EIGENVALUES AND EIGENVECTORS

[1] Is λ an eigenvalue of A? why or why not? If so find the corresponding eigenvector.

(a) $A = \begin{pmatrix} 3 & 2 \\ 3 & 8 \end{pmatrix}$, $\lambda = 2$

To check wether $\lambda = 2$ is eigenvalue of A or not we have characteristic equation

$$|A - \lambda I| = 0$$

$$\Rightarrow \left| \begin{bmatrix} 3 & 2 \\ 3 & 8 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \right| = 0$$

$$\Rightarrow \begin{vmatrix} 3-\lambda & -2 \\ 3 & 8-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (3-\lambda)(8-\lambda) - 6 = 0$$

$$\Rightarrow 24 - 3\lambda - 8\lambda + \lambda^2 - 6 = 0$$

$$\Rightarrow \lambda^2 - 11\lambda + 18 = 0$$

$$\Rightarrow \lambda^2 - 9\lambda - 2\lambda + 18 = 0$$

$$\Rightarrow \lambda(\lambda - 9) - 2(\lambda - 9) = 0$$

$$\Rightarrow (\lambda - 2)(\lambda - 9) = 0$$

$$\Rightarrow \lambda = 2, \lambda = 9$$

$\lambda = 2$ satisfies the characteristic equation. Hence it is eigenvalue for matrix A.

For eigen vector

Let $v = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ be eigenvector corresponding to eigenvalue $\lambda = 2$

Then from eigen equation,

$$[A - \lambda I]x = 0$$

$$= \left\{ \begin{bmatrix} 3 & 2 \\ 3 & 8 \end{bmatrix} - \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \right\} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

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$$\Rightarrow \begin{bmatrix} 1 & 2 \\ 3 & -6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\Rightarrow$ $x_1 + 2x_2 = 0$

$3x_1 - 6x_2 = 0$

Hence $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

Again, let $u = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ be eigenvector corresponding to $d = 9$

from eigenequation,

$[A - dI]y = 0$

$$\Rightarrow \left\{\begin{bmatrix} 3 & 2 \\ 3 & 8 \end{bmatrix} - \begin{bmatrix} 9 & 0 \\ 0 & 9 \end{bmatrix}\right\}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -6 & 2 \\ 3 & -1 \end{bmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$\Rightarrow$ $-6y_1 + 2y_2 = 0$

$3y_1 - y_2 = 0$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

$\therefore$ eigen vectors are $v = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$ and $u = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ corresponding to eigen values $d = 2$ and $d = 9$.

(b) $A = \begin{bmatrix} 7 & 3 \\ 3 & -1 \end{bmatrix}$, $d = -2$

Characteristic equation is

$|A - dI| = 0$

$$\text{or, } \left|\begin{bmatrix} 7 & 3 \\ 3 & -1 \end{bmatrix} - \begin{bmatrix} d & 0 \\ 0 & d \end{bmatrix}\right| = 0$$

$$\text{or, } \begin{vmatrix} 7-d & 3 \\ 3 & -1-d \end{vmatrix} = 0$$

or, $(7-\lambda)(-1-\lambda) = 0$

or, $\lambda = 7, \lambda = -1$

Hence $\lambda = -2$ is not root of eigen equation. Hence it is not eigen value for matrix A.

[2] Is x an eigen vector of A if so, Find the Corresponding eigen value.

a) $A = \begin{bmatrix} -3 & 1 \\ -3 & 8 \end{bmatrix}, \quad x = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$

=) If x is eigen vector of A then we can express it in terms of eigen equation as follows,

$AX = \lambda x \quad — (*)$

L.H.S

$$\begin{bmatrix} -3 & 1 \\ -3 & 8 \end{bmatrix}\begin{bmatrix} 1 \\ 4 \end{bmatrix} = \begin{bmatrix} -3+4 \\ -3+32 \end{bmatrix} = \begin{bmatrix} 1 \\ 29 \end{bmatrix} \neq \lambda \begin{bmatrix} 1 \\ 4 \end{bmatrix}$$

$\therefore \quad x = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$ is not eigenvector for $A = \begin{bmatrix} -3 & 1 \\ -3 & 8 \end{bmatrix}$

b) $A = \begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix} \quad x = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

If x is eigenvector of A then we can express it in term of eigenequation as,

$Ax = \lambda x \quad — (*)$

L.H.S

$$\begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -4+2 \\ -2+4 \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \end{bmatrix} = 2\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \lambda \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$= \lambda x = R.H.S$

Hence eigenvalue is $\lambda = 2$.

