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Exercise A

1. Show that A has $\lambda = 0$ as an eigenvalues if and only if A is not invertible.

Soln If $\lambda = 0$ is an eigenvalue of A, then there exists non-zero vector V such that $AV = 0$

Thus $|A| = 0$ i.e A is not invertible.

Conversely,

If A is not invertible, then the equation $AV = 0$ has non-trivial solution V. Since $AV = 0 = 0V$, then 0 is eigenvalue of A.

~~To~~ To Conform

let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

A is Singular matrix iff $|A| = 0$
i.e $ad - bc = 0$

$\therefore$ Characteristic eqn of A is $|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix} = 0$$

$\Rightarrow (a-\lambda)(d-\lambda) - bc = 0$

$\Rightarrow ad - a\lambda - d\lambda + \lambda^2 - bc = 0$

$\Rightarrow \lambda^2 - a\lambda - d\lambda = 0$ $[\because |A| = 0]$

$\Rightarrow \lambda(\lambda - a - d) = 0$

$\therefore \lambda = 0$ is eigenvalue of A.

[2] Find eigenvalues, eigenvector & if possible an invertible matrix P such that $P^{-1}AP$ is diagonal.

(a) $\begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}$ Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}$

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Characteristic polynomial of A is,

$|A - \lambda I| = 0$

$$\begin{vmatrix} 1-\lambda & 2 \\ 3 & 2-\lambda \end{vmatrix} = 0$$

$\Rightarrow (1-\lambda)(2-\lambda) - 6 = 0$

$\Rightarrow \lambda^2 - 3\lambda - 4 = 0$

$\Rightarrow \lambda^2 - 4\lambda + \lambda - 4 = 0$

$\Rightarrow \lambda(\lambda - 4) + 1(\lambda - 4) = 0$

$\Rightarrow (\lambda + 1)(\lambda - 4) = 0$

$\Rightarrow \lambda = -1, 4$

For $\lambda = -1$

$(A + 1 \cdot I)x = 0$

$$\Rightarrow \begin{bmatrix} 2 & 2 \\ 3 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\Rightarrow 2x_1 + 2x_2 = 0$

$$\therefore V_{\lambda=-1} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

For $\lambda = 4$

$(A - 4I)x = 0$

$$\Rightarrow \begin{bmatrix} -3 & 2 \\ 3 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\Rightarrow -3x_1 + 2x_2 = 0 \Rightarrow x_2 = \frac{3}{2} x_1$

$$V_{\lambda=4} = \begin{bmatrix} 1 \\ 3/2 \end{bmatrix}$$

Thus, $P = \begin{bmatrix} 1 & 1 \\ -1 & 3/2 \end{bmatrix}$ $D = \begin{bmatrix} -1 & 0 \\ 0 & 4 \end{bmatrix}$

For P^{-1}

$|P| = \frac{3}{2} + 1 = \frac{5}{2}$ $\quad P^{-1} = \frac{1}{|P|} \operatorname{adj}(P)$

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$$= \frac{2}{5}\begin{bmatrix} 3/2 & -1 \\ 1 & 1 \end{bmatrix}$$

$$\therefore P^{-1} = \begin{bmatrix} 3/5 & -2/5 \\ 2/5 & 2/5 \end{bmatrix}$$

Thus, $D = P^{-1}AP$

$$= \begin{bmatrix} 3/5 & -2/5 \\ 2/5 & 2/5 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -1 & 3/2 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{3}{5} - \frac{6}{5} & 6/5 - 4/5 \\ 2/5 + 6/5 & 4/5 + 4/5 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -1 & 3/2 \end{bmatrix}$$

$$= \begin{bmatrix} -3/5 & 2/5 \\ 8/5 & 8/5 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -1 & 3/2 \end{bmatrix} = \begin{bmatrix} -3/5 - 2/5 & -3/5 + 3/5 \\ 8/5 - 8/5 & 8/5 + 12/5 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 0 \\ 0 & 4 \end{bmatrix} = D \;\#$$

(b) $\begin{bmatrix} 2 & -4 \\ -1 & -1 \end{bmatrix}$

Let $A = \begin{bmatrix} 2 & -4 \\ -1 & -1 \end{bmatrix}$ Characteristic eqn is

$|A - dI| = 0$

$$\Rightarrow \begin{vmatrix} 2-d & -4 \\ -1 & -1-d \end{vmatrix} = 0$$

$\Rightarrow (2-d)(-1-d) - 4 = 0$

$\Rightarrow -2 - 2d + d + d^2 - 4 = 0$

$\Rightarrow d^2 - d - 6 = 0$

$\Rightarrow d^2 - 3d + 2d - 6 = 0$

$\Rightarrow d(d-3) + 2(d-3) = 0 \Rightarrow (d+2)(d-3) = 0 \Rightarrow d = -2, 3$

