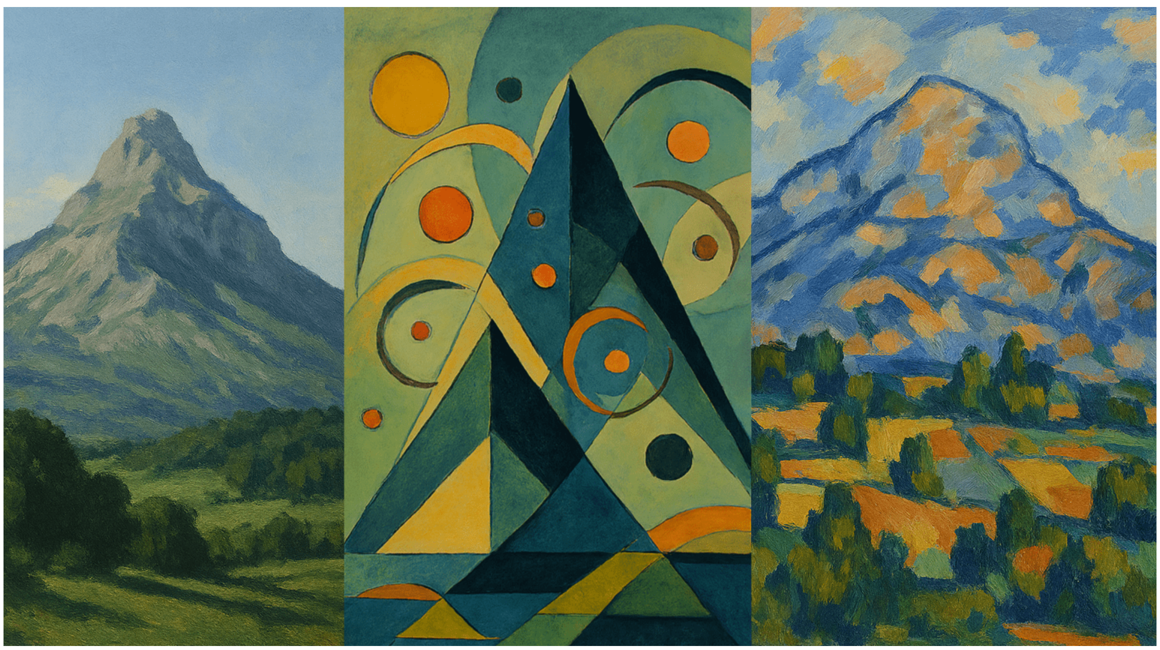
Seeing the Mountain

Keyao Peng

keyao.peng@ube.fr



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Seeing the mountain as the mountain;
Seeing the mountain as not a mountain;
Seeing the mountain as still a mountain.

QINGYUAN XINGSI

Contents

Chapter 0

Introduction

0.1 What is Algebraic Geometry?

Algebraic geometry is the study of geometric spaces that locally arise as solutions to polynomial equations. This study can be approached at two levels:

- **Local:** This involves studying the geometric properties of solution sets to polynomial equations. Since we work not only over $\mathbb{R}$ or $\mathbb{C}$ but potentially over arbitrary fields or rings, we must construct a more intrinsic notion of geometry associated with algebra—namely, the spectrum of a ring. This leads to the study of affine varieties and affine schemes, allowing a translation between algebra and geometry.

- **Synthetic:** Analogous to how manifolds generalize open subsets of $\mathbb{R}^n$, we study spaces that are locally isomorphic to affine varieties. This broader perspective leads to the study of varieties and schemes through various formal frameworks.

This lecture is part of an algebraic geometry course that emphasizes the second, synthetic level. However, the underlying question—"How can we construct and classify generalized spaces from certain building blocks?"—is relevant across many branches of geometry. Thus, this part can stand alone as a form of *post-modern*[1] synthetic geometry.

[1] A commonly accepted definition is "after World War II."

