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Assignment 2

Tangent planes and linear approximation

1. Find the eqn of tangent plane to the given surface at the specified point.

(a) $z = 3y^2 - 2x^2 + x$, $(2, -1, -3)$

Soln

given $z = 3y^2 - 2x^2 + x$

let $F(x,y) = z = 3y^2 - 2x^2 + x$

$F_x(x,y) = -4x + 1$

$F_x(2,-1) = -4 \times 2 + 1 = -8 + 1 = -7$

$F_y(x,y) = 6y$

$F_y(2,-1) = -6$

So equation of tangent plane at point $(2,-1,-3)$ is,

$z + 3 = -7(x-2) - 6(y+1)$

$\Rightarrow z = -7x + 14 - 6y - 6 - 3$

$\Rightarrow z = -7x - 6y + 5$

(2) $z = 3(x-1)^2 + 2(y+3)^2 + 7$, $(2, -2, 12)$

Soln

Here, $z = 3(x-1)^2 + 2(y+3)^2 + 7$

Let $F(x,y) = 3(x-1)^2 + 2(y+3)^2 + 7$

$F_x(x,y) = 3 \cdot 2(x-1) \cdot 1$

$F_x(2,-2) = 6 \cdot 1 \cdot 1 = 6$

$F_y(x,y) = 4(y+3)\cdot 1$

$F_y(2,-2) = 4(-2+3) = 4\cdot 1 = 4$

$z_0 = 12$

Hence, Eqn of tangent plane at point $(2,-2,12)$ is,

$z - 12 = 6(x-2) + 4(y+2)$

$\Rightarrow z = 6x - 12 + 4y + 8 + 12$

$\Rightarrow z = 6x + 4y + 8$

(3) $z = \sqrt{xy}$, $(1,1,1)$

Soln Here, $z = \sqrt{xy}$

Let $F(x,y) = z = \sqrt{xy}$

$F_x(x,y) = \frac{1}{2}(xy)^{-1/2}\cdot y = \frac{y}{2\sqrt{xy}}$

$F_x(1,1) = \frac{1}{2}$

$F_y(x,y) = \frac{1}{2}(xy)^{-1}\cdot x = \frac{x}{2\sqrt{xy}}$

$F_y(1,1) = \frac{1}{2}$

$z_0 = 1$

Eqn of tangent plane at point $(1,1,1)$ is,

$z - 1 = \frac{1}{2}(x-1) + \frac{1}{2}(y-1)$

$\Rightarrow z = \frac{1}{2}x - \frac{1}{2} + \frac{1}{2}y - \frac{1}{2} + 1$

$\Rightarrow 2z = x + y$

$\Rightarrow x + y - 2z = 0$

[11] Explain why the function is differential at the given point. Then find the ~~part~~ linearization $L(x,y)$ of the function at point

(a) $f(x,y) = x\sqrt{y}$ $(1,4)$

Soln

Here given $f(x,y) = x\sqrt{y}$

$f_x(x,y) = \sqrt{y}$ $f_x(1,4) = 2$

$f_y(x,y) = \dfrac{x}{2\sqrt{y}}$ $f_y(1,4) = \dfrac{1}{2\times2} = \dfrac{1}{4}$

Both f_x and f_y are continious function, so f is differentiable at given point.

The linearization is,

$L(x,y) = f(1,4) + f_x(1,4)(x-1) + f_y(1,4)(y-4)$

$= 2 + 2(x-1) + \frac{1}{4}(y-4)$

$= 2 + 2x - 2 + \frac{1}{4}y - 1$

$= 2x + \frac{1}{4}y - 1$

(12) $F(x,y) = x^3y^4$ $(1,1)$

Here given $F(x,y) = x^3y^4$

$F_x(x,y) = 3x^2y^4$ $F_x(1,1) = 3$

$F_y(x,y) = 4x^3y^3$ $F_y(1,1) = 4$

$F(1,1) = 1$

Here both $F_x(x,y)$, $F_y(x,y)$ are continious function because both are polynomial function. Hence

Function is differential at given point.

