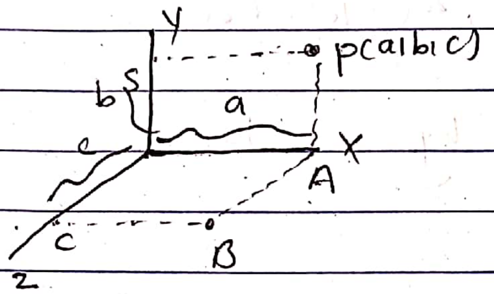
Analytic geometry

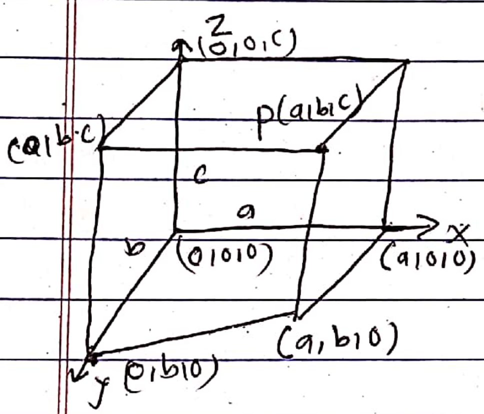
(I) plane geometry

(II) space geometry

(III) Solid geometry

Position of point in space :-





In space $\mathbb{R}^3$

$$(x,y,z) = \{(x,y,z) : \forall x,y,z \in \mathbb{R}\}$$

Distance between two points

$P_1(x_1\, x_2\, x_3)$ & $P_2(y_1\, y_2\, y_3)$

$$|P_1P_2| = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}$$

Sphere

Equation of sphere with centre (h, k, l) and radius r is,

$$(x-h)^2+(y-k)^2+(z-l)^2=r^2 \quad \text{--- (1)}$$

Also,

$$(x-h)^2+(y-k)^2+(z-l)^2<r^2 \text{ (open disk)}$$

$$(x-h)^2+(y-k)^2+(z-l)^2\leq r^2 \text{ (closed disk)}$$

The intersection of sphere (1) by $x=0$ i.e. yz plane is,

$$(0-h)^2+(y-k)^2+(z-l)^2=r^2$$

$$\Rightarrow (y-k)^2+(z-l)^2=r^2-h^2$$

$$\Rightarrow (y-k)^2+(z-l)^2=\left(\sqrt{r^2-h^2}\right)^2 \quad \text{--- (*)}$$

$\therefore$ (*) is circle on yz plane.

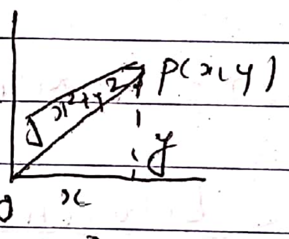
Vectors:

Position vector

A vector which is defined with respect to some reference point is called position vectors.

$\vec{OP} = (x, y)$ plane vector

$\vec{OP} = (x, y, z)$ space vector.



$|\vec{OP}| = \sqrt{x^2+y^2}$

$\vec{OP} = (x, y)$

Modulus of vector $\vec{OP} = (x, y)$ is the distance from initial point to final point.

$$|\vec{OP}| = \sqrt{x^2+y^2}$$

Similarly For Space vector

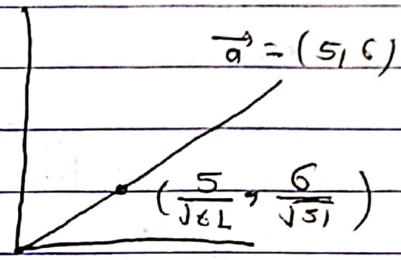
$\overrightarrow{op} = (x, y, z)$

$|\overrightarrow{op}| = \sqrt{x^2+y^2+z^2}$

Unit vector :- Vector having modulus 1 is unit vector.

Thus for a vector $\vec{a} = (a_1, a_2)$ the unit vector is,

$$\hat{a} = \frac{\vec{a}}{|\vec{a}|}$$



(8) Given the point $P(100, 200)$ in the plane find a point on $\overrightarrow{op}$ whose length is 5.

Hint: Required $= 5 \times \left[\frac{\overrightarrow{op}}{|\overrightarrow{op}|}\right]$

Every plane vectors (x, y) can be written as,

$(x, y) = x\vec{i} + y\vec{j}$ where $\vec{i} = (1, 0)$ & $\vec{j} = (0, 1)$

$= x(1, 0) + y(0, 1)$

$= (x, 0) + (0, y)$

$= (x+0, 0+y) = (x, y)$

Thus every plane vector (x, y) can be written as linear combination of $\vec{i}$ and $\vec{j}$.

Similarly every space vector (x, y, z) can be written as linear combination of $\vec{i}, \vec{j}, \vec{k}$,

i.e; $(x, y, z) = x\vec{i} + y\vec{j} + z\vec{k}$

where $\vec{i} = (1, 0, 0)$ and $k = (0, 0, 1)$

$\vec{j} = (0, 1, 0)$

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Like vector:

Two vectors $\vec{a}$ and $\vec{b}$ are like if
$\vec{a} = k\vec{b}$ for $k>0$

Unlike vector:

$\vec{a} = -k\vec{b}$ where $k>0$

Example:

Let $\vec{a} = \vec{OA}$
$\vec{b} = \vec{AB}$

Then, $\vec{OB} = \vec{OA} + \vec{AB}$

$$\vec{BA} = \vec{BO} + \vec{OA}$$
$$= -\vec{OB} + \vec{OA}$$
$$= \vec{OA} - \vec{OB}$$

Find $\vec{AB}$

$$\vec{AB} = \vec{OB} - \vec{OA}$$

Let $\vec{OA} = (1,2)$
$\vec{OB} = (4,6)$

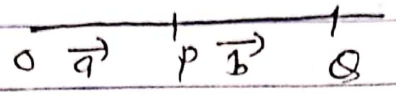
Then, $AB = \vec{OB} - \vec{OA}$
$= (4,6) - (1,2)$
$= (3,4)$

Triangle Inequality:

If three point are lie in same ~~point~~ line they are called Collinear point.

For any vectors $\vec{a}$ and $\vec{b}$
$$|\vec{a}+\vec{b}| \le |\vec{a}| + |\vec{b}|$$

Case I:



If O, P, Q lie on Same line.

$\vec{OQ} = \vec{OP} + \vec{PQ}$

$|\vec{OQ}| = |\vec{OP}| + |\vec{PQ}|$

$|\vec{OP} + \vec{PQ}| = |\vec{a}| + |\vec{b}|$

or $|\vec{a} + \vec{b}| = |\vec{a}| + |\vec{b}|$

Case II: If O, P, Q do not lie on Same plane.

$|\vec{a} + \vec{b}| < |\vec{a}| + |\vec{B}|$

Let P, Q be two vectors in $\mathbb{R}^n$

Also, Let c be a number $\vec{P} = c\vec{Q}$

Then, $PQ = cQ \cdot Q = c\|Q\|^2$

where c is real number

Let c be a Complex number

$P = cQ$

$|P - cQ| = 0$

$\|P - cQ\|^2 = 0$

$(P - cQ) \cdot (P - cQ) = 0$

$\Rightarrow P \cdot P - P \cdot cQ - cQ \cdot P + cQ \cdot cQ = 0$

$\Rightarrow \|P\|^2 - c(P \cdot Q) - c(P \cdot Q) + c^2\|Q\|^2 = 0$

$\Rightarrow \|P\|^2 - 2c(P \cdot Q) + c^2\|Q\|^2$

which is quadratic in c and value of c is Imaginary

$4(p\cdot Q)^2 - 4\|Q\|^2\|p\|^2 < 0$

$\Rightarrow \quad |p\cdot Q|^2 < \|p\|^2\|Q\|^2$

$\Rightarrow \quad |p\cdot Q| < \|p\|\|Q\|$

Angle between two vectors :-

Let $\vec{a}$ and $\vec{b}$ be two vectors

Then the cosine of the angle between $\vec{a}$ and $\vec{b}$ is,

$\cos\theta = \dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}$ where $|\vec{a}| \neq 0$, $|\vec{b}| \neq 0$

Thus, this can be written as,

$\cos\theta = \hat{a}\cdot\hat{b}$

Two vectors are orthogonal if

$\theta = 90^\circ$

$\cos\theta = \cos 90^\circ = 0$

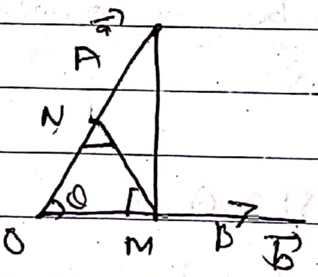
$\vec{a}\cdot\vec{b} = 0$

Geometrical meaning of dot product :-

Let θ be the angle betn

$\vec{a} = \vec{OA}$

& $\vec{b} = \vec{OB}$



Draw $AM \perp OB$

OM is projection of $\vec{OA}$ on $\vec{OB}$

$\cos\theta = \dfrac{OM}{OA} = \dfrac{OM}{|\vec{a}|}$

$OM = \cos\theta\,|\vec{a}|$

$= \dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}|\cdot|\vec{b}|}\cdot|\vec{a}|$

$= \dfrac{\vec{a}\cdot\vec{b}}{|\vec{b}|}$

$\vec{a}\cdot\vec{b} = |\vec{b}| \cdot OM$

$=$ magnitude of $\vec{b}$ x projection of $\vec{a}$ on $\vec{b}$.

$\cos\theta = \dfrac{ON}{|\vec{b}|}$

$ON = \cos\theta \, |\vec{b}|$

$= \dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|} \cdot |\vec{b}|$

$= \dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}|}$

$\vec{a}\cdot\vec{b} = |ON| \cdot |\vec{a}|$

$=$ magnitude of $\vec{a}$ x projection of $\vec{b}$ on $\vec{a}$.

<u>vector projection of $\vec{a}$ on $\vec{b}$</u>

$= \dfrac{\vec{a}\cdot\vec{b}}{|\vec{b}|} \cdot \dfrac{\vec{b}}{|\vec{b}|}$

Similarly vector projection of $\vec{b}$ on $\vec{a}$

$= \dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}|} \cdot \dfrac{\vec{a}}{|\vec{a}|}$

Two projection are equal if two vectors have equal length.

Let $\vec{a} = (5,0)$

$\vec{b} = (7,1)$

$\vec{a}\cdot\vec{b} = (5,0)\cdot(7,1)$

$= 35.$

$\vec{a} = \sqrt{5^2+0^2} = 5$

vector projection of $\vec{a}$ on $\vec{b}$

$$= \frac{35}{5} \cdot \frac{(5,0)}{5}$$

$$= (7,0)$$

Scalar product of unit vector

$\hat{i}\cdot\hat{j} = 0$

$\hat{j}\cdot\hat{k} = 0$

$\hat{k}\cdot\hat{j} = 0$

Fact: Dot product is commutative

Vector product:

Let $\vec{a} = (a_1, a_2, a_3)$ & $\vec{b} = (b_1, b_2, b_3)$ be two vectors,

Then their vector product $\vec{a}\times\vec{b}$ is defined by,

$$\vec{a}\times\vec{b} = \begin{matrix} a_1 & a_2 & a_3 & a_1 & a_2 \\ b_1 & b_2 & b_3 & b_1 & b_2 \end{matrix}$$

$$= (a_2 b_3 - b_2 a_3,\ a_3 b_1 - b_3 a_1,\ a_1 b_2 - b_1 a_2)$$

which is again vector.

Fact $\vec{a}\times\vec{b}$ & $\vec{b}\times\vec{a}$ are not equal

But $\vec{a}\times\vec{b} = -(\vec{b}\times\vec{a})$

Note: $\vec{a}\times\vec{b}$ is always perpendicular to $\vec{a}$ & $\vec{b}$.

Example

$\vec{a} = (7,0,0)$
$\vec{B} = (0,2,0)$

$$\vec{a} \times \vec{B} = \begin{vmatrix} i & j & k \\ 7 & 0 & 0 \\ 0 & 2 & 0 \end{vmatrix}$$

$= (0,0,14)$

$\vec{a} \cdot (\vec{a} \times \vec{b}) = (7,0,0) \cdot (0,0,14)$
$= 0$

Example

let $\vec{a} = (1,5,2)$
$\vec{B} = (2,4,0)$

find r
where $\vec{a} \cdot \vec{r} = 0$
$\vec{b} \cdot \vec{r} = 0$

Soln Here $\vec{r}$ is perpendicular to both $\vec{a}$ and $\vec{b}$ Hence r Cross product of $\vec{a}$ and $\vec{b}$.

Cross product of

$\vec{i} \times \vec{i} = 0$
$\vec{i} \times \vec{j} = \vec{k}$
$\vec{j} \times \vec{k} = \vec{i}$
$\vec{j} \times \vec{j} = 0$
$\vec{k} \times \vec{k} = 0$
$\vec{k} \times \vec{i} = 0$ J

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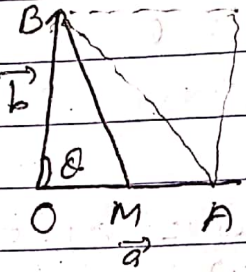
Angle between two vectors using cross product

Let $\vec{a}$ and $\vec{b}$ two non-zero vectors
Then the sine of angle between $\vec{a}$ and $\vec{b}$ is,

$$\sin\theta = \frac{|\vec{a} \times \vec{b}|}{|\vec{a}|\,|\vec{b}|}$$ ← To get Scalar

Geometrical meaning of $\vec{a} \times \vec{b}$



Let
$\vec{a} = \vec{OA}$
$\vec{b} = \vec{OB}$

Draw $\perp^r$ BM from $\vec{OB}$ on $\vec{OA}$

Then $\sin\theta = \frac{BM}{|\vec{b}|}$

$BM = |\vec{b}|\sin\theta$

Area of $\triangle OBA = \frac{1}{2} \times OA \times BM$

$= \frac{1}{2} \times |\vec{a}|\,|\vec{b}|\sin\theta$

$= \frac{1}{2}\,|\vec{a}|\,|\vec{b}| \cdot \frac{|\vec{a} \times \vec{b}|}{|\vec{a}|\,|\vec{b}|}$

$= \frac{1}{2}\,|\vec{a} \times \vec{b}|$

Area of parallelogram $= 2 \times \frac{1}{2}\,|\vec{a} \times \vec{b}|$

$= |\vec{a} \times \vec{b}|$

$\therefore$ $\vec{a} \times \vec{b}$ always represent the area of parallelogram with side $\vec{a}$ and $\vec{b}$.

example

If $\vec{a} = (1,2,0)$

$\vec{b} = (2,1,3)$

$$\vec{a}\times\vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2 & 0 \\ 2 & 1 & 3 \end{vmatrix}$$

$$= (6,-3,-3)$$

Hence

area $= |\vec{a}\times\vec{b}|$

$= \sqrt{6^2+3^2+3^2}$

$= \sqrt{54}$.

Direction Angles :-

If α, β, γ be the angles made by line with positive x y and z axis then

$\cos\alpha, \cos\beta, \cos\gamma$ are called direction Cosine of line and α, β, γ are called direction angles.

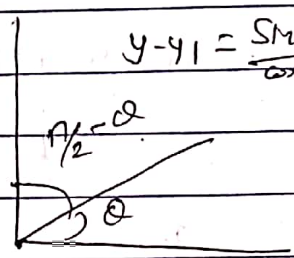
If $\vec{a} = (a_1, a_2, a_3)$ then unit Vector along $\vec{a}$ is,

$$\hat{a} = \left(\frac{\vec{a_1}}{|a_1|}, \frac{\vec{a_2}}{|\vec{a}|}, \frac{\vec{a_3}}{|\vec{a}|}\right)$$

$$= \left(\frac{a_1}{|a|}\vec{i} + \frac{\vec{a_2}}{|\vec{a}|}\vec{j} + \frac{\vec{a_3}}{|\vec{a}|}\vec{k}\right)$$

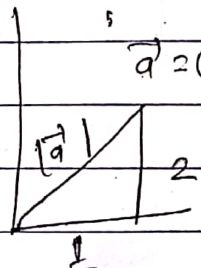
$$= \cos\alpha\,\vec{i} + \cos\beta\,\vec{j} + \cos\gamma\,\vec{k}.$$

$y - y_1 = \frac{\sin\theta}{\cos\theta}(x - x_1)$



$$\frac{x - x_1}{\cos\theta} = \frac{y - y_1}{\cos(\pi/2 - \theta)}$$

$\vec{a} = (1,2)$



$\cos\alpha = \frac{1}{|\vec{a}|}$

$\cos\beta = \frac{2}{|\vec{a}|}$

~~Here~~ Example

Let $\vec{OP} = 2\vec{i} + 3\vec{j} + 1\vec{k}$

Find unit vector along $\vec{OP}$ & direction Cosine & direction angles:

$$\hat{a} = \frac{\vec{a}}{|\vec{a}|}$$

$$= \frac{2}{\sqrt{14}}\vec{i} + \frac{3}{\sqrt{14}}\vec{j} + \frac{1}{\sqrt{14}}\vec{k}$$

D.C.R $l = \cos\alpha = \frac{2}{\sqrt{14}}$, $m = \cos\beta = \frac{3}{\sqrt{14}}$, $n = \cos\gamma = \frac{1}{\sqrt{14}}$

$\alpha = \cos^{-1}\left(\frac{2}{\sqrt{14}}\right)$, $\beta = \cos^{-1}\left(\frac{3}{\sqrt{14}}\right)$ $\gamma = \cos^{-1}\left(\frac{1}{\sqrt{14}}\right)$

Note:- $\boxed{\text{D.C Satisfied the relation, } l^2+m^2+n^2=1}$ (*)

Fact: $|\vec{a}| = a$, $\vec{a}^2 = \vec{a}\cdot\vec{a}$, $\vec{a}^3$ is meaningless.

Theorem 1:
For any two vectors $\vec{a}$ and $\vec{b}$
$$|\vec{a}+\vec{b}|^2 = |\vec{a}|^2 + 2\vec{a}\cdot\vec{b} + |\vec{b}|^2$$

proof:-

$$|\vec{a}+\vec{b}|^2 = (\vec{a}+\vec{b})\cdot(\vec{a}+\vec{b}) = (\vec{a}+\vec{b})^2$$

$[\because \vec{a}^2 = |\vec{a}|^2]$

$$= \vec{a}\cdot\vec{a} + \vec{a}\cdot\vec{b} + \vec{b}\cdot\vec{a} + \vec{b}\cdot\vec{b}$$ $[\because \vec{a}^2 = \vec{a}\cdot\vec{a}]$

$$= \vec{a}^2 + 2\vec{a}\cdot\vec{b} + \vec{b}^2$$

$$= |\vec{a}|^2 + 2\vec{a}\cdot\vec{b} + |\vec{b}|^2 \quad -(**)$$

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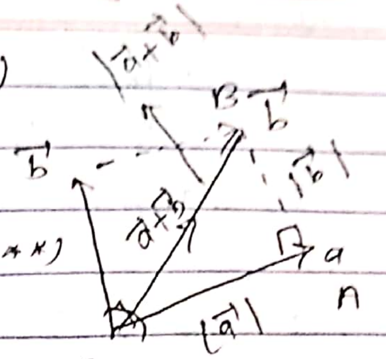
Fact :- If $\vec{a}\cdot\vec{b} = 0$ then by (* *)

$|\vec{a}+\vec{b}|^2 = |\vec{a}|^2 + |\vec{b}|^2$

$\boxed{\text{In } \Delta OAB \ (OA)^2 + (AB)^2 = (OB)^2}$ — (* * *)



This is Femous pythagorian theorem.