For $d = 3$

$(A - 3I)x = 0$

$$\Rightarrow \begin{bmatrix} -1 & -4 \\ -1 & -4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\Rightarrow -x_1 - 4x_2 = 0$ when $x_2 = -1$ $x_1 = 4$

$-x_1 = 4x_2$

$V_{d=3} = \begin{bmatrix} 4 \\ -1 \end{bmatrix}$

For $\lambda = -2$

$(A - 2I)x = 0$

$$\Rightarrow \begin{bmatrix} 4 & -4 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$4x_1 - 4x_2 = 0$

$\Rightarrow x_1 = x_2$

$\therefore V_{\lambda=-2} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Now, $P = \begin{bmatrix} 4 & -1 \\ -1 & 1 \end{bmatrix}$ for $D = \begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix}$

For P^{-1}

$|P| = 4 - 1 = 3$

$adj(P) = \begin{bmatrix} 1 & 1 \\ 1 & 4 \end{bmatrix}$

$$P^{-1} = \frac{1}{|P|} adj(P) = \frac{1}{3}\begin{pmatrix} 1 & 1 \\ 1 & 4 \end{pmatrix} = \begin{bmatrix} 1/3 & 1/3 \\ 1/3 & 4/3 \end{bmatrix}$$

we know that,

$D = P^{-1} A P$

$$= \begin{bmatrix} 1/3 & 1/3 \\ 1/3 & 4/3 \end{bmatrix} \begin{bmatrix} 2 & -4 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} 4 & -1 \\ -1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2/3 - 1/3 & -4/3 - 1/3 \\ 2/3 - 4/3 & -4/3 - 4/3 \end{bmatrix} \begin{bmatrix} 4 & -1 \\ -1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1/3 & -5/3 \\ -2/3 & -8/3 \end{bmatrix} \begin{bmatrix} 4 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 4/3 + 5/3 & -1/3 + 1/3 \\ -8/3 + 8/3 & 2/3 - 8/3 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix}$$

(c) $\begin{bmatrix} 7 & 0 & -4 \\ 0 & 5 & 0 \\ 5 & 0 & -2 \end{bmatrix}$

characteristic eqn is

$|A - \lambda I| = 0$

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$$\begin{vmatrix} 7-\lambda & 0 & -4 \\ 0 & 5-\lambda & 0 \\ 5 & 0 & -2-\lambda \end{vmatrix} = 0$$

$\Rightarrow \lambda^3 - 10\lambda^2 + 41\lambda - 30 = 0$

~~Hence~~ Here $\lambda = 5, 3, 2$

When $\lambda = 5$,

$(A - 5I)x = 0$

$$\Rightarrow \begin{bmatrix} 2 & 0 & -4 \\ 0 & 0 & 0 \\ 5 & 0 & -7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Finding reduce row echon form

$R_1 \leftarrow R_1/2$

$$\sim \begin{bmatrix} 1 & 0 & -2 \\ 0 & 0 & 0 \\ 5 & 0 & -7 \end{bmatrix}$$

Interchanging 2nd & 3rd row.

$$\sim \begin{bmatrix} 1 & 0 & -2 \\ 5 & 0 & -7 \\ 0 & 0 & 0 \end{bmatrix}$$

$R_2 \leftarrow R_2 - 5 \times R_1$

$$\sim \begin{bmatrix} 1 & 0 & -2 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

The solution is of vector form

$$\begin{bmatrix} 2s \\ 0 \\ s \end{bmatrix} = s \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \quad \text{where } s \in \mathbb{R}, \; s \neq 0$$

$\therefore$ Basic eigenvector for $\lambda = 5$ is $\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$

For $\lambda = 3$,

$(A - 3I)x = 0$

$$\Rightarrow \begin{bmatrix} 5 & 0 & -4 \\ 0 & 2 & 0 \\ 5 & 0 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Converting $A-3I$ to reduced row echlon form.

$$A-3I \sim \begin{bmatrix} 5 & 0 & -4 \\ 0 & 2 & 0 \\ 5 & 0 & -5 \end{bmatrix}$$

$R_3 \leftarrow R_3 - R_1$

$$\sim \begin{bmatrix} 5 & 0 & -4 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$\therefore$ Basic eigen vector is $\begin{bmatrix} 4 \\ 0 \\ 5 \end{bmatrix}$

For $\lambda = 2$

$(A - \lambda I)x = 0$

$$\Rightarrow \begin{bmatrix} 5 & 0 & -4 \\ 0 & 3 & 0 \\ 5 & 0 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow$ $5x_1 - 4x_3 = 0$

$x_1 = 4/5 \, x_3$

$\therefore$ eigenvector is given as

$$x_3 \begin{bmatrix} 4/5 \\ 0 \\ 1 \end{bmatrix} = x_3 \begin{bmatrix} 4 \\ 0 \\ 5 \end{bmatrix}$$

$$v_{\lambda=2} = \frac{1}{\sqrt{41}} \begin{bmatrix} 4 \\ 0 \\ 5 \end{bmatrix}$$

Now eigenvector are given as,

$$P = \begin{bmatrix} 2 & 4 & 4 \\ 0 & 0 & 0 \\ 1 & 5 & 5 \end{bmatrix}$$

Since $|P| = 0$, The matrix A is not diagonizable.

