mathrm{Ab}$ and $R_p : \mathcal{N}_{\div}^{\mathrm{op}} \to \mathrm{Ab}$ where $I(n) = R(n) = \mathbb{Z}/n\mathbb{Z}$, etc. Show that $\mathrm{colim}_{\mathcal{N}_{\div}} I(n) \cong \mathbb{Q}/\mathbb{Z}$ and

$$\lim_{\mathcal{N}_{\div}} R(n) := \hat{\mathbb{Z}} \cong \prod_p \mathbb{Z}_p \cong \mathrm{Hom}_{\mathrm{Ab}}(\mathbb{Q}/\mathbb{Z}, \mathbb{Q}/\mathbb{Z}).$$

3. (Yoneda lemma)

   (a) Let C be a category, and let $\mathrm{PSh}(C) = \mathrm{Fun}(C^{\mathrm{op}}, \mathrm{Set})$. Check that we can define a functor $y : C \to \mathrm{PSh}(C), X \mapsto \mathrm{Hom}(-, X)$.

   (b) For $A \in \mathrm{PSh}(C)$ and and $X \in C$, show that there is a bijection $\mathrm{Hom}_{\mathrm{PSh}(C)}(y(X), A) \cong A(X)$.

   (c) Show that y is fully faithful.

4. (the category RMod)

   (a) Show that $- \otimes_R M$ is the left adjoint of $\mathrm{Hom}(M, -)$, use this to conclude that $- \otimes_R M$ commutes with all colimits.

   (b) Let S be a multiplicative system, define the filtered system $R_s \cong R, \forall s \in S$ and $a_{s,st} = t\cdot : R_s \to R_{st}$ given by multiplying t. Show that

   $$\underset{S}{\mathrm{colim}} R_s \cong \{(r, s) \in R \times S\}/\{(r, s) \sim (r', s'), \text{iff } \exists t \in S, \text{s.t. } trs' = tr's \in R\}$$

   and $\mathrm{colim}_S R_s \cong S^{-1}R$. Use this to show $S^{-1}R$ is flat in RMod

   (c) Show that the filtered colimit of flat modules is again flat. In particular, the filtered colimits of finite free modules are flat. (In fact, the converse is also true.)

   (d) Show that $\mathbb{Z}/n\mathbb{Z}$ is not flat.

5. (Compact object)

   (a) Let A_i be a system indexed by I in a category C and let $X \in C$, construct the natural morphism $d : \mathrm{colim}_I\, y(A_i) \to y(\mathrm{colim}_I A_i)$, in particular,

   $$d(X) : \mathrm{colim}_I \mathrm{Hom}_C(X, A_i) \to \mathrm{Hom}_C(X, \mathrm{colim}_I A_i)$$

   (b) Let C be the category of topological space Top, and let $I = \mathcal{N}_{\leq}$ where $A_n = (-n, n) \subset \mathbb{R}$. Show that $d(X)$ is a bijection if X is compact. What if X is not compact?

   (c) Let C be the RMod and let A_i be a filtered system. Show that $d(M)$ are bijections for all filtered systems, iff M is finitely presented (i.e. there is a short exact sequence $R^n \to R^m \to M \to 0$ with n, m finite). (Such M is also called a compact object.)

6. (Snake lemma) Let

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r, dashed] & \ker(a) \arrow[r] \arrow[d] & \ker(b) \arrow[r] \arrow[d] & \ker(c) \arrow[d] \arrow[dddll, rounded corners, "\partial"] & \\
0 \arrow[r, dashed] & A \arrow[r, "f"] \arrow[d, "a"] & B \arrow[r, "g"] \arrow[d, "b"] & C \arrow[r] \arrow[d, "c"] & 0 \\
0 \arrow[r] & A' \arrow[r, "f'"] \arrow[d] & B' \arrow[r, "g'"] \arrow[d] & C' \arrow[r, dashed] \arrow[d] & 0 \\
 & \operatorname{coker}(a) \arrow[r] \arrow[d] & \operatorname{coker}(b) \arrow[r] \arrow[d] & \operatorname{coker}(c) \arrow[r, dashed] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

be a commuting diagram in RMod such that the middle two rows are exact sequences. Then prove that there is a long exact sequence of kernels and cokernels of the form

$$0 \dashrightarrow \ker(a) \to \ker(b) \to \ker(c) \xrightarrow{\partial} \operatorname{coker}(a) \to \operatorname{coker}(b) \to \operatorname{coker}(c) \dashrightarrow 0.$$