Parallelogram Law

If $\vec{a}$ and $\vec{b}$ be two vectors

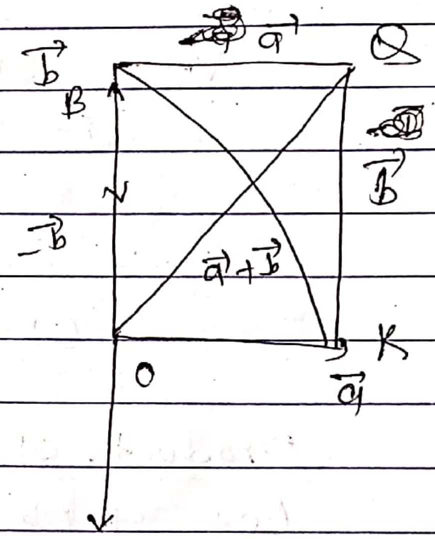
$|\vec{a}+\vec{b}|^2 + |\vec{a}-\vec{b}|^2$

$= (\vec{a}+\vec{b})\cdot(\vec{a}+\vec{b}) + (\vec{a}-\vec{b})\cdot(\vec{a}-\vec{b})$

$= |\vec{a}|^2 + \cancel{2\vec{a}\vec{b}} + |\vec{b}|^2 + |\vec{a}|^2 \cancel{- 2\vec{a}\vec{b}} + |\vec{b}|^2$

$= 2|\vec{a}|^2 + 2|\vec{b}|^2$

$= 2(|\vec{a}|^2 + |\vec{b}|^2)$



$\vec{RB} = \vec{BO} + \vec{OR}$

$= -\vec{b} + \vec{a}$

$= \vec{a} - \vec{b}$

Here $|\vec{a}+\vec{b}| = \vec{OQ}$

$|\vec{a}-\vec{b}| = BR$

2 times Sum of the square of the side is equals to sum of the diagonal of parallelogram.

Fact :- If $\vec{a}$ and $\vec{b}$ be two vectors then the difference of $\vec{b}$ and its vector projection onto $\vec{a}$ is orthogonal to $\vec{a}$.

Soln The vector $\vec{b} - \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|}\cdot\frac{\vec{a}}{|\vec{a}|}$ is orthogonal to $\vec{a}$.

Now,

$$\vec{a}\cdot\left[\vec{b}-\frac{\vec{a}\cdot\vec{b}}{|\vec{a}|}\cdot\frac{\vec{a}}{|\vec{a}|}\right]$$

$$\Rightarrow \vec{a}\cdot\vec{b}-\frac{\vec{a}\cdot\vec{b}\,(\vec{a}\cdot\vec{a})}{|\vec{a}|^2}$$

$$\Rightarrow \vec{a}\cdot\vec{b}-\frac{a^2}{a^2}[\vec{a}\cdot\vec{b}]$$

$$\Rightarrow \vec{a}\cdot\vec{b}-\vec{a}\cdot\vec{b}=0$$

Example Let $\vec{a}=(1,0,0)$, $\vec{b}=(0,1,0)$.

Find the vector orthogonal to $\vec{b}$.

Here $\vec{a}=(1,0,0)$, $\vec{b}=(0,1,0)$

$\vec{a}\cdot\vec{b}=(1,0,0)\cdot(0,1,0)=(0,0,0)$

Hence $|\vec{a}|=\sqrt{1^2+0+0}=1$

$|\vec{b}|=\sqrt{0^2+1^2+0}=1$

Vector orthogonal to $\vec{b}$ is,

$$\vec{a}-\frac{\vec{a}\cdot\vec{b}}{|\vec{b}|}\cdot\frac{\vec{b}}{|\vec{b}|}$$

$$=(1,0,0)-\frac{(0,0,0)\cdot(0,1,0)}{1}$$

$$=(1,0,0)$$

Product of three vectors

Let $\vec{a},\vec{b},\vec{c}$ be three vectors, then,

(i) $\vec{a}\cdot(\vec{b}\times\vec{c})\rightarrow$ Scalars

(ii) $\vec{a}\times(\vec{b}\times\vec{c})\rightarrow$ vectors.

But $\vec{a}(\vec{a}\times\vec{c})$, $\vec{a}\times(\vec{b}-\vec{c})$ are undefined.

Let $\vec{a},\vec{b},\vec{c}$ be three vectors, Then their Scalar triple product is denoted by, $[\vec{a}\ \vec{b}\ \vec{c}]$ or $(\vec{a},\vec{b},\vec{c})$ and is defined by

$[\vec{a}\ \vec{b}\ \vec{c}] = \vec{a}\cdot(\vec{b}\times\vec{c})$

Fact:- unit vectors $\hat{i}, \hat{j}, \hat{k}$

$= \hat{i}\cdot(\hat{j}\times\hat{k})$

$= \hat{i}\cdot\hat{i} = 1$

Evaluate $\vec{a} = (a_1, a_2, a_3)$, $\vec{b} = (b_1, b_2, b_3)$ &

$\vec{c} = (c_1, c_2, c_3)$

Then $[\vec{a}\ \vec{b}\ \vec{c}] = \vec{a}\cdot(\vec{b}\times\vec{c})$

$$\vec{b}\times\vec{c} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$$= (b_2c_3 - c_2b_3)\vec{i} + (c_1b_3 - b_1c_3)\vec{j} + (b_1c_2 - b_2c_1)\vec{k}$$

$$\vec{a}\cdot(\vec{b}\times\vec{c}) = a_1(b_2c_3 - c_2b_3) + a_2(c_1b_3 - b_1c_3) + a_3(b_1c_2 - b_2c_1)$$

$$= \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

This shows that Scalar triple product can be expressed in terms of determinant.

Properties:-

(1) The value of STP changes its sign if the position of two vectors are interchange.

Geometrical meaning of Scalar Triple product

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Geometrically the value of scalar triple product $[\vec{a}, \vec{b}, \vec{c}]$ always represent the volume of parallelopiped with edge $\vec{a}, \vec{b}, \vec{c}$.

Fact:- Find the volume of the parallelopiped with edge having 3 vectors.
$(2, -3, 4)$, $(1, 2, -1)$ & $(3, -1, 2)$

Let $\vec{a} = (2, -3, 4)$, $\vec{b} = (1, 2, -1)$ and $\vec{c} = (3, -1, 2)$

Then,
$$\vec{a} \cdot (\vec{b} \times \vec{c}) = \begin{vmatrix} 2 & -3 & 4 \\ 1 & 2 & -1 \\ 3 & -1 & 2 \end{vmatrix} = |-7| = 7$$

we take positive value so value is 7.

Note:- If three vectors are coplanar then Volume of ppd $= 0$
i.e $[\vec{a}\ \vec{b}\ \vec{c}] = 0$.

Example Find the value of d such that three vectors
$$2\vec{i} - \vec{j} + \vec{k},\quad \vec{i} + 2\vec{j} - 3\vec{k} \ \ \&\ \ 3\vec{i} + d\vec{j} + 5\vec{k}.$$
are coplaner.

Soln

Here, $\vec{a} = 2\vec{i} - \vec{j} + \vec{k}$
$\vec{b} = \vec{i} + 2\vec{j} - 3\vec{k}$
$\vec{c} = 3\vec{i} + d\vec{j} + 5\vec{k}$

$$\vec{a} \cdot (\vec{b} \times \vec{c}) = \begin{vmatrix} 2 & -1 & 1 \\ 1 & 2 & -3 \\ 3 & d & 5 \end{vmatrix} = 0$$

$$\Rightarrow 2\begin{vmatrix} 2 & -3 \\ d & 5 \end{vmatrix} + 1\begin{vmatrix} 1 & -3 \\ 3 & 5 \end{vmatrix} + 1\begin{vmatrix} 1 & 2 \\ 3 & d \end{vmatrix} = 0$$

$\Rightarrow 2(10+3d) + (5+9) + (5-d) = 0$

$\Rightarrow 20 + 6d + 14 + d - 6 = 0$

$\Rightarrow 7d = -28$

$d = -4$

Vector triple product

Let $\vec{a}, \vec{b}, \vec{c}$ be three vectors, then their vector triple product is,

$\vec{a} \times (\vec{b} \times \vec{c})$

$$\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a}\cdot\vec{c})\vec{b} - (\vec{a}\cdot\vec{b})\vec{c}$$

This show that vector triple product can be expressed as linear combination of $\vec{b}$ and $\vec{c}$.

Example: Let $\vec{a} = (1, 2, 0)$

$\vec{b} = (0, 7, 2)$

$\vec{c} = (1, 1, 1)$

Find $\vec{a} \times \vec{b} \times \vec{c}$ & verify the formula.

Soln: Here $\vec{a} = (1, 2, 0)$

$\vec{b} = (0, 7, 2)$

$\vec{c} = (1, 1, 1)$

$$\vec{b} \times \vec{c} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 7 & 2 \\ 1 & 1 & 1 \end{vmatrix} = (7-2)\vec{i} + (2-0)\vec{j} + (-7)\vec{k}$$

$= 5\vec{i} + 2\vec{j} - 7\vec{k}$.

again,

$$\vec{a} \times (\vec{b} \times \vec{c}) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2 & 0 \\ 5 & 2 & -7 \end{vmatrix} = (-14)\vec{i} + 7\vec{j} + (2-10)\vec{k}$$

$= -14\vec{i} + 7\vec{j} - 8\vec{k}$ — (1)

Also,

$(\vec{a}\cdot\vec{c}) = (1,2,0)\cdot(1,1,1)$
$= 1+2+0 = 3$

$(\vec{a}\cdot\vec{c})\vec{b} = 3(0,7,2)$
$= (0,21,6)$

$(\vec{a}\cdot\vec{b}) = (1,2,0)\cdot(0,7,2) = 14$

$(\vec{a}\cdot\vec{b})\vec{c} = 14(1,1,1)$
$= (14,14,14)$

Now, $(\vec{a}\cdot\vec{c})\vec{b} - (\vec{a}\cdot\vec{b})\vec{c}$
$= (0,21,6) - (14,14,14)$
$= (-14,7,-8)$
$= -14\vec{i} + 7\vec{j} - 8\vec{k}$

Hence, $\vec{a}\times(\vec{b}\times\vec{c}) = (\vec{a}\cdot\vec{c})\vec{b} - (\vec{a}\cdot\vec{b})\vec{c}$ proved

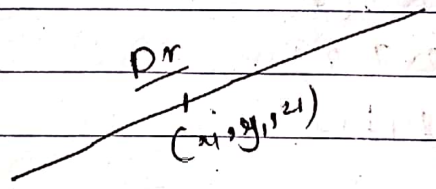
we had :- $\vec{a}\times(\vec{b}\times\vec{c}) = (\vec{a}\cdot\vec{c})\vec{b} - (\vec{a}\cdot\vec{b})\cdot\vec{c}$

Eqn of Stright line in space

$y - y_1 = \frac{y_2-y_1}{x_2-x_1}(x-x_1)$

$\frac{x-x_1}{x_2-x_1} = \frac{y-y_1}{y_2-y_1}$

$\cos\alpha = \cos(\pi/2 - \theta)$



Any three numbers that are proportional to the direction Cosine l, m, n are called direction ratio.

Thus, if a, b, c be three numbers satisfied,

$$\frac{l}{a} = \frac{m}{b} = \frac{n}{c} = \frac{\sqrt{l^2+m^2+n^2}}{\sqrt{a^2+b^2+c^2}} = \frac{1}{\sqrt{a^2+b^2+c^2}}$$

Then,

$$l = \frac{a}{\sqrt{a^2+b^2+c^2}}, \quad m = \frac{b}{\sqrt{a^2+b^2+c^2}}, \quad n = \frac{c}{\sqrt{a^2+b^2+c^2}}$$

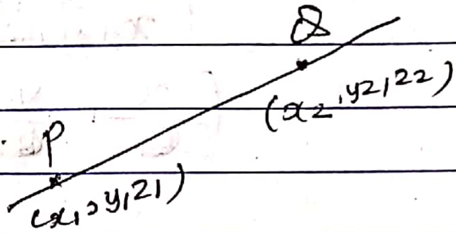
~~Proofs~~,

Formula 1

(i) The equation of straight line passing through (x_1, y_1, z_1) with direction cosine l, m, n is,

$$\frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n}$$

(ii) Two point form.

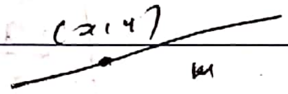


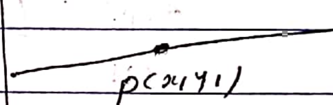
$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1}$$

~~$a(x - x_1) +$~~

b

where $x_2 - x_1$, $y_2 - y_1$, $z_2 - z_1$ are called direction ratio of PQ.



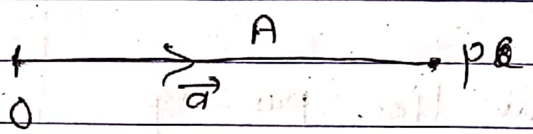


$y - y_1 = m(x - x_1)$

point (x, y) is moving point

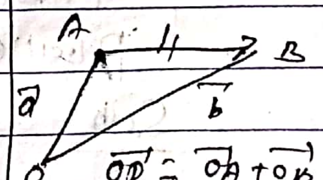
Vector equation of a line :-

Vector equation of a line passing through origin and with direction of $\vec{a}$.



Let $\vec{OA} = \vec{a}$ passing through origin.

Let, $\vec{op}$ be any point on the line such that $\vec{op} = \vec{p}$.



$\vec{op} = \vec{oA} + \vec{oB}$

Then $\overrightarrow{OA}$ & $\overrightarrow{OP}$ are Collinear.

~~or~~ $\overrightarrow{OP} = t\,\overrightarrow{OA}$

$\therefore$ $\vec{r} = t\vec{a}$, which is vector equation of Straight line passing through origin.

Verification

Changing into Cartesian form,

$\vec{r} = (x, y, z)$, $\vec{a} = (a_1, a_2, a_3)$, Then eqn (x) gives,

$$(\vec{x}, \vec{y}, \vec{z}) = t(\vec{a_1}, \vec{a_2}, \vec{a_3})$$

$$\left(\frac{x}{a_1}, \frac{y}{a_2}, \frac{z}{a_3}\right) = t\vec{a_1}, t\vec{a_2}, t\vec{a_3}$$

=> Equating the Corresponding Component,

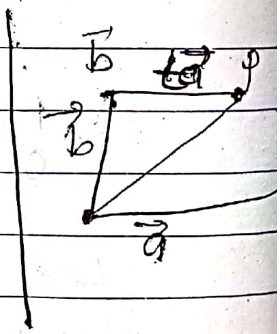
$$x = ta_1, \quad y = ta_2, \quad z = ta_3$$

$$\boxed{\frac{x}{a_1} = \frac{y}{a_2} = \frac{z}{a_3} = t} \quad \text{— (i)}$$

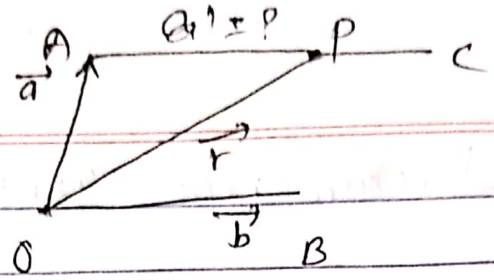
eqn (i) is in Cartesian form. This is the eqn of Straight line with direction ratio a_1, a_2, a_3.

Example

~~F~~ vector eqn of Straight line passing through a point A (where $\overrightarrow{OA} = \vec{a}$) & parallel to vector $\vec{b}$.



Soln



Let $\vec{OA} = \vec{a}$, and $\vec{OB} = \vec{b}$ be the given vectors.
Let p be any point on the required line AC,
So that $\vec{OP} = \vec{r}$.
From traingle low of vector addition.

$$\vec{OP} = \vec{OA} + \vec{AP}$$

or $\vec{r} = \vec{a} + \vec{AP}$ Since $\vec{AP}$ is parallel to $\vec{OB} = \vec{b}$

$\therefore \vec{AP} = t\vec{b}$

or $\boxed{\vec{r} = \vec{a} + t\vec{b}.}$ — (ii)

which is equation of Stright line passing from $\vec{a}$ and parallel to $\vec{b}$.

Varification from Cartision Coordinates,

$\vec{r} = (x, y, z)$, $\vec{a} = (x_1, y_1, z_1)$ and $\vec{b} = (x_2, y_2, z_2)$

$\Rightarrow (x, y, z) = (x_1, y_1, z_1) + t(x_2, y_2, z_2)$

$\Rightarrow (x, y, z) = (x_1, y_1, z_1) + (tx_2, ty_2, tz_2)$

$\Rightarrow (x - x_1, y - y_1, z - z_1) = (tx_2, ty_2, tz_2)$ — (iii)

$$\Rightarrow \boxed{\frac{x - x_1}{x_2} = \frac{y - y_1}{y_2} = \frac{z - z_1}{z_2} = t} \text{ — (iv)}$$

Eqn (iii) is Called parametric equation
$\therefore$ Eq (iv) is eqn of Stright line passing through

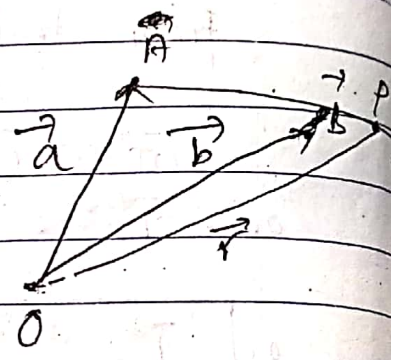
(x_1, y_1, z_1) with direction ratio x_2, y_2, z_2.

~~Vect~~

Example 2

Vector eqn of Straight lines passing through ~~2~~ two vectors $\vec{A}$ & $\vec{B}$ with points $\vec{A}$ and $\vec{B}$.

Soln :- Let $\vec{OA} = \vec{a}$, $\vec{OB} = \vec{b}$ be given vectors, let P be any point on the required line AC, So that $\vec{OP} = \vec{r}$



~~From triangle~~

$$\vec{OP} = \vec{OA} + \vec{AB} \quad — (1)$$

$$\vec{AB} = \vec{b} - \vec{a}$$

From traingle low vector addition,

From (1)

$$\cancel{\vec{OP} = \vec{OA} + (\vec{OB} - \vec{OA})}$$

$$\vec{OP} = \vec{OA} + \vec{AP}$$

$$\vec{r} = \vec{OA} + t(\vec{AB})$$

Where AP is Collinear to $\vec{AB}$

$$\vec{r} = \vec{OA} + t(\vec{b} - \vec{a})$$

$$\vec{r} = \vec{a} + t(\vec{b} - \vec{a})$$

which is equation of straight line passing through two points A & B with position vectors $\vec{a}$ and $\vec{b}$.

Verification

Let $\vec{r} = (x, y, z)$, $\vec{a} = (x_1, y_1, z_1)$, $\vec{b} = (x_2, y_2, z_2)$

$(x, y, z) = (x_1, y_1, z_1) + t(x_2 - x_1, y_2 - y_1, z_2 - z_1)$

$$\Rightarrow \boxed{\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1} = t} \quad \text{--- } (*)$$

Question :-

Find the eqn of line which is parallel to $2\vec{i} - \vec{j} + 3\vec{k}$ and passes through $(5, -2, 4)$.

Soln Here, Direction ratio are $x_1 = 5$, $x_2 = -2$, $x_3 = 4$

also, it can be written as,

$\vec{a} = (5, -2, 4) = 5\vec{i} - 2\vec{j} + 4\vec{k}$

$\vec{b} = 2\vec{i} - \vec{j} + 3\vec{k}$

$\therefore$ Vector equation of straight line parallel to $\vec{b}$ and passes through $(5, -2, 4)$

$$\vec{r} = \vec{a} + t\vec{b} \quad \text{--- (i)}$$

$(x, y, z) = (5, -2, 4) + t(2, -1, 3)$

$\Rightarrow (x - 5, y + 2, z - 4) = (2t, -t, 3t)$

$$\Rightarrow \frac{x - 5}{2} = \frac{y + 2}{-1} = \frac{z - 4}{3} = t$$

This is true for all t $\therefore$

$$\frac{x - 5}{2} = \frac{y + 2}{-1} = \frac{z - 4}{3}$$

Example :- Find the vector equation of straight line whose Cartesian eqn is

$$\frac{x-1}{2} = \frac{y+1}{-3} = \frac{z+2}{4}$$

Let $\frac{x-1}{2} = \frac{y+1}{-3} = \frac{z+2}{4} = t$ (say)

$x-1 = 2t \qquad y+1 = -3t \qquad z+2 = 4t$

$x = 2t+1 \qquad y = -3t-1 \qquad z = 4t-2$

Let $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$ be the vector eqn.

or $\vec{r} = (2t+1)\vec{i} + (-3t-1)\vec{j} + (4t-2)\vec{k}$

or $\vec{r} = (2t\vec{i} - 3t\vec{j} + 4t\vec{k}) + \vec{i} - \vec{j} - 2\vec{k}$

$\vec{r} = \vec{i} - \vec{j} - 2\vec{k} + t(2\vec{i} - 3\vec{j} + 4\vec{k})$

$= a + tb$

Where,

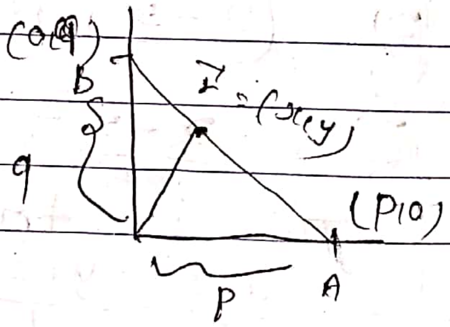
$\vec{a} = (\vec{i} - \vec{j} - 2\vec{k})$

$\vec{B} = (2\vec{i} - 3\vec{j} + 4\vec{k})$

Example Find the vector or use vector method to find the equation of straight line in the form $\frac{x}{p} + \frac{y}{p} = 1$ (i.e. Double Intercept Form.)