(d) $\begin{bmatrix} 1 & 1 & -3 \\ 2 & 0 & 6 \\ 1 & -1 & 5 \end{bmatrix}$

$\Rightarrow$ characteristic eqn is given as

$|A - \lambda I| = 0$

$\Rightarrow \begin{vmatrix} 1-\lambda & 1 & -3 \\ 2 & 0-\lambda & 6 \\ 1 & -1 & 5-\lambda \end{vmatrix} = 0$

$\lambda = 2, 2, 2$

i.e $\lambda = 2$ (with multipeicity of 3)

Now Orthogonal eigenvectors P will have three column same as λ is same thus, $|P| = 0$. Therefore, P^{-1} does not exist.

Hence, A is not diagonizable.

(e) $\begin{bmatrix} 1 & -2 & 3 \\ 2 & 6 & -6 \\ 1 & 2 & -1 \end{bmatrix}$

Here, characteristic polynomial is given as,

$|A - \lambda I| = 0$

$\Rightarrow \begin{vmatrix} 1-\lambda & -2 & 3 \\ 2 & 6-\lambda & -6 \\ 1 & 2 & -1-\lambda \end{vmatrix} = 0$

$\Rightarrow \lambda^3 - 6\lambda^2 + 12\lambda - 8 = 0$

$\lambda = 2, 2, 2$

For 3X3 matrix, eigenvalue is 2 with multipeicity of 3 matrix can not be diagonalized.

(f) $\begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & -1 & 2 \end{bmatrix}$

Let $A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & -1 & 2 \end{bmatrix}$ $\therefore$ Characteristic eqn is,

$|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 2-\lambda & 1 & 1 \\ 0 & 1-\lambda & 0 \\ 1 & -1 & 2-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \cancel{\lambda^3 - 5\lambda^2 + 7\lambda - 3 = 0}$$

$$\Rightarrow (2-\lambda)\begin{vmatrix} 1-\lambda & 0 \\ -1 & 2-\lambda \end{vmatrix} - 1\begin{vmatrix} 0 & 0 \\ 1 & 2-\lambda \end{vmatrix} + 1\begin{vmatrix} 0 & (1-\lambda) \\ 1 & -1 \end{vmatrix} = 0$$

$$\Rightarrow (2-\lambda)(1-\lambda)(2-\lambda) - 1(1-\lambda) = 0$$

$$\Rightarrow (2-\lambda)^2(1-\lambda) - (1-\lambda) = 0$$

$$\Rightarrow (1-\lambda)[(2-\lambda)^2 - 1] = 0$$

~~either~~ $(1-\lambda) = 0 \Rightarrow \lambda = 1$

$$(2-\lambda)^2 - 1 = 0$$

$$\Rightarrow 4 - 4\lambda + \lambda^2 - 1 = 0$$

$$\Rightarrow \lambda^2 - 4\lambda + 3 = 0$$

$$\Rightarrow \lambda^2 - 3\lambda - \lambda + 3 = 0$$

$$\Rightarrow \lambda(\lambda-3) - 1(\lambda-3) = 0$$

$$\Rightarrow (\lambda-1)(\lambda-3) = 0$$

$$\lambda = 1, 3$$

Hence $\lambda = 1, 1, 3$

When $\lambda = 1$

$$(A - 1I)x = 0$$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 1 & -1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The reduced echelon form for $A - I$ is

$$\sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 1 & -1 & 1 \end{bmatrix}$$

Exchange $R_2 \leftrightarrow R_3$

$$\sim \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$R_2 \leftarrow R_2 - R_1$

$$\sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Basis for eigen ~~value~~ vector can be form as

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow x_1 + x_2 + x_3 = 0$

$\Rightarrow x_1 = -x_2 - x_3$

Basis for eigen space

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -x_2 - x_3 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\Rightarrow x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

For $\lambda = 1$

$$V_{\lambda=1} = \left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

For $\lambda = 3$

$$\begin{bmatrix} -1 & 1 & 1 \\ 0 & -2 & 0 \\ 1 & -1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

using Gauss elimination method

$$A - 3I \sim \begin{bmatrix} -1 & 1 & 1 \\ 0 & -2 & 0 \\ 1 & -1 & -1 \end{bmatrix}$$