0.2 Essentialism vs. Structuralism

Synthetic geometry formalizes geometric concepts through axioms that directly address fundamental entities—such as points and lines—rather than relying on a background space like Cartesian coordinates. In contrast to the analytic viewpoint, where every geometric object is composed of points, synthetic geometry treats lines, curves, and other entities as primitive. The focus shifts to the **relationships** between these objects, such as "a point lies on a line" or "two lines intersect."

This structuralism perspective emphasizes understanding objects through their interactions. To formalize these relationships, we use **category theory** and **sheaf theory**. Moreover, by viewing categories themselves as spaces, we uncover deeper geometric structures. This formalism, developed through algebraic geometry, now plays a central role in various fields: differential geometry, topology, quantum field theory, and beyond.

0.3 What is Geometry in Physics?

Quantum physics connects to both the local and synthetic levels of algebraic geometry:

- **Local:** The concept of the spectrum of a commutative ring originates from C^*-algebras and operator theory, both fundamental in quantum physics. The duality between algebra and geometry is already present in the Heisenberg and Schrödinger formulations.

- **Synthetic:** In quantum field theory, we study the space of field configurations (histories), which is infinite-dimensional and behaves irregularly. Nevertheless, we aim to treat it as a "manifold" to define metrics, path integrals, and differential forms. Synthetic geometry provides a rigorous framework for these constructions (e.g., via smooth sets). Gauge theory, supergeometry, and related topics can also be unified within this framework [1].

A concrete example is mirror symmetry, which links Gromov–Witten invariants with Hodge theory.

0.4 Course Plan

We will focus on three types of geometry and study them comparatively using synthetic methods:

- **Smooth Sets / Manifolds:** Built from open subsets of $\mathbb{R}^n$, these provide intuitive and geometric examples, serving as a bridge to classical geometry.
- **Simplicial Sets:** Constructed from simplices Δ, these offer the simplest and most abstract examples.
- **Schemes:** Built from spectra of commutative rings, these are the central objects of study in this course.

Chapter 1

Category

There are many references available for category theory; in this course, we follow [2]

1.1 Definition and Examples

Definition 1.1.1 (Category)**.** A **category** $\mathcal{C}$ consists of:

- A class of objects $\mathrm{Ob}(\mathcal{C})$.
- For each pair A, B, a set of morphisms $\mathrm{Hom}_{\mathcal{C}}(A, B)$.
- A composition operation $\circ$ of morphisms and identity morphisms id_A for each object A.

Subject to:

- **Associativity:** $h \circ (g \circ f) = (h \circ g) \circ f$
- **Identity:** $\mathrm{id}_B \circ f = f = f \circ \mathrm{id}_A$

For simplicity, we denote objects in a category $\mathcal{C}$ by $A \in \mathcal{C}$ and morphisms by $f : A \to B$. There are various ways to understand what a category is; let us explore this through examples:

Example 1.1.2 (Concrete Category)**.** A concrete category can be viewed as a collection of mathematical structures, where morphisms are maps that preserve those structures.

- **Set:** Objects are sets[1]; morphisms are functions between sets.
- **Grp:** Objects are groups; morphisms are group homomorphisms.
- **Vect:** Objects are vector spaces; morphisms are linear maps.
- **Top:** Objects are topological spaces; morphisms are continuous maps.

You can construct infinitely many examples from the mathematical structures you are familiar with.

At first glance, this abstraction may seem unnecessary. However, we can generalize familiar notions from set theory. For instance, consider a morphism $f : A \to B$ in a category $\mathcal{C}$:

- **Isomorphism:** f is an *isomorphism* if there exists a morphism $g : B \to A$ such that:
$$g \circ f = \mathrm{id}_A \quad \text{and} \quad f \circ g = \mathrm{id}_B (\text{i.e.} g = f^{-1})$$
In this case, A and B are said to be *isomorphic*.

- **Monomorphism:** f is a *monomorphism* (or *mono*) if for all morphisms $g_1, g_2 : X \to A$, we have:
$$f \circ g_1 = f \circ g_2 \Rightarrow g_1 = g_2$$
That is, f is left-cancellable.

- **Epimorphism:** f is an *epimorphism* (or *epi*) if for all morphisms $h_1, h_2 : B \to Y$, we have:
$$h_1 \circ f = h_2 \circ f \Rightarrow h_1 = h_2$$
That is, f is right-cancellable.

