Unit 3: Spectral Theory

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1 Eigenvectors and eigenvalues

The scalar multiplication of a vector x by a scalar λ is

$$\lambda x,$$

which simply multiplies each element (component) of x by λ. This operation does not change the direction of the vector x and simply *striches* out the vector.

Observe that there are several ways that a vector can be stretched out:

- The stretch is an expansion or shrinkage in the same direction if the stretch factor $\lambda > 0$.
- The stretch shrink the vector all the way to the origin if the stretch factor $\lambda = 0$.
- The stretch is in the opposite direction if $\lambda < 0$.

Scalar multiplication can be replicated by a special matrix multiplication:

$$\lambda I x = \lambda x.$$

However, there are many other matrices that, when applied to the appropriate vector or collection of vectors, act exactly like scalar multiplication.

If one chooses an arbitrary $n \times n$ matrix A and an arbitrary n-dimensional vector x, then the vector $y = Ax$ may point in some other direction than the original vector. There are, however, certain vectors x for which the matrix-vector product points in the same "direction" as that vector, i.e., where there exists a $\lambda \in \mathbb{R}$ such that $y = \lambda x$.

Note that if we view a matrix as a linear transformation of the input vector, then this says that the direction of the input vector is unchanged by the action of the linear transformation.

Consider the matrix

$$A = \begin{pmatrix} -3 & 4 \\ 4 & 3 \end{pmatrix}.$$

- If one applies A to the first coordinate vector e_1, then A changes the direction of it:

$$Ae_1 = \begin{pmatrix} -3 & 4 \\ 4 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -3 \\ 4 \end{pmatrix} \neq \lambda \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

  for any $\lambda \in \mathbb{R}$.

- Similarly, if one applies A to the vector $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$, then A changes the direction of it too:

$$A \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -3 & 4 \\ 4 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 7 \end{pmatrix} \neq \lambda \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

  for any $\lambda \in \mathbb{R}$.

- However, if one applies A to the vector $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$, then one gets the following:

$$A \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} -3 & 4 \\ 4 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 5 \\ 10 \end{pmatrix} = 5 \begin{pmatrix} 1 \\ 2 \end{pmatrix},$$

  i.e., the vector is simply rescaled, and the direction of the vector is unchanged.

From the above discussion, in general, the question of interest is:

Given an $n \times n$ matrix A, for what vectors $x \in \mathbb{R}^n$ and numbers $\lambda \in \mathbb{R}$ does $Ax = \lambda x$?

These vectors and scalars are *eigenvectors* and *eigenvalues* !

Definition. An **eigenvector** of an $n \times n$ matrix A is a nonzero vector x such that $Ax = \lambda x$ for some scalar λ. A scalar λ is called an **eigenvalue** of A if there is a nontrivial solution x of $Ax = \lambda x$; such an x is called an **eigenvector corresponding to** λ.

Note that

- If x is an eigenvector of A, then so is any rescaled vector ax for $a \in \mathbb{R}, a \neq 0$. Moreover, ax still has the same eigenvalue as x has. For this reason, we usually look only for unit eigenvectors.
- From the definition, it is clear that an eigenvector is a nonzero vector.

2 Some simple examples

Before describing eigenvectors and eigenvalues in more detail, let's go through several examples of matrices and their eigenvectors and eigenvalues, which will illustrate more general properties that we will discuss in more detail below.

Example 1 (Diagonal matrix).

Consider the following matrix:

$$A = \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}.$$

Q1. *What can we say about the action of A on an arbitrary vector ?*

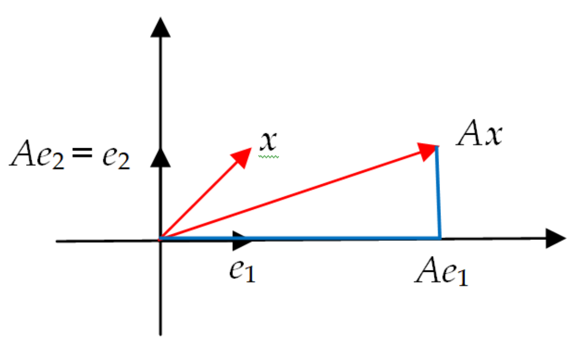
Let us apply A to an arbitrary vector $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$. We have

$$Ax = \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 3x_1 \\ x_2 \end{pmatrix}.$$

We also observe that there is no $\lambda \in \mathbb{R}$ such that

$$\begin{pmatrix} 3x_1 \\ x_2 \end{pmatrix} = \lambda \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

if x_1 and x_2 both are nonzero. So, in this case, the vector x cannot be eigenvector of the matrix A. The following figure illustrates this when $x = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$:



Note that in this case

$$Ax = A\begin{pmatrix} 1 \\ 1 \end{pmatrix} = A\begin{pmatrix} 1 \\ 0 \end{pmatrix} + A\begin{pmatrix} 0 \\ 1 \end{pmatrix} = Ae_1 + Ae_2.$$

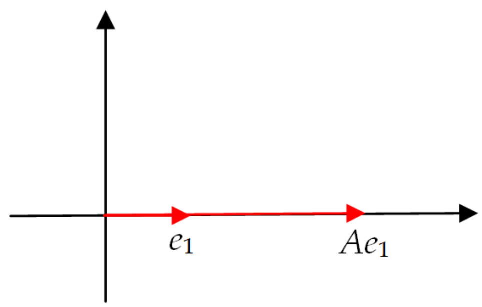
Q2. *What are examples of eigenvalues/eigenvectors of this matrix?*

Let us apply A to the coordinate vectors e_1 and e_1. We have

$$Ae_1 = \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 3 \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Thus,

Eigenvalue: $\lambda = 3$, Eigenvector: $v_{\lambda=3} = e_1$.

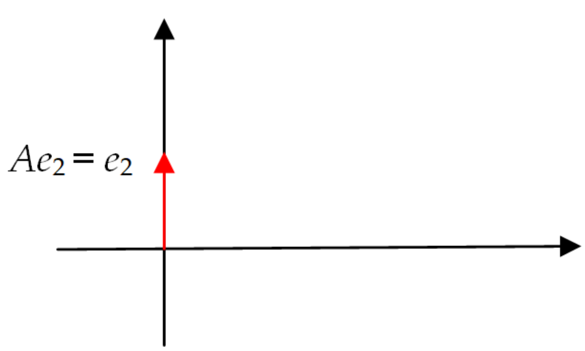


Similarly, if we apply A to the coordinate vectors e_2, we have

$$Ae_2 = \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Thus,

Eigenvalue: $\lambda = 1$, Eigenvector: $v_{\lambda=1} = e_2$.



Here, we have identified two eigenvalue-eigenvector pairs.

Q3. *What does the discussion for this example illustrate*?

1. The matrix A has two distinct eigenvalues and each is associated with an eigenvector.
2. The two eigenvectors corresponding to distinct eigenvalues are orthogonal to each other. Hence they are linearly independent and span all of $\mathbb{R}^2$.
3. Each of the two directions or subspaces defined by one of the eigenvectors is stretched out by a different amount under the action of A.
4. Let $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ be a vector on $\mathbb{R}^2$ not lying along one of the coordinate axes, i.e., if x_1 and x_2 both are nonzero. Then

$$x = x_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Clearly, from this decomposition in view of (3), under the action of A the vector x will move to some other vector on $\mathbb{R}^2$ that is not just a stretch, but instead combines both directions. So, x is not an eigenvector of A.