Soln we have equation of straight line in two point methods,

$\vec{r} = \vec{a} + t(\vec{B} - \vec{a})$



$\Rightarrow \quad x\vec{i} + y\vec{j} + z\vec{k} = p\vec{i} + 0\vec{j} + 0\vec{k} + t(0\vec{i} - q\vec{j} + z\vec{k} - (p\vec{i} + 0\vec{j} + z\vec{k}))$

$\Rightarrow \quad x\vec{i} + y\vec{j} + z\vec{k} = p\vec{i} + 0\vec{j} + 0\vec{k} + t(-p\vec{i} - q\vec{j})$

$\Rightarrow \quad (x, y) = (p - tp, tq)$

$\Rightarrow \quad p - tp = x$

$\quad\quad y = tq$

$\Rightarrow \quad -tp = x - p$

$\Rightarrow \quad t = \dfrac{x-p}{-p}, \quad t = \dfrac{y}{q}$

$\Rightarrow \quad \dfrac{y}{q} = \dfrac{x-p}{-p}$

$\Rightarrow \quad \dfrac{x-p}{-p} = \dfrac{y}{q}$

$\Rightarrow \quad \boxed{\dfrac{x}{p} + \dfrac{y}{q} = 1}$

Another

~~PA = OA~~

$AP = OP - OA$

$= (x - p, y)$

$PB = OB - OP$

$= (x, y - q)$

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Page

$AP = t(PB)$

$\Rightarrow (x-p, y) = t(x, y-q)$

$\Rightarrow (x-p, y) = (tx, t(y-q))$

Equating,

$x-p = tx \Rightarrow t = \frac{x-p}{x}$

$t(y-q) = y$

$t = \frac{y}{y-q}$

$\frac{x-p}{x} = \frac{y}{y-q}$

Or, $1 - \frac{p}{x} = \frac{y}{y-q}$

$\Rightarrow \frac{p}{x} = -\frac{y}{y-q} - 1$

$\Rightarrow \frac{p}{x} = \frac{-y-y+q}{y-q}$

$\Rightarrow \frac{p}{x} = \frac{-2y+q}{y-q}$

$\Rightarrow p(y-q) = x(-2y+q)$

$\Rightarrow yp - pq = -2xy + xq$

$\Rightarrow \frac{y}{q} - 1 = -\frac{2xy}{pq} + \frac{x}{p}$

$\Rightarrow \frac{x}{p} - \frac{y}{q} = -\frac{2xy}{pq} +$

$xy = (x-p)(y-q)$

$\Rightarrow \cancel{xy} = \cancel{xy} - xq - yq + pq$

$\Rightarrow xq + yq = pq$

$\Rightarrow \frac{x}{p} + \frac{y}{q} = 1$

Equation of plane

A plane is the locus of the ~~plane~~ points Such that any two points P, Q on the locus, the straight line joining P & Q i.e. PQ lies wholly on the locus.

* note the general equation of a plane passes through origin is,

$$ax+by+cz=0$$

This can be written as,

$$(a,b,c)\cdot(x,y,z)=0 \quad -(*)$$

This shows that vector $(a,b,c)=\vec{n}$ is perpendicular to $\vec{r}=(x,y,z)$.

This show that from $(*)$

$$\vec{n}\cdot\vec{r}=0$$

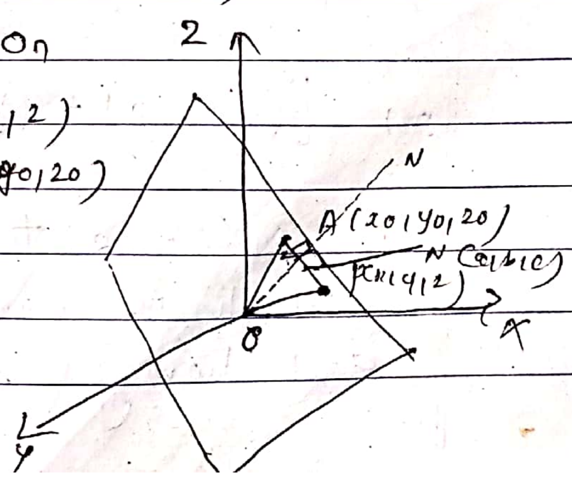
Thus, a plane is completely determine by a point $\vec{r}=(x,y,z)$ on plane normal vector or $\vec{n}=(a,b,c)$ which is normal to the plane.

Determination of equation of plane

Find the eqn of plane which passes through $A(x_0,y_0,z_0)$ and having normal vector $\vec{n}=(a,b,c)$.

Let $P(x,y,z)$ be any points on the plane where $\vec{r}=\vec{OP}=(x,y,z)$ given point on the plane $A(x_0,y_0,z_0)$ i.e $\vec{a}=\vec{OA}=(x_0,y_0,z_0)$.

Then, the line $\vec{AP}$ lies on the plane.



$\therefore \overrightarrow{AP} = \overrightarrow{OP} - \overrightarrow{OA} = \vec{r} - \vec{a}$

(x, y, z)

But PN is orthogonal to $\overrightarrow{AP}$

~~where~~ $\cancel{\vec{n} = (a, b, c)}$ i.e $\overrightarrow{AP} \cdot \overrightarrow{PN} = 0$

where $\overrightarrow{ON} = \vec{n} = (a, b, c)$

Or $(\vec{r} - \vec{a}) \cdot \vec{n} = 0$ — $(* \alpha)$

Note

If we shift PN to ZN then direction ratio of both line are same

Eqn $(* *)$ is the equation of plane which passes through $\vec{a} = \overrightarrow{OA} = (x_0, y_0, z_0)$.

If the plane passes through origin, then,

$A(x_0, y_0, z_0) \to O(0, 0, 0)$

i.e $\vec{a} = 0$

Hence eqn $(* *)$ becomes,

$\vec{r} \cdot \vec{n} = 0 \Rightarrow \vec{n} \cdot \vec{r} = 0$

When, $\vec{n} = (a, b, c)$, $\vec{r} = (x, y, z)$ and $\vec{a} = (x_0, y_0, z_0)$

Then from $(* *)$

$\Rightarrow [(x, y, z) - (x_0, y_0, z_0)] \cdot (a, b, c) = 0$

$\Rightarrow [(x - x_0) + (y - y_0) + (z - z_0)](a \cdot b \cdot c) = 0$

$\Rightarrow a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$ — $(* * \alpha)$

which is equation of plane passes through (x_0, y_0, z_0) and having normal vector (a, b, c).

$\vec{r}\cdot\vec{n} = \vec{a}\cdot\vec{n}$

$a\vec{x} + b\vec{y} + c\vec{z} = ax_0 + by_0 + cz_0$ — (x x x x)

In the eqⁿ (x x x x) If $ax_0 + by_0 + cz_0 = d$

Then eqⁿ (x x x x) becomes,

$$\boxed{ax + by + cz = d}$$

If the plane passes through origin

i.e $(x_0, y_0, z_0) = (0, 0, 0)$

Then eqⁿ becomes,

$$\boxed{ax + by + cz = 0}$$

<u>Example</u> Find the equation of plane which passes through $A(-1, 2, 3)$ with normal vector $\vec{n} = (4, 1, 2)$

<u>Soln</u> we have,

$(\vec{r} - \vec{a})\cdot\vec{n} = 0$

where $\vec{r} = (x, y, z)$, $\vec{a} = (-1, 2, 3)$, $\vec{n} = (4, 1, 2)$

$\Rightarrow [(x, y, z) - (-1, 2, 3)]\cdot(4, 1, 2) = 0$

$\Rightarrow (x+1) + (y-2) + (z-3)\cdot(-1, 2, 3) = 0$

$\Rightarrow -1(x+1) + 2(y-2) + 3(z-3) = 0$

$\Rightarrow -x + 2y + 3z - 1 - 4 - 9 = 0$

$\Rightarrow -x + 2y + 3z = 14$ — (i)

Eq (i) is required equation of plane.

Let $\vec{ON} = \vec{a} = (-1, 2, 3)$

Normal vector $\vec{n} = (4, 1, 2)$

Then the vector equation of plane is,

$$(\vec{r} - \vec{a}) \cdot \vec{n} = 0$$

$\Rightarrow \vec{r} \cdot \vec{n} = \vec{a} \cdot \vec{n}$

$\Rightarrow (x, y, z) \cdot (4, 1, 2) = (-1, 2, 3) \cdot (4, 1, 2)$

Example Find the intercept of the plane $4x + y + 2z = 4$

& Sketch the graph.

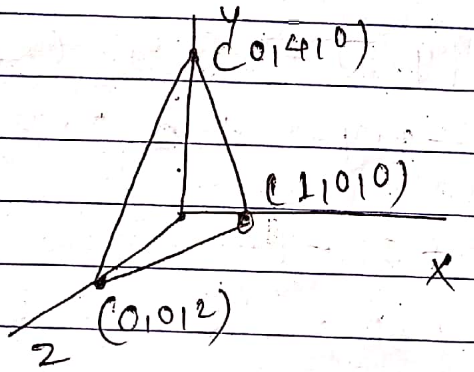
Soln put $y = 0,\ z = 0$, then $x = 1$

put $y = 0,\ x = 0$ then $z = 2$

put $x = 0,\ z = 0$ then $y = 4$.

$\therefore$ ~~The plane cut~~

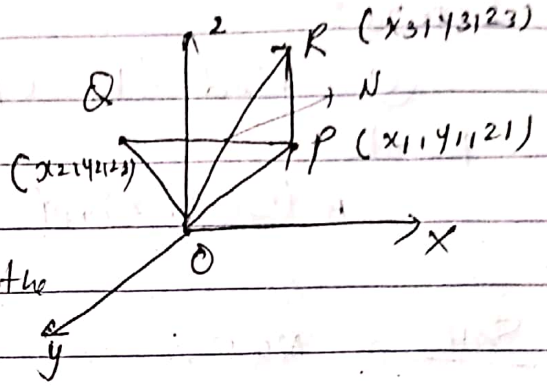
x, y and z intercept are $1, 2$ & 2.



Plane through three points

Find the equation of plane through three points, $P(x_0, y_0, z_0)$, $Q(x_2, y_2, z_2)$ & $R(x_3, y_3, z_3)$.

Hint: Find $\vec{PR}$ also find $\vec{PQ}$.



Step I Find $\vec{PR}$ & $\vec{PQ}$

Step II Find normal vector to the plane.

$\vec{n} = \vec{PQ} \times \vec{PR}$

PA is perpendicular to N

So, $\vec{PQ} \cdot \vec{N} = 0$

$(\vec{r} - \vec{a}) \cdot [(\vec{b} - \vec{a}) \times (\vec{c} - \vec{a})] = 0$

Here $P = (x_0, y_0, z_0)$, $Q = (x_1, y_1, z_1)$ & $R = (x_2, y_2, z_2)$

Let (x, y, z) be the coordinate of point A on the plane whose position vector $\vec{r}$.

Then,

$\vec{PA} = (x - x_0)\vec{i} + (y - y_0)\vec{j} + (z - z_0)\vec{k}$

$\vec{PQ} = (x_1 - x_0)\vec{i} + (y_1 - y_0)\vec{j} + (z_1 - z_0)\vec{k}$

$\vec{PR} = (x_2 - x_0)\vec{i} + (y_2 - y_0)\vec{j} + (z_2 - z_0)\vec{k}$

Now requared eqn of plane is obtained through

$$\begin{vmatrix} (x - x_0) & (y - y_0) & (z - z_0) \\ (x_1 - x_0) & (y_1 - y_0) & (z_1 - z_0) \\ (x_2 - x_0) & (y_2 - y_0) & (z_2 - z_0) \end{vmatrix} = 0 \quad \text{--- (i)}$$

Plane (I) passes through the point (x_0, y_0, z_0), (x_1, y_1, z_1) and (x_2, y_2, z_2)

Example :-

(V) Find the equation of plane through three points

A $(3, -1, 2)$, B $(8, 2, 4)$ & C $(-1, -2, -3)$

Soln Now,

$$\vec{AB} = \vec{OB} - \vec{OA}$$
$$= (8, 2, 4) - (3, -1, 2)$$
$$= (5, 3, 2)$$

$$\vec{AC} = \vec{OC} - \vec{OA}$$
$$= (-1, -2, -3) - (3, -1, 2)$$
$$= (-4, -1, -5)$$

Now $\vec{AB} \times \vec{AC} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 5 & 3 & 2 \\ -4 & -1 & -5 \end{vmatrix}$

$$= \vec{i}(-15+2) + \cancel{(\ldots)}\ (-8+25)\vec{j} + (-5+12)\vec{k}$$

$$= -13\vec{i} + 17\vec{j} + 7\vec{k}$$

$\therefore$ Equations of plane passes through $(3, -1, 2)$ with normal vector $\vec{n} = (-13, 17, 7)$ is,

$a(x-x_1)+b(y-y_1)+c(z-z_1)=0$

$\Rightarrow -13(x-3)+17(y+1)+7(z-2)=0$

$\Rightarrow -13x+13\times3+17y+17+7z-14=0$

$\Rightarrow -13x+17y+7z=-13\times3-17+14$ — (*)

Point of intersection of a line & plane

Example: Find the point of intersection of a line

$$\frac{x-2}{3}=\frac{y-0}{-4}=\frac{z-5}{1}$$

with a plane $4x+5y-2z=18$

Soln:-

Let $\frac{x-2}{3}=\frac{y-0}{-4}=\frac{z-5}{1}=t$ say

$x=3t+2,\ y=-4t,\ z=t+5$ — (*)

The plane $4x+5y-2z=18$ —— (* *)

If the line (*) intersect a plane (* *) then, (* *) must be satisfy by the value of x,y,z in (*)

$\therefore 4(3t+2)+5(-4t)-2(t+5)=18$

$\Rightarrow 12t+8+-20t-2t-10=18$

$\Rightarrow$ ~~20t~~ $-8t-2t=18+10-8$

$\Rightarrow t=-2$

Substituting $t = -2$ on (*)
required point is $x = -4, y = 8, z = 3$
$\therefore (x, y, z) = (-4, 8, 3)$

Angle between two plane :-

Angle between two planes is defined as the angle between their normal vector.

Formula : The angle between two planes

$P_1: a_1x + b_1y + c_1z = d_1 \quad \text{--- (I)}$
$P_2: a_2x + b_2y + c_2z = d_2 \quad \text{--- (II)}$

The normal vector to the plane P_1
$\vec{n_1} = (a_1, b_1, c_1)$

normal vector to the plane P_2,
$\vec{n_2} = (a_2, b_2, c_2)$

$\therefore$ Angle between them

$$\cos\theta = \frac{\vec{n_1} \cdot \vec{n_2}}{|\vec{n_1}| |\vec{n_2}|}$$

Example

Find the angle between two planes
$x + y + z = 1, \quad x - 2y + 3z = 1$
--- (*) --- (**)

The normal vector of plane (*) and (**) are respectively, $\vec{n_1} = (1, 1, 1)$, $\vec{n_2} = (1, -2, 3)$

∴ Angle between them is

$$\cos\theta = \frac{\vec{n_1}\cdot\vec{n_2}}{|\vec{n_1}|\,|\vec{n_2}|}$$

$$|\vec{n_1}| = \sqrt{1^2+1^2+1^2} = \sqrt{3}$$
$$|\vec{n_2}| = \sqrt{1+4+9} = \sqrt{14}$$

$$\cos\theta = \frac{(1,1,1)\cdot(1,-2,3)}{\sqrt{3}\cdot\sqrt{14}}$$

$$= \frac{1+(-2)+3}{\sqrt{42}} = \frac{2}{\sqrt{42}}$$

$$\therefore \theta = \cos^{-1}\left(\frac{2}{\sqrt{42}}\right)$$

$\boxed{\text{Equation of St. line through the intersection of two planes}}$

Let $P_1 : a_1x + b_1y + c_1z = d_1$ —(i)

$P_2 : a_2x + b_2y + c_2z = d_2$ —(ii)

be two planes.

Step I Find point on (i) put $x=0$, and ~~y=0~~

then we get $z = \frac{d_1}{c_1}$ on (i) and (ii), $\left.\begin{matrix} b_1y + c_1z = d_1 \\ b_2y + c_2z = d_2 \end{matrix}\right\}$ solve

∴ Required point $(x_1, y_1, z_1) = \left(0, 0, \frac{d_1}{c_1}\right)$

The required line L_1 lies on both planes,

So the line L is $\perp^r$ to both the planes having normal vector $\vec{n_1} = (a_1, b_1, c_1)$, $\vec{n_2} = (a_2, b_2, c_2)$

$\therefore$ required D.r $= \vec{n_1} \times \vec{n_2}$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}$$

$$= (a\vec{i} + b\vec{j} + c\vec{k}) \text{ say}$$

$$= (a, b, c)$$

$$\frac{(x-x_1)}{a} = \frac{y-y_1}{b} = \frac{z-z_1}{c}$$

Example Find the Symmetrical eqn of line through the intersection of two planes

$2x + y + z = 3$

$x + y + 3z = 6$

Soln

Here, given two planes are,

$2x + y + z = 3$ — (i)

$x + y + 3z = 6$ — (ii)

Let $(0, 0, 3)$ be the point in plane (i)

Also, normal vectors of two planes are,

$\vec{n_1} = (2, 1, 1)$, $\vec{n_2} = (1, 1, 3)$

required direction ratio is,

$$= \vec{n_1} \times \vec{n_2} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 1 & 1 \\ 1 & 1 & 3 \end{vmatrix}$$

$$= (3-1)\vec{i} + (1-6)\vec{j} + (2-1)\vec{k}$$

$$= 2\vec{i} - 5\vec{j} + 1\vec{k} = (2, -5, 1)$$

Hence required line is,

$$\frac{x-0}{2} = \frac{y}{-5} = \frac{z-3}{1}$$

$$\Rightarrow \frac{x}{2} = \frac{y}{-5} = \frac{z-3}{1} = t$$

$x = 2t - 3,\ y = 5t + 9,\ z = t$

$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$= (2t - 3)\hat{i} + (-5t + 9)\hat{j} + t\vec{k}$

$= (-3\vec{i} + 9\vec{j}) + t(2\hat{i} - 5\hat{j} + 1k)$

$= \vec{a} + t\vec{b}$

Alternatively, put $x = 0$,

$y + z = 3$
$y + 3z = 6$
$-2z = -3$

$z = 3/2$

$y = 0$

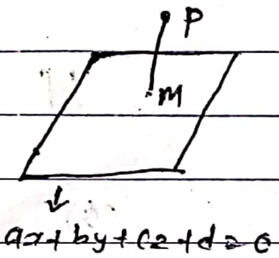
$\left(0, 0, \frac{3}{2}\right)$

$$\Rightarrow \frac{x-0}{2} = \frac{y}{-5} = \frac{z-\frac{3}{2}}{1}$$

Formula

Length of $\perp^r$ from $P(x_1, y_1, z_1)$ to a plane $ax + by + cz + d = 0$

$$PM = \left|\frac{ax_1 + by_1 + cz_1 + d}{\sqrt{a^2 + b^2 + c^2}}\right|$$



Example

Find the distance between two parallel planes,

$$x+y+z=9 \quad -(I)$$

$$2x+2y+2z=10 \quad -(II)$$

~~y+z=9~~

~~2y+2z=10~~

Soln

Put, $x=0, y=0$ then, $z=9$. $(x_1, y_1, z_1) = (0,0,9)$

also, normal vector to the plane (II) is,

$(2,2,2)$

Hence distance between plane (I) and (II) is,

$$d = \frac{ax_1+by_1+cz_1+d}{\sqrt{a^2+b^2+c^2}}$$

$$= \frac{18-10}{\sqrt{4+4+4}}$$

$$= \frac{8}{\sqrt{12}} = \frac{8}{2\sqrt{3}} = \frac{4}{\sqrt{3}}$$

Fact:- If two lines are neither parallel nor intersect, then they are called skewed line.

