[S.N.1] Let $A = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 3 & 1 \\ 1 & 1 & 4 \end{bmatrix}$. Find the maximum value of quadratic form x^TAx subject to the constraint $x^Tx = 1$ and find a unit vector at which this maximum value is attached.

Soln

Here given matrix is $A = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 3 & 1 \\ 1 & 1 & 4 \end{bmatrix}$ To find the maximum value of quardatic form x^TAx subject to constrant $x^Tx = I$. First find the eigenvalues associated with A. for this we have characteristic equation,

$$|A - dI| = 0$$

$$\left| \begin{bmatrix} 3 & 2 & 1 \\ 2 & 3 & 1 \\ 1 & 1 & 4 \end{bmatrix} - \begin{bmatrix} d & 0 & 0 \\ 0 & d & 0 \\ 0 & 0 & d \end{bmatrix} \right| = 0$$

$$\Rightarrow \begin{vmatrix} 3-d & 2 & 1 \\ 2 & 3-d & 1 \\ 1 & 1 & 4-d \end{vmatrix} = 0$$

$$\Rightarrow (3-d)\begin{vmatrix} 3-d & 1 \\ 1 & 4-d \end{vmatrix} - 2\begin{vmatrix} 2 & 1 \\ 1 & 4-d \end{vmatrix} + 1\begin{vmatrix} 2 & 3-d \\ 1 & 1 \end{vmatrix} = 0$$

$$\Rightarrow (3-d)[12-3d-4d+d^2] - 3+d-14+4d-1+d = 0$$

$$\Rightarrow 18 - 27d + 10d^2 - d^3 = 0$$

$$\Rightarrow d^3 - 10d^2 + 27d - 18 = 0$$

$$\Rightarrow d^2(d-1) - 11d(d-1) + 18(d-1) = 0$$

$$\Rightarrow (d^2 - 11d + 18)(d-1) = 0$$

$$\Rightarrow (d-2)(d-9)(d-1) = 0$$

$$\Rightarrow d = 2, 9, 1$$

Here maximum eigen value is 9 so maximum value associ

ated with quadratic form is $J_g = 3$.
Also to find the unit vector we need to find ~~unit vector~~ eigenvector associated with $\lambda = 9$.

So, $A \cdot x = \lambda x$

$$\begin{bmatrix} -6 & 2 & 1 \\ 2 & -6 & 1 \\ 1 & 1 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow$ $-6x_1 + 2x_2 + x_3 = 0$

$2x_1 - 6x_2 + x_3 = 0$

$x_1 + x_2 - 5x_3 = 0$

Here all the equations are different so by cross multiplication method, from first two equation,

$$\frac{x_1}{2+6} = \frac{x_2}{2+6} = \frac{x_3}{36-4}$$

$$\Rightarrow \frac{x_1}{8} = \frac{x_2}{8} = \frac{x_3}{32}$$

$\therefore x_1 = 1, x_2 = 1, x_3 = 4$

$\therefore$ unit vector ~~associated~~ at which this maximum value is attached is $\begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix}$.

[Q.N.2] Find maximum and minimum value of $Q(x) = 7x_1^2 + 4x_2^2 + 2x_3^2$ subject to Constraint $x^Tx = 1$

Soln Since x_2^2 and x_3^2 are non-negative note that,

we have

$$Q(x) = 7x_1^2 + 4x_2^2 + 2x_3^2 \le 7x_1^2 + 7x_2^2 + 7x_3^2$$
$$= 7(x_1^2 + x_2^2 + x_3^2)$$
$$= 7$$

Whenever $x_1^2 + x_2^2 + x_3^2 = x^Tx = 1$, so the maximum value of $Q(x)$ can not exceed 7. When $x = (1,0,0)$. Thus 7 is maximum value of $Q(x)$.