$R_3 \leftarrow R_3 + R_1$

$$\sim \begin{bmatrix} -1 & 1 & 1 \\ 0 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Thus eigen Space is represented by:-

$$\begin{bmatrix} -1 & 1 & 1 \\ 0 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$\Rightarrow -x_1 + x_2 + x_3 = 0$

$x_1 = x_2 + x_3$

Basis for eigen space:-

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_2 + x_3 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$\therefore V_{\lambda=3} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

Now P can be represented as orthogonal matrix as,

$$P = \begin{bmatrix} -1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

and $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

matrix A can be diagonalized as

$P^{-1}AP = D$

Finding P^{-1}

using Gauss Jordan method

$$\sim \left[\begin{array}{ccc|ccc} -1 & -1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$

$R_1 \rightarrow R_1 \times -1$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 1 & -1 & -1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$

$R_2 \leftarrow R_2 - R_1$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 1 & -1 & -1 & 0 & 0 \\ 0 & -1 & 2 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$

$R_3 \leftarrow R_3 - R_2$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 1 & -1 & -1 & 0 & 0 \\ 0 & -1 & 2 & 1 & 1 & 0 \\ 0 & 0 & -2 & -1 & -1 & 1 \end{array}\right]$$

$\sim \; R_2 \leftarrow R^*(-)$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 1 & -1 & -1 & 0 & 0 \\ 0 & 1 & -2 & -1 & -1 & 0 \\ 0 & 0 & -2 & -1 & -1 & 1 \end{array}\right]$$

$R_3 \leftarrow R_3 / -2$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 1 & -1 & -1 & 0 & 0 \\ 0 & 1 & -2 & -1 & -1 & 0 \\ 0 & 0 & 1 & 1/2 & 1/2 & -1/2 \end{array}\right]$$

$R_1 \leftarrow R_1 - R_2$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & -2 & -1 & -1 & 0 \\ 0 & 0 & 1 & 1/2 & 1/2 & -1/2 \end{array}\right]$$

$R_1 \leftarrow R_1 - R_3$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1/2 & 1/2 & 1/2 \\ 0 & 1 & -2 & -1 & -1 & 0 \\ 0 & 0 & 1 & 1/2 & 1/2 & -1/2 \end{array}\right]$$

$R_2 \leftarrow R_2 - 2R_3$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1/2 & 1/2 & 1/2 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1/2 & 1/2 & -1/2 \end{array}\right]$$

$$P^{-1} = \begin{bmatrix} -1/2 & 1/2 & 1/2 \\ 0 & 0 & 1 \\ 1/2 & 1/2 & -1/2 \end{bmatrix}$$

Now A is diagonalized as,

$$P^{-1}AP = \begin{bmatrix} -1/2 & 1/2 & 1/2 \\ 0 & 0 & 1 \\ 1/2 & 1/2 & -1/2 \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & -1 & 2 \end{bmatrix} \begin{bmatrix} -1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} -1+1/2 & -1/2+1/2-1/2 & -1/2+1 \\ 1 & -1 & 2 \\ 1-1/2 & 1/2+1/2+1/2 & 1/2-1 \end{bmatrix} \begin{bmatrix} -1 & -1 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} -1/2 & -1/2 & 1/2 \\ 1 & -1 & 2 \\ 1/2 & 3/2 & -1/2 \end{bmatrix} \begin{bmatrix} -1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

(3) If A is $n \times n$ matrix show that A is diagonalizable iff A^T is diagonalizable.

→ suppose A is diagonalizable; Hence,

$$P^{-1}AP = D$$

taking transpose on both side,

$$(P^{-1}AP)^T = D^T$$

$$\Rightarrow P^T A^T (P^{-1})^T = D^T = D \quad (\because D^T = D)$$

Let $Q = (P^{-1})^T$

$$Q = (P^T)^{-1}$$

$$Q^{-1} = [(P^T)^{-1}]^{-1}$$

$$\therefore P^T = Q^{-1}$$

Thus $Q^{-1} A^T Q = D$

Hence, A^T is diagonalizable if $(A^T)^T = A$ follows A is also diagonalizable.

(4) If A is diagonalizable and 1 and −1 are the only eigen values show that $A^{-1} = A$.