Set IV Determine whether two lines L_1 & L_2 are parallel, skewed or intersect.

If intersecting find common point.

(a) $\dfrac{x-3}{2} = \dfrac{y-4}{-1} = \dfrac{z-1}{3} \quad -(I)$

& $\dfrac{x-1}{4} = \dfrac{y-3}{-2} = \dfrac{z-4}{5} \quad -(II)$

Soln :- Here given eqn are,

$$\frac{x-3}{2} = \frac{y-4}{-1} = \frac{z-1}{3} = t \quad (\text{say})$$

$$\frac{x-1}{4} = \frac{y-3}{-2} = \frac{z-4}{5} = s \quad (\text{say})$$

then any point on first,

$$x = 2t+3, \quad y = -t+4, \quad z = 3t+1$$

$$(x, y, z) = (2t+3, -t+4, 3t+1)$$

Also, Any point on second,

$$x = 4s+1, \quad y = -2s+3, \quad z = 5s+4$$

$$(x, y, z) = (4s+1, -2s+3, 5s+4)$$

If they meet each other then for certain t & s they are equal.

$$\therefore \quad 2t+3 = 4s+1 \Rightarrow 2t - 4s = -2 \quad \text{— (iii)}$$

$$-t+4 = -2s+3 \Rightarrow -t + 2s = -1 \quad \text{— (iv)}$$

$$3t+1 = 5s+4 \Rightarrow 3t - 5s = 3 \quad \text{— (v)}$$

From (iii) and (iv)

$$2t - 4s = -2$$

$$-2t + 4s = -2$$

$$-4s = 0$$

$$s = 0$$

$$t = 1$$

$s = 0$, $t = 1$ satisfy eqn (v) then it satisfy.

we can not solve (III) and (IV), Also, solve (III) & (V).

If they are parallel or not,

$$\frac{2}{4} = \frac{-1}{-2} = \frac{3}{5}$$

ratio are not proportional. Hence they are skiewed.

Vector Function

Note: This is extension of vector ~~function~~ with constant coordinates.

Let t scalar variable defined on a interval (a, b) and let $\vec{r}$ be a vector function depends on scalar variable t. The we say that $\vec{r}$ is called vector function of scalar variable t and write

$$\vec{r} = \vec{r}(t)$$

example let $x = a\cos t$, $y = a\sin t$, $z = 0$

Then,

$$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$$

$$= a\cos t\,\vec{i} + a\sin t\,\vec{j} + 0.\vec{k}$$

$$\vec{r} = a\cos t\,\vec{i} + a\sin t\,\vec{j} \quad \text{--- (x)}$$

On squaring and adding,

$$r^2 = x^2 + y^2 = a^2, \ z = 0$$

This shows that (*) represents a circle on xy plane.

Example 2. Let $x = a\cos t$, $y = b\sin t$, $z = 0$

Then, $\frac{x}{a} = \cos t$, $\frac{y}{b} = \sin t$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \quad z = 0$$

$\therefore$ $\vec{r} = a\cos t\,\vec{i} + b\sin t\,\vec{j} + 0\vec{k}$ is a vector equation of ellipse on xy plane.

It is also called vector function of ellipse on xy plane.

Similarly, The equation $x = at^2$, $y = 2at$ represent a parabola.

i.e; $\vec{r} = at^2\,\vec{i} + 2at\,\vec{j} + 0.\vec{k}$ is a parabola

Example: Find the domain of vector function,

$\vec{r} = \sqrt{4-t^2}\,\vec{i} + e^{-3t}\vec{j} + \ln(t+1)\vec{k}$. [$\because \ln 0 = \infty$]

Soln:- Here,

$\vec{r} = f(t)\,\vec{i} + g(t)\,\vec{j} + h(t)\,\vec{k}$

Where, $f(t) = \sqrt{4-t^2}$ is defined on $[-2 \le t \le 2]$

$g(t) = e^{-3t}$ is defined for all finite value of t.

& $h(t) = \ln(1+t)$ is defined for all $t > -1$

$\therefore$ The common domain $(-1, 2]$

or $-1 < t \le 2$.

$\frac{0}{0}$ - Indeterminate form.

$\frac{\infty}{\infty}$, $\infty - \infty$

$\frac{0}{0}, \frac{\infty}{\infty}, \infty - \infty$

Limit of a Vector Function

A ~~function~~ vector function

$\vec{r} = \vec{r}(t)$ is said to have limit $\vec{L}$ as $t \to t_0$ if

$$\lim_{t \to t_0} \vec{r}(t) = \vec{L}$$

example :- Evaluate ~~limit~~

$$\lim_{t \to 0} \left(e^{-3t} \vec{i} + \frac{t^2}{\sin^2 t} \vec{j} + \cos 2t \vec{k} \right)$$

Soln

$$\lim_{t \to 0} \left(e^{-3t} \vec{i} \right) + \lim_{t \to 0} \frac{t^2}{\sin^2 t} \vec{j} + \lim_{t \to 0} \cos 2t \vec{k}$$

$$= 1 \vec{i} + 1 \vec{j} + 1 \cdot \vec{k}$$

$$= \vec{i} + \vec{j} + \vec{k}$$

Note:- If $\vec{r}(t) = f(t)\vec{i} + g(t)\vec{j} + h(t)\vec{k}$

Then $\lim_{t \to t_0} \vec{r}(t) = \lim_{t \to t_0} f(t)\vec{i} + \lim_{t \to t_0} g(t)\vec{j} + \lim_{t \to t_0} h(t)\vec{k}$.

This shows that limit of a vector function $r(t)$ exist if each of its component ~~has~~ $f(t), g(t), h(t)$ exists.

Continuity of vector function :-

A Vector Function $\vec{r}(t)$ of Scalar variable t is said to be continious at point $t = t_0$ if

$$\lim_{t \to t_0} \vec{r}(t) = \vec{r}(t_0).$$

i.e; limiting value = Functional value (of vector function)

Discuss the Continuity:

$$\vec{r}(t) = 2t\vec{i} + 3t\vec{j} + 4t\vec{k}$$

Soln.

$$\lim_{t \to 1} r(t) = \lim_{t \to 1} 2t\vec{i} + \lim_{t \to 1} 3t\vec{j} + \lim_{t \to 1} 4t\vec{k}$$

$$= 2\vec{i} + 3\vec{j} + 4\vec{k} \quad \text{--- } (*)$$

Also, Functional value,

$$\vec{r}(1) = 2\vec{i} + 3\vec{j} + 4\vec{k} \quad \text{--- } (**)$$

From $(*)$ and $(**)$

$$\lim_{t \to 1} r(t) = \vec{r}(1).$$

$\therefore$ $\vec{r}(t)$ is Continious at $t = 1$.

Derivatives of Vector function

Let $\vec{r} = \vec{r}(t)$ be a vector Function of Scalar variable t. Let δt be small change in t. Then the derivatives of $\vec{r} = \vec{r}(t)$ is denoted by, $\frac{d\vec{r}(t)}{d(t)}$ & is defined by

$$\frac{d\vec{r}(t)}{d(t)} = \lim_{\delta t \to 0} \frac{\vec{r}(t+\delta t) - \vec{r}(t)}{\delta(t)},$$

provided the limit on right hand side exists.

Fact If $\vec{r} = f(t)\vec{i} + g(t)\vec{j} + h(t)\vec{k}$ then

$$\frac{d\vec{r}}{dt} = \frac{df(t)}{dt}\vec{i} + \frac{dg(t)}{dt}\vec{j} + \frac{dh(t)}{dt}\vec{k}$$

$$\frac{d\vec{r}}{dt} = \left(\frac{df(t)}{dt}, \frac{dg(t)}{dt}, \frac{dh(t)}{d(t)}\right)$$

Basic Rules on differentiation

If $\vec{r}, \vec{r_1}$ and $\vec{r_2}$ be three vectors of Scalar variable t. then,

i) If $\vec{r} = \vec{r_1} \pm \vec{r_2}$ then,

$$\frac{d\vec{r}}{dt} = \frac{d\vec{r_1}}{dt} \pm \frac{d\vec{r_2}}{dt}$$

) If $\vec{r}$ is Constant vector,

$$\frac{d\vec{r}}{dt} = 0$$

) $\frac{d}{dt}(\phi\vec{r}) = \phi\frac{d\vec{r}}{dt} + \vec{r}\frac{d\phi}{dt}$ where ϕ Function of t.

$$\frac{d}{dt}(\vec{r_1}\cdot\vec{r_2}) = \vec{r_1}\cdot\frac{d\vec{r_2}}{dt} + \frac{d\vec{r_1}}{dt}\cdot\vec{r_2}$$

(5) $\frac{d}{dt}(\vec{r_1} \times r_2) = \vec{r_1} \times \frac{d\vec{r_2}}{dt} + \frac{d\vec{r_1}}{dt} \times \vec{r_2}$

(6) $\frac{d}{dt}[r_1 r_2 r_3] = \left[\frac{d\vec{r_1}}{dt}\ \vec{r_2}\ \vec{r_3}\right] + \left[\vec{r_1}\ \frac{d\vec{r_2}}{dt}\ \vec{r_3}\right]$
$+ \left[\vec{r_1}\ \vec{r_2}\ \frac{d\vec{r_3}}{dt}\right]$

Proof

We know that,

$[r_1 r_2 r_3] = r_1 \cdot r_2 \times r_3$

$\frac{d}{dt}(r_1 \cdot r_2 \times r_3) = \frac{dr_1}{dt}(r_2 \times r_3) +$
$\vec{r_1} \cdot \frac{d(r_2 \times r_3)}{dt}$

$= \frac{dr_1}{dt}(r_2 \times r_3) + \vec{r_1} \cdot \left[\vec{r_2} \times \frac{dr_3}{dt} + \frac{d\vec{r_2}}{dt} \times r_3\right]$

$= \left[\frac{dr_1}{dt} \cdot r_2 \times r_3 + r_1 \cdot \vec{r_2} \times \frac{dr_3}{dt} + \vec{r_1} \cdot \frac{d\vec{r_2}}{dt} \times r_3\right]$

$= \left[\frac{d\vec{r_1}}{dt}, \vec{r_1}\ \vec{r_2}\right) + \left[\vec{r_1}\ \frac{d\vec{r_2}}{dt}\ \vec{r_3}\right]$
$+ \left[\vec{r_1}\ r_2\ \frac{dr_3}{dt}\right]$

(7) Derivative for vector triple product.

$$\frac{d}{dt}(\vec{r_1}\times\vec{r_2}\times\vec{r_3}) = \frac{d\vec{r_1}}{dt}\times(\vec{r_2}\times\vec{r_3}) + \vec{r_1}\times\frac{d(\vec{r_2})}{dt}\times\vec{r_3} + \vec{r_1}\times\vec{r_2}\times\frac{d\vec{r_3}}{dt}$$

example

(1) If $\vec{r} = t^2\vec{i} - t\vec{j} + (2t+1)\vec{k}$ find

(i) $\frac{d\vec{r}}{dt}\cdot\frac{d^2r}{dt}$ (ii) $\frac{d\vec{r}}{dt}\times\frac{d^2r}{dt^2}$

Soln

$$\vec{r} = t^2\vec{i} - t\vec{j} + (2t+1)\vec{k}$$

$$\frac{d\vec{r}}{dt} = 2t\vec{i} - \vec{j} + 2\vec{k}$$

$$\frac{d^2\vec{r}}{dt^2} = 2\vec{i}$$

(i) $\frac{d\vec{r}}{dt}\cdot\frac{d^2r}{dt} = (2t, -1, 2)\cdot(2, 0, 0)$

$= 4t = 0$

(ii) $\frac{d\vec{r}}{dt}\times\frac{d^2r}{dt^2} = (2t, -1, 2)\times(2, 0, 0)$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2t & -1 & 2 \\ 2 & 0 & 0 \end{vmatrix} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & -1 & 2 \\ 2 & 0 & 0 \end{vmatrix}$$

$= 0\vec{i} + 4\vec{j} + 2\vec{k}.$

example If $\vec{r_1} = t^3\vec{i} + t^2\vec{j} + t\vec{k}$

$\vec{r_2} = (t+1)\vec{i} + (t+2)\vec{j} - 3t\vec{k}$

Find

$\frac{d}{dt}(\vec{r_1} \times r_2)$ at $t = 2$

Soln $\frac{d}{dt}(\vec{r_1} \times r_2) = \vec{r_1} \times \frac{d\vec{r_2}}{dt} + \frac{d\vec{r_1}}{dt} \times \vec{r_2}$

$= \left(t^3\vec{i} + t^2\vec{j} + t\vec{k}\right) \times \left(\vec{i} + \vec{j} - 3\vec{k}\right)$

$+ \left(3t^2\vec{i} + 2t\vec{j} + \vec{k}\right) \times$
$(t+1)\vec{i} + (t+2)\vec{j} - 3t\vec{k}$

$\frac{d}{dt}(\vec{r_1} \times r_2)$ at $t = 2$

$= (8\vec{i} + 4\vec{j} + 2\vec{k}) \times (\vec{i} + \vec{j} - 3\vec{k})$
$+ (12\vec{i} + 4\vec{j} + \vec{k}) \times (3\vec{i} + 4\vec{j} - 6\vec{k})$

$\vec{i}$	$\vec{j}$	$\vec{k}$	i	j	k
8	4	2	12	4	1
1	1	-3	3	4	6

$= (-12-2)\vec{i} + (2+24)\vec{j} + (8-4)\vec{k}$
$+ (24-4)\vec{i} + (3-12\times6)\vec{k} + (12\times4-12)\vec{k}$

$= \left(12\vec{i} + 26\vec{j} + 4\vec{k}\right) + 20\vec{i} - 69\vec{j}$
$+ 24\vec{k}$

$= 8\vec{i} + \quad + 28\vec{k}$

(Q) If $\dfrac{d\vec{a}}{dt} = \vec{c} \times \vec{a}$

& $\dfrac{d\vec{b}}{dt} = \vec{c} \times \vec{b}$

then show that $\dfrac{d}{dt}(\vec{a} \times \vec{b}) = \vec{c} \times (\vec{a} \times \vec{b})$

Soln

$$\frac{d\vec{a}}{dt} \times \frac{d\vec{b}}{dt} = \frac{d}{dt}(\vec{a} \times \vec{b})$$

$$= \vec{a} \times \frac{d\vec{b}}{dt} + \frac{d\vec{a}}{dt} \times \vec{b}$$

$$= \vec{a} \times (\vec{c} \times \vec{b}) + (\vec{c} \times \vec{a}) \times \vec{b}$$

$$= (\vec{a} \times (\vec{c} \times \vec{b}) + \vec{c} \times (\vec{a} \times \vec{b})$$

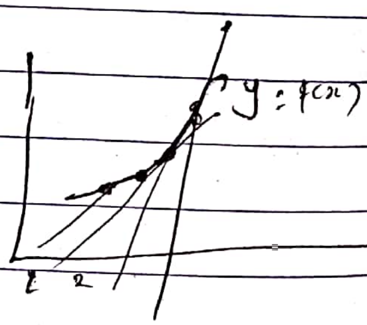
$$= -[\vec{b} \times (\vec{c} \times \vec{a})] + \vec{a} \times (\vec{c} \times \vec{b})$$

$$= -[\cancel{(b.a)c} - (b.c)\vec{a}]$$
$$+ \cancel{(a.b)\vec{c}} - (a.c)\vec{b}$$

$$= (b.c)\vec{a} - (a.c)\vec{b}$$

$$= (\vec{c}.\vec{b})\vec{a} - (\vec{c}.\vec{a})\vec{b}$$

$$= \vec{c} \times (\vec{a} \times \vec{b})$$



Note: derivative represent Slope of tangent to the curve

$r = 2at\,i + [illegible]\,j$

$\frac{dr}{dt} \to$ vector along the tangent

Geometrical meaning

If $\vec{r} = \vec{r}(t)$ be a vector function of scalar variable t then $\frac{d\vec{r}}{dt}$ always represent the vector along the tangent in the sense of t increasing.

Example Let $\vec{r} = 2t^2\vec{i} + 3t^2\vec{j}$. Find the unit vector along the tangent at point $t = 2$

Soln

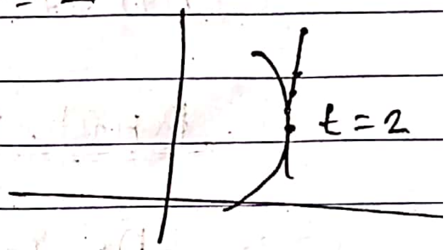
$$\vec{r} = 2t^2\vec{i} + 3t^2\vec{j}$$

$$\frac{d\vec{r}}{dt} = 4t\,\vec{i} + 6t\,\vec{j}$$

$$= 8\vec{i} + 12\vec{j} \quad \text{at } t = 2$$

$$\left|\frac{d\vec{r}}{dt}\right| = \sqrt{64 + 144}$$

$$= \sqrt{208}$$



$t = 2$

$$\hat{r} = \frac{8}{\sqrt{208}}\vec{i} + \frac{12}{\sqrt{208}}\vec{j}.$$

Vector along the tangent having length 5

$$= 5 \times [$$

Vector Integration:-

Integration of a vector function is the reverse process of differentiation.

Consider a vector $\vec{r} = \vec{r}(t)$ then its integration is denoted by

$$\int \vec{r}(t)\, dt.$$

Note:- If $\vec{r} = f(t)\vec{i} + g(t)\vec{j} + h(t)\vec{k}$ then its Vector Integration is,

$$\int \vec{r}(t)\, dt = \left(\int f(t)\, dt\right)\vec{i} + \left(\int g(t)\, dt\right)\vec{j} + \left(\int h(t)\, dt\right)\vec{k}$$

Definite Integral

As in the real valued function we can evaluate the definite integral for vector valued function. If $\vec{r} = \vec{r}(t)$ be a vector function and $\int \vec{r}(t) = \vec{R}(t) + \vec{c}$. Then definite integral of $\vec{r}(t)$ w.r.t. t from $t = t_1$ to $t = t_2$ is,

Indefinite $\int_{t_1}^{t_2} \vec{r}(t)\, dt = \left[\vec{R}(t)\right]_{t_1}^{t_2}$

Integrand   Integral

$$= \vec{R}(t_2) - \vec{R}(t_1)$$

Q) If $\vec{r} = t^2\vec{i} + t\vec{j} + t^2\vec{k}$ Find $\int_0^2 \left(\vec{r} \times \frac{d^2\vec{r}}{dt^2}\right) dt$

Soln Here $\vec{r}(t) = t^2\vec{i} + t\vec{j} + t^2\vec{k}$

$$\frac{d\vec{r}}{dt} = 2t\vec{i} + 1\vec{j} + 2t\vec{k}$$

$$\frac{d^2 r}{dt^2} = 2\vec{i} + 0\vec{j} + 2\vec{k}$$

$$\vec{r} \times \frac{d^2r}{dt^2} = \begin{vmatrix} i & j & k \\ t^2 & t & t^2 \\ 2 & 0 & 2 \end{vmatrix}$$

$$= 2t\,\vec{i} + (2t^2 - 2t^2)\vec{j} - 2t\,\vec{k}$$

$$= 2t\,\vec{i} + 0\vec{j} - 2t\,\vec{k}$$

$$\int_0^2 \vec{r} \times \frac{d^2r}{dt^2} = \int_0^2 2t\,\vec{i} - \int_0^2 2t\,\vec{k}$$

$$= [t^2]_0^2\,\vec{i} - [t^2]_0^2\,\vec{k}$$

$$= 4\vec{i} - 4\vec{k}$$

Arc length of a curve

Recall that the arc length of a Cartesian curve $y = f(x)$ from $x = a$ to $x = b$ is

$$s = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

Also from differentials calculus,

$x = f(t)$ & $y = g(t)$

$$\frac{ds}{dx} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

Then,

$$\frac{ds}{dt} = \frac{ds}{dx} \cdot \frac{dx}{dt} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \cdot \left(\frac{dx}{dt}\right)$$

$$= \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$$

If we extending this concept to vector function $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$, where x, y, z are functions of t, then,

$$\frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}$$

Integrating w.r.t t from $t = t_1$ to $t = t_2$

$$S = \int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt \quad \text{—} (*)$$

is arc length of vector valued function from t_1 to t_2.