Similarly,

$Q(x) = 7x_1^2 + 4x_2^2 + 2x_3^2 \ge 3(x_1^2 + x_2^2 + x_3^2) \ge 3.1$

whenever $x_1^2 + x_2^2 + x_3^2 = x^Tx = 1$; $Q(x) = 3$ when $x = (0,0,1)$

So, 3 is minimum value of $Q(x)$.

[Q.N.3] Is $Q(x) = 3x_1^2 + 2x_2^2 + x_3^2 + 4x_1x_2 + 4x_2x_3$. positively definite?

Soln Here $Q(x) = 3x_1^2 + 2x_2^2 + x_3^2 + 4x_1x_2 + 4x_2x_3^2$. So, associated matrix corresponding to $Q(x)$ is

$$A = \begin{bmatrix} 3 & 2 & 0 \\ 2 & 2 & 2 \\ 0 & 2 & 1 \end{bmatrix}.$$

No deciede weather A is positive or not First find eigen values of A. For this we have characteristic equation,

$$|A - \lambda I| = 0$$

$$\left\| \begin{matrix} 3-\lambda & 2 & 0 \\ 2 & 2-\lambda & 2 \\ 0 & 2 & 1-\lambda \end{matrix} \right\| = 0$$

$$\Rightarrow (3-d)\begin{vmatrix} 2-d & 2 \\ 2 & 1-d \end{vmatrix} - 2\begin{vmatrix} 2 & 2 \\ 0 & 1-d \end{vmatrix} = 0$$

$$\Rightarrow (3-d)[(2-d)(1-d)-4] - 2(2-2d) = 0$$

$$\Rightarrow (3-d)(2-d)(1-d) - 12 + 4d - 4 + 4d = 0$$

$$\Rightarrow 6 - 6d - 3d + 3d^2 - 2d + 2d^2 + d^2 - d^3 - 12 + 4d - 4 + 4d = 0$$

$$\Rightarrow -10 - 3d + 6d^2 - d^3 = 0$$

$$\Rightarrow d^3 - 6d^2 + 3d + 10 = 0$$

$$\Rightarrow d^2(d+1) - 5d(d-1) - 2d + 10 = 0$$

$$\Rightarrow (d+1)(d^2 - 7d + 10) = 0$$

$$\Rightarrow (d+1)(d^2 - 5d - 2d + 10) = 0$$

$$\Rightarrow (d+1)(d-2)(d-5) = 0$$

$$\Rightarrow d = -1, d = 2, d = 5$$

Hence, ~~the~~ eigen values of A are not all positive. Hence we can say that it is not positively definite. Actually it is indefinite.

Q.N.4] Make a change of variable $x = py$ that transform the quardatic form $x_1^2 + 10x_1x_2 + x_2^2$ into quadratic form with no-cross product term. Give p and the new quardatic form.

Soln $Q(x) = x^T Ax$, with $A = \begin{bmatrix} 1 & 5 \\ 5 & 1 \end{bmatrix}$

Now, calculating the orthogonal diagonalization of A

Characteristic eqn of A is,

$$|A - dI| = 0$$

$$(1-d)(1-d) - 25 = 0$$

$$\Rightarrow 1 - 2d + d^2 - 25 = 0$$

$$\Rightarrow d^2 - 2d - 24 = 0$$

$$\Rightarrow d^2 - 6d + 4d - 24 = 0$$

$$(d-6)(d+4) = 0$$

$$\Rightarrow d = 6, -4$$

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Corresponding eigenvectors:

$\lambda = 6$

$$\begin{bmatrix} -5 & 5 \\ 5 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\Rightarrow -5x_1 + 5x_2 = 0$

$5x_1 - 5x_2 = 0$

$\Rightarrow x_1 = x_2 \qquad V_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

when $\lambda = -4$

$$\begin{bmatrix} 5 & 5 \\ 5 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\Rightarrow x_1 = -x_2 \quad \therefore V_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Now, the normalized eigenvectors are,

$\lambda = 6, : \quad \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\lambda = -4 : \quad \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

