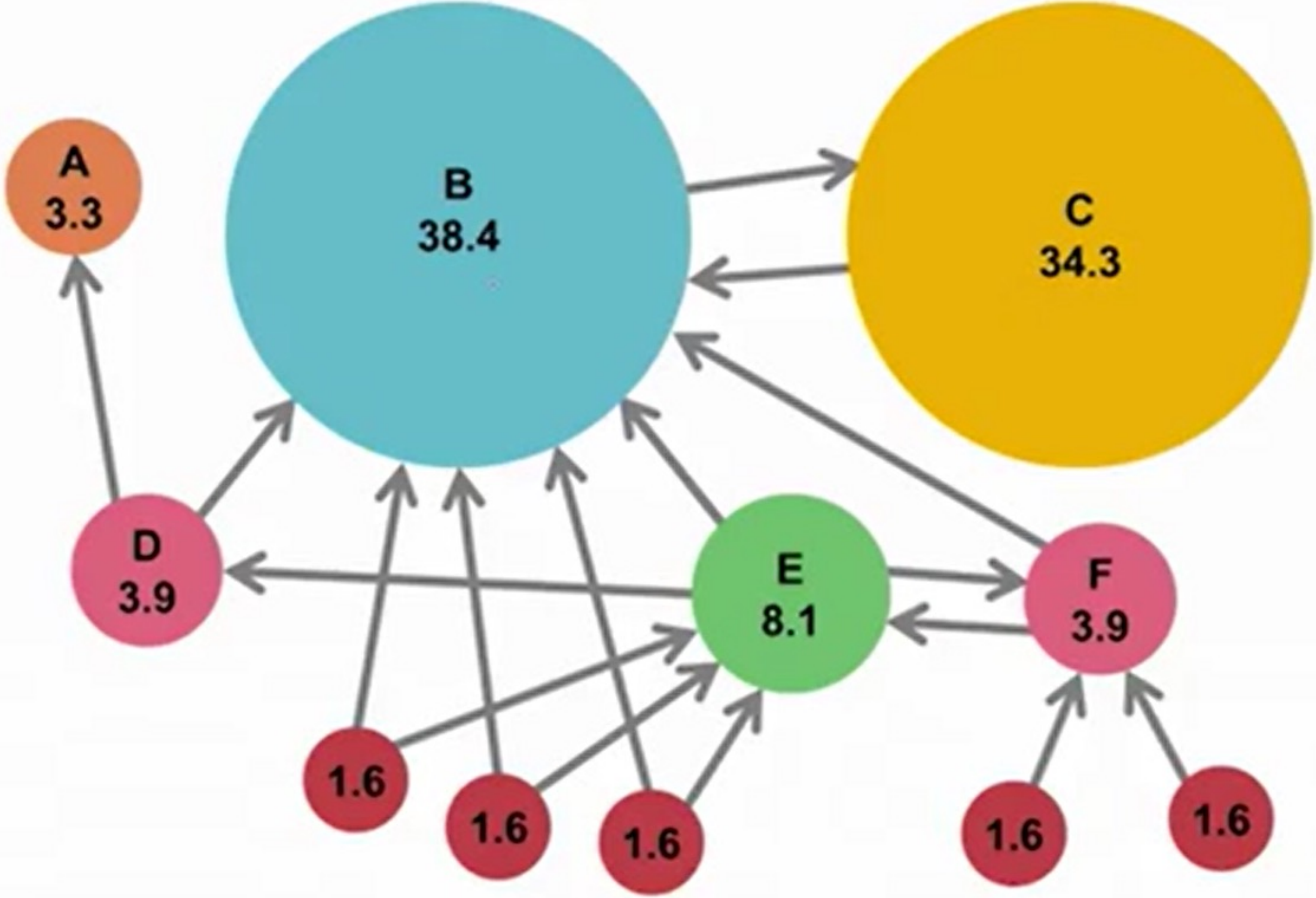
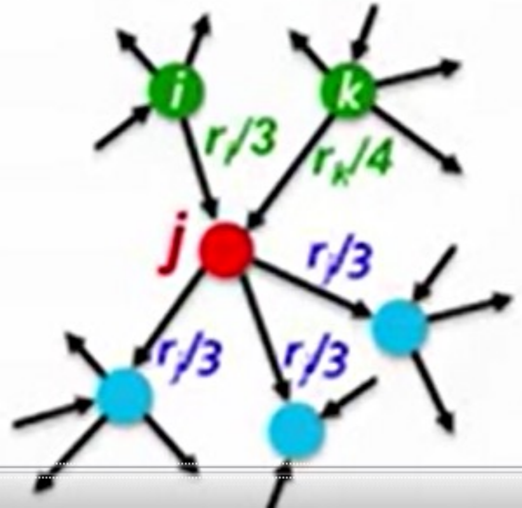
Page Rank