Example 2 (Identity matrix).

Consider the identity matrix:

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Q1. *What can we say about the action of A on an arbitrary vector* ?

Let us apply A to an arbitrary vector $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$. We have

$$Ax = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} .$$

What this says is that any vector on $\mathbb{R}^2$ is an eigenvector of this A, with eigenvalue 1. If we are only considering normalized unit-length vectors, then what this says is that any vector on the L_2 unit ball is an eigenvector of this A, with eigenvalue 1. By the way, we can choose any pair of linearly independent vectors as eigen vectors.

Q2. *What are examples of eigenvalues/eigenvectors of this matrix*?

Let us apply A to the coordinate vectors e_1. We have

$$Ae_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} .$$

Thus,

Eigenvalue: $\lambda = 1$, Eigenvector: $v_{\lambda=1} = e_1$.

Similarly, if we apply A to the coordinate vectors e_2, we have

$$Ae_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} .$$

Thus,

Eigenvalue: $\lambda = 1$, Eigenvector: $v_{\lambda=1} = e_2$.

So, here are two vectors along the two coordinate axes, that are eigenvectors. Both have the same eigenvalue 1. When there are multiple eigenvectors associated with a given eigenvalue, it is common to call that a *degenerate eigenvalue*.

Q3. *What does the discussion for this example illustrate*?

- For this matrix, there are multiple eigenvectors for a given eigenvalue, whixh are not scalar multiples of one another. One can choose those 2 eigenvectors to be linearly independent, and even to be orthogonal to each other.
- The span of any two orthogonal eigenvectors of A is $\mathbb{R}^2$.

Example 3 (Reflection matrix).

Consider the identity matrix:

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Q1. *What can we say about the action of A on an arbitrary vector* ?

Let us apply A to an arbitrary vector $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$. We have

$$Ax = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ x_1 \end{pmatrix}.$$

This matrix perfroms a reflection about the line $x_2 = x_1$. If one considers a vector in the subspace defined by $x_2 = x_1$, then the vector is unchanged by the matrix; and if one considers a vector in the subspace perpendicular to $x_2 = x_1$, then the vector is "reflected through" the line $x_2 = x_1$.

Q2. *What are examples of eigenvalues/eigenvectors of this matrix?*

Let us apply A to the vectors $\begin{pmatrix} \alpha \\ \alpha \end{pmatrix}$. We have

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha \\ \alpha \end{pmatrix} = \begin{pmatrix} \alpha \\ \alpha \end{pmatrix}.$$

Thus,

Eigenvalue: $\lambda = 1$

Eigenvector: $v_{\lambda=1} = \begin{pmatrix} \alpha \\ \alpha \end{pmatrix}$.

Similarly, if we apply A to the vector $\begin{pmatrix} \alpha \\ -\alpha \end{pmatrix}$, we have

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha \\ -\alpha \end{pmatrix} = \begin{pmatrix} -\alpha \\ \alpha \end{pmatrix} = (-1) \begin{pmatrix} \alpha \\ -\alpha \end{pmatrix}.$$

Thus,

Eigenvalue: $\lambda = -1$

Eigenvector: $v_{\lambda=1} = \begin{pmatrix} \alpha \\ -\alpha \end{pmatrix}$.

So, here are two vectors along the two coordinate axes, that are eigenvectors. In this case, we can choose the orthonormal vectors $\frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

as eigen vectors relative to $\lambda = 1$ and $\lambda = -1$ respectively.

Q3. *What does the discussion for this example illustrate*?

- Eigenvalues can be positive or negative; and associated with each distinct eigenvalue, there is an eigenvector.
- The two eigenvectors are perpendicular to each other.

Example 4 (**Projection matrix**).

Consider the following matrix:

$$A = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}.$$

Q1. *What can we say about the action of A on an arbitrary vector* ?

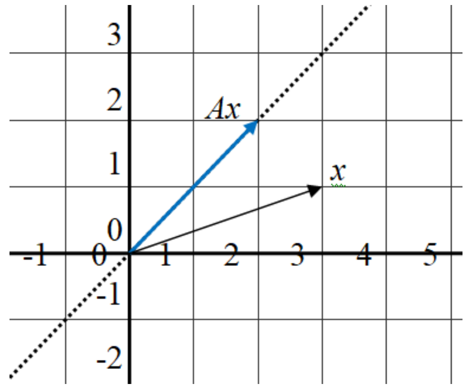
Let us apply A to an arbitrary vector $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$. We have

$$Ax = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} (x_1 + x_2)/2 \\ (x_1 + x_2)/2 \end{pmatrix}.$$

That is, A "projects" the vector x onto the line $x_1 = x_2$. For example, we have

$$A \begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix} \begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \end{pmatrix}.$$

It is illustrated in the following figure.



Q2. *What are examples of eigenvalues/eigenvectors of this matrix?*

Let us apply A to the vector $\begin{pmatrix}1\\1\end{pmatrix}$. We have

$$A\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}1/2 & 1/2\\1/2 & 1/2\end{pmatrix}\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}1\\1\end{pmatrix}.$$

Thus,

Eigenvalue: $\lambda = 1$

Eigenvector: $v_{\lambda=1} = \begin{pmatrix}1/\sqrt{2}\\1/\sqrt{2}\end{pmatrix}$, normalised vector relative to $\begin{pmatrix}1\\1\end{pmatrix}$.

Similarly, if we apply A to the vector $\begin{pmatrix}1\\-1\end{pmatrix}$, we have

$$A\begin{pmatrix}1\\-1\end{pmatrix} = \begin{pmatrix}1/2 & 1/2\\1/2 & 1/2\end{pmatrix}\begin{pmatrix}1\\-1\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix} = 0\begin{pmatrix}1\\-1\end{pmatrix}.$$

Thus,

Eigenvalue: $\lambda = 0$

Eigenvector: $v_{\lambda=0} = \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix}$, normalised vector relative to $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

Observe that these are the same eigenvectors as for the reflection matrix given in Example 3, but the eigenvalues are different. These are also the same eigenvectors as one possible choice for the identity matrix given in Example 2, but again the eigenvalues are different.

Q3. *What does the discussion for this example illustrate*?

- Since the two columns are linearly-dependent, A is a rank-deficient matrix. Thus, the output must lie in a lower-dimensional subspace, and that subspace is the span of the one linearly-independent column.
- Under the action of A on a vector, the vector is unchanged along one direction, while along the other direction, the vector is shrunk to the origin.
- This is an example of a "projection matrix," and so it really is doing a projection. (The reason for this is that the non-zero eigenvalue equals unity – more generally, an $n \times n$ matrix in which all the eigenvalues equal 0 or 1 is a projection matrix onto the subspace spanned by its columns.)

3 Computing eigenvectors and eigenvalues

The examples of the previous section were sufficiently simple that we were able just to write down eigenvalue-eigenvector pairs by inspection, but in general they must be computed with some algorithm. There are several ways to compute eigenvalues and eigenvectors.

