Seeing the Mountain

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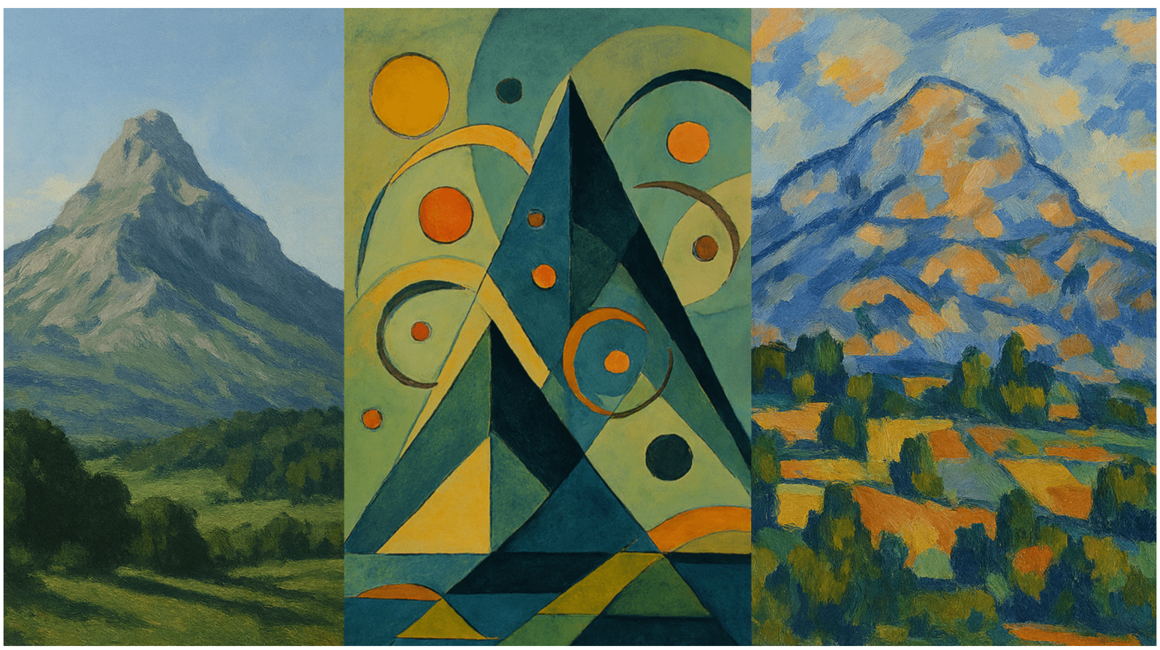

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Seeing the mountain as the mountain;
Seeing the mountain as not a mountain;
Seeing the mountain as still a mountain.

QINGYUAN XINGSI

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Chapter 0

Introduction

0.1 What is Algebraic Geometry?

Algebraic geometry is the study of geometric spaces that locally arise as solutions to polynomial equations. This study can be approached at two levels:

- **Local:** This involves studying the geometric properties of solution sets to polynomial equations. Since we work not only over $\mathbb{R}$ or $\mathbb{C}$ but potentially over arbitrary fields or rings, we must construct a more intrinsic notion of geometry associated with algebra—namely, the spectrum of a ring. This leads to the study of affine varieties and affine schemes, allowing a translation between algebra and geometry.

- **Synthetic:** Analogous to how manifolds generalize open subsets of $\mathbb{R}^n$, we study spaces that are locally isomorphic to affine varieties. This broader perspective leads to the study of varieties and schemes through various formal frameworks.

This lecture is part of an algebraic geometry course that emphasizes the second, synthetic level. However, the underlying question—"How can we construct and classify generalized spaces from certain building blocks?"—is relevant across many branches of geometry. Thus, this part can stand alone as a form of *post-modern*[1] synthetic geometry.

[1] A commonly accepted definition is "after World War II."