<u>example</u> Find the arc length of the vector function $\vec{r}(t) = \cos t\,\vec{i} + \sin t\,\vec{j} + t\vec{k}$ from $t = 0$ to $t = 2\pi$

<u>Soln</u>:

$$\vec{r}(t) = \cos t\,\vec{i} + \sin t\,\vec{j} + t\vec{k}$$

$x = \cos t \qquad y = \sin t \qquad z = t$

$\frac{dx}{dt} = -\sin t \qquad \frac{dy}{dt} = \cos t \qquad \frac{dz}{dt} = 1$

$$\frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}$$

$$[s]_0^{2\pi} = \int_0^{2\pi} \sqrt{(-\sin t\,\vec{i})^2 + (\cos t)^2 + 1}\; dt$$

$$= \int_0^{2\pi} \sqrt{2}\, dt$$

$$= \sqrt{2}\,[t]_0^{2\pi}$$

$$= 2\sqrt{2}\pi.$$

Facts: If $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$

$$\frac{d\vec{r}}{dt} = \frac{dx}{dt}\vec{i} + \frac{dy}{dt}\vec{j} + \frac{dz}{dt}\vec{k}$$

$$\left|\frac{d\vec{r}}{dt}\right| = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}$$

* Can be written as

$$[s]_{t_1}^{t_2} = \int_{t_1}^{t_2} \left|\frac{d\vec{r}}{dt}\right| dt$$

$$= \int_{t_1}^{t_2} |r'(t)|\, dt$$

Curvature:-

We had the derivatives of $\vec{r}$ w.r.t t i.e $\frac{d\vec{r}}{dt}$ represent the vector along the tangent.

Also, unit vector,

$$T(t) = \frac{\frac{d\vec{r}}{dt}}{\left|\frac{d\vec{r}}{dt}\right|} = \frac{\vec{r}'(t)}{|r'(t)|}$$

The Curvature of a curve $\vec{r} = \vec{r}(t)$ is the rate of change of $T(t)$ w.r.t arc length s. i.e It is denoted by K.

Thus, $K = \left|\frac{d\vec{T}}{ds}\right|$

This can be evaluated as,

$$K = \left|\frac{\frac{dT}{dt}}{\frac{ds}{dt}}\right|$$

Q) Find the Curvature of a circle of radius a.

Sol) we know circle with radius $x = a\cos t$
$y = a\sin t$.

$\therefore \vec{r} = a\cos t\,\vec{i} + a\sin t\,\vec{j}$

$$\frac{d\vec{r}}{dt} = -a\sin t\,\vec{i} + a\cos t\,\vec{j}$$

$$T(t) = \frac{r'(t)}{|r'(t)|} = \frac{-a\sin t\,\vec{i} + a\cos t\,\vec{j}}{|a|}$$

$$= -\sin t\,\vec{i} + \cos t\,\vec{j}$$

$$T(t) = \cos t\,\vec{i} - \sin t\,\vec{j}$$

$$\frac{ds}{dt} = |r'(t)| = a$$

$$K = \frac{\left|\frac{dT}{dt}\right|}{\left|\frac{ds}{dt}\right|} = \left|\frac{-\cos t\,\vec{i} - \sin t\,\vec{j}}{a}\right| = \frac{1}{a} \#$$

we had.

$T(t) = \dfrac{r'(t)}{|r'(t)|}$, unit vector in the direction of tangent.

$$K = \frac{dT}{ds} = \frac{\frac{dT}{dt}}{\frac{ds}{dt}}, \text{ where } \frac{ds}{dt} = |r'(t)|$$

Alternative Formula to Find Curvature

The Curvature of a vector function $\vec{r} = \vec{r}(t)$ is,

$$k(t) = \frac{|r'(t) \times r''(t)|}{|r'(t)|^3}$$

Example Find the Curvature of a Curvature $\vec{r}(t) = (t^2, \ln t, t\ln t)$ at $(1,0,0)$

$$r'(t) = 2t + \frac{1}{t} + \ln t + t \cdot \frac{1}{t}$$

$$= 2t\vec{i} + \left(\frac{1}{t}\right)\vec{j} + [1 + \ln t]k$$

$$r''(t) = 2 - \frac{1}{t^2} + \frac{1}{t}$$

$$r'(t) \times r''(t) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2t & \frac{1}{t} & 1+\ln t \\ 2 & -\frac{1}{t^2} & \frac{1}{t} \end{vmatrix}$$

$$= \left[\frac{1}{t^2} + \frac{1}{t^2}(1+\ln t)\right]\vec{i} +$$

$$\left[2 + \cancel{2\ln t} \cancel{-2}\right]\vec{j} + \left(-\frac{2}{t} - \frac{2}{t}\right)\vec{k}$$

$$= (\cancel{1} + 2)\vec{i} + 2\vec{j} + -4\vec{k}$$

$$= 2\vec{i} + 0\vec{j} - 4\vec{k}$$

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$|\vec{r}'(t) \times r''(t)| = \sqrt{2^2+0^2+4^2} = \sqrt{20}$

$|r'(t)| = |2\vec{i}+\vec{j}+\vec{k}|$

$= \sqrt{4+1+1}$

$= \sqrt{6}$

$k(t) = \dfrac{\sqrt{20}}{(\sqrt{6})^3}$

$=$

Curvature with Cartesian eqn $y = f(x)$

If vector function $\vec{r} = x\vec{i} + y\vec{j}$ is expressed in plane curve $y = f(x)$. Then

$\vec{r} = x\vec{i} + f(x)\vec{j}$

$\vec{r}' = 1\vec{i} + f'(x)\vec{j}$

$\vec{r}'' = f''(x)\vec{j}$

Hence, above formula becomes

$k(x) = \dfrac{|r'(x) \times r''(x)|}{|r'(x)|^3} =$

$r'(x) \times r''(x) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & f'(x) & 0 \\ 0 & f''(x) & 0 \end{vmatrix}$

$= 0\cdot\vec{i} + 0\cdot\vec{j} + F''(x)\vec{k}$

$|r'(x)| = \sqrt{1 + F'(x)^2}$

$$k(x) = \frac{\sqrt{[F''(x)]^2}}{\{1 + (F'(x))^2\}^{3/2}}$$

$$= \frac{|F''(x)|}{(1 + [F'(x)]^2)^{3/2}}$$

example Find the curvature of the curve,

$y = x^4$ at $x = 2$

Soln Here,

$y = x^4$

$y' = 4x^3 \qquad y'|_{x=2} = 32$

$y'' = 12x^2 \qquad y''|_{x=2} = 12 \times 4 = 48$

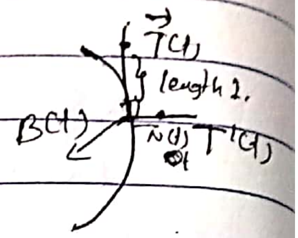
$$= \frac{|48|}{|1 + (32)^2|^{3/2}}$$

$$= \frac{|48|}{[1 + 1024]^{3/2}}$$

$$= \frac{48}{(1025)^{3/2}}$$

Normal & Binormal Vector

~~Fact~~ For a vector function $\vec{r}(t)$ we have unit tangent vector



$$T(t) = \frac{\vec{r}'(t)}{|\vec{r}'(t)|}$$

There are infinitely many vectors in the direction of $T(t)$ but we have only one single vector along the direction of t with $|T(t)| = 1$.

Fact: For every costant unit vector $\vec{r}$.

$$\vec{r} \cdot \frac{d\vec{r}}{dt} = 0$$

This means that

$$T(t) \cdot \frac{dT}{dt} = 0$$

i.e $T(t) \cdot T'(t) = 0$

The principal normal vector $N(t)$ is defined ~~fig~~ by

$$N(t) = \frac{T'(t)}{|T'(t)|}$$

The vector perpendicular to both $T(t)$ & $N(t)$ is called Binormal vector and is denoted by

Thus,

$$\vec{B}(t) = \vec{T}(t) \times \vec{N}(t)$$

It is $\perp^r$ to both vector $\vec{T}(t)$ & $\vec{N}(t)$.

C.W Find the tangent vector, normal vector and binormal vector of the vector function,

$$\vec{r}(t) = (t^2, \tfrac{2}{3}t^3, t) \text{ at } t = (1, 2/3, 1)$$

Soln

Here $\vec{r}(t) = (t^2, \frac{2}{3}t^3, t)$ at $t = (1, 2/3, 1)$

$$r'(t) = 2t\vec{i} + \frac{2}{3}\cdot 3t^2\vec{j} + 1\vec{k}$$
$$= 2t\vec{i} + 2t^2\vec{j} + \vec{k}$$

$$r''(t) = 2\vec{i} + 4t\vec{j} + 0$$

$$T(t) = \frac{r'(t)}{|r'(t)|} = \frac{2t\vec{i} + 2t^2\vec{j} + \vec{k}}{\sqrt{2^2 + 2^2 + 1}}$$

$$= \frac{2\vec{i} + 2\vec{j} + \vec{k}}{\sqrt{9}}$$

$$= \frac{2}{3}\vec{i} + \frac{2}{3}\vec{j} + \frac{1}{3}\vec{k}$$

$$= \frac{2t\vec{i}}{3} + \frac{2t^2}{3}\vec{j} +$$

$$T'(t) = \frac{d}{dt}\left(\frac{2t}{3}, \frac{2t^2}{3}, \frac{1}{3}\right)$$

$$\left(\frac{2}{3}, \frac{4t}{3}, 0\right)$$

$$= \left(\frac{2}{3}, \frac{4}{3}, 0\right)$$

Normal vector $N(t) = \dfrac{T(t)}{|T'(t)|} = \dfrac{\frac{2}{3}t\vec{i} + \frac{2t^2}{3}\vec{j} + \frac{1}{3}\vec{k}}{\sqrt{(\frac{2}{3})^2 + (\frac{4}{3})^2 + 0^2}}$

$$= \frac{\frac{2}{3}\vec{i} + \frac{2}{3}\vec{j} + \frac{1}{3}\vec{k}}{\sqrt{\frac{(2)^2 + (4)^2}{(3)^2}}}$$

$$= \frac{3}{\sqrt{20}}\frac{2}{3}\vec{i} + \frac{3}{\sqrt{20}}\frac{2}{3}\vec{j} + \frac{3}{\sqrt{20}}\frac{1}{3}\vec{k}$$

$$= \frac{2}{\sqrt{20}}\vec{i} + \frac{2}{\sqrt{20}}\vec{j} + \frac{1}{\sqrt{20}}\vec{k}$$

Now, Binormal vector is,

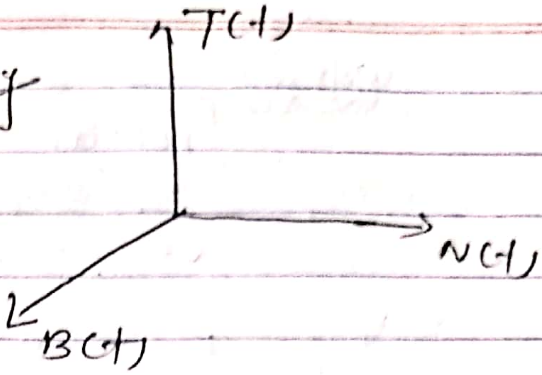
$$B(t) = T(t) \times N(t)$$

$$= \left(\frac{2}{3}\vec{i} + \frac{2}{3}\vec{j} + \frac{1}{3}\vec{k}\right) \times \left(\frac{2}{\sqrt{20}}\vec{i} + \frac{2}{\sqrt{20}}\vec{j} + \frac{1}{\sqrt{20}}\vec{k}\right)$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2/3 & 2/3 & 1/3 \\ 2/\sqrt{20} & 2/\sqrt{20} & 1/\sqrt{20} \end{vmatrix}$$

$$= (\quad)\vec{i} + (\quad)\vec{j} + (\quad)\vec{k}$$

Def^n, The plane determined by the vectors $N(t)$ & $T(t)$ is called osculating plane.



If $\vec{r} = \vec{r}(t)$ be a position of a moving particle at time t. Then, the velocity (v) is $\frac{d\vec{r}}{dt}$ and acceleration $\vec{a}$ is $\frac{d^2\vec{r}}{dt^2}$

example The acceleration of a particle at time t is $\vec{a}(t) = e^{2t}\vec{i}\,t + e^t\vec{j} + 2\vec{k}$. Find the velocity $\vec{v}$ and displacement $\vec{r}$ at time t. Given $\vec{v} = \vec{i} + \vec{j}$ if $\vec{r} = 0$ at t

Here, $\vec{a}(t) = e^{2t}\vec{i} + e^t\vec{j} + 2\vec{k}$

$$\int a(t)\,dt = \int (e^{2t}\vec{i} + e^t\vec{j} + 2\vec{k})\,dt$$

$$= \int e^{2t}\vec{i}\,dt + \int e^t\vec{j}\,dt + \int 2\vec{k}\,dt$$

$$= \frac{e^{2t}}{2}\vec{i} + e^t\vec{j} + 2t\vec{k} + c \quad -(*)$$

given, At $t=0$, $v = \vec{i} + \vec{j}$

$$\vec{i} + \vec{j} = \frac{1}{2}\vec{i} + \vec{j} + c$$

$\Rightarrow$ $c = \frac{1}{2}\vec{i}$

put c in eq $(*)$

$$\int a(t)\,dt = \frac{e^{2t}}{2}\vec{i} + e^t\vec{j} + 2t\vec{k} + \frac{1}{2}\vec{i}$$

against

$$\int r(t)\,dt = \int a(t)\,dt$$

$$= \int \left[\frac{e^{2t}}{2}\vec{i} + e^{t}\vec{j} + 2tk + \frac{1}{2}\vec{i}\right]dt$$

$$= \left[\frac{e^{2t}}{4}\vec{i} + e^{t}\vec{j} + t^{2}k + \frac{1}{2}t + c\right] \quad -(*)(v)$$

when $c(t) = 0$ $r(t) = 0$

$$\Rightarrow \quad 0 = \frac{1}{4}\vec{i} + \vec{j} + c$$

$$\Rightarrow \quad c = -\frac{1}{4}\vec{i} - \vec{j}$$

put in $(* \times)$

$$= \frac{e^{2t}}{4}\vec{i} + e^{t}\vec{j} + t^{2}k + \frac{1}{2}t - \frac{1}{4}\vec{i} - \vec{j} \quad -(* \times v)$$

$(* \times v)$ is required displacement.

<u>Vector Calculus</u>

<u>Smooth curve</u> :-

If a curve $\vec{r} = \vec{r}(t)$ is determine in space depending on the parameter(t). Such that

$$\vec{r} = x(t)\vec{i} + y(t)\vec{j} + z(t)\vec{k}$$

Then, it is called curve in space. If it is Continious and differentiable on the ~~integrable~~ interval then it is called Smooth Curve.

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eg The curve $\vec{r} = \vec{r}(t) = 2t\,\vec{i} + \frac{2}{t}\vec{j} + \frac{3}{t^2}\vec{k}$ is not smooth $[-2, 3]$.

Line Integral :-

$f(t) = 2t\vec{i} + 3t\vec{j} + 4t\vec{k}$.

Any integral which is evaluated along a curve is called line integral. Line Integral $\int_C \vec{F}\cdot d\vec{r}$.

Line Integral $\int_C \vec{F}\cdot d\vec{r}$:

Let us Consider vector point Function.

$\vec{F} = F_1(x,y,z)\vec{i} + F_2(x,y,z)\vec{j} + F_3(x,y,z)\vec{k}$.

Defined on a smooth curve C which is represented by $\vec{r}$. on the interval $[a,b]$.

Then line integral of F along the curve C is given by,

$\int_C \vec{F}\cdot d\vec{r}$ where $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$.

example Evaluate the line integral. $\int_C \vec{F}\cdot d\vec{r}$ where $x^2\vec{i} + y^3\vec{j}$ and C is arc of the parabola $y = x^2$ From $(0,0)$ to $(1,1)$.

Soln Here $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$

$d\vec{r} = dx\vec{i} + dy\vec{j} + dz\vec{k}$

$F\cdot dr = (x^2\vec{i} + y^3\vec{j})\cdot(dx\vec{i} + dy\vec{j} + dz\vec{k})$

$\int_C F\cdot dr = \int_C (x^2 dx + y^3 dy)\, dx$

$$= \int_0^1 x^2\,dx + \int_0^1 2x^7\,dx$$

$$= \left[\frac{x^3}{3}\right]_0^1 + \left[\frac{2x^8}{8}\right]_0^1$$

$$= \frac{1}{3} + \frac{2}{8} = \frac{4+3}{12} = \frac{7}{12}$$

[Q] Evaluate $\int_C \vec{F}\cdot d\vec{r}$ where $\vec{F} = y^2\vec{i} + x^2\vec{j}$, C is the straight line joining (0,0) to (1,0) and then from (1,0) to (1,1)

Soln St. line joining (0,0) to (1,0)

$$y - 0 = \frac{0-0}{1-0}(x-0) \Rightarrow y = 0$$

St. line joing (1,0) to (1,1)

$$y - 1 = \frac{1-0}{1-1}(x-1) \Rightarrow x = 1$$

$$r = x\vec{i} + y\vec{j} + z\vec{k}$$
$$dr = dx\vec{i} + dy\vec{j} + dz\vec{k}$$

$$\vec{F}\cdot dr = (y^2\vec{i} + x^2\vec{j})\cdot(dx\vec{i} + dy\vec{j})$$
$$= y^2dx + x^2dy$$

$$\int_C \vec{F}\cdot d\vec{r} =$$

$\therefore$ required integral $= \int_C \vec{F}\cdot d\vec{r}$

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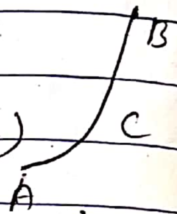
$\nabla\phi$ - represent Normal vector.

$\boxed{\text{Line Integration Independent of path}}$.

<u>Theorem</u>: The line integral $\int_C \vec{F}\cdot d\vec{r}$ of a Continious Function F is Independent of path (C) If $\vec{F}$ Can be expressed as $\vec{F} = \nabla\phi$ where ϕ is ~~Scalar~~ Some Scalar valued Function.

<u>Proof</u>:- We proved that if $\vec{F} = \nabla\phi$, then $\int_C \vec{F}\cdot d\vec{r}$ is Independent of path (C).

Let C be a path joining A to B.



Now, $\int_C \vec{F}\cdot d\vec{r} = \int_C \nabla\phi\cdot(\vec{i}dx + \vec{j}dy + \vec{k}dz)$

$$= \int_C \left(\vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z}\right)\cdot(\vec{i}dx + \vec{j}dy + \vec{k}dz)$$

$$= \int_C \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz$$

$$= \int_C d\phi$$

$$= [\phi]_A^B$$

$$= \phi(B) - \phi(A).$$

This Shows that $\int_C \vec{F}\cdot d\vec{r}$ is Independent of path.

<u>Defination</u>:- (Irrotational vector)

We know that If $\vec{F} = \nabla\phi$ for Some Scalar Valued Function ϕ, then $\int_C \vec{F}\cdot d\vec{r}$ is Independent of path in Such case $\vec{F}$ is called Irrotational vector Function.

For Such Function $\vec{F}$

$$\nabla\times\vec{F} = 0$$

i.e F is irrotational if $\nabla \times \vec{F} = 0$

$$\nabla \times \vec{F} = \nabla \times (\nabla \phi) = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \frac{\partial \phi}{\partial x} & \frac{\partial \phi}{\partial y} & \frac{\partial \phi}{\partial z} \end{vmatrix}$$

$$= \left(\frac{\partial^2 \phi}{\partial y \partial z} - \frac{\partial^2 \phi}{\partial z \partial y}\right)\vec{i} + \left(\frac{\partial^2 \phi}{\partial x \partial z} - \frac{\partial^2 \phi}{\partial z \partial x}\right)\vec{j} + \left(\frac{\partial^2 \phi}{\partial x \partial y} - \frac{\partial^2 \phi}{\partial y \partial x}\right)\vec{k} = 0.$$

__example:-__

Show that the vector function $\vec{F} = (x^2 - yz)\,i + (y^2 - zx)\,\vec{j} + (z^2 - xy)\,\vec{k}$ is irrotational. Also find a scalar function ϕ such that $\vec{F} = \nabla \phi$.