Now, linearization $L(x,y)$ is given by

$$L(x,y) = f(1,1) + f_x(1,1)(x-1) + f_y(1,1)(y-1)$$
$$= 1 + 3(x-1) + 4(y-1)$$
$$= 1 + 3x - 3 + 4y - 4$$
$$L(x,y) = 3x + 4y - 6$$

(13) $f(x,y) = \dfrac{x}{x+y}$ $(2,1)$

Soln Here,

$$f(x,y) = \frac{x}{x+y}$$

$$f_x(x,y) = \frac{x + (x+y)}{(x+y)^2} = \frac{2x+y}{(x+y)^2}$$

$$f_x(2,1) = \frac{2+(1+2)}{(1+2)^2} = \frac{5}{9}$$

$$f_y(x,y) = \frac{y + (x+y)}{(x+y)^2} = \frac{x+2y}{(x+y)^2}$$

$$f_y(2,1) = \frac{1+(1+2)}{(1+2)^2} = \frac{4}{9}$$

Here, both $f_x(x,y)$ and $f_y(x,y)$ are rational function we know that every rational functions are continious. Hence $f(x,y)$ is continious and differential at given point.

Linearization $L(x,y)$ is given by,

$$L(x,y) = f(2,1) + f_x(2,1)(x-2) + f_y(2,1)(y-1)$$
$$L(x,y) = \frac{2}{3} + \frac{5}{9}(x-2) + \frac{4}{9}(y-1)$$

[14] $f(x,y) = \sqrt{x + e^{4y}}$ $(3,0)$

Soln

Here, $f(x,y) = \sqrt{x + e^{4y}}$

$$fx(x,y) = \frac{1}{2}(x + e^{4y})^{-1/2} \cdot 1 = \frac{1}{2\sqrt{x+e^{4y}}}$$

$$fx(3,0) = \frac{1}{2\sqrt{3+1}} = \frac{1}{4}.$$

$$fy(x,y) = \frac{1}{2\sqrt{x+e^{4y}}} \cdot e^{4y} \cdot 4$$

$$fy(3,0) = \frac{1}{4} \cdot 4 = 1$$

Here, both $fx(x,y)$ and $fy(x,y)$ are rational function. We know that every rational functions are Continious hence given function is differential at given point.

Now Linearization is,

$$L(x,y) = f(3,0) + fx(3,0)(x-3) + fy(3,0)(y-0)$$

$$= 2 + \frac{1}{4}(x-3) + 1y$$

$$= 2 + \frac{1}{4}(x-3) + y$$

[17] Given that f is a differential function with $f(2,5) = 6$, $fx(2,5) = 1$ and $fy(2,5) = -1$ Use linear approximate to ~~find~~ ~~f(x)~~ estimate $f(2.2, 4.9)$

Soln

f is differential function with $f(2,5) = 6$, $f_x(2,5) = 1$ and $f_y(2,5) = -1$

$$\boxed{\begin{aligned} L(x,y) &= F(2,5) + f_x(2,5)(x) + f_y(2,5)(y) \\ &= 6 + x - y \\ L(2.2, 4.9) &= 6 + 2.2 - 4.9 = 3.3 \end{aligned}}$$

$$L(x,y) = F(2,5) + F_x(2,5)(x-2) + F_y(2,5)(y-5)$$

$$L(x,y) = 6 + 1(x-2) - 1(y-5) = 6 + x - 2 - y + 5$$

$$= x - y + 9$$

$$L(2.2, 4.9) = 2.2 - 4.9 + 9 = 6.3$$

(Q3) Find the differential of the function,

$$z = x^3 \ln(y^2)$$

Soln Here $z = f(x,y) = x^3 \ln y^2$

$$dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy$$

$$= 3x^2 \ln(y^2) + \frac{x^3}{y^2} \cdot 2y$$

$$= 3x^2 \ln(y^2) + \frac{2x^3}{y}$$

Q4] ~~u = e^s~~ $u = e^{-t} \sin(s + 2t)$

Here, $u = F(s,t) = e^{-t} \sin(s + 2t)$

$$du = \frac{\partial u}{\partial s} ds + \frac{\partial u}{\partial t} dt$$

$$= e^{-t} \cos(s+2t) + e^{-t} \cos(s+2t) \cdot 2 + e^{-t} \cdot (-1) \sin(s+2t$$

$= e^{-t}\cos(s+2t) + 2e^{-t}\cos(s+2t) - e^{-t}\sin(s+2t)$

$= 3e^{-t}\cos(s+2t) - e^{-t}\sin(s+2t)$

$= e^{-t}[3\cos(2+2t) - \sin(s+2t)]$

[25] $m = p^5q^5$

Soln Here, $m = F(p,q) = p^5q^5$

$$dm = \frac{\partial m}{\partial p}.dp + \frac{\partial m}{\partial q}dq$$

$= 5p^4q^5 + 5p^5q^4$

$= 5p^4q^4(q+p)$

$= 5p^4q^4(p+q)$

(30) If $z = x^2 - xy + 3y^2$ and (x,y) changes from $(3,-1)$ to $(2.96, -0.95)$, compare the value of Δz and dz.