Orthogonal diagonalization

$A = P D P^T$

with

$$P = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 1/\sqrt{2} \end{bmatrix}$$

$$D = \begin{bmatrix} 6 & 0 \\ 0 & -4 \end{bmatrix}$$

changing variable $x = Py$

$x_1^2 + 10x_1x_2 + x_2^2 = x^T A x = y^T D y = (y_1, y_2) \begin{bmatrix} 6 & 0 \\ 0 & -4 \end{bmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$

$= 6y_1^2 - 4y_2^2$

[Q.N.5] Let A be the matrix of the quardatic form

$9x_1^2+7x_2^2+11x_3^2-8x_1x_2+8x_1x_3$

It can be shown that the eigen values of A are 3, 9, 15. Find an orthogonal matrix P such that the change of variable $x=Py$ transform x^TAx into a quardatic form with no crossproduct term. Given P and the new quardatic form.

Solution Matrix associated with quardatic form is,

$9x_1^2+7x_2^2+13x_3^2-8x_1x_2+8x_1x_3$ is

$$A=\begin{bmatrix} 9 & -4 & 4 \\ -4 & 7 & 0 \\ 4 & 0 & 13 \end{bmatrix}$$

From question eigen values associated with A are 3, 9, 15.

To find orthogonal matrix P such that the change of variable $x=Py$ transform x^TAx into a quardatic form we need to find eigenvectors for eigen values 3, 9, 15 and then normalized vectors. So,

when $\lambda=3$

$AV=\lambda V$

$$\Rightarrow \begin{bmatrix} 6 & -4 & 4 \\ -4 & 4 & 0 \\ 4 & 0 & 10 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \quad 6x_1-4x_2+4x_3=0$$
$$-4x_1+4x_2=0$$
$$4x_1+10x_3=0$$

all the equations are different so appeying Crossmultipeication method, from (I) and (II) eqn

$$\frac{x_1}{-2}=\frac{x_2}{-2}=\frac{x_3}{1} \quad \text{So, } V_1=\begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}$$

Corresponding normalized vector is $\frac{1}{3}\begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}$

Similarly for $d=9$

$$\begin{bmatrix} 0 & -4 & 4 \\ -4 & -2 & 0 \\ 4 & 0 & 9 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow -4x_2 + 4x_3 = 0$

$\Rightarrow -4x_1 - 2x_2 = 0$

$\Rightarrow 4x_1 - 9x_3 = 0$

all eqn are different so applying cross multipeication method on eqn (1) and (11)

$$\frac{x_1}{1} = \frac{x_2}{-2} = \frac{x_3}{-2}$$

$V_2 = \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}$ Correoponding normalized vector is $\frac{1}{3}\begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}$

also, for $d = 15$

$$\begin{bmatrix} -6 & -4 & 4 \\ -4 & -8 & 0 \\ 4 & 0 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow -6x_1 - 4x_2 + 4x_3 = 0$

$-4x_1 - 8x_2 = 0$

$4x_1 - 2x_3 = 0$

all equations are different so appiying crossmuctipeication method to solve eqn (1) and (11)

$$\frac{x_1}{32} = \frac{x_2}{-16} = \frac{x_3}{32}$$

$\therefore V_3 = \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}$

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Hence, matrix p is

$$\begin{bmatrix} \frac{2}{3} & -\frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} & \frac{2}{3} \end{bmatrix}$$ with orthogonal diagonalization,

$A = PDP^T$

where $D = \begin{bmatrix} 13 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 3 \end{bmatrix}$.

matrix p changes the matrix A ~~with~~ into quardatic Form with no quardatic term.

changing the variable $x = Py$

$$9x_1^2 + 7x_2^2 + 13x_3^2 - 8x_1x_2 + 8x_1x_3 = x^TAx = y^TDy$$

$$= (y_1\, y_2\, y_3) \begin{bmatrix} 13 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$$= (y_1\, y_2\, y_3) \begin{bmatrix} 13y_1 \\ 9y_2 \\ 3y_3 \end{bmatrix}$$

$$= 13y_1^2 + 9y_2^2 + 3y_3^2.$$

[Q.N.6] Classify the following quardatic forms Then make a change of variable, $x = Py$, that transforms the quardatic form into one with no cross-product term. write the new quardatic form construct P,

