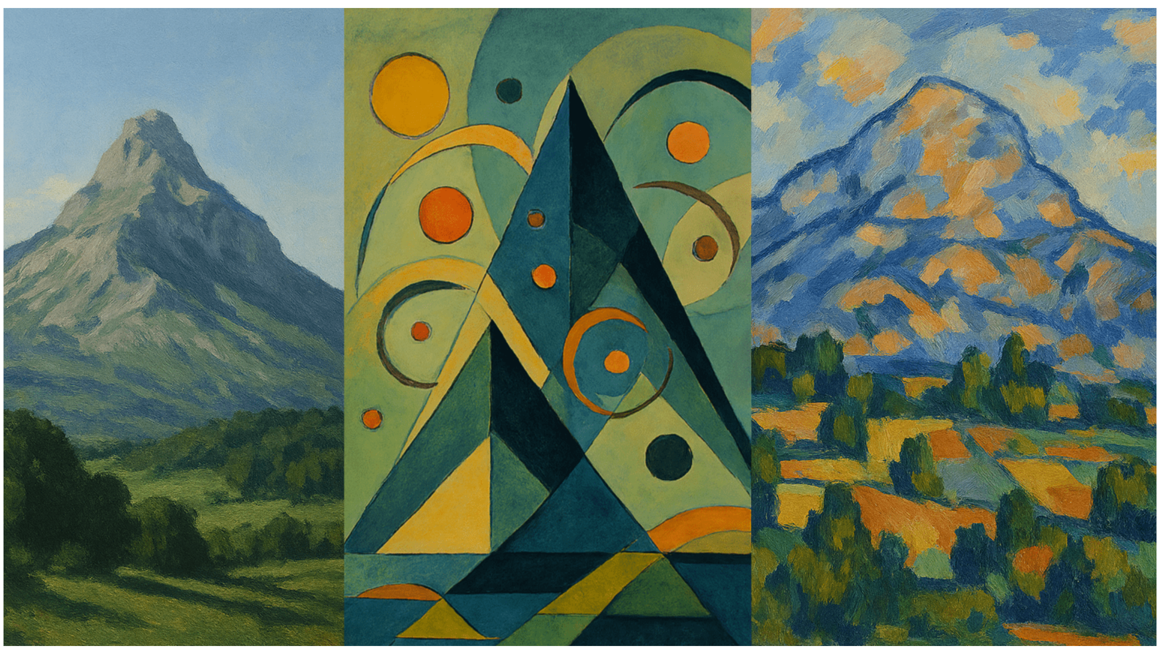
Seeing the Mountain

Keyao Peng
keyao.peng@ube.fr



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Seeing the mountain as the mountain;
Seeing the mountain as not a mountain;
Seeing the mountain as still a mountain.

QINGYUAN XINGSI

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Chapter 0

Introduction

0.1 What is Algebraic Geometry?

Algebraic geometry is the study of geometric spaces that locally arise as solutions to polynomial equations. This study can be approached at two levels:

- **Local:** This involves studying the geometric properties of solution sets to polynomial equations. Since we work not only over $\mathbb{R}$ or $\mathbb{C}$ but potentially over arbitrary fields or rings, we must construct a more intrinsic notion of geometry associated with algebra—namely, the spectrum of a ring. This leads to the study of affine varieties and affine schemes, allowing a translation between algebra and geometry.

- **Synthetic:** Analogous to how manifolds generalize open subsets of $\mathbb{R}^n$, we study spaces that are locally isomorphic to affine varieties. This broader perspective leads to the study of varieties and schemes through various formal frameworks.

This lecture is part of an algebraic geometry course that emphasizes the second, synthetic level. However, the underlying question—"How can we construct and classify generalized spaces from certain building blocks?"—is relevant across many branches of geometry. Thus, this part can stand alone as a form of *post-modern*[1] synthetic geometry.

[1] A commonly accepted definition is "after World War II."

0.2 Essentialism vs. Structuralism

Synthetic geometry formalizes geometric concepts through axioms that directly address fundamental entities—such as points and lines—rather than relying on a background space like Cartesian coordinates. In contrast to the analytic viewpoint, where every geometric object is composed of points, synthetic geometry treats lines, curves, and other entities as primitive. The focus shifts to the **relationships** between these objects, such as "a point lies on a line" or "two lines intersect."