- **With determinants.** This approach is most appropriate for small 2×2 and 3×3 matrices, and it can be understood in terms of ideas like linear dependence and independence. It is not a practically-appropriate method for larger more realistic problems.
- **With quadratic forms.** This approach illustrates the basic algebraic and geometric ideas of how to compute eigenvectors and eigenvalues for larger more realistic problems, in a way that conveys the basic ideas more generally.
- **In actual numerical practice.** The actual practical computation of eigenvectors is quite involved, and we won't discuss it.

Here, we will consider the first two ways. In many cases, understanding the basic ideas is sufficient, basically since one typically calls existing software as a black box. Note, however, that while the actual practical computation of eigenvectors is quite involved, many of the ideas have roots in the quadratic form perspective we will discuss.

4 Basic properties of determinants

Let

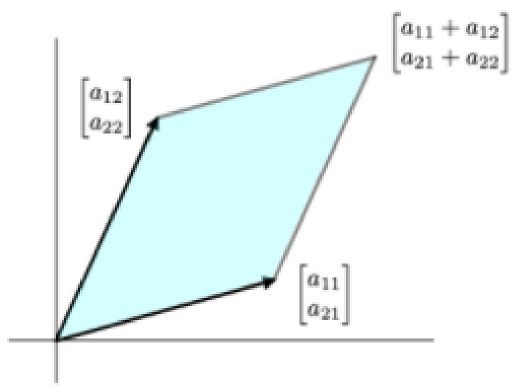
$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}.$$

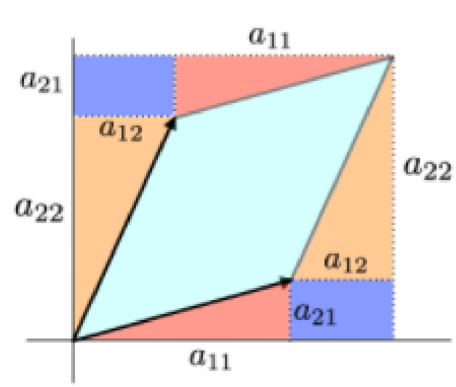
The determinant of A is given by

$$|A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{21}a_{12}.$$

Question: What does the determinant of a matrix geometrically mean?

Answer: The determinant $|A|$ represents the area of the parallelogram spanned by the columns of the matrix A. See the figure given below.





From the figure, we obtain

$$\begin{aligned} \text{Area of parrallelogram } &= (a_{11}+a_{12})(a_{21}+a_{22}) - a_{12}a_{22} \\ &\quad - a_{11}a_{21} - 2a_{21}a_{12} \\ &= a_{11}a_{21} + a_{11}a_{22} - a_{12}a_{21} + a_{11}a_{22} \\ &\quad - a_{11}a_{22} - a_{11}a_{21} - 2a_{21}a_{12} \\ &= a_{11}a_{22} - a_{21}a_{12}. \end{aligned}$$

Therfore, by definition,

$$\text{Area of parrallelogram} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}.$$

Question: What does "the area = 0" mean?

Answer: When $\begin{pmatrix} a_{11} \\ a_{21} \end{pmatrix}$ is linearly dependent on $\begin{pmatrix} a_{12} \\ a_{22} \end{pmatrix}$, i.e., when

$$\begin{pmatrix} a_{11} \\ a_{21} \end{pmatrix} = \alpha \begin{pmatrix} a_{12} \\ a_{22} \end{pmatrix} = \begin{pmatrix} \alpha a_{12} \\ \alpha a_{22} \end{pmatrix},$$

we have

$$a_{11}a_{22} - a_{12}a_{21} = (\alpha a_{12})a_{22} - a_{12}(\alpha a_{22}) = 0.$$

Conversely, if Area $= 0$, then

$$\begin{aligned} & a_{11}a_{22} - a_{12}a_{21} = 0 \\ \Rightarrow \quad & \frac{a_{11}}{a_{12}} = \frac{a_{21}}{a_{22}} \\ \Rightarrow \quad & \begin{cases} a_{11} = ka_{12} \\ a_{21} = ka_{22} \end{cases} \\ \Rightarrow \quad & \begin{pmatrix} a_{11} \\ a_{21} \end{pmatrix} = \begin{pmatrix} ka_{12} \\ ka_{22} \end{pmatrix} = k \begin{pmatrix} a_{12} \\ a_{22} \end{pmatrix} \end{aligned}$$

for some number k. Hence the columns of A are linearly dependent.

Note that these ideas can easily be generalized to 3×3 matrices and even to $n \times n$ matrices.

Some properties of determinants

- A square matrix A is invertible if and only if $|A| \neq 0$.

- Equivalently, a square matrix A is not invertible if and only if $|A| = 0$.
- $|A| = 0$ if and only if the columns/rows of A are linearly dependent.
- If A is triangular, then $|A|$ is the product of the entries on the main diagonal of A.
- Recall that A is invertible if and only if the equation $Ax = 0$ has only the trivial solution. Notice that the number 0 is an eigenvalue of A if and only if there is a nonzero vector x such that $Ax = 0x = 0$, which happens if and only if $0 = |A - 0I| = |A|$. Hence A is invertible if and only if 0 is not an eigenvalue.

5 Using determinants to understand eigendecompositions

Let's now see how to use determinants to understand eignedecompositions of matrices. To do so, recall that the basic eigenvalue equation,

$$Ax = \lambda x$$

can also be written as

$$(A - \lambda I)x = 0.$$

If A is a 3×3 matrix, then

$$A - \lambda I = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} a_{11} - \lambda & a_{12} & a_{13} \\ a_{21} & a_{22} - \lambda & a_{23} \\ a_{31} & a_{32} & a_{33} - \lambda \end{pmatrix}.$$

Thus,

λ is an eigenvalue of an $n \times n$ matrix A if and only if the equation

$$(A - \lambda I)x = 0$$

has a nontrivial solution. The set of all solutions of this equation together with the *zero vector* is a subspace of $\mathbb{R}^n$ and is called the **eigenspace** of A corresponding to the eigenvalue λ.

Example 5.

Find the eigenvalues of $A = \begin{pmatrix} 2 & 3 \\ 3 & -6 \end{pmatrix}$.

Solution. We must find all scalars λ such that the matrix equation

$$(A - \lambda I)x = 0$$

has a nontrivial solution. This problem is equivalent to finding all λ such that the matrix $A - \lambda I$ is not invertible, i.e., $|A - \lambda I| = 0$. We have

$$\begin{aligned}
|A - \lambda I| &= \begin{vmatrix} 2-\lambda & 3 \\ 3 & -6-\lambda \end{vmatrix} \\
&= (2-\lambda)(-6-\lambda) - 3 \times 3 \\
&= \lambda^2 + 4\lambda - 21 \\
&= (\lambda - 3)(\lambda + 7).
\end{aligned}$$

If $|A - \lambda I| = 0$, then

$$\lambda = 3 \text{ or } \lambda = -7.$$

Therefore, the eigenvalues of A are 3 and -7.

This example motivates the following result.

A scalar λ is an eigenvalue of an $n \times n$ matrix A if and only if λ satisfies the characteristic equation

$$|A - \lambda I| = 0.$$

Example 6.

Find the eigenvalues of $A = \begin{pmatrix} 5 & -2 & 6 & -1 \\ 0 & 3 & -8 & 0 \\ 0 & 0 & 5 & 4 \\ 0 & 0 & 0 & 1 \end{pmatrix}$.