__Soln__ Curl $\vec{F}$ = ~~$\nabla \times \phi(\nabla \phi)$~~ $\nabla \times F$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \frac{\partial \phi}{\partial x} & \frac{\partial \phi}{\partial y} & \frac{\partial \phi}{\partial z} \end{vmatrix}$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ (x^2 - yz) & (y^2 - zx) & z^2 - xy \end{vmatrix}$$

$$= \left[\frac{\partial}{\partial y}(z^2 - xy) - \frac{\partial}{\partial z}(y^2 - zx)\right] i + \left[\frac{\partial}{\partial z}(x^2 - yz) - \frac{\partial}{\partial x}(z^2 - xy)\right]\vec{j} + \left[\frac{\partial}{\partial x}(y^2 - zx) - \frac{\partial}{\partial y}(x^2 - yz)\right]\vec{k}$$

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$$\left[-x+x\right]\vec{i}+\left[-y+y\right]\vec{j}+\left[-z+z\right]\vec{k}$$

$$= (0\vec{i}+0\vec{j}+0\vec{k})$$

$= \vec{0}$ vector

$=$ Curl $\vec{F}$ is zero, Hence $\vec{F}$ is irrotational.

For second part,
we have $\vec{F} = \nabla\phi$

$$\Rightarrow (x^2-yz)\vec{i}+(y^2-zx)\vec{j}+(z^2-xy)\vec{k} = \frac{\partial\phi}{\partial x}\vec{i}+\frac{\partial\phi}{\partial y}\vec{j}+\frac{\partial\phi}{\partial y}\vec{k}$$

$$\Rightarrow x^2-yz = \frac{\partial\phi}{\partial x} \Rightarrow \cancel{2x = \partial\phi}$$

$$y^2-zx = \frac{\partial\phi}{\partial y} \Rightarrow \cancel{2y = \partial\phi}$$

$$z^2-xy = \frac{\partial\phi}{\partial z} \Rightarrow \cancel{2z = \partial\phi}$$

Integrating w.r.t x.

$$\phi = \frac{x^3}{3} - xyz + C_1, \quad \phi = \frac{y^3}{3} - xyz + C_2,$$

$$\phi = \frac{z^3}{3} - xyz + C_3$$

$\therefore$ Hence, ϕ is,

$$\phi = \frac{x^3}{3}+\frac{y^3}{3}+\frac{z^3}{3} - 3xyz + C$$

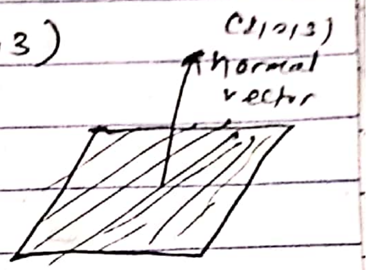
Surface Integral :-

Fact : If $\phi = \cancel{\text{con}}\ c$ be a plane or surface. Then, $\nabla\phi$ always represent vector normal to the surface

eg $\phi = x+2y+3z = 0$ be a plane.

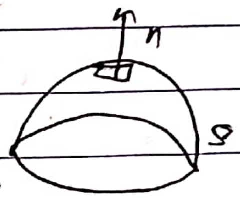
Then,

$$\nabla\phi = \vec{i}.1 + 2.\vec{j} + 3\vec{k} = (1,2,3)$$



Defn Any integral which is evaluated over a surface is called surface integral.

Let S be the surface $\vec{n}$ be the vector normal to the surface of small rectangle surface ds in the surface. Then the integral ~~of~~ over the surface S of a vector function $\vec{F}$



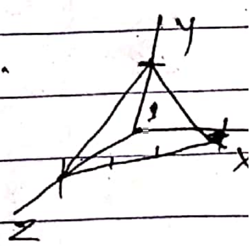
$$= \iint_S \vec{F}.\vec{n}\, ds$$

where $\vec{n}$ is the unit vector normal to the given surface $\vec{S}$. If $S = \phi(x,y,z)$

$$\vec{n} = \frac{\nabla\phi}{|\nabla\phi|} \quad \& \quad ds = \frac{dx\,dy}{|\vec{n}.\vec{k}|}$$

where $\vec{k} = (0,0,1)$.

Evaluate the surface integral $\iint_S \vec{F}.\vec{n}\, ds$ where $\vec{F} = x^2\vec{i} + y^2\vec{j} + z^2\vec{k}$. and S be the finite plane $x+y+z=1$.



Soln $\vec{F} = x^2\vec{i} + y^2\vec{j} + z^2\vec{k}$

S ~~ϕs~~, $x+y+z=1$

$\phi = (x+y+z-1 = 0$

Now, $\nabla\phi = \vec{i}\frac{\partial}{\partial x} \cdot x + \vec{j}\frac{\partial}{\partial y} y + \frac{\partial}{\partial z} z$

$= (\vec{i} + \vec{j} + \vec{k})$

$\vec{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{\vec{i}+\vec{j}+\vec{k}}{\sqrt{3}}$

Now,

$S = \iint_S \vec{F} \cdot \vec{n}\, ds$

$= \iint_S \vec{F} \cdot \vec{n}\, ds$

where

$d\vec{s} = \frac{dx\,dy}{|\vec{n}\cdot\vec{k}|} = \frac{dx\,dy}{\left|\frac{1}{\sqrt{3}}\right|} = \sqrt{3}\,dx\,dy$

Also,

$z = 1-x-y$

$\therefore$ Surface integral.

$(S) = \iint_S \vec{F} \cdot \vec{n}\, ds$

$= \iint_S (x^2\vec{i} + y^2\vec{j} + z^2\vec{k}) \cdot \left(\frac{\vec{i}+\vec{j}+\vec{k}}{\sqrt{3}}\right) \cdot \sqrt{3}\,dx\,dy$

$= \int_{x=0}^{1}\int_{y=0}^{1-x} [x^2\vec{i} + y^2\vec{j} + (1-x-y)^2\vec{k}] \cdot [\vec{i}+\vec{j}+\vec{k}]\, dx\,dy.$

$$= \int_{x=0}^{1}\int_{y=0}^{1-x}\left\{\left[x^2+y^2+(1-x-y)^2\right]dy\right\}dx$$

$$= \int_{x=0}^{1}\left[x^2y+\frac{y^3}{3}+\frac{(1-x-y)^3}{3\cdot(-1)}\right]_0^{1-x}dx$$

$$= \int_{x=0}^{1} x^2(1-x)+\frac{(1-x)^3}{3}+\left[\left\{\frac{(1-x)-(1-x)^3}{-3}\right\}-\left\{\frac{(1-x)^3}{-3}\right\}\right]dx$$

$$= \int_{x=0}^{1}\left[x^2-x^3+\frac{(1-x)^3}{3}+\frac{(1-x)^3}{3}\right]dx$$

$$= \left[\frac{x^3}{3}-\frac{x^4}{4}+\frac{2}{3}\frac{(1-x)^4}{4(-1)}\right]_0^1$$

$$= \left[\frac{1}{3}-\frac{1}{4}+\frac{2}{3}\cdot\frac{1}{-4}(0-1)\right]$$

$$= \left[\frac{1}{3}-\frac{1}{4}+\frac{2}{12}\right]$$

$$= \frac{4-3+2}{12}$$

$$= \frac{3}{12}$$

$$= \frac{1}{4}.$$

Example:- Find the Surface Integral of the vector function $\vec{F} = yz\vec{i}+zx\vec{j}+xy\vec{k}$ through that of sphere $x^2+y^2+z^2=1$ in the first octant.

Soln given the surface is $x^2+y^2+z^2=1$, we know that $\nabla\varphi$ is a vector normal to the surface $\varphi(x,y,z)=C$

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taking $\phi(x,y,z) = x^2+y^2+z^2$

$$\nabla\phi = \frac{\partial\phi}{\partial x}\vec{i} + \frac{\partial\phi}{\partial y}\vec{j} + \frac{\partial\phi}{\partial z}k$$

$$= 2xi + 2yj + 2zk$$

$\therefore$ unit vector normal

$$\hat{n} = \frac{\nabla\phi}{|\nabla\phi|}$$

$$\hat{n} = \frac{2(x\vec{i}+y\vec{j}+z\vec{k})}{\sqrt{2^2(x^2+y^2+z^2)}}$$

$$= \frac{xi+yj+zk}{\sqrt{x^2+y^2+z^2}}$$

$$= \frac{xi+yj+zk}{a} \quad (\because x^2+y^2+z^2=1)$$

Also If $\vec{F} = yz\vec{i} + zx\vec{j} + xy\vec{k}$

$$\vec{F}\cdot\hat{n} = \frac{1}{a}(xyz + yzx + zxy)$$

$$= \frac{3xyz}{a} \quad -(1)$$

$$ds = \frac{dx\,dy}{\hat{n}\cdot\hat{k}} = \frac{dx\,dy}{z/a} = \frac{a\,dx\,dy}{z}$$

From (1) and (2)

$$\iint_S \vec{F}\cdot\hat{n}\,ds = \iint_R \frac{3xyz}{a}\cdot\frac{a\,dx\,dy}{z} = \iint_R 3xy\,dx\,dy$$

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The region r of Integration is the quadrant of the circle $x^2+y^2=1$

$$\iint_S \vec{F}\cdot\hat{n}\,ds = 3\int_{x=0}^{1}\int_{y=0}^{\sqrt{1-x^2}} xy\,dy\,dx$$

$$= 3\int_{x=0}^{1} x\left[\frac{y^2}{2}\right]_{x=0}^{\sqrt{1-x^2}} dx$$

$$= \frac{3}{2}\int_0^1 x(1-x^2)\,dx$$

$$= \frac{3}{2}\left[\frac{x^2}{2}-\frac{x^4}{4}\right]_0^1$$

$$= \frac{3}{2}\left[\frac{1}{2}-\frac{1}{4}\right]$$

$$= \frac{3}{8}$$

Thus $\iint_S (yz\vec{i}+zx\vec{j}+xyk)\cdot\hat{n}\,ds$

$$= \frac{3a^4}{8}$$

Gradient field :-

If $\phi(x,y,z)$ be surface function of three variables x, y & z then its gradient function or gradient field $\nabla\phi$ is. ie $\nabla\phi = \vec{i}\frac{\partial\phi}{\partial x}+\vec{j}\frac{\partial\phi}{\partial y}+\vec{k}\frac{\partial\phi}{\partial z}$

(Q) Find the gradient field of $f(x,y) = x^2y^2-x^2-y^2$

Soln: Here $F(x,y) = x^2y^2-x^2-y^2$

$$\text{Gradient } \vec{F} = \vec{i}\frac{\partial F}{\partial x}+\vec{j}\frac{\partial F}{\partial y}+\vec{k}\frac{\partial F}{\partial z}$$

$$= \vec{i}(2xy^2-2x)+(2yx^2-2y)\vec{k}$$

Note:. Gradi f = vector normal to the surface.

If we have to find line integral of scalar valued function

Then the line integral of the scalar function $f(x,y)$ over the curve c is given by the formula,

$\int_c f(x,y)\, ds$ where ds = arc length.

$$= \int_c f(x,y) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

where $t \in (a,b)$

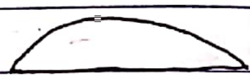
$$= \int_a^b f(x,y) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

Q) Evaluate the line ~~int~~ integral $\int_c f(x,y)\, ds$ where, $f(x,y) = 2 + x^2 y$ & c is upper half of unit circle.

Soln:- Clearly, unit circle is $x = \cos t$, $y = \sin t$

$[x^2 + y^2 = \cos^2 t + \sin^2 t = 1]$

clearly t lies in between 0 & π.



$t=\pi$ $t=0$

i.e $0 \le t \le \pi$.

$\because$ required line integral $= \int_c f(x,y)\, ds$

$$= \int_0^\pi (2 + x^2 y) \sqrt{(-\sin t)^2 + (\cos t)^2}\, dt$$

$$= \int_0^\pi (2 + x^2 y)\, dt$$

$$= \int_0^\pi (2 + \cos^2 t \cdot \sin t)\, dt$$

$$= \int_0^\pi 2\, dt + \int_0^\pi \cos^2 t \sin t\, dt$$

put $\cos t = y$

$-\sin t\, dt = dy$

If $t=0$, $y=1$, $t=2\pi$, $y=-1$,

$$= [2t]_0^{\pi} + \left[\frac{-y^3}{3}\right]_1^{-1}$$

$$= 2\pi + \frac{1}{3} + \frac{1}{3} = 2\pi + 2/3$$

Workdone by force [Application of line Integral]

Soln If a force $\vec{F}$ act on a particle which is moved along a curve C. Then the workdone W.D of the Force F along the tangent to the curve C is,

$$W.D = \vec{F} \cdot \frac{d\vec{r}}{dt}$$

$\therefore$ Total workdone $= \int_c \vec{F} \cdot \frac{d\vec{r}}{dt}$.

Q Find the workdone in moving a particle once along the circle $x^2+y^2=4$, $z=0$ if the force field $\vec{F}$ is defined by $\vec{F} = (2x-y+2z)\vec{i} + (x+y-z)\vec{j} + (3x-2y+5z)\vec{k}$.

Soln Here circle C is $x^2+y^2=4$, $z=0$ which is given by $x=2\cos t$, $y=2\sin t$,

$$F = (2x-y)\vec{i} + (x+y)\vec{j} + (3x-2y)\vec{k}$$

$$F = (2.2\cos t - 2\sin t)\vec{i} + (2\cos t + 2\sin t)\vec{j} + (3.2\cos t - 2.2\sin t)\vec{k}$$

$$= (4\cos t - 2\sin t)\vec{i} + (2\cos t + 2\sin t)\vec{j} + (6\cos t - 4\sin t)\vec{k}$$

Also, $d\vec{r} = \vec{i}dx + \vec{j}dy$ where $dx = -2\sin t\, dt$

$dy = 2\cos t\, dt$.

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$\therefore$ Workdone by Force$(\vec{F}) = \int_C \vec{F} \cdot d\vec{r}$.

$= \int_0^{2\pi} (4\cos t - 2\sin t)\vec{i} + (2\cos t + 2\sin t)\vec{j} + (6\cos t - 4\sin$
$\cdot (-2\sin t)\vec{i} + 2\cos t \, \vec{j} \cdot dt$

$= \int_0^{2\pi} (-8\cos t \sin t + 4\sin^2 t)\, dt + (4\cos^2 t + 4\cos t)\, dt$

$= \int_0^{2\pi} -8\cos t \sin t \, dt + \int_0^{2\pi} 4\sin^2 t \, dt + \int_0^{2\pi} 4\cos^2 t \, dt$
$+ \int_0^{2\pi} 4\cos t \sin t \, dt$

$= \int_0^{2\pi} [-4\cos t \sin t \, dt + 4] \, dt$

$= \int_0^{2\pi} [-2\sin 2t + 4] \, dt$

$= \left[-2(-1)\frac{\cos 2t}{2}\right]_0^{2\pi} + [4t]_0^{2\pi}$

$= [\cos 2t]_0^{2\pi} + [4]_0^{2\pi}$.

Green theorem :

Relation Between Line Integral and Surface Integral (Double Integral)

If D be a closed region in the (x,y) plane, bounded by a simple closed curve C. If P(x,y) and Q(x,y) are two continious curve in D.

Then, $\int_C [P dx + Q dy] = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\, dy.$

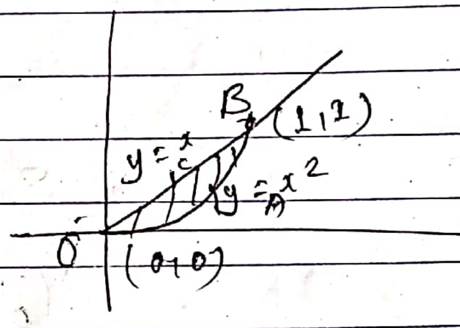
example verify Green theorem for the Function $P = xy + y^2$ & $Q = x^2$ where c is the closed curve of the region bounded by the line $y = x$ & parabola $y = x^2$

Soln $\frac{\partial Q}{\partial x} = 2x$, $\frac{\partial P}{\partial y} = x + 2y$

the line $y = x$ and $y = x^2$ meets at O(0,0) & A(1,1)

∴ ~~L.H.S =~~

The parabola & line Intersect at O(0,0) & A(1,1)



Now, $\int_C (xy + y^2)dx + x^2 dy$

Along $y = x^2$

$$I_1 = \int_{x=0}^{x=1} x \cdot x^2 + (x^2)^2 dx + \int_0^1 x^2 2x\, dx$$

$$= \frac{x^4}{4}\Big|_0^1 + \frac{x^5}{5}\Big|_0^1 + 2\left|\frac{x^4}{4}\right|_0^1$$

$$= \frac{1}{4} + \frac{1}{5} + \frac{2}{4} = \frac{1}{4} + \frac{1}{5} + \frac{1}{2} = \frac{19}{20}$$

Again, along the curve $y = x$ i.e BCO

$= \int_c (xy + y^2)\,dx + x^2\,dy$

$= \int_1^0 (x^2 + x^2)\,dx + \int_1^0 x^2\,dx$

$= \left.\frac{x^3}{3}\right|_1^0 + \left.\frac{x^3}{3}\right|_1^0 + \left.\frac{x^3}{3}\right|_1^0$

$= \frac{-1}{3} - \frac{1}{3} - \frac{1}{3}$

$= \frac{-1-1-1}{3}$

$I_2 = -1$

$\therefore$ line Integral $= I_1 + I_2$

$= \frac{19}{20} - 1$

$= \frac{-1}{20}$ —— (A)

Again, Surface integral

$= \iint_D \left[\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right] dx\,dy$

$= \iint_D \left[\frac{\partial x^2}{\partial x} - \frac{\partial}{\partial y}(xy + y^2)\right] dx\,dy$

$= \int_0^1 \int_{x^2}^{x} \left[\frac{\partial x^2}{\partial x} - \frac{\partial}{\partial y}(xy + y^2\right] dx\,dy$

$= \int_0^1 \int_{x^2}^{x} 2x - (x + 2y) \,]\, dx\,dy$

$= \int_0^1 [\int_{x^2}^{x} (x - 2y)\,dy]\,dx$

$$= \int_0^1 \left[xy\Big|_{x^2}^{x} - \frac{2y^2}{2}\Big|_{x^2}^{x} \right] dx$$

$$= \int_0^1 \left[x.x - x^3 - x^2 + x^4 \right] dx$$

$$= \cancel{\int_0^1 (-x^3 + 2x^4 + x)\, dx}$$

$$= \int_0^1 (x^4 - x^3)\, dx$$

$$= \frac{x^5}{5} - \frac{x^4}{4}\Big|_0^1$$

Total area $= \frac{1}{2}\int_C (x\,dy - y\,dx)$

$$= \frac{1}{5} - \frac{1}{4} = -\frac{1}{20} \quad \text{--- (B)}$$

$\therefore$ from (A) & (B)

$$\int_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\, dy$$

Application :-

Statement : The area enclosed by the curve C is given by $\frac{1}{2}\int_C x\,dy - y\,dx$

Proof :- By Green theorem in the plane,

$$\int_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\, dy \quad \text{--- (X)}$$

In particular, Let us choose

$$P = -y \ \& \ Q = x \ \text{in (X)}$$

$$\int_C x\,dy - y\,dx = \iint_D \left[\frac{\partial x}{\partial x} - \frac{\partial(-y)}{\partial y} \right] dx\, dy$$

or, $\int_C x\,dy - y\,dx = \iint_D (1+1)\, dx\, dy$

or $\frac{1}{2}\int_C x\,dy - y\,dx = \iint_D dx\, dy$

= Total area of region D.

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Divergence & Curl of a Vector Function

Divergence: If $\vec{V} = P\vec{i} + Q\vec{j} + R\vec{k}$ where P, Q, R are function of x, y, z, then divergence of $\vec{V}$ is denoted by $\nabla\cdot\vec{V}$ and is defined by

$\nabla\cdot\vec{V} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$, provided all the partial derivatives exists.

