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Exponentiated Power Lindley Distribution : A Bayes Study using MCMC Approach

By

RAMESH KUMAR JOSHI AND VIJAY KUMAR

Abstract

This paper deals with the Bayesian analysis of three-parameter exponentiated power Lindley distribution. The parameters are estimated using likelihood based inferential procedures. The maximum likelihood(ML) estimates and asymptotic confidence intervals based on ML estimates are obtained. We have also obtained the bootstrap confidence intervals for the parameters. The Bayesian estimates of the parameters of exponentiated power Lindley distribution are obtained using Markov chain Monte Carlo (MCMC) simulation method. We have obtained the probability intervals for parameters, hazard and reliability functions. The posterior predictive check procedure is used for evaluating model fit. All the Bayesian computations are performed in OpenBUGS and R software. A real data set is analyzed for illustration of the proposed Bayesian approach.

Keywords : Exponentiated power Lindley distribution, Markov chain Monte Carlo, Bayesian estimation, Reliability function, OpenBUGS.
2010 AMS Subject Classification : 62F15, 65C05.

1. Introduction

Recently, some attempts have been made to define new families of distributions to extend well known models and at the same time provide great flexibility in modeling data in practice. Several techniques could be employed to form a larger family from an existing distribution by incorporating extra parameters. So, several classes by adding one or more parameters to generate new models have been proposed in the statistical literature. These methods include the generalization of a distribution by exponentiation procedure. Method of exponentiation is an of the important and commonly used techniques to add a parameter to a lifetime model, the new model becomes more flexible and can accommodate both monotones as well as non-monotone failure rate functions.

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The Lindley distribution was originally proposed by Lindley [17] in the context of Bayesian statistics, as a counter example of fudicial statistics. Ghitany et al. [12] developed different properties and the necessary inferential procedure for the Lindley distribution. Because of having only one parameter, the Lindley distribution does not provide enough flexibility for analysing different types of lifetime data. To increase the flexibility for modelling purposes it will be useful to consider further alternatives of this distribution. Ghitany et al. [11] developed a two-parameter weighted Lindley distribution and discussed its applications to survival data. Nadarajah et al. [19] obtained a generalized Lindley distribution and discussed its various properties and applications. Bakouch et al. [2] obtained an extended Lindley distribution and discussed its various properties and applications. Using the power transformation method Ghitany et al.[10] introduced the power Lindley (PL) distribution and discussed its statistical properties including maximum likelihood estimation of the parameters. Ashour and Eltehiwy [1] defined the exponentiated power Lindley distribution using the method of exponentiation and studied the various properties.

The main objective of this paper is to explore the inferential procedures, classical as well as Bayesian, for the exponentiated power Lindley distributions. It is to be noted that most of the cited literature is confined to classical developments and any systematic development on Bayesian results are rarely seen for the exponentiated power Lindley distribution.

The aim of this paper is to provide full Bayesian analysis of three-parameter exponentiated power Lindley distribution. Markov chain Monte Carlo (MCMC) techniques have become the preferred computational tools for Bayesian inference, Christensen et al.[4]. The freely available software package known as Bayesian inference using Gibbs sampling(BUGS) has been in the forefront of this abundance. However, more recent advances in this software, leading first to WinBUGS and now to an open-source version OpenBUGS, Thomas et al. [27], Thomas [26] and Lunn et al. [18], including interfaces to the open-source statistical package R, (R Development Core Team, [23]), have brought MCMC to a wider audience. We shall use OpenBUGS and R software in the present study.

The rest of the article is arranged as follows. The statistical properties of the three-parameter exponentiated power Lindley distribution are presented in Section 2. The Section 3 deals with the maximum likelihood estimation procedure to estimate the model parameters and associated confidence intervals using the observed information matrix are discussed. We also propose parametric bootstrap confidence intervals. In Section 4, we have developed the Bayesian estimation procedure under independent gamma and uniform priors for the parameters. It is observed that the Bayes estimates cannot be obtained analytically. We use Gibbs sampling technique in OpenBUGS, to generate MCMC samples from the posterior density function. The Bayes estimates of the parameters and associated probability intervals are computed based on the generated posterior samples. We have also estimated the reliability and hazard functions. A real data set has been analysed in Section 5 to illustrate the proposed methodologies. In this section, the ML estimators of the parameters, approximate confidence intervals are presented. The Bayesian analysis using the MCMC simulation is discussed in Section 6. In this section, the Bayes estimates and credible intervals of parameters, hazard and reliability functions are presented. In Section 7 we have applied the predictive check method in order to give an assessment of the performance of the model for the given data. Finally, Section 8 ends up with some general concluding remarks.

2. The Model: structural analysis

Lindley [17] proposed the following probability density function(PDF)

$$f(t;\theta)=\frac{\theta^2}{(1+\theta)}\,(1+t)\,e^{-\theta t}\;;\quad t>0.\ \theta>0$$

is known as Lindley distribution. Using the transformation $Y=T^{1/\alpha}$ Ghitany et al. [10] derived the power Lindley (PL)distribution given by

$$f\,(y;\beta.\theta)=\frac{\beta\theta^2y^{\beta-1}}{(\theta+1)}\left(1+y^{\beta}\right)e^{-\theta y^{\beta}};\ y>0.\beta>0.\theta>0,$$

The cumulative distribution function (CDF) of the power Lindley distribution is

$$F\,(y;\beta.\theta)=\,1-\left(1+\frac{\theta y^{\beta}}{\theta+1}\right)\,e^{-\theta y^{\beta}},\ y>0.\beta>0.\theta>0.$$

Ashour and Eltehiwy [1] applied the method of exponentiation to obtain the three parameter exponentiated power Lindley distribution and studied the various properties. The cumulative distribution function(CDF) of the exponentiated power Lindley distribution with three parameters is given by

$$F\,(x;\alpha.\beta.\theta)=\left[1-\left(1+\frac{\theta x^{\beta}}{\theta+1}\right)e^{-\theta x^{\beta}}\right]^{\alpha};\ x>0. \tag{2.1}$$

where $\alpha>0$, $\beta>0$ are the shape and $\theta>0$ is the scale parameters. It is denoted by $EPL(\alpha.\beta.\theta)$.

The probability density function(PDF) is given by

$$f(x;\alpha,\beta,\theta) = \frac{\alpha\beta\theta^2 x^{\beta-1}}{(\theta+1)}\left(1+x^\beta\right)e^{-\theta x^\beta}\left[1-\left(1+\frac{\theta x^\beta}{\theta+1}\right)e^{-\theta x^\beta}\right]^{\alpha-1};\ x>0 \tag{2.2}$$

The reliability/survival function is

$$R(x) = 1-\left[1-\left(1+\frac{\theta x^\beta}{\theta+1}\right)e^{-\theta x^\beta}\right]^{\alpha}\quad ;\ x\geqslant 0 \tag{2.3}$$

The hazard rate function(HRF) is

$$h(x;\alpha,\beta,\theta) = \frac{\alpha\beta\theta^2 x^{\beta-1}}{(\theta+1)}\left(1+x^\beta\right)\left[1-\left(1+\frac{\theta x^\beta}{\theta+1}\right)e^{-\theta x^\beta}\right]^{\alpha-1} e^{-\theta x^\beta}\left\{1-\left[1-\left(1+\frac{\theta x^\beta}{\theta+1}\right)e^{-\theta x^\beta}\right]^{\alpha-1}\right\}^{-1} \tag{2.4}$$

Shapes of PDF and HRF:

Ashour and Eltehiwy discussed the shape of the pdf $f(x)$ and hazard function $h(x)$ based on the result of power Lindley distribution given in Ghitany et al. [10].

- Decreasing if
  - (i) $\left\{0<\beta\leqslant\frac{1}{2},\ \theta>0,\ \alpha\leqslant 1\right\}$;
  - (ii) $\left\{\frac{1}{2}<\beta<1,\ \theta\geqslant\rho(\beta),\ \alpha\leqslant 1\right\}$, where $\rho(\beta)=\frac{1}{\beta}\left(1-2\sqrt{\beta(1-\beta)}\right)$,
  - (iii) $\{\beta=1,\ \theta\geqslant 1,\ \alpha=1\}$;
  - (iv) $\{\beta\geqslant 1,\ \theta>0,\ \alpha<1\}$.
- Unimodal if
  - (i) $\{\beta=1,\ \theta>0,\ \alpha>1\}$;
  - (ii) $\{\beta=1,\ 0<\theta<1,\ \alpha=1\}$;
  - (iii) $\{\beta>1,\ \theta>0,\ \alpha\geqslant 1\}$.
- Decreasing-increasing-decreasing if $\left\{\frac{1}{2}<\beta<1,\ 0<\theta<\rho(\beta),\ \alpha=1\right\}$.

Figure 1(left panel) illustrates some of the possible shapes of the density function of $EPL(\alpha,\beta,\theta)$ for selected values of the parameters (α,β,θ). We note that this model is quite flexible for modelling positive data.

The hazard rate function of the exponentiated power Lindley distribution may be inverse bathtub shaped, decreasing and constant. The behaviour of HRF is as follows:

- Decreasing if

(i) $\left\{0 < \beta \leqslant \frac{1}{2},\ \theta > 0,\ \alpha \leqslant 1\right\}$;
(ii) $\left\{\frac{1}{2} < \beta < 1,\ \theta \geqslant \eta(\beta),\ \alpha \leqslant 1\right\}$, where $\eta(\beta) = \frac{(2\beta-1)^2}{4\beta(1-\beta)}$; ,

- Increasing if $\{\beta \geqslant 1,\ \theta > 0,\ \alpha \geqslant 1\}$.
- Decreasing-increasing-decreasing if $\left\{\frac{1}{2} < \beta < 1,\ 0 < \theta < \eta(\beta),\ \alpha = 1\right\}$.