Exercise 1.1.1. Verify that in the category **Set**, these definitions correspond to bijections, injections, and surjections, respectively.

[1]Since the collection of all sets does not itself form a set, we refer instead to a *class* of objects. However, if we restrict our attention to *small* sets, then the collection of objects can be treated as a set. In this course, we will ignore set-theoretic subtleties and proceed informally.

Example 1.1.3 (Classifying Category)**.** A category can be viewed as an algebraic structure generalizing a monoid[2]. Indeed, for any object $A \in \mathcal{C}$, the set of endomorphisms $\mathrm{Hom}_{\mathcal{C}}(A, A)$ forms a monoid. Conversely, any monoid M can be associated with a **classifying category** BM, which has a single object $\bullet$ and morphisms $\mathrm{Hom}_{BM}(\bullet, \bullet) = M$. In particular, for any group G, we obtain a **classifying space** BG, where all morphisms are isomorphisms.

We refer to BG as a space because we can interpret its isomorphisms as paths in a certain topological space:

Example 1.1.4 (Groupoid)**.** A **groupoid** is a category in which every morphism is an isomorphism. Given a groupoid $\mathcal{X}$, we can construct its **geometric realization** $|\mathcal{X}|$ as follows:

- Take the objects $x \in \mathcal{X}$ as points.
- For each morphism $f : x \to y$, attach a segment from point x to point y.
- For each relation $f \circ g = h$, attach a triangle with edges labeled by f, g, and h.
- Continue this process for higher-dimensional cells ...

We will formalize this construction when we introduce simplicial sets.

Exercise 1.1.2. Show that the set of connected components $\pi_0(|\mathcal{X}|, x)$ corresponds bijectively to the isomorphism class of x in $\mathcal{X}$. Furthermore, observe that $\mathrm{Hom}_{\mathcal{X}}(x, x)$ is a group, and prove that the fundamental group $\pi_1(|\mathcal{X}|, x) \cong \mathrm{Hom}_{\mathcal{X}}(x, x)$.

Conversely, for a topological space S, we can define the **fundamental groupoid** $\Pi_1(S)$, where:

- Objects are points of S.
- Morphisms $\mathrm{Hom}_{\Pi_1(S)}(a, b)$ are homotopy classes of paths from a to b.

Exercise 1.1.3. Describe the composition law in $\Pi_1(S)$ and verify that it satisfies the axioms of a groupoid.

The examples above illustrate that morphisms in a category can carry rich structure. On the other hand, if morphisms are trivial (i.e., at most one between any two objects), we obtain a partially ordered set:

[2] An algebraic structure similar to a group, but without requiring inverses

Example 1.1.5 (Poset)**.** Let $(P, \leq)$ be a partially ordered set. We can regard P as a category as follows:

- **Objects:** Elements of P.
- **Morphisms:** For $x, y \in P$, there exists a unique morphism $f : x \to y$ if and only if $x \leq y$.
- **Composition:** If $x \leq y$ and $y \leq z$, then $x \leq z$, so the morphism $x \to z$ is the composition of $x \to y$ and $y \to z$.
- **Identity:** For each $x \in P$, the identity morphism $\mathrm{id}_x : x \to x$ corresponds to the reflexivity $x \leq x$.

This category is called *thin*, meaning there is at most one morphism between any two objects.

A frequently used example is the poset of open subsets of a topological space S, denoted $(\mathrm{Op}(S), \subseteq)$.

Note that a set A can be viewed as a category in two distinct ways: either as a trivial groupoid or as a trivial poset. And a category can be viewed as a combination of this two case[3].

$$\begin{tikzcd}
 & \textbf{Category} & \\
\textbf{Groupoid} \arrow[ur, hook] & & \textbf{Poset} \arrow[ul, hook'] \\
 & \textbf{Set} \arrow[ul, hook'] \arrow[ur, hook] &
\end{tikzcd}$$

