Seeing the Mountain

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Seeing the mountain as the mountain;
Seeing the mountain as not a mountain;
Seeing the mountain as still a mountain.

QINGYUAN XINGSI

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Chapter 0

Introduction

0.1 What is Algebraic Geometry?

Algebraic geometry is the study of geometric spaces that locally arise as solutions to polynomial equations. This study can be approached at two levels:

- **Local:** This involves studying the geometric properties of solution sets to polynomial equations. Since we work not only over $\mathbb{R}$ or $\mathbb{C}$ but potentially over arbitrary fields or rings, we must construct a more intrinsic notion of geometry associated with algebra—namely, the spectrum of a ring. This leads to the study of affine varieties and affine schemes, allowing a translation between algebra and geometry.

- **Synthetic:** Analogous to how manifolds generalize open subsets of $\mathbb{R}^n$, we study spaces that are locally isomorphic to affine varieties. This broader perspective leads to the study of varieties and schemes through various formal frameworks.

This lecture is part of an algebraic geometry course that emphasizes the second, synthetic level. However, the underlying question—"How can we construct and classify generalized spaces from certain building blocks?"—is relevant across many branches of geometry. Thus, this part can stand alone as a form of *post-modern*[1] synthetic geometry.

[1] A commonly accepted definition is "after World War II."

0.2 Essentialism vs. Structuralism

Synthetic geometry formalizes geometric concepts through axioms that directly address fundamental entities—such as points and lines—rather than relying on a background space like Cartesian coordinates. In contrast to the analytic viewpoint, where every geometric object is composed of points, synthetic geometry treats lines, curves, and other entities as primitive. The focus shifts to the **relationships** between these objects, such as “a point lies on a line” or “two lines intersect.”

This structuralism perspective emphasizes understanding objects through their interactions. To formalize these relationships, we use **category theory** and **sheaf theory**. Moreover, by viewing categories themselves as spaces, we uncover deeper geometric structures. This formalism, developed through algebraic geometry, now plays a central role in various fields: differential geometry, topology, quantum field theory, and beyond.

0.3 What is Geometry in Physics?

Quantum physics connects to both the local and synthetic levels of algebraic geometry:

- **Local:** The concept of the spectrum of a commutative ring originates from C^*-algebras and operator theory, both fundamental in quantum physics. The duality between algebra and geometry is already present in the Heisenberg and Schrödinger formulations.

- **Synthetic:** In quantum field theory, we study the space of field configurations (histories), which is infinite-dimensional and behaves irregularly. Nevertheless, we aim to treat it as a “manifold” to define metrics, path integrals, and differential forms. Synthetic geometry provides a rigorous framework for these constructions (e.g., via smooth sets). Gauge theory, supergeometry, and related topics can also be unified within this framework [2].

A concrete example is mirror symmetry, which links Gromov–Witten invariants with Hodge theory.

0.4 Course Plan

We will focus on three types of geometry and study them comparatively using synthetic methods:

- **Smooth Sets / Manifolds:** Built from open subsets of $\mathbb{R}^n$, these provide intuitive and geometric examples, serving as a bridge to classical geometry.
- **Simplicial Sets / Kan Complexes:** Constructed from simplices Δ, these offer the simplest abstract examples.
- **Algebraic Sets / Schemes:** Built from spectra of commutative rings, these are the central objects of study in this course.

Chapter 1

Category

There are many references available for category theory; in this course, we follow [3]

1.1 Definition and Examples

Definition 1.1.1 (Category)**.** A **category** $\mathcal{C}$ consists of:

- A class of objects $\mathrm{Ob}(\mathcal{C})$.
- For each pair A, B, a set of morphisms $\mathrm{Hom}_{\mathcal{C}}(A, B)$.
- A composition operation $\circ$ of morphisms and identity morphisms id_A for each object A.

Subject to:

- **Associativity:** $h \circ (g \circ f) = (h \circ g) \circ f$
- **Identity:** $\mathrm{id}_B \circ f = f = f \circ \mathrm{id}_A$

For simplicity, we denote objects in a category $\mathcal{C}$ by $A \in \mathcal{C}$ and morphisms by $f : A \to B$. There are various ways to understand what a category is; let us explore this through examples:

Example 1.1.2 (Concrete Category)**.** A concrete category can be viewed as a collection of mathematical structures, where morphisms are maps that preserve those structures.

- **Set:** Objects are sets[1]; morphisms are functions between sets.
- **Grp:** Objects are groups; morphisms are group homomorphisms.
- **Vect:** Objects are vector spaces; morphisms are linear maps.
- **Top:** Objects are topological spaces; morphisms are continuous maps.

You can construct infinitely many examples from the mathematical structures you are familiar with.

At first glance, this abstraction may seem unnecessary. However, we can generalize familiar notions from set theory. For instance, consider a morphism $f : A \to B$ in a category $\mathcal{C}$:

- **Isomorphism:** f is an *isomorphism* if there exists a morphism $g : B \to A$ such that:
$$g \circ f = \mathrm{id}_A \quad \text{and} \quad f \circ g = \mathrm{id}_B (\text{i.e.} g = f^{-1})$$
In this case, A and B are said to be *isomorphic*.
- **Monomorphism:** f is a *monomorphism* (or *mono*) if for all morphisms $g_1, g_2 : X \to A$, we have:
$$f \circ g_1 = f \circ g_2 \Rightarrow g_1 = g_2$$
That is, f is left-cancellable.
- **Epimorphism:** f is an *epimorphism* (or *epi*) if for all morphisms $h_1, h_2 : B \to Y$, we have:
$$h_1 \circ f = h_2 \circ f \Rightarrow h_1 = h_2$$
That is, f is right-cancellable.

Exercise 1.1.1. Verify that in the category **Set**, these definitions correspond to bijections, injections, and surjections, respectively.

Besides the concrete categories, in which object have its own meaning, we can have abstract categories than objects are meaningless without in the context of category.

[1] Since the collection of all sets does not itself form a set, we refer instead to a *class* of objects. However, if we restrict our attention to *small* sets, then the collection of objects can be treated as a set. In this course, we will ignore set-theoretic subtleties and proceed informally.

Example 1.1.3 (Classifying Category)**.** A category can be viewed as an algebraic structure generalizing a monoid[2]. Indeed, for any object $A \in \mathcal{C}$, the set of endomorphisms $\mathrm{Hom}_{\mathcal{C}}(A, A)$ forms a monoid. Conversely, any monoid M can be associated with a **classifying category** BM, which has a single object $\bullet$ and morphisms $\mathrm{Hom}_{BM}(\bullet, \bullet) = M$. In particular, for any group G, we obtain a **classifying space** BG, where all morphisms are isomorphisms.

We refer to BG as a space because we can interpret its isomorphisms as paths in a certain topological space:

Example 1.1.4 (Groupoid)**.** A **groupoid** is a category in which every morphism is an isomorphism. Given a groupoid $\mathcal{X}$, we can construct its **geometric realization** $|\mathcal{X}|$ as follows:

- Take the objects $x \in \mathcal{X}$ as points.
- For each morphism $f : x \to y$, attach a segment from point x to point y.
- For each relation $f \circ g = h$, attach a triangle with edges labeled by f, g, and h.
- Continue this process for higher-dimensional cells ...

We will formalize this construction when we introduce simplicial sets.

Exercise 1.1.2. Show that the set of connected components $\pi_0(|\mathcal{X}|, x)$ corresponds bijectively to the isomorphism class of x in $\mathcal{X}$. Furthermore, observe that $\mathrm{Hom}_{\mathcal{X}}(x, x)$ is a group, and prove that the fundamental group $\pi_1(|\mathcal{X}|, x) \cong \mathrm{Hom}_{\mathcal{X}}(x, x)$.

Conversely, for a topological space S, we can define the **fundamental groupoid** $\Pi_1(S)$, where:

- Objects are points of S.
- Morphisms $\mathrm{Hom}_{\Pi_1(S)}(a, b)$ are homotopy classes of paths from a to b.

Exercise 1.1.3. Describe the composition law in $\Pi_1(S)$ and verify that it satisfies the axioms of a groupoid.

The examples above illustrate that morphisms in a category can carry rich structure. On the other hand, if morphisms are trivial (i.e., at most one between any two objects), we obtain a partially ordered set:

[2] An algebraic structure similar to a group, but without requiring inverses

Example 1.1.5 (Poset)**.** Let $(P, \leq)$ be a partially ordered set. We can regard P as a category as follows:

- **Objects:** Elements of P.
- **Morphisms:** For $x, y \in P$, there exists a unique morphism $f : x \to y$ if and only if $x \leq y$.
- **Composition:** If $x \leq y$ and $y \leq z$, then $x \leq z$, so the morphism $x \to z$ is the composition of $x \to y$ and $y \to z$.
- **Identity:** For each $x \in P$, the identity morphism $\mathrm{id}_x : x \to x$ corresponds to the reflexivity $x \leq x$.

This category is called *thin*, meaning there is at most one morphism between any two objects.

A frequently used example is the poset of open subsets of a topological space S, denoted $(\mathrm{Op}(S), \subseteq)$.

Note that a set A can be viewed as a category in two distinct ways: either as a trivial groupoid or as a trivial poset. And a category can be viewed as a combination of this two case[3].

$$
\begin{tikzcd}
\mathbf{Monoid} \arrow[r, hook] & \mathbf{Category} & \mathbf{Poset} \arrow[l, hook'] \\
\mathbf{Group} \arrow[u, hook] \arrow[r, hook] \arrow[ur, phantom, very near start, "\urcorner"] & \mathbf{Groupoid} \arrow[u, hook] & \mathbf{Set} \arrow[u, hook] \arrow[l, hook'] \arrow[ul, phantom, very near start, "\ulcorner"]
\end{tikzcd}
$$

> *Remark* 1.1.6. In the definition of a category, if we allow the morphism set $\mathrm{Hom}_{\mathcal{C}}(A, B)$ to carry additional structure—such as an Abelian group, vector space, groupoid, topological space, or even another category—we obtain an **enriched category**. In fact, it is often more natural to think of $\mathrm{Hom}_{\mathcal{C}}(A, B)$ as the set of connected components of a space:
>
> $$\mathrm{Hom}_{\mathcal{C}}(A, B) \cong \pi_0 \mathrm{Map}_{\mathcal{C}}(A, B).$$
>
> This reflects a general principle of the **Univalence Foundation**: mathematical structures should be treated as spaces (or types) from the outset, and the classical set-theoretic version can be recovered by taking the set of connected components. We will explore how to identify and work with these underlying

[3]It is useful to think category as an oriented graph

geometric structures later in the course.

Give an object X in a category, we can also consider the category of relative objects. In some sense those are "families parameterized by X".

Definition 1.1.7 (Slice Category)**.** $\mathcal{C}$ be a category and let X be an object of $\mathcal{C}$. The *slice category* $\mathcal{C}/X$ is defined as follows:

- **Objects:** Morphisms $f : A \to X$ in $\mathcal{C}$.
- **Morphisms:** A morphism from $f : A \to X$ to $g : B \to X$ is a morphism $h : A \to B$ in $\mathcal{C}$ such that $g \circ h = f$.

Dually, the *coslice category* $X/\mathcal{C}$ has objects morphisms $f : X \to A$ and morphisms $h : A \to B$ satisfying $h \circ f = g$.

Exercise 1.1.4. Let $X \in \mathbf{Set}$, then $\mathbf{Set}/X$ is isomorphic to the category of X indexed sets: objects are families of set $\{S_x\}_{x\in X}$, morphisms are families of maps $\{f : S_x \to T_x\}_{x\in X}$.

1.2 Functor

To define a map between two categories, it is natural to require that such a map respects the structure of morphisms.

Definition 1.2.1 (Functor)**.** Let $\mathcal{C}$ and $\mathcal{D}$ be categories. A **functor** $F : \mathcal{C} \to \mathcal{D}$ consists of:

- A function that assigns to each object $A \in \mathcal{C}$ an object $F(A) \in \mathcal{D}$.
- A function that assigns to each morphism $f : A \to B$ in $\mathcal{C}$ a morphism $F(f) : F(A) \to F(B)$ in $\mathcal{D}$.

such that:

- **Identity preservation:** $F(\mathrm{id}_A) = \mathrm{id}_{F(A)}$.

- **Composition preservation:** For all $f : A \to B$ and $g : B \to C$ in $\mathcal{C}$,

$$F(g \circ f) = F(g) \circ F(f).$$

We denote the set (later, the category) of functors from $\mathcal{C}$ to $\mathcal{D}$ by $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$.

Example 1.2.2 (Functors Between Concrete Categories)**.** Functors between concrete categories respect the underlying structures:

- **Forgetful Functor:**

$$U : \mathbf{Grp} \to \mathbf{Set}$$

assigns to each group its underlying set, and to each group homomorphism the same function viewed as a map of sets. Similar forgetful functors exist for **Vect**, **Top**, etc.

- **Free Functor:**

$$\mathrm{Free} : \mathbf{Set} \to \mathbf{Grp}$$

assigns to each set the free group it generates, and to each function between sets the induced group homomorphism.

Exercise 1.2.1. Verify that Free is indeed a functor. Then show:

$$\mathrm{Hom}_{\mathbf{Grp}}(\mathrm{Free}(A), G) \cong \mathrm{Hom}_{\mathbf{Set}}(A, U(G)).$$

How would you define Free for **Vect**?

- **Discrete Functor:**

$$\mathrm{Disc} : \mathbf{Set} \to \mathbf{Top}$$

assigns to each set the discrete topological space, and to each function the same map viewed as continuous.

- **Connected Components:**

$$\pi_0 : \mathbf{Top} \to \mathbf{Set}$$

assigns to each topological space its set of connected components.

Exercise 1.2.2. Show:

$$\mathrm{Hom}_{\mathbf{Top}}(\mathrm{Disc}(A), S) \cong \mathrm{Hom}_{\mathbf{Set}}(A, U(S)), \quad \mathrm{Hom}_{\mathbf{Top}}(S, \mathrm{Disc}(A)) \cong \mathrm{Hom}_{\mathbf{Set}}(\pi_0(S), A).$$

Exercise 1.2.3. 1. Let G, H be groups (or more generally, monoids). Show that functors $F : BG \to BH$ correspond bijectively to group (monoid) homomorphisms $f : G \to H$.

2. Show that a functor $F : BG \to \mathbf{Vect}$ correspond bijectively to representation of G.

Exercise 1.2.4. 1. Let $\mathcal{X}, \mathcal{Y}$ be groupoids. Show that a functor $F : \mathcal{X} \to \mathcal{Y}$ induces a continuous map between their geometric realizations:

$$|F| : |\mathcal{X}| \to |\mathcal{Y}|.$$

Let **Grpd** be the category of groupoids with functors as morphisms. Show that geometric realization defines a functor:

$$|\cdot| : \mathbf{Grpd} \to \mathbf{Top}.$$

2. Similarly, show that the fundamental groupoid construction defines a functor:

$$\Pi_1 : \mathbf{Top} \to \mathbf{Grpd}.$$

3. * Show the adjunction:

$$\mathrm{Hom}_{\mathbf{Top}}(S, |\mathcal{X}|) \cong \mathrm{Hom}_{\mathbf{Grpd}}(\Pi_1(S), \mathcal{X}).$$

Exercise 1.2.5. Let P, Q be posets. Show that a functor $F : P \to Q$ is equivalent to an order-preserving function.

Example 1.2.3 (Ring–Space Correspondence)**.** Given a topological space X, its ring of real continuous functions $C(X)$ defines a contravariant functor. A map $p : X \to Y$ induces a pullback:

$$p^* : C(Y) \to C(X), \quad f \mapsto f \circ p.$$

To formalize this, we introduce the **opposite category** $\mathcal{C}^{\mathrm{op}}$, which has the same objects as $\mathcal{C}$ but reverses the direction of morphisms:

$$\mathrm{Hom}_{\mathcal{C}^{\mathrm{op}}}(A, B) = \mathrm{Hom}_{\mathcal{C}}(B, A).$$

Then,

$$C(-) : \mathbf{Top}^{\mathrm{op}} \to \mathbf{Ring}$$

is a functor.

Remark 1.2.4. A central theme in algebraic geometry is reversing this functor: constructing a space from a commutative ring. That is, defining a functor:

$$\mathrm{Spec} : \mathbf{Ring}^{\mathrm{op}} \to \mathbf{Top},$$

such that we have *adjunction*:

$$\mathrm{Hom}_{\mathbf{Ring}}(R, C(X)) \cong \mathrm{Hom}_{\mathbf{Top}}(X, \mathrm{Spec}(R)).$$

Exercise 1.2.6. * Show that the underlying set of $\mathrm{Spec}(R)$ is:

$$U(\mathrm{Spec}(R)) = \mathrm{Hom}_{\mathbf{Ring}}(R, \mathbb{R}).$$

What topology should be given to $\mathrm{Spec}(R)$?

In practice, we impose additional structure to make this correspondence well-behaved:

- Between C^*-algebras and locally compact Hausdorff spaces, via the Gelfand representation theorem—where the term "spectrum" originates.
- Between commutative rings and locally ringed spaces, which is foundational in algebraic geometry.

Remark 1.2.5 (Adjoint Functors). As we have already seen in several examples, functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ are called adjoint functors or an *adjunction*[a] if there is a natural bijection for all $A \in \mathcal{C}$, $B \in \mathcal{D}$:

$$\mathrm{Hom}_{\mathcal{D}}(F(A), B) \cong \mathrm{Hom}_{\mathcal{C}}(A, G(B)).$$

As a simple example, let X, Y be two sets viewed as categories. For a map $f : X \to Y$, the adjoint $g : Y \to X$ is just the inverse of f, since for $x \in X$, $y \in Y$:

$$\mathrm{Hom}_Y(f(x), y) \cong \mathrm{Hom}_X(x, g(y)) \quad \Leftrightarrow \quad (f(x) = y \Leftrightarrow x = g(y)).$$

So, adjoint functors can be thought of as a generalization or weakening of the notion of inverse.