0.2 Essentialism vs. Structuralism

Synthetic geometry formalizes geometric concepts through axioms that directly address fundamental entities—such as points and lines—rather than relying on a background space like Cartesian coordinates. In contrast to the analytic viewpoint, where every geometric object is composed of points, synthetic geometry treats lines, curves, and other entities as primitive. The focus shifts to the **relationships** between these objects, such as “a point lies on a line” or “two lines intersect.”

This structuralism perspective emphasizes understanding objects through their interactions. To formalize these relationships, we use **category theory** and **sheaf theory**. Moreover, by viewing categories themselves as spaces, we uncover deeper geometric structures. This formalism, developed through algebraic geometry, now plays a central role in various fields: differential geometry, topology, quantum field theory, and beyond.

0.3 What is Geometry in Physics?

Quantum physics connects to both the local and synthetic levels of algebraic geometry:

- **Local:** The concept of the spectrum of a commutative ring originates from C^*-algebras and operator theory, both fundamental in quantum physics. The duality between algebra and geometry is already present in the Heisenberg and Schrödinger formulations.

- **Synthetic:** In quantum field theory, we study the space of field configurations (histories), which is infinite-dimensional and behaves irregularly. Nevertheless, we aim to treat it as a “manifold” to define metrics, path integrals, and differential forms. Synthetic geometry provides a rigorous framework for these constructions (e.g., via smooth sets). Gauge theory, supergeometry, and related topics can also be unified within this framework [1].

A concrete example is mirror symmetry, which links Gromov–Witten invariants with Hodge theory.

0.4 Course Plan

We will focus on three types of geometry and study them comparatively using synthetic methods:

- **Smooth Sets / Manifolds:** Built from open subsets of $\mathbb{R}^n$, these provide intuitive and geometric examples, serving as a bridge to classical geometry.
- **Simplicial Sets:** Constructed from simplices Δ, these offer the simplest and most abstract examples.
- **Schemes:** Built from spectra of commutative rings, these are the central objects of study in this course.

Chapter 1

Category

There are many references available for category theory; in this course, we follow [2]

1.1 Definition and Examples

Definition 1.1.1 (Category)**.** A **category** $\mathcal{C}$ consists of:

- A class of objects $\mathrm{Ob}(\mathcal{C})$.
- For each pair A, B, a set of morphisms $\mathrm{Hom}_{\mathcal{C}}(A, B)$.
- A composition operation $\circ$ of morphisms and identity morphisms id_A for each object A.

Subject to:

- **Associativity:** $h \circ (g \circ f) = (h \circ g) \circ f$
- **Identity:** $\mathrm{id}_B \circ f = f = f \circ \mathrm{id}_A$

For simplicity, we denote objects in a category $\mathcal{C}$ by $A \in \mathcal{C}$ and morphisms by $f : A \to B$. There are various ways to understand what a category is; let us explore this through examples:

Example 1.1.2 (Concrete Category)**.** A concrete category can be viewed as a collection of mathematical structures, where morphisms are maps that preserve those structures.

- **Set:** Objects are sets; morphisms are functions between sets.
- **Grp:** Objects are groups; morphisms are group homomorphisms.
- **Vect:** Objects are vector spaces; morphisms are linear maps.
- **Top:** Objects are topological spaces; morphisms are continuous maps.

You can construct infinitely many examples from the mathematical structures you are familiar with.

At first glance, this abstraction may seem unnecessary. However, we can generalize familiar notions from set theory. For instance, consider a morphism $f : A \to B$ in a category $\mathcal{C}$:

- **Isomorphism:** f is an *isomorphism* if there exists a morphism $g : B \to A$ such that:
$$g \circ f = \mathrm{id}_A \quad \text{and} \quad f \circ g = \mathrm{id}_B (\text{i.e.} g = f^{-1})$$
In this case, A and B are said to be *isomorphic*.
- **Monomorphism:** f is a *monomorphism* (or *mono*) if for all morphisms $g_1, g_2 : X \to A$, we have:
$$f \circ g_1 = f \circ g_2 \Rightarrow g_1 = g_2$$
That is, f is left-cancellable.
- **Epimorphism:** f is an *epimorphism* (or *epi*) if for all morphisms $h_1, h_2 : B \to Y$, we have:
$$h_1 \circ f = h_2 \circ f \Rightarrow h_1 = h_2$$
That is, f is right-cancellable.