Remark 1.1.6. In the definition of a category, if we allow the morphism set $\mathrm{Hom}_{\mathcal{C}}(A, B)$ to carry additional structure—such as an Abelian group, vector space, groupoid, topological space, or even another category—we obtain an **enriched category**. In fact, it is often more natural to think of $\mathrm{Hom}_{\mathcal{C}}(A, B)$ as the set of connected components of a space:

$$\mathrm{Hom}_{\mathcal{C}}(A, B) \cong \pi_0 \mathrm{Map}_{\mathcal{C}}(A, B).$$

[3]It is useful to think category as an oriented graph

This reflects a general principle of the **Univalence Foundation**: mathematical structures should be treated as spaces (or types) from the outset, and the classical set-theoretic version can be recovered by taking the set of connected components. We will explore how to identify and work with these underlying geometric structures later in the course.

1.2 Functor and Presheaf

1.2.1 Functor

To define a map between two categories, it is natural to require that such a map respects the structure of morphisms.

Definition 1.2.1 (Functor)**.** Let $\mathcal{C}$ and $\mathcal{D}$ be categories. A **functor** $F : \mathcal{C} \to \mathcal{D}$ consists of:

- A function that assigns to each object $A \in \mathcal{C}$ an object $F(A) \in \mathcal{D}$.
- A function that assigns to each morphism $f : A \to B$ in $\mathcal{C}$ a morphism $F(f) : F(A) \to F(B)$ in $\mathcal{D}$.

such that:

- **Identity preservation:** $F(\mathrm{id}_A) = \mathrm{id}_{F(A)}$.
- **Composition preservation:** For all $f : A \to B$ and $g : B \to C$ in $\mathcal{C}$,

$$F(g \circ f) = F(g) \circ F(f).$$

We denote the set (later, the category) of functors from $\mathcal{C}$ to $\mathcal{D}$ by $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$.

Example 1.2.2 (Functors Between Concrete Categories)**.** Functors between concrete categories respect the underlying structures:

- **Forgetful Functor:**
$$U : \mathbf{Grp} \to \mathbf{Set}$$
assigns to each group its underlying set, and to each group homomorphism the same function viewed as a map of sets. Similar forgetful functors exist for **Vect**, **Top**, etc.

- **Free Functor:**
$$\mathrm{Free} : \mathbf{Set} \to \mathbf{Grp}$$
assigns to each set the free group it generates, and to each function between sets the induced group homomorphism.

  Exercise 1.2.1. Verify that Free is indeed a functor. Then show:
$$\mathrm{Hom}_{\mathbf{Grp}}(\mathrm{Free}(A), G) \cong \mathrm{Hom}_{\mathbf{Set}}(A, U(G)).$$
How would you define Free for **Vect**?

- **Discrete Functor:**
$$\mathrm{Disc} : \mathbf{Set} \to \mathbf{Top}$$
assigns to each set the discrete topological space, and to each function the same map viewed as continuous.

- **Connected Components:**
$$\pi_0 : \mathbf{Top} \to \mathbf{Set}$$
assigns to each topological space its set of connected components.

  Exercise 1.2.2. Show:
$$\mathrm{Hom}_{\mathbf{Top}}(\mathrm{Disc}(A), S) \cong \mathrm{Hom}_{\mathbf{Set}}(A, U(S)), \quad \mathrm{Hom}_{\mathbf{Top}}(S, \mathrm{Disc}(A)) \cong \mathrm{Hom}_{\mathbf{Set}}(\pi_0(S), A).$$

Exercise 1.2.3. Let G, H be groups (or more generally, monoids). Show that functors $F : BG \to BH$ correspond bijectively to group (monoid) homomorphisms $f : G \to H$.

Exercise 1.2.4.

1. Let $\mathcal{X}, \mathcal{Y}$ be groupoids. Show that a functor $F : \mathcal{X} \to \mathcal{Y}$ induces a continuous map between their geometric realizations:
$$|F| : |\mathcal{X}| \to |\mathcal{Y}|.$$
Let **Grpd** be the category of groupoids with functors as morphisms. Show that geometric realization defines a functor:
$$|\cdot| : \mathbf{Grpd} \to \mathbf{Top}.$$

2. Similarly, show that the fundamental groupoid construction defines a functor:
$$\Pi_1 : \mathbf{Top} \to \mathbf{Grpd}.$$

3. * Show the adjunction:

$$\mathrm{Hom}_{\mathbf{Top}}(S, |\mathcal{X}|) \cong \mathrm{Hom}_{\mathbf{Grpd}}(\Pi_1(S), \mathcal{X}).$$

Exercise 1.2.5. Let P, Q be posets. Show that a functor $F : P \to Q$ is equivalent to an order-preserving function.