[8] Find a basis for the eigenspace corresponding to each listed eigenvalue.

a) $A = \begin{bmatrix} 9 & 0 \\ 2 & 3 \end{bmatrix}$ $\lambda = 3, 9$

Let $u = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ be the eigenvector corresponding to $\lambda = 3$

From eigen equation,

$[A - \lambda I]x = 0$

$\Rightarrow \left\{ \begin{bmatrix} 9 & 0 \\ 2 & 3 \end{bmatrix} - \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} \right\} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\Rightarrow \left\{ \begin{bmatrix} 6 & 0 \\ 2 & 0 \end{bmatrix} \right\} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\Rightarrow$ $6x_1 + 0 \cdot x_2 = 0$

$2x_1 + 0 \cdot x_2 = 0$

$\therefore u = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Here $x_2 = 1$ is because eigenvector never be zero vector. Hence, multiple of any real number and zero is always zero. In order to ~~take~~ make eigenvector non-zero is chosed $x_2 = 1$.

Again Let $v = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ be the eigenvector corresponding eigenvalue $\lambda = 9$,

Then, we have eigen equation,

$[A - \lambda I]v = 0$

$\Rightarrow \left[\begin{bmatrix} 9 & 0 \\ 2 & 3 \end{bmatrix} - \begin{bmatrix} 9 & 0 \\ 0 & 9 \end{bmatrix} \right] \cdot \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} 0 & 0 \\ 2 & -6 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

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Corresponding equations are

$2y_1 - 6y_2 = 0$

$y_1 - 3y_2 = 0$

$\therefore V = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

Hence two vectors are $u = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, V = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

To find the basis for eigenspace we need to ~~do~~ show

(i) u and v are linearly Independent

(ii) u and v span every vectors of eigenspace.

To show u and v are linearly Independent,

Let

$a_1 u + a_2 V = 0 \quad \forall a_1, a_2 \in \mathbb{R}$

$\Rightarrow a_1(0,1) + a_2(3,1) = (0,0)$

$\Rightarrow (0 \cdot a_1 + 3 \cdot a_2, a_1 + a_2) = (0,0)$

$\Rightarrow 0 \cdot a_1 + 3 \cdot a_2 = 0 \quad -(*)$

$a_1 + a_2 = 0 \quad -(**)$

~~from~~ By solving eq^n $(*)$ and $(**)$ we get $a_1 = 0, a_2 = 0$

Hence u and v are linearly Independent.

By Theorem "Any linearly independent set of n elements of V must span V and thus form a basis".

Hence basis for eigenspace is $B = \left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \end{bmatrix} \right\}$

(b) $A = \begin{bmatrix} 14 & -4 \\ 16 & -2 \end{bmatrix} \quad \lambda = 6$

Let $v = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ be eigenvector corresponding to eigenvalue $\lambda = 6$

From eigenequation,

$[A - \lambda I] x = 0$

$$\Rightarrow \left\{ \begin{bmatrix} 14 & -4 \\ 16 & -2 \end{bmatrix} - \begin{bmatrix} 6 & 0 \\ 0 & 6 \end{bmatrix} \right\} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 14-6 & -4 \\ 16 & -2-6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 8 & -4 \\ 16 & -8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\Rightarrow 8x_1 - 4x_2 = 0$ $\quad \therefore V = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$16x_1 - 8x_2 = 0$.

To find basis for eigenspace:

(9) A set containing only one vector, say V is linearly independent if and only if V is not the zero vectors. Hence our vectors $V = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. Hence it is linearly independent

also from theorem " Any linearly independent set of n elements of V must span V and thus form a basis.

Hence basis for eigenspace is $B = \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$

(c) $A = \begin{bmatrix} 4 & -2 \\ -3 & 9 \end{bmatrix}, \lambda = 10$

Let $V = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ be eigenvector of A corresponding to eigenvalue $\lambda = 10$,

From eigen equation,

$$|A - \lambda I| V = 0$$

$$\Rightarrow [A - \lambda I] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$\Rightarrow \left\{ \begin{bmatrix} 4 & -2 \\ -3 & 9 \end{bmatrix} - \begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix} \right\} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -6 & -2 \\ -3 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Corresponding equations are,

$-6x_1 - 2x_2 = 0$

$-3x_1 - x_2 = 0$

$\therefore v = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$

A set containing only one vector, say v is linearly independent if and only if v is not the zero vectors. Hence vector $v = \begin{bmatrix} -1 \\ 3 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. Hence it is linearly independent.