Exercise 1.1.1. Verify that in the category **Set**, these definitions correspond to bijections, injections, and surjections, respectively.

Example 1.1.3 (Classifying Category)**.** A category can be viewed as an algebraic structure generalizing a monoid[1]. Indeed, for any object $A \in \mathcal{C}$, the set of endomorphisms $\mathrm{Hom}_{\mathcal{C}}(A, A)$ forms a monoid. Conversely, any monoid M can be associated with a **classifying category** BM, which has a single object $\bullet$ and morphisms

[1] An algebraic structure similar to a group, but without requiring inverses

$\mathrm{Hom}_{BM}(\bullet, \bullet) = M$. In particular, for any group G, we obtain a **classifying space** BG, where all morphisms are isomorphisms.

We refer to BG as a space because we can interpret its isomorphisms as paths in a certain topological space:

Example 1.1.4 (Groupoid)**.** A **groupoid** is a category in which every morphism is an isomorphism. Given a groupoid $\mathcal{X}$, we can construct its **geometric realization** $|\mathcal{X}|$ as follows:

- Take the objects $x \in \mathcal{X}$ as points.
- For each morphism $f : x \to y$, attach a segment from point x to point y.
- For each relation $f \circ g = h$, attach a triangle with edges labeled by f, g, and h.
- Continue this process for higher-dimensional cells ...

We will formalize this construction when we introduce simplicial sets.

Exercise 1.1.2. Show that the set of connected components $\pi_0(|\mathcal{X}|, x)$ corresponds bijectively to the isomorphism class of x in $\mathcal{X}$. Furthermore, observe that $\mathrm{Hom}_{\mathcal{X}}(x, x)$ is a group, and prove that the fundamental group $\pi_1(|\mathcal{X}|, x) \cong \mathrm{Hom}_{\mathcal{X}}(x, x)$.

Conversely, for a topological space S, we can define the **fundamental groupoid** $\Pi_1(S)$, where:

- Objects are points of S.
- Morphisms $\mathrm{Hom}_{\Pi_1(S)}(a, b)$ are homotopy classes of paths from a to b.

Exercise 1.1.3. Describe the composition law in $\Pi_1(S)$ and verify that it satisfies the axioms of a groupoid.

The examples above illustrate that morphisms in a category can carry rich structure. On the other hand, if morphisms are trivial (i.e., at most one between any two objects), we obtain a partially ordered set:

Example 1.1.5 (Poset)**.** Let $(P, \leq)$ be a partially ordered set. We can regard P as a category as follows:

- **Objects:** Elements of P.

- **Morphisms:** For $x, y \in P$, there exists a unique morphism $f : x \to y$ if and only if $x \leq y$.

- **Composition:** If $x \leq y$ and $y \leq z$, then $x \leq z$, so the morphism $x \to z$ is the composition of $x \to y$ and $y \to z$.

- **Identity:** For each $x \in P$, the identity morphism $\mathrm{id}_x : x \to x$ corresponds to the reflexivity $x \leq x$.

This category is called *thin*, meaning there is at most one morphism between any two objects.

A frequently used example is the poset of open subsets of a topological space S, denoted $(\mathrm{Op}(S), \subseteq)$.