Example 1.2.3 (Ring–Space Correspondence)**.** Given a topological space X, its ring of real continuous functions $C(X)$ defines a contravariant functor. A map $p : X \to Y$ induces a pullback:

$$p^* : C(Y) \to C(X), \quad f \mapsto f \circ p.$$

To formalize this, we introduce the **opposite category** $\mathcal{C}^{\mathrm{op}}$, which has the same objects as $\mathcal{C}$ but reverses the direction of morphisms:

$$\mathrm{Hom}_{\mathcal{C}^{\mathrm{op}}}(A, B) = \mathrm{Hom}_{\mathcal{C}}(B, A).$$

Then,

$$C(-) : \mathbf{Top}^{\mathrm{op}} \to \mathbf{Ring}$$

is a functor.

Remark 1.2.4. A central theme in algebraic geometry is reversing this functor: constructing a space from a commutative ring. That is, defining a functor:

$$\mathrm{Spec} : \mathbf{Ring}^{\mathrm{op}} \to \mathbf{Top},$$

such that:

$$\mathrm{Hom}_{\mathbf{Ring}}(R, C(X)) \cong \mathrm{Hom}_{\mathbf{Top}}(X, \mathrm{Spec}(R)).$$

Exercise 1.2.6. * Show that the underlying set of $\mathrm{Spec}(R)$ is:

$$U(\mathrm{Spec}(R)) = \mathrm{Hom}_{\mathbf{Ring}}(R, \mathbb{R}).$$

What topology should be given to $\mathrm{Spec}(R)$?

In practice, we impose additional structure to make this correspondence well-behaved:

- Between C^*-algebras and locally compact Hausdorff spaces, via the Gelfand representation theorem—where the term “spectrum” originates.

- Between commutative rings and locally ringed spaces, which is foundational in algebraic geometry.

Example 1.2.5 (Hom Functor)**.** For a fixed object A in a category $\mathcal{C}$, the **Hom functor** is defined as:

$$h^A := \mathrm{Hom}_{\mathcal{C}}(A, -) : \mathcal{C} \to \mathbf{Set},$$

which assigns to each object B the set $\mathrm{Hom}_{\mathcal{C}}(A, B)$, and to each morphism $f : B \to C$ the function:

$$h^A(f) : \mathrm{Hom}_{\mathcal{C}}(A, B) \to \mathrm{Hom}_{\mathcal{C}}(A, C), \quad g \mapsto f \circ g.$$

Similarly, we define the contravariant version:

$$h_A := \mathrm{Hom}_{\mathcal{C}}(-, A) : \mathcal{C}^{\mathrm{op}} \to \mathbf{Set}.$$

Exercise 1.2.7. Show that:

- f is a monomorphism if and only if $h^A(f)$ is injective for all A.
- f is an epimorphism if and only if $h_A(f)$ is injective for all A.
- f is an isomorphism if and only if both $h^A(f)$ and $h_A(f)$ are bijective for all A.

Remark 1.2.6. We can think of objects in $\mathcal{C}$ as "test objects," and a functor as a way of encoding data about how these tests behave. For example, in a sigma model with target space M, let $\mathcal{C}$ be the category of space-times. For each space-time Σ, the collection of fields $\mathrm{Map}(\Sigma, M)$ defines such a functor. The fundamental question is: given such a functor, can we reconstruct the underlying "space"?

1.2.2 Presheaf

Now we introduce the central conception for "generalized space" of the course

Definition 1.2.7 (Presheaf)**.** Let $\mathcal{C}$ be a category. A **presheaf** on $\mathcal{C}$ is a functor:

$$F : \mathcal{C}^{\mathrm{op}} \to \mathbf{Set}$$

That is, F assigns:

- To each object U in $\mathcal{C}$, a set $F(U)$.

