Assignment 2

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1. Recall the definition of the L1-norm. $\|x\|_1 = \Sigma_i^n |x_i|$. By showing that it ~~st~~ satisfies each of the conditions in the definition of a norm prove this is a vector norm. First do this for $\mathbb{R}^2$, and then do this for $\mathbb{R}^n$

Soln L1 norm is one of the popular norm in $\mathbb{R}$. It is ~~sev~~ simply calculate by taking ~~sub~~ sum of the absolute value of the given point.

i.e;

$L1 = \|x\|_1 = |x|$ in $\mathbb{R}$.

To show L1 is vector norm it should satisfy following property.

Given a mapping $\|\cdot\| : \mathbb{R}^2 \to \mathbb{R}$

N1: positivity/non-negativity :-

i.e;

$L1 = \|x\|_1 = |x| \in \mathbb{R} \ \forall . x \in \mathbb{R}^2$

L1 norm defined the absolute length of vector so it is always positive.

N2:

$L1 = \|x\|_1 = |x| = 0 \in \mathbb{R}$ iff $x = 0 \ \forall x \in \mathbb{R}^2$

It can gives zero value if given point in vector is zero.

N3: positive scalability :-

Let

$L1 = \|x\|_1 = \|ax\|_1 = |a| \|x\|_1 = |a||x|$ ∈

we already know that $|x| \geq 0$ always and

absolute value of any Scalar is always positive.
So $|a||x| > 0 \in \mathbb{R} \; \forall \; x \in \mathbb{R}^2$

N4:- Triangle inequality:-

Let $x = (x_1, y_1), y = (x_2, y_2) \in \mathbb{R}^2$

Now, $x+y \in \mathbb{R}^2$

L_1: $\|x+y\|_1 = |x+y| \leq |x| + |y| = \|x\|_1 + \|y\|_1$

$\therefore \|x+y\|_1 \leq \|x\|_1 + \|y_1\|$

Hence L_1 norm satisfies all the properties of vector norm so it is a vector norm in $\mathbb{R}^2$

Similarly, let us assume the mapping

$\|\cdot\| : \mathbb{R}^n \to \mathbb{R}$

N1: positivity / non-negativity:-

$L_1 = \|xi\|_1 = \sum_{i=1}^{n} |xi| \geq 0 \;\; \forall \; xi \in \mathbb{R}^n$

Since, ~~abs~~ Sum of absolute value of any number is always positive.

N2:

$L_1 = \|xi\|_1 = \sum_{i=1}^{n} |xi| = 0 \;\; \text{iff} \;\; xi = 0$
$\forall \; xi = (0, 0 \ldots 0) \in \mathbb{R}^n$

N3: Positive Scalability:-

Let

$L_1 = \|axi\|_1 = \|axi\| = |a| \|xi\|$
$= |a| |xi| \;\; \forall \; xi \in \mathbb{R}^n$

We already verify that $|x_i|>0$ also, absolute value of any scalar is always positive. So product of two positive number is always positive.

N4: Triangle inequality:-

Let, $x=(x_1, x_2 \cdots x_n)$, $y=(y_1, y_2 \cdots y_n) \in \mathbb{R}^n$
also, $x+y \in \mathbb{R}^n$

L1: $\|x+y\|_1 = |x+y| < |x|+|y| = \|x\|_1+\|y\|_1$
$\therefore \|x+y\|_1 \leq \|x\|_1+\|y\|_1$

Hence, L1 norm satisfies all the properties of vector norm so it is a vector norm in $\mathbb{R}^n$.

[Q.] Recall the defination of L_∞ norm $\|x\|_\infty = \max\limits_{i \in \{1, \cdots n\}} |x_i|$. By showing that it satisfies each of the conditions in the defination of a norm prove this is a vector norm. first do for $\mathbb{R}^2$ and then do this for $\mathbb{R}^n$.