Also from theorem "Any linearly independent set of n elements of V must span V and thus form a basis. Hence basis for eigenspace is $B = \left\{ \begin{bmatrix} -1 \\ 3 \end{bmatrix} \right\}$.

(d) $A = \begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix}$, $\lambda = -2, 5$

Let $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ be the eigenvector of matrix A corresponding to $\lambda = -2$,

Then from eigen equation,

$[A - \lambda I]X = 0$

$$\Rightarrow \left\{ \begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix} - \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix} \right\} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 3 & 4 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Corresponding equations are,

$3x_1 + 4x_2 = 0$

$3x_1 + 4x_2 = 0$

$\therefore X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -4 \\ 3 \end{bmatrix}$

When $\lambda = 5$

Then,

$$[A - \lambda I] X = 0$$

$$\Rightarrow \left\{ \begin{bmatrix} -4 & 4 \\ 3 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right\} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow -4x_1 + 4x_2 = 0 \quad \text{so } X' = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$3x_1 - 3x_2 = 0$$

To find basis for eigen space we need to show X and X' are linearly independent and every vector from eigen space belongs to span(X, X').

Hence

(i) Let us write $X = \begin{bmatrix} -4 \\ 3 \end{bmatrix}$ and $X' = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ as linear combination as follows,

$$a_1 X + a_2 X' = 0$$

$$\Rightarrow a_1 \begin{bmatrix} -4 \\ 3 \end{bmatrix} + a_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow -4a_1 + a_2 = 0 \quad -(*)$$

$$3a_1 + a_2 = 0 \quad -(*)(*)$$

By solving eqn $(*)$ and $(*)(*)$ we get,

$$a_1 = 0, a_2 = 0$$

Hence vectors X and X' are linearly independent. Also from theorem " Any linearly Independent set of n elements of V must span V and thus form a basis. Hence basis for eigen-space be $B = \{X, X'\} = \left\{ \begin{bmatrix} -4 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$.

[4] For each of the following matrices, find all eigenvalues and associated eigenvectors then find an orthormal basis of each eigenspace.

(9) $\begin{bmatrix} 6 & 3 \\ 2 & 7 \end{bmatrix}$

characteristic corresponding to given matrix A is

$$|A - \lambda I| = 0$$

$$\Rightarrow \left| \begin{bmatrix} 6 & 3 \\ 2 & 7 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \right| = 0$$

$$\Rightarrow \begin{vmatrix} 6-\lambda & 3 \\ 2 & 7-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (6-\lambda)(7-\lambda) - 6 = 0$$

$$\Rightarrow 42 - 6\lambda - 7\lambda + \lambda^2 - 6 = 0$$

$$\Rightarrow \lambda^2 - 13\lambda - 36 = 0$$

$$\Rightarrow \lambda^2 - 9\lambda - 4\lambda - 36 = 0$$

$$\Rightarrow \lambda(\lambda - 9) - 4(\lambda - 9) = 0$$

$$\Rightarrow (\lambda - 4)(\lambda - 9) = 0$$

$$\Rightarrow \lambda = 4, \quad \lambda = 9$$

Let $v = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ be the eigenvector corresponding to A when $\lambda = 4$,

Then, $[A - \lambda I] v = 0$

$$\Rightarrow \left\{ \begin{bmatrix} 6 & 3 \\ 2 & 7 \end{bmatrix} - \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} \right\} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2 & 3 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow 2 \cdot x_1 + 3x_2 = 0$$

$$2x_1 + 3x_2 = 0$$

$$\therefore v = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$$

when

$\lambda = 9$,

$$[A - \lambda I] v = 0$$

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$$\left\{\begin{bmatrix} 6 & 3 \\ 2 & 7 \end{bmatrix} - \begin{bmatrix} 9 & 0 \\ 0 & 9 \end{bmatrix}\right\}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -3 & 3 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\Rightarrow -3x_1 + 3x_2 = 0$