[a] We also say F is the left adjoint of G, and G is the right adjoint of F.

Example 1.2.6 (Hom Functor)**.** For a fixed object A in a category $\mathcal{C}$, the **Hom functor** is defined as:

$$h^A := \mathrm{Hom}_{\mathcal{C}}(A, -) : \mathcal{C} \to \mathbf{Set},$$

which assigns to each object B the set $\mathrm{Hom}_{\mathcal{C}}(A, B)$, and to each morphism $f : B \to C$ the function:

$$h^A(f) : \mathrm{Hom}_{\mathcal{C}}(A, B) \to \mathrm{Hom}_{\mathcal{C}}(A, C), \quad g \mapsto f \circ g.$$

Similarly, we define the contravariant version:

$$h_A := \mathrm{Hom}_{\mathcal{C}}(-, A) : \mathcal{C}^{\mathrm{op}} \to \mathbf{Set}.$$

Exercise 1.2.7. Show that:

- f is a monomorphism if and only if $h^A(f)$ is injective for all A.
- f is an epimorphism if and only if $h_A(f)$ is injective for all A.
- f is an isomorphism if and only if both $h^A(f)$ and $h_A(f)$ are bijective for all A.

Remark 1.2.7. We can think of objects in $\mathcal{C}$ as "test objects," and a functor as a way of encoding data about how these tests behave. For example, in a sigma model with target space M, let $\mathcal{C}$ be the category of space-times. For each space-time Σ, the collection of fields $\mathrm{Map}(\Sigma, M)$ defines such a functor. The fundamental question is: given such a functor, can we reconstruct the underlying "space"?

1.3 Presheaf

We now introduce a central concept for understanding "generalized spaces" in this course.

Definition 1.3.1 (Presheaf)**.** Let $\mathcal{C}$ be a category. A **presheaf** on $\mathcal{C}$ is a functor:

$$F : \mathcal{C}^{\mathrm{op}} \to \mathbf{Set}$$

That is, F assigns:

- To each object U in $\mathcal{C}$, a set $F(U)$.
- To each morphism $f : V \to U$ in $\mathcal{C}$, a function:

$$F(f) : F(U) \to F(V)$$

such that:

- $F(\mathrm{id}_U) = \mathrm{id}_{F(U)}$
- $F(g \circ f) = F(f) \circ F(g)$ for composable morphisms f and g in $\mathcal{C}$

Let $\mathrm{PSh}(\mathcal{C}) = \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathbf{Set})$ denote the category of presheaves on $\mathcal{C}$.

Remark 1.3.2. To understand why presheaves represent generalized spaces, consider the analogy with distributions as generalized functions:

- We begin with a space of test functions. Let $\mathcal{D}(\Omega)$ denote the space of smooth functions with compact support in Ω. There is a natural pairing:

  $$\langle f, g \rangle := \int_\Omega fg$$

  This allows us to associate to each test function $f \in \mathcal{D}(\Omega)$ a continuous functional $T_f \in \mathcal{D}'(\Omega)$ defined by:

  $$T_f(\phi) := \int_\Omega f\phi.$$

  The space $\mathcal{D}'(\Omega)$ of distributions is thus a space of generalized functions.

- Similarly, starting from a category of test spaces $\mathcal{C}$, we consider the Hom functor:

  $$\mathrm{Hom}_{\mathcal{C}}(-, -) : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathbf{Set}.$$

  For each object $A \in \mathcal{C}$, the presheaf $h_A := \mathrm{Hom}_{\mathcal{C}}(-, A)$ is *representable*. Presheaves generalize these representable ones, just as distributions generalize smooth functions.

- In solving differential equations, we often first obtain a distributional solution, and then study its *regularity*—how closely it resembles a smooth function locally—allowing us to conclude that the distribution is an actual function.

- In geometry, for example in moduli problems, we may first define a (pre)sheaf[a] and then study its *representability*—whether it locally resembles a test space—allowing us to conclude that we have constructed an actual space.

[a]sheaf to presheaf, is like continuous functional to functional, we ask some continuous condition on presheaf

As we've seen, for any $A \in \mathcal{C}$, the presheaf h_A is called representable. Just as there are distributions that are not smooth functions, there are presheaves that are not representable. Here are some concrete examples:

Example 1.3.3 (Presheaf of Functions)**.** Let X be a topological space. For each open set $U \in \mathrm{Op}(X)$, the set of continuous functions $C(U)$ defines a presheaf $C \in \mathrm{PSh}(\mathrm{Op}(X))$. For $U \subseteq V$, we have a restriction map $C(V) \to C(U)$. More generally, for any topological space Y, we can define a presheaf $C(-, Y) \in \mathrm{PSh}(\mathrm{Op}(X))$. Similarly, one can define presheaves of smooth, analytic, or locally constant functions.

Example 1.3.4 (Presheaf of Sections)**.** Let $E \to X$ be a vector bundle over a topological space X. Define the presheaf of sections Γ_E by assigning to each open set $U \subseteq X$ the set of continuous (or smooth) sections of E over U:

$$\Gamma_E(U) = \{s : U \to E \mid s \text{ is a section of } E \text{ over } U\}.$$

Example 1.3.5 (Smooth Set)**.** Let **Cart** be the category of Cartesian spaces, with objects $\mathbb{R}^n$ and morphisms given by smooth maps. A presheaf on **Cart** is called a **smooth set**.

- **Manifolds:** For any smooth manifold M, define a presheaf $M(\mathbb{R}^n) := \{f : \mathbb{R}^n \to M \mid f \text{ is smooth.}\}$. Thus, every manifold defines a smooth set.

- **Mapping Spaces:** For any smooth manifolds M, N, define a presheaf

$$\mathbf{C}^\infty(M, N)(\mathbb{R}^n) := \{f : M \times \mathbb{R}^n \to N \mid f \text{ is smooth.}\}.$$

  And we have $\mathrm{Hom}_{\mathrm{PSh}(\mathbf{Cart})}(L, \mathbf{C}^\infty(M, N)) \cong \mathrm{Hom}_{\mathrm{PSh}(\mathbf{Cart})}(L \times M, N)$.

- **Differential Forms:** Consider the presheaf Ω^k assigning to each $\mathbb{R}^n$ the space of k-forms $\Omega^k(\mathbb{R}^n)$, with pullbacks along smooth maps. Show that Ω^k is not representable by a manifold.

Example 1.3.6 (Simplicial Set)**.** Let Δ be the category of simplices, whose objects are the n-simplices $[n]$, defined as the subset of $\mathbb{R}^{n+1}$:

$$[n] = \left\{ (t_0, t_1, \dots, t_n) \in \mathbb{R}^{n+1} \;\middle|\; t_i \geq 0,\ \sum_{i=0}^{n} t_i = 1 \right\},$$

with each vertex $(t_i = 1, t_{j \neq i} = 0)$ marked by i. The morphisms in Δ are linear maps that send vertices to vertices while preserving the order of the markings. For example, we have:

- Face maps $\delta_i : [n-1] \to [n]$, which are injections corresponding to the face given by $t_i = 0$.
- Degeneracy maps $\sigma_i : [n+1] \to [n]$, which are surjections corresponding to projections of simplices,

for $0 \leq i \leq n$.

Exercise 1.3.1. Show that Δ is equivalent to the category of finite totally ordered sets, whose objects are $[n] = \{0 < 1 < \cdots < n\}$, and whose morphisms are order-preserving maps.

A presheaf on Δ is called a **simplicial set**. More explicitly, a simplicial set X is a sequence of sets $\{X_n = X([n])\}_{n \geq 0}$ together with:

- Face maps $d_i = X(\delta_i) : X_n \to X_{n-1}$,
- Degeneracy maps $s_i = X(\sigma_i) : X_n \to X_{n+1}$,

for $0 \leq i \leq n$.

Intuitively, we can think of X_n as encoding the data of a triangulated space: X_0 consists of points, X_1 of segments, X_2 of triangles, and so on. The face maps d_i describe how these simplices are glued together via their boundaries.

- Let Δ^n be the underlying topological space of $[n]$. For any topological space X, we can define a simplicial set $\mathrm{Sing}(X)$ where:

$$\mathrm{Sing}(X)_n := \mathrm{Hom}_{\mathbf{Top}}(\Delta^n, X).$$

- For any small category $\mathcal{C}$, we can define the *nerve* simplicial set $N(\mathcal{C})$, where:

$$N(\mathcal{C})_n := \mathrm{Fun}([n], \mathcal{C}) = \{C_0 \xrightarrow{f_1} C_1 \xrightarrow{f_2} \cdots \xrightarrow{f_n} C_n\},$$

  viewing $[n]$ as a category via its poset structure.

  More concretely we have:

  - $N(\mathcal{C})_0 = \{C_0\} = \mathrm{Ob}(\mathcal{C})$,
  - $N(\mathcal{C})_1 = \{C_0 \xrightarrow{f_1} C_1\} = \mathrm{Mor}(\mathcal{C})$, the face maps $d_0(C_0 \xrightarrow{f_1} C_1) = C_0, d_1(C_0 \xrightarrow{f_1} C_1) = C_1$, the degeneracy map $s_0(C_0) = [C_0 \xrightarrow{\mathrm{id}} C_0]$,
  - $N(\mathcal{C})_1 = \{C_0 \xrightarrow{f_1} C_1 \xrightarrow{f_2} C_2\}$, the face maps $d_0(C_0 \xrightarrow{f_1} C_1 \xrightarrow{f_2} C_2) = [C_0 \xrightarrow{f_1} C_1], d_1(C_0 \xrightarrow{f_1} C_1 \xrightarrow{f_2} C_2) = [C_1 \xrightarrow{f_2} C_2], d_2(C_0 \xrightarrow{f_1} C_1 \xrightarrow{f_2} C_2) = [C_0 \xrightarrow{f_2 \circ f_1} C_2]$, the degeneracy map $s_0(C_0 \xrightarrow{f_1} C_1) = [C_0 \xrightarrow{\mathrm{id}} C_0 \xrightarrow{f_1} C_1], s_1(C_0 \xrightarrow{f_1} C_1) = [C_0 \xrightarrow{f_1} C_1 \xrightarrow{\mathrm{id}} C_1]$,

- In fact, $\mathrm{Sing} : \mathbf{Top} \to \mathrm{PSh}(\Delta)$ is a functor. We aim to define its adjoint functor $|\cdot| : \mathrm{PSh}(\Delta) \to \mathbf{Top}$, called the *geometric realization*, such that:

$$\mathrm{Hom}_{\mathbf{Top}}(|S|, X) \cong \mathrm{Hom}_{\mathrm{PSh}(\Delta)}(S, \mathrm{Sing}(X)).$$

  Clearly, we have $|h_{[n]}| = \Delta^n$, and the intuition is that we glue together topological simplices according to the data encoded in the simplicial set. A formal definition will be given after we introduce colimits.

Example 1.3.7 (Algebraic Set)**.** In algebraic geometry, we treat $\mathbf{Ring}^{\mathrm{op}}$ as the category of test spaces. A presheaf $\mathcal{F} \in \mathrm{PSh}(\mathbf{Ring}^{\mathrm{op}}) = \mathrm{Fun}(\mathbf{Ring}, \mathbf{Set})$ is then a functor from rings to sets, which we call an **algebraic set**.

- **Affine Variety:** We begin with the geometry of zero set of polynomials. Let $P_1, \ldots, P_m \in \mathbb{Z}[x_1, \ldots, x_n]$ be polynomials. Define a functor $V_P : \mathbf{Ring} \to \mathbf{Set}$ by:

$$V_P(R) = \{(r_1, \ldots, r_n) \in R^n \mid P_i(r_1, \ldots, r_n) = 0 \text{ for all } i\}.$$

  Exercise 1.3.2. Let $R_P = \mathbb{Z}[x_1, \ldots, x_n]/(P_1, \ldots, P_m)$. Show that $V_P = h^{R_P}$, i.e., $V_P(R) = \mathrm{Hom}_{\mathbf{Ring}}(R_P, R)$.

- **Projective Space:** Then we consider a non-representable example. Define a functor $\mathbb{P}^n : \mathbf{Ring} \to \mathbf{Set}$ by:

$$\mathbb{P}^n(R) = \{(r_0, \dots, r_n) \in R^{n+1} \mid \exists (u_0, \dots, u_n) \in R^{n+1} \text{ s.t. } \sum u_i r_i = 1\}/\sim,$$

where $\sim$ identifies tuples under scalar multiplication by units in R.

Exercise 1.3.3. Show that this defines a functor. Compare this with the definition of projective space in differential geometry when $R = \mathbb{R}$.

As we indicated earlier, $\mathrm{PSh}(\mathcal{C})$ should itself form a category. Let us now determine what the morphisms between presheaves ought to be.

Since representable presheaves h_A and h_B are thought of as generalized spaces, it is natural to expect that morphisms between them should reflect the morphisms in the original category $\mathcal{C}$. Indeed, given a morphism $f : A \to B$ in $\mathcal{C}$, we can define a map of sets:

$$f \circ - : h_A(C) \to h_B(C), \quad \text{for each } C \in \mathcal{C},$$

However, for this to define a morphism of presheaves, these maps must be compatible with the structure of the presheaves—i.e., they must commute with the restriction maps for all choices of C and morphisms between them.

This leads us naturally to the definition of morphisms between presheaves as *natural transformations*, which ensure such compatibility across the entire category.

Definition 1.3.8 (Morphisms of Presheaves)**.** A *morphism of presheaves* $\varphi : \mathcal{F} \to \mathcal{G}$ is a natural transformation between the functors $\mathcal{F}$ and $\mathcal{G}$. That is, for each object U in $\mathcal{C}$, there is a function $\varphi_U : \mathcal{F}(U) \to \mathcal{G}(U)$ such that for every morphism $f : V \to U$ in $\mathcal{C}$, the following diagram commutes:

$$\begin{tikzcd}
\mathcal{F}(U) \arrow[r, "\varphi_U"] \arrow[d, "\mathcal{F}(f)"'] & \mathcal{G}(U) \arrow[d, "\mathcal{G}(f)"] \\
\mathcal{F}(V) \arrow[r, "\varphi_V"] & \mathcal{G}(V)
\end{tikzcd}$$

Exercise 1.3.4. 1. Show that a morphism $f : A \to B$ in $\mathcal{C}$ induces a morphism of presheaves $よ(f) : h_A \to h_B$. This defines a functor:

$$よ : \mathcal{C} \to \mathrm{PSh}(\mathcal{C}), \quad A \mapsto h_A,$$

called the *Yoneda embedding.*

2. * (Yoneda Lemma) Show that, for $A \in \mathcal{C}$ and $F \in \mathrm{PSh}(\mathcal{C})$ there is a bijection:

$$F(A) \cong \mathrm{Hom}_{\mathrm{PSh}(\mathcal{C})}(h_A, F).$$

As a corollary, for any $A, B \in \mathcal{C}$ there is a bijection of morphisms:

$$よ(-) : \mathrm{Hom}_{\mathcal{C}}(A, B) \cong \mathrm{Hom}_{\mathrm{PSh}(\mathcal{C})}(h_A, h_B).$$

This lemma is extremely important. It justifies embedding the category $\mathcal{C}$ into the category of presheaves $\mathrm{PSh}(\mathcal{C})$ in a way that fully preserves its morphism structure. In other words, $\mathcal{C}$ can be faithfully represented inside $\mathrm{PSh}(\mathcal{C})$ via the Yoneda embedding.

Exercise 1.3.5.

1. Let M, N be smooth manifolds viewed as smooth sets. Show that a smooth map of manifold $f : M \to N$ induces a morphism of presheaves.

2. Show that every differential form $\omega \in \Omega^k(M)$ induces a morphism of smooth sets:

$$\omega : M \to \Omega^k.$$

3. * Then prove that this defines a bijection:

$$\Omega^k(M) \cong \mathrm{Hom}_{\mathrm{sm}}(M, \Omega^k).$$

Exercise 1.3.6 (Smooth Algebras)**.** Let $C^\infty\mathbf{Alg}$ be the category of product-preserving functors $A : \mathbf{Cart} \to \mathbf{Set}$.

1. Show that $A(\mathbb{R})$ has a ring structure.

2. Show that for any smooth manifold M, we can define a smooth algebra:

$$C^\infty(M) : \mathbb{R}^i \mapsto C^\infty(M, \mathbb{R}^i).$$

Thus, every manifold contravariantly defines a smooth algebra.