Solution: L_∞ norm is one of the popular norm of $\mathbb{R}$. It gives the largest magnitude among each element of a vector.
i.e

$$L_\infty = \|x\|_\infty = \max_{i \in \{1,2 \cdots n\}} |x_i| = \max(|x_1|, |x_2| \cdots |x_i|) \quad \forall x \in \mathbb{R}^n.$$

To show L_∞ is a vector norm under a mapping

$\|\circ\|: \mathbb{R}^2 \to \mathbb{R}$ it should satisfy each of the following properties.

N1. positivity / non-negativity :-

Let $\forall x = (x_1, y_1) \in \mathbb{R}^2$

$$L_\infty = \max_{i \in (1,2)} |x_i| = x > 0$$

N2:

$$L_\infty = \max_{i \in (1,2)} |x_i| = 0 \quad \text{Iff} \quad x = (x_1, y_1) \in \mathbb{R}^2 = 0$$

N3: positive Scalability :-

$$L_\infty = \|x\|_\infty = \|ax\|_\infty = \max_{i \in (1,2)} |a x_i|$$

$$= ax \in \mathbb{R}$$

~~$\forall x = (x_1, x_2) \in \mathbb{R}^2$~~

$\forall x = (x_1, x_2) \in \mathbb{R}^2$

N4: Trangle inequality :-

Let $x = (x_1, x_2)$, $y = (y_1, y_2) \in \mathbb{R}^2$

also, $(x+y) \in \mathbb{R}^2$

$$L_\infty = \|x+y\|_\infty = \max_{i \in (1,2)} |x+y|$$

$$= x + y$$

$$= \|x\|_\infty + \|y\|_\infty$$

Hence, L_∞ norm satisfy all the required properties of vector norm so, it is a vector norm in $\mathbb{R}^2$

Similarly, Let us suppose a mapping $\|\circ\| : \mathbb{R}^n \to \mathbb{R}$
s.t $\forall\ x = (x_1, x_2, x_3 \cdots x_n)$ and $y = (y_1, y_2 \cdots y_n) \in \mathbb{R}^n$

N1:

$$L_\infty = \max_{i \in \{1,2 \cdots n\}} |x_i| = x > 0$$

N2:

$$L_\infty = \max_{i \in \{1,2 \cdots n\}} |x_i| = 0 \text{ iff } x = 0$$

where $x = (0, 0 \cdots 0) \in \mathbb{R}^n$

N3:

Let $x \in \mathbb{R}^n$ so $ax \in \mathbb{R}^n$

Now,

$$L_\infty = \|x\|_\infty = \|ax\|_\infty$$

$$= \max_{i \in \{1,2 \cdots n\}} |ax| = ax \in \mathbb{R}$$

N4:

Let $x = (x_1, x_2 \cdots x_n)$, $y = (y_1, y_2 \cdots y_n) \in \mathbb{R}^n$
So $x + y \in \mathbb{R}^n$

Now, $L_\infty = \|x+y\|_\infty = \max_{i \in \{1,2 \cdots n\}} |x+y|$

$$= |x+y| \in \mathbb{R}$$

$$= \|x\|_\infty + \|y_\infty\|$$

Hence, L_∞ norm satisfies all the properties of vector norm in $\mathbb{R}^n$ so we can say it is vector norm.

[3] Let $n=3$ and Let $x \in \mathbb{R}^n$ be a vector with $x_i = i^{-1}$. By hand, Compute 1-norm, the 2nd norm and the ∞-norm. Do the same for the all-ones vector, i.e. $x_i \in \mathbb{R}^n$, where $x_i = 1$ for $i \in \{1, \dots n\}$

Soln:- Here $n=3$ and $x \in \mathbb{R}^n$ with

$$x_i = i^{-1} = \left(\frac{1}{i}\right) = \left(\frac{1}{1}, \frac{1}{2}, \frac{1}{3}\right) \in \mathbb{R}^3$$