$2x_1 - 2x_2 = 0$

$$\therefore \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Vectors Vectors $\begin{bmatrix} -3 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ are linearly independent. Also any linearly independent set of n elements of V must span V and thus form a basis. Hence Basis of eigenspace is $B = \left\{\begin{bmatrix} -3 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}$. To find orthonormal basis for eigensp

$$V_1' = \frac{1}{\sqrt{15}}\begin{bmatrix} -3 \\ 2 \end{bmatrix}$$

$$V_2' = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Hence orthonormal basis for eigenspace $B' = \left\{\begin{pmatrix} -3/\sqrt{15} \\ 2/\sqrt{15} \end{pmatrix}, \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 1/\sqrt{2} \end{pmatrix}\right.$

(b) $\begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$

$|A - \lambda I| = 0$

$$\Rightarrow \left|\begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right| = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1-\lambda & 2 \\ 4 & 3-\lambda \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\Rightarrow (1-\lambda)(3-\lambda) - 8 = 0$

$\Rightarrow 3 - \lambda - 3\lambda + \lambda^2 - 8 = 0$

$\Rightarrow \lambda^2 - 4\lambda - 5 = 0$

$\Rightarrow \lambda^2 - 5\lambda + \lambda - 5 = 0$

$\Rightarrow \lambda(\lambda - 5) + 1(\lambda - 5) = 0$

$\Rightarrow (\lambda + 1)(\lambda - 5) = 0$

$\Rightarrow \lambda = -1, \lambda = 5,$

Let $v = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ be eigenvector for $A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$ when $\lambda = -1$

Then $|A - \lambda I| v = 0$

$$\left\{\begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} - \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\right\}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

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$$\begin{bmatrix} 2 & 2 \\ 4 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\Rightarrow$ $2x_1 + 2x_2 = 0$

$4x_1 + 4x_2 = 0$

$\therefore \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} = V$

Again when $\lambda = 5$,

$$\left\{\begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} - \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}\right\}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \left\{\begin{bmatrix} -4 & 2 \\ 4 & -2 \end{bmatrix}\right\}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\Rightarrow$ $-4x_1 + 2x_2 = 0$

$4x_1 - 2x_2 = 0$

$\therefore \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

Vectors $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ are linearly independent. Also any linearly independent set of n elements of V must span V and thus form a basis. Hence basis of eigenspace is $B = \left\{\begin{bmatrix} -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix}\right\}$

To find orthonormal basis for eigenspace

$v_1' = \frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$v_2' = \frac{1}{\sqrt{5}}\begin{bmatrix} 1 \\ 2 \end{bmatrix}$

Hence, orthonormal basis for eigenspace is $B' = \left\{\begin{pmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}, \begin{pmatrix} \frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{pmatrix}\right\}$

[Q.N.5] prove that the eigenvalue of a traingular matrix is diagonal element.

Soln

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$A \in M_{n \times n}(\mathbb{C})$, A is upper traingular matrix so,
A Can be written as

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ 0 & \cdots 0 & \cdots & a_{nn} \end{bmatrix}$$

The eigenvalues of A are the n roots of
$\det(A - \lambda I) = 0$
Infact, determinant of uppor traingular matrix is product of its diagonal element so,

$$\left| \begin{bmatrix} a_{11} & a_{12} \cdots a_{1n} \\ 0 & a_{22} \cdots a_{2n} \\ \vdots & \vdots \\ 0 & 0 \cdots a_{nn} \end{bmatrix} - \begin{bmatrix} \lambda & 0 \cdots 0 \\ 0 & \lambda \cdots 0 \\ \vdots & \\ 0 & 0 \cdots \lambda \end{bmatrix} \right| = 0$$

$$\Rightarrow \begin{vmatrix} a_{11}-\lambda & a_{12} \cdots a_{1n} \\ 0 & a_{22}-\lambda \cdots a_{2n} \\ \vdots & \\ 0 & 0 \cdots a_{nn}-\lambda \end{vmatrix} = 0$$

$\Rightarrow (a_{11}-\lambda)(a_{22}-\lambda)(a_{nn}-\lambda) = 0$
Hence eigen values of traingular matrix are its diagonal element.

(6) What does it means for a matrix A to have an eigen value of 0?

Soln It means that some nonzero vector is mapped to zero times itself that is to the zero vector. By linearity every scalar multiple of this vector is also mapped to the zero vector. As a Consequence the matrix is not invertible.

(7) Explain why 2x2 matrix Can have at most two distinct eigenvalue.

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Soln:- 2x2 matrix have 2 dimension. Also number of basis vectors is equals to dimension. Also, If vectors are greater than 3 in 2 dimensional space then they never be linearly independent and hence they do not form basis for vector space. Hence we need at most two linearly independent vectors. Because of that reason 2x2 matrix have at most two distinct eigen vectors.