Plots of the hazard rate function of the $EPL(\alpha, \beta, \theta)$ some choices of values of the parameters are displayed in Figure 1(right panel).

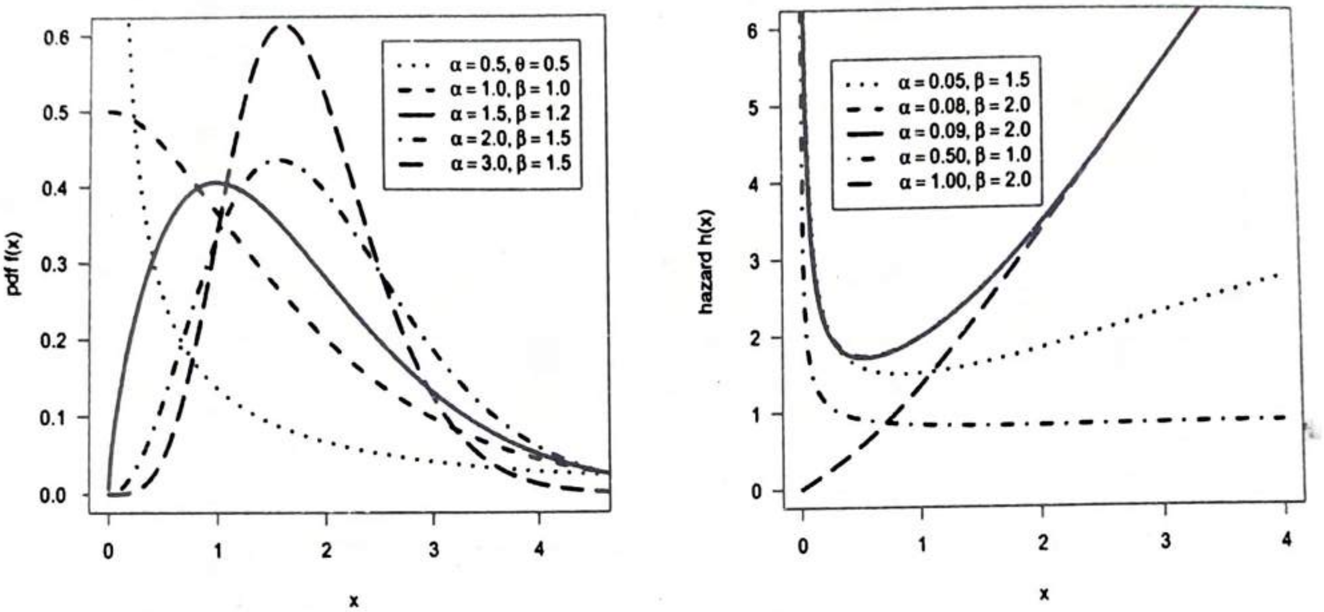


Figure 1. Plots of the probability density function(left panel) and hazard function (right panel), for θ=1 and different values of α and β.

Quantile function:

Let X denotes a random variable with PDF. The quantile function, say $Q(p)$, defined by$F(Q(p)) = p$ is the root of the equation

$$\left(1 + \frac{\theta[Q(p)]^{\beta}}{1+\theta}\right) \exp\left\{-\theta[Q(p)]^{\beta}\right\} = 1 - p^{1/\alpha},\ \ 0 < p < 1. \tag{2.5}$$

Simulation:

Here, we consider simulating values of a random variable X with the probability density function 2. Let U denote a uniform random variable on the interval $(0, 1)$. One way to simulate values of X is to set

$$\left(1 + \frac{\theta x^{\beta}}{1+\theta}\right) \exp\left(-\theta x^{\beta}\right) = 1 - U^{1/\alpha}$$

and solve for X, i.e. use the inversion method. Alternatively, we can generate random data from $EPL(\alpha, \beta, \theta)$ using the mixture of power Lindley distribution as follows:

- Generate $U_i \sim Unifrom(0,\ 1)\ ; i = 1, \ldots, n$;
- Generate $V_i \sim Exponential\,(\theta)\ ,\ i = 1, \ldots, n$;
- Generate $G_i \sim Gamma\,(2, \theta)\ ,\ i = 1, \ldots, n$;
- If $U_i^{1/\alpha} \leqslant p = \frac{\theta}{1+\theta}$, then set $X_i = V_i^{1/\beta}$, otherwise, set $X_i = G_i^{1/\beta}$.

3. Maximum Likelihood Estimation

In this section, we determine the maximum likelihood estimates of the model parameters and asymptotic confidence intervals. Let $\underline{x} = (x_1, \ldots, x_n)$ be a sample from a distribution with probability density function (2.2). The log likelihood function of the parameter $\ell(\alpha, \beta, \theta)$ is given by

$$\begin{aligned} \ell\,(\alpha, \beta, \theta|\underline{x}) &= n \log \alpha + 2n \log \theta + n \log \beta - n \log\,(\theta + 1) \\ &\quad + (\beta - 1) \sum_{i=1}^{n} \log x_i + \sum_{i=1}^{n} \log \left(1 + x_i^{\beta}\right) - \theta \sum_{i=1}^{n} x_i^{\beta} \\ &\quad + (\alpha - 1) \sum_{i=1}^{n} \log \left[1 + \left(1 + \frac{\theta x_i^{\beta}}{\theta + 1}\right) e^{-\theta x_i^{\beta}}\right]. \end{aligned} \tag{3.1}$$

The maximum likelihood estimators of the parameters have obtained by differentiating the log of likelihood function (3.1) w.r.t.to parameters and equating to zero. Thus three normal equations have been obtained as

$$\begin{aligned} \frac{\partial \ell}{\partial \alpha} &= \frac{n}{\alpha} + \sum_{i=1}^{n} \log \left[1 - \left(1 + \frac{\theta x_i^{\beta}}{\theta + 1}\right) e^{-\theta x_i^{\beta}}\right] = 0, \\ \frac{\partial \ell}{\partial \beta} &= \frac{n}{\beta} + \sum_{i=1}^{n} \log\,(x_i) + \sum_{i=1}^{n} \frac{x_i^{\beta}}{1 + x_i^{\beta}} \log\,(x_i) - \theta \sum_{i=1}^{n} x_i^{\beta} \log\,(x_i) \\ &\quad + \frac{\theta\,(\alpha - 1)}{(\theta + 1)} \sum_{i=1}^{n} \frac{e^{-\theta x_i^{\beta}} x_i^{\beta} \log\,(x_i) \left(2 + \theta + \theta x_i^{\beta}\right)}{\left[1 - \left(1 + \frac{\theta x_i^{\beta}}{\theta+1}\right) e^{-\theta x_i^{\beta}}\right]} = 0 \quad , \\ \frac{\partial \ell}{\partial \theta} &= \frac{2n}{\theta} - \frac{n}{\theta + 1} - \sum_{i=1}^{n} x_i^{\beta} + \frac{\theta\,(\alpha - 1)}{(\theta + 1)^2} \sum_{i=1}^{n} \frac{e^{-\theta x_i^{\beta}} x^{2\beta}}{\left[1 - \left(1 + \frac{\theta x_i^{\beta}}{\theta+1}\right) e^{-\theta x_i^{\beta}}\right]} = 0. \end{aligned} \tag{3.2}$$

The maximum likelihood estimator (MLE) $\hat{\delta} = (\hat{\alpha}, \hat{\beta}, \hat{\theta})^{\top}$ of $\delta = (\alpha, \beta, \theta)^{\top}$ is obtained by solving simultaneously the nonlinear equations (3.2). These equations cannot be solved analytically and statistical software can be used to solve them numerically. We can use iterative techniques such as a Newton-Raphson type algorithm to calculate the estimate $\hat{\delta}$. For example, *optim*() function in R software can be used to compute $\hat{\delta}$ numerically.

The $(1-\gamma)\,\%$ confidence intervals for α, β and θ can be obtained from the usual asymptotic normality of the maximum likelihood estimators with $var\,(\hat{\alpha})$, $var\left(\hat{\beta}\right)$ and $var\left(\hat{\theta}\right)$ estimated from the inverse of the observed Fisher information matrix, that is, the inverse of the matrix of second derivatives of the log-likelihood function locally at $\hat{\alpha}$, $\hat{\beta}$ and $\hat{\theta}$. Hence, from the asymptotic normality of MLEs, approximate $(1-\gamma)\,\%$ confidence intervals for α, β and θ can be constructed as

$$\hat{\alpha} \pm z_{\gamma/2}\sqrt{var(\hat{\alpha})};\quad \hat{\beta} \pm z_{\gamma/2}\sqrt{var(\hat{\beta})} \text{ and } \hat{\theta} \pm z_{\gamma/2}\sqrt{var(\hat{\theta})}$$

where $z_{\gamma/2}$ is the upper percentile of standard normal variate.

3.1 Bootstrap Confidence Intervals(BCI)

In this section we discuss the bootstrap confidence intervals. We have used the percentile bootstrap (Boot-p) method, proposed by Efron and Tibshirani [5], to construct confidence intervals for the parameters. To construct the *Boot-p* confidence interval, we proceed as follows, Soliman et al. [24]:

(i) Compute the ML estimates $\hat{\alpha}$, $\hat{\beta}$ and $\hat{\theta}$ of the parameters: α, β and θ from the original data $\underline{x} = (x_1, \ldots, x_n)$ by solving the nonlinear equations (3.2).