- To each morphism $f : V \to U$ in $\mathcal{C}$, a function:

$$F(f) : F(U) \to F(V)$$

such that:

- $F(\mathrm{id}_U) = \mathrm{id}_{F(U)}$
- $F(g \circ f) = F(f) \circ F(g)$ for composable morphisms f and g in $\mathcal{C}$

Remark 1.2.8. Let $\mathrm{PSh}(\mathcal{C}) = \mathrm{Fun}(\mathcal{C}, \mathbf{Set})$ be the collection of presheaf.

To understand why is notion of generalized space, let us compare to how do we define distribution as generalized function:

- We begin with some test function, Let $\mathcal{D}(\Omega)$ denotes the space of test functions, i.e., smooth functions with compact support in Ω, notice that we have a pairing
$$< -, - >: \mathcal{D}(\Omega) \times \mathcal{D}(\Omega) \to \mathbb{R}, (f, g) \mapsto \int_\Omega fg$$
By which we can send a test function $f \in \mathcal{D}(\Omega)$ be a functional $T_f : \mathcal{D}'(\Omega) := \mathcal{D}(\Omega) \to \mathbb{R}, \phi \mapsto \int_\Omega f\phi$. Then the we define the *distribution* as $\mathcal{D}'(\Omega)$, which are generalized functions.

- Now we begin with category of test spaces $\mathcal{C}$, we have the hom functor
$$\mathrm{Hom}_{\mathcal{C}}(-, -) : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathbf{Set}$$
For each object $A \in \mathcal{C}$ we can define presheaf $h_A : \mathcal{C}^{\mathrm{op}} \to \mathbf{Set}$, then we think presheaf as generalized spaces. Just like case of distribution, we have more presheaf than h_A.

- When we solve differntial equation, we first get a distribution solution, then we discuss the *regularity* of distribution, i.e. locally how closed is it to test functions, to conclude the solution is an actual function.

- When we solve some geometric problem, for example moduli problem, we need to construct a space M, we can first define a (pre)sheaf, then we can discuss the *repsentablity* of this (pre)sheaf, i.e. locally the sheaf is look like the test spaces, we can conclude we construct an actual space.

As we have seen, for any $A \in \mathcal{C}$, h_A is a presheaf, such presheaf we called *representable*. Just like there are non-smooth distribution, we have more presheaf than representable ones. Aside from h_A, now we give more concrete examples of presheaves.

Example 1.2.9 ((Pre)sheaf of function)**.** Let X be a topological space, for each open set $U \in \mathrm{Op}(X)$, the set of continuous function $C(U)$ define a presheaf $C \in \mathrm{PSh}(\mathrm{Op}(X))$: for $U \subseteq V$, we have the restriction $C(V) \to C(U)$. In general, for any other topological space Y we can have a presheaf $C(-, Y) \in \mathrm{PSh}(\mathrm{Op}(X))$. Similarly, you can define presheaf of all kinds of function: smooth, analytic, locally constant, etc.

Example 1.2.10 ((Pre)sheaf of section)**.** Let $E \to X$ be a vector bundle over a topological space X. Define a presheaf $\mathcal{F}$ by assigning to each open set $U \subseteq X$ the set of continuous (or smooth) sections of E over U:

$$\mathcal{F}(U) = \{s : U \to E \mid s \text{ is a section of } E \text{ over } U\}.$$

The following is one of the main example

Example 1.2.11 (Smooth Set)**.** Let category **Cartesian space Cart** have the objects $\mathbb{R}^n, n \in \mathbb{N}$, morphism $\mathrm{Hom}_{\mathbf{Cart}}(\mathbb{R}^n, \mathbb{R}^m)$ are smooth functions. We call (pre)sheaf of Cartesian space as smooth set.

- **Manifold.** For any smooth manifold M define a presheaf $M(\mathbb{R}^n) = \mathrm{Map}_{\mathrm{sm}}(\mathbb{R}^n, M)$ defined by smooth map. So all manifold are smooth set.
- **Differential forms.** Consider differential from $\Omega^k(\mathbb{R}^n)$ and its pullback along smooth functions, we have a smooth set Ω^k. Show that Ω^k is not a manifold.