3. We have a functor from smooth sets to smooth algebras:

$$C^\infty : \mathrm{PSh}(\mathbf{Cart}) \to C^\infty\mathbf{Alg}^{\mathrm{op}}, \quad X \mapsto \left(\mathbb{R}^i \mapsto \mathrm{Hom}_{\mathrm{PSh}(\mathbf{Cart})}(X, よ\mathbb{R}^i)\right),$$

and a functor from smooth algebras to smooth sets:

$$\mathrm{Spec} : C^\infty\mathbf{Alg}^{\mathrm{op}} \to \mathrm{PSh}(\mathbf{Cart}), \quad A \mapsto \left(\mathbb{R}^i \mapsto \mathrm{Hom}_{C^\infty\mathbf{Alg}}(A, C^\infty(\mathbb{R}^i))\right).$$

Show that we have the adjunction:

$$\mathrm{Hom}_{C^\infty\mathbf{Alg}}(A, C^\infty(X)) \cong \mathrm{Hom}_{\mathrm{PSh}(\mathbf{Cart})}(X, \mathrm{Spec}(A)).$$

4. Define the smooth algebra of *dual numbers*:

$$C^\infty(\mathbb{D}) = \mathbb{R}[\epsilon]/(\epsilon^2) = \{a + b\epsilon \mid \epsilon^2 = 0\},$$

which can be thought of as smooth functions on an *infinitesimal interval*:

$$\mathbb{D} = \text{“}\{x \in \mathbb{R} \mid x^2 = 0\}.\text{”}$$

Verify the functor structure of $C^\infty(\mathbb{D})$: For any smooth map $f : \mathbb{R}^n \to \mathbb{R}$, we have:

$$C^\infty(\mathbb{D})(f) : C^\infty(\mathbb{D})^n \to C^\infty(\mathbb{D}), \quad (\cdots, a_i + b_i\epsilon, \cdots) \mapsto f(\cdots, a_i, \cdots) + \sum_{i=1}^n b_i \frac{\partial f}{\partial x_i}\epsilon.$$

5. For any smooth set X, define the tangent bundle TX to be the smooth set:

$$TX(\mathbb{R}^i) = \mathrm{Hom}_{C^\infty\mathbf{Alg}}(C^\infty(X), C^\infty(\mathbb{D}) \otimes_{\mathbb{R}} C^\infty(\mathbb{R}^i)).$$

Compare this definition to the usual definition of the tangent bundle of a manifold.

Exercise 1.3.7. Let **Bool** be the category consisting of two sets: $\perp = \emptyset$ and $\top = \{*\}$, with morphisms given by set maps. For any category $\mathcal{C}$, define the category of 0-presheaves as:

$$\mathrm{PSh}_0(\mathcal{C}) := \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathbf{Bool}),$$

i.e., contravariant functors from $\mathcal{C}$ to **Bool**.

1. Show that the morphisms of **Bool** are:

$$\mathrm{Hom}_{\mathbf{Bool}}(\perp, \perp) = \top, \mathrm{Hom}_{\mathbf{Bool}}(\top, \perp) = \perp, \mathrm{Hom}_{\mathbf{Bool}}(\perp, \top) = \top, \mathrm{Hom}_{\mathbf{Bool}}(\top, \top) = \top.$$

2. Let $(P, \leq)$ be a thin category defined from a poset. For any object $a \in P$, show that the hom functor $h_a := \mathrm{Hom}_P(-, a) \in \mathrm{PSh}_0(P)$ is a 0-presheaf defined by:

$$h_a(b) = \begin{cases} \top, & \text{if } b \leq a, \\ \perp, & \text{otherwise.} \end{cases}$$

3. Consider the poset of rational numbers $(\mathbb{Q}, \leq)$. Define a 0-presheaf $\sqrt{2} \in \mathrm{PSh}_0(\mathbb{Q})$ by:
$$\sqrt{2}(q) = \begin{cases} \top, & \text{if } q^2 \leq 2, \text{ or } q < 0, \\ \bot, & \text{otherwise.} \end{cases}$$
Verify that $\sqrt{2}$ is indeed a 0-presheaf, and that $\sqrt{2} \neq h_q$ for any $q \in \mathbb{Q}$.

4. Show that $\mathrm{PSh}_0(\mathbb{Q})$ is isomorphic to the extended real line (as a thin category from poset):
$$[-\infty, \infty] = \{-\infty\} \sqcup \mathbb{R} \sqcup \{\infty\}.$$

Exercise 1.3.8. Let X be a set, and let $\mathrm{Sub}(X)$ be the poset of subsets of X ordered by inclusion $\subseteq$, viewed as a thin category.

1. Let $(-)^c$ denote taking the complement. Show that:
$$(-)^c : \mathrm{Sub}(X)^{\mathrm{op}} \to \mathrm{Sub}(X)$$
is a functor.

2. For $A, B \in \mathrm{Sub}(X)$, define:
$$A \Rightarrow B := A^c \cup B \in \mathrm{Sub}(X).$$
Show that for any $A, B, C \in \mathrm{Sub}(X)$, there is a bijection:
$$\mathrm{Hom}_{\mathrm{Sub}(X)}(A \cap B, C) \cong \mathrm{Hom}_{\mathrm{Sub}(X)}(A, B \Rightarrow C).$$

3. Let $f : X \to Y$ be a function between sets. Define:
$$f^{-1} : \mathrm{Sub}(Y) \to \mathrm{Sub}(X), \quad f_! : \mathrm{Sub}(X) \to \mathrm{Sub}(Y), \quad f_!(A) := f(A).$$
Verify that f^{-1} and $f_!$ are functors, and show that for any $A \in \mathrm{Sub}(X)$, $B \in \mathrm{Sub}(Y)$, we have:
$$\mathrm{Hom}_{\mathrm{Sub}(Y)}(f_!(A), B) \cong \mathrm{Hom}_{\mathrm{Sub}(X)}(A, f^{-1}(B)).$$

4. Define:
$$f_* : \mathrm{Sub}(X) \to \mathrm{Sub}(Y), \quad f_*(A) := f(A^c)^c.$$
Verify that f_* is a functor, and show that for any $A \in \mathrm{Sub}(Y)$, $B \in \mathrm{Sub}(X)$, we have:
$$\mathrm{Hom}_{\mathrm{Sub}(X)}(f^{-1}(A), B) \cong \mathrm{Hom}_{\mathrm{Sub}(Y)}(A, f_*(B)).$$

Chapter 2

Sheaf

As we mentioned early, presheaf is analogue of "linear functional", to get a category of generalized space, we need to impose the "continuous" condition, And sheaf is such "continuous" presheaf.

In analysis, continuous means preserve the limit, i.e. $f(\lim x_i) = \lim f(x_i)$. So we should also define limit in category, in some sense it describes how to approximate an object by others.

Some reference can be found in [1].

2.1 Limit and Colimit

Let's begin with the easiest example of analysis: the limit of an increasing sequence.

If we view the real number poset $(\mathbb{R}, \leq)$ as a category. Then an increasing sequence is an order preserving map from $(\mathbb{N}, \leq)$ to $\mathbb{R}$. i.e a functor $a_{(-)} : \mathbb{N} \to \mathbb{R}$. Let us unwind the definition of the limit $\lim a_i$: it is the supremum of $\{a_i\}$, i.e.

$$\forall b \in \mathbb{R}, \forall i \in \mathbb{N}, a_i \leq b \Leftrightarrow \lim a_i \leq b$$

Recall for the poset category, morphism $\mathrm{Hom}_{\mathbb{R}}(a, b)$ can be seen as the proofs of propsition $a \leq b$: if there a morphism, $a \leq b$ is true, otherwise the proofs is empty, it is false. So we can rewrite it as

$$\forall b \in \mathbb{R}, \prod_{i \in \mathbb{N}} \mathrm{Hom}_{\mathbb{R}}(a_i, b) \cong \mathrm{Hom}_{\mathbb{R}}(\lim a_i, b)$$

So we can think the limit is formally defined as a pre(co)sheaf $b \mapsto \lim h^{a_i}(b)$, and then if we can find an object who represent this pre(co)sheaf as $h^{\lim a_i}$, the limit exists as this object.

Then next example we consider a functor $X_{(-)} : \mathbb{N} \to \mathbf{Set}$, we should intuitively think it limit is $\bigcup_{i\in\mathbb{N}} X_i$, in this case it is called **Colimit**. But if we compare to the previous formula, we just get an inclusion:

$$\forall A \in \mathbf{Set}, \prod_{i\in\mathbb{N}} \mathrm{Hom}_{\mathbf{Set}}(X_i, A) \supseteq \mathrm{Hom}_{\mathbf{Set}}(\bigcup_{i\in\mathbb{N}} X_i, A)$$

The reason for this is we also need to ask the morphisms $f_i \in \mathrm{Hom}_{\mathbf{Set}}(X_i, A)$ compactible which the morphism from functor $X_{i\leq j} : X_i \to X_j$, that is to say $f_i = f_j \circ X_{i\leq j}$.

$$\begin{tikzcd}
\cdots \arrow[r, "X_{i-1\leq i}"] \arrow[dr] & X_i \arrow[r, "X_{i\leq i+1}"] \arrow[d, dashed, "f_i"] & X_{i+1} \arrow[r, "X_{i+1\leq i+2}"] \arrow[dl, dashed, "f_{i+1}"] & \cdots \arrow[dll] \\
& A & &
\end{tikzcd}$$

This motivates us to give the definition of Limit and Colimit:

Definition 2.1.1 (Limit and Colimit of Set)**.** Let $D : J \to \mathbf{Set}$ be a diagram(functor) of sets indexed by a small category J. The *limit* of D, denoted $\mathrm{Lim}_J\, D$, is the subset of the product

$$\prod_{j\in J} D(j)$$

consisting of all families $(x_j)_{j\in J}$ such that for every morphism $f : i \to j$ in J, we have $D(f)(x_i) = x_j$.

The *colimit* of D, denoted $\mathrm{Colim}_J\, D$, is the quotient of the disjoint union

$$\bigsqcup_{j\in J} D(j)$$

by the equivalence relation generated by $x \sim D(f)(x)$ for every morphism $f : i \to j$ in J and every $x \in D(i)$.

We will omit J sometimes.

Remark 2.1.2. The intuition of limit is gluing functions, of colimit is gluing space. Image there is a covering of spaces $\bigcup X_i \to X$, to gluing space we start from $\bigsqcup X_i$ then we identify the intersections $X_i \cap X_j \hookrightarrow X_i, X_j$; To gluing function on $C(X_i)$ we start with $\prod C(X_i)$ then we impose compactibility condition when restrict to

$C(X_i \cap X_j)$.

Exercise 2.1.1. If we view $i \mapsto \mathrm{Hom}_{\mathbf{Set}}(X_i, A)$ as functor $h_A(X_{(-)}) : \mathbb{N}^{\mathrm{op}} \to \mathbf{Set}$, Then we have the relation between limit and colimit:

$$\forall A \in \mathbf{Set}, \operatorname*{Lim}_{i \in \mathbb{N}^{\mathrm{op}}} \mathrm{Hom}_{\mathbf{Set}}(X_i, A) \cong \mathrm{Hom}_{\mathbf{Set}}(\operatorname*{Colim}_{i \in \mathbb{N}} X_i, A)$$

Show that this is hold in general for all category J

$$\forall A \in \mathbf{Set}, \operatorname*{Lim}_{j \in J^{\mathrm{op}}} \mathrm{Hom}_{\mathbf{Set}}(X_j, A) \cong \mathrm{Hom}_{\mathbf{Set}}(\operatorname*{Colim}_{j \in J} X_j, A)$$

$$\forall A \in \mathbf{Set}, \operatorname*{Lim}_{j \in J} \mathrm{Hom}_{\mathbf{Set}}(A, X_j) \cong \mathrm{Hom}_{\mathbf{Set}}(A, \operatorname*{Lim}_{j \in J} X_j)$$

Exercise 2.1.2. Consider the category of functor $\mathrm{Fun}(J, \mathbf{Set})$ where the morphism is natural transformation. For $A \in \mathbf{Set}$ let $c(A) \in \mathrm{Fun}(J, \mathbf{Set})$ be the const functor. Show that

$$\mathrm{Hom}_{\mathrm{Fun}(J,\mathbf{Set})}(X_{(-)}, c(A)) \cong \operatorname*{Lim}_{j \in J^{\mathrm{op}}} \mathrm{Hom}_{\mathbf{Set}}(X_j, A)$$

$$\mathrm{Hom}_{\mathrm{Fun}(J,\mathbf{Set})}(c(A), X_{(-)}) \cong \operatorname*{Lim}_{j \in J} \mathrm{Hom}_{\mathbf{Set}}(A, X_i)$$

Example 2.1.3.

- **Product and Coproduct:**
  Consider the set J viewed as a discrete category and no non-identity morphisms. A functor $A : J \to \mathbf{Set}$ is simply a family of sets $\{A_j\}_{j \in J}$. The limit of A is the product set $\prod_{j \in J} A_j$ and the colimit is the coproduct (disjoint union) $\bigsqcup_{j \in J} A_j$.

- **Equalizer and Coequalizer:**
  Let J be the category with two objects α, β and two parallel morphisms $f, g : \alpha \to \beta$. A functor $D : J \to \mathbf{Set}$ consists of sets A, B and functions $f, g : A \to B$. The limit (equalizer) is:

  $$\mathrm{Lim}\, D = \mathrm{Eq}(f, g) := \{a \in A \mid f(a) = g(a)\}.$$

  And the colimit (coequalizer) is the quotient set:

  $$\mathrm{Colim}\, D = \mathrm{Coeq}(f, g) := B/\sim$$

  where $b \sim b'$ if there exists $a \in A$ such that $f(a) = b$ and $g(a) = b'$.

- **Pullback and Pushout:**
  Let J be the diagram $\alpha \xrightarrow{f} \gamma \xleftarrow{g} \beta$. A functor $D : J \to \mathbf{Set}$ assigns sets X, Y, Z and functions $f : X \to Z$, $g : Y \to Z$. The limit (pullback) is:
  $$\operatorname{Lim} D = X \times_Z Y := \{(x, y) \in X \times Y \mid f(x) = g(y)\}.$$
  And for J^{op} be the diagram $\alpha \xleftarrow{f} \gamma \xrightarrow{g} \beta$. A functor $D : J^{\mathrm{op}} \to \mathbf{Set}$ assigns sets X, Y, Z and functions $f : Z \to X$, $g : Z \to Y$. the colimit (pushout) is:
  $$\operatorname{Colim} D = X \sqcup_Z Y := (X \sqcup Y)/\sim$$
  where $\sim$ is the equivalence relation generated by $f(z) \sim g(z)$ for all $z \in Z$.

Remark 2.1.4. Actually these are all essential limit and colimit: limit can be seen as equalizer of product and colimit can be seen as coequalizer of coproduct:
Limit:

- Consider the product $\prod_{j \in J} D(j)$.
- For each morphism $f : i \to j$ in J, define two morphisms:
  $$\alpha, \beta : \prod_{j \in J} D(j) \to \prod_{f : i \to j} D(j)$$
  where α sends $(x_j)_{j \in J}$ to $(D(f)(x_i))_{f:i\to j}$ and β sends $(x_j)_{j \in J}$ to $(x_j)_{f:i\to j}$.
- The limit $\operatorname{Lim} D$ is the equalizer of α and β:
  $$\operatorname{Lim} D = \operatorname{Eq}(\alpha, \beta).$$

Colimit:

- Consider the coproduct $\coprod_{j \in J} D(j)$.
- For each morphism $f : i \to j$ in J, define two morphisms:
  $$\alpha', \beta' : \bigsqcup_{f : i \to j} D(i) \to \bigsqcup_{j \in J} D(j)$$
  where α' sends $x \in D(i)$ (in the $f : i \to j$ summand) to $D(f)(x) \in D(j)$, and β' sends $x \in D(i)$ to x viewed in $D(i)$.

- The colimit $\operatorname{Colim} D$ is the coequalizer of α' and β':

$$\operatorname{Colim} D = \operatorname{Coeq}(\alpha', \beta').$$

Remark 2.1.5 (Homotopy (Co)limit). As we mentioned earlier, everything should be a priori a space, for (co)limit, it is called homotopy (co)limit. so even for sets as discrete spaces, the homotopy colimit can be non discrete (limit are still discrete) and the colimit we have here is just set of connected components.

We give an example how homotopy colimit is like for coequalizer: given $f, g : A \to B$ of sets, we construct a space of graph $\operatorname{hoCoeq}(f, g)$, begin with elements $b \in B$ as points, then for every element $a \in A$, add a segment between $f(a)$ and $g(a)$. We can see $\pi_0 \operatorname{hoCoeq}(f, g) = \operatorname{Coeq}(f, g)$, and contain more information, for example $\pi_1 \operatorname{hoCoeq}(f, g)$.

To definite limit and colimit for general category, we can make use of morphism:

Definition 2.1.6 (Limit and Colimit)**.** Let $\mathcal{C}$ be a category, J a small category, and $D : J \to \mathcal{C}$ a functor (called a diagram in $\mathcal{C}$). A *limit* of D is an object $\operatorname{Lim}_J D$ of $\mathcal{C}$ such that

$$\forall A \in \mathcal{C}, \operatorname*{Lim}_{j \in J} \operatorname{Hom}_{\mathcal{C}}(A, D(j)) \cong \operatorname{Hom}_{\mathcal{C}}(A, \operatorname*{Lim}_J D)$$

A *colimit* of D is an object $\operatorname{Colim}_J D$ of $\mathcal{C}$ such that

$$\forall A \in \mathcal{C}, \operatorname*{Lim}_{j \in J^{\mathrm{op}}} \operatorname{Hom}_{\mathcal{C}}(D(j), A) \cong \operatorname{Hom}_{\mathcal{C}}(\operatorname*{Colim}_J D, A)$$

Example 2.1.7 (Poset and Lattice)**.** In a thin category come from poset P we have the $\operatorname{Lim} p$ is the meet $\bigwedge p$, and the colimit $\operatorname{Colim} p$ is the joint $\bigvee p$. Then the thin category admitted all limit and colimit is a complete lattice.