Solution. We have

$$\begin{aligned}
|A - \lambda I| &= \begin{vmatrix} 5-\lambda & -2 & 6 & -1 \\ 0 & 3-\lambda & -8 & 0 \\ 0 & 0 & 5-\lambda & 4 \\ 0 & 0 & 0 & 1-\lambda \end{vmatrix} \\
&= (5-\lambda)^2(3-\lambda)(1-\lambda).
\end{aligned}$$

If $|A - \lambda I| = 0$, then

$$(5-\lambda)^2(3-\lambda)(1-\lambda)0 \Rightarrow \lambda = 5, 3, 1.$$

Therefore, the eigenvalues of A are 5, 3, and 1.

In the last two examples , $|A - \lambda I|$ is a polynomial in λ. It can be shown that if A is an $n \times n$ matrix, then $|A - \lambda I|$ is a polynomial of degree n called the **characteristic polynomial** of A.

The eigenvalue 5 in the last example is said to have **multiplicity 2** because $5 - \lambda$ occurs two times as a factor of the characteristic polynomial. In general, the (algebraic) multiplicity of an eigenvalue is its multiplicity as a root of the characteristic equation.

Each eigenvalue determines the corresponding eigenvector x by solving the system

$$(A - \lambda I)x = 0.$$

Example 7.

Find the eigenvectors of $A = \begin{pmatrix} 2 & 3 \\ 3 & -6 \end{pmatrix}$.

Solution. We have found that the eigenvalues of A are 3 and -7.

$\underline{\lambda = 3}$

Let $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ be the eigenvector corresponding to $\lambda = 3$.

Then

$$(A - \lambda I)x = 0 \Rightarrow \left(\begin{pmatrix} 2 & 3 \\ 3 & -6 \end{pmatrix} - 3 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$$
$$\Rightarrow \begin{pmatrix} -1 & 3 \\ 3 & -9 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$$

Therefore, we obtain a system of equations

$$\begin{cases} -x_1 + 3x_2 = 0 \\ 3x_1 - 9x_2 = 0 \end{cases}.$$

However, both the equations represent the same equation:

$$-x_1 + 3x_2 = 0.$$

The simplest solution is $x_1 = 3$ and $x_2 = 1$. Therefore, an eigenvector relative to the eigenvalue $\lambda = 3$ is $\begin{pmatrix} 3 \\ 1 \end{pmatrix}$.

$\underline{\lambda = -7}$

Let $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ be the eigenvector corresponding to $\lambda = -7$. Then

$$(A - \lambda I)x = 0 \Rightarrow \left(\begin{pmatrix} 2 & 3 \\ 3 & -6 \end{pmatrix} + 7 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$$
$$\Rightarrow \begin{pmatrix} 9 & 3 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$$

Therefore, we obtain a system of equations

$$\begin{cases} 9x_1 + 3x_2 = 0 \\ 3x_1 + x_2 = 0 \end{cases}.$$

However, both the equations represent the same equation:

$$3x_1 + x_2 = 0.$$

The simplest solution is $x_1 = -1$ and $x_2 = 3$. Therefore, an eigenvector relative to the eigenvalue $\lambda = -7$ is $\begin{pmatrix} -1 \\ 3 \end{pmatrix}$.

6 Using determinants to compute simple eigendecompositions

Taken together, the eigenvectors and eigenvalues of a matrix A are sometimes called the eigendecomposition of A. The reason is that we can use them to construct a decomposition of A, where by "decomposition" we mean an equivalent form for A that is simpler or more convenient in some sense. Basically, this will correspond to expressing A in terms of linear combinations of these special vectors and numbers.

Eigendecompositions of symmetric matrices are particularly well-behaved and have particularly nice properties, while the eigendecompositions of general matrices may or may not have these nice properties. By nice properties, we mean that for a given $n \times n$ matrix A we can find a full set of eigenvectors that form an orthonormal basis of $\mathbb{R}^n$.

Several examples of "nice" matrices.

Example 8.

Let $A = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$. Then

$$A - \lambda I = \begin{pmatrix} 1-\lambda & 2 \\ 2 & 4-\lambda \end{pmatrix}.$$

It follows that

$$\begin{aligned} |A - \lambda I| &= (1-\lambda)(4-\lambda) - 4 \\ &= \lambda^2 - 5\lambda. \end{aligned}$$

Clearly, if $\lambda = 0$ or 5, then

$$|A - \lambda I| = 0.$$

So, the eigenvalues are $\lambda = 0, 5$. Let's compute the eigenvectors.

$\underline{\lambda = 0}$

$$\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow v_{\lambda=0} = \frac{1}{\sqrt{5}} \begin{pmatrix} -2 \\ 1 \end{pmatrix}.$$

$\underline{\lambda = 5}$

$$\begin{pmatrix} -4 & 2 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow v_{\lambda=5} = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

This is similar to the example we saw previously, with two distinct eigenvalues and two orthogonal eigenvectors. One of the eigenvalues equals 0, and this is a consequence of the fact that the columns of A are not linearly independent.

Example 9.

Let $A = \begin{pmatrix} -3 & 4 \\ 4 & 3 \end{pmatrix}$. Then

$$A - \lambda I = \begin{pmatrix} -3-\lambda & 4 \\ 4 & 3-\lambda \end{pmatrix}.$$

It follows that

$$\begin{aligned} |A - \lambda I| &= (3-\lambda)(-3+\lambda) - 16 \\ &= \lambda^2 - 25. \end{aligned}$$

Clearly, if $\lambda = -5$ or 5, then

$$|A - \lambda I| = 0.$$

So, the eigenvalues are $\lambda = -5, 5$. Let's compute the eigenvectors.

$\underline{\lambda = -5}$

$$\begin{pmatrix} 2 & 4 \\ 4 & 8 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow v_{\lambda=-5} = \frac{1}{\sqrt{5}} \begin{pmatrix} -2 \\ 1 \end{pmatrix}.$$

$\underline{\lambda = 5}$

$$\begin{pmatrix} -8 & 4 \\ 4 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow v_{\lambda=5} = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

Here, the eigenvectors are the same as in the previous example, but the eigenvalues are different.

Example 10.

Let $A = \begin{pmatrix} 9 & -2 \\ -2 & 6 \end{pmatrix}$. Then

$$A - \lambda I = \begin{pmatrix} 9-\lambda & -2 \\ -2 & 6-\lambda \end{pmatrix}.$$

It follows that

$$\begin{aligned} |A - \lambda I| &= (9-\lambda)(6+\lambda) - 4 \\ &= \lambda^2 - 15\lambda + 50. \end{aligned}$$

Clearly, if $\lambda = 5$ or 10, then

$$|A - \lambda I| = 0.$$

So, the eigenvalues are $\lambda = 5, 10$. Let's compute the eigenvectors.

$\underline{\lambda = 10}$

$$\begin{pmatrix} -1 & -2 \\ -2 & -4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow v_{\lambda=10} = \frac{1}{\sqrt{5}} \begin{pmatrix} -2 \\ 1 \end{pmatrix}.$$

$\underline{\lambda = 5}$

$$\begin{pmatrix} 4 & -2 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow v_{\lambda=5} = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

Here, the eigenvectors are the same as in the last two examples, but the eigenvalues are different.

These matrices are "nice" in the sense that they have n real-valued eigenvalues (perhaps counting multiplicity) and a set of n orthogonal real-valued eigenvectors. As we will see later, this is extremely important for many applications.