A
3.3
B
38.4
C
34.3
D
3.9
E
8.1
F
3.9
1.6
1.6
1.6
1.6
1.6

Simple Recursive Formulation

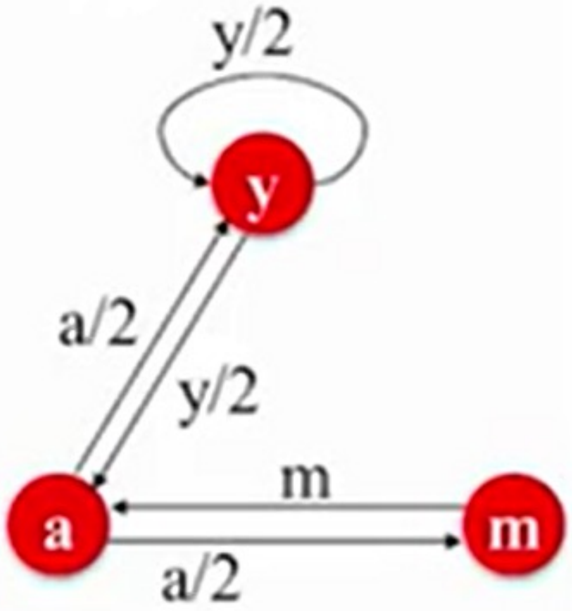
- Each link's vote is proportional to the **importance** of its source page
- If page j with importance r_j has n out-links, each link gets r_j / n votes
- Page j's own importance is the sum of the votes on its in-links

$$r_j = r_i/3 + r_k/4$$



PageRank: The "Flow" Model

- A "vote" from an important page is worth more
- A page is important if it is pointed to by other important pages
- Define a "rank" r_j for page j



$$r_j = \sum_{i \to j} \frac{r_i}{d_i}$$

d_i … out-degree of node i

What is the flow equation?

- **3 equations, 3 unknowns, no constants**
  - No unique solution
  - All solutions equivalent modulo the scale factor
- **Additional constraint forces uniqueeness:**
  - $r_y + r_a + r_m = 1$
  - Solution: $r_y = \frac{2}{5},\ r_a = \frac{2}{5},\ r_m = \frac{1}{5}$
- **Gaussian elimination method works for small examples, but we need a better method for large web-size graphs**
- **We need a new formulation!**

Flow equations:

$$r_y = r_y/2 + r_a/2$$

$$r_a = r_y/2 + r_m$$

$$r_m = r_a/2$$

PageRank: Matrix Formulation

- **Stochastic adjacency matrix M**
  - Let page i has d_i out-links
  - If $i \to j$, then $M_{ji} = \frac{1}{d_i}$ else $M_{ji} = 0$
    - M is a **column stochastic matrix**
      - Columns sum to **1**

- **Rank vector r:** vector with an entry per page
  - r_i is the importance score of page i
  - $\sum_i r_i = 1$
- **The flow equations can be written**