Soln Here $z = x^2 - xy + 3y^2$

Let $F(x,y) = z = x^2 - xy + 3y^2$

$F_x(x,y) = 2x - y$

$F_y(x,y) = -y + 6y$

Now, $dz = (2x-y)dx + (\cancel{3} - y + 6y)dy$

Also, (x,y) changes from $(3,-1)$ to $(2.96, -0.95)$

so, $\Delta x = 2.96 - 3 = -0.040$

$\Delta y = -0.95 + 1 = 0.050$

$\therefore dz = (2\times3+1) - 0.040 + (1 - 6y)\, 0.050$

$= -0.28$ ...

$$\Delta z = f(x+\Delta x, y+\Delta y) - f(x,y)$$
$$= (2.96)^2 - (2.96)\times(-0.95) + 3\times(-0.95)^2 - (3)^2 + 3\cdot(-1) - 3\cdot(-1)^2$$

$$= 8.76 + 2.812 + 3\times 0.9025 - 9 - 3 - 3$$
$$= 11.572 + 2.7075 - 15$$
$$= 14.279 - 15$$
$$= -0.720$$
$$\therefore \Delta z \simeq dz.$$

[31] The length and width of a rectangle are measured as 30cm and 24cm, respectively, with an error in measurement of at most 0.1cm in each. Use differentials to estimate the maximum error in the calculated area of the rectangle.

Soln

We know that area of rectangle is,

$A = l \times b$ where L is length of rectangle and b is width of rectangle.

$$dA = A_l\,dl + A_b\,db$$

where $l = 30$ cm and $b = 24$ cm

$$\Delta A = db = 0.1$$

To determine the maximum error in the calculated area of the rectangle,

$$dl = db = 0.1 \text{ cm}$$

$$dA = 24\times 0.1 + 30\times 0.1$$
$$= 2.40 + 3.0$$
$$= 5.40$$

[2] Chain rule

Use chain Rule to find dz/dt or dw/dt.

1. $z = x^2+y^2+xy$ $x = \sin t$, $y = e^t$

Soln

Here $z = x^2+y^2+xy$, $x = \sin t$, $y = e^t$

$$\frac{\partial z}{\partial x} = 2x+y$$

$$\frac{\partial z}{\partial y} = 2y+x$$

$$\frac{\partial x}{\partial t} = \cos t$$

$$\frac{\partial y}{\partial t} = e^t$$

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

$$= (2x+y)\cos t + (2y+x)e^t$$ #

[2] $z = \cos(x+4y)$, $x = 5t^4$, $y = \frac{1}{t}$

Soln

$z = \cos(x+4y)$, $x = 5t^4$, $y = 1/t$

$$\frac{\partial z}{\partial x} = \frac{\partial \cos(x+4y)}{\partial(x+4y)} \cdot \frac{\partial(x+4y)}{\partial x}$$

$$= -\sin(x+4y) \cdot 1$$

$$\frac{\partial z}{\partial y} = \frac{\partial \cos(x+4y)}{\partial(x+4y)} \cdot \frac{\partial(x+4y)}{\partial(y)}$$

$$= -\sin(x+4y) \cdot 4$$

$$\frac{\partial x}{\partial t} = \frac{\partial}{\partial t} 5t^4 = 20t^3$$

$$\frac{\partial y}{\partial t} = \frac{\partial}{\partial t}\left(-\frac{1}{t}\right) = -1\,(-t)^{-2} = \frac{-1}{t^2}$$

$$\therefore \frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} = -\sin(x+4y)\cdot 20t^3 + 4\sin(x+4y)\cdot -\frac{1}{t^2}$$

$$= -20\sin(x+4y)t^3 - 4\sin(x+4y)\frac{1}{t^2}$$

(5) $z = \sqrt{1+x^2+y^2} \quad x = \ln t,\ y = \cos t$

soln

$$z = \sqrt{1+x^2+y^2},\ x = \ln t,\ y = \cos t$$

$$\frac{\partial z}{\partial x} = \frac{\partial \sqrt{1+x^2+y^2}}{\partial(1+x^2+y^2)} \cdot \frac{d(1+x^2+y^2)}{dx}$$

$$= \frac{1}{2}(1+x^2+y^2)^{-1/2} \cdot 2x$$

$$= \frac{x}{\sqrt{1+x^2+y^2}}$$

$$\frac{\partial z}{\partial y} = \frac{\partial(\sqrt{1+x^2+y^2})}{\partial(1+x^2+y^2)} \cdot \frac{d(1+x^2+y^2)}{dy}$$