Soln (a) $4x_1^2 - 4x_1x_2 + 4x_2^2$

$\Rightarrow Q(x) = x^TAx$

matrix associated with quardatic form is,

$$A = \begin{bmatrix} 4 & -2 \\ -2 & 4 \end{bmatrix}$$

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Now,

$$|A - dI| = \begin{vmatrix} 4-d & -2 \\ -2 & 4-d \end{vmatrix} = (4-d)(4-d) - 4 = 0$$

$$\Rightarrow 16 - 4d - 4d + d^2 - 4 = 0$$

$$\Rightarrow d^2 - 8d + 12 = 0$$

$$\Rightarrow (d-6)(d-2) = 0 \Rightarrow d = 2, 6$$

Now, calculating crossponding eigenvectors

$d = 6$

$$\begin{bmatrix} -2 & -2 \\ -2 & -2 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow -2x_1 - 2x_2 = 0$$

$$-2x_1 - 2x_2 = 0$$

$$x_1 = -x_2$$

Corresponding eigenvector $V_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

when $d = 2$

$$\begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow 2x_1 - 2x_2 = 0$$

$$-2x_1 + 2x_2 = 0$$

$$\Rightarrow x_1 = x_2$$

eigenvector is, $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Now, normalized eigenvectors are,

when $d = 6$: $\frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

When $d = 2$ $\frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

orthogonal diagonalization,

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$A = PDP^T$

with

$$P = \begin{bmatrix} \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$D = \begin{bmatrix} 6 & 0 \\ 0 & 2 \end{bmatrix}$$

Changing variable $x = Py$

$$4x_1^2 - 4x_1x_2 + 4x_2^2 = x^TAx = y^TDy$$

$$= (y_1\ y_2)\begin{bmatrix} 6 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$$= 6y_1^2 + 2y_2^2$$

(b) $2x_1^2 + 6x_1x_2 - 6x_2^2$

Corresponding matrix associated with quardatic eqn is

$$A = \begin{bmatrix} 2 & 3 \\ 3 & -6 \end{bmatrix}$$

To find eigen vector we have characteristic eqn is,

$$|A - \lambda I| = 0$$

$$\Rightarrow \begin{vmatrix} 2-\lambda & 3 \\ 3 & -6-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (2-\lambda)(-6-\lambda) - 9 = 0$$

$$\Rightarrow -12 - 2\lambda + 6\lambda + \lambda^2 - 9 = 0$$

$$\Rightarrow \lambda^2 + 4\lambda - 21 = 0$$

$$\Rightarrow \lambda^2 + 7\lambda - 3\lambda - 21 = 0$$

$$\Rightarrow (\lambda + 7)(\lambda - 3) = 0$$

$$\lambda = 3, -7$$

Now, calculating the corresponding eigen vector,

$\lambda = 3$ $$\begin{bmatrix} -1 & 3 \\ 3 & -9 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$-x_1 + 3x_2 = 0$

$3x_1 - 9x_2 = 0$

$\Rightarrow x_1 = 3x_2$

Corresponding eigenvector is $\begin{pmatrix} 3 \\ 1 \end{pmatrix}$

When $\lambda = -7$,

$$\begin{bmatrix} 9 & 3 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$9x_1 + 3x_2 = 0$

$3x_1 + x_2 = 0$

$\Rightarrow x_2 = -3x_1$

Corresponding eigenvector is, $\begin{bmatrix} 1 \\ -3 \end{bmatrix}$

Now, Normalized eigenvectors are,

$\lambda = 3 : \frac{1}{\sqrt{10}} \begin{pmatrix} 3 \\ 1 \end{pmatrix}$

$\lambda = -7 : \frac{1}{\sqrt{10}} \begin{bmatrix} 1 \\ -3 \end{bmatrix}$

orthogonal diagonalization

$A = PDP^T$

with,

$$P = \begin{bmatrix} \frac{3}{\sqrt{10}} & \frac{1}{\sqrt{10}} \\ \frac{1}{\sqrt{10}} & -3/\sqrt{10} \end{bmatrix}, D = \begin{bmatrix} 3 & 0 \\ 0 & -7 \end{bmatrix}$$