Several examples of "not nice" matrices.

Let's move onto a few examples which will illustrate several fundamental "non-nice" behaviors of eigenvalues and/or eigenvectors. These matrices are "not nice" in the sense that they do not have have a full set of n orthogonal real-valued eigenvectors and associated real-valued eigenvalues.

Example 11.

A matrix with non-orthogonal eigenvectors.

Let $A = \begin{pmatrix} 3 & 2 \\ 4 & 1 \end{pmatrix}$. Then

$$A - \lambda I = \begin{pmatrix} 3-\lambda & 2 \\ 4 & 1-\lambda \end{pmatrix}.$$

It follows that

$$\begin{aligned} |A - \lambda I| &= (3-\lambda)(1-\lambda) - 8 \\ &= \lambda^2 - 4\lambda - 5 \\ &= (\lambda - 5)(\lambda + 1). \end{aligned}$$

Clearly, if $\lambda = 5$ or -1, then

$$|A - \lambda I| = 0.$$

So, the eigenvalues are $\lambda = 5, -1$. Let's compute the eigenvectors.

$\underline{\lambda = 5}$

$$\begin{pmatrix} -2 & 2 \\ 4 & -4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow v_{\lambda=5} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

$\underline{\lambda = -1}$

$$\begin{pmatrix} 4 & 2 \\ 4 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow v_{\lambda=-1} = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 \\ -2 \end{pmatrix}.$$

Observe that $v_{\lambda=5}$ and $v_{\lambda=-1}$ are linearly independent, but they are not orthogonal to each other. Thus, this matrix is an example of a matrix with two distinct eigenvalues, each of which has an associated eigenvector, where the two eigenvectors are linearly independent but not orthogonal.

Example 12.

An example with complex eigenvalues and eigenvectors. Let $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Then

$$A - \lambda I = \begin{pmatrix} -\lambda & 1 \\ -1 & -\lambda \end{pmatrix}.$$

It follows that

$$|A - \lambda I| = \lambda^2 + 1.$$

Clearly, if $\lambda = \pm i$, then

$$|A - \lambda I| = 0.$$

So, the eigenvalues are $\lambda = \pm i$. Let's compute the eigenvectors.

$\underline{\lambda = i}$

$$\begin{pmatrix} -i & 1 \\ -1 & -i \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow v_{\lambda=i} = \begin{pmatrix} -i \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ i \end{pmatrix}.$$

$\underline{\lambda = -i}$

$$\begin{pmatrix} -i & 1 \\ -1 & i \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow v_{\lambda=-i} = \begin{pmatrix} 1 \\ -i \end{pmatrix} = \begin{pmatrix} i \\ 1 \end{pmatrix}.$$

Thus, the given matrix is an example of a real-valued matrix, i.e., a matrix whose elements consist of only real numbers, that has eigenvalues that are imaginary/complex numbers and eigenvectors that contain imaginary/complex entries.

Example 13.

An example with fewer than two eigenvectors. Let $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. Then

$$A - \lambda I = \begin{pmatrix} -\lambda & 1 \\ 0 & -\lambda \end{pmatrix}.$$

It follows that

$$|A - \lambda I| = \lambda^2.$$

Clearly, if $\lambda = 0$, then

$$|A - \lambda I| = 0.$$

Thus, $\lambda = 0$ is a degenerate eigenvalue with degeneracy equal to 2. Let's compute the eigenvectors. Since

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ 0 \end{pmatrix},$$

there is only one corresponding eigenvector, which is

$$v_{\lambda=0} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Thus, the given matrix is an example of a real-valued $n \times n$ matrix that has fewer than n eigenvectors.

7 Expressing matrices in terms of their eigendecompositions

Many mathematical objects can be understood better by breaking them into constituent parts, or finding some properties of them that are universal, not caused by the way we choose to represent them.

For example, integers can be decomposed into prime factors. The way we represent the number will change depending on whether 12 we write it in base ten or in binary, but it will always be true that $12 = 2 \times 2 \times 3$. From this representation we can conclude useful properties, for example, that 12 is not divisible by 5, and that any integer multiple of 12 will be divisible by 3.

Much as we can discover something about the true nature of an integer by decomposing it into prime factors, we can also decompose matrices in ways that show us information about their functional properties that is not obvious from the representation of the matrix as an array of elements.

The reason that the set of eigenvalues and eigenvectors of a matrix is known as the **eigendecomposition** of that matrix is that one can use them to decompose the matrix into a simpler form. For the time being, we will describe it in the context of the simple 2×2 matrices.

To do this, let's enumerate the eigenvectors and eigenvalues in a way that will make it easier to generalize beyond 2×2 matrices.

We enumerate eigenvalues in decreasing order. If a matrix has two (or more) identical eigenvalues, then we repeat i in λ_i according to the multiplicity, and we can enumerate their associated orthogonal eigenvectors arbitrarily.

Given this numbering convention, recall that for the 2×2 symmetric matrices we have been discussing, we have seen that there are two eigenvalue-eigenvector pairs (λ_i, v_i) that satisfy:

$$Av_i = \lambda_i v_i, \tag{1}$$

where $i \in \{1, 2\}$. The LHS and RHS of this equation are both vectors, i.e., 2×1 matrices. Let's write the two equations for $i \in \{1, 2\}$ as a single matrix equation. To do so, put

$$\Lambda = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \qquad V = (v_1 \quad v_2).$$

Here, both are 2×2 matrices. If the ith eigenvalue is Λ_{ii}, then the corresponding ith eigenvector is the ith column of V.

Note that, since the eigenvectors are unit-length and pair-wise orthogonal, V is an orthogonal matrix. Hence

$$VV^T = V^T V = I \in \mathbb{R}^{2\times 2}$$

with $V^T = \begin{pmatrix} v_1^T \\ v_2^T \end{pmatrix}$.

Spectral decomposition: Expressing A as a product of three matrices.

Theorem 7.1 (Spectral decomposing I)**.** *Let v_1, v_2 be the eigenvectors associated with the eigenvalues λ_1, λ_2 of a 2×2 symmetric matrix A respectively. If*

$$\Lambda = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \qquad V = (v_1 \quad v_2),$$

then

$$A = V \Lambda V^T. \tag{2}$$

Proof. We have

$$\begin{aligned} AV &= \begin{pmatrix} Av_1 & Av_2 \end{pmatrix} \\ &= \begin{pmatrix} \lambda_1 v_1 & \lambda_2 v_2 \end{pmatrix} \\ &= \begin{pmatrix} \lambda_1 v_{11} & \lambda_2 v_{12} \\ \lambda_1 v_{21} & \lambda_2 v_{22} \end{pmatrix} \\ &= \begin{pmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{pmatrix} \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \\ &= V\Lambda. \end{aligned}$$

Therefore, from this in view of $VV^T = I$, we obtain

$$A = V\Lambda V^T. \qquad \square$$

This theorem provides a decomposition of the matrix A into the product of three matrices:

> an orthogonal matrix V (consisting of the eigenvectors of A),
>
> a diagonal matrix Λ (consisting of the eigenvalues of A), and
>
> the transpose of that orthogonal matrix, V^T.

We can write Equation (2) in terms of individual elements as follows:

$$A = V\Lambda V^T = \begin{pmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{pmatrix} \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} v_{11} & v_{21} \\ v_{12} & v_{22} \end{pmatrix}.$$

This simple eigendecomposition is the first example of the *spectral theorem*.

