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System of linear eqn

A linear equation in the variables $x_1, \dots, x_n$ is an equation that can be written in the form

$$a_1x_1 + a_2x_2 + \dots + a_nx_n = b \quad — (1)$$

where b is the coefficients $a_1, \dots, a_n$ are real and complex numbers. The subscript n may be any positive integer.

The equations

$$4x_1 - 5x_2 + 2 = x_1 \text{ and } x_2 = 2(\sqrt{6} - x_1) + x_3$$

are both linear because they can be rearrange algebrafically as in equation (1)

$$3x_1 - 5x_2 = -2 \text{ and } 2x_1 + x_2 - x_3 = 2\sqrt{6}$$

The equations

$$4x_1 - 5x_2 = x_1x_2 \text{ and } x_2 = 2\sqrt{x_1} - 6$$

both are not linear because of presence of x_1x_2 in first equation and $\sqrt{x}$ in the second eqn.

Defination :-

A system of linear equation is a collection of one or more linear equations involving the same variables — say $x_1, \dots, x_n$. An example is,

$$\left.\begin{aligned} 2x_1 - x_2 + 1.5x_3 &= 8 \\ x_1 - 4x_3 &= -7 \end{aligned}\right\} — (2)$$

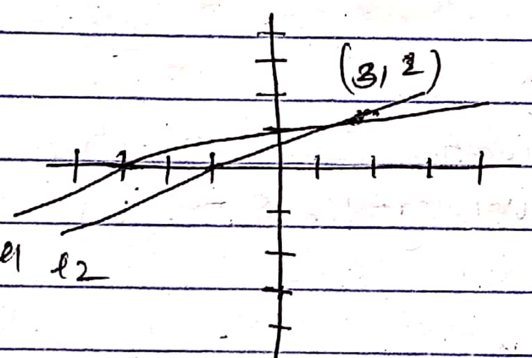
A solution of the system is a list $(s_1, s_2, \dots, s_n)$ of numbers that makes each equation a true statement when the values $s_1, \dots, s_n$ are substituted for $x_1, \dots, x_n$ respectivey. For instance $(5, 6.5, 3)$ is a solution of system (2) because when we substitute the values in (2) for x_1, x_2, x_3 respectivey the equations simplyfy to $8 = 8$ and $-7 = -7$.

The set of all possible solutions is called the Solution set of the linear system. Two linear systems are called equivalent If they have the same solution set.

Finding the solution set of a system of two linear equations in two variables is easy because it amounts to finding the interaction of two lines. A typical problem is,

$$x_1 - 2x_2 = -1$$
$$-x_1 + 3x_2 = 3$$

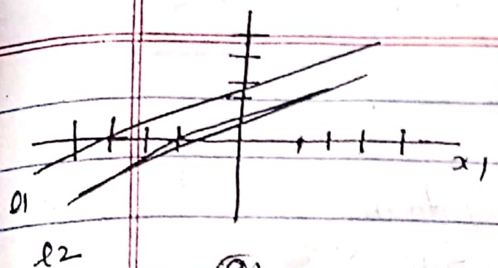
The graph of these equations are lines, which are denoted by l_1 and l_2. A pair of numbers (x_1, x_2) satisfies both equations in the system If and only if the point (x_1, x_2) lies on both l_1 and l_2. The solution for above equations is the single point $(3,2)$.



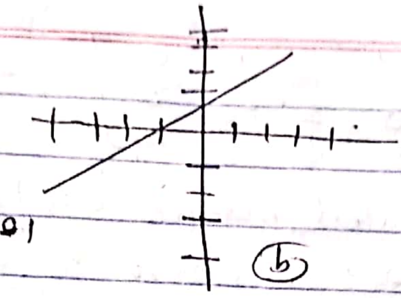
Of cource, two linear lines need not Intersect in a single point. they could be parallel. or they could be coincide and hence "Intersect" at every point on the line.
Following graphs that correspond to the following systems

(a)
$$x_1 - 2x_2 = -1$$
$$-x_1 + 2x_2 = 3$$

(b)
$$x_1 - 2x_2 = -1$$
$$-x_1 + 2x_2 = 1$$



(a)



(b)

No Solution

Infinite many Solutions.