Soln Given A is diagonalizable then,

$P^{-1}AP = D$ where D is diagonal matrix

$$\Rightarrow A = PDP^{-1}$$

Given $D = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

then $D^{-1} = D^T = D = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

Here $P^{-1}AP = D$

taking inverse on both side

$(P^{-1}AP)^{-1} = D^{-1}$ $\quad [(AB)^{-1} = B^{-1}A^{-1}$

$\Rightarrow P^{-1}A^{-1}P = D^{-1} = D$

multiply left by P and right by P^{-1}, then

$\Rightarrow PP^{-1}A^{-1}PP^{-1} = PDP^{-1}$

$\Rightarrow \quad A^{-1} = PDP^{-1} = A$

(5) If $P^{-1}AP$ and $P^{-1}BP$ are both diagonal show that $AB = BA$

Soln Since $P^{-1}AP$ & $P^{-1}BP$ are diagonal

i.e $P^{-1}AP = D_1$

$P^{-1}BP = D_2$

$\Rightarrow A = PD_1P^{-1}$ &

$B = PD_2P^{-1}$

$AB = PD_1P^{-1}PD_2P^{-1}$

$= PD_1D_2P^{-1} \quad [P^{-1}P = I]$

$= PD_2D_1P^{-1}$

$= PD_2P^{-1}PD_1P^{-1}$

$= BA$

$\therefore AB = BA$ H

Exercise B

I Show that A and A^T have the same non-zero singular values. How are their singular value decomposition are related.

$\Rightarrow$ The singular value decomposition of A is given as:-

$A = U\Sigma V^T$

where Σ is a diagonal containing singular value U & V are orthogonal matrices / singular vectors.

Taking transpose:-

$A^T = (U\Sigma V^T)^T = (V^T)^T\Sigma^T U^T = V\Sigma^T U^T$

Here $\Sigma = \Sigma^T$ only when A is a square matrix. The singular value of A and A^T are equal only for m×n matrix $\Sigma \neq \Sigma^T$. The transpose of singular value decomposition of A is equal to singular value decomposition of A^T.

(5) Find the SVD of each of following matrices.

(a) $\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$

Let $A = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$

and $A^T = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$ $A^TA = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix}$

Here characteristic polynomial for A^TA is,

$$\begin{vmatrix} 2-d & -2 \\ -2 & 2-d \end{vmatrix} = 0$$

$\Rightarrow (2-d)(2-d) - 4 = 0$

$\Rightarrow 4 - 2d - 2d + d^2 - 4 = 0$

$\Rightarrow d^2 - 4d = 0$

$\Rightarrow d(d-4) = 0$

$d = 0, 4$

Thus $\sigma_1 = \sqrt{d_1} = \sqrt{4} = 2$

Calculating orthogonal right singular vector (v) as:-

For $d = 4$

$$\begin{bmatrix} -2 & -2 \\ -2 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\Rightarrow -2x_1 - 2x_2 = 0$

$\Rightarrow -x_1 = x_2$

$\Rightarrow x_1 = -x_2$

$\therefore V_{d=4} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

For $\lambda = 0$

$$\begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$V_{\lambda=0} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Now $u_1 = \frac{Av_1}{\sigma_1} = \frac{\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix}}{2} = \frac{1}{2}\begin{bmatrix} 2 \\ -2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

For u_2

$u_1^T x = 0$

$$\Rightarrow \begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$\Rightarrow x_1 - x_2 = 0 \quad x_1 = x_2$

$$u_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Thus SVD for A is

$A = U \Sigma V^T$

$$= \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}^T$$

$$= \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

2(b) $\begin{bmatrix} 5 & -3 \\ 0 & 4 \end{bmatrix}$

Let $A = \begin{bmatrix} 5 & -3 \\ 0 & 4 \end{bmatrix}$

$$A^T A = \begin{bmatrix} 5 & 0 \\ -3 & 4 \end{bmatrix} \begin{bmatrix} 5 & -3 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} 25 & -15 \\ -15 & 25 \end{bmatrix}$$

Characteristic eqn for $A^T A$ is:

$$\begin{vmatrix} 25-\lambda & -15 \\ -15 & 25-\lambda \end{vmatrix} = 0$$

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$\Rightarrow (25-\lambda)(25-\lambda) - 225 = 0$

$\Rightarrow 625 - 25\lambda - 25\lambda + \lambda^2 - 225 = 0$

$\Rightarrow \lambda^2 - 50\lambda + 400 = 0$

$\Rightarrow \lambda^2 - 40\lambda - 10\lambda + 400 = 0$

$\Rightarrow \lambda(\lambda-40) - 10(\lambda-40) = 0$

$\Rightarrow (\lambda-10)(\lambda-40) = 0 \quad \lambda = 40, 10$

$\sigma_1 = \sqrt{40}$ and $\sigma_2 = \sqrt{10}$

For $\lambda = 40$

$$\begin{bmatrix} 25-40 & -15 \\ -15 & 25-40 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\Rightarrow -15x_1 - 15x_2 = 0$