In Symbolic notation

$$\nabla\cdot\vec{V} = \left(\vec{i}\cdot\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)\cdot \cancel{(P\vec{i} + Q\vec{j} + R\vec{k})}\ \vec{V}$$

$$= \left(\vec{i}\frac{\partial v}{\partial x} + \vec{j}\frac{\partial v}{\partial y} + \vec{k}\frac{\partial v}{\partial z}\right)$$

$$= \Sigma\ \vec{i}\cdot\frac{\partial v}{\partial x}$$

Curl: If $\vec{V} = P\vec{i} + Q\vec{j} + R\vec{k}$, where P, Q, R are function of x, y, z, Then curl of $\vec{V}$ is denoted by $\nabla\times\vec{V}$ and is defined by

$\nabla\times V =$ and is given by

$$= \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)\times(P\vec{i} + Q\vec{j} + R\vec{k})$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ P & Q & R \end{vmatrix}$$

$$= \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\vec{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\vec{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\vec{k}$$

provided that all partial derivatives exist

In Symbolic Notation

$$\nabla \times \vec{V} = \left(\hat{i}\frac{\partial}{\partial x} + \vec{J}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right) \times \vec{V}$$

$$= \vec{i} \times \frac{\partial \vec{V}}{\partial x} + \vec{J} \times \frac{\partial \vec{V}}{\partial x} + \vec{k} \times \frac{\partial \vec{V}}{\partial z}$$

$$= \Sigma\, \vec{i} \times \frac{\partial \vec{V}}{\partial x}$$

example:-

If $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$

Find divergence $\vec{r}$ & curl $\vec{r}$.

Sol^n Here, $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$

divergence of $\vec{r}$ is given by,

$$\nabla \cdot \vec{r} = \frac{\partial}{\partial x} x + \frac{\partial}{\partial y} y + \frac{\partial}{\partial z} z$$

$$= 1 + 1 + 1$$

$$= 3$$

also

$$\nabla \times \vec{r} = \left(\frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{J} + \frac{\partial}{\partial z}\vec{k}\right) \times \left(x\vec{i} + y\vec{j} + z\vec{k}\right)$$

$$= \begin{vmatrix} \vec{i} & \vec{J} & \vec{k} \\ \frac{\partial}{\partial x} & \partial/\partial y & \partial/\partial z \\ x & y & z \end{vmatrix}$$

$$= \left(\frac{\partial z}{\partial x} - \frac{\partial x}{\partial z}\right)\vec{i} + \left(\frac{\partial x}{\partial z} - \frac{\partial z}{\partial x}\right)\vec{J} + \left(\frac{\partial z}{\partial x} - \frac{\partial x}{\partial z}\right)\vec{k}$$

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$= 0+0+0 = 0$

Example If $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$ & $r = |\vec{r}|$
prove that (i) $\nabla\cdot(r^3\vec{r}) = 6r^3$ (ii) $Div\left(\frac{\vec{r}}{r}\right) = \frac{2}{r}$

Soln $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$

$$\frac{\partial r}{\partial x} = \frac{\partial}{\partial x}\left(\sqrt{x^2+y^2+z^2}\right)^{1/2}$$

$$= \frac{x}{\sqrt{x^2+y^2+z^2}} = \frac{x}{r}$$

Similarly,

$$\frac{\partial r}{\partial y} = \frac{y}{r}$$

$$\frac{\partial r}{\partial z} = \frac{z}{r}$$

Now, (i) $\nabla\cdot(r^3\vec{r})$

$$= \Sigma i \frac{\partial}{\partial x}\cdot(r^3\vec{r})$$

$$= \Sigma i\cdot\frac{\partial}{\partial x}(r^3\vec{r})$$ [$\because$ By defination of divergence]

$$= \Sigma i\cdot\left\{r^3\frac{\partial\vec{r}}{\partial x} + \frac{\partial r^3}{\partial x}\vec{r}\right)$$

$$= \boxed{\Sigma i\cdot\left\{r^3\cdot\frac{x}{r} + 3r^2\frac{\partial r}{\partial x}i\right\}}$$

$$= \Sigma\vec{i}\cdot\left\{r^3\vec{i} + 3r^2\frac{\partial r}{\partial x}\vec{r}\right\}$$

$$= \Sigma\vec{i}\cdot\left\{r^3\vec{i} + 3rx\vec{r}\right)$$

$$= \Sigma\left[r^3(\vec{i}\cdot\vec{i}) + 3rx(\vec{i}\cdot\vec{r})\right]$$

$$= \cancel{\Sigma\left[r^3 +\right.}$$

$= \Sigma r^3(1) + \Sigma 3r x \cdot x$

$= r^3 \Sigma(1) + 3r \Sigma x^2$

$= 3r^3 + 3r r^2$

$= 3r^3 + 3r^3$

$= 6r^3$

(ii) $Div\left(\frac{\vec{r}}{r}\right)$

(iii) $\nabla^2\left(\frac{1}{r}\right) = 0$

Notes:- A vector function v is said to be

(i) Solenoidal if $\nabla \cdot \vec{v} = 0$ i.e $div \vec{v} = 0$

(ii) Irrotational $\nabla \times \vec{v} = 0$ i.e $curl \vec{v} = 0$.

$Div\left(\frac{\vec{r}}{r}\right) = \nabla \cdot \left(\frac{\vec{r}}{|r|}\right) = \nabla \cdot \left(\frac{x+y+z}{r}\right)$

$= \Sigma i \frac{\partial}{\partial x} \cdot \frac{x}{\sqrt{x^2+y^2+z^2}}$

$= \Sigma i \cdot \frac{\partial}{\partial x}\left\{\frac{x}{\sqrt{x^2+y^2+z^2}}\right\}$

$= \Sigma i \cdot \frac{1\sqrt{x^2+y^2+z^2} - x \cdot \frac{1}{\sqrt{x^2+y^2+z^2}} \cdot x}{\left(\sqrt{x^2+y^2+z^2}\right)^2}$

$= \Sigma i \cdot \frac{\cancel{x^2}+y^2+z^2-\cancel{x^2}}{(x^2+y^2+z^2) \cdot \sqrt{x^2+y^2+z^2}}$

$= \Sigma i \cdot \frac{y^2+z^2}{r^3}$

$= \frac{2(x^2+y^2+z^2)}{r^3}$

$= \frac{2r^2}{r^3}$

$= \frac{2}{r}$

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example Find the value of Constant p so that the vector

$$\vec{V} = (x+3y,\ y-2z,\ x+pz) \text{ is Solenoidal,}$$

Soln

$$\nabla \cdot \vec{V} = 0$$

$$\Rightarrow \frac{\partial}{\partial x}(x+3y)\vec{i} + \frac{\partial}{\partial y}(y-2z)\vec{j} + \frac{\partial}{\partial z}(x+pz)\vec{k} = (0+0+0)$$

$$\Rightarrow \cancel{\vec{i} + \vec{j} + p\vec{k} = (0\vec{i} + 0\vec{j} + 0\vec{k})}$$

$$1+1+p = 0$$

$$\Rightarrow p = -2$$

H.W Find the value of the constant p, q and r so that the vector,

$$\vec{V} = (x+2y+pz)\vec{i} + (qx-3y-z)\vec{j} + (4x+ry+2z)\vec{k} \text{ is Irrotational}$$

Soln

$$\nabla \times \vec{V} = \left(\frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}\right) \times \left\{ (x+2y+pz)\vec{i} + (qx-3y-z)\vec{j} + (4x+ry+2z)\vec{k} \right\}$$

$$\begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ (x+2y+pz) & (qx-3y-z) & (4x+ry+2z) \end{vmatrix}$$

Ans
$p = 4$
$q = 2$
$r = -1$

$$= \left\{\frac{\partial}{\partial y}(4x+ry+2z) - \frac{\partial}{\partial z}(qx+3y-z)\right\}\vec{i} + \left\{\frac{\partial}{\partial z}(x+2y+pz) - \frac{\partial}{\partial x}(4x+ry+2z)\right\}\vec{j} + \left\{\frac{\partial}{\partial x}(qx-3y-z) - \frac{\partial}{\partial y}(x+2y+qz)\right\}\vec{k}$$

Conservative & Non-Conservative Field :-

A vector field $\vec{F} = P\vec{i} + Q\vec{j}$ is called Conservative if

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

& Non-Conservative,

If, $\frac{\partial P}{\partial y} \neq \frac{\partial Q}{\partial x}$

eg Does the vector field $\vec{F} = (x-y)\vec{i} + (x-2)\vec{j}$ is Conservative?

Soln

$$\boxed{\cancel{\frac{\partial F}{\partial x} = 1\vec{i} + 1\vec{j}} \qquad \cancel{\frac{\partial F}{\partial y} = -1\vec{i} + 0\vec{j}}}$$

$P = (x-y)$ $\quad Q = (x-2)$

$$\frac{\partial P}{\partial y} = -1$$

$$\frac{\partial Q}{\partial x} = 1$$

$$\therefore \frac{\partial P}{\partial y} \neq \frac{\partial Q}{\partial x}$$

example Check whether the vector function or field $\vec{F}(x,y) = (3+2xy)\vec{i} + (x^2-3y^2)\vec{j}$ is Conservative

Soln

given function

$\vec{F}(x,y) = P\vec{i} + Q\vec{j}$ where $P = (3+2xy)$

$Q = (x^2-3y^2)$

$\frac{\partial P}{\partial y} = 2x$

$\frac{\partial Q}{\partial x} = 2x$

$\therefore \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ Hence is Conservative.

Note:- If $\vec{F} = P\vec{i} + Q\vec{j}$ find curl $\vec{F}$

Soln

$$\text{curl}\,\vec{F} = \nabla \times \vec{F}$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ P & Q & 0 \end{vmatrix}$$

$$= \cancel{\frac{\partial Q}{\partial z}\vec{i}}\ 0\vec{i} + 0\vec{j} + \left\{\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right\} k$$

$$= \vec{k}\left\{\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right\} \quad -(*)$$

Note:- From $(*)$ we can say that

(i) $\vec{F}$ is Conservative if $\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$

i.e curl $\vec{F} = 0$

(ii) $\vec{F}$ is non conservative if $\frac{\partial Q}{\partial x} \neq \frac{\partial P}{\partial y}$

i.e curl $\vec{F} \neq 0$ i.e F is not irrotational

Question:- Show that the vector field $\vec{F} = xz\vec{i} + xyz\vec{j} - y^2\vec{k}$ Can not be written as the curl of another function G such that $\vec{F} \neq$ Curl G.

Soln:- div $\vec{f}$

$$= \frac{\partial}{\partial x}(xz) + \frac{\partial}{\partial y}(xyz) + \frac{\partial}{\partial z}(-y^2)$$

$$= z + xz + 0$$

$$= z(1+x) \quad \text{--- } (*)$$

$$\text{Div}\vec{F} \neq 0$$

If $\vec{F} = \text{Curl}\vec{G}$ For some vector $\vec{G}$.

Div F = 0

But By (*) Div (Curl G) $\neq 0$

So our supposition is false.

Evaluate: div curl $\vec{V} = ?$, where $\vec{V}$ is any vector.

Soln Let $\vec{V} = V_1\vec{i} + V_2\vec{j} + V_3\vec{k}$

$$\text{div curl } V = \nabla\cdot(\nabla\times\vec{V})$$

$$\begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_1 & V_2 & V_3 \end{vmatrix} = 0$$

$$= \frac{\partial}{\partial x}\left[\frac{\partial V_3}{\partial y} - \frac{\partial V_2}{\partial z}\right) - \frac{\partial}{\partial y}\left[\frac{\partial V_3}{\partial x} - \frac{\partial V_1}{\partial z}\right] +$$

$$\frac{\partial}{\partial z}\left[\frac{\partial V_2}{\partial x} - \frac{\partial V_1}{\partial y}\right]$$

$$= \cancel{\frac{\partial^2 V_3}{\partial x\partial y}} - \cancel{\frac{\partial^2 V_2}{\partial x\partial z}} - \cancel{\frac{\partial^2 V_3}{\partial y\partial x}} + \cancel{\frac{\partial^2 V_1}{\partial y\partial z}}$$

$$+ \cancel{\frac{\partial^2 V_2}{\partial z\partial x}} - \cancel{\frac{\partial^2 V_1}{\partial z\partial y}}$$

$$= 0$$

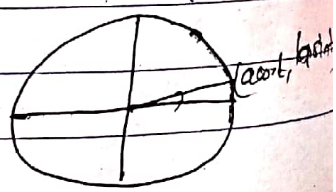
Green theorem: Used Green theorem to find the area of the ellipse. $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Soln For the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Parametric represent,

$x = a\cos t$, $y = b\sin t$

where θ varies, $0 \le \theta \le 2\pi$



$dx = -a\sin t\, dt$

$dy = b\cos t\, dt$

$\therefore$ By green theorem

$$= \frac{1}{2}\int_C (x\,dy - y\,dx)$$

$$= \frac{1}{2}\int_0^{2\pi} (a\cos t\; b\cos t\, dt + b\sin t\; a\sin t\, dt)$$

$$= ab\frac{1}{2}\int_0^{2\pi} dt$$

$$= \frac{ab}{2}[t]_0^{2\pi}$$

$$= \frac{ab}{2} \times 2\pi$$

$$= ab\pi$$

Q10 used Green theorem to find the area of the circle $x^2+y^2=a^2$.

Sol^n for circle $x^2+y^2=a^2$ parametric representation is $x = a\cos t$, $y = a\sin t$.

where $0 \leq \theta \leq 2\pi$

$dx = -a\sin t\, dt$

$dy = a\cos t\, dt$

By green theorem

$$= \frac{1}{2}\int_0^{2\pi} (x\,dy - y\,dx)$$

$$= \frac{1}{2}\int_0^{2\pi} (a\cos t\; a\cos t\, dt + a\sin t \sin t\, dt)$$

$$= \frac{1}{2}\int_0^{2\pi} a^2(\cos^2 t + \sin^2 t)\, dt$$

$$= \frac{a^2}{2}\int_0^{2\pi} dt = \frac{a^2}{2} t\Big|_0^{2\pi} = \frac{a^2}{2} \times 2\pi = \pi a^2$$

Stokes Theorem (Relation between line Integral & Surface Integral in Vector function)

If F is continious & differential vector Function defined over the surface S, Then,

$$\underbrace{\int_C \vec{F} \cdot d\vec{r}}_{\text{line Integral}} = \underbrace{\iint_S (\text{Curl } \vec{F}) \cdot \vec{n}\, ds}_{\text{Surface integral}}$$

where $ds = \dfrac{dx\,dy}{\vec{n} \cdot \vec{k}}$

$$\int_C \vec{F} \cdot d\vec{r} = \iint_S (\nabla \times \vec{F}) \cdot \vec{n}\, ds$$

example

Special Case of Stokes's theorem

In plane Surface, prove that Green theorem is the particular Case of Stock's theorem.

Proof:- If S is the plane Surface lying in xy plane, so that z axis is the direction of unit normal vector $\vec{n}$. so that, $\vec{n} = \vec{k}$.

Also in xy plane $z = 0$

Let,

Let $\vec{F} = P\vec{i} + Q\vec{j} + R\vec{k}$

& $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$

$\therefore d\vec{r} = dx\vec{i} + dy\vec{j} + dz\vec{k}$

L.H.S $\int_C \vec{F} \cdot d\vec{r}$

$$= \int_c (P\vec{i} + Q\vec{j} + R\vec{k}) \cdot (dx\vec{i} + dy\vec{j} + dz\vec{k})$$

$$= \int_c P\,dx + Q\,dy \quad — (x)$$

But $z = 0$
$dz = 0$

From R.H.S

$$\iint_S (\nabla \times \vec{F}) \cdot \vec{n}\, ds \quad — (x\,x)$$

$$\nabla \times F = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ P & Q & R \end{vmatrix}$$

$$= \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\vec{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\vec{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\vec{k}$$

from $(x\,x)$

$$\iint_S \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\vec{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\vec{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\vec{k} \cdot \vec{k}$$

$dx\,dy$

$$= \iint_S \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy. \quad — (x\,x\,x)$$

From (x) and $(x\,x\,x)$

$$\int_c P\,dx + Q\,dy = \iint_S \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy.$$

which green theorem.

Consequence of Stoke law :-

Prove that $\int_c d\vec{r}\cdot\vec{F} = \iint_s (\vec{n}\times\nabla)\cdot\vec{F}\,ds$

Proof :- From Stock's theorem,

$$\int_c \vec{F}\cdot d\vec{r} = \iint_s (\nabla\times\vec{F})\cdot\vec{n}\,ds \quad \text{--- } (*)$$

Now, $(\nabla\times\vec{F})\cdot\vec{n} = \vec{n}\cdot(\nabla\times\vec{F})$

$= \cancel{(\vec{n}\times\nabla\vec{F})}$

$= (\vec{n}\times\nabla)\cdot\vec{F}$

Problem :- verify Stock's theorem for the vector function $\vec{F} = x^2\vec{i} + xy\vec{j}$ for the square region the xy plane bounded by line $x=0, y=0$, $x=a$, $y=a$.

proof Here, $\vec{F} = x^2\vec{i} + xy\vec{j}$

We verify $\int_c \vec{F}\cdot d\vec{r} = \iint_s (\text{curl}\vec{F})\cdot\vec{k}\,dx\,dy$

$$\text{curl}\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^2 & xy & 0 \end{vmatrix}$$

$= (0-0)\vec{i} + (0-0)\vec{j} + \vec{k}(y-0)$

$= \vec{k}y$

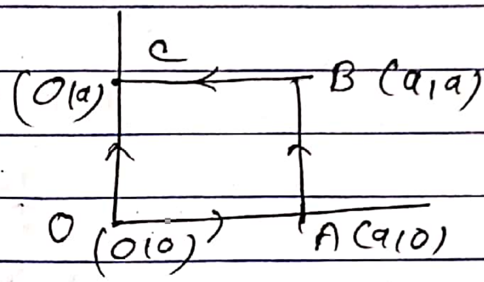
So $\iint_S (\text{curl}\,\vec{F})\cdot\hat{k}\,dx\,dy = \iint_S \vec{k}\,y\cdot\vec{k}\,dx\,dy$

$$= \iint_S y\,dx\,dy$$

$$= \int_0^a\left[\int_0^a y\,dy\right]dx$$

$$= \int_0^a \frac{a^2}{2}\,dx$$

$$= \frac{a^3}{2}$$



Also,

$\int_C \vec{F}\cdot d\vec{r} =$

we denote the closed curve C by OABC

$$\int_C \vec{F}\cdot d\vec{r} = \int_{OA}\vec{F}\cdot d\vec{r} + \int_{AB}\vec{F}\cdot d\vec{r} + \int_{BC}\vec{F}\cdot d\vec{r} + \int_{CO}\vec{F}\cdot d\vec{r} \quad \text{--- (1)}$$

Now, $\int_{OA}\vec{F}\cdot d\vec{r} = \int_{OA}(x^2\vec{i} + xy\,\vec{j})\cdot(\hat{i}\,dx + \hat{j}\,dy)$

But eqn OA
$y = 0$
$dy = 0$

$$= \int_0^a x^2\,dx$$

$$= \left.\frac{x^3}{3}\right|_0^a$$

$$= \frac{a^3}{3}$$

$\int_{AB}\vec{F}\cdot d\vec{r} = \int_{OA}(x^2\vec{i} + xy\,\vec{j})\cdot(\hat{i}\,dx + \hat{j}\,dy)$

But eqn AB
$x = a$
$dx = 0$

$$= \int (a^2 i + ay\,\vec{j})\cdot(\vec{j}\,dy) = \int_0^a a\,y\,dy$$

$$= a \frac{y^2}{2}\Big|_0^a$$

$$= a \cdot \frac{a^2}{2} = \frac{a^3}{2}$$

$$\int_{BC} \vec{F} \cdot d\vec{r} = \int_{BC} (x^2\vec{i} + xy\vec{j}) \cdot (\vec{i}dx + \vec{j}dy)$$

But at BC $y = a$

$dy = 0$

$$= \int_{B} (x^2\vec{i} + ax\vec{j}) \cdot (\vec{i}dx + 0\,dy)$$

$$= \int_a^0 x^2\,dx$$

$$= -\frac{a^3}{3}$$

$$\int_{CO} \vec{F} \cdot d\vec{r} = \int_{CO} (x^2\vec{i} + xy\vec{j}) \cdot (\vec{i}dx + \vec{j}\,dy)$$

But at CO $y = 0$

$dy = 0$

$$\int (x^2\vec{i} + 0\vec{j}) \cdot (\vec{i}dx + 0\,dy)$$

$$= \int_0^0 x^2 dx$$

$$= \frac{x^3}{3}\Big|_0^0$$

$$= 0$$

put all value in (1)

$$= \cancel{\frac{a^3}{3}} + \frac{a^3}{2} - \cancel{\frac{a^3}{3}} + 0$$

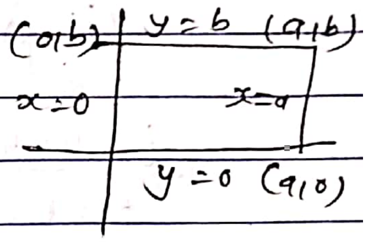
$$= \frac{a^3}{2}$$

(Q) Verify Stock's theorem for

(i) $\vec{F} = (x^2y + y^2)\vec{i} - xy\vec{J}$

(ii) $\vec{F} = (x^2 - y^2)\vec{i} + 2xy\vec{J}$

taken around the rectangle in the xy plane bounded by $x=0$ & $x=a$, $y=0$, $y=b$



Example Use stock's theorem to prove

$$\int_C \vec{r}\cdot d\vec{r} = 0$$

Soln $\vec{r} = x\vec{i} + y\vec{J} + z\vec{k}$

$d\vec{r} = dx\vec{i} + dy\vec{J} + dz\vec{k}$.

we have,

$$\int_C \vec{F}\cdot d\vec{r} = \iint_S (\text{Curl}\,\vec{F})\cdot\vec{n}\,ds \quad \text{--- (*)}$$

Replace $\vec{F} = \vec{r} = x\vec{i} + y\vec{J} + z\vec{k}$

Now, $\text{curl}\,\vec{F} = \begin{vmatrix} i & J & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x & y & z \end{vmatrix}$

$$= \left(\frac{\partial z}{\partial y} - \frac{\partial y}{\partial z}\right)\vec{i} + \left(\frac{\partial x}{\partial z} - \frac{\partial z}{\partial x}\right)\vec{J} + \left(\frac{\partial x}{\partial y} - \frac{\partial y}{\partial x}\right)\vec{k}$$

$$= \vec{0}$$

From (*)

$$\int_C \vec{r}\cdot d\vec{r} = 0$$ Hence proved

Divergence Theorem (Gauss's Theorem)

If $\vec{F}$ be vector differentiable function then normal surface integral of $\vec{F}$ over the closed surface S enclosing the volume v is equal to volume integral of divergence of $\vec{F}$ over V.

$$\iint_S \vec{F}\cdot\vec{n}\,ds = \iiint_V \nabla\cdot F\,dv$$

Evaluate $\iint_S \vec{F}\cdot\vec{n}\,ds$ where S is the closed surface enclosing the volume V.