This decomposition of the form $A = V\Lambda V^T$ holds for general $n \times n$ symmetric matrices, and generalizations of it hold more generally. In particular, if we consider computing $y = Ax$, this is the same as

$$y = V\Lambda V^T x = V(\Lambda(V^T x))$$

Spectral decomposition: expressing A as a sum of outer products.

We can write Equation (2) in terms of the columns of V (which recall are the eigenvectors, which are also the rows of V^T) and elements of Λ (which are the eigenvalues). This gives the following theorem:

Theorem 7.2 (Spectral decomposition II)**.** *Let v_1, v_2 be the eigenvectors associated with the eigenvalues λ_1, λ_2 of a 2×2 symmetric matrix A respectively. Then*

$$A = \lambda_1 v_1 v_1^T + \lambda_2 v_2 v_2^T. \tag{3}$$

Proof. Let

$$\Lambda = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \qquad V = (v_1 \quad v_2),$$

Then by Spectral theorem I, we have

$$\begin{aligned} A &= V\Lambda V^T \\ &= (v_1 \quad v_2) \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} v_1^T \\ v_2^T \end{pmatrix} \\ &= (v_1 \quad v_2) \begin{pmatrix} \lambda_1 v_1^T \\ \lambda_2 v_2^T \end{pmatrix}. \end{aligned}$$

This gives

$$A = \lambda_1 v_1 v_1^T + \lambda_2 v_2 v_2^T. \qquad \square$$

Equation (3) says that if we do this for every i and sum up the corresponding matrices, then we get the original matrix A. Thus, the decomposition given by Equation (3) expresses the matrix A in terms of the sum of 2 terms, each of which is the outer product of an eigenvector with its transpose, multiplied/scaled by the corresponding eigenvalue.

This decomposition holds for general $n \times n$ symmetric matrices, in which case

$$A = \sum_{i=1}^{n} \lambda_i v_i v_i^T,$$

and generalizations of it hold more generally. We will discuss this later.

In particular, by the spectral theorem II, the equation $y = Ax$ is equivalent to the equation

$$y = \lambda_1 v_1 v_1^T x + \lambda_2 v_2 v_2^T x. \tag{4}$$

Example 14.

> **Diagonal matrix.** How do we express the matrix $A = \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}$ in two standard forms?

Solution. For this matrix, we have computed

Eigenvalues: $\lambda_1 = 3$, $\lambda_2 = 1$,

Eigenvectors: $v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $v_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

Expressing A as a sum of 2 outer products. We have

$$\begin{aligned}\sum_{i=1}^{2}\lambda_i v_i v_i^T &= \lambda_1 v_1 v_1^T + \lambda_2 v_2 v_2^T \\ &= 3\begin{pmatrix}1\\0\end{pmatrix}\begin{pmatrix}1 & 0\end{pmatrix} + 1\begin{pmatrix}0\\1\end{pmatrix}\begin{pmatrix}0 & 1\end{pmatrix} \\ &= \begin{pmatrix}3 & 0\\0 & 0\end{pmatrix} + \begin{pmatrix}0 & 0\\0 & 1\end{pmatrix} \\ &= \begin{pmatrix}3 & 0\\0 & 1\end{pmatrix} = A.\end{aligned}$$

Therefore,

$$A = 3\begin{pmatrix}1\\0\end{pmatrix}\begin{pmatrix}1 & 0\end{pmatrix} + 1\begin{pmatrix}0\\1\end{pmatrix}\begin{pmatrix}0 & 1\end{pmatrix}.$$

Expressing A as a product of 3 matrices. We have

$$\begin{aligned}V\Lambda V^T &= \begin{pmatrix}v_1 & v_2\end{pmatrix}\begin{pmatrix}\lambda_1 & 0\\0 & \lambda_2\end{pmatrix}\begin{pmatrix}v_1^T\\v_2^T\end{pmatrix} \\ &= \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}\begin{pmatrix}3 & 0\\0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix} \\ &= \begin{pmatrix}3 & 0\\0 & 1\end{pmatrix} = A.\end{aligned}$$

Therefore,

$$A = \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}\begin{pmatrix}3 & 0\\0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}.$$

Example 15.

Identity matrix. How do we express the matrix $A = \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}$ in two standard forms?

Solution. For this matrix, we have computed

Eigenvalues: $\lambda_1 = 1$, $\lambda_2 = 1$,

Eigenvectors: $v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $v_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

Expressing A as a sum of 2 outer products. We have

$$\begin{aligned}
\sum_{i=1}^{2} \lambda_i v_i v_i^T &= \lambda_1 v_1 v_1^T + \lambda_2 v_2 v_2^T \\
&= 1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \end{pmatrix} + 1 \begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = A.
\end{aligned}$$

Therefore,

$$A = 1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \end{pmatrix} + 1 \begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \end{pmatrix}.$$

Alternatively, if we choose $v_1 = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$, and $v_2 = \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix}$, then we obtain

$$\begin{aligned}
\sum_{i=1}^{2} \lambda_i v_i v_i^T &= \lambda_1 v_1 v_1^T + \lambda_2 v_2 v_2^T \\
&= 1 \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} \\
&\quad + 1 \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix} \\
&= \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} + \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = A.
\end{aligned}$$

Therefore,

$$A = 1 \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} + 1 \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix}.$$

if, on the other hand, we choose $v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, and $v_2 = \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix}$, then we obtain

$$\begin{aligned}
\sum_{i=1}^{2} \lambda_i v_i v_i^T &= \lambda_1 v_1 v_1^T + \lambda_2 v_2 v_2^T \\
&= 1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \end{pmatrix} + 1 \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix} \begin{pmatrix} 1/2 & 1/2 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix} \\
&= \begin{pmatrix} 3/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix} \neq A.
\end{aligned}$$

Expressing A as a product of 3 matrices. If we choose $v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, and $v_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, then we obtain

$$
\begin{aligned}
V\Lambda V^T &= (v_1 \quad v_2) \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} v_1^T \\ v_2^T \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = A.
\end{aligned}
$$

Therefore,

$$
A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}
$$

If we choose $v_1 = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$, and $v_2 = \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix}$, then we obtain

$$
\begin{aligned}
V\Lambda V^T &= (v_1 \quad v_2) \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} v_1^T \\ v_2^T \end{pmatrix} \\
&= \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix} \\
&= \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = A.
\end{aligned}
$$

Therefore,

$$
A = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix}.
$$

If we choose $v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, and $v_2 = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$, then we obtain

$$
\begin{aligned}
V\Lambda V^T &= (v_1 \quad v_2) \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} v_1^T \\ v_2^T \end{pmatrix} \\
&= \begin{pmatrix} 1 & 1/\sqrt{2} \\ 0 & 1/\sqrt{2} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} \\
&= \begin{pmatrix} 1 & 1/\sqrt{2} \\ 0 & 1/\sqrt{2} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} \\
&= \begin{pmatrix} 3/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix} \neq A.
\end{aligned}
$$

Example 16.

Reflection matrix. How do we express the matrix $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ in two standard forms?

Solution. For this matrix, we have computed

Eigenvalues: $\lambda_1 = 1$, $\lambda_2 = -1$,

Eigenvectors: $v_1 = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$, $v_2 = \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix}$.