A system of linear equation has
1. no Solutions
2. exactly one solution
3. Infinitely many Solutions.

A System of linear equations is said to be consistent if it has either one solution or infinitely many solutions. a System is inconsistant if it has no solution.

Matrix notation :-

The essential Information of a linear system can be recorded compactly in a rectangular array called a matrix. Given the System.

$$\left.\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ 2x_2 - 8x_3 &= 8 \\ 5x_1 - 5x_3 &= 10 \end{aligned}\right\} - (3)$$

with the Coefficients of each variable aligned in columns, the matrix.

$$\begin{bmatrix} 1 & -2 & 1 \\ 0 & 2 & -8 \\ 5 & 0 & -5 \end{bmatrix}$$

is called the Coefficient matrix (or matrix of Coefficients) of the System (3) and

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$$\begin{bmatrix} 1 & -2 & 1 \\ 0 & 2 & -8 \\ 5 & 0 & -5 \end{bmatrix} \quad \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 5 & 0 & -5 & 10 \end{bmatrix} \quad — (4)$$

is called augmented matrix of the system.
Second row here contain a zero because the second equation could be written as $0 \cdot x_1 + 2x_2 - 8x_3 = 8$. An Augmented matrix of a system consists of the coefficient matrix with an added columns containing the ~~coeff~~ constants from the right sides of the equations.

The size of a matrix tells how many rows and columns it has. an $m \times n$ matrix is a rectangular array of numbers with m rows and n columns.

Solving linear System

The basic strategy is to replace one system with an equivalent system (i.e one with the same solution set) that is easier to solve.

example Solve System (3)

$$\left.\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ 2x_2 - 8x_3 &= 8 \\ 5x_1 - 5x_3 &= 10 \end{aligned}\right\} — (1)$$

The ellimination procedure is shown here,
The augmented matrix corresponding to system (1) is

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 5 & 0 & -5 & 10 \end{bmatrix}$$

keep x_1 from (1) first equation and elimate it from the other equations. To do so add -5 times equation (1) to equation (3)

$$\begin{aligned} -5 \cdot [\text{equation } 1] \quad & -5x_1 + 10x_2 - 5x_3 = 0 \\ + [\text{equation } 3] \quad & 5x_1 - 5x_3 = 10 \\ \hline & 10x_2 - 10x_3 = 10 \end{aligned}$$

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The result of this calculation is written in place of the original third equation.

$$x_1 - 2x_2 + x_3 = 0$$
$$2x_2 - 8x_3 = 8$$
$$10x_2 - 10x_3 = 10$$

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 0 & 10 & -10 & 10 \end{bmatrix}$$

Now multiplying equation ② by $\frac{1}{2}$ in order to obtain 1 as the coefficient for x_2.

$$x_1 - 2x_2 + x_3 = 0$$
$$x_2 - 4x_3 = 4$$
$$10x_2 - 10x_3 = 10$$

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 10 & 10 & 10 & 10 \end{bmatrix}$$

Use x_2 in equation ② to eliminate $10x_2$ in eqn ③. The mental computation is,

$$-10\cdot[\text{equation } 2] \quad -10x_2 + 40x_3 = -40$$
$$+[\text{equation } 3] \quad 10x_2 - 10x_3 = 10$$
$$30x_3 = -30$$

The result of this calculation is written in place of the previous third equation (row).

$$x_1 - 2x_2 + x_3 = 0$$
$$x_2 - 4x_3 = 4$$
$$30x_3 = -30$$

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 30 & -30 \end{bmatrix}$$

Now, multiply eqn ③ by $\frac{1}{30}$ in order to obtain 1 as the coefficient for x_3.

$$x_1 - 2x_2 + x_3 = 0$$
$$x_2 - 4x_3 = 4$$
$$x_3 = -1$$

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

The new system has a triangular form (the intutive term traingular will be replaced by a precise term in the next section)

$$x_1 - 2x_2 + x_3 = 0$$
$$x_2 - 4x_3 = 4$$
$$x_3 = -1$$

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

Eventually, you want to eliminate the $-2x_2$ term from eqn 1. but it is more efficient to use the x_3 in eqn 3 first to eliminate the $-4x_3$ and $+x_3$ term in equation 2 and 1.