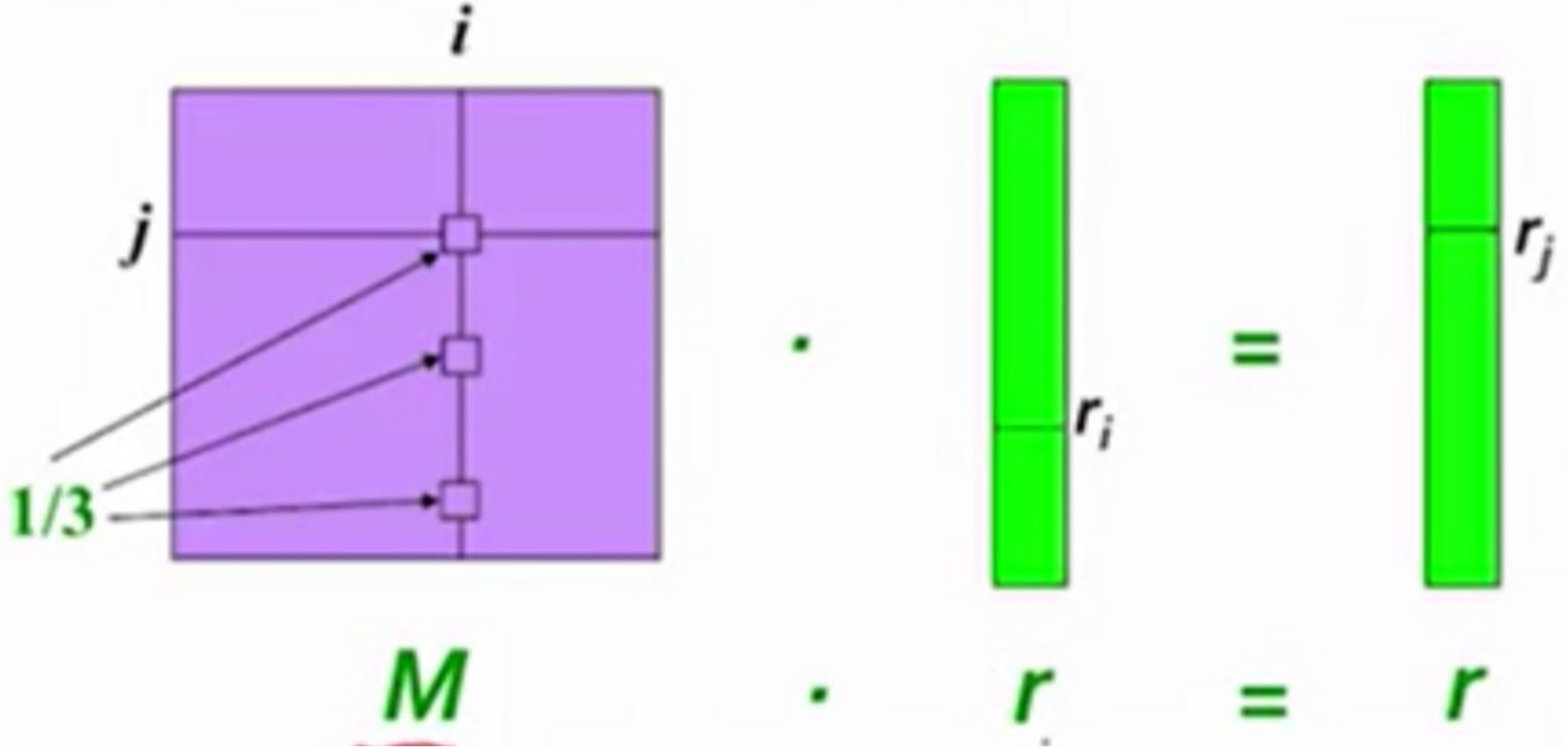
$$r_j = \sum_{i \rightarrow j} \frac{r_i}{d_i}$$

$$\boldsymbol{r} = \boldsymbol{M} \cdot \boldsymbol{r}$$

- Remember the flow equation: $r_j = \sum_{i \to j} \frac{r_i}{\mathrm{d_i}}$
- Flow equation in the matrix form