This structuralism perspective emphasizes understanding objects through their interactions. To formalize these relationships, we use **category theory** and **sheaf theory**. Moreover, by viewing categories themselves as spaces, we uncover deeper geometric structures. This formalism, developed through algebraic geometry, now plays a central role in various fields: differential geometry, topology, quantum field theory, and beyond.

0.3 What is Geometry in Physics?

Quantum physics connects to both the local and synthetic levels of algebraic geometry:

- **Local:** The concept of the spectrum of a commutative ring originates from C^*-algebras and operator theory, both fundamental in quantum physics. The duality between algebra and geometry is already present in the Heisenberg and Schrödinger formulations.

- **Synthetic:** In quantum field theory, we study the space of field configurations (histories), which is infinite-dimensional and behaves irregularly. Nevertheless, we aim to treat it as a "manifold" to define metrics, path integrals, and differential forms. Synthetic geometry provides a rigorous framework for these constructions (e.g., via smooth sets). Gauge theory, supergeometry, and related topics can also be unified within this framework [1].

A concrete example is mirror symmetry, which links Gromov–Witten invariants with Hodge theory.

0.4 Course Plan

We will focus on three types of geometry and study them comparatively using synthetic methods:

- **Smooth Sets / Manifolds:** Built from open subsets of $\mathbb{R}^n$, these provide intuitive and geometric examples, serving as a bridge to classical geometry.
- **Simplicial Sets:** Constructed from simplices Δ, these offer the simplest and most abstract examples.
- **Schemes:** Built from spectra of commutative rings, these are the central objects of study in this course.

Chapter 1

Category

There are a lot of references for category theory, we follow the [2]

1.1 Definition and Examples

Definition 1.1.1 (Category)**.** A **category** $\mathcal{C}$ consists of:

- A class of objects $\mathrm{Ob}(\mathcal{C})$.
- For each pair A, B, a set of morphisms $\mathrm{Hom}_{\mathcal{C}}(A, B)$.
- A composition operation $\circ$ of morphisms and identity morphisms id_A for each object A.

Subject to:

- **Associativity:** $h \circ (g \circ f) = (h \circ g) \circ f$
- **Identity:** $\mathrm{id}_B \circ f = f = f \circ \mathrm{id}_A$

For simplicity, we just use $A \in \mathcal{C}$ to refer objects, and $f : A \to B$ to refer morphisms. There are varies way to think about what is a category, let's discuss by examples:

Example 1.1.2 (Concrete Category)**.** We can view it as a container of math structures and morphisms is the map perserve the structures.

- **Set:** Objects are sets, morphisms are maps between sets.
- **Grp:** Objects are groups, morphisms are group homomorphisms.
- **Vect:** Objects are vector spaces, morphisms are linear homomorphisms.
- **Top:** Objects are topological spaces, morphisms are continuous maps.

You can create infinite many of examples from the math structures you know.

At this point it seems useless to definition such abstraction. But we can try to genealize the notion from set theory. For example, let $f : A \to B$ be a morphism in $\mathcal{C}$.

- **Isomorphism:** f is an *isomorphism* if there exists a morphism $g : B \to A$ such that:
$$g \circ f = \mathrm{id}_A \quad \text{and} \quad f \circ g = \mathrm{id}_B$$
In this case, A and B are said to be *isomorphic*.
- **Monomorphism:** f is a *monomorphism* (or *mono*) if for all morphisms $g_1, g_2 : X \to A$, we have:
$$f \circ g_1 = f \circ g_2 \Rightarrow g_1 = g_2$$
That is, f is left-cancellable.
- **Epimorphism:** f is an *epimorphism* (or *epi*) if for all morphisms $h_1, h_2 : B \to Y$, we have:
$$h_1 \circ f = h_2 \circ f \Rightarrow h_1 = h_2$$
That is, f is right-cancellable.

Exercise 1.1.1. Verify in **Set** these definitions give bijection, injection and surjection.