(ii) Using $\hat{\alpha}$, $\hat{\beta}$ and $\hat{\theta}$ generate a bootstrap sample $\underline{x}^* = (x_1^*, \ldots, x_n^*)$ of size n from (2.1). As in (i), compute the ML estimates of α, β and θ say $\hat{\alpha}^*$, $\hat{\beta}^*$ and $\hat{\theta}^*$, using the bootstrap sample.

(iii) Repeat (ii), K-times. Obtain the bootstrap estimates $(\hat{\alpha}_1^*, \ldots, \hat{\alpha}_K^*)$, $(\hat{\beta}_1^*, \ldots, \hat{\beta}_K^*)$ and $(\hat{\theta}_1^*, \ldots, \hat{\theta}_K^*)$.

(iv) Let $(\hat{\alpha}_{(1)}^*, \ldots, \hat{\alpha}_{(K)}^*)$ be the ordered values of the estimates $(\hat{\alpha}_1^*, \ldots, \hat{\alpha}_K^*)$. The $100\,(1-\gamma)\,\%$ two-sided boot-p confidence interval (BCI) for α can be obtained by $(\hat{\alpha}_{([K\gamma/2])}^*,\ \hat{\alpha}_{([K(1-\gamma/2)])}^*)$, where $[\chi]$ denotes the largest integer less than or equal to χ. Similarly, we can obtain the $100\,(1-\gamma)\,\%$ BCI for β and θ.

4. The Bayesian Model

In Bayesian analysis, parameters of the models are considered to be a random variable and following certain distribution. This distribution is called prior distribution. If prior information available to us which may be used for selection of prior distribution. But in many real situation it is very difficult to select a prior distribution. Therefore selection of prior distribution plays an important role in estimation of the parameters. A natural choice would be the gamma priors for $\alpha \sim G(a_1, b_1)$, $\beta \sim G(a_2, b_2)$ and uniform prior for $\theta \sim U(a_3, b_3)$. It is important to mention that Gamma prior has flexible nature as a non-informative prior in particular when the values of hyper parameters are considered to be zero. Thus the proposed independent priors for α, β and θ may be considered as

$$p(\alpha) = \frac{b_1^{a_1}}{\Gamma(a_1)} e^{-b_1\alpha} \alpha^{a_1-1} \; ; a_1 > 0,\, b_1 > 0,\, \alpha \geqslant 0$$

$$p(\beta) = \frac{b_2^{a_2}}{\Gamma(a_2)} e^{-b_2\beta} \beta^{a_2-1} \quad ; a_2 > 0,\, b_2 > 0,\, \beta \geqslant 0$$

$$\text{and } p(\theta) = \frac{1}{b_3 - a_3} \quad ; a_3 < \theta < b_3$$

Thus, prior distribution of α, β and θ can be written as

$$p(\alpha, \beta, \theta) = p(\alpha)\, p(\beta)\, p(\theta).$$

The posterior distribution is obtained by multiplying $p(\alpha, \beta, \theta)$ with the likelihood function $L(\alpha, \beta, \theta|\underline{x})$ for the given data $\underline{x} = (x_1, \ldots, x_n)$. The likelihood function $L(\alpha, \beta, \theta|\underline{x})$ is

$$L(\alpha, \beta, \theta|\underline{x}) = \left(\frac{\alpha\beta}{\theta+1}\right)^n \theta^{2n} \exp\left(-\theta \sum_{i=1}^{n} x_i^\beta\right) \left(\prod_{i=1}^{n} \left(1 + x_i^\beta\right)\right)$$
$$\prod_{i=1}^{n} \left[1 - \left(1 + \frac{\theta x_i^\beta}{\theta+1}\right) e^{-\theta x_i^\beta}\right]^{\alpha-1}$$

The joint posterior, up to proportionality, is given by

$$p(\alpha, \beta, \theta|\underline{x}) \;\propto\; L(\alpha, \beta, \theta|\underline{x})\, p(\alpha, \beta, \theta),$$

and may be written as

$$p(\alpha, \beta, \theta|\underline{x}) \propto \alpha^n \beta^n \theta^{2n} (\theta+1)^{-n} \left(\prod_{i=1}^{n} \left(1 + x_i^\beta\right)\right) T$$
$$\exp\left(-b_1\alpha - b_2\beta - \theta \sum_{i=1}^{n} x_i^\beta\right)$$

where $T = \prod_{i=1}^{n} \left\{1 - \left(1 + \frac{\theta x_i^\beta}{\theta+1}\right) e^{-\theta x_i^\beta}\right\}^{\alpha-1}$

It can be seen that the above expression cannot be expressed in nice closed form. The approximate Bayes estimators of α, β and θ can be obtained using the Lindley approximation, but it is not possible to construct the highest posterior density (HPD) confidence intervals using the same method. Therefore, the following Markov chain Monte Carlo (MCMC) method is suggested to generate samples from the posterior density function,and in turn to obtain the Bayes estimators, and the HPD confidence intervals. The Gibbs sampler, is an important MCMC technique, provides a way for extracting samples from the posteriors. This sampling scheme was first introduced by Geman and Geman [8], but the applicability to statistical modelling for Bayesian computation was demonstrated by Gelfand and Smith [6]. Readers are referred to Gelman et al. [8] and Suess and Trumbo [25] for details of the procedure and the related convergence diagnostic issues. For Gibbs sampler implementation, the full conditionals for α, β and θ up to proportionality can be specified as

- Full conditional distribution of the parameter α for given β, θ and $\underline{x}$

$$p\left(\alpha|\beta,\theta,\underline{x}\right) \quad \propto \quad \alpha^n \exp\left(-b_1\alpha\right) T$$

- Full conditional distribution of the parameter β for given α, θ and $\underline{x}$

$$p\left(\beta|\alpha,\theta,\underline{x}\right) \propto \beta^n \exp\left(-b_2\beta - \theta\sum_{i=1}^{n} x_i^{\beta}\right)\left(\prod_{i=1}^{n}\left(1+x_i^{\beta}\right)\right) T$$

- Full conditional distribution of the parameter θ for given α, β and $\underline{x}$

$$p\left(\theta|\alpha,\beta,\underline{x}\right) \quad \propto \quad \theta^{2n}(\theta+1)^{-n}\exp\left(-\theta\sum_{i=1}^{n} x_i^{\beta}\right) T$$

The procedure, to draw samples from the posterior density function and then to compute the Bayes estimators under a given loss function and the HPD confidence intervals of α, β and θ, is as follows:

(i) Take some initial value of α, β and θ, such as $\alpha^{(0)}$, $\beta^{(0)}$ and $\theta^{(0)}$.
(ii) Generate $\alpha^{(i+1)}$, $\beta^{(i+1)}$ and $\theta^{(i+1)}$ from $p\left(\alpha|\beta^{(i)},\theta^{(i)},\underline{x}\right)$, $p\left(\beta|\alpha^{(i+1)},\theta^{(i)},\underline{x}\right)$ and $p\left(\theta|\alpha^{(i+1)},\beta^{(i+1)},\underline{x}\right)$.
(iii) Repeat Step 2, N times.
(iv) Obtain Bayes estimators of ξ with respect to a squared error loss function:

$$\hat{\xi} = \frac{1}{M}\sum_{i=1}^{M}\xi^{(i)}$$

where ξ may be α, β or θ, $M = N - B$ and B is the burn-in period.

(v) Obtain the HPD confidence interval of α :
Order $\alpha^{(1)}, \ldots, \alpha^{(M)}$ as $\alpha_{(1)} < \ldots < \alpha_{(M)}$ and construct all the 100(1 $-\eta$)% confidence intervals of α, as:

$$\left(\alpha_{(1)}, \alpha_{([M(1-\eta)])}\right), \ldots, \left(\alpha_{([M\eta])}, \alpha_{(M)}\right),$$

where $[M]$ symbolizes the largest integer less than or equal to M. The HPD confidence interval of α is the shortest length interval. The highest posterior density (HPD) confidence interval, which are the optimum and shortest intervals can be obtained by numeral method easily. Similarly, we can compute a $100\,(1-\eta)\,\%$ HPD confidence interval of β and θ.

We have used the OpenBUGS software for the implementation of MCMC technique. The software offers a user-interface, based on dialogue boxes and menu commands, through which the model may then be analyzed using Markov Chain Monte Carlo methods. It is a fully extensible modular framework for constructing and analyzing probability models in Bayesian setup, provided the probability model defined in OpenBUGS. As the $EPL(\alpha, \beta, \theta)$ is not available in OpenBUGS. Thus a module $dexpo.pow.lindley(alpha, beta, lambda)$ is written for $EPL(\alpha, \beta, \theta)$ model to perform full Bayesian analysis in OpenBUGS using the method described in Thomas et al.[27], Thomas [26], Kumar et al. [15] and Lunn et al.[18]. The computer program can be obtained from the author.