4·[equation 1] $4x_3 = 4$

+[equation 2] $x_2 - 4x_3 = 4$

$x_2 = 0$

−1[equation 3] $-x_3 = 1$

+[equation 1]. $x_1 - 2x_2 + x_3$

$x_1 - 2x_2 = 1$

It is convenient to combine the results of these two operations.

$$x_1 - 2x_2 = 1$$
$$x_2 = 0$$
$$x_3 = -1$$

$$\begin{bmatrix} 1 & -2 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

Now having cleared ~~back~~ out column above the x_3 in eq 3 move back to x_2 in eq 2. and use it to eliminate $-2x_2$ above it.

$$x_1 = 1$$
$$x_2 = 0$$
$$x_3 = -1$$

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

So, $(1, 0, -1)$ is a solution of the system.

Elementary row operations

1. (Replacement) Replace one rows by the sum of itself and a multiple of another row.
2. (Interchange) Interchange two rows
3. (Scaling) multiply all entries in a row by non zero constant.

Here given three eqn are

$x+y+z=1$

$x+y-2z=3$

$2x+y+z=2$

The augmented matrix of linear system is,

$$\left[\begin{array}{ccc:c} 1 & 1 & 1 & 1 \\ 1 & 1 & -2 & 3 \\ 2 & 1 & 1 & 2 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc:c} 1 & 1 & 1 & 1 \\ 0 & 0 & -3 & 2 \\ 0 & -1 & -1 & 0 \end{array}\right] \quad \begin{array}{l} R_2 \leftarrow R_2 - R_1 \\ R_3 \leftarrow R_3 - 2R_1 \end{array}$$

$$\sim \left[\begin{array}{ccc:c} 1 & 1 & 1 & 1 \\ 0 & 1 & -2 & 2 \\ 0 & -1 & -1 & 0 \end{array}\right] \quad R_2 \leftarrow R_2 - R_3$$

$$\sim \left[\begin{array}{ccc:c} 1 & 1 & 1 & 1 \\ 0 & 1 & -2 & 2 \\ 0 & 0 & -3 & 2 \end{array}\right] \quad R_3 \leftarrow R_3 + R_2$$

$$\sim \left[\begin{array}{ccc:c} 1 & 1 & 1 & 1 \\ 0 & 1 & -2 & 2 \\ 0 & 0 & 1 & -2/3 \end{array}\right] \quad R_3 \leftarrow -\tfrac{1}{3} R_3$$

$$\sim \left[\begin{array}{ccc:c} 1 & 0 & 3 & -1 \\ 0 & 1 & -2 & 2 \\ 0 & 0 & 1 & -2/3 \end{array}\right] \quad \begin{array}{l} R_1 \leftarrow R_1 - R_2 \\ R_2 \leftarrow R_2 + 2R_3 \end{array}$$

$$\sim \left[\begin{array}{ccc:c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 2/3 \\ 0 & 0 & 1 & -2/3 \end{array}\right] \quad R_2 \leftarrow R_2 + 2R_3.$$

Hence values of x, y, z are $-1, 2/3, -2/3$

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Two Fundmental questions about A Linear system

1. Is the system Consistent, that is, does at least one soln existe

2. If a solution exists, is it the only one, that is, is the solution unique.

examples Determine if the following system is Consistent

$$x_1 - 2x_2 + x_3 = 0$$
$$2x_2 - 8x_3 = 0$$
$$5x_1 - 5x_3 = 10$$

Soln This is the system from example 1. suppose that we have performed the row operations necessary to obtained the traingle Form

$$x_1 - 2x_2 + x_3 = 0 \qquad x_2 - 4x_3 = 4 \qquad x_3 = -1$$

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

At this point we know x_3. Were we to ~~obtai~~ substitute the value of x_3 into equation 2, we could compute x_2 and hence could determine x_1 from eq (1). So a solution exist the system is consistent. (In fact x_2 is uniquely determined by equation 2 since x_3 has only one possible value, and x_1 is therefore uniquely determined by eq (1) So the solution is unique).