Example 1.1.3 (classifying category)**.** The category can be viewed as an algebra structure which generalize "monoid", something like a group but not to ask elements to have inverse. In fact, for all $A \in \mathcal{C}$ endmorphism $\mathrm{Hom}_{\mathcal{C}}(A, A)$ is a monoid, and a monoid M can be associated with a **classifying category** BM with only one object $\bullet$ and the morphisms $\mathrm{Hom}_{BM}(\bullet, \bullet) = M$. In particular, for any group G, we have a **classifying space** BG, such that all morphisms are isomorphisms.

The reason we call BG is that we can view it as a space, we can regard the isomorphisms as paths in certin space:

Example 1.1.4 (Groupoid). A **groupoid** is a category such that all morphisms are isomorphisms. For groupoid $\mathcal{X}$, we can build a space **geometric realization** $|\mathcal{X}|$ as following:

- take the objects of $x \in \mathcal{X}$ as points;
- for each morphism $f : x \to y$, attach a segment from point x to point y;
- for each possible relation $f \circ g = h$, attach a triangle with edge f, g, h;
- for the higher dimensional cells...

We will make this process more precise when we introduce simplicial sets.

Exercise 1.1.2. First show that the connection component $\pi_0(|\mathcal{X}|, x)$ is bijective to the isomorphic class of $\mathcal{X}$. Notice that $\mathrm{Hom}_{\mathcal{X}}(x, x)$ is a group, show that the fundamental group $\pi_1(|\mathcal{X}|, x) \cong \mathrm{Hom}_{\mathcal{X}}(x, x)$

Conversely, for a topological space S, we can define the **fundamental groupoid** $\Pi_1(S)$: the objects are points of S, morphisms $\mathrm{Hom}_{\Pi_1(S)}(a, b)$ are homotopy class of path from a to b.

Exercise 1.1.3. What is the composition for $\Pi_1(S)$? Verify this is indeed a groupoid.

The examples above we see for a category, the morphisms can have a lot of structures. On other hand if we let the morphism be trivial, then we get the partial order set:

Example 1.1.5 (Poset). Let $(P, \leq)$ be a partially ordered set. We can regard P as a category as follows:

- **Objects:** Elements of P.
- **Morphisms:** For $x, y \in P$, there exists a unique morphism $f : x \to y$ if and only if $x \leq y$.
- **Composition:** If $x \leq y$ and $y \leq z$, then $x \leq z$, so the morphism $x \to z$ is the composition of $x \to y$ and $y \to z$.
- **Identity:** For each $x \in P$, the identity morphism $\mathrm{id}_x : x \to x$ corresponds to the reflexivity $x \leq x$.

This category is *thin*, meaning there is at most one morphism between any two objects.

An example we will use a lot is for a topological space S, let $(Op(S), \subseteq)$ be the poset of open subsets of S.

Notice that a set A can be viewed as a category in both way: as a trivial groupoid or a trivial poset.

$$\begin{tikzcd}
& Cateogry & \\
Groupoid \arrow[ur, hook] & & Poset \arrow[ul, hook'] \\
& Set \arrow[ul, hook'] \arrow[ur, hook] &
\end{tikzcd}$$

Remark 1.1.6. In the definition of category, if we allow the morphism $\mathrm{Hom}_{\mathcal{C}}(A, B)$ to be something more that a set (e.g. Abelian group, vector space, groupoid/topological space, category etc.) we get the **enriched category**. In fact, it is more natural to think $\mathrm{Hom}_{\mathcal{C}}(A, B)$ as the connected components of some space $\pi_0 \mathrm{Map}_{\mathcal{C}}(A, B)$.

This is a general principle of **Univalence Foundation** , The math structures should be a priori "space"(type), and we can recover the set version by taking the connected components. We will discuss how to recognize the a priori geometric structure later.

1.2 Functor and Presheaf

If we want to define map between to category, it is natural to ask respect the morphism:

Definition 1.2.1 (Functor)**.** Let $\mathcal{C}$ and $\mathcal{D}$ be categories. A **functor** $F : \mathcal{C} \to \mathcal{D}$ consists of:

- A function that assigns to each object $A \in \mathcal{C}$ an object $F(A) \in \mathcal{D}$.
- A function that assigns to each morphism $f : A \to B$ in $\mathcal{C}$ a morphism $F(f) : F(A) \to F(B)$ in $\mathcal{D}$.