$\Rightarrow x_1 = -x_2$

$$V_{\lambda=40} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

For $\lambda = 10$

$$\begin{bmatrix} 25-10 & -15 \\ -15 & 25-10 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 15 & -15 \\ -15 & 15 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\Rightarrow x_1 = x_2$

$$V_{\lambda=10} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Thus $$V = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

$$\Sigma = \begin{bmatrix} \sqrt{40} & 0 \\ 0 & \sqrt{10} \end{bmatrix}$$

For U (i.e. left singular vector)

$$u_1 = \frac{Av_1}{\sigma_1}$$

$$= \frac{\begin{bmatrix} 5 & -3 \\ 0 & 4 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix}}{\sqrt{40}} = \frac{\begin{bmatrix} 8 \\ -4 \end{bmatrix}}{\sqrt{40}} = \frac{1}{2\sqrt{10}}\begin{bmatrix} 8 \\ -4 \end{bmatrix} = \frac{1}{\sqrt{10}}\begin{bmatrix} 4 \\ -2 \end{bmatrix}$$

$$u_2 = \frac{Av_2}{\sigma_2} = \frac{1}{\sqrt{10}}\begin{bmatrix} 5 & -3 \\ 0 & 4 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{10}}\begin{bmatrix} 2 \\ 4 \end{bmatrix}$$

SVD for A

$$A = D \Sigma V^T$$

$$= \begin{bmatrix} 4/\sqrt{10} & 2/\sqrt{10} \\ -2/\sqrt{10} & 4/\sqrt{10} \end{bmatrix}\begin{bmatrix} \sqrt{40} & 0 \\ 0 & \sqrt{10} \end{bmatrix}\begin{bmatrix} 1/\sqrt{2} & \frac{1}{\sqrt{2}} \\ -1/\sqrt{2} & \frac{1}{\sqrt{2}} \end{bmatrix}^T \quad H$$

2.c) $\begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \end{bmatrix}$

$$\Rightarrow A^TA = \begin{bmatrix} 1 & -1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & -1 & 1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

Characteristic eq^n for A^TA

$$\begin{vmatrix} 2-d & -1 & 1 \\ -1 & 1-d & 0 \\ 1 & 0 & 1 \end{vmatrix} = 0$$

$\Rightarrow d^3 - 4d^2 + 4d - 0 = 0$

$\Rightarrow d^3 - 4d^2 + 4d = 0$

$\Rightarrow d(d^2 - 4d + 4) = 0$

$d = 0, 2, 2$

For $d = 2, 2$

$\sigma_1 = \sqrt{2}$ $\qquad \Sigma = \begin{bmatrix} \sqrt{2} & 0 & 0 \\ 0 & \sqrt{2} & 0 \end{bmatrix}$

$\sigma_2 = \sqrt{2}$

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For $\lambda=2$

$(A-2I)x=0$

$$\Rightarrow \begin{bmatrix} 0 & -1 & 1 \\ -1 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left.\begin{aligned} &\Rightarrow 0\cdot x_1 - x_2 + x_3 = 0 \\ &\Rightarrow -x_1 - x_2 + 0\cdot x_3 = 0 \end{aligned}\right\}$$

$\Rightarrow 1\cdot x_1 + 0\cdot x_2 - x_3 = 0$

Using Cramer's rule:-

$$\frac{x_1}{\begin{vmatrix} -1 & 1 \\ -1 & 0 \end{vmatrix}} = \frac{-x_2}{\begin{vmatrix} 0 & 1 \\ -1 & 0 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 0 & -1 \\ -1 & 1 \end{vmatrix}}$$

$$\Rightarrow \frac{x_1}{1} = \frac{-x_2}{1} = \frac{x_3}{-1}$$

$$\therefore V_{\lambda=2} = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$$

For $\lambda=0$

$|A-0I|x=0$

$$\Rightarrow \begin{bmatrix} 2 & -1 & 1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Reduce $A-0I$ into row echleon matrix

$$A-0I \sim \begin{bmatrix} 2 & -1 & 1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$R_2 \leftarrow 2R_2 + R_1$

$$\sim \begin{bmatrix} 2 & -1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

$R_3 \leftarrow 2R_3 - R_1$

$$\sim \begin{bmatrix} 2 & -1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$R_3 \leftarrow R_3 - R_2$

$$\sim \begin{bmatrix} 2 & -1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{bmatrix}$$

$$\therefore \begin{bmatrix} 2 & -1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$2x_1 - x_2 + x_3 = 0$

$x_2 + x_3 = 0$

$\Rightarrow x_3 = 1 \quad x_2 = -1$

then $2x_1 + 1 + 1 = 0$

$\Rightarrow 2x_1 + 2 = 0$

$x_1 = -1$

$$\therefore V_{d=0} = \frac{1}{\sqrt{3}} \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$$