5. Data Analysis:

In this section we present the analysis of one real data set for illustration of the proposed methodology. The data extracted from Lee and Wang [16], represents the remission times (in months) of a random sample of 128 bladder cancer patients. The data set is presented below:

0.08, 0.20, 0.40, 0.50, 0.51, 0.81, 0.90, 1.05, 1.19, 1.26, 1.35, 1.40, 1.46, 1.76, 2.02, 2.02, 2.07, 2.09, 2.23, 2.26, 2.46, 2.54, 2.62, 2.64, 2.69, 2.69, 2.75, 2.83, 2.87, 3.02, 3.25, 3.31, 3.36, 3.36, 3.48, 3.52, 3.57, 3.64, 3.70, 3.82, 3.88, 4.18, 4.23, 4.26, 4.33, 4.34, 4.40, 4.50, 4.51, 4.87, 4.98, 5.06, 5.09, 5.17, 5.32, 5.32, 5.34, 5.41, 5.41, 5.49, 5.62, 5.71, 5.85, 6.25, 6.54, 6.76, 6.93, 6.94, 6.97, 7.09, 7.26, 7.28, 7.32, 7.39, 7.59, 7.62, 7.63, 7.66, 7.87, 7.93, 8.26, 8.37, 8.53, 8.65, 8.66, 9.02, 9.22, 9.47, 9.74, 10.06, 10.34, 10.66, 10.75, 11.25, 11.64, 11.79, 11.98, 12.02, 12.03, 12.07, 12.63, 13.11, 13.29, 13.80, 14.24, 14.76, 14.77, 14.83, 15.96, 16.62, 17.12, 17.14, 17.36, 18.10, 19.13, 20.28, 21.73, 22.69, 23.63, 25.74, 25.82, 26.31, 32.15, 34.26, 36.66, 43.01, 46.12, 79.05

5.1. MLE and Asymptotic Confidence Intervals

We have used *optim*() function in R software with option Newton-Raphson method. We have started the iterative procedure by maximizing the log-likelihood function given in equation (3.1) directly with an initial guess for $\alpha = 1.0$, $\beta = 0.1$ and $\theta = 0.1$, far away from the solution. We obtain $\hat{\alpha} = 2.7683$, $\hat{\beta} = 0.5663$ and $\hat{\theta} = 0.8191$, and the corresponding log-likelihood value is -410.4335. The profile log-likelihood plots of α, β and θ are displayed in Figure 3. More information is provided by a visual comparison via Q-Q plot Figure 2(left panel) and P-P plot in Figure 2(right panel), Kumar and Ligges [14]. The straight line pattern of PP plot suggests that the $EPL(\alpha, \beta, \theta)$ model fits the data extremely well.

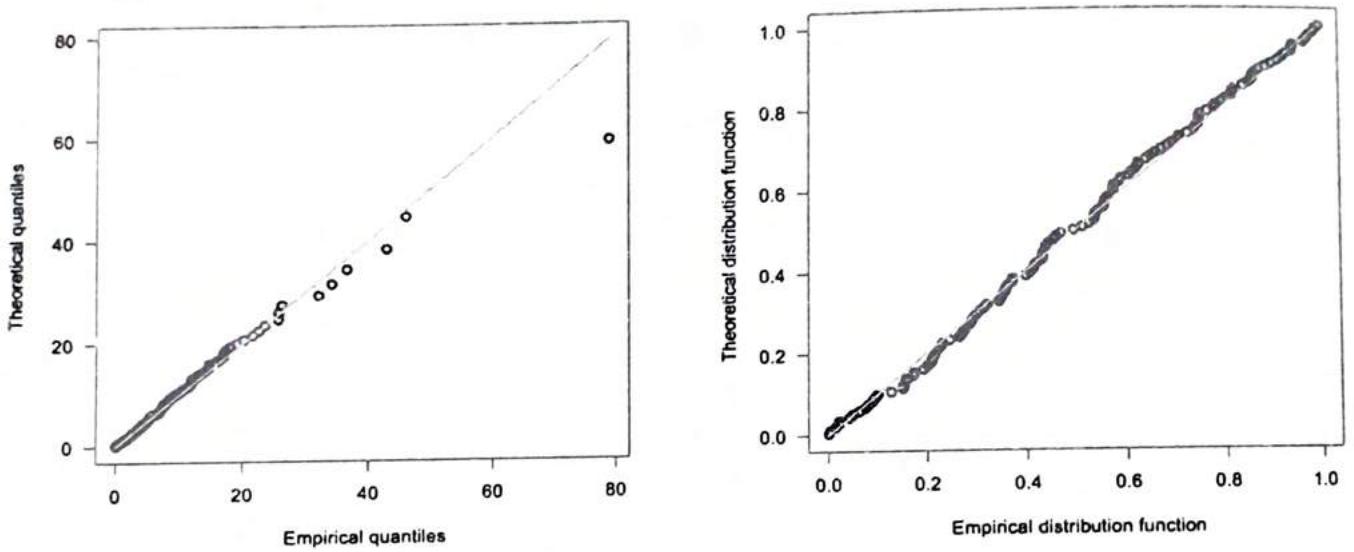


Figure 2. The QQ plot (left panel)and PP plot(right panel).

The result of K-S test is $D = 0.0428$, with the corresponding *p-value* = 0.9726, Therefore, the high *p-value* clearly indicates that $EPL(\alpha, \beta, \theta)$ can be used to analyze the given data. The Akaike information criterion (AIC) and Bayesian information criterion(BIC) can be used to determine which model is most appropriate for the given data. For the given data set AIC = 825.579 and BIC = 834.136. Using the method described in Section 3, we can construct the approximate confidence intervals(ACI) based on MLE's and bootstrap confidence intervals(BCI) for the parameters of $EPL(\alpha, \beta, \theta)$. Table 1 shows the MLE's with their standard errors(SE) and 95% confidence intervals for α, β and θ.

Table 1
MLE, SE and 95% confidence intervals

Parameter	MLE	SE	95% ACI	95% BCI
α	2.7683	1.26616	(0.2867, 5.2500)	(1.0202, 6.5553)
β	0.5663	0.09997	(0.3704, 0.7623)	(0.3302, 0.7261)
θ	0.8191	0.30594	(0.2194, 1.4187)	(0.5295, 2.0822)

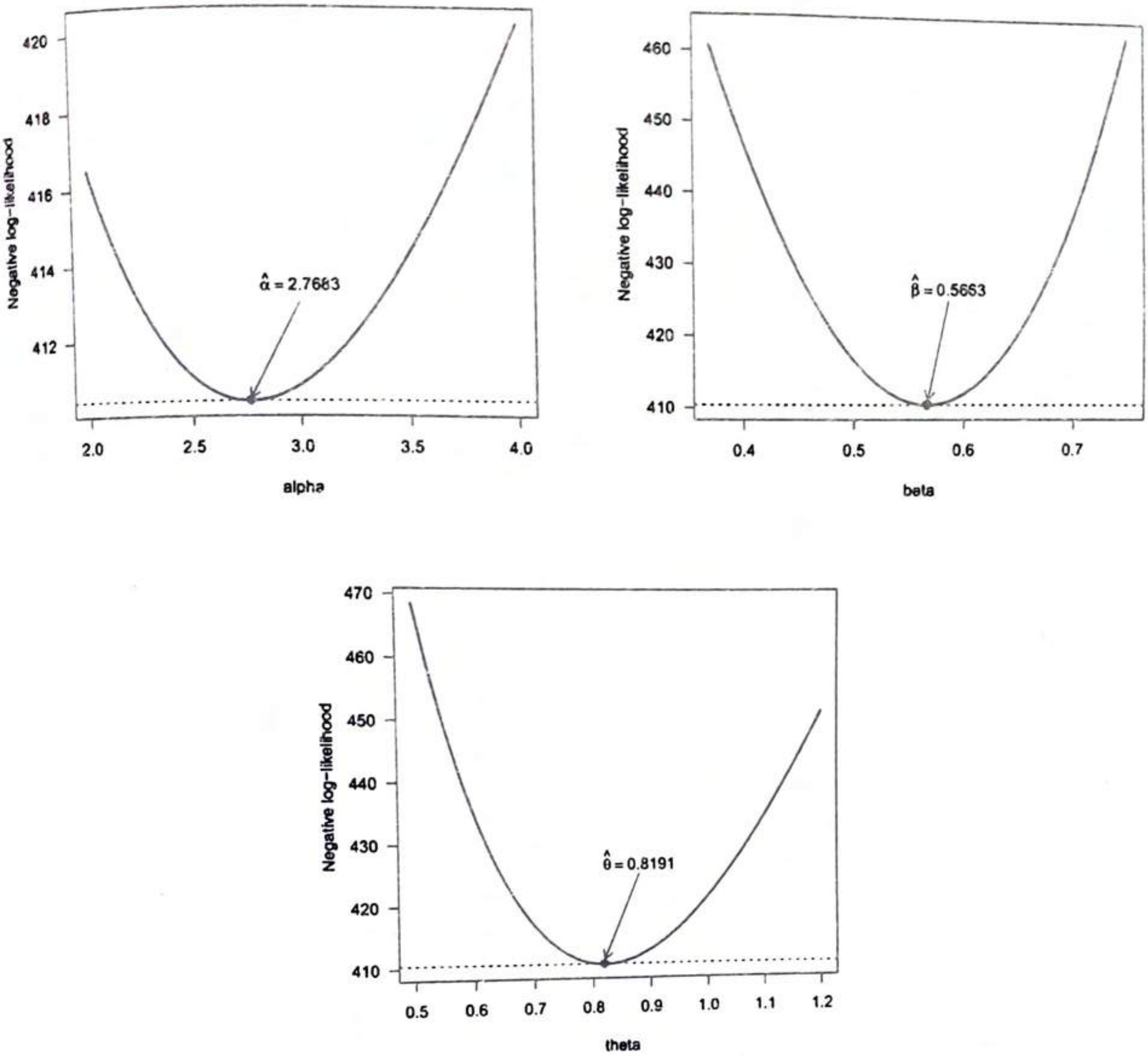


Figure 3. Profile log-likelihood functions of α, β and θ.