Such that:

- **Preservation of identities:** $F(\mathrm{id}_A) = \mathrm{id}_{F(A)}$
- **Preservation of composition:** For all $f : A \to B$ and $g : B \to C$ in $\mathcal{C}$,

$$F(g \circ f) = F(g) \circ F(f)$$

Example 1.2.2. For concrete categories, we can have the functor with respect their structures

- **Forgetful Functor:**
$$U : \mathbf{Grp} \to \mathbf{Set}$$
assigns to each group its underlying set, and to each group homomorphism the same map viewed as a map of sets. Same for **Vect**, **Top**, etc.

- **Free Functor:**
$$Free : \mathbf{Set} \to \mathbf{Grp}$$
assigns to each set the free group it generates, and to each map of sets the free group homomorphism it generates.

  Exercise 1.2.1. Verify $Free$ is indeed a functor, then show that $\mathrm{Hom}_{\mathbf{Grp}}(Free(A), G) \cong \mathrm{Hom}_{\mathbf{Set}}(A, U(G))$. Then how to define $Free$ for **Vect**?

- **Discrete Functor:**
$$Disc : \mathbf{Set} \to \mathbf{Top}$$
assigns to each set the discrete topological space, and to each map of sets the same map viewed as a continuous map.

- **Connected components:**
$$\pi_0 : \mathbf{Top} \to \mathbf{Set}$$
assigns to each topological space its set of connected component.

  Exercise 1.2.2. Show that $\mathrm{Hom}_{\mathbf{Top}}(Disc(A), S) \cong \mathrm{Hom}_{\mathbf{Set}}(A, U(S))$ and $\mathrm{Hom}_{\mathbf{Top}}(S, Disc(A)) \cong \mathrm{Hom}_{\mathbf{Set}}(\pi_0 S, A)$

Exercise 1.2.3. For groups (more general monoids) G, H, show that there is a bijection between functors $F : BG \to BH$ and group(monoid) homomorphism $f : G \to H$.

Exercise 1.2.4. 1. For groupoid $\mathcal{X}, \mathcal{Y}$, show that a functor $F : \mathcal{X} \to \mathcal{Y}$ induce a continuous map of geometric realization $|F| : |\mathcal{X}| \to |\mathcal{Y}|$. Now let **Grpd** be the category of groupoid, with functor as morphism, then show that geometric realization $|\cdot| : \mathbf{Grpd} \to \mathbf{Top}$ is actually a functor.

2. Similarly, show that $\Pi_1 : \mathbf{Top} \to \mathbf{Grpd}$ is also a functor.

3. * Moreover, show that $\mathrm{Hom}_{\mathbf{Top}}(S, |\mathcal{X}|) \cong \mathrm{Hom}_{\mathbf{Grpd}}(\Pi_1(S), \mathcal{X})$

- **Power Set Functor:**

$$\mathcal{P} : \mathbf{Set} \to \mathbf{Set}$$

assigns to each set X its power set $\mathcal{P}(X)$, and to each function $f : X \to Y$ the function $\mathcal{P}(f) : \mathcal{P}(X) \to \mathcal{P}(Y)$ defined by:

$$\mathcal{P}(f)(A) = f(A) = \{f(a) \mid a \in A\}$$

- **Hom Functor:** For a fixed object A in a category $\mathcal{C}$, the functor

$$\mathrm{Hom}_{\mathcal{C}}(A, -) : \mathcal{C} \to \mathbf{Set}$$

assigns to each object B the set of morphisms $\mathrm{Hom}_{\mathcal{C}}(A, B)$, and to each morphism $f : B \to C$ the function:

$$\mathrm{Hom}_{\mathcal{C}}(A, f) : \mathrm{Hom}_{\mathcal{C}}(A, B) \to \mathrm{Hom}_{\mathcal{C}}(A, C), \quad g \mapsto f \circ g$$

1.3 Yoneda Lemma

1.4 Universal Property and (Co)limit

Bibliography

[1] nLab authors. geometry of physics. https://ncatlab.org/nlab/show/geometry+of+physics, September 2025. Revision 201.

[2] nLab authors. geometry of physics – categories and toposes. https://ncatlab.org/nlab/show/geometry+of+physics+--+categories+and+toposes, September 2025. Revision 128.