Remark 2.1.8 (Adjoint Preserving (Co)limit). Let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ be adjoint functors, i.e. we have the bijection for all $A \in \mathcal{C}, B \in \mathcal{D}$:

$$\operatorname{Hom}_{\mathcal{D}}(F(A), B) \cong \operatorname{Hom}_{\mathcal{C}}(A, G(B))$$

Then if (co)limit exist in these categories, then by definition

$$\begin{aligned}&\mathrm{Hom}_{\mathcal{D}}(F(\mathrm{Colim}\, A_i), B) \cong \mathrm{Hom}_{\mathcal{C}}(\mathrm{Colim}\, A_i, G(B))\\ &\cong \mathrm{Lim}\, \mathrm{Hom}_{\mathcal{C}}(A_i, G(B)) \cong \mathrm{Lim}\, \mathrm{Hom}_{\mathcal{D}}(F(A_i), B) \cong \mathrm{Hom}_{\mathcal{D}}(\mathrm{Colim}\, F(A_i), B)\end{aligned}$$

By Yoneda lemma this means $F(\mathrm{Colim}\, A_i) \cong \mathrm{Colim}\, F(A_i)$. Similarly, we also have $G(\mathrm{Lim}\, B_i) \cong \mathrm{Lim}\, G(B_i)$.

Thanks to Yoneda lemma, to understand stand (co)limit of general category, we only need to understand its presheaf, and thus limit of **Set**.

Then we introduce an important kind of colimit, *filtered colimit*.

Definition 2.1.9 (Filtered Category)**.** A small category J is called *filtered* if:

1. It is non-empty.
2. For every pair of objects $j_1, j_2 \in \mathrm{Ob}(J)$, there exists an object $k \in \mathrm{Ob}(J)$ and morphisms $j_1 \to k$ and $j_2 \to k$.
3. For every pair of parallel morphisms $f, g : j \to j'$ in J, there exists an object k and a morphism $h : j' \to k$ such that $h \circ f = h \circ g$.

Example 2.1.10. For the category of open set $\mathrm{Op}(X)$, both $\mathrm{Op}(X)$ and $\mathrm{Op}(X)^{\mathrm{op}}$ are filtered

Exercise 2.1.3. A category I is finite if it has only finite many objects and morphisms. Show that J is filtered iff for any finite category I and any functor $F : I \to J$, there exists an object $j_F \in J$ and a natural transformation $F \to c(j_F)$.

Intuitively, a filtered category allows us to "coherently glue" data indexed by J. The colimit definite from $D : J \to \mathcal{C}$ is called *filtered colimit*. Filtered colimit can be defined by more explicit quotient instead of that of general colimit.

Exercise 2.1.4. Show that for J filtered $D : J \to \mathbf{Set}$, the filtered colimit $\mathrm{Colim}_J\, D = \left(\coprod_{j\in J} F(j)\right) / \sim$, where

$$(x, j) \sim (y, k) \quad \text{if there exist morphisms } f : j \to l,\ g : k \to l \text{ in } J \text{ such that } D(f)(x) = D(g)(y).$$

Exercise 2.1.5.
1. Let I, J be any small categories, consider $D : I \times J \to \mathcal{C}$. Then show that: $\mathrm{Lim}_I\, \mathrm{Lim}_J\, D \cong \mathrm{Lim}_{I\times J}\, D \cong \mathrm{Lim}_J\, \mathrm{Lim}_I\, D$, same for the colimit. (You just need prove it for **Set**.)

2. Let I be finite, and J be filtered category, consider $D : I \times J \to \mathbf{Set}$. Then show that $\mathrm{Lim}_I \, \mathrm{Colim}_J \, D \cong \mathrm{Colim}_J \, \mathrm{Lim}_I \, D$. That is to say filtered colimit preserve finite limit.

Limit and Colimit are not necessarily always existing for all $D : J \to \mathcal{C}$, but just like for a space X we can define its completion $\hat{X}$ to make limit exist, we can define a (co)completion of a category $\mathcal{C}$ to make (co)limit exists. An easy observation is limit and colimit are interchanged in $\mathcal{C}$ and $\mathcal{C}^{\mathrm{op}}$, so let us be focus on the case for colimit. Notice that limit and colimit are admitted for **Set**, then so do presheaf category $\mathrm{PSh}(\mathcal{C})$

We first introduce the category of elements for a presheaf, which will be the diagram for a colim to assembly spaces.

Definition 2.1.11 (Category of Elements)**.** Let $\mathcal{C}$ be a category and let $\mathcal{F} : \mathcal{C}^{\mathrm{op}} \to \mathbf{Set}$ be a presheaf. The *category of elements* of $\mathcal{F}$, denoted $\int_{\mathcal{C}} \mathcal{F}$, is defined as follows:

- **Objects:** Pairs (C, x) where C is an object of $\mathcal{C}$ and $x \in \mathcal{F}(C)$.
- **Morphisms:** A morphism $(C, x) \to (D, y)$ is a morphism $f : C \to D$ in $\mathcal{C}$ such that
$$\mathcal{F}(f)(y) = x.$$
(Note: since $\mathcal{F}$ is contravariant, the direction of f is $C \to D$, but the induced map goes $\mathcal{F}(D) \to \mathcal{F}(C)$.)
- **Composition and identities:** Inherited from the category $\mathcal{C}$.

Exercise 2.1.6. 1. Let $\mathcal{F} = \text{よ}(D)$ for $D \in \mathcal{C}$, show that $\int_{\mathcal{C}} \text{よ}(D) \cong \mathcal{C}/D$.

2. On the other hand, let $\mathcal{C}/_{\text{よ}}\mathcal{F} \subset \mathrm{PSh}(\mathcal{C})/\mathcal{F}$ be the subcategory consists of morphisms $x : \text{よ}(C) \to \mathcal{F}$, then show that $\int_{\mathcal{C}} \mathcal{F} \cong \mathcal{C}/_{\text{よ}}\mathcal{F}$

Example 2.1.12 (Category of Elements of Simplicial Set)**.** Let $X : \Delta^{\mathrm{op}} \to \mathbf{Set}$ be a simplicial set, Then the category of elements $\int_{\Delta} X$ will be

- **Objects:** X_0 collection of points;X_1 collection of segments;X_2 collection of triangles, ...
- **Morphisms:** A morphism $x \in X_n \to y \in X_{n+1}$ if $d_i(y) = x$ such that x is the i-th face of y, etc.

Intuitively we should think $\int_{\mathcal{C}} \mathcal{F}$ give us the blueprint to reassembly. For any functor $R : \mathcal{C} \to \mathcal{D}$, we should think

$$\int_{C \in \mathcal{C}} \mathcal{F}(C) \times R(C) \left(\text{or } \int_{\mathcal{C}} \mathcal{F} \times R \right) := \operatorname*{Colim}_{(C,x) \in \int_{\mathcal{C}} \mathcal{F}} R(C) \in \mathcal{D}$$

Is the assembly of $\mathcal{F}$ inside $\mathcal{D}$.

Exercise 2.1.7. Take the Yoneda embedding よ $: \mathcal{C} \to \mathrm{PSh}(\mathcal{C})$, we have the $\int_{\mathcal{C}} \mathcal{F} \times$ よ $\in \mathrm{PSh}(\mathcal{C})$.

1. Let $| - | : \Delta \to \mathbf{Top}$ be defined as $|[n]| = \Delta^n$, then we can extend it to all simplicial set $| - | : \mathrm{PSh}(\Delta) \to \mathbf{Top}$:

   $$|X| = \int_{[n] \in \Delta} X_n \times \Delta^n \in \mathbf{Top}$$

   Verify this is adjoint to Sing:

   $$\mathrm{Hom}_{\mathbf{Top}}(|X|, T) \cong \mathrm{Hom}_{\mathrm{PSh}(\Delta)}(X, \mathrm{Sing}(T)).$$

2. Show that for $B \in \mathcal{C}$, we have a map

   $$\mathcal{F}(B) \to \int_{C \in \mathcal{C}} \mathcal{F}(C) \times よ(C)(B) = \operatorname*{Colim}_{(C,x) \in \int_{\mathcal{C}} \mathcal{F}} \mathrm{Hom}_{\mathcal{C}}(B, C)$$

   defined by

   $$b \in \mathcal{F}(B) \mapsto ((B, b), \mathrm{id}_B) \in \operatorname*{Colim}_{(C,x) \in \int_{\mathcal{C}} \mathcal{F}} \mathrm{Hom}_{\mathcal{C}}(B, C)$$

   which is a bijection. (This is analogue to $f(b) = \int_{c \in \mathbb{R}} f(c)\delta(c - b)$)

3. This extended to a natural transformation $\mathcal{F} \to \int_{\mathcal{C}} \mathcal{F} \times$ よ which is an isomorphism.

4. Let $\mathcal{D}$ be a cocomplete category(i.e. admitted all colimit), then there is a map between

   $$\mathrm{Fun}(\mathcal{C}, \mathcal{D}) \to \mathrm{Fun}^{\mathrm{cocont}}(\mathrm{PSh}(\mathcal{C}), \mathcal{D})$$

   where $\mathrm{Fun}^{\mathrm{cocont}}$ means cocontinuous functor i.e. preserve colimit, the map is defined by

   $$R \in \mathrm{Fun}(\mathcal{C}, \mathcal{D}) \mapsto \left(\mathcal{F} \in \mathrm{PSh}(\mathcal{C}) \mapsto \int_{\mathcal{C}} \mathcal{F} \times R \in \mathcal{D} \right)$$

   Show that this is a bijection. And in other words, よ $: \mathcal{C} \to \mathrm{PSh}(\mathcal{C})$ is the *free cocompletion* of $\mathcal{C}$.

Notice that even if $\mathcal{C}$ is cocompletion, which dosen't means $\mathcal{C} \cong \mathrm{PSh}(\mathcal{C})$, because $よ : \mathcal{C} \cong \mathrm{PSh}(\mathcal{C})$ is not cocontinuous, i.e $よ(\operatorname{Colim} C) \neq \operatorname{Colim} よ(C)$ in general. This can also been seen from $\mathcal{F} \in \mathrm{PSh}$ is not continuous in general:

$$\mathcal{F}(\operatorname{Colim} C) = \mathrm{Hom}_{\mathrm{PSh}(\mathcal{C})}(よ(\operatorname{Colim} C), \mathcal{F}) \neq \mathrm{Hom}_{\mathrm{PSh}(\mathcal{C})}(\operatorname{Colim} よ(C), \mathcal{F}) = \operatorname{Lim} \mathcal{F}(C)$$

So if we impose some continuous condition, we can get subcategory of continuous presheaf, which are called sheaf, behaves more close to the test category $\mathcal{C}$.

Exercise 2.1.8. Show that for $B \in \mathcal{C}$, $よ(B)$ are continuous, i.e. $よ(B)(\operatorname{Colim}_{j\in J} C_j) \cong \operatorname{Lim}_{j\in J} よ(B)(C_j)$. In others words if $\mathcal{C}$ is cocomplete, $よ(B)$ is a sheaf.

Remark 2.1.13 (Presheaf as LEGO set). There is an analogy between presheaf and LEGO set:

Presheaf	**LEGO**
Category $\mathcal{C}$	Types of bricks
Morphism $f : C \to D$	Way of bricks connected
Presheaf $\mathcal{F} : \mathcal{C} \to \mathbf{Set}$	LEGO set
$\mathcal{F}(C)$	Instances of bricks of certain type in set
$\mathrm{Obj}(\mathcal{F}) : \mathrm{Obj}(\mathcal{C}) \to \mathrm{Obj}(\mathbf{Set})$	Catalog of bricks in set
$\mathcal{F}(f) : \mathcal{F}(C) \to \mathcal{F}(D)$	Bricks connected in certain way in set
Category of elements $\int_{\mathcal{C}} \mathcal{F}$	Instruction book of set
Colimit $\operatorname{Colim}_{\int_{\mathcal{C}} \mathcal{F}}$	Build the set following the instruction

Notice that there are some bricks can be also built from combination of smaller bricks. And the sheaf we will discuss later is we identify the bricks and those combinations.

2.2 Sheaf and Coverage

If we ask to presheaf be continuous for all colimit we will get nothing more that representable ones (this is an analogue to the Riesz representation theorem), instead we can restrict which kinds of colimit that it preserve.

Example 2.2.1 (Ind-Object)**.** Let $\mathcal{C}$ have all finite colimit (i.e for $C : J \to \mathcal{C}$ with J finite), Then the ind-object is the presheaf that preserve finite colimit, i.e. $\mathcal{F}(\operatorname{Colim}_J C) \cong \operatorname{Lim}_J \mathcal{F}(C)$ for J finite, this is also called *left exact*. Let $\mathrm{ind}(\mathcal{C}) \subset \mathrm{PSh}(\mathcal{C})$ be the subcategory of ind-object.

Exercise 2.2.1. 1. For any J filtered and $D : J \to \mathcal{C}$, show that $\mathrm{Colim}_J \, よ(D) \in \mathrm{PSh}(\mathcal{C})$ is actually an ind-object.

2. * For any $\mathcal{F} \in \mathrm{ind}(\mathcal{C})$, show that the category of elements $\int_{\mathcal{C}} \mathcal{F}$ is filtered. This shows that $\mathcal{F} \cong \int_{\mathcal{C}} \mathcal{F} よ$ is a filter colimit, and $\mathrm{ind}(\mathcal{C})$ is generated by representable presheaf with filtered colimit.

In geometry setting, the colimit we want to preserve are come from coverage. We should first decide a class of open morphism in $\mathcal{C}$, then we decide which collection of open morphism is a covering. Now for example consider a covering of $j_1 : U_1 \hookrightarrow X \hookleftarrow U_2 : j_2$, intuitively, X should identify to the pushout colimit $|U_\bullet| := U_1 \sqcup_{U_1 \cap U_2} U_2$. The presheaf preserve this colimit will satisfy

$$\mathcal{F}(X) = \mathcal{F}(U_1 \sqcup_{U_1 \cap U_2} U_2) \cong \mathcal{F}(U_1) \times_{\mathcal{F}(U_1 \cap U_2)} \mathcal{F}(U_2)$$

The reason of consider this sheaf is because of locality, we want the generalized space Y have can be tested locally, that is

$$\mathrm{Map}(X, Y) \cong \mathrm{Map}(U_1, Y) \times_{\mathrm{Map}(U_1 \cap U_2, Y)} \mathrm{Map}(U_2, Y) = \left\{ \substack{\text{maps from } U_1, U_2 \text{ to } Y \\ \text{that coincide on } U_1 \cap U_2} \right\}$$

Notice that the gluing $|U_\bullet| := U_1 \sqcup_{U_1 \cap U_2} U_2$ might not exist in $\mathcal{C}$, but it always exists in $\mathrm{PSh}(\mathcal{C}) \ni |U_\bullet| := よU_1 \sqcup_{よU_1 \cap よU_2} よU_2$.

Let us define it more formally

Definition 2.2.2 (Coverage(Grothendieck Topology))**.** Let $\mathcal{C}$ be a category. Given a class of *basic open morphisms* of $\mathcal{C}$, a *coverage* τ on $\mathcal{C}$ assigns to each object $X \in \mathcal{C}$ a collection of families of basic open morphisms $\{U_i \to X\}_{i \in I}$, called *covering families*, satisfying the following condition:

If $\{U_i \to X\}_{i \in I}$ is a covering family and $f : Y \to X$ is any morphism in $\mathcal{C}$, then there exists a covering family $\{V_j \to Y\}_{j \in J}$ such that for each j, the morphism $V_j \to Y \xrightarrow{f} X$ factors through some $U_i \to X$.

A category $\mathcal{C}$ equip with a coverage τ is call a *site* $(\mathcal{C}, \tau)$. Given a covering families, $\{U_i \to X\}_{i \in I}$, let $U_{ij} := U_i \times_X U_j$, then the *Cech Nerve* is the presheaf $|U_\bullet| := \bigsqcup_{i \in I} よ(U_i) / \sim_{よ(U_{ij})} \in \mathrm{PSh}(\mathcal{C})$.

With all these definitions, we can finally define sheaf

Definition 2.2.3 (Sheaf)**.** Let $(\mathcal{C}, T)$ be a site, then sheaf $\mathcal{F} \in \mathrm{Sh}_T(\mathcal{C}) \subset \mathrm{PSh}(\mathcal{C})$ is the presheaf preserve gluing, i.e. for any covering families, $\{U_i \to X\}_{i \in I}$

$$\mathcal{F}(X) \cong \mathcal{F}(|U_\bullet|) := \mathrm{Hom}_{\mathrm{PSh}(\mathcal{C})}(|U_\bullet|, \mathcal{F}) = \mathrm{Lim}\,\mathrm{Hom}_{\mathrm{PSh}(\mathcal{C})}(U_\bullet, \mathcal{F}) = \mathrm{Lim}\,\mathcal{F}(U_\bullet)$$

Here

$$\mathrm{Lim}\,\mathcal{F}(U_\bullet) := \left\{ (f_i) \in \prod_{i \in I} \mathcal{F}(U_i) \,\middle|\, f_i|_{U_{ij}} = f_j|_{U_{ij}} \in \mathcal{F}(U_{ij}) \right\}$$

And the morphisms of sheaves are morphisms of them as presheaves.

When the coverage is clear, we can omit it to just write $\mathrm{Sh}(\mathcal{C})$. Such category is called *Topos*.

We have a lot of examples.

Example 2.2.4 (Sheaf on a Space)**.** Let X be a topological space, consider the usual coverage for $\mathrm{Op}(X)$: $\{U_i \to U\}_{i \in I}$, is a covering family iff $\bigcup_{i \in I} U_i = U$. Then we have the sheaf category $\mathrm{Sh}(X) := \mathrm{Sh}(\mathrm{Op}(X))$.

Exercise 2.2.2.

1. Show that the continuous function presheaf $C(U)$ is a sheaf.
2. Show that for a vector bundle, the presheaf of section Γ_E is a sheaf.
3. Show that for non-compact space X, the bound continuous function $C^b(U)$ is just a presheaf but not a sheaf.