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Example 4 Determine if the following system is consistent

$$\left.\begin{aligned} x_2 - 4x_3 &= 8 \\ 2x_1 - 3x_2 + 2x_3 &= 1 \\ 4x_1 - 8x_2 + 12x_3 &= 1 \end{aligned}\right\} - (5)$$

Soln The augmented matrix is,

$$\begin{bmatrix} 0 & 1 & -4 & 8 \\ 2 & -3 & 2 & 1 \\ 4 & -8 & 12 & 1 \end{bmatrix}$$

To obtain an x_1 in the first equation, interchange rows 1 and 2.

$$\begin{bmatrix} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 4 & -8 & 12 & 1 \end{bmatrix}$$

To eliminate the $4x_1$ term in the third eqn add -2 times row 1 to row (3)

$$\begin{bmatrix} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 0 & -2 & 8 & -1 \end{bmatrix} - (6)$$

next use x_2 term in the second equation to ~~obtained~~ eliminate $-2x_2$ term from the third equations add 2 times row 2 to row 3.

$$\begin{bmatrix} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 0 & 0 & 0 & 15 \end{bmatrix} - (7)$$

The augment matrix now in traingular form. To interpret it correctly go back to eqn notations

$$\left.\begin{aligned} 2x_1 - 3x_2 + 2x_3 &= 1 \\ x_2 - 4x_3 &= 8 \\ 0 &= 15 \end{aligned}\right\} - (8)$$

The eqn $0 = 15$ is a short form of $0 \cdot x_1 + 0x_2 + 0x_3 = 15$

There are no values of $x_1\ x_2\ x_3$ that satisfies (8)

Hence it is inconsistance.

Echelon, Matrices, Row Reduction pivoting

Defination:- An $m \times n$ matrix A is said to be in reduced row echelon form if it ~~is~~ satisfies the following properties.

1. All zero rows if there are any appear at the bottom of the matrix.
2. The first non zero entry from the left of a non zero row is 1. This entry is called a leading one of its row.
3. For each non-zero row, the leading one appears to the right and below any one's in preceding rows.
4. If a column contains a leading one, then all other entries in that column are zero.

An $m \times n$ matrix is satisfying properties (1), (2) and (3) is said to be in row echelon form.

eg. $A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ matrix A is in reduced row echelon form.

Defination:- Let A be an $m \times n$ matrix. Any one of the following three operation on the rows (columns) of A is called an elementry row (column) operation.

1. Interchanging any two rows (columns) of A.
2. multiply any row (columns) of A by a non zero scalar.
3. adding any scalar multiple of a row (column) of A to another row (column).

1. Every matrix is row equivalent to itself 2. If A is row equivalent to B, then B is row equivalent to A.
3. If A is row equivalent to c then A is row equivalent to c.

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Defination pivot

A pivot position in matrix A is a location in A that corresponding to a leading in the reduced echelon form of A. A pivot column is a column of A contain a pivot position.

$$\text{Let } A = \begin{bmatrix} 0 & 2 & 3 & -4 & 1 \\ 0 & 0 & 2 & 3 & 4 \\ 2 & 2 & -5 & 2 & 4 \\ 2 & 0 & -6 & 9 & 7 \end{bmatrix}$$

procedure for transforming a matrix to row echelon form as follows.