Expressing A as a sum of 2 outer products. We have

$$\begin{aligned}\sum_{i=1}^{2}\lambda_i v_i v_i^T &= \lambda_1 v_1 v_1^T + \lambda_2 v_2 v_2^T \\ &= 1\begin{pmatrix}1/\sqrt{2}\\1/\sqrt{2}\end{pmatrix}\begin{pmatrix}1/\sqrt{2} & 1/\sqrt{2}\end{pmatrix} \\ &\quad - 1\begin{pmatrix}1/\sqrt{2}\\-1/\sqrt{2}\end{pmatrix}\begin{pmatrix}1/\sqrt{2} & -1/\sqrt{2}\end{pmatrix} \\ &= \begin{pmatrix}1/2 & 1/2\\1/2 & 1/2\end{pmatrix} - \begin{pmatrix}1/2 & -1/2\\-1/2 & 1/2\end{pmatrix} \\ &= \begin{pmatrix}0 & 1\\1 & 0\end{pmatrix} = A.\end{aligned}$$

Therefore,

$$A = 1\begin{pmatrix}1/\sqrt{2}\\1/\sqrt{2}\end{pmatrix}\begin{pmatrix}1/\sqrt{2} & 1/\sqrt{2}\end{pmatrix} - 1\begin{pmatrix}1/\sqrt{2}\\-1/\sqrt{2}\end{pmatrix}\begin{pmatrix}1/\sqrt{2} & -1/\sqrt{2}\end{pmatrix}.$$

Expressing A as a product of 3 matrices. We have

$$\begin{aligned}V\Lambda V^T &= \begin{pmatrix}v_1 & v_2\end{pmatrix}\begin{pmatrix}\lambda_1 & 0\\0 & \lambda_2\end{pmatrix}\begin{pmatrix}v_1^T\\v_2^T\end{pmatrix} \\ &= \begin{pmatrix}1/\sqrt{2} & 1/\sqrt{2}\\1/\sqrt{2} & -1/\sqrt{2}\end{pmatrix}\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}\begin{pmatrix}1/\sqrt{2} & 1/\sqrt{2}\\1/\sqrt{2} & -1/\sqrt{2}\end{pmatrix} \\ &= \begin{pmatrix}1/\sqrt{2} & 1/\sqrt{2}\\1/\sqrt{2} & -1/\sqrt{2}\end{pmatrix}\begin{pmatrix}1/\sqrt{2} & 1/\sqrt{2}\\-1/\sqrt{2} & 1/\sqrt{2}\end{pmatrix} \\ &= \begin{pmatrix}0 & 1\\1 & 0\end{pmatrix} = A.\end{aligned}$$

Therefore,

$$A = \begin{pmatrix}1/\sqrt{2} & 1/\sqrt{2}\\1/\sqrt{2} & -1/\sqrt{2}\end{pmatrix}\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}\begin{pmatrix}1/\sqrt{2} & 1/\sqrt{2}\\1/\sqrt{2} & -1/\sqrt{2}\end{pmatrix}$$

Example 17.

How do we express the matrix $A = \begin{pmatrix} 9 & -2 \\ -2 & 6 \end{pmatrix}$ in two standard forms?

Solution. For this matrix, we have computed

Eigenvalues: $\lambda_1 = 5$, $\lambda_2 = 10$,

Eigenvectors: $v_1 = \begin{pmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{pmatrix}$, $v_2 = \begin{pmatrix} -2/\sqrt{5} \\ 1/\sqrt{5} \end{pmatrix}$.

Expressing A as a sum of 2 outer products. We have

$$\begin{aligned}
\sum_{i=1}^{2} \lambda_i v_i v_i^T &= \lambda_1 v_1 v_1^T + \lambda_2 v_2 v_2^T \\
&= 5 \begin{pmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{pmatrix} \begin{pmatrix} 1/\sqrt{5} & 2/\sqrt{5} \end{pmatrix} \\
&\quad + 10 \begin{pmatrix} -2/\sqrt{5} \\ 1/\sqrt{5} \end{pmatrix} \begin{pmatrix} -2/\sqrt{5} & 1/\sqrt{5} \end{pmatrix} \\
&= \begin{pmatrix} 1 \\ 2 \end{pmatrix} \begin{pmatrix} 1 & 2 \end{pmatrix} + \begin{pmatrix} -2 \\ 1 \end{pmatrix} \begin{pmatrix} -2 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} + \begin{pmatrix} 8 & -4 \\ -4 & 2 \end{pmatrix} \\
&= \begin{pmatrix} 9 & -2 \\ -2 & 6 \end{pmatrix} = A.
\end{aligned}$$

Therefore,

$$A = 5 \begin{pmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{pmatrix} \begin{pmatrix} 1/\sqrt{5} & 2/\sqrt{5} \end{pmatrix} + 10 \begin{pmatrix} -2/\sqrt{5} \\ 1/\sqrt{5} \end{pmatrix} \begin{pmatrix} -2/\sqrt{5} & 1/\sqrt{5} \end{pmatrix}.$$

Expressing A as a product of 3 matrices. We have

$$\begin{aligned}
V\Lambda V^T &= (v_1 \quad v_2)\begin{pmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{pmatrix}\begin{pmatrix}v_1^T\\ v_2^T\end{pmatrix}\\
&= \begin{pmatrix}1/\sqrt{5} & -2/\sqrt{5}\\ 2/\sqrt{5} & 1/\sqrt{5}\end{pmatrix}\begin{pmatrix}5 & 0\\ 0 & 10\end{pmatrix}\begin{pmatrix}1/\sqrt{5} & 2/\sqrt{5}\\ -2/\sqrt{5} & 1/\sqrt{5}\end{pmatrix}\\
&= \frac{1}{5}\begin{pmatrix}1 & -2\\ 2 & 1\end{pmatrix}\begin{pmatrix}5 & 0\\ 0 & 10\end{pmatrix}\begin{pmatrix}1 & 2\\ -2 & 1.\end{pmatrix}\\
&= \frac{1}{5}\begin{pmatrix}1 & -2\\ 2 & 1\end{pmatrix}\begin{pmatrix}5 & 10\\ -20 & 10\end{pmatrix}\\
&= \frac{1}{5}\begin{pmatrix}45 & -10\\ -10 & 30\end{pmatrix}\\
&= \begin{pmatrix}9 & 2\\ -2 & 6\end{pmatrix} = A
\end{aligned}$$

Therefore,

$$A = \begin{pmatrix}1/\sqrt{5} & -2/\sqrt{5}\\ 2/\sqrt{5} & 1/\sqrt{5}\end{pmatrix}\begin{pmatrix}5 & 0\\ 0 & 10\end{pmatrix}\begin{pmatrix}1/\sqrt{5} & 2/\sqrt{5}\\ -2/\sqrt{5} & 1/\sqrt{5}\end{pmatrix}.$$

8 A larger example

So far, we have focused on 2×2 matrices. The same ideas hold for larger matrices. Here, we will illustrate this with a 3×3 example. Let's consider

$$A = \begin{pmatrix}1 & 4 & 3\\ 4 & 1 & 0\\ 3 & 0 & 1\end{pmatrix}$$

Note that this matrix is symmetric. We will show that for this matrix, we can compute 3 eigenvalues and 3 corresponding eigenvectors that form an orthonormal and that thus form an orthonormal basis for $\mathbb{R}^3$.