Soln: Here we use Gauss's Theorem.

$$\iint_S \vec{F}\cdot\vec{n}\,ds = \iiint_V \nabla\cdot f\,dv$$

replacing $\vec{F}$ by $\vec{r}$ where $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$

Then, $\iint_S \vec{r}\cdot\vec{n}\,ds = \iiint_V \nabla\cdot r\,dv$

$$= \iiint \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)\cdot(x\vec{i} + y\vec{j} + z\vec{k})$$

$$= \iiint_V \left(\frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z}\right) dv$$

$$= \iiint_V 3\,dv$$

$$= 3V$$

Q) Prove that $\iint_S \nabla r^2\cdot\vec{n}\,ds = 6V$

Soln where V is the volume enclosed by the surface S.

$$\nabla r^2 = \vec{i}\,\frac{\partial r^2}{\partial x} + \vec{j}\,\frac{\partial r^2}{\partial y} + \vec{k}\,\frac{\partial r^2}{\partial z}$$

$$= 2r\,\vec{i}\,\frac{x}{r} + 2r\,\vec{j}\,\frac{y}{r} + \vec{k}\,2r\,\frac{z}{r}$$

$$= 2r$$

Replacing $\vec{F}$ by $2\vec{r}$

$$\iint_S 2\vec{r}\cdot n\,ds = \iiint_V \nabla\cdot 2r\,dv$$

$$= 2\iiint\left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)(\vec{i}x + \vec{j}y + \vec{k}z)\,dv$$

$$= 2\iiint_V\left(\frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z}\right)$$

$$= 6.V$$

(Q) Evaluate the Surface integral $\iint_S \vec{F}\cdot\vec{n}\,ds$ where $\vec{F} = (2x+3z)\vec{i} - (xz+y)\vec{j} + (y^2+2z)\vec{k}$ and S be the Surface of sphere of $(x-3)^2+(y+1)^2+(z-2)^2=9$

Soln:- Here $\vec{F} = (2x+3z)\vec{i} - (xz+y)\vec{j} + (y^2+2z)\vec{k}$

and given surface is $(x-3)^2+(y+1)^2+(z-2)^2=9$.

We have,

$$\iint_S \vec{F}\cdot\vec{n}\,ds = \iiint_V \nabla\cdot\vec{F}\,d\vec{v}$$

$$= \iiint_V \left(\frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}\right)\cdot\left((2x+3z)\vec{i} - (xz+y)\vec{j} + (y^2+2z)\vec{k}\right)$$

$$= \iiint_V (2-1+2)\,dv$$

$$= 3\iiint_V dv$$

$$= 3V = 3\times\tfrac{4}{3}\pi r^3 = 3\times\tfrac{4}{3}\pi(3)^3 = 108\pi$$

(Q) If $\vec{F} = 2x\vec{i} + 3y\vec{j} + 4z\vec{k}$ prove that normal ~~Solution~~ Surface of unit sphere.

Soln, Here $\vec{F} = 2x\vec{i} + 3y\vec{j} + 4z\vec{k}$

$$\iiint_V \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right) \cdot (2x\vec{i} + 3y\vec{j} + 4z\vec{k})\, dv$$

$$= \iiint_V \frac{\partial}{\partial x}(2x) + \frac{\partial}{\partial y}(3y) + \frac{\partial}{\partial z}(4z)\, dv$$

$$= \iiint_V (2 + 3 + 4)\, dv$$

$$= 9V = 9 \times \tfrac{4}{3}\pi r^3 = 9 \times \tfrac{4}{3}\pi (1)^3$$

(Q) with the help of Gauss divergence theorem show that

$$\iint_S (ax\vec{i} + by\vec{j} + cz\vec{k}) \cdot \vec{n}\, ds = \tfrac{4}{3}\pi (a+b+c)$$

where S is the surface of sphere $x^2+y^2+z^2=1$

Soln Here $\vec{F} = (ax\vec{i} + by\vec{j} + cz\vec{k})$

$$\iiint_V \left(\frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}\right) \cdot (ax\vec{i} + by\vec{j} + cz\vec{k})\, dv$$

$$= \iiint_V \left(\frac{\partial ax}{\partial x} + \frac{\partial by}{\partial y} + \frac{\partial cz}{\partial z}\right) dv$$

$$= \iiint_V (a+b+c)\, dV$$

$$= (a+b+c)\, V$$

$$= (a+b+c)\frac{4}{3}\pi$$

Some extra problem

(1) For any closed surface S prove that,

(i) $\iint_S \text{curl}\vec{F}\cdot\vec{n}\,ds = 0$

(ii) $\iint_S \vec{n}\cdot ds = 0$

(i) From Divergence theorem,

$$\iint_S \vec{F}\cdot n\,ds = \iiint_V \nabla\cdot f\,dv \quad \text{—}(\alpha)$$

Here $\vec{F} = \text{curl}\vec{F}$

put in (α)

$$\iint_S \vec{F}\cdot n\,ds = \iiint_V \nabla\cdot \text{curl}\vec{f}\,dv$$

$$= 0 \qquad [\because \nabla\cdot\text{curl}\vec{f} = 0]$$

(ii) Here $\vec{F} = 1$

$$\iint_S \vec{n}\cdot ds = \iiint_V \nabla\cdot 1\,dv$$

$$= \iiint_V 0\,dv$$

$$= 0$$

If S be the closed surface of volume V &

$\vec{F} = x\vec{i} + y\vec{j} + z\vec{k}$

Show that $\iint_S \vec{F}\cdot\vec{n}\,ds = 6V$.

Parametric Surface

The eqn of sphere is written in parametric form,

$\vec{r} = f(t)\vec{i} + g(t)\vec{j} + h(t)\vec{k}$ where $a \le t \le b$

eg. $\vec{r} = at^2\vec{i} + 2at\vec{j}$ is the parametric eqn of parabola $a \le t \le b$.

Also, If p is expressed as function of u & v then,
$\vec{r} = f(u,v)\vec{i} + g(u,v)\vec{j} + h(u,v)\vec{k}$

example Find the Cartesian eqn of surface given by the parametric representation,

$\vec{r}(u,v) = u\vec{i} + u\cos v\vec{j} + u\sin v\vec{k}$

Soln:- Here $x = u$, $y = u\cos v$, $z = u\sin v$
Squaring and adding
$y^2 + z^2 = u^2\cos^2 v + u^2\sin^2 v = u^2 = x^2$
$\therefore$ $x^2 = y^2 + z^2$ as required surface.

example Find the parametric representation of
$x = 5y^2 + 2z^2 - 10$

Soln Here, $x = 5y^2 + 2z^2 - 10$, $y = y$, $z = z$
$\therefore$ parametric representation,
$\vec{r} = (5y^2 + 2z^2 - 10)\vec{i} + y\vec{j} + z\vec{k}$

Some Application:-
We can find the equation of tangent plane to the surface $\vec{r}(u,v) = F(u,v)\vec{i} + g(u,v)\vec{j} + h(u,v)\vec{k}$ — (1)
At point (u_0, v_0)

Step I Find partial derivatives $\vec{r_u}$ and $\vec{r_v}$ which gives tangent vector to the surface

Step II Find normal vector to the surface $\vec{n}$
$\vec{n} = \vec{r_u} \times \vec{r_v}$

Step III eqn of tangent plane at point (u_0, v_0, w_0)
$(\vec{r} - \vec{a}) \cdot \vec{n} = 0$ where, $\vec{a} = (u_0, v_0, w_0)$
$\vec{r} = (x, y, z)$.

Example : Find the equation of tangent plane to the surface given by ~~Funch~~

(1) $F(u,v) = u\vec{i} + 2v^2\vec{j} + (u^2+v)\vec{k}$ at point $(2,2,3)$.

Soln Differenciating partially w.r.t u and v.

$F_u = \vec{i} + 0\cdot\vec{j} + 2u\vec{k}$
$F_u = \vec{i} + 2u\vec{k}$

$F_v = 0\cdot\vec{i} + 4v\vec{j} + \vec{k}$
$= 4v\vec{j} + \vec{k}$

~~Now the normal vector~~ These represent the direction of tangent,

$$\vec{n} = \vec{F_u} + \vec{F_v}$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & 2u \\ 0 & 4v & 1 \end{vmatrix}$$

$= -8v^2\vec{i} + -\vec{j} + 4v\vec{k}$
$= -8v\vec{i} - \vec{j} + 4v\vec{k} = -8uv\vec{i} - \vec{j} + 4v\vec{k}$. (*)

But For point $(2,2,3)$

$$2u\vec{i} + 2v^2\vec{j} + (u^2+v)\vec{k} = 2\vec{i} + 2\vec{j} + 3\vec{k}$$

$\Rightarrow$ $u = 2$

$2v^2 = 2 \Rightarrow v = \pm 1$

$u^2 + v = 3$

Here, $v = 1$ does not satisfied remaining eqn

So we take $v = -1$.

From (*)

$$-8 \cdot 2 \cdot -1\vec{i} - \vec{j} + 4 \cdot -1 k$$

$$= 16\vec{i} - \vec{j} - 4k$$

$$\vec{n} = (16, -1, -4)$$

$\therefore$ eqn of ~~plane~~ tangent plane through $(2,2,3) = (x_1, y_1, z_1)$ with normal $(16, -1, -4)$ is,

$$a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$$

$$\Rightarrow 16(x-2) - 1(y-2) - 4(z-3) = 0$$

$$\Rightarrow 16x - 16 - y + 2 - 4z + 12 = 0$$

$$\Rightarrow 16x - y - 4z - 32 + 2 + 12 = 0$$

$$\Rightarrow 16x - y - 4z - 18 = 0$$

Q10 Find the equation of tangent plane to the surface with parametric eqn.

$$x = u^2, y = v^2 \;\&\; z = u + 2v$$

Sol^n Here parametric curve is,

$$\vec{r}(u,v) = x\vec{i} + y\vec{j} + z\vec{k}$$
$$= u^2\vec{i} + v^2\vec{j} + (u+2v)\vec{k}$$

$$\vec{r}(u) = 2u\cdot\vec{i} + 0\cdot\vec{j} + \vec{k}$$
$$\vec{r}_v = 0\cdot\vec{i} + 2v\vec{j} + 2\vec{k}$$

Now,

$$\vec{n} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2u & 0 & 1 \\ 0 & 2v & 2 \end{vmatrix} = -2v\vec{i} - 4u\vec{j} + 4uv\vec{k}$$

$$u^2\vec{i} + v^2\vec{j} + (u+2v)\vec{k} = \vec{i} + \vec{j} + 3\vec{k}$$
$$u^2 = \pm 1$$
$$v^2 = \pm 1$$

$$u + 2v = 3$$
$$u = 1, \quad v = 1$$

$$\vec{n} = -2\vec{i} - 4\vec{j} + 4\vec{k}$$

Now eq^n of plane

$$a(x-x_1) + b(y-y_1) + c(z-z_1) = 0$$

$$\Rightarrow -2(x-1) - 4(y-1) + 4(z-3) = 0$$
$$\Rightarrow -2x + 2 - 4y + 4 + 4z - 12 = 0$$
$$\Rightarrow -2x - 4y + 4z - 6 = 0$$
$$\Rightarrow x + 2y \pm 2z + 3 = 0$$

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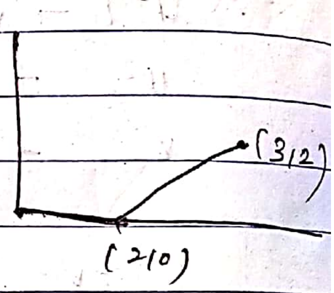
Problem related to line Integral

Evaluate the line integral $\int_C \vec{F}\cdot d\vec{r}$ where
$\vec{F} = (2x+y)\vec{i} + (3y-x)\vec{j}$
& C is the curve in xy plane consisting the Stright line from (0,0) to (2,0) and then to (3,2)

Soln Here.
$\vec{F} = (2x+y)\vec{i} + (3y-x)\vec{j}$
$\vec{r} = x\vec{i} + y\vec{j}$
$dr = i\,dx + \vec{j}\,dy$

$\vec{F}\cdot d\vec{r} = (2x+y)\,dx + (3y-x)\,dy$



line integral $= \int \vec{F}\cdot d\vec{r}$
$= \int_{OA} \vec{F}\cdot d\vec{r} + \int_{AB} \vec{F}\cdot d\vec{r}$ — (8)

For OA
$x = 2$
$y = 0$

For AB
$y - 0 = \frac{2-0}{3-2}(x-2)$

i.e, $y = 2x-4$
$dy = 2\,dx$

$= \int_0^2 (2x+y)dx + (3y-x)\,dy + \int_2^3 (2x+y)\,dx + (3y-x)$

$= \int_0^2 2x\,dx + \int_2^3 (2x + 2x - 4)\,dx + (8(2x-4) - x)\,2\,dx$

$= \int_0^2 2x\,dx + \int_2^3 (4x-4)\,dx + (6x - 12 - x)\,2\,dx$

$= \int_0^2 2x\,dx + \int_2^3 (4x-4)\,dx + (12x - 24 - 2x)\,dx$

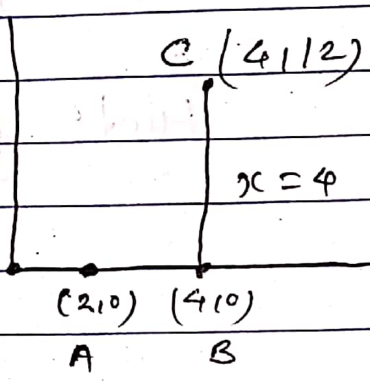
$= \frac{2x^2}{2}\Big|_0^2 + \frac{4x^2}{2} - 4x\Big|_2^3 + \frac{12x^2}{2} - 24x - \frac{2x^2}{2}\Big|_2^3$

$= 11.$

Evaluate $\int_C \vec{F} \cdot d\vec{r}$

where $\vec{F} = xy\,\vec{i} + (x^2 + y^2)\,\vec{j}$

C is the x-axis from $x = 2$ to $x = 4$ and straight line $x = 4$ from $y = 0$ to $y = 12$.



Soln Here,

$\vec{F} = xy\,\vec{i} + (x^2 + y^2)\,\vec{j}$

$d\vec{r} = \vec{i}\,dx + \vec{j}\,dy$

$\vec{F} \cdot d\vec{r} = (xy\,\vec{i} + (x^2+y^2)\vec{j}) \cdot (\vec{i}\,dx + \vec{j}\,dy)$

$= xy\,dx + (x^2+y^2)\,dy$

The line integral $= \int_C \vec{F} \cdot d\vec{r}$

$= \int_{AB} \vec{F} \cdot d\vec{r} + \int_{BC} \vec{F} \cdot d\vec{r}$

$= \int_2^4 x \cdot 0\,dx + (x^2+y^2)\,dy + \int_{BC} \vec{F} \cdot d\vec{r}$

$y = 0$

$dy = 0$

$= 0.$

$\int_0^{12} xy\,dx + (x^2+y^2)\,dy$

where ~~x=~~ equation of BC $\quad x=4$

$dx=0$

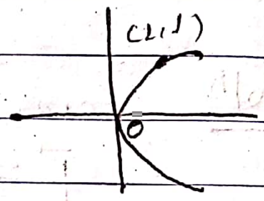
$\int_0^{12} (16+y^2)\,dy$

$= 16y + \frac{y^3}{3}\Big|_0^{12}$

$16\times 12 + \frac{12^3}{3}$

H.W Evaluate the line integral $\int_C \vec{F}\cdot d\vec{r}$ where

(a) $\vec{F} = x^2\vec{i} - xy\vec{j}$ & c is the curve from (0,0) to (1,1) along parabola $y^2 = x$.



Hint:- $\int_C \vec{F}\cdot d\vec{r} = \int_{OA} \vec{F}\cdot d\vec{r}$

$= \int_{y=0}^{y=1} x^2\,dx - xy\,dy$

$= \int_{y=0}^{y=1} \cancel{x^2dx} - y^4\, 2y\,dy - y^2 . y\,dy$

$= \int_{y=0}^{y=1} 2y^5\,dy - y^3\,dy.$

$= 2.\frac{y^6}{6}\Big|_0^1 - \frac{y^4}{4}\Big|_0^1$

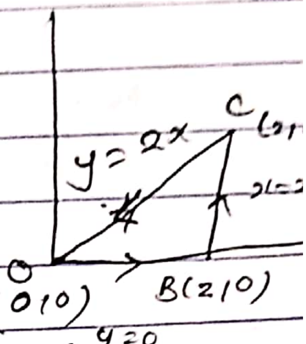
$= \frac{2.1}{6} - \frac{1}{4}$

$=$

(b) $\vec{F} = (2x^2+y^2)\vec{i} + (3y-4x)\vec{j}$
& around the triangle ABC whose vertices are $A(0,0,0)$, $B(2,0)$ & $C(2,1)$

Soln Here,

$$\vec{F} = (2x^2+y^2)\vec{i} + (3y-4x)\vec{j}$$
$$\vec{r} = x\vec{i} + y\vec{j}$$
$$d\vec{r} = \vec{i}\,dx + \vec{j}\,dy.$$



$$\vec{F}\cdot d\vec{r} = \left((2x^2+y^2)\vec{i} + (3y-4x)\vec{j}\right)\cdot\left(\vec{i}\,dx+\vec{j}\,dy\right)$$
$$= (2x^2+y^2)\,dx + (3y-4x)\,dy$$

Now, line integral

$y = 2x$
$dy = 2\,dx$

$$= \int_C \vec{F}\cdot d\vec{r}$$
$$= \int_{OB} (2x^2+y^2)\,dx + (3y-4x)\,dy + \int_{BC} (2x^2+y^2)\,dx + (3y-4x)$$
$$+ \int_{CO} (2x^2+y^2)\,dx + (3y-4x)\,dy$$
$$= \int_0^2 2x^2\,dx + \int_0^1 (3y-8)\,dy + \int_2^0 (2x^2+4x^2)\,dx + (3\cdot 2x-8)\,2dx$$

$$= \frac{2x^3}{3}\Big|_0^2 + \frac{3y^2}{2}\Big|_0^1 - 8y\Big|_0^1 + \frac{2x^3}{3}\Big|_0^2 + \frac{4x^3}{3}\Big|_2^0$$
$$+ \frac{6x^2}{2}\Big|_0^2 - 16x\Big|_0^2$$