$$= \frac{1}{2}(1+x^2+y^2)^{-1/2}\cdot 2y$$

$$= \frac{y}{\sqrt{1+x^2+y^2}}$$

$$\frac{\partial x}{\partial t} = \frac{d\ln t}{dt} = \frac{1}{t}$$

$$\frac{\partial y}{\partial t} = \frac{d\cos t}{dt} = -\sin t$$

$$\therefore \frac{dz}{dt} = \left(\frac{x}{\sqrt{1+x^2+y^2}}\right)\frac{1}{t} - \left(\frac{y}{\sqrt{1+x^2+y^2}}\right)\sin t$$

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Use the Chain Rule to find $\partial z/\partial s$ and $\partial z/\partial t$.

[7] $z = x^2y^3$ $\quad x = s\cos t,\ y = s\sin t$

Soln

$z = x^2y^3 \quad x = s\cos t \quad y = s\sin t$

$\frac{\partial z}{\partial x} = 2xy^3 \quad \frac{\partial x}{\partial s} = \cos t \quad \frac{\partial x}{\partial t} = -s\sin t$

$\frac{\partial z}{\partial y} = 3x^2y^2 \quad \frac{\partial y}{\partial s} = \sin t \quad \frac{\partial y}{\partial t} = s\cos t$

$\therefore \frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}$

$= 2xy^3\cos t + 3x^2y^2\sin t$

$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$

$= 2xy^3 \cdot -s\sin t + 3x^2y^2 \cdot s\cos t$

$= -2xy^3s\sin t + 3x^2y^2s\cos t$

(8) $z = \arcsin(x-y),\ x = s^2+t^2,\ y = 1-2st$

Soln. Here,

$z = \arcsin(x-y),\ x = s^2+t^2,\ y = 1-2st$

$\frac{\partial z}{\partial x} = \frac{1}{\sqrt{1-(x-y)^2}} \cdot 1$

$\frac{\partial z}{\partial y} = \frac{1}{\sqrt{1-(x-y)^2}} \cdot (-1) = \frac{-1}{\sqrt{1-(x-y)^2}}$

$\frac{\partial x}{\partial t} = 2t$

$\frac{\partial x}{\partial s} = 2s$

$\frac{\partial y}{\partial t} = -2s$

$\frac{\partial y}{\partial s} = -2t$

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$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial t}$$

$$= \frac{1}{\sqrt{1-(x-y)^2}}\, 2t \,\cancel{\phi} + \frac{2s}{\sqrt{1-(x-y)^2}}$$

$$= \frac{2\ 1}{\sqrt{1-(x-y)^2}}\left[t+s\right]$$

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}$$

$$= \frac{2s}{\sqrt{1-(x-y)^2}} + \frac{-1}{\sqrt{1-(x-y)^2}}\cdot(-2t)$$

$$= \frac{2s}{\sqrt{1-(x-y)^2}} + \frac{2t}{\sqrt{1-(x-y)^2}} = \frac{2}{\sqrt{1-(x-y)^2}}\left[s+t\right]$$

[10] $z = e^{x+2y}$, $x = s/t$, $y = t/s$

Soln

Here $z = e^{x+2y}$, $x = s/t$, $y = t/s$

$$\frac{\partial z}{\partial x} = e^{x+2y}$$

$$\frac{\partial z}{\partial y} = e^{x+2y}\cdot 2$$

$$\frac{\partial x}{\partial t} = \frac{-s}{t^2} \qquad \frac{\partial x}{\partial s} = \frac{1}{t}$$

$$\frac{\partial y}{\partial s} = \frac{-t}{s^2} \qquad \frac{\partial y}{\partial t} = \frac{1}{s}$$

Now,

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$

$$= e^{x+2y}\cdot\frac{-s}{t^2} + 2e^{x+2y}\cdot\frac{1}{s}$$

$$= e^{x+2y}\left(\frac{s}{t^2} + \frac{2}{s}\right)$$

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$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\,\frac{\partial y}{\partial s}$$

$$= e^{x+2y}\cdot\frac{1}{t} + 2e^{x+2y}\,\frac{-t}{s^2}$$

$$= e^{x+2y}\left(\frac{1}{t} - \frac{t}{s^2}\right)$$

[21] Use the chain Rule to find the Indicated partial derivatives,

(a) $z = x^2 + xy^3 \quad x = uv^2 + w^3,\; y = u + ve^w$

$\frac{\partial z}{\partial u}, \frac{\partial z}{\partial v}, \frac{\partial z}{\partial w}$ when $u = 2,\; v = 1,\; w = 0$

