Exercises 4

[1]

1. For any topological space X, we think of it as a ringed space, with the structure sheaf $C^0(U) = \{f : U \to \mathbb{R}, continuous\}$ of continuous functions.
   (a) Show that (X, C^0) is a locally ringed space, and any continuous map $f : X \to Y$ induces a morphism of locally ringed spaces $(f, f^\#) : (X, C^0) \to (Y, C^0)$
   (b) Let $X = [0, 1]$ for multiplicative system $S_U = \{f \in C^0(X), s.t. \forall x \in U, f(x) \neq 0\}$. Then show that there is an isomorphism $S_U^{-1}C^0(X) \cong C^0(U)$, thus $\operatorname{Spm} C^0(X) \cong (X, C^0)$ as locally ringed spaces.
   (c) Let $X = \mathbb{R}^n, Y = \mathbb{R}^m$(or more generally differential manifolds), we can define the structure to be $C^\infty(U) = \{f : U \to \mathbb{R}, smooth\}$. Show that any smooth map $f : X \to Y$ induces a morphism of locally ringed spaces $(f, f^\#) : (X, C^\infty) \to (Y, C^\infty)$
   (d) Let $X = \mathbb{R}^1, Y = \mathbb{R}^2$, let $f : X \to Y$ be $f(t) = (t, |t|)$. Verify that f only induces morphism between $(X, C^0) \to (Y, C^0)$ rather than $(X, C^\infty) \to (Y, C^\infty)$. (Actually, f is smooth, iff it induces $(f, f^\#) : (X, C^\infty) \to (Y, C^\infty)$)

2. Let LRS be the category of locally ringed space. And let (X, Γ) be a locally ringed space and R be a ring.
   (a) Show that there is a canonical map $\operatorname{Hom}_{LRS}(X, \operatorname{Spec} R) \to \operatorname{Hom}_{Ring}(R, \Gamma(X))$.
   (b) Let $f \in \Gamma(X)$, and let $D(f) = \{x \in X, s.t. f_x \in \Gamma_x \text{ is invertible}\}$. Show that $D(f)$ is open in X.
   (c) Recall the fact that for local ring A, the $A \backslash \mathfrak{m} = A^*$. Show that $V(f) = \{x \in X, s.t. f_x \in \mathfrak{m}_x \subset \Gamma_x\}$ is closed in X.
   (d) For $x \in X$, let $I_x = \{f \in \Gamma(X), s.t. f_x \in \mathfrak{m}_x \subset \Gamma_x\}$. Show that I_x is a prime ideal of $\Gamma(X)$. Construct a canonical morphism $i \in \operatorname{Hom}_{LRS}(X, \operatorname{Spec} \Gamma(X))$. (Generally, i is not injective nor surjective)
   (e) Let $h \in \operatorname{Hom}_{Ring}(R, \Gamma(X))$, and let $\mu(h)(x \in X) = h^{-1}(I_x) \in \operatorname{Spec} R$. Show that $\mu(h)$ induces a morphism $\in \operatorname{Hom}_{LRS}(X, \operatorname{Spec} R)$.
   (f) Show that there is a bijection $\operatorname{Hom}_{Ring}(R, \Gamma(X)) \cong \operatorname{Hom}_{LRS}(X, \operatorname{Spec} R)$

[1]find also these exercises on https://github.com/iamcxds/AG-exercise, you can skip a question with * if it is difficult.