Changing variable $x = Py$

$$2x_1^2 + 6x_1x_2 - 6x_2^2 = x^TAx = y^TDy$$

$$= (y_1 \; y_2) \begin{bmatrix} 3 & 0 \\ 0 & -7 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$$= 3y_1^2 - 7y_2^2$$

(c) $2x_1^2 - 4x_1x_2 - x_2^2$

$\Rightarrow Q(x) = x^T A x$

matrix associated with quardatic equation is,

$$A = \begin{bmatrix} 2 & -2 \\ -2 & -1 \end{bmatrix}$$

First find the eigenvalues of A, For this we have Characteristic equation

$$|A - \lambda I| = 0$$

$$\Rightarrow \begin{vmatrix} 2-\lambda & -2 \\ -2 & -1-\lambda \end{vmatrix} = 0$$

$\Rightarrow (2-\lambda)(-1-\lambda) - 4 = 0$

$\Rightarrow (-2 - 2\lambda + \lambda + \lambda^2 - 4) = 0$

$\Rightarrow \lambda^2 - \lambda - 6 = 0$

$\Rightarrow \lambda^2 - 3\lambda + 2\lambda - 6 = 0$

$\Rightarrow (\lambda - 3)(\lambda + 2) = 0$

$\Rightarrow \lambda = 3, -2$

Now, calculating the corresponding eigenvectors of λ

When, $\lambda = 3$; we have eigen eqn $Av = \lambda v$

$$\begin{bmatrix} -1 & -2 \\ -2 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\Rightarrow -x_1 - 2x_2 = 0$

$-2x_1 - 4x_2 = 0$

$\Rightarrow x_1 = 2x_2$

Corresponding eigenvector is,

$$\begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

when $\lambda = -2$

we have the eigen eqn $Av = \lambda v$

$$\begin{bmatrix} 4 & -2 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\Rightarrow 4x_1 - 2x_2 = 0$

$\Rightarrow -2x_1 + x_2 = 0$

$\Rightarrow x_2 = 2x_1$

Corresponding eigen vector is $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$

Now, normalized eigenvectos are,

$\lambda = 3 : \frac{1}{\sqrt{5}}\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ $\quad \lambda = -2 : \frac{1}{\sqrt{5}}\begin{pmatrix} 1 \\ 2 \end{pmatrix}$

Orthogonal diagonalization,

$A = PDP^T$

with,

$P = \begin{bmatrix} \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & 2/\sqrt{5} \end{bmatrix}$ and $D = \begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix}$

Changing the variable $x = Py$

$2x_1^2 - 4x_1x_2 - x_1^2 = x^TAx = y^TDy$

$= (y_1 \ y_2)\begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$

$= 3y_1^2 - 2y_2^2$

(d) $-x_1^2 - 2x_1x_2 - x_2^2$

$\Rightarrow Q(x) = x^TAx$

matrix associated with quardatic eqn $-x_1^2 - 2x_1x_2 - x_2^2$ is

$A = \begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

First find eigenvalues associated with A, For this we have

$|A - \lambda I| = 0$

$\Rightarrow \left|\begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right| = 0$

$$\begin{vmatrix} -1-d & -1 \\ -1 & -1-d \end{vmatrix} = 0$$

$\Rightarrow (-1-d)(-1-d) - 1 = 0$

$\Rightarrow 1 + d + d + d^2 - 1 = 0$

$\Rightarrow d^2 + 2d = 0$

$\Rightarrow d(d+2) = 0$

$\Rightarrow d = 0, d = -2$

Corresponding eigen vectors are,

When $d = 0$

$$\begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\Rightarrow -x_1 - x_2 = 0$