Step1:- Find the first (counting from left to right) column in A not all of whose entries are zero.

$$A = \begin{bmatrix} 0 & 2 & 3 & -4 & 1 \\ 0 & 0 & 2 & 3 & 4 \\ 2 & 2 & -5 & 2 & 4 \\ 2 & 0 & -6 & 9 & 7 \end{bmatrix}$$

This element 2 is called pivot which we circle in A.

Step2:- Interchanging if necessary, the first row with the row where the pivot occurs so that the pivot is now in the first row. call the new matrix A_1.

$$A_1 = \begin{bmatrix} 2 & 2 & -5 & 2 & 4 \\ 0 & 0 & 2 & 3 & 4 \\ 0 & 2 & 3 & -4 & 1 \\ 2 & 0 & -6 & 9 & 7 \end{bmatrix} \quad R_3 \leftrightarrow R_1$$

Step3 multiply the first row of A_1 by reciprocal of the pivot. Thus the entry in the first row and pivot column.

$$A_2 = \begin{bmatrix} 1 & 1 & -5/2 & 1 & 2 \\ 0 & 0 & 2 & 3 & 4 \\ 0 & 2 & 3 & -4 & 1 \\ 2 & 0 & -6 & 9 & 7 \end{bmatrix} \quad R_1 \leftarrow \tfrac{1}{2} R_1$$

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$$A_3 = \begin{bmatrix} 1 & 1 & -5/2 & 1 & 2 \\ 0 & 0 & 2 & 3 & 4 \\ 0 & 2 & 3 & -4 & 1 \\ 0 & -2 & -1 & 7 & 3 \end{bmatrix} \quad R_4 \leftarrow R_4 - 2R_1$$

Step6

$$\begin{array}{ccccc} 1 & 1 & -5/2 & 1 & 2 \end{array}$$

$$B = \begin{bmatrix} 0 & 0 & 2 & 3 & 4 \\ 0 & 2 & 3 & 4 & 1 \\ 0 & -2 & -1 & 7 & 3 \end{bmatrix}$$

$$B_1 = \begin{bmatrix} 0 & 1 & 3/2 & -2 & 1/2 \\ 0 & 0 & 2 & 3 & 4 \\ 0 & -2 & -1 & 7 & 3 \end{bmatrix} \quad R_1 \leftarrow 1/2 R_1$$

$$B_3 = \begin{bmatrix} 0 & 1 & 3/2 & -2 & 1/2 \\ 0 & 0 & 2 & 3 & 4 \\ 0 & 0 & 2 & 3 & 4 \end{bmatrix} \quad R_3 \leftarrow R_3 + 2R_1$$

$$\begin{array}{ccccc} 1 & 1 & -5/2 & 1 & 2 \\ 0 & 1 & 3/2 & -2 & 1/2 \end{array}$$

$$C = \begin{bmatrix} 0 & 0 & 2 & 3 & 4 \\ 0 & 0 & 2 & 3 & 4 \end{bmatrix}$$

$$C_1 = \begin{bmatrix} 0 & 0 & 1 & 3/2 & 2 \\ 0 & 0 & 2 & 3 & 4 \end{bmatrix} \quad R_1 \leftarrow 1/2 R_1$$

$$C_2 = \begin{bmatrix} 0 & 0 & 1 & 3/2 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad R_2 \leftarrow R_2 - 2R_1$$

Setp8

$$D = (0\ 0\ 0\ 0\ 0$$

$$H = \begin{bmatrix} 1 & 1 & -5/2 & 1 & 2 \\ 0 & 1 & 3/2 & -2 & 1/2 \\ 0 & 0 & 1 & 3/2 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The matrix, obtained by H Consisting of D and H is the echelon form.

Appeication of linear system

~~Solution to the System of linear~~ equation

Solutions of Linear system

The row reduction algorithm leads directly to the expeicit description of the solution set of a linear system when the algorithm is appeied to the augmented matrix of the system.