Note that a set A can be viewed as a category in two distinct ways: either as a trivial groupoid or as a trivial poset. And a category can be viewed as a combination of this two case[2].

$$
\begin{tikzcd}
& \textbf{Category} & \\
\textbf{Groupoid} \arrow[ur, hook] & & \textbf{Poset} \arrow[ul, hook'] \\
& \textbf{Set} \arrow[ul, hook] \arrow[ur, hook'] &
\end{tikzcd}
$$

Remark 1.1.6. In the definition of a category, if we allow the morphism set $\mathrm{Hom}_{\mathcal{C}}(A, B)$ to carry additional structure—such as an Abelian group, vector space, groupoid, topological space, or even another category—we obtain an **enriched category**. In fact, it is often more natural to think of $\mathrm{Hom}_{\mathcal{C}}(A, B)$ as the set of connected components of a space:

$$\mathrm{Hom}_{\mathcal{C}}(A, B) \cong \pi_0 \mathrm{Map}_{\mathcal{C}}(A, B).$$

This reflects a general principle of the **Univalence Foundation**: mathematical structures should be treated as spaces (or types) from the outset, and the classical set-theoretic version can be recovered by taking the set of connected

[2]It is useful to think category as an oriented graph

components. We will explore how to identify and work with these underlying geometric structures later in the course.

1.2 Functor and Presheaf

If we want to define map between to category, it is natural to ask respect the morphism:

Definition 1.2.1 (Functor)**.** Let $\mathcal{C}$ and $\mathcal{D}$ be categories. A **functor** $F : \mathcal{C} \to \mathcal{D}$ consists of:

- A function that assigns to each object $A \in \mathcal{C}$ an object $F(A) \in \mathcal{D}$.
- A function that assigns to each morphism $f : A \to B$ in $\mathcal{C}$ a morphism $F(f) : F(A) \to F(B)$ in $\mathcal{D}$.

Such that:

- **Preservation of identities:** $F(\mathrm{id}_A) = \mathrm{id}_{F(A)}$
- **Preservation of composition:** For all $f : A \to B$ and $g : B \to C$ in $\mathcal{C}$,

$$F(g \circ f) = F(g) \circ F(f)$$

We denote the set (later will be category) of functor $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$

Example 1.2.2. For concrete categories, we can have the functor with respect their structures

- **Forgetful Functor:**
$$U : \mathbf{Grp} \to \mathbf{Set}$$
assigns to each group its underlying set, and to each group homomorphism the same map viewed as a map of sets. Same for **Vect**, **Top**, etc.

- **Free Functor:**
$$Free : \mathbf{Set} \to \mathbf{Grp}$$
assigns to each set the free group it generates, and to each map of sets the free group homomorphism it generates.

Exercise 1.2.1. Verify $Free$ is indeed a functor, then show that $\mathrm{Hom}_{\mathbf{Grp}}(Free(A), G) \cong \mathrm{Hom}_{\mathbf{Set}}(A, U(G))$. Then how to define $Free$ for **Vect**?

- **Discrete Functor:**

$$Disc : \mathbf{Set} \to \mathbf{Top}$$

assigns to each set the discrete topological space, and to each map of sets the same map viewed as a continuous map.

- **Connected components:**

$$\pi_0 : \mathbf{Top} \to \mathbf{Set}$$

assigns to each topological space its set of connected component.

Exercise 1.2.2. Show that $\mathrm{Hom}_{\mathbf{Top}}(Disc(A), S) \cong \mathrm{Hom}_{\mathbf{Set}}(A, U(S))$ and $\mathrm{Hom}_{\mathbf{Top}}(S, Disc(A)) \cong \mathrm{Hom}_{\mathbf{Set}}(\pi_0 S, A)$

Exercise 1.2.3. For groups (more general monoids) G, H, show that there is a bijection between functors $F : BG \to BH$ and group (monoid) homomorphism $f : G \to H$.

Exercise 1.2.4.