6. Bayesian Estimation

In this section we provide the Bayes estimates of the parameters under independent gamma priors for α and β, and uniform prior for θ. In OpenBUGS, we have generated two parallel independent runs of the Gibbs sampler chain with size 30000, for each parameter discarding the first 5000 iterations, to eliminate the effect of the initial values and to avoid correlation problems, we considered a spacing of size 5, obtaining a sample of size 5000 from each chain, we monitored the convergence of the Gibbs samples using the trace and ergodic mean plots, we find that the Markov Chain converge together after approximately 4000 observations. Therefore, burn-in of 5000 samples is more than enough to erase the effect of starting point(initial values). We have chosen initial values for the parameters, wide spread over the parameter space $\alpha = 1.5$, $\beta = 0.1$ and $\theta = 0.1$ for chain 1 and $\alpha = 5.0$, $\beta = 1.0$ and $\theta = 1.0$ for chain 2. Therefore, we have the posterior sample from chain 1 and chain 2 as $(\alpha_i^{(j)}, \beta_i^{(j)}, \theta_i^{(j)})$, $j = 1, \ldots, 5000$ $i = 1, 2$. The posterior analysis of chain 1 is presented. It is also observed that chain 2 produces the similar results.

The convergence is monitored via trace and ergodic mean plots. The trace plot is shown in Figure 4(left panel). The ergodic mean is computed as the mean of all sampled values up to and including that at a given iteration.

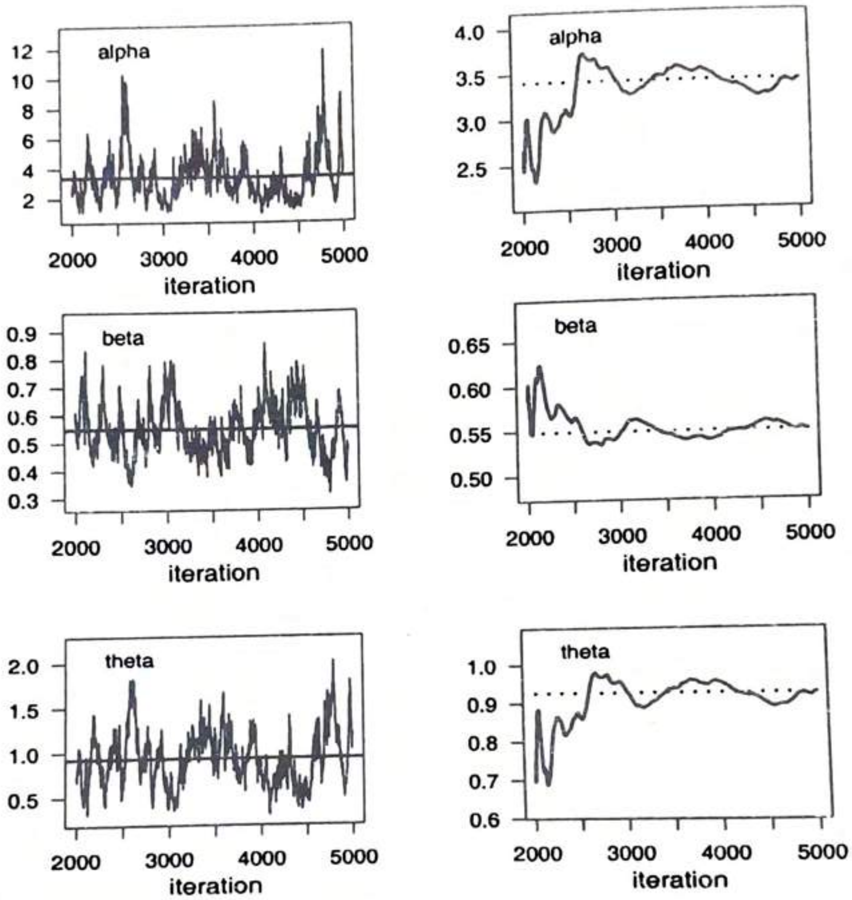


Figure 4. The trace and Ergodic mean plots for α, β and θ.

The convergence pattern based on ergodic average as shown in Figure 4(right panel) is obtained after generating a time series (iteration number) plot of the running mean for each parameter in the chain. The plots show steady convergence to mean(dotted horizontal line).

We have considered various quantities of interest and their numerical values based on posterior sample. The computed values of the posterior mean, mode, standard deviation(SD), 2.5^{th} percentile, first quartile, median, third quartile, 97.5^{th} percentile, mode, skewness and kurtosis of the parameters α and β are presented in Table 2. The Bayes estimates under absolute and zero-one loss eunctions are posterior median and mode, respectively. The numerical summary is based on posterior sample $(\alpha_1^{(j)}, \beta_1^{(j)}, \theta_1^{(j)})$, $j = 1, \ldots, 5000$ for α, β and θ.

Table 2

Numerical summaries based on posterior sample

Characteristics	alpha	beta	theta
Mean	3.8108	0.5354	0.9869
Standard Deviation	2.1362	0.1005	0.3625
2.5th Percentile($P_{2.5}$)	1.3509	0.3565	0.4122
First Quartile (Q_1)	2.3820	0.4645	0.7255
Median	3.2705	0.5285	0.9436
Third Quartile (Q_3)	4.6175	0.5994	1.2003
97.5th Percentile($P_{97.5}$)	9.7273	0.7471	1.8351
Mode	2.6564	0.5161	0.9280
Skewness	1.8963	0.3838	0.6927
Kurtosis	5.3467	0.1063	0.4720

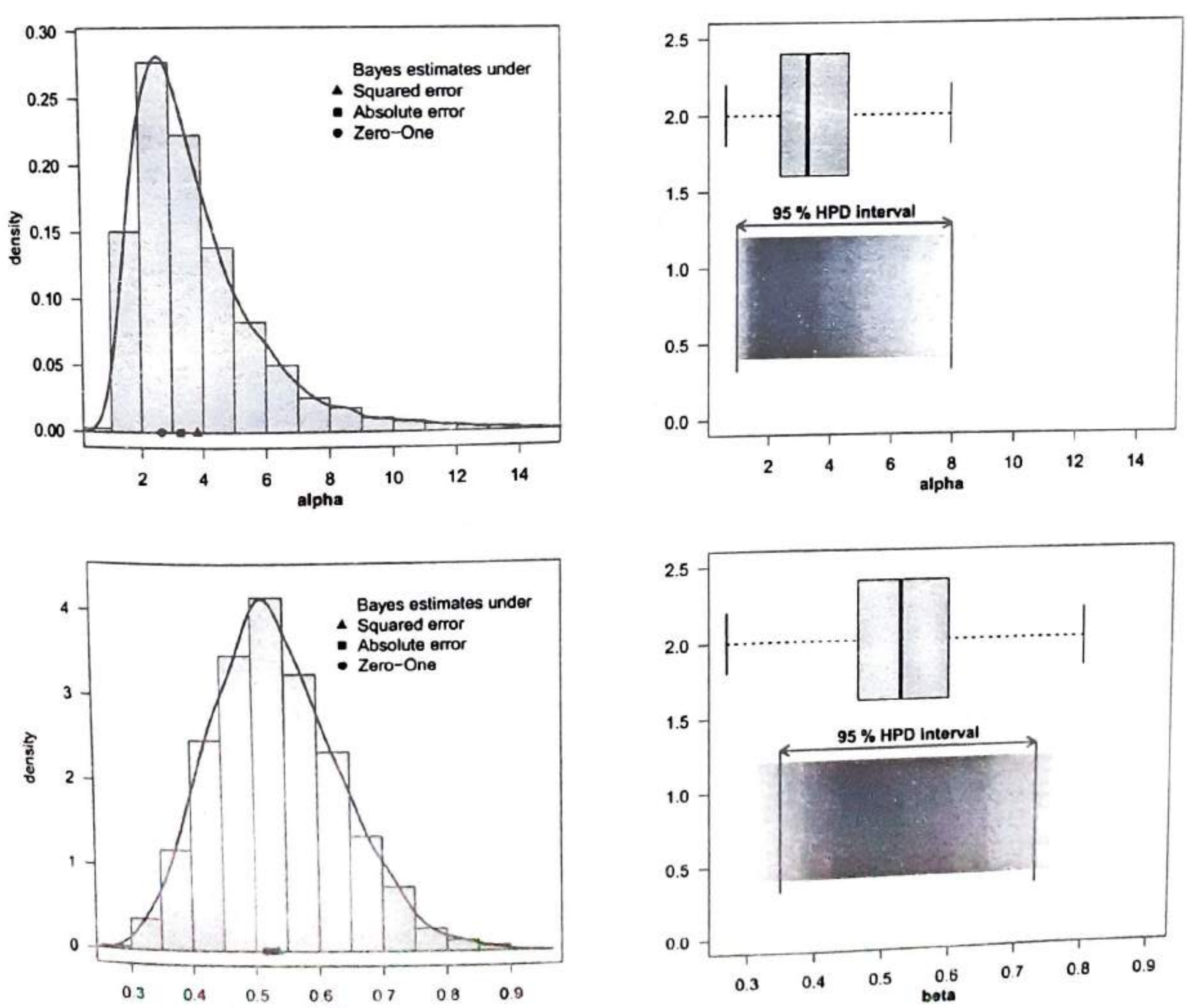
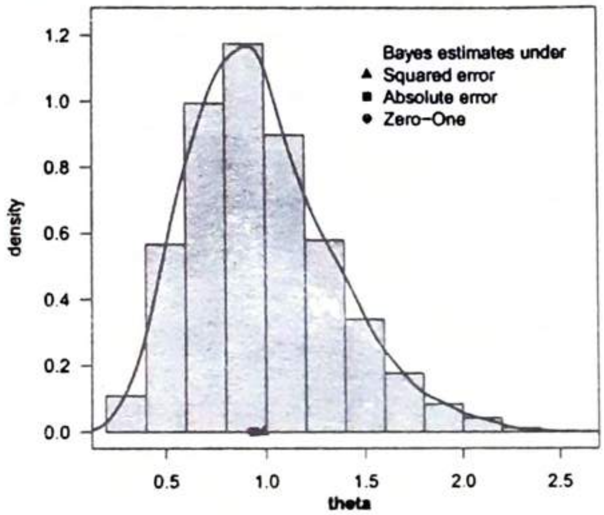


Figure 5. Histogram and kernel density estimate of α and β.