Example Augmented matrix of a linear system has been changed into the equivalent reduced echelon form.

$$\begin{bmatrix} 1 & 0 & -5 & 1 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

There are three variables because the augmented matrix has four columns. The associated system of equation is,

$$x_1 - 5x_3 = 1$$
$$x_2 + x_3 = 4$$
$$0 = 0$$

The variables x_1 and x_2 corresponding to pivot columns in the matrix are called basic variables. The other variable x_3 is called free variable.

$$\begin{cases} x_1 = 1 + 5x_3 \\ x_2 = 4 - x_3 \\ x_3 \text{ is free} \end{cases}$$

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Conditions for a system of linear equation to have
(i) a unique Solution
(ii) no solution
(iii) Infinite number of solutions.

If system of linear equations are consistent they have unique solution and infinite solution.

If the system of linear equation is inconsistent then it has no solution.

Remarks :- If Rank of augmented matrix = rank of matrix
In this case system is consistent.

Using Rank to test consistency of matrix

i) A system of linear equation to have a unique solution if rank of coefficient matrix equals to rank of augmented matrix = number of variables (unknown) i.e Rank(A) = rank[A:B] = Number of unknowns.

ii) A system of linear eqn have no solution if Rank of coefficient matrix does not equal to rank of augmented matrix i.e Rank(A) ≠ Rank[A:B]

iii) A system of linear eqn to have infinite number of solution if Rank of coefficient matrix equals to rank of augmented matrix which is less than the no. of variable (unknowns).
i.e Rank (A) = n-1 = rank [A:B]
But variable number is n.

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Appoication of Linear system :-

- Set up and Solve appolcations Involving relationships between numbers.
- Set up and solve appoication involving Interest and money.
- set up and solve mixture problem.
- set up and Solve uniform motion problems.

Cramer's Rule, Volume, and Linear transformation

Theorem:- Let $Ax=b$ be the matrix form of a system of n linear equations in n unknowns where $x=(x_1, x_2, \dots x_n)^T$. If $\det(A) \neq 0$, then this system has a unique Solution, and for each k ($k=1,2 \dots n$), $x_k = \frac{\det(M_k)}{\det(A)}$, where M_k is the $n \times n$ matrix obtained from A by replacing Column k of A by b.

Example. By using Cramer's rule solve

$x_1 + 2x_2 + 3x_3 = 2$

$x_1 + x_3 = 3$

$x_1 + x_2 - x_3 = 1$

Solution: The matrix form of this system of linear eqn is $Ax=b$ i.e the given system of linear equations Can be written as

$$\begin{bmatrix} 1 & 2 & 3 \\ 1 & 0 & 1 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$$

Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 0 & 1 \\ 1 & 1 & -1 \end{bmatrix} = 6.$

Then $x_1 = \frac{\det(M_1)}{\det(A)} = \dfrac{\begin{vmatrix} 2 & 2 & 3 \\ 3 & 0 & 1 \\ 1 & 1 & -1 \end{vmatrix}}{\begin{vmatrix} 1 & 2 & 3 \\ 1 & 0 & 1 \\ 1 & 1 & -1 \end{vmatrix}} = \frac{15}{6} = 5/2$

$x_2 = \frac{\det(M_2)}{\det(A)} = \dfrac{\begin{vmatrix} 1 & 2 & 3 \\ 1 & 3 & 1 \\ 1 & 1 & -1 \end{vmatrix}}{\begin{vmatrix} 1 & 2 & 3 \\ 1 & 0 & 1 \\ 1 & 1 & -1 \end{vmatrix}} = \frac{-6}{6} = -1$

$x_3 = \frac{\det(M_3)}{\det(A)} = \dfrac{\begin{vmatrix} 1 & 2 & 3 \\ 1 & 0 & 3 \\ 1 & 1 & 1 \end{vmatrix}}{\begin{vmatrix} 1 & 2 & 3 \\ 1 & 0 & 1 \\ 1 & 1 & -1 \end{vmatrix}} = \frac{3}{6} = 1/2.$

Hence required solution is $(x_1, x_2, x_3) = (5/2, -1, 1/2)$

<u>Linear transformation</u>