1. For groupoid $\mathcal{X}, \mathcal{Y}$, show that a functor $F : \mathcal{X} \to \mathcal{Y}$ induce a continuous map of geometric realization $|F| : |\mathcal{X}| \to |\mathcal{Y}|$. Now let **Grpd** be the category of groupoid, with functor as morphism, then show that geometric realization $|\cdot| : \mathbf{Grpd} \to \mathbf{Top}$ is actually a functor.
2. Similarly, show that $\Pi_1 : \mathbf{Top} \to \mathbf{Grpd}$ is also a functor.
3. * Moreover, show that $\mathrm{Hom}_{\mathbf{Top}}(S, |\mathcal{X}|) \cong \mathrm{Hom}_{\mathbf{Grpd}}(\Pi_1(S), \mathcal{X})$

Exercise 1.2.5. For posets P, Q, show that a functor $F : P \to Q$ is the same thing as an order preserving function.

Example 1.2.3 (Ring-Space Correspondence)**.** Given a topological space X, assign to its ring of (real or complex) continuous function $C(X)$ seems to be a functor, but there is an issue: the map $p : X \to Y$ induce a pullback of function $p^* : C(Y) \to C(X), f \mapsto f(p(\cdot))$. For this reason, we introduce the notion of **Opposite Category** $\mathcal{C}^{\mathrm{op}}$: it has the same objects as $\mathcal{C}$, but reverse direction of all morphism, i.e. $\mathrm{Hom}_{\mathcal{C}^{\mathrm{op}}}(A, B) = \mathrm{Hom}_{\mathcal{C}}(B, A)$. Then $C(\cdot) : \mathbf{Top}^{\mathrm{op}} \to \mathbf{Ring}$ is a functor.

Remark 1.2.4. The core of algebraic geometric is how to reverse this functor, to construct a space out of a commutative ring. That to say, to define a functor Spec $: \mathbf{Ring}^{\mathrm{op}} \to \mathbf{Top}$, and we have

$$\mathrm{Hom}_{\mathbf{Ring}}(R, C(X)) \cong \mathrm{Hom}_{\mathbf{Top}}(X, \mathrm{Spec}\ R)$$

But in reality we need to add some restriction in category to make this well define:

- between C^*-algbra and locally compact Hausdorff space, which is Gelfand representation theorem, and where the name Spec spectrum origin from;
- between commutative ring and locally ringed space, which is a fundamental theorem in algebraic geometry.

Here is another important example:

Example 1.2.5 (Hom Functor)**.** For a fixed object A in a category $\mathcal{C}$, the functor

$$h^A := \mathrm{Hom}_{\mathcal{C}}(A, -) : \mathcal{C} \to \mathbf{Set}$$

assigns to each object B the set of morphisms $\mathrm{Hom}_{\mathcal{C}}(A, B)$, and to each morphism $f : B \to C$ the function:

$$\mathrm{Hom}_{\mathcal{C}}(A, f) : \mathrm{Hom}_{\mathcal{C}}(A, B) \to \mathrm{Hom}_{\mathcal{C}}(A, C), \quad g \mapsto f \circ g$$

Remark 1.2.6 (Opposite category).

Now we introduce the central object of the course

- **Power Set Functor:**
$$\mathcal{P} : \mathbf{Set} \to \mathbf{Set}$$
assigns to each set X its power set $\mathcal{P}(X)$, and to each function $f : X \to Y$ the function $\mathcal{P}(f) : \mathcal{P}(X) \to \mathcal{P}(Y)$ defined by:
$$\mathcal{P}(f)(A) = f(A) = \{f(a) \mid a \in A\}$$
- **Hom Functor:** For a fixed object A in a category $\mathcal{C}$, the functor
$$\mathrm{Hom}_{\mathcal{C}}(A, -) : \mathcal{C} \to \mathbf{Set}$$
assigns to each object B the set of morphisms $\mathrm{Hom}_{\mathcal{C}}(A, B)$, and to each morphism $f : B \to C$ the function:
$$\mathrm{Hom}_{\mathcal{C}}(A, f) : \mathrm{Hom}_{\mathcal{C}}(A, B) \to \mathrm{Hom}_{\mathcal{C}}(A, C), \quad g \mapsto f \circ g$$

1.3 Yoneda Lemma

1.4 Universal Property and (Co)limit

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