Different types of graphs are presented for qualitative analysis of the marginal posteriors of the parameters. These graphs include the boxplot, density strip plot, histogram and marginal posterior density estimate for the parameters. We have also superimposed the 95% HPD intervals. These graphs provide almost complete picture of the posterior uncertainty about the parameters. We have used the posterior sample $(\alpha_1^{(j)}, \beta_1^{(j)}, \theta_1^{(j)})$; $j = 1, \ldots, 5000$ to draw these graphs. Figure 5 and Figure 6 represents the histogram, marginal posterior density, rug plot and 95% HPD interval for the parameters. The kernel density estimates have been drawn using R software with the assumption of Gaussian kernel and properly chosen values of the bandwidths. Table 3 shows the 95% symmetric credible interval and HPD intervals.



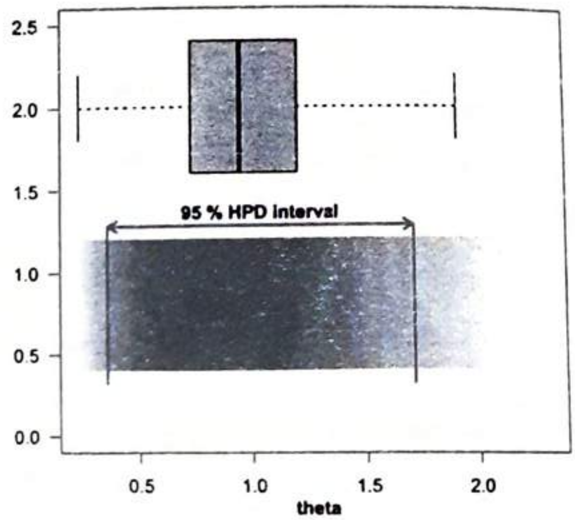


Figure 6. Histogram and kernel density estimate of θ.

Jackson [13] introduced the density strip plot for a univariate distribution as a shaded rectangular strip, whose darkness at a point is proportional to the probability density. It may be noted from Figures 6 that density strip plots are more informative as compared to corresponding boxplot.

Table 3

Bayesian intervals

Parameter	95% SCI	95% HPD
α	(1.3509, 9.7273)	(0.9598, 8.082)
β	(0.3565, 0.7471)	(0.3464, 0.7311)
θ	(0.4122, 1.8351)	(0.3323, 1.7050)

6.1. Comparison with MLE

For the comparison with ML estimates, we have computed the density function at each observed data point for 5000 posterior samples, $(\alpha_1^{(j)}, \beta_1^{(j)}, \theta_1^{(j)})$; $j = 1, \ldots, 5000$ as $f^{(j)}(x_i; \alpha_1^{(j)}, \beta_1^{(j)}, \theta_1^{(j)})$; $i = 1, \ldots, 128$.

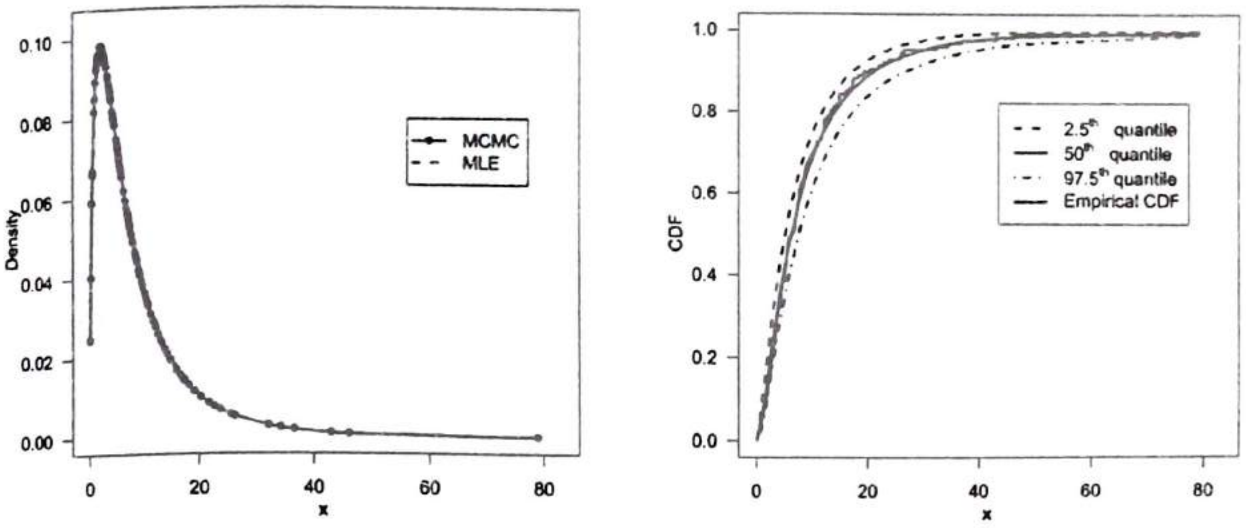


Figure 7. Density and CDF estimate based on MCMC samples.

In Figure 7 we have plotted 2.5^{th}, 50^{th} and 97.5^{th} quantiles of the estimated density, it can be considered as evaluation of model fit, based on posterior sample. The density corresponding to MLE has been plotted using the ML estimates of the parameters. We observe in the Figure 7, the MLEs and the Bayes estimates are quite close.

6.2 Estimation of Hazard and Reliability Functions

The posterior samples may be used to completely summarize the posterior uncertainty about the functions of parameters e.g. reliability and hazard functions. Suppose we wish to give point and interval estimates for reliability and hazard functions at the mission time $t = 2.26$ (at the 20th observed data point).

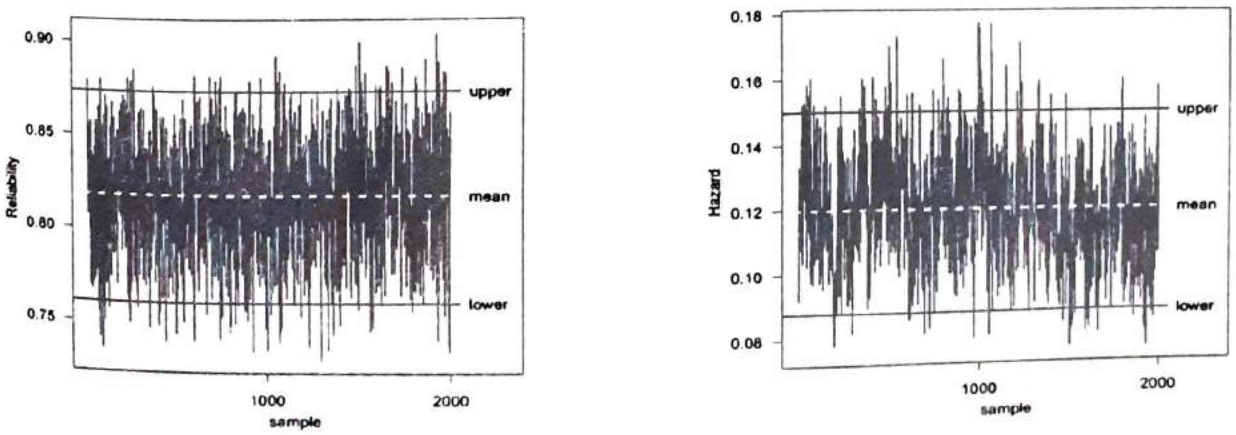


Figure 8. MCMC output of $R(t = 2.26)$ and $h(t = 2.26)$. Dashed line(...) represents the posterior median and solid lines(-) represent lower and upper bounds of 95% probability intervals (HPD)