(8) ~~explain~~ Construct an example of 2x2 matrix with only one distinct eigenvalue. Explain why your example must have only one distinct eigenvalue.

Soln Let $A = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$

Hence, $|A - \lambda I| = 0$

$\Rightarrow \left| \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \right| = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\Rightarrow \begin{vmatrix} 1-\lambda & -1 \\ 0 & 1-\lambda \end{vmatrix} = 0$

$\Rightarrow (1-\lambda)(1-\lambda) = 0$

$(\lambda - 1)^2 = 0$

$\Rightarrow \lambda = 1$

Here in matrix A there is one distinct eigen value. But corresponding to this eigenvalue there exist two different eigen vectors they are linearly Independent.

[9] If λ is an eigenvalue of an $n \times n$ matrix A, show that 3λ is an eigenvalue of 3A.

Soln Let $A = \begin{bmatrix} 3 & 2 \\ 3 & 8 \end{bmatrix}$

Now, characteristic equation is

$|A - \lambda I| = 0$

$$\left|\begin{bmatrix}3 & 2\\3 & 8\end{bmatrix}-\begin{bmatrix}\lambda & 0\\0 & \lambda\end{bmatrix}\right|=0$$

$$\Rightarrow \begin{vmatrix}3-\lambda & 2\\3 & 8-\lambda\end{vmatrix}=0$$

$\Rightarrow (3-\lambda)(8-\lambda)-6=0$

$\Rightarrow 24-3\lambda-8\lambda+\lambda^2-6=0$

$\Rightarrow \lambda^2-11\lambda+18=0$

$\Rightarrow \lambda^2-9\lambda-2\lambda+18=0$

$\Rightarrow \lambda(\lambda-9)-2(\lambda-9)=0$

$\Rightarrow (\lambda-2)(\lambda-9)=0$

$\Rightarrow \lambda_1=2;\ \lambda_2=9$

Again Let,

$B = 3A$

$$= 3\begin{bmatrix}3 & 2\\3 & 8\end{bmatrix}=\begin{bmatrix}9 & 6\\9 & 24\end{bmatrix}$$

Now, characteristic equation is,

$|A-\lambda I|=0$

$$\Rightarrow \left|\begin{bmatrix}9 & 6\\9 & 24\end{bmatrix}-\begin{bmatrix}\lambda & 0\\0 & \lambda\end{bmatrix}\right|=0$$

$$\Rightarrow \left|\begin{bmatrix}9-\lambda & 6\\9 & 24-\lambda\end{bmatrix}\right|=0$$

$\Rightarrow (9-\lambda)(24-\lambda)-54=0$

$\Rightarrow 216-9\lambda-24\lambda+\lambda^2-54=0$

$\Rightarrow \lambda^2-33\lambda+162=0$

$\Rightarrow \lambda^2-27\lambda-6\lambda+162=0$

$\Rightarrow \lambda(\lambda-27)-6(\lambda-27)=0$

$\Rightarrow (\lambda-6)(\lambda-27)=0$

$\Rightarrow \lambda_1'=6,\ \lambda_2'=27$

Hence

$\lambda_1' = 3\cdot\lambda_1$ and $\lambda_2' = 3\cdot\lambda_2'$

[10] For $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix}$, Find one eigenvalue, with no Calculation. Justify your answer.

Soln Here given matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix}$ Let $V = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ be one eigenvector for A then

$Av = \lambda v$

$$\Rightarrow \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1+2+3 \\ 1+2+3 \\ 1+2+3 \end{bmatrix} = 6 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \lambda V$$

Hence eigenvalue $\lambda = 6$.

[11] without Calculation, Find one eigenvalue and two linearly Independent eigenvectors of $A = \begin{bmatrix} 4 & 4 & -4 \\ 4 & 4 & -4 \\ 4 & 4 & -4 \end{bmatrix}$. Justify your answer.

Soln Here given matrix is,

$A = \begin{bmatrix} 4 & 4 & -4 \\ 4 & 4 & -4 \\ 4 & 4 & -4 \end{bmatrix}$ Let $V = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ be the one eigenvector for A

then, $Av = \lambda V$

$$\Rightarrow \begin{bmatrix} 4 & 4 & -4 \\ 4 & 4 & -4 \\ 4 & 4 & -4 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \\ 4 \end{bmatrix} = 4 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \lambda V$$

Hence $\lambda = 4$.

Another vector is $\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$