To do so, consider

$$A - \lambda I = \begin{pmatrix} 1-\lambda & 4 & 3 \\ 4 & 1-\lambda & 0 \\ 3 & 0 & 1-\lambda \end{pmatrix}$$

In this case,

$$\begin{aligned} |A - \lambda I| &= (1-\lambda)((1-\lambda)^2 - 0) - 4(4(1-\lambda) - 0) \\ &\quad + 3(0 - 3(1-\lambda)) \\ &= (1-\lambda)^3 - 25(1-\lambda) \\ &= (1-\lambda)((1-\lambda)^2 - 25) \\ &= (1-\lambda)(\lambda - 6)(\lambda + 4). \end{aligned}$$

We see that if $\lambda = -4, 1, 6$, then

$$|A - \lambda I| = 0.$$

So, the eigenvalues are $\lambda = -4, 1, 6$.

To compute the eigenvectors for each of these eigenvalues, we can use Gauss-Jordan reduction method that we won't cover here. Instead, we'll simply state the eigenvectors and verify that they are eigenvectors. Here they are:

$$\lambda_1 = 6,\ v_1 = \begin{pmatrix} 5 \\ 4 \\ 3 \end{pmatrix} : \begin{pmatrix} 1 & 4 & 3 \\ 4 & 1 & 0 \\ 3 & 0 & 1 \end{pmatrix} \begin{pmatrix} 5 \\ 4 \\ 3 \end{pmatrix} = \begin{pmatrix} 30 \\ 24 \\ 18 \end{pmatrix} = 6 \begin{pmatrix} 5 \\ 4 \\ 3 \end{pmatrix}.$$

$$\lambda_1 = 1,\ v_2 = \begin{pmatrix} 0 \\ -3 \\ 4 \end{pmatrix} : \quad \begin{pmatrix} 1 & 4 & 3 \\ 4 & 1 & 0 \\ 3 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ -3 \\ 4 \end{pmatrix} = 1 \begin{pmatrix} 0 \\ -3 \\ 4 \end{pmatrix}.$$

$$\lambda_1 = -4,\ v_3 = \begin{pmatrix} -5 \\ 4 \\ 3 \end{pmatrix} : \qquad \begin{pmatrix} 1 & 4 & 3 \\ 4 & 1 & 0 \\ 3 & 0 & 1 \end{pmatrix} \begin{pmatrix} -5 \\ 4 \\ 3 \end{pmatrix} = -4 \begin{pmatrix} -5 \\ 4 \\ 3 \end{pmatrix}.$$

<u>To verify that the eigenvectors v_1, v_2, v_3 are orthogonal.</u> To do so, let $V_O = (v_1\ v_2\ v_3)$ and let's multiply it by it's transpose:

$$V_O^T V_O = \begin{pmatrix} 5 & 4 & 3 \\ 0 & -3 & 4 \\ -5 & 4 & 3 \end{pmatrix} \begin{pmatrix} 5 & 0 & -5 \\ 4 & -3 & 4 \\ 3 & 4 & 3 \end{pmatrix} = \begin{pmatrix} 50 & 0 & 0 \\ 0 & 25 & 0 \\ 0 & 0 & 50 \end{pmatrix}$$

So, these three vectors are eigenvectors, and they are orthogonal, and so they provide a basis for $\mathbb{R}^3$. To normalize them, we divide by their norms, the square of which are the diagonal elements of $V_O^T V_O$. Here are the normalized eigenvectors.

$$\lambda_1 = 6 : \qquad v_1 = \frac{1}{\sqrt{50}} \begin{pmatrix} 5 \\ 4 \\ 3 \end{pmatrix}$$

$$\lambda_1 = 1 : \qquad v_2 = \frac{1}{\sqrt{25}} \begin{pmatrix} 0 \\ -3 \\ 1 \end{pmatrix}$$

$$\lambda_1 = -4 : \qquad v_1 = \frac{1}{\sqrt{50}} \begin{pmatrix} -5 \\ 4 \\ 3 \end{pmatrix}.$$

Let's now work with these normalized eigenvectors. In this case, the 3×3 matrix of normalized eigenvectors (the columns of which form an orthonormal basis for $\mathbb{R}^3$) is

$$V = (v_1\ v_2\ v_3) = \begin{pmatrix} 5/\sqrt{50} & 0 & -5/\sqrt{50} \\ 4/\sqrt{50} & -3/\sqrt{25} & 4/\sqrt{50} \\ 3/\sqrt{50} & 4/\sqrt{25} & 3/\sqrt{50} \end{pmatrix}.$$

Clearly, $V^T V = I$. Moreover,

$$AV = V\Lambda.$$

Here is how do we express the matrix in two standard forms.

Expressing A as a sum of 3 outer products. We have

$$\begin{aligned}
\sum_{i=1}^{2} \lambda_i v_i v_i^T &= \lambda_1 v_1 v_1^T + \lambda_2 v_2 v_2^T + \lambda_3 v_3 v_3^T \\
&= 6 \begin{pmatrix} 5/\sqrt{50} \\ 4/\sqrt{50} \\ 3/\sqrt{50} \end{pmatrix} \begin{pmatrix} 5/\sqrt{50} & 4/\sqrt{50} & 3/\sqrt{50} \end{pmatrix} \\
&+ 1 \begin{pmatrix} 0 \\ -3/\sqrt{25} \\ 4/\sqrt{25} \end{pmatrix} \begin{pmatrix} 0 & -3/\sqrt{25} & 4/\sqrt{25} \end{pmatrix} \\
&- 4 \begin{pmatrix} -5/\sqrt{50} \\ 4/\sqrt{50} \\ 3/\sqrt{50} \end{pmatrix} \begin{pmatrix} -5/\sqrt{50} & 4/\sqrt{50} & 3/\sqrt{50} \end{pmatrix} \\
&= A.
\end{aligned}$$

Expressing A as a product of 3 matrices. We have

$$\begin{aligned}
VAV^T &= (v_1 \; v_2 \; v_3) \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} \begin{pmatrix} v_1^T \\ v_2^T \\ v_3^T \end{pmatrix} \\
&\cdots \\
&= A
\end{aligned}$$

Theorem 8.1. *If a matrix A is symmetric, then any two distinct eigenvectors corresponding to different eigenvalues are orthogonal.*

Proof.

Let v_1 and v_2 be eigenvectors that correspond to eigenvalues λ_1 and λ_2. To show that $v_1 \cdot v_2 = 0$, compute

$$\begin{aligned}
\lambda_1 v_1 \cdot v_2 &= (\lambda_1 v_1)^T v_2 = (Av_1)^T v_2 \\
&= (v_1^T A^T) v_2 = v_1^T (A^T v_2) = v_1^T (A v_2) \\
&= v_1^T (\lambda_2 v_2) = \lambda_2 v_1^T v_2 \\
&= \lambda_2 v_1 \cdot v_2.
\end{aligned}$$

Hence $(\lambda_1 - \lambda_2) v_1 v_2 = 0$, but $\lambda_1 - \lambda_2 \neq 0$, so

$$v_1 \cdot v_2 = 0.$$

We mention the following result without proof.

Theorem 8.2. *If A is a symmetric matrix, then all eigenvalues of A are real.*