Example 2.2.5 (Smooth Set)**.** Let us define the coverage on **Cart**: The basic open morphism is just open embedding of balls, *good open covers* is $\{U_i \to \mathbb{R}^n\}_{i \in I}$ such that for all finite subset $J \subset I$, the intersection $\bigcap_{i \in J} U_i \cong \mathbb{R}^n$ are also open balls. We always consider the sheaf of this coverage

Exercise 2.2.3.

1. Show that good open covers gives a coverage.
2. Show that a manifold M is a sheaf.
3. Show that the differential form Ω^k is a sheaf (but not a manifold).

Example 2.2.6 (Zariski Site)**.** Let us define the *Zariski Topology* on $\mathbf{Ring}^{\mathrm{op}}$: Then base open morphism is given by localization, for $f \in R, R \to R_f$. Now a *Zariski cover* is $\{R \to R_{f_i} := R[x]/(xf_i - 1)\}_{i \in I}$ such that $(f_i)_{i \in I} = 1$, or more explicitly, for all $i \in I$,there exist some $a_i \in R$ such that $\sum_{i \in I} a_i f_i = 1$

The intuition is, if R is the function ring of some space $X = \operatorname{Spec} R$, then R_f is the function ring of subspace $\{f \neq 0\} \subset X$. The condition of $\bigcup_{i\in I}\{f_i \neq 0\} = X$ is equivalent to $(f_i)_{i\in I} = 1$.

Exercise 2.2.4.

1. Show that Zariski covers gives a coverage.
2. Recall $\mathbb{A}^1 = U : \mathbf{Ring} \to \mathbf{Set}$ send the ring to its underlining set. Show that the functor $\mathbb{A}^1$ is a sheaf.
3. More general, for polynomials $P = (f_1, \dots, f_m)$, V_P is a sheaf.

We give another example which is not exactly a sheaf of set, but very closed.

Example 2.2.7 (Kan complex)**.** For each $k \in \mathbb{N}, 0 \leq k \leq n$, let $\{n \setminus k\} = \{i \in \mathbb{N} \mid 0 \leq i \leq n, i \neq k\}$. For simplex $[n]$, we consider covering "up to homotopy" : $\{\delta_i : [n-1] \to [n]\}_{i\in\{n\setminus k\}}$, the Cech nerve for the covering is called (n,k)-horn $\Lambda^k_n \in \mathrm{PSh}(\Delta)$. We have a canonical morphism $\Lambda^k_n \hookrightarrow よ[n]$.

Now we define a simplicial set $S\mathrm{PSh}(\Delta)$ is a *Kan Complex*, if for all k, n

$$S([n]) \twoheadrightarrow S(\Lambda^k_n) := \mathrm{Hom}_{\mathrm{PSh}(\Delta)}(\Lambda^k_n, S)$$

is surjective.

Exercise 2.2.5. Show that for topological space X, $\mathrm{Sing}(X)$ is a Kan complex.

Remark 2.2.8. The reason we only ask a surjection is also because of it is isomorphism up to homotopy. We can define a space version of morphism of simplicial set $\mathrm{Map}_{\mathrm{PSh}(\Delta)}(A,B)$: we first definite a simplicial set $s\mathrm{Map}_{\mathrm{PSh}(\Delta)}(A,B) : [n] \mapsto \mathrm{Hom}_{\mathrm{PSh}(\Delta)}(A \times [n], B)$, and taking the geometric realization $\mathrm{Map}_{\mathrm{PSh}(\Delta)}(A,B) = |s\mathrm{Map}_{\mathrm{PSh}(\Delta)}(A,B)|$

Then the condition of Kan complex is equivalent to

$$\mathrm{Map}_{\mathrm{PSh}(\Delta)}([n], S) \xrightarrow{\cong} \mathrm{Map}_{\mathrm{PSh}(\Delta)}(\Lambda^k_n, S)$$

is a homotopic equivalence.

2.3 Sheafification

We have seen sheaf is a subcategory of presheaf $\iota : \mathrm{Sh}(\mathcal{C}) \hookrightarrow \mathrm{PSh}(\mathcal{C})$, actually for any presheaf we can associate sheaf as the "best approximate". That is the sheafification functor $L : \mathrm{PSh}(\mathcal{C}) \to \mathrm{Sh}(\mathcal{C})$ such that we have the bijection

$$\mathrm{Hom}_{\mathrm{PSh}(\mathcal{C})}(\mathcal{F}, \iota\mathcal{G}) \cong \mathrm{Hom}_{\mathrm{Sh}(\mathcal{C})}(L\mathcal{F}, \mathcal{G})$$

Remark 2.3.1. Take the analogue of distribution, Let $P = D \to \mathbb{R}$ be the space of all linear functional, and we have an inclusion $\text{よ} : D \to P, f \mapsto < -, f >$. Now consider the subspace $K = \mathrm{Span} < \lim \text{よ}(x_i) - \text{よ}(\lim x_i) >$ generated by all converge sequence x_i, then the continuous functional is just $K^{\perp}$:

$$\iota : D' = K^{\perp} = \{f \in P \mid \forall x_i, \lim < x_i, f > - < \lim x_i, f >= 0\} \hookrightarrow P$$

On other hand we can understand $D' \cong P/K$ as a quotient. The is give the projective map $L : P \to D'$ such that $< f, \iota(g) >_P =< L(f), g >_{D'}$

To definition the sheafification, we want to identify all Cech nerves $|U_\bullet|$ of a covering $\{U_i \to X\}_{i \in I}$ with $X \in \mathcal{C}$, i.e. $L|U_\bullet| \cong L\text{よ}X = \text{よ}X$, in other words we want to invert the morphism $|U_\bullet| \to \text{よ}X$ in $\mathrm{PSh}(\mathcal{C})$. Therefore, abstractly, we have the Bousfield localization

$$L : \mathrm{PSh}(\mathcal{C}) \to \mathrm{PSh}(\mathcal{C})[(|U_\bullet| \to \text{よ}X)^{-1}] \cong \mathrm{Sh}(\mathcal{C})$$

To give more explicit construction of sheafification, let us consider again the simplest example: a covering of space $j_1 : U_1 \hookrightarrow X \hookleftarrow U_2 : j_2$, if we compare $\text{よ}X$ and $|U_\bullet| := \text{よ}U_1 \sqcup_{\text{よ}(U_1 \cap U_2)} \text{よ}U_2$, for any space Y, we have

$$|U_\bullet|(Y) = \mathrm{Map}(Y, U_1) \sqcup_{\mathrm{Map}(Y, U_1 \cap U_2)} \mathrm{Map}(Y, U_1) \subsetneq \mathrm{Map}(Y, X) = \text{よ}X(Y)$$

This is not equal in general, since we only have the map land in U_1 or U_2. To solve this problem we consider a cover of $V_1 \hookrightarrow Y \hookleftarrow V_2$, we define

$$\begin{aligned}
&L_0|U_\bullet|(Y) := |U_\bullet|(V_1) \times_{|U_\bullet|(V_1 \cap V_2)} |U_\bullet|(V_2) \\
=&|U_\bullet|(Y) \sqcup (\mathrm{Map}(V_1, U_1)' \times_{\mathrm{Map}(V_1 \cap V_2, U_1 \cap U_2)} \mathrm{Map}(V_2, U_2)') \\
&\sqcup (\mathrm{Map}(V_2, U_1)' \times_{\mathrm{Map}(V_1 \cap V_2, U_1 \cap U_2)} \mathrm{Map}(V_1, U_2)')/ \sim \subsetneq \mathrm{Map}(Y, X) \\
&\text{where } \mathrm{Map}(V_i, U_j)' = \{f \in \mathrm{Map}(V_i, U_j) \mid f(V_1 \cap V_2) \subset U_1 \cap U_2\}
\end{aligned}$$

This is closer to $\mathrm{Map}(Y, X)$, in fact for any $f \in \mathrm{Map}(Y, X)$, we have the cover $f^{-1}(U_i) \hookrightarrow Y$ and $f \in \mathrm{Map}(f^{-1}(U_1), U_1)' \times_{\mathrm{Map}(f^{-1}(U_1\cap U_2), U_1\cap U_2)} \mathrm{Map}(f^{-1}(U_i), U_2)'$. That is to say if we take colimit all possible covering, we can finally recover $\mathrm{Map}(Y, X)$:

$$L|U_\bullet|(Y) := \operatorname*{Colim}_{\{V_i\hookrightarrow Y\}} |U_\bullet|(V_1) \times_{|U_\bullet|(V_1\cap V_2)} |U_\bullet|(V_2) \cong \mathrm{Map}(Y, X)$$

This motivates us to give following definitions

Definition 2.3.2 (Plus construction)**.** Let $(\mathcal{C}, T)$ be a site, and let $\mathcal{F} : \mathcal{C}^{\mathrm{op}} \to \mathbf{Set}$ be a presheaf. The *plus construction* $\mathcal{F}^+$ is a new presheaf defined as follows:

For each object $X \in \mathcal{C}$, define $\mathcal{F}^+(X) =$ to be the colimit over covering families(c.f.) $\{U_i \to U\}_{i\in I}$

$$\mathcal{F}^+(X) := \operatorname*{Colim}_{\text{c.f.}\{U_i\to X\}_{i\in I}} \mathrm{Hom}_{\mathrm{PSh}(\mathcal{C})}(|U_\bullet|, \mathcal{F})$$

more explicitly it is the set of equivalence classes of *matching families* over covering families $\{U_i \to U\}_{i\in I}$:

$$\mathcal{F}^+(X) = \bigsqcup_{\text{c.f.}\{U_i\to X\}_{i\in I}} \left\{ (s_i) \in \prod_{i\in I} \mathcal{F}(U_i) \;\middle|\; s_i|_{U_{ij}} = s_j|_{U_{ij}} \right\} / \sim$$

where two such families are equivalent if they agree on a common refinement.

The *sheafification* of a presheaf $\mathcal{F}$ is the sheaf $L\mathcal{F}$ obtained by applying the plus construction twice:

$$L\mathcal{F} = (\mathcal{F}^+)^+.$$

Remark 2.3.3. We can see the + construction is a filter colimit, thus it commutes with finite limit. As a consequence, functor $(-)^+$ and L is left exact, i.e. preserving finite limit.

Example 2.3.4 (Sheafification of C^b)**.** For the bound continuous function sheaf C^b on $\mathbb{R}$, we have

$$LC^b(\mathbb{R}) = (C^b)^+(\mathbb{R}) = \bigsqcup_{\text{c.f.}\{U_i\to \mathbb{R}\}_{i\in I}} \left\{ (s_i) \in \prod_{i\in I} C^b(U_i) \;\middle|\; s_i|_{U_{ij}} = s_j|_{U_{ij}} \right\} / \sim$$

If we take a specific covering family $\{]n, n+2[\hookrightarrow \mathbb{R}\}_{n\in\mathbb{Z}}$. Then

$$LC^b(\mathbb{R}) \supset \left\{ (s_i) \in \prod_{i\in\mathbb{Z}} C^b(]i, i+2[) \;\middle|\; s_i|_{]i,i+1[} = s_{i-1}|_{]i,i+1[} \right\} \cong C(\mathbb{R})$$

Similarly we have $LC^b(\mathbb{R}) \supset C(U)$. On other hand we have $C^b(U) \subset C(U)$, thus $LC^b(U) \subset LC(U) = C(U)$. This showes we have $LC^b \cong C$.

In the case of sheaf $\mathrm{Sh}(X)$ on a topological space X, we have even more concrete construction: the idea is every sheaf is a sheaf of sections. The category of open subset $\mathrm{Op}(X)$ is actually category of *relative spaces*: Open subsets $U \subset X$ are actually injective local homeomorphism map. So we can think it can be used to build general local homeomorphism $E \to X$, which is called *étale space*.

Definition 2.3.5 (Étale Space)**.** Let $\mathrm{Et}/X \subset \mathbf{Top}/X$ be the category of étale space over X, whose objects are local homeomorphism $E \to X$. There is clear a functor $\Gamma_{(-)} : \mathrm{Et}/X \to \mathrm{Sh}(X)$ by taking the continuous sections. Conversely, for any presheaf $\mathcal{F}$ we can associate an étale space $\widetilde{\mathcal{F}}$:

1. **Étale Space:** Define the étale space $\widetilde{\mathcal{F}}$ of $\mathcal{F}$ as the disjoint union

$$\widetilde{\mathcal{F}} = \bigsqcup_{x\in X} \mathcal{F}_x,$$

   where $\mathcal{F}_x$ is the *stalk* of $\mathcal{F}$ at x, i.e.,

$$\mathcal{F}_x = \operatorname*{Colim}_{x\in U\in \mathrm{Op}(X)} \mathcal{F}(U).$$

2. **Topology on $\widetilde{\mathcal{F}}$:** A basis for the topology on $\widetilde{\mathcal{F}}$ is given by sets of the form

$$\widetilde{s}(U) = \{s_x \in \mathcal{F}_x \mid x \in U\},$$

   where $s \in \mathcal{F}(U)$ and s_x is the germ of s at x.

3. **Projection Map:** Define a map $\pi : \widetilde{\mathcal{F}} \to X$ by sending each germ $s_x \in \mathcal{F}_x$ to the point x. This map is a local homeomorphism.

4. **Sheaf of Sections:** The sheafification $L\mathcal{F} = \Gamma_{\widetilde{\mathcal{F}}}$ is defined as the sheaf of continuous sections of π:

$$L\mathcal{F}(U) = \{\sigma : U \to \widetilde{\mathcal{F}} \mid \pi \circ \sigma = \mathrm{id}_U,\ \sigma \text{ is continuous}\}.$$

This actually lead to an equivalence between $\mathrm{Et}/X \cong \mathrm{Sh}(X)$

Exercise 2.3.1 ((Co)Limit of Sheaf and Presheaf)**.** Recall the (co)limit of presheaf is taking objectwise, i.e. $(\mathrm{Lim}_{\mathrm{PSh}} \mathcal{F}_i)(X) = \mathrm{Lim}_{\mathrm{PSh}} \mathcal{F}_i(X), (\mathrm{Colim}_{\mathrm{PSh}} \mathcal{F}_i)(X) = \mathrm{Colim}_{\mathrm{PSh}} \mathcal{F}_i(X)$. Let $\mathcal{F}_- : I \to \mathrm{Sh}(\mathcal{C})$ be some diagram of sheaves.

1. Show that the presheaf limit $\mathrm{Lim}_{\mathrm{PSh}} \iota\mathcal{F}_i$ is still a sheaf. Therefore $\iota(\mathrm{Lim}_{\mathrm{Sh}} \mathcal{F}_i) \cong \mathrm{Lim}_{\mathrm{PSh}} \iota\mathcal{F}_i$ we can compute sheaf limit by presheaf limit.

2. Use the example above to show $\iota(\mathrm{Colim}_{\mathrm{Sh}} \mathcal{F}_i) \ncong \mathrm{Colim}_{\mathrm{PSh}} \iota\mathcal{F}_i$ in general, but rather we have $\mathrm{Colim}_{\mathrm{Sh}} \mathcal{F}_i \cong L(\mathrm{Colim}_{\mathrm{PSh}} \iota\mathcal{F}_i)$, that is to say the colimit of sheaves is the sheafification of its presheaves colimit.

We usually omit ι and think $\mathrm{Sh}(\mathcal{C}) \subset \mathrm{PSh}(\mathcal{C})$ as a subcategory.

Now we can define several natural things. Recall the open subset is define by the union of open base, here we replace union by the sheaf colimit

Definition 2.3.6 (Open Morphism)**.** For $X \in \mathcal{C}$ and $\mathcal{U} \in \mathrm{Sh}(\mathcal{C})$, a morphism $j : \mathcal{U} \to \text{よ}X$ is open if there is family of basic open morphism $\{U_i \to \mathcal{U} \to X\}_{i\in I}$ (not necessarily be a covering) factor through $\mathcal{U}$ such that $L|U_\bullet| \cong \mathcal{U}$.

More generally, a morphism of sheaf $j : \mathcal{F} \to \mathcal{G}$ is open if for all $X \in \mathcal{C}, f \in \mathcal{G}(X) \cong \mathrm{Hom}_{\mathrm{Sh}(\mathcal{C})}(\text{よ}X, \mathcal{G})$ the pullback $\mathcal{F} \times_{\mathcal{G}} \text{よ}X \to \text{よ}X$ are open.

Here we can see a general way of definition in sheaf: define by pullback along all $X \in \mathcal{C}, f \in \mathcal{G}(X) \cong \mathrm{Hom}_{\mathrm{Sh}(\mathcal{C})}(\text{よ}X, \mathcal{G})$.

Definition 2.3.7 (Open Covering)**.** For $X \in \mathcal{C}$ and a family of open morphisms $\{\mathcal{U}_j \to \text{よ}X\}_{j\in J}$ is an *open covering* if there is cover family of basic open morphism $\{\text{よ}U_i \to \mathcal{U}_{j(i)} \to \text{よ}X\}_{i\in I}$ factor through some $\mathcal{U}_j$.

More generally, a family of open morphisms $\{\mathcal{U}_j \to \mathcal{F}\}_{j\in J}$ is an open covering if for all $X \in \mathcal{C}, f \in \mathcal{F}(X) \cong \mathrm{Hom}_{\mathrm{Sh}(\mathcal{C})}(\text{よ}X, \mathcal{F})$ the pullback $\{\mathcal{U}_j \times_{\mathcal{F}} \text{よ}X \to$

$よX\}_{j\in J}$ are open coverings.