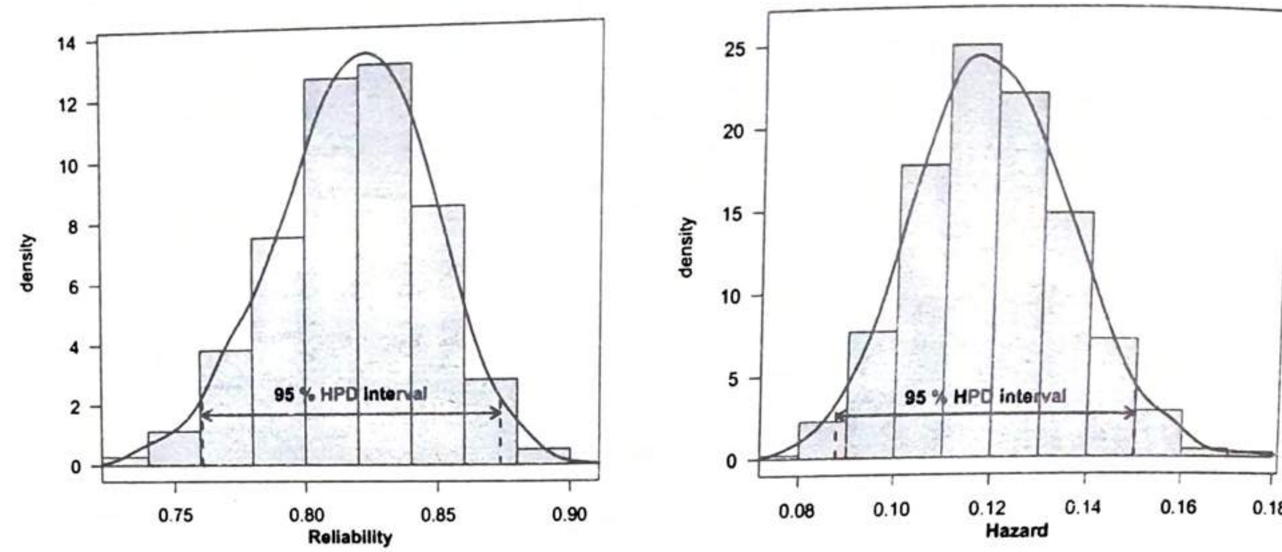


Figure 9. Histogram and density estimate of reliablity and hazard functions

For the given posterior sample $(\alpha_1^{(j)}, \beta_1^{(j)}, \theta_1^{(j)})$; $j = 1, \ldots, 5000$, we can obtain the posterior sample for the reliability and hazard functions at $t = 2.26$ as

$$h^{(j)}\left(x = 2.26\,; \alpha_1^{(j)}, \beta_1^{(j)}, \theta_1^{(j)}\right) \quad ; j = 1, \ldots, 5000 \text{ and}$$
$$R^{(j)}\left(x = 2.26\,; \alpha_1^{(j)}, \beta_1^{(j)}, \theta_1^{(j)}\right) \; ; j = 1, \ldots, 5000.$$

The MCMC results of the posterior mean, mode, standard deviation (SD), first quartile, median, third quartile, mode, skewness of reliability and hazard functions are displayed in Table 4. A trace plot is a plot of the iteration number against the value of the draw of the parameter at each iteration. Figure 8 displays 5000 chain values for the hazard $h(t = 2.26)$ and reliability $R(t = 2.26)$ functions, with their sample median and 95% HPD credible intervals.

Figure 9 represents the histogram and marginal posterior density of reliability and hazard functions at $t = 2.26$. The ML estimates of reliability and hazard function at $t = 2.26$ are computed using invariance property of the MLE. The ML estimates are $\hat{h}(t = 2.26) = 0.1139$ and $\hat{R}(t = 2.26) = 0.8251$.

Table 4

Posterior summary for Reliability and Hazard functions at $t = 2.26$

Parameters	Mean	Median	Mode	SD	Skewness	Kurtosis
R(t = 2.26)	0.8183	0.8197	0.8231	0.0289	-0.2216	-0.1362
h(t = 2.26)	0.1198	0.1192	0.1152	0.0160	0.2354	0.0029

We have computed the 95% BCI for the parameters α, β and θ. The Highest probability density (HPD) credible intervals are computed using the algorithm described by Chen and Shao [3] under the assumption of unimodal marginal posterior distribution and are presented in Table 4.

Now we shall demonstrate the effectiveness of proposed methodology for the entire data set. Since we have an effective MCMC technique, we can estimate any function of the parameters. For this, we have computed the reliability function at each data point, using posterior sample $\left(\alpha_1^{(j)}, \beta_1^{(j)}, \theta_1^{(j)}\right); j = 1, \ldots, 5000$

$$R^{(j)}\left(x_i ; \alpha_1^{(j)}, \beta_1^{(j)}, \theta_1^{(j)}\right) \; ; i = 1, \ldots, 128; j = 1, \ldots, 5000.$$

$$h^{(j)}\left(x_i ; \alpha_1^{(j)}, \beta_1^{(j)}, \theta_1^{(j)}\right) \; ; i = 1, \ldots, 128; j = 1, \ldots, 5000.$$

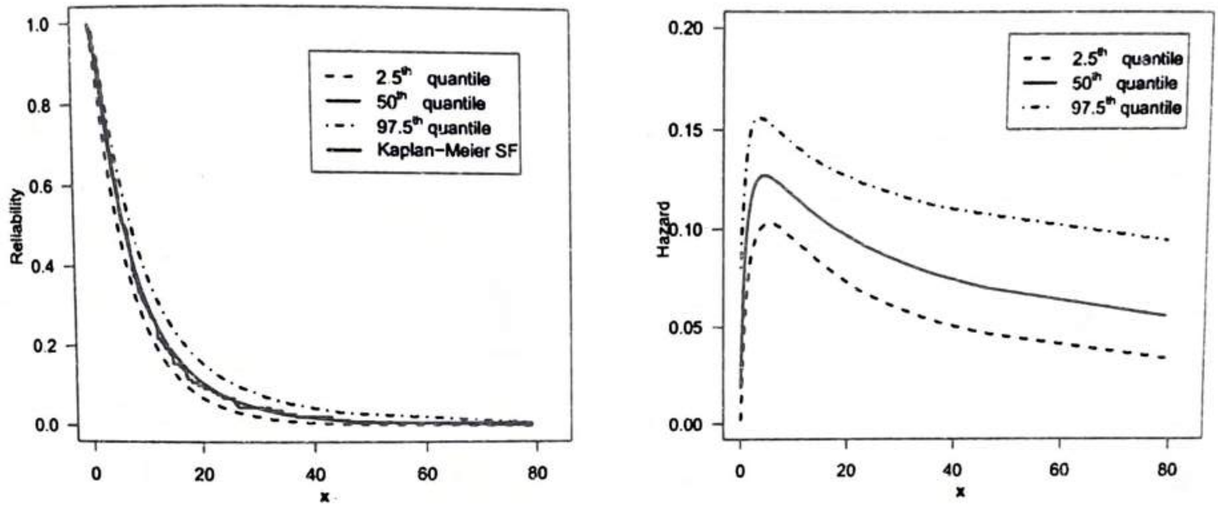


Figure 10. Posterior sample based estimated reliability function(left panel) and hazard function (right panel)

The Figure 10(left panel), exhibits the estimated reliability function, dashed line for 2.5^{th} and 97.5^{th} quantiles and solid line for 50^{th} quantile, using Bayes estimate based on MCMC output. We have superimposed the Kaplan-Meier estimate of the reliability function to make the comparison more meaningful. We observe that reliability estimate based on MCMC is very close to the empirical reliability estimates. The Figure 10(right panel) shows estimated hazard function.

7. Posterior Predictive Analysis

A Bayesian approach for checking whether the model fits the data is known as posterior predictive checking. To do posterior predictive checking, we generate replicates of the dataset from the predictive distribution and compare these replicate data sets to the sample. If the replicate data sets and the sample are similar, we conclude that the model fits the data, Gelman [7] and Gelman et al. [8]. Modern Bayesian computational tools, however, provide straightforward solutions as one can easily simulate predictive samples if MCMC outputs are available from the posterior corresponding to the assumed model, Christensen et al.[4] and Ntzoufras [20].

Let $\underline{x} = (x_1, \ldots, x_n)$ is a vector of n observations from the model $EPL(\alpha, \beta, \theta)$. We can simulate the posterior predictive distribution as follows:

- Obtain posterior sample $\delta^{(j)} = \left(\alpha^{(j)}, \beta^{(j)}, \theta^{(j)}\right)$; $j = 1, 2, \ldots, M$
- For each posterior sample $\delta^{(j)}$, simulate n data points as $x_i^{rep,j} \sim EPL(\alpha^{(j)}, \beta^{(j)}, \theta^{(j)})$; $j = 1, 2, \ldots, M$ and $i = 1, 2, \ldots, n$
- Thus, for each sampled value, $(\alpha^{(j)}, \beta^{(j)}, \theta^{(j)})$, we obtain M replicated data set $\underline{x}^{rep,j} = (x_1^{rep,j}, \ldots, x_n^{rep,j})$.

The predictive analysis is based on 2000 posterior samples. For this purpose, 2000 samples have been drawn from the posterior using MCMC procedure and then obtained predictive samples from the model under consideration using each simulated posterior sample. In fact, we have 2000 replicates for each data point x_i ; $i = 1, \ldots, 128$.