Now

$$L_1 = \|x_i\|_1 = \sum_{i=1}^{3} |x_i| = |1| + \left|\frac{1}{2}\right| + \left|\frac{1}{3}\right|$$

$$= 1.833$$

$$L_2 = \|x_i\|_2 = \sum_{i=1}^{3} \|x_i^2\|^{1/2}$$

$$= \left[1 + \frac{1}{4} + \frac{1}{9}\right]^{1/2} = \sqrt{1.361}$$

$$= 1.16$$

$$L_\infty = \max_{i \in \{1,2,3\}} |x_i| = \max_{i \in \{1,2,3\}} \left\{1, \frac{1}{4}, \frac{1}{9}\right\} = 1$$

$\therefore L_\infty \leq L_2 \leq L_1$

Similarly,

$n=3$ and $x \in \mathbb{R}^n$ with

$x_i = 1 = (1,1,1) \in \mathbb{R}^3$

$L1: \|x\|_1 = \sum_{i=1}^{3} |x_i| = |1| + |1| + |1| = 3$

$L2: \|x\|_2 = \left(\sum_{i=1}^{3} \|x_i^2\|\right)^{1/2} = \left(1 + \frac{1}{1} + \frac{1}{1}\right)^{1/2}$

$= 1.73$

$L\infty: \|x\|_\infty = \max_{i \in \{1,2,3\}} |x_i| = 1$

$\therefore \; L\infty \leq L_2 \leq L_1$

[4] ⓐ Let $n = 10$, and Let $x \in \mathbb{R}^n$ be a vector with $x_i = i^{-1}$. By hand, Compute the 1-norm, the 2-norm and the ∞-norm. Do the Same for the

[4] ⓐ Let $n = 10$, and Let $x \in \mathbb{R}^n$ be a vector with $x_i = i^{-1}$. In R, Compute the 1-norm, the 2-norm and the 1-norm of this vector. Confirm that you obtain the same results as when you Compute these norms by hand.

Soln By hand Solution,

Here $n = 10$ and $x \in \mathbb{R}^n$ with vector $x = i^{-1}$

i.e $x = i^{-1} = \left(\frac{1}{i}\right) = \left(\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \frac{1}{7}, \frac{1}{8}, \frac{1}{9}, \frac{1}{10}\right) \in \mathbb{R}^{10}$

Now,

$L1 = \|x_i\|_1 = \sum_{i=1}^{10} |x_i|$

$= \left[\left|\frac{1}{1}\right| + \left|\frac{1}{2}\right| + \left|\frac{1}{3}\right| + \cdots \left|\frac{1}{10}\right|\right]$

$= 2.92$

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$$L_2 = ||x||_2 = \sum_{i=1}^{10} |x_i^2|^{1/2}$$

$$= \left[1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \cdots \frac{1}{100}\right]^{1/2}$$

$$= \sqrt{1.54} = 1.24$$

$$L_\infty = \max_{i \in \{1,2..10\}} |x_i| = \max_{i \in \{1,2..10\}} |1, 1/2, 1/3, \cdots \frac{1}{10}|$$

$$= 1$$

$$\therefore L_\infty \le L_2 \le L_1$$

(b) In R, Compute all three norms for the all-ones vector in $\mathbb{R}^n$, i.e where $x_i = 1$, for $i \in \{1, \cdots n\}$, where again $n = 10$. Again, confirm that you obtain the same results as when you compute norms by hand.

Soln By hand.

Here $n=10$ and $x \in \mathbb{R}^n$ with $x_i = 1$

i.e $x = (1,1,1,1,1,1,1,1,1,1) \in \mathbb{R}^{10}$

Now,

$$L_1 = ||x_i||_1 = \sum_{i=1}^{10} |x_i| = 10$$

$$L_2 = ||x_i||_2 = \sum_{i=1}^{10} |x_i^2|^{1/2} = \sqrt{10} = 3.16$$

$$L_\infty = \max_{i \in \{1,2..10\}} |x_i| = 1$$

$\therefore L_\infty \le L_2 \le L_1$.

(c) In R, do the same for the vector the vector $x \in \mathbb{R}^{10}$, where each x_i is a random number from $[0, 1)$.