Remark 2.3.8. The advantage of considering sheaf rather than presheaf is we can reconstruct it via a relative small colimit along open covering. That is to say for an open covering $\{\mathcal{U}_i \to \mathcal{F}\}_{i\in I}$, we can define the Cech nerve same as before: let $\mathcal{U}_{ij} := \mathcal{U}_i \times_X \mathcal{U}_j$, and the presheaf $|\mathcal{U}_\bullet| := \bigsqcup_{i\in I} \mathcal{U}_i / \sim_{\mathcal{U}_{ij}} \in \mathrm{PSh}(\mathcal{C})$, the *Cech Nerve* is the sheafification $L|\mathcal{U}_\bullet|$. We can show that $L|\mathcal{U}_\bullet| \cong \mathcal{F}$ in $\mathrm{Sh}(\mathcal{C})$. If the open covering is giving by the representable objects $よU_i$, this is even simpler.

This motivates us to give following definition

Definition 2.3.9 (Locally Representable Sheaves)**.** A sheaf $\mathcal{F}$ is locally representable if there is an open covering of representable sheaves $\{よU_i \to \mathcal{F}\}_{i\in I}$. In other words we can think $\mathcal{F} \cong L|U_\bullet|$ is gluing by test spaces $U_i \in \mathcal{C}$. Let $\mathrm{Sch}(\mathcal{C}) \subset \mathrm{Sh}(\mathcal{C})$ be the subcategory of locally representable sheaves.

In smooth set $\mathrm{Sh}(\mathbf{Cart})$ locally representable sheaf are *smooth manifold* **Mnfd**. In algebraic set $\mathrm{Sh}_{\mathrm{Zar}}(\mathbf{Ring}^{\mathrm{op}})$ locally representable sheaves are called *scheme* $\mathrm{Sch}_{\mathbb{Z}}$.

Example 2.3.10 (S^1 as a gluing space)**.** Let S^1 is covered by $U_1 \cong U_2 \cong \mathbb{R}$, and $U_1 \cap U_2 = V_1 \sqcup V_2$ where $V_1 \cong V_2 \cong \mathbb{R}$. Then we have $|U_\bullet| \in \mathrm{PSh}(\mathbf{Cart})$, where

$$|U_\bullet|(\mathbb{R}^n) = \mathrm{Hom}_{\mathbf{Cart}}(\mathbb{R}^n, U_1) \sqcup_{\mathrm{Hom}_{\mathbf{Cart}}(\mathbb{R}^n,V_1)\sqcup \mathrm{Hom}_{\mathbf{Cart}}(\mathbb{R}^n,V_2)} \mathrm{Hom}_{\mathbf{Cart}}(\mathbb{R}^n, U_2)$$

As we have seen before, these are not all smooth map $\mathrm{Map}(\mathbb{R}^n, S^1)$. But if we sheafify it, we found $L|U_\bullet|(\mathbb{R}^n) \cong \mathrm{Map}(\mathbb{R}^n, S^1)$. Therefore S^1 is a manifold.

Example 2.3.11 ($\mathbb{P}^1$ as a gluing space)**.** Recall We define $\mathbb{P}^1_{pre} \in \mathrm{PSh}(\mathbf{Ring}^{\mathrm{op}})$ as for $R \in \mathbf{Ring}$:

$$\mathbb{P}^1_{pre}(R) = \{(r_0, r_1) \in R^2 \mid \exists u_i, u_0 r_0 + u_1 r_1 = 1\}/\sim_{R^\times}$$

Recall the representable (pre)sheaves $\mathbb{A}^1 = h^{\mathbb{Z}[x]}, R \mapsto U(R)$ and $\mathbb{G}_m = h^{\mathbb{Z}[x,x^{-1}]}, R \mapsto R^\times$. Now we have morphisms $\phi_i : U_i \cong \mathbb{A}^1 \to \mathbb{P}^1_{pre}$ given by

$$\phi_0 : r \in R \mapsto (1, r)/\sim, \phi_1 : r \in R \mapsto (r, 1)/\sim .$$

This is an open covering since $(r_0, r_1)/\sim \in \mathbb{P}^1_{pre}(R)$, the pullback $V_i := h^R \times_{\mathbb{P}^1_{pre}} U_i :$ $\mathbf{Ring} \to \mathbf{Set}$ is the functor: For $S \in \mathbf{Ring}$

$$V_0(S) = \{f : R \to S \mid \exists s \in S, (1, s) \sim (f(r_0), f(r_1))\} = \{f : R \to S \mid \exists t \in S, f(r_0)t = 1\}$$

By the universal property of localization R_{r_0}, we find $V_0 \cong h^{R_{r_0}}$. Similarly, $V_1 \cong h^{R_{r_1}}$, and by definition we have $u_0r_0+u_1r_1 = 1$, then $\{V_i \to h^R\}$ is a Zariski covering family, and thus $\{\phi_i : U_i \to \mathbb{P}^1_{pre}\}$ is an open covering.

And we have that $U_{01} = U_0 \times_{\mathbb{P}^1_{pre}} U_1 \cong \mathbb{G}_m$. Then the presheaf $|U_\bullet| : \mathbf{Ring} \to \mathbf{Set}$

$$|U_\bullet|(R) = \{a \in R\} \sqcup \{b \in R\}/u \in R^\times, u = a \sim b = u^{-1}$$

Let $\mathbb{P}^1 = L\mathbb{P}^1_{pre}$, and we have $\mathbb{P}^1 \cong L|U_\bullet|$ is a scheme.

2.4 Moduli Spaces via Sheaves

As we mentioned earlier, we want to give natrual space structure for collection of mathematics objects. Now this goal can be achieved by definite a sheaf which characterize the "families of objects".

Example 2.4.1 (Moduli Space of Triangles)**.** If we classify triangles inside a circle, then we can define a smooth set $\mathrm{Bun}_\Delta : \mathbf{Cart}^{\mathrm{op}} \to \mathbf{Set}$ as the families of triangles:

$$\mathrm{Bun}_\Delta(\mathbb{R}^n) = \{a, b, c : \mathbb{R}^n \to \mathbb{C} \mid |a| = |b| = |c| = 1, a \neq b, b \neq c, c \neq a\}/S_3$$

Where permutation group S_3 permute a, b, c.

We can extend this to general smooth set $\mathrm{Bun}_\Delta(M) := \mathrm{Hom}_{\mathrm{Sh}(\mathbf{Cart})}(M, \mathrm{Bun}_\Delta)$. Concretely, for manifold M, taking a good covering $\{U_i\}_{i \in I}$ of M, by the sheaf condition, we have:

$$\mathrm{Bun}_\Delta(M) = \mathrm{Lim}\,\mathrm{Bun}_\Delta(U_\bullet) = \left\{ (T_i) \in \prod_{i \in I} \mathrm{Bun}_\Delta(U_i) \,\middle|\, T_i|_{U_{ij}} = g_{ij} \cdot T_j|_{U_{ij}}, g_{ij} \in S_3 \right\}$$

Notice that we have global $U(1)(M)$ symmetry.

Example 2.4.2 (Grassmannian/Moduli Space of Linear Subspaces)**.** Give a vector space of dimension n, let $Gr(k, n)$ be the moduli space of k-dimensional linear subspaces. We can define it as a smooth set (manifold):

$$Gr(k, n)(\mathbb{R}^m) = \{M : \mathbb{R}^m \to \mathrm{Mat}_{k \times n} \mid \mathrm{rk}(M) = k\}/\mathrm{GL}_k(C^\infty(\mathbb{R}^m))\}$$

For general manifold X, we find this classifies rank k sub vector bundle of $X \times \mathbb{R}^n \to X$

$$Gr(k, n)(X) = \{E \to \mathbb{R}^m \text{ vector bundle } \mid \mathrm{rk}(E) = k, E \subset \mathbb{R}^m \times \mathbb{R}^n\}$$

Notice that we have global $\mathrm{GL}_n(C^\infty(X))$ symmetry.

This can also be defined as a scheme

Example 2.4.3 (Configuation Space of Points/Hilbert Scheme)**.** For a manifold M, we can consider the configuration space of d unordered points $\mathrm{Sym}^d M$. This can be defined as a smooth set:

$$\mathrm{Sym}^d M(\mathbb{R}^m) := \{N \subset \mathbb{R}^m \times M \mid N \to \mathbb{R}^m \text{ is } d\text{-etale space}\}$$

The problem of this is it is only an open manifold, we want something compact and smooth, so we need to deal with the diagonal, i.e. where points collide. The idea is when two points collide to one point, this one point should remember the data of two points by equipping with 2-dimensional function. This will be more clear in later locally ringed spaces.

Following this idea, we can define an algebraic version of compact configuration space, *Hilbert space*. For simplicity, we consider Hilbert scheme of d points in affine $\mathbb{A}^n$ space. We first define presheaf $\mathrm{Hilb}^d_{\mathbb{A}^n,pre} : \mathbf{Ring} \to \mathbf{Set}$ as:

$$\mathrm{Hilb}^d_{\mathbb{A}^n,pre}(R) = \{I \subset R[x_1, \ldots, x_n] \text{ ideal} \mid R[x_1, \ldots, x_n]/I \cong R^d\}$$

We can prove similarly as $\mathbb{P}^1$, $\mathrm{Hilb}^d_{\mathbb{A}^n,pre}$ is covered by representable sheaves (affine scheme), thus it $\mathrm{Hilb}^d_{\mathbb{A}^n} := L\mathrm{Hilb}^d_{\mathbb{A}^n,pre}$ is scheme.

Chapter 3

Realization and Invariant

We have seen how to define a generalized space via sheaf, which is data of testing spaces map into it. In this chapter we reconstruct space from the sheaf. That is a (fully) faithful[1] functor of *realization* $R : \mathrm{Sh}(\mathcal{C}) \to \mathcal{S}$ to some concrete category of some kind of spaces.

On the other hand, sometimes it is also useful to construct functor $I : \mathrm{Sh}(\mathcal{C}) \to \mathcal{V}$ to a simpler category $\mathcal{V}$ (value), even thought we lose some information. This is called *invariant*.

3.1 Relative Topos/Geometric morphism of Topos

We have seen a topos $\mathcal{T}$ (category of sheaf) as a collection of generalized spaces. Sometimes we can seen it as generalized spaces relative certain space X, in that case $\mathcal{T}_X$ also reflex the geometry of X.

Example 3.1.1. For topological spaces X, we have seen $\mathcal{SH}(X) \cong \mathrm{Et}/X$ étale space over X. Then for any continuous map $f : X \to Y$, we have the following adjoint functors:

$$f^{-1} : \mathrm{Sh}(Y) \leftrightarrows \mathrm{Sh}(X) : f_*$$

Where for $U \in \mathrm{Op}(Y)$, we ask $f^{-1}(\text{よ}(U)) := \text{よ}(f^{-1}(U))$. Then by the property of adjoint functors we have for $\mathcal{F} \in \mathrm{Sh}(X)$

$$f_*\mathcal{F}(U) = \mathrm{Hom}_{\mathrm{Sh}(Y)}(\text{よ}(U), f_*\mathcal{F}) \cong \mathrm{Hom}_{\mathrm{Sh}(X)}(f^{-1}(\text{よ}(U)), \mathcal{F}) = \mathcal{F}(f^{-1}(U))$$

[1]Fully means surjective on morphism, faithful means injective on morphism.

To define $f^{-1}\mathcal{G}$ for general $\mathcal{G} \in \mathrm{Sh}(Y)$, recall $\mathcal{G} \cong \int_{U\in \mathrm{Op}(Y)} \mathcal{G}(U) \times よ(U)$, then since f^{-1} preserve colimit, we have

$$f^{-1}\mathcal{G} \cong L \int_{U\in \mathrm{Op}(X)} \mathcal{G}(U) \times f^{-1}(よ(U))$$

Exercise 3.1.1. * Show that the presheaf $\int_{U\in \mathrm{Op}(X)} \mathcal{G}(U) \times f^{-1}(よ(U)) \in \mathrm{PSh}(X)$ is given by $V \in \mathrm{Op}(X) \mapsto \mathrm{Colim}_{f(V)\subset U \in \mathrm{Op}(Y)} \mathcal{G}(U)$

If we look at some example of continuous map to a point $p : X \to *$.
Then we consider a point $x \in X$, viewed as a map $x : * \to X$.

Example 3.1.2 (Slice Category)**.** Let $S \in \mathrm{Sh}(\mathcal{C})$, then the slice category $\mathrm{Sh}(\mathcal{C})/S$ ($\cong \mathrm{Sh}(\int_{\mathcal{C}} S)$ exercise) consists of morphisms $p : X \to S$. For a morphism $f : S \to T$, we can simply define $f^*(p : X \to S) := f^*p : X \times_S T \to T$. In fact f^* has both left and right adjoint functors $f_\#, f_*$.

point of topos
points and prime ideals.
classifying topos of local ring

3.2 Ringed Space

Recall for $\mathcal{D}$ cocomplete, if we have a realization functor on test objects $R : \mathcal{C} \to \mathcal{D}$, we can extend it to all presheaf $R : \mathrm{PSh}(\mathcal{C}) \to \mathcal{D}$ by using category of elements

$$R : \mathcal{F} \in \mathrm{PSh}(\mathcal{C}) \mapsto \int_{\mathcal{C}} \mathcal{F} \times R \in \mathcal{D}$$

This means we only need to consider the realization functor at level of test objects $\mathcal{C}$.

Recall for the simplicial set, smooth set, algebraic set, we can have realization functor in topological space **Top**:

Example 3.2.1.

1. For simplex category, we can take the underlying space $|-| : \Delta \to \mathbf{Top}$, thus we have $|-| : \mathrm{PSh}(\Delta) \to \mathbf{Top}$
2. Similarly for Cartesian spaces, we can take the underlying space $|-| : \mathbf{Cart} \to \mathbf{Top}$, thus we have $|-| : \mathrm{PSh}(\mathbf{Cart}) \to \mathbf{Top}$

3. For the category of ring **Ring**, we can take spectrum $\operatorname{Spec} : \mathbf{Ring}^{\mathrm{op}} \to \mathbf{Top}$: The spectrum of a ring A is the topological space $\operatorname{Spec} A$ whose points are prime ideals of A, topological base is given by $f \in A, D(f) := \{\mathfrak{p} \in \operatorname{Spec} A \mid f \notin \mathfrak{p}\}$. Thus we have $\operatorname{Spec} : \mathrm{PSh}(\mathbf{Ring}^{\mathrm{op}}) \to \mathbf{Top}$.

But this is not yet what we want: the realization functor is not fully, that is, for example there are much more continuous morphisms that smooth morphisms for manifold. To solve this problem, we need to update the target category.

The spaces we consider they all have function ring on it, and the morphisms of function rings will reflect the property of morphisms of space

Exercise 3.2.1. Consider a map $f : \mathbb{R} \to \mathbb{R}^2$,

1. if f is continuous, this induces a ring homomorphism $f^{\#} : C(\mathbb{R}^2) \to C(\mathbb{R})$.
2. if f is smooth, this induces a ring homomorphism $f^{\#} : C^{\infty}(\mathbb{R}^2) \to C^{\infty}(\mathbb{R})$.
3. if f is algebraic, this induces a ring homomorphism $f^{\#} : \mathbb{R}[x, y] \to \mathbb{R}[t]$.

This motivates us of following definition:

Definition 3.2.2 (Locally Ringed Space)**.** A *locally ringed space* is a pair $(X, \mathcal{O}_X) \in LRS$ where:

- X is a topological space,
- $\mathcal{O}_X \in \mathrm{Sh}(X, \mathbf{Ring})$ is a sheaf of rings on X, called *structure sheaf*,
- for every point $x \in X$, the stalk $\mathcal{O}_{X,x}$ is a local ring (i.e., it has a unique maximal ideal).

A morphism of locally ringed spaces

$$(f, f^{\#}) : (X, \mathcal{O}_X) \longrightarrow (Y, \mathcal{O}_Y)$$

consists of:

- a continuous map $f : X \to Y$,
- a morphism of sheaves of rings

$$f^{\#} : \mathcal{O}_Y \longrightarrow f_*\mathcal{O}_X,$$

such that for every $x \in X$, the induced map on stalks

$$f_x^{\#} : \mathcal{O}_{Y,f(x)} \longrightarrow \mathcal{O}_{X,x}$$

is a local homomorphism (i.e., it sends the maximal ideal of $\mathcal{O}_{Y,f(x)}$ into the maximal ideal of $\mathcal{O}_{X,x}$).

Even thought we first ask f to be continuous, to define $f^{\#}$ it turns out f have to be the map preserve some structure.

Example 3.2.3. 1. **Topological Space with Continuous Functions:** For any topological space X, (X, C_X), where C_X is the sheaf of continuous real-valued functions, is a locally ringed space. Each stalk is a local ring of germs of continuous functions.

2. **Smooth Manifold:** Let M be a smooth manifold. The pair (M, C_M^∞), where C_M^∞ is the sheaf of smooth real-valued functions on M, is a locally ringed space. Each stalk consists of germs of smooth functions at a point, forming a local ring.

   For smooth manifolds M and N and a smooth map $f : M \to N$, we have:

   $$(f, f^{\#}) : (M, \mathcal{C}_M^\infty) \longrightarrow (N, \mathcal{C}_N^\infty)$$

   where $f^{\#}$ sends a smooth function h on N to $h \circ f$ on M.

3. **Complex Analytic Space:** For a complex manifold X, the pair $(X, \mathcal{O}_X)$, where $\mathcal{O}_X$ is the sheaf of holomorphic functions, is a locally ringed space. Each stalk is a local ring of germs of holomorphic functions.

   For complex manifolds X and Y and a holomorphic map $f : X \to Y$, the morphism:

   $$(f, f^{\#}) : (X, \mathcal{O}_X) \longrightarrow (Y, \mathcal{O}_Y)$$

   where $f^{\#}$ sends a holomorphic function h on Y to $h \circ f$ on X.