Example 1.2.12 (Algebraic Set)**.** As we mention before, in algebraic geometry, we should think $\mathbf{Ring}^{\mathrm{op}}$ as "test spaces", Then presheaf $\mathcal{F} \in \mathrm{PSh}(\mathbf{Ring}^{\mathrm{op}}) = \mathrm{Fun}(\mathbf{Ring}, \mathbf{Set})$ is nothing but a functor from **Ring** to **Set**. Let us call such functor algebraic set.

- **Affine Variety** As we pointed out, we should study the geometry of zero set of polynomials. Let $P_i \in \mathbb{Z}[x_1, \ldots, x_n]$ for $i = 1, \ldots, m$ be some polynomails, now we define functor $V_P : \mathbf{Ring} \to \mathbf{Set}$

$$V_P(R) = \{(r_1, \ldots, r_n) \in R^n \mid \forall i, P_i(r_1, \ldots, r_n) = 0\}.$$

Exercise 1.2.8. Let $R_P = \mathbb{Z}[x_1, \ldots, x_n]/(P_1, \ldots, P_m)$, show that $V_P = h^{R_P}$, i.e. $V_P(R) = \mathrm{Hom}_{\mathbf{Ring}}(R_P, R)$.

- **Projective Space** Let us consider a non-representable alebraic set: let functor $\mathbb{P}^n : \mathbf{Ring} \to \mathbf{Set}$ to be

$$\mathbb{P}^n(R) = \{(r_0, \ldots, r_n) \in R^{n+1} \mid \exists (u_0, \ldots, u_n) \in R^{n+1} \text{ s.t. } \sum_{i=0}^{n} u_i r_i = 1\} / \sim$$

where $\sim$ is $[\forall a \in R^\times, (r_0, \ldots, r_n) \sim (ar_0, \ldots, ar_n)]$.

Exercise 1.2.9. Show that this defines a functor. And compare this to projective space in differential geometry when take $R = \mathbb{R}$.

As we indicated, $\mathrm{PSh}(\mathcal{C})$ should be a category, let us found out what morphism they should have: Since h_A, h_B is thought as generalized space, it should be inherent the morphism in $\mathcal{C}$. We can check that for $f : A \to B$, we have the map of set $f \circ - : h_A(C) \to h_B(C)$ for each C. Nevertheless, those map should be compactible for the chose of C. This lead to our definition:

Definition 1.2.13. A *morphism of presheaves* $\varphi : \mathcal{F} \to \mathcal{G}$ is a natural transformation between the functors $\mathcal{F}$ and $\mathcal{G}$; that is, for each object U in $\mathcal{C}$, there is a function $\varphi_U : \mathcal{F}(U) \to \mathcal{G}(U)$ such that for every morphism $f : V \to U$ in $\mathcal{C}$, the following diagram commutes:

$$\begin{tikzcd} \mathcal{F}(U) \arrow[r, "\varphi_U"] \arrow[d, "\mathcal{F}(f)"'] & \mathcal{G}(U) \arrow[d, "\mathcal{G}(f)"] \\ \mathcal{F}(V) \arrow[r, "\varphi_V"] & \mathcal{G}(V) \end{tikzcd}$$

Exercise 1.2.10. Show that a morphism $f : A \to B$ induce a morphism of presheaf $Y(f) : h_A \to h_B$. This means we have in fact a functor $Y : \mathcal{C} \to \mathrm{PSh}(\mathcal{C}), A \mapsto h_A$. This is called *Yoneda Embedding.*

Exercise 1.2.11. * Let M be a smooth manifold and seen as smooth set, Show that every differential form $\Omega^k(M)$ on M induce a morphism of smooth sets $\mathrm{Hom}_{sm}(M, \Omega^k)$. Then show that is a bijection $\Omega^k(M) \to \mathrm{Hom}_{sm}(M, \Omega^k)$.

1.3 Yoneda Lemma

1.4 Universal Property and (Co)limit

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