$-x_1 - x_2 = 0$

$x_1 = -x_2$

$$v_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

When $d = -2$

$$\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\Rightarrow x_1 - x_2 = 0$

$-x_1 + x_2 = 0$

$x_1 = x_2$

$$v_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Corresponding normalized eigenvectors are,

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$\lambda = 0 : \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$\lambda = -2 : \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

orthogonal diagonalization

$A = PDP^T$

with

$$P = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

changing the variable $x = Py$

$$-x_1^2 - 2x_1x_2 - x_2^2 = x^T A x = y^T D y$$

$$= (y_1\ y_2) \begin{bmatrix} 0 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$$= -y_2^2$$

(e) $x_1^2 - 6x_1x_2 + 9x_2^2$

$Q(x) = x^T A x$ Corresponding a matrix associated with quadratic eqn

$x_1^2 - 6x_1x_2 + 9x_2^2$

is $A = \begin{bmatrix} 1 & -3 \\ -3 & 9 \end{bmatrix}$

First find eigen values associated with A. For this we have characteristic eqn $|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 1-\lambda & -3 \\ -3 & 9-\lambda \end{vmatrix} = 0$$

$\Rightarrow (1-\lambda)(9-\lambda) - 9 = 0$

$\Rightarrow 9 - \lambda - 9\lambda + \lambda^2 - 9 = 0$

$\Rightarrow \lambda^2 - 10\lambda = 0$

$\Rightarrow \lambda(\lambda - 10) = 0$

$\Rightarrow \lambda = 0, \lambda = 10$

Now find corresponding eigenvectors of λ

when $\lambda = 0$, we have eigen eq^n $Av = \lambda v$

$$\begin{bmatrix} 1 & -3 \\ -8 & 9 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\Rightarrow$ $x_1 - 3x_2 = 0$

$\Rightarrow$ $-3x_1 - 9x_2 = 0$

$\Rightarrow$ $x_1 = 3x_2$

$\therefore v_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

Similarly when $\lambda = 10$

$$\begin{bmatrix} -9 & 3 \\ -3 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$-9x_1 + 3x_2 = 0$

$-3x_1 + x_2 = 0$

$\Rightarrow$ $-3x_1 + x_2 = 0$

$\Rightarrow$ $-3x_1 + x_2 = 0$

$\Rightarrow$ $x_2 = 3x_1$

$\Rightarrow$ $v_2 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

Hence ~~ortho~~ normalized eigenvectors are,

$\lambda = 0$: $\frac{1}{\sqrt{10}}\begin{bmatrix} 3 \\ 1 \end{bmatrix}$

$\lambda = 10$: $\frac{1}{\sqrt{10}}\begin{bmatrix} 1 \\ 3 \end{bmatrix}$

c) Orthogonal diagonalization,

$A = PDP^T$

with $P = \begin{bmatrix} 3/\sqrt{10} & 1/\sqrt{10} \\ 1/\sqrt{10} & 3/\sqrt{10} \end{bmatrix}$

Changing the variable $x = Py$

$x_1^2 - 6x_1x_2 + 9x_2^2 = x^T A x = y^T D y = (y_1 \; y_2) \begin{bmatrix} 10 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$

$= y_1^2$

[Q.N.7] What is the largest value of the quardatic form $5x_1^2 - 3x_2^2$ if $x^T x = 1$?

Soln Here, x_1^2 and x_2^2 both are positive,

also, $5x_1^2 \geq -3x_2^2$

So $Q(x) = 5x_1^2 - 3x_2^2 \geq 5x_1^2 + 5x_2^2$

$\geq 5(x_1^2 + x_2^2)$

≥ 5

Whenever $x_1^2 + x_2^2 + x_3^2 = x^T x = 1$. So largest value of $Q(x)$ can not exceed 5 when $x = (1,0,0)$. Thus 5 is largest value of $Q(x)$.