4. **Affine Scheme:** For a commutative ring A, the spectrum $\operatorname{Spec} A$ with the Zariski topology and the structure sheaf $\mathcal{O}_{\operatorname{Spec} A}$ is a locally ringed space. Each stalk $\mathcal{O}_{\operatorname{Spec}(A),\mathfrak{p}}$ is the localization $A_{\mathfrak{p}}$, which is a local ring. This locally ringed space is called *affine scheme*, we also use $\operatorname{Spec} A$ to refer the affine scheme.

For rings A and B and a ring homomorphism $\varphi : A \to B$, we get a morphism of locally ringed spaces:

$$(f, f^{\#}) : (\mathrm{Spec}(B), \mathcal{O}_{\mathrm{Spec}(B)}) \longrightarrow (\mathrm{Spec}(A), \mathcal{O}_{\mathrm{Spec}(A)})$$

given by:

- Continuous map: $f : \mathrm{Spec}(B) \to \mathrm{Spec}(A)$, $f(\mathfrak{q}) = \varphi^{-1}(\mathfrak{q})$.
- Sheaf morphism: $f^{\#}$ induced by φ on localizations.

5. **Point and Double point:** We study more examples for affine scheme: $\mathrm{Spec}\,\mathbb{C}$ and $D = \mathrm{Spec}\,\mathbb{C}[x]/(x^2)$, they have the same underlying topological space, one point. But they have different structure sheaves:$\mathbb{C}$ and $\mathbb{C}[x]/(x^2)$. Image D is the limit of two points $\mathrm{Spec}\,\mathbb{C} \sqcup \mathrm{Spec}\,\mathbb{C} \cong \mathrm{Spec}\,\mathbb{C}[x]/(x(x-c))$ collide together.

Moreover, one can show the realization functors $R_{sm} : \mathbf{Cart} \to LRS, R_{alg} : \mathbf{Ring}^{\mathrm{op}} \to LRS$ are fully faithful, which means they extend to a functor on sheaves, in particular a fully faithful functor on locally representable sheaves:

$$R_{sm} : \mathrm{Sch}(\mathbf{Cart}) \to LRS, R_{alg} : \mathrm{Sch}(\mathbf{Ring}^{\mathrm{op}}) \to LRS$$

Therefore, we can define smooth manifold (scheme) as a locally ringed space which locally isomorphic to Cartesian spaces (affine scheme).

3.3 Geometry of Affine Scheme

The idea of spectrum is to think the ring A as functor ring on $\mathrm{Spec}\,A$: for $f \in A$ and point $\mathfrak{p} \in \mathrm{Spec}\,A$, we should think $f(\mathfrak{p}) = ev(f) \in A/\mathfrak{p}$, where $ev : A \to A/\mathfrak{p}$ is the quotient.

Notice that the spectrum of ring form an unusual space, it is only T_0. So not all points are closed. It's helpful to think in following ways: It contains closed points, which is usual points; then for each irreducible algebraic curve C, we have a corresponding generic point ξ_C. Then for every closed point x on that curve, ξ_C is in the neighborhood of x, i.e. $x \in \overline{\{\xi_C\}}$, we denote this as $\xi_C \rightsquigarrow x$; Similarly, for each irreducible surface S, we have a generic point ξ_S, such that for the curve it contains, we have $\xi_S \rightsquigarrow \xi_C$.

Another interesting fact is the spectrum is a filter limit of finite T_0 space: You can consider a finite stratification of space by points, curve, surface, etc., then take the limit by have more and more of them. This is called *Spectral space.*

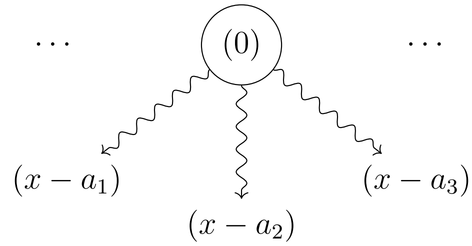
We have the following diagram of corresponding geometric and algebraic objects.

Geometry	**Algebra**	**Connection**
Locally Ringed Space $(X, \mathcal{O}_X)$	Ring A	$X = \operatorname{Spec} A$; $A = \mathcal{O}_X(X)$.
Point $x \in X$	Prime ideal $\mathfrak{p} \subset A$	
Closed point $x \in X$	Maximal ideal $\mathfrak{m} \subset A$	
Function $f \in \mathcal{O}_X(X)$, eval $f(x)$	$ev_{\mathfrak{p}} : A \to A/\mathfrak{p}$	$f(\mathfrak{p}) := ev_{\mathfrak{p}}(f) \in A/\mathfrak{p}$
Open subset $U \subseteq X$	Localization A_f	$U = D(f) = \operatorname{Spec} A_f = \{\mathfrak{p} \mid f(\mathfrak{p}) \neq 0 \Leftrightarrow f \notin \mathfrak{p}\}$.
Closed subset $Z \subseteq X$	Ideal $I \subset A$	$Z = V(I) = \operatorname{Spec} A/I = \{\mathfrak{p} \mid \forall f \in I, f(\mathfrak{p}) = 0 \Leftrightarrow I \subseteq \mathfrak{p}\}$; $I(Z) = \{f \in A \mid \forall \mathfrak{p} \in Z, f(\mathfrak{p}) = 0\} = \bigcap_{\mathfrak{p} \in Z} \mathfrak{p}$.
Structure sheaf $\mathcal{O}_X$	$\operatorname{Lim} A_f$	$\mathcal{O}_{\operatorname{Spec} A}(D(f)) = A_f$ and $\mathcal{O}_{\operatorname{Spec} A}(U) = \operatorname{Lim}_{D(f) \subseteq U} A_f$.
Stalk $\mathcal{O}_{X,x}$	Local ring $A_{\mathfrak{p}}$	$\mathcal{O}_{X,x} = A_{\mathfrak{p}} = (A \setminus \mathfrak{p})^{-1} A$.
Irreducible closed subset F	Prime ideal $\mathfrak{p}$	$F = \overline{\{\mathfrak{p}\}}$; generic point of F is $\mathfrak{p}$.
Specialization $\xi \rightsquigarrow x$ ($x \in \overline{\{\xi\}}$)	$\mathfrak{p}_\xi \subset \mathfrak{p}_x$	
Morphism $(X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$	Ring homomorphism $B \to A$	$\varphi^{-1} : \operatorname{Spec} A \to \operatorname{Spec} B$ corresponds to $\varphi : B \to A$.
Fiber product $X \times_Z Y$	Tensor product $A \otimes_C B$	$\operatorname{Spec}(A \otimes_C B) = \operatorname{Spec} A \times_{\operatorname{Spec} C} \operatorname{Spec} B$
Fiber $f^{-1}(y)$ over $y \in Y$	Tensor product $A \otimes_B \kappa(y)$	$f^{-1}(y) = \operatorname{Spec}(A \otimes_B \kappa(y))$ where residue field $\kappa(y) = \operatorname{Frac}(B/\mathfrak{p}_y)$.
Cotangent space $T_x^* X$	$\kappa(\mathfrak{m})$-Vector space $\mathfrak{m}/\mathfrak{m}^2$	
Quasi-Coherent sheaf $\mathcal{F} \in \operatorname{Sh}(X)$	A-module M	$\mathcal{F} = \widetilde{M}; M = \mathcal{F}(X)$
Vector bundle E	Locally free module M	$M = \Gamma(X, E)$
Section $\mathcal{F}(U)$	Localization M_f	$\mathcal{F}(D(f)) = M_f = M \otimes_A A_f$ and $\mathcal{F}(U) = \operatorname{Lim}_{D(f) \subseteq U} M_f$

Example 3.3.1. • **Spectrum of** $\mathbb{C}[x]$ Prime ideals:

$$\operatorname{Spec}(\mathbb{C}[x]) = \{(0)\} \cup \{(x - a) \mid a \in \mathbb{C}\}.$$

Generic point: (0); closed points: $(x - a)$ for $a \in \mathbb{C}$.

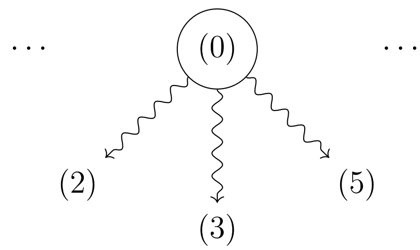


Affine line over $\mathbb{C}$

- **Spectrum of** $\mathbb{Z}$

$$\mathrm{Spec}(\mathbb{Z}) = \{(0)\} \cup \{(p) \mid p \text{ prime}\}.$$

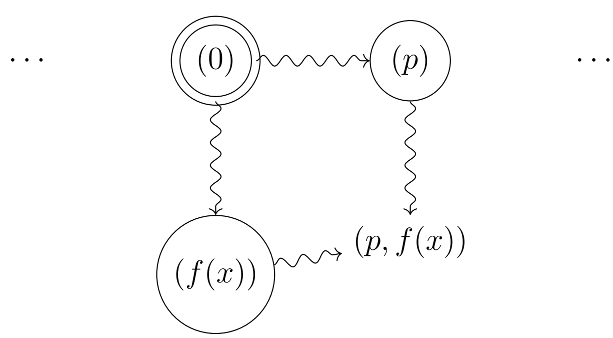
Generic point: (0); closed points: (p) for $p \in \mathbb{N}$ prime.



Points indexed by primes

- **Spectrum of** $\mathbb{Z}[x]$ Prime ideals:

$$\begin{aligned}
&(0),\\
&(p), p \text{ prime}\\
&(f(x)) \text{ irreducible over } \mathbb{Q},\\
&(p, f(x)) \text{ irreducible mod } p.
\end{aligned}$$



Affine line over $\mathrm{Spec}(\mathbb{Z})$

For affine scheme, we have an analogue result of Nullstellensatz, which is easier to prove that original one.

Exercise 3.3.1 (Generalized Nullstellensatz)**.** Let A be any ring.

1. Show that for non-nilpotent $f \notin \sqrt{(0)}$(i.e. $\forall n, f^n \neq 0$), $A_f \neq 0$, therefore $\operatorname{Spec} A_f = D(f) \neq \emptyset$. As a consequence, there exist a prime ideal $\mathfrak{p} \in D(f)$, i.e. $f \notin \mathfrak{p}$.

2. Show that $\bigcap_{\mathfrak{p} \in \operatorname{Spec} A} \mathfrak{p} \subseteq \sqrt{(0)}$.

3. Show that for nilpotent $g \in \sqrt{(0)}$, For all prime ideals $\mathfrak{p}$, $g \in \mathfrak{p}$.

4. Show that $\bigcap_{\mathfrak{p} \in \operatorname{Spec} A} \mathfrak{p} = \sqrt{(0)}$, and thus $I(V(J)) = \{f \in A \mid \forall \mathfrak{p} \in V(J), f(\mathfrak{p}) = 0\} = \bigcap_{\mathfrak{p} \supseteq J} \mathfrak{p} = \sqrt{J}$. (The original Nullstellensatz is $\bigcap_{\text{maximal ideal } \mathfrak{m} \supseteq J} \mathfrak{m} = \sqrt{J}$, which is stronger that this.)

3.4 Quasi-Coherent Sheaf and Vector Bundle

Over a locally ringed space

Definition 3.4.1. Let $(X, \mathcal{O}_X)$ be a ringed space. A sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$ on X is called *quasi-coherent* if for every point $x \in X$, there exists an open neighborhood U of x and an exact sequence of $\mathcal{O}_X|_U$-modules

$$\mathcal{O}_X|_U^{\oplus I} \longrightarrow \mathcal{O}_X|_U^{\oplus J} \longrightarrow \mathcal{F}|_U \longrightarrow 0,$$

where I and J are (possibly infinite) index sets.

To understand what quasi-coherent sheaf looks like, we introduce the support of sheaf:

Definition 3.4.2. Let X be a topological space and $\mathcal{F} \in \operatorname{Sh}(X, \mathbf{Ab})$ a sheaf of Abelian group. The *support* of $\mathcal{F}$ is the subset

$$\operatorname{Supp}(\mathcal{F}) = \{x \in X \mid \mathcal{F}_x \neq 0\},$$

where $\mathcal{F}_x$ denotes the stalk of $\mathcal{F}$ at x. It is a closed subset of X.

We give some example of quasi-coherent sheaf on affine scheme.

Example 3.4.3 (Quasi-Coherent Sheaf).

Chapter 4

To Infinity and Beyond

Up to now we only considered **Set**-value sheaves, and we can easily extend to other concrete 1-category: Abelian group , Ring, Vector space, etc. But the more interesting is to extend to higher category: Groupoid, Category, in that case we need to extend the Cech nerves and replace (co)limit by homotopy (co)limit. For example, to define $\mathcal{F} \in \mathrm{Sh}(\mathcal{C}, \mathbf{Grpd}) \subset \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathbf{Grpd})$, for any covering families, $\{U_i \to X\}_{i\in I}$, we have $\mathcal{F}(X) = \mathrm{holim}\mathcal{F}(U_\bullet) \in \mathbf{Grpd}$, where

$$\mathrm{Obj}(\mathrm{holim}\mathcal{F}(U_\bullet)) := \left\{ (f_i) \in \prod_{i\in I} \mathcal{F}(U_i), g_{ij} : f_i|_{U_{ij}} \xrightarrow{\cong} f_j|_{U_{ij}} \in \mathcal{F}(U_{ij}) \;\middle|\; g_{ij}g_{jk}|_{U_{ijk}} = g_{ik}|_{U_{ijk}} \right\}$$

$$\mathrm{Hom}_{\mathrm{holim}\mathcal{F}(U_\bullet)}((f_\bullet, g_{\bullet\bullet}), (h_\bullet, g'_{\bullet\bullet})) := \left\{ \varphi_i \in \prod_{i\in I} \mathrm{Hom}_{\mathcal{F}(U_i)}(f_i, h_i) \;\middle|\; \begin{array}{ccc} f_i|_{U_{ij}} & \xrightarrow{\varphi_i} & h_i|_{U_{ij}} \\ {\scriptstyle g_{ij}}\downarrow & & \downarrow{\scriptstyle g'_{ij}} \\ f_j|_{U_{ij}} & \xrightarrow{\varphi_j} & h_j|_{U_{ij}} \end{array} \text{ commutes} \right\}$$

The sheaf of groupoid is called *stack* $\mathrm{Stk}(\mathcal{C})$.

4.1 Moduli Space via Stack

Definition 4.1.1 (Action Groupoid)**.** Given a group G acting on a set X, the *action groupoid* $X /\!\!/ G$ has:

- Objects: elements of X.

- Morphisms: for $x, y \in X$, a morphism $x \to y$ is an element $g \in G$ such that $g \cdot x = y$, i.e. $\mathrm{Hom}_{X /\!\!/ G}(x, y) = \{g \in G \mid g \cdot x = y\}$.

Notice that the isomorphic class $\pi_0(X /\!\!/ G) = X/G$.

Example 4.1.2. 1. $* /\!\!/ G \cong \mathrm{B}G$

2. $\mathbb{R} /\!\!/ \mathbb{Z}$: $\mathbb{Z}$ acting on $\mathbb{R}$ by translations

$$n \cdot x = x + n, \quad n \in \mathbb{Z}, x \in \mathbb{R}.$$

   Objects: points of $\mathbb{R}$.

   Morphisms: $(x \xrightarrow{n} x + n)$ for $n \in \mathbb{Z}$.

   In fact, we have equivalence of category between $\mathbb{R}//\mathbb{Z} \cong U(1)$

3. $\mathbb{C} /\!\!/ \mathbb{C}^\times$: $\mathbb{C}^\times$ acting on $\mathbb{C}$ by scaling

$$\lambda \cdot z = \lambda z, \quad \lambda \in \mathbb{C}^\times, z \in \mathbb{C}.$$

   Objects: points of $\mathbb{C}$.

   Morphisms: $(z \xrightarrow{\lambda} \lambda z)$ for $\lambda \in \mathbb{C}^\times$.

Now for a set sheaf $X \in \mathrm{Sh}(\mathcal{C})$ and a group sheaf $G \in \mathrm{Sh}(\mathcal{C}, \mathbf{Grp})$ acts on it. We have a presheaf of groupoid $X /\!\!/_{pre} G(C) := X(C) /\!\!/ G(C)$, which reflex global symmetry, but it is not a sheaf in general. We can define the action stack as the sheafification $X /\!\!/ G = L(X /\!\!/_{pre} G)$, which reflex local symmetry. Similarly, we can define the quotient sheaf $X/G = L(X/_{pre}G)$

Example 4.1.3. Let $\mathbb{R}, \mathbb{Z} \in \mathrm{Sh}(\mathrm{Mnfd})$, where $\mathbb{R}(U) = C^\infty(U)$ and $\mathbb{Z}(\mathbb{R}^m) = \mathbb{Z}$. Now the presheaf quotient $\mathbb{R}/_{pre}\mathbb{Z}(U) := C^\infty(U)/\mathbb{Z}$. Now we show that this is not a sheaf: consider $\mathbb{R}/_{pre}\mathbb{Z}(S^1) := C^\infty(S^1)/\mathbb{Z}$, take the standard covering $U_1, U_2 \subset S^1$ and $U_1 \cap U_2 = V_1 \sqcup V_2$, we have

$$\mathrm{Lim}\, \mathbb{R}/_{pre}\mathbb{Z}(U_\bullet) = \{f_1 \in C^\infty(U_1), f_2 \in C^\infty(U_2) \mid f_1|_{V_1} = n_1 + f_2|_{V_1}, f_1|_{V_2} = n_2 + f_2|_{V_2}\}/\mathbb{Z}$$

In fact we have $\mathbb{R}/\mathbb{Z}(M) = \mathrm{Map}_{sm}(M, U(1))$.