Calculation for u

$$u_1 = \frac{Av_1}{\sigma_1} = \frac{\begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{3}} \\ -1/\sqrt{3} \\ -1/\sqrt{3} \end{bmatrix}}{\sqrt{2}} = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ -2/\sqrt{3} \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{-2}{\sqrt{6}} \end{bmatrix}$$

$$u_2 = u_1 = \begin{bmatrix} 0 \\ \frac{-2}{\sqrt{6}} \end{bmatrix}$$

$u_3^T x = 0$

$$\Rightarrow \begin{bmatrix} 0 & \frac{-2}{\sqrt{6}} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

Thus SVD for A is :- $U \Sigma V^T$

$$\begin{bmatrix} 0 & 0 \\ -2/\sqrt{6} & -2/\sqrt{6} \end{bmatrix} \begin{bmatrix} \sqrt{2} & 0 & 0 \\ 0 & \sqrt{2} & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{3}} \\ -1/\sqrt{3} & -1/\sqrt{3} \\ -1/\sqrt{3} & 1/\sqrt{3} \end{bmatrix}$$

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[Q3] prove that if A is symmetric matrix with eigen values $d_1, d_2 \dots d_n$ then singular values of A are $|d_1|, |d_2|, \dots$

→ Since A is symmetric matrix with eigenvalue d then by theorem A is orthogonally diagonalizable such that $A = PDP^{-1}$ where D is diagonal matrix with diagonal elements of eigen value of A.

For symmetric matrix $A^T = A$

$A^T A = A^2$

$= (PDP^{-1})(PDP^{-1}) = PD(P^{-1}P)DP^{-1} = PDIDP^{-1}$

$= PD^2P^{-1}$

Since, $P^{-1} = P^T$ [P is orthogonal]

$\therefore A^T A = PD^2P^T$

Here $A^T A$ is symmetric matrix with eigen value of d^2

Since, singular values for matrix A, are square root of eigen values of $A^T A$ then A has singular value of $\sqrt{d^2} = |d|$ #

[4] Let A be a $m \times n$ matrix with SVD $U \Sigma V^T$ and suppose that A has rank r where $r < n$, show that $\{v_1, \dots v_r\}$ is orthogonal basis for row(A).

⇒ we know that

SVD of $A = U \Sigma V^T$

where U represent orthonormal singular vectors whose columns represents basis for R^n

$\Sigma = diag(\sigma_1, \dots \sigma_n)$

Now, $A = U \Sigma V^T$

taking transpose both sides,

$A^T = (U \Sigma V^T)^T$

$A^T = V \Sigma^T U^T \quad (\because \Sigma^T = \Sigma)$

$\therefore A^T = V \Sigma U^T$

multiplying both sides by U.

$A^T U = V\Sigma$

For ith row:-

$A^T u_i = v_i \sigma_i$

$\therefore v_i = A^T(\sigma_i^{-1} u_i) \in Row(A^T)$

Here

$\sigma_i > 0$ for $i = 1, \dots r$

and

$\sigma_{r+1}, \sigma_{r+2}, \dots \sigma_n = 0$

The rank of Row(A^T) = r and $\{v_1, \dots v_r\}$ is orthonormal set thus it follows that it is orthonormal basis for Row(A).

(5) Let A be $n \times n$ matrix with singular values $\sigma_1, \dots \sigma_n$ and eigen values $d_1, \dots d_n$ show that,

$|d_1, d_2, \dots d_n| = \sigma_1, \sigma_2, \dots \sigma_n$.

$\Rightarrow$ Given,

A is square matrix with singular values as;

$\Sigma^T = \Sigma$

$Ax = dx$

multipy both side by A^T

$A^T A x = d A^T x$

$\Rightarrow A^T A x = d A x$ $[A^T = A]$ $[\because Ax = da]$

$\Rightarrow A^T A x = d(da)$

$\Rightarrow A^T A x = d^2 x$ $[\because \sigma = \sqrt{d^2}$

eigen value of $A^T A$ is d^2 and singular value of A is d, thus singular value of A $= \sqrt{d^2} = |d| = \sigma$

$\therefore |d_1, \dots d_n| = \sigma_1, \sigma_2 \dots \sigma_n$.

[Q.N 6] Show that if σ is a singular values of A then there exists a non-zero vector x such that

$$\sigma = \frac{\|Ax\|_2}{\|x\|_2}$$

$\Rightarrow$ Here

$$\|Ax_2\|^2 = \langle Ax, Ax \rangle$$

$$\therefore \|Ax_2\|^2 = (Ax)^T (Ax)$$

$$\therefore \|Ax_2\|^2 = x^T A^T A x$$

Here $[A^T A x = d x]$

$$\therefore \|Ax_2\|^2 = x^T d x$$

$$= d x^T x$$

$$= d \|x\|_2^2$$

$$\therefore \|Ax_2\|^2 = \sqrt{d} \|x\|_2$$

Since $\sigma = \sqrt{d}$

$$\Rightarrow \|Ax_2\| = \sigma \|x\|_2$$

$$\Rightarrow \sigma = \frac{\|Ax\|_2}{\|x\|_2} \#$$

8) If A is $m\times n$ matrix with all singular value positive, what is rank A?