$$M \cdot r = r$$

  - Suppose page *i* links to 3 pages, including *j*



Eigenvector Formulation

- **The flow equations can be written**

$$r = M \cdot r$$

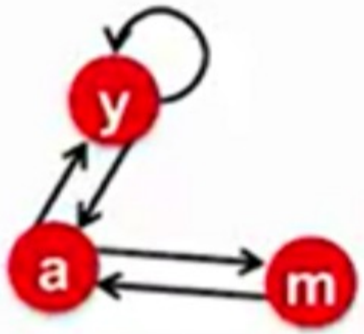
  - So the **rank vector** *r* is an **eigenvector** of the stochastic web matrix *M*
    - In fact, its first or principal eigenvector, with corresponding eigenvalue *1*
      - Largest eigenvalue of *M* is **1** since *M* is column stochastic
        - **Why?** *We know* **r** *is a stochastic vector and each column of* **M** *sums to one, so* $\mathbf{Mr} \le 1$

NOTE: *x* is an eigenvector with the corresponding eigenvalue **λ** if:

$$Ax = \lambda x$$

- **We can now efficiently solve for *r*!**
  The method is called Power iteration

Flow equations and Matrix



$$M = \begin{array}{c|ccc} & y & a & m \\ \hline y & \frac{1}{2} & \frac{1}{2} & 0 \\ a & \frac{1}{2} & 0 & 1 \\ m & 0 & \frac{1}{2} & 0 \end{array}$$

$$r = M \cdot r$$

$$r_y = r_y/2 + r_a/2$$
$$r_a = r_y/2 + r_m$$
$$r_m = r_a/2$$

$$\begin{bmatrix} y \\ a \\ m \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & 1 \\ 0 & \frac{1}{2} & 0 \end{bmatrix} \begin{bmatrix} y \\ a \\ m \end{bmatrix}$$

Power Iteration Method

- **Given a web graph with *N* nodes, where the nodes are pages and edges are hyperlinks**
- **Power iteration:** a simple iterative scheme
  - Suppose there are ***N*** web pages
  - Initialize: $\mathbf{r}^{(0)} = [1/N, \ldots, 1/N]^T$
  - Iterate: $\mathbf{r}^{(t+1)} = \mathbf{M} \cdot \mathbf{r}^{(t)}$
  - Stop when $|\mathbf{r}^{(t+1)} - \mathbf{r}^{(t)}|_1 < \varepsilon$

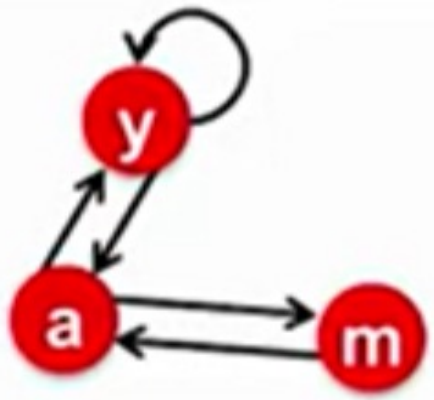
$$r_j^{(t+1)} = \sum_{i \to j} \frac{r_i^{(t)}}{d_i}$$

d_i …. out-degree of node i

$|\mathbf{x}|_1 = \sum_{1 \le i \le N} |x_i|$ is the $\mathbf{L_1}$ norm
Can use any other vector norm, e.g., Euclidean

PageRank: How to solve?

- **Power Iteration:**
  - Set $r_j = 1/N$
  - **1:** $r'_j = \sum_{i \to j} \frac{r_i}{d_i}$
  - **2:** $r = r'$
  - If not converged: goto **1**



	y	a	m
y	½	½	0
a	½	0	1
m	0	½	0

$r_y = r_y/2 + r_a/2$

$r_a = r_y/2 + r_m$

$r_m = r_a/2$

- **Example:**

$$\begin{bmatrix} r_y \\ r_a \\ r_m \end{bmatrix} = \begin{matrix} 1/3 & 1/3 & 5/12 & 9/24 & & 6/15 \\ 1/3 & 3/6 & 1/3 & 11/24 & \ldots & 6/15 \\ 1/3 & 1/6 & 3/12 & 1/6 & & 3/15 \end{matrix}$$

Iteration 0, 1, 2, …