As we mentioned earlier, we want to give natural space structure for collection of mathematics objects. Now this goal can be achieved by definite a sheaf which characterize the "families of objects A over Σ" $\mathrm{Bun}_A(\Sigma)$, with additional symmetry by using action groupoid, which should be thought as moduli space $\mathcal{M}_A$ containing A, i.e. $\mathrm{Map}(\Sigma, \mathcal{M}_A) := \mathrm{Bun}_A(\Sigma)$

Example 4.1.4 (Moduli Space of Triangles)**.** If we classify triangles up to similarity, then we can define a smooth stack $\mathcal{M}_\Delta : \mathbf{Cart}^{\mathrm{op}} \to \mathbf{Grpd}$ as the families of triangles:

$$\mathcal{M}_\Delta(\mathbb{R}^n) = \{a, b, c : \mathbb{R}^n \to \mathbb{R} \mid a + b + c = 2, 0 < a, b, c < 1\} /\!\!/ S_3$$

Notice that the triangle have the symmetry group of permutation group S_3.

We can extend this to general smooth set $\mathcal{M}_\Delta(M) := \mathrm{Hom}_{\mathrm{Stk}(\mathbf{Cart})}(M, \mathcal{M}_\Delta)$. Concretely, for manifold M, taking a good covering $\{U_i\}_{i\in I}$ of M, by the sheaf condition, we have:

$$\mathcal{M}_\Delta(M) = \mathrm{holim}\mathcal{M}_\Delta(U_\bullet) =_{\mathrm{Obj}} \left\{ (T_i) \in \prod_{i\in I} \mathcal{M}_\Delta(U_i) \;\middle|\; T_i|_{U_{ij}} = g_{ij} \cdot T_j|_{U_{ij}}, g_{ij} \in S_3 \right\}$$

And the morphism is given by isomorphism of bundles.

Example 4.1.5 (Moduli Space of Vector Bundles)**.** Let $\mathcal{M}_{Vect,k} = \mathrm{BGL}_k$ be the moduli space of k-dimensional vector space. We can define it as a smooth set (manifold):

$$\mathcal{M}_{Vect,k}(\mathbb{R}^m) = \{V : \mathbb{R}^k \times \mathbb{R}^m \to \mathbb{R}^m\} /\!\!/ \mathrm{GL}_k(C^\infty(\mathbb{R}^m)) = \mathrm{BGL}_k(C^\infty(\mathbb{R}^m))$$

For general manifold M, we find $\mathcal{M}_{Vect,k}(M) = \mathrm{Hom}_{\mathrm{Stk}(\mathbf{Cart})}(M, \mathcal{M}_{Vect,k})$ classifies the isomorphic groupoid of rank k vector bundles $E \to M$: taking a good covering $\{U_i\}_{i\in I}$ of M,

$$\begin{aligned}\mathcal{M}_{Vect,k}(M) &=_{\mathrm{Obj}} \left\{ g_{ij} \in \mathrm{GL}_k(C^\infty(U_{ij})), \mid g_{ij}|_{U_{ijk}} g_{jk}|_{U_{ijk}} = g_{ik}|_{U_{ijk}} \right\} \\ &= \{E \to M \text{ vector bundle } \mid \mathrm{rk}(E) = k\}\end{aligned}$$

And the morphisms are isomorphism of vector bundle. In other world the stack BGL_k classify the rank k vector bundle.

Example 4.1.6 (Configuation Space of Points/Hilbert Scheme)**.** For a manifold M, we can consider the compact configuration space of d unordered points Conf^d_M. This can be defined as a smooth stack $M^d /\!\!/ S_d$, which is sheafification of prestack.

$$\mathrm{Conf}^d_{M,pre}(\mathbb{R}^m) := \{(f_i) \in \mathrm{Map}_{sm}(\mathbb{R}^m, M^d)\} /\!\!/ S_d$$

We can see $\mathrm{Sym}^d_M = (M^d \setminus \mathrm{diag})/S_d \subset \mathrm{Conf}^d M$ as subspace of non-stacky points.

Following this idea, we can define an algebraic version of compact configuration space, *Hilbert space*, which is blow up at the diagonal. For simplicity, we consider

Hilbert scheme of d points in affine $\mathbb{A}^n$ space. We first define presheaf $\mathrm{Hilb}^d_{\mathbb{A}^n,pre}$: **Ring** $\to$ **Set** as:

$$\mathrm{Hilb}^d_{\mathbb{A}^n,pre}(R) = \{I \subset R[x_1, \ldots, x_n] \text{ ideal} \mid R[x_1, \ldots, x_n]/I \cong R^d\}$$

We can prove similarly as $\mathbb{P}^1$, $\mathrm{Hilb}^d_{\mathbb{A}^n,pre}$ is covered by representable sheaves (affine scheme), thus it $\mathrm{Hilb}^d_{\mathbb{A}^n} := L\mathrm{Hilb}^d_{\mathbb{A}^n,pre}$ is scheme.

4.2 Quasi-Coherent Sheaf and Vector Bundle via Stack of Category

We want to define a category of vector bundle or more general other bundle over a space which is defined by a sheaf $\mathrm{Sch}(\mathcal{C})$, the nicer way is to define a category-value sheaf. That is to say, we first define functor $\mathcal{C}^{\mathrm{op}} \to \mathrm{Cat}$, then extend it to $\mathrm{Sch}(\mathcal{C}) \to \mathrm{Cat}$.

Example 4.2.1 (Vector bundle)**.** We first define the category of vector bundle on Cartesian space: $VB(\mathbb{R}^n)$, the objects are trivial vector bundles$V_k := \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}^n$, and the morphisms $f : V_k \to V_l$ are smooth map $f : \mathbb{R}^n \to \mathrm{Mat}(\mathbb{R}^k, \mathbb{R}^l)$. This is indeed a functor $VB : \mathbf{Cart}^{\mathrm{op}} \to \mathrm{Cat}$. For $g : \mathbb{R}^n \to \mathbb{R}^m$, we have the pull back functor $g^* : VB(\mathbb{R}^m) \to VB(\mathbb{R}^n)$. We can show this is sheaf of category.

Now this functor can extend to all manifolds $VB : \mathrm{Sch}(\mathbf{Cart})^{\mathrm{op}} \to \mathrm{Cat}, X \mapsto \mathrm{Nat}(X, VB)$

Example 4.2.2 (Module and Quasi-Coherent Sheaf)**.** For each ring A, we can define the category of Mod_A. For $\varphi : A \to B$, we have the functor $\varphi_* : \mathrm{Mod}_A \to \mathrm{Mod}_B, M \mapsto B \otimes_A M$. We can show this is sheaf of category. Recall that we have the realization functor $\widetilde{(-)} : \mathrm{Mod}_A \xrightarrow{\cong} \mathrm{QC}(\mathrm{Spec}\, A) \subset \mathrm{Sh}(\mathrm{Spec}\, A)$.

Now this functor can extend to all schemes $\mathrm{Mod}_{(-)}\mathrm{Sch}^{\mathrm{op}} \to \mathrm{Cat}, X \mapsto \mathrm{Nat}(X, \mathrm{Mod}_{(-)})$. We have also the realization functor $\widetilde{(-)} : \mathrm{Mod}_X \xrightarrow{\cong} \mathrm{QC}(X) \subset \mathrm{Sh}(X)$

4.3 What is a Field Theory ?

Recall for sheaf $\mathcal{F}, \mathcal{G} \in \mathrm{Sh}(\mathcal{C})$ we can define the *inner hom* $[\mathcal{F}, \mathcal{G}] \in \mathrm{Sh}(\mathcal{C})$ to be again a sheaf, where $[\mathcal{F}, \mathcal{G}](C) := \mathrm{Hom}_{\mathrm{PSh}(\mathcal{C})}(\mathcal{F} \times よ(C), \mathcal{G})$.

A physics field theory contain following thing:

1. a spacetime manifold Σ of dimensional d,
2. a smooth sheaf (stack) of fields configuration $\mathcal{F}$,
3. an action functional $S : [\Sigma, \mathcal{F}] \to \mathbb{R}$ or $U(1)$, where $[\Sigma, \mathcal{F}]$ is the field history,

(4.) usually the action is defined from Lagrangian $\mathcal{L} : \mathcal{F}|_\Sigma \to \Omega^d|_\Sigma \in \mathrm{Sh}(\Sigma)$, and

$$S : [\Sigma, \mathcal{F}] \xrightarrow{\mathcal{L}(\Sigma)} \Omega^d(\Sigma) \xrightarrow{\int_\Sigma} \mathbb{R}.$$

In particular, for every point $x \in \Sigma$ we have

$$\mathcal{L}(x) : \mathcal{F}_x \xrightarrow{\mathcal{L}_x} \Omega^d_x \xrightarrow{ev} \mathbb{R}$$

For example let $\phi_x \in \mathcal{F}_x$, we can define $\mathcal{L}(x)(\phi_x) = a_0\phi_x(x) + a_1\partial_i\phi_x(x) + \cdots$

So the core field theory is to study the geometry of field history space $[\Sigma, \mathcal{F}]$ pair with action functional $S \in C^\infty([\Sigma, \mathcal{F}])$. For example the classic solution space is $\mathrm{Sol}([\Sigma, \mathcal{F}]) = \{dS = 0\}$.

Example 4.3.1 (Mechanics and Sigma Model)**.** Let $\Sigma = \mathbb{R}$, we start the trajectory of a particle in a space X. Now the field history space $[\mathbb{R}, X]$. Let $q \in [\mathbb{R}, X]$, we define $\mathcal{L}(t)(q) := \mathcal{L}(q, \dot{q}, t)$ and $S : [\mathbb{R}, X] \to \mathbb{R}, q \mapsto \int_\mathbb{R} \mathcal{L}\mathrm{d}t$.

Similarly, if we take general Σ, we got the sigma model.

Example 4.3.2 (Vector Bundle)**.** Let $E \to \Sigma$ be a rank k vector bundle, we consider the space of section $\Gamma(\Sigma, E)$, let $\{U_i \to \Sigma\}_{i\in I}$ be a covering, and E is given by $\{g_{ij} \in \mathrm{GL}_k(C^\infty(U_{ij}))\}$, Then the section space is the limit:

$$\Gamma(\Sigma, E) = \mathrm{Lim}[U_\bullet, \mathbb{R}^k] = \left\{ \phi_i \in \prod_{i\in I} [U_i, \mathbb{R}^k] \,\middle|\, g_{ij} \cdot \phi_i|_{U_{ij}} = \phi_j|_{U_{ij}} \right\}$$

As we saw earlier, E can be classified by a morphism of stack $f_E : \Sigma \to \mathrm{BGL}_k$, and we have universal vector bundle $p_k : \mathcal{U}_k \to \mathrm{BGL}_k$, such that E is the pullback $f_E^*\mathcal{U}_k$. That is to say, the setion space can be also seen as collection of commutive diagram:

$$\Gamma(\Sigma, E) = \left\{ \begin{tikzcd} \Sigma \arrow[r] \arrow[dr, "f_E"'] & \mathcal{U}_k \arrow[d, "p_k"] \\ & \mathrm{BGL}_k \end{tikzcd} \right\}$$

To define the Lagrangian $\mathcal{L} : \Gamma(-, E) \to \Omega^d$, we can define $\mathcal{L}_i : [U_i, \mathbb{R}^k] \to \Omega^d(U_i)$ such that $g_{ij}^*\mathcal{L}_j = \mathcal{L}_i$. This is in general difficult to construct, since g_{ij}^* can be very complicated. For example $\partial_\mu : [U_i, \mathbb{R}^k] \to C^\infty(U_i), \phi \mapsto \partial_\mu\phi$, $g^*\partial_\mu\phi = \partial_\mu(g\phi) = (\mu, g\mathrm{d}\phi) + (\partial_\mu g)\phi$. The introducing of gauge field can help to define such morphism.

4.4 Gauge Theory

The word "gauge" used in Physics are usually associated with fields defined by an action stack $X /\!\!/ G$.

We begin with a gauge linear sigma model(GLSM). We can consider sigma model of following target space $\mathbb{CP}^n := \{(z_\mu) \in \mathbb{C}^{n+1} | \sum_{\mu=0}^{n} |z_\mu|^2 = 1\}/U(1)$. This identification can be seen as quotient of manifold, we can also interpret as sheaf, let $U_i \to \Sigma_{i\in I}$ be a good covering:

$$\mathbb{CP}^n(\Sigma) = \mathrm{Lim}\,\mathbb{CP}^n(U_\bullet) = \left\{ (\boldsymbol{\phi}_i) \in \prod_{i\in I} \mathrm{Map}(U_i, \{\sum_{\mu=0}^{n} |z_\mu|^2 = 1\}) \;\middle|\; \boldsymbol{\phi}_i|_{U_{ij}} = g_{ij} \cdot \boldsymbol{\phi}_j|_{U_{ij}}, g_{ij} \in U(1)(U_{ij}) \right\}$$

Let G be a Lie group, viewed as a smooth group, let $\Omega^1_{\mathfrak{g}}$ be the sheaf of Lie algebra value 1-form. Then G acts on $\Omega^1_{\mathfrak{g}}$ by $g \cdot A = \mathrm{d}gg^{-1} + \mathrm{ad}_g A$. We define an action stack $\mathrm{B}G_{conn} = \Omega^1_{\mathfrak{g}} /\!\!/ G$, this is fields configuation of *Yang-Mills Theory*. In the following, we just discuss the $U(1)$ case for simplicity, the action is simply $z \cdot A = \frac{\mathrm{d}z}{z} + A = \mathrm{dlog}(z) + A$. $\mathrm{B}U(1)_{conn} = \Omega^1 /\!\!/ U(1)$ is also called differential/Deligne cohomology.

Explicitly, let Σ be a manifold, and a good covering $\{U_i \to \Sigma\}_{i\in I}$, then by definition

$$\mathrm{Obj}(\mathrm{B}U(1)_{conn}(\Sigma)) := \left\{ (A_i) \in \prod_{i\in I} \Omega^1(U_i), g_{ij} \in C^\infty(U_{ij}, U(1)) \;\middle|\; g_{ij}g_{jk}|_{U_{ijk}} = g_{ik}|_{U_{ijk}}, \frac{\mathrm{d}g_{ij}}{g_{ij}} + A_i|_{U_{ij}} = A_j \right.$$

$$\mathrm{Hom}_{\mathrm{B}U(1)_{conn}(\Sigma)}((A_\bullet, g_{\bullet\bullet}), (B_\bullet, g'_{\bullet\bullet})) := \left\{ \varphi_i \in \prod_{i\in I} \mathrm{Hom}_{\mathcal{F}(U_i)}(A_i, B_i) \;\middle|\; \begin{array}{c} \begin{array}{ccc} A_i|_{U_{ij}} & \xrightarrow{\varphi_i} & B_i|_{U_{ij}} \\ {\scriptstyle +\mathrm{dlog}(g_{ij})}\downarrow & & \downarrow{\scriptstyle +\mathrm{dlog}(g'_{ij})} \\ A_j|_{U_{ij}} & \xrightarrow{\varphi_j} & B_j|_{U_{ij}} \end{array} \\ \\ \text{commutes} \end{array} \right\}$$

We can have two morphism of stack $p : \mathrm{B}U(1)_{conn} \to \mathrm{B}U(1)$ which forget the connection A_i, and curvature $F : \mathrm{B}U(1)_{conn} \to \Omega^2_{cl}$ by taking $\mathrm{d}A_i$. Moreover, by composing with $U(1) \cong \mathbb{R}/\mathbb{Z} \to \mathrm{B}\mathbb{Z}$, we can get a further morphism Chern class $\mathbf{c} : \mathrm{B}U(1)_{conn} \to \mathrm{B}U(1) \to \mathrm{B}^2\mathbb{Z}$. In fact $\pi_0\mathrm{B}^2\mathbb{Z}(M) = \mathrm{H}^2(M, \mathbb{Z})$, thus $\pi_0\mathbf{c} : \pi_0\mathrm{B}U(1)_{conn}(M) \to \mathrm{H}^2(M, \mathbb{Z})$ is the usual Chern class.

In fact $\mathrm{B}U(1)_{conn}$ have both geometric and topological information: the F extract the geometric data, gives the curvature form and $\mathbf{c}$ gives the topological twist of the bundle. We can link those two by the $\mathrm{B}^2\mathbb{R}_{disc}$ which classify real coefficient cohomology group $\mathrm{H}^2(M, \mathbb{R})$:

$$\begin{tikzcd}
 & \Omega^2_{cl} \arrow[dr, "\text{De Rham}"] & \\
\mathrm{B}U(1)_{conn} \arrow[ur, "F"] \arrow[dr, "\mathbf{c}"'] \arrow[rr, phantom, "\lrcorner" very near start] & & \mathrm{B}^2\mathbb{R}_{disc} \\
 & \mathrm{B}^2\mathbb{Z} \arrow[ur] &
\end{tikzcd}$$

We even get a pullback diagram, which means $\mathrm{B}U(1)_{conn}$ is gluing of geometric data Ω^2_{cl} and topological data $\mathrm{B}^2\mathbb{Z}$. That is why it is also called *differential cohomology.*

To define Lagrangian, we chose a manifold Σ of dimension d with metric g, and we find that $\mathcal{L} : \mathrm{B}U(1)_{conn}|_\Sigma \to \Omega^d|_\Sigma$ have to factor though $\pi_0 \mathrm{B}U(1)_{conn}$ since the latter is just a sheaf of set. In practice, we define $\mathcal{L} : \mathrm{B}U(1)_{conn}|_\Sigma \xrightarrow{F} \Omega^2_{cl}|_\Sigma \xrightarrow{F \wedge *_g F} \Omega^d|_\Sigma$. This gives us electromagnetic field theory.

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