Soln Rank of matrix A is equal to the number of non-zero eigenvalue (i.e no. of singular value)

A has order of $m\times n$.

A^TA will have order of $n\times n$

Thus, there will be n eigenvalues of A^TA. Given all singular values are positive i.e all eigenvalues are positive & non-zero i.e $\text{rank}(A)=n$.

9) If A is square, show that $|A|$ is the product of singular values of A.

$\Rightarrow$ SVD of A is:-

$$A = U\Sigma V^T$$

Taking determinant both sides,

$$\det(A) = \det(U\Sigma V^T)$$

$[\because \det(AB) = \det(A)\cdot\det(B)]$

$$\Rightarrow \det(A) = \det(U)\det(\Sigma).\det(V^T)$$

$$= 1.\det(\Sigma).1$$

[U & V are orthogonal matrix thus $\det(U) = \det(V^T) = 1$]

$$\therefore \det(A) = \det(\Sigma)$$

Since, Σ is diagonal matrix containing singular values in its diagonal for A as square matrix:-

$$\det(A) = \sigma_1 \times \sigma_2 \times \dots \sigma_r \quad [\sigma_1 \dots \sigma_r > 0]$$

[Q.N.10] If A is square & real, show that $A=0$ IFF every eigenvalues of A is 0.

$\Rightarrow$ If every eigenvalue of matrix A is equal to 0,

it follows that every eigen value of matrix A^k is also 0
Since, all eigen values of matrix A are real, there exist orthogonal matrix P, such that $P^TAP = D$, this implies
$P^TAP = 0$ [i.e all eigenvalues are 0]
matrix P^T and P are invertible thus can not be zero.
thus $A \neq 0$ means A is nilpotent. [i.e orthogonal]

[Q.N. 11] Given a SVD for invertible matrix A find one for A^{-1}. How are Σ_A and $\Sigma_{A^{-1}}$ related?

Soln We know that SVD of A;

$$A = U \Sigma V^T$$

taking inverse both sides;

$$A^{-1} = (U \Sigma V^T)^{-1}$$

2) $A^{-1} = (V^T)^{-1} \Sigma^{-1} U^{-1}$

$$= (V^T)^T \Sigma^{-1} U^T$$

$$= V \Sigma^{-1} U^T \quad [\because V^{-1} = U^T]$$

$[\because U^{-1} = U^T]$

$\therefore A^{-1} = V \Sigma^{-1} U^T$

Since U & V are orthogonal matrix. Let us check if $V\Sigma^{-1}U^T$ is SVD of A^{-1}

$$(A^TA)^{-1} = A^{-1}(A^T)^{-1} = A^{-1}(A^{-1})^T$$

Let d be eigen value of A^TA then

$$(A^TA)x = dx \text{ and } d \neq 0$$

multiply both side by A

$$(AA^T)(Ax) = d(Ax)$$

Since $d \neq 0$ and x is non-zero vector thus, A^TA and AA^T has same eigen value, Thus, σ is singular value of A then singular value of A^{-1} is $\frac{1}{\sigma}$.

Thus,

$$\Sigma_A = \begin{bmatrix} \sigma_1 & 0 & 0 & 0 & \cdots \\ 0 & \sigma_2 & 0 & 0 & \cdots \\ \vdots & 0 & \sigma_3 & 0 & \cdots \\ \vdots & & & & \\ 0 & 0 & \cdots & 0 & 0 & \cdots \\ 0 & 0 & \cdots & & 0 \end{bmatrix} \quad \text{Rank}_A = r$$

Then

$$\Sigma_{A^+} = \begin{bmatrix} \frac{1}{\sigma_1} & 0 & 0 & \cdots & \cdots \\ 0 & \cdots & \frac{1}{\sigma_2} & 0 & \cdots \\ 0 & \cdots & 0 & \cdots \frac{1}{\sigma_r} & 0 \cdots \\ 0 & \cdots & & 0 & \\ 0 & \cdots & & 0 & \end{bmatrix} \#$$

Q.N.12 If $A = U\Sigma_A V^T$ is SVD for A find SVD for A^T.

Soln SVD for A is given as,

$$A = U\Sigma V^T$$

$$A^T = (U\Sigma V^T)^T$$

$$\Rightarrow A^T = (V^T)^T \Sigma^T U^T = V\Sigma^T U^T$$

Thus, transpose of SVD of A is equal to SVD of A^T.

Here for square matrix

$$\Sigma^T = \Sigma \quad \#$$