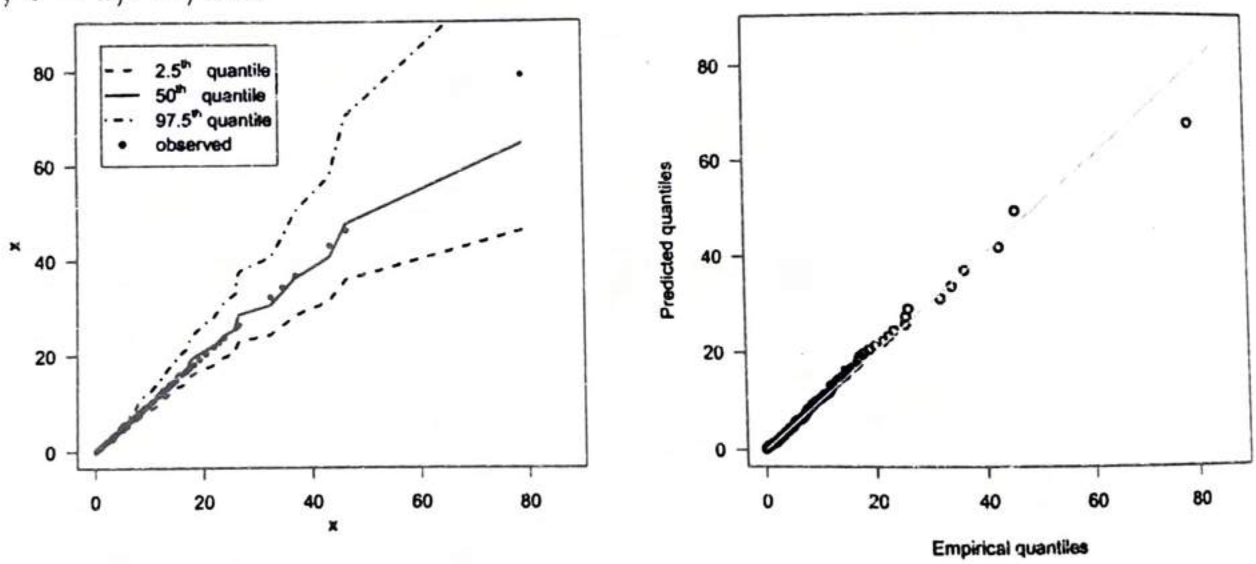


Figure 11. The model fit based on predicted values(left panel) and Q-Q plot of predicted quantiles versus observed quantiles (right panel)

The graphical method is one of the best way to assess model adequacy based on posterior predictive distributions. We view the model-checking as a comparison of the data with the replicated data given by the model, which includes exploratory graphics, Pandey and Kumar [22]. Figure 11(right panel) represents the Q-Q plot of predicted quantiles versus observed quantiles. The graph of model fit based on replicated data given by the model is displayed in Figure 11(left panel). Perhaps it may be considered as posterior predictive check for the model adequacy graphically, solid line(-) represents the posterior median and dashed lines(...) represent lower and upper bounds of 95% probability intervals.

empirical distribution function is superimposed. We, therefore, conclude that the exponentiated power Lindley distribution is compatible with the given data set.

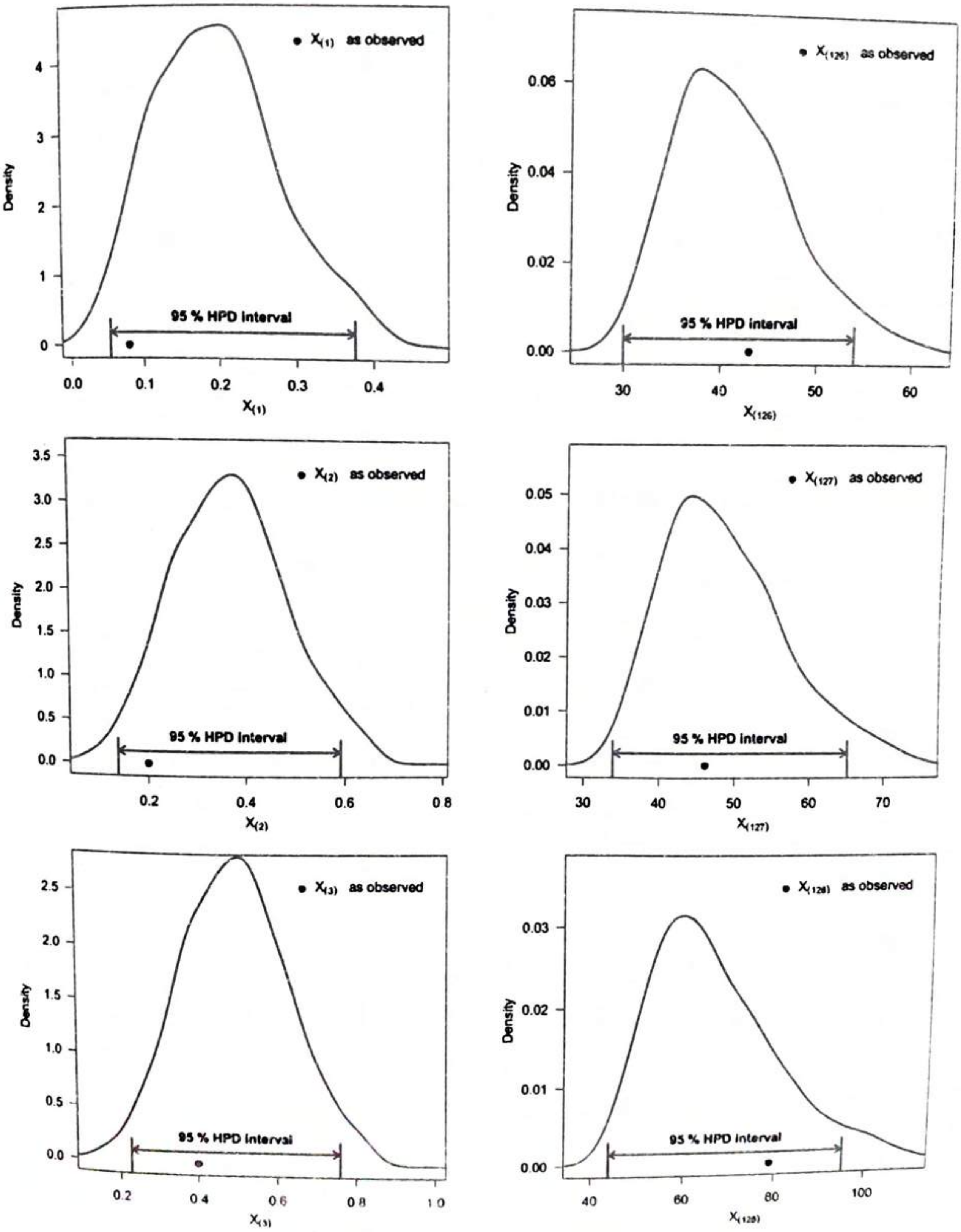


Figure 12. Posterior predictive densities of $x_{(1)}^{rep}$, $x_{(2)}^{rep}$, $x_{(3)}^{rep}$, $x_{(126)}^{rep}$, $x_{(127)}^{rep}$ and $x_{(128)}^{rep}$, points on x-axis represent corresponding observed value.

To obtain further clarity on our conclusion for the study of model compatibility, we have considered plotting of density estimates of

$$\left(x^{rep}_{(1)}, x^{rep}_{(2)}, x^{rep}_{(3)}, x^{rep}_{(126)}, x^{rep}_{(127)} \text{ and } x^{rep}_{(128)}\right)$$

replicated future observations from the model with superimposed corresponding observed data, Figure 12. As the Figure 12 shows, the posterior predictive distributions are centered over the observed values, which indicate good fit. In general, the distribution of replicated data appears to match that of the observed data fairly well.

The Table 5 shows the MCMC results of the posterior mean, median, mode and 95% HPD credible intervals for $\left(x^{rep}_{(1)}, x^{rep}_{(2)}, x^{rep}_{(3)}, x^{rep}_{(20)}, x^{rep}_{(126)}, x^{rep}_{(127)} \text{ and } x^{rep}_{(128)}\right)$. Also, from the visual posterior predictive checks, Figure 11 and Figure 12, we observe no systematic deviations of the observed data from the predicted data. Overall, the results of the posterior predictive simulation indicate that model fits the data particularly well.

Table 5

Results on posterior predictive simulation

Data points	Observed	Based on MCMC			
		Mean	Median	Mode	HPD
$X_{(1)}$	0.08	0.21	0.20	0.19	(0.054, 0.376)
$X_{(2)}$	0.20	0.38	0.37	0.36	(0.136, 0.591)
$X_{(3)}$	0.40	0.51	0.50	0.50	(0.228, 0.760)
$X_{(20)}$	2.26	1.95	1.98	1.98	(1.440, 2.510)
$X_{(126)}$	43.01	37.44	41.56	40.60	(29.97, 55.07)
$X_{(127)}$	46.12	43.87	49.16	47.67	(34.56, 67.23)
$X_{(128)}$	79.05	59.43	67.54	64.56	(43.36, 98.30)

8. Conclusion

In this paper we have considered the inferential procedures for three-parameter exponentiated power Lindley distribution. The methods described to implement recent computational-based classical as well as Bayesian approaches related to exponentiated power Lindley distribution. The MCMC procedure provides a flexible environment for fitting a wide range of models. The MCMC sample completely summarizes posterior distribution about the parameters. We have used exploratory data analysis techniques for the posterior analysis. We have shown that it is true for any function of the parameters such as hazard function.

reliability etc. We have obtained the probability intervals for parameters, hazard and reliability functions. We used the parametric bootstrap to compute the confidence intervals. We have also presented the model compatibility analysis via the posterior predictive check method. It is observed that in estimating the posterior predictive density function at any point, the MCMC samples can be used effectively. Finally, we have applied the developed techniques on a real data set. Therefore, MCMC procedure can easily be applied for Bayesian estimation and prediction related to exponentiated power Lindley model.

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