Soln

Here $z = x^2 + xy^3,\; x = uv^2 + w^3,\; y = u + ve^w$

$$\frac{\partial z}{\partial u} = \frac{d}{du}\left[(uv^2+w^3)^2 + (uv^2+w^3)(u+ve^w)\right]$$

$$= \frac{d}{du}(uv^2+w^3)^2 + \frac{d}{du}(uv^2+w^3)(u+ve^w)$$

$$= 2(uv^2+w^3)\cdot v^2 + (uv^2+w^3)\cdot 1 + (u+ve^w)\cdot v$$

$$= 2(uv^2+w^3)v^2 + (uv^2+w^3)(u+ve^w)v^2$$

$$\frac{\partial z}{\partial v} = \frac{d}{dv}\left\{(uv^2+w^3)^2 + (uv^2+w^3)(u+ve^w)\right\}$$

$$= \frac{d}{dv}(uv^2+w^3)^2 + \frac{d}{dv}(uv^2+w^3)(u+ve^w)$$

$$= 2(uv^2+w^3)\,2v + (uv^2+w^3)\cdot e^w + (u+ve^w)\cdot u$$

$$= 2(uv^2+w^3)\,2v + (uv^2+w^3)e^w + (u+ve^w$$

$$\frac{\partial z}{\partial w} = \frac{d}{dw}\left\{(uv^2+w^3)^2 + (uv^2+w^3)(u+ve^w)\right\}$$

$$= \frac{d}{dw}(uv^2+w^3)^2 + \frac{d}{dw}(uv^2+w^3)(u+ve^w)$$

$= 2(uv^2+w^3)^2 \cdot 3w^2 + (uv^2+w^3) \cdot ve^w \cdot 2 + (u+ve^w)$

$= 6(uv^2+w^3)^2 \cdot w^2 + (uv^2+w^3) ve^w + 3(u+ve^w) w^2$

when $u=2, v=1, w=0$

$\frac{\partial z}{\partial u} = 2(2+0)\cdot 1^2 + (2\times1+0)(2+1)\cdot 1$

$= 4 + 6 = 10$

$\frac{\partial z}{\partial v} = 4(2\cdot1+0) + (2\cdot1+0)1 + (2+1)2$

$= 8 + 2 + 6 = 16$

$\frac{\partial z}{\partial w} = 6(2+0)\cdot 0 + (2+0)1 + 3(2+1)\cdot 0$

$= 2$

(22) $u = \sqrt{r^2+s^2}$, $r = y + x\cos t$, $s = x + y\sin t$

$\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial t}$ when $x=1, y=2, t=0$

Soln - Here, $u = \sqrt{r^2+s^2}$, $r = y + x\cos t$, $s = x + y\sin t$

$u = \sqrt{(y+x\cos t)^2 + (x+y\sin t)^2}$

$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\left(\sqrt{(y+x\cos t)^2 + (x+y\sin t)^2}\right)$

$= \frac{1}{2}\left[(y+x\cos t)^2 + (x+y\sin t)^2\right]^{-1/2} \{2(y+x\cos t)\cdot \cos t + 2(x+y\sin t)\cdot 1\}$

$= \frac{1}{2\sqrt{(y+x\cos t)^2+(x+y\sin t)^2}} \cdot 2 \cdot [\cos t(y+x\cos t) + (x+y\sin t)]$

$= \frac{\cos t\,(y+x\cos t) + (x+y\sin t)}{\sqrt{(y+x\cos t)^2 + (x+y\sin t)^2}}$

$\left.\frac{\partial u}{\partial x}\right|_{x=1, y=2, t=0} = \frac{1(2+1) + (1+2)}{\sqrt{(2+1)^2 + (1+0)^2}} = \frac{3+3}{\sqrt{9+1}} = \frac{6}{\sqrt{10}}$

$$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}\left(\sqrt{(y+x\cos t)^2+(x+y\sin t)^2}\right)$$

$$= \frac{1}{2}\left[(y+x\cos t)^2+(x+y\sin t)^2\right]^{-1/2}\{2(y+x\cos t)\cdot 1 + 2(x+y\sin t)\cdot\sin t\}$$

$$= \frac{1\cdot 2\{(y+x\cos t)+\sin t(x+y\sin t)\}}{2\sqrt{(y+x\cos t)^2+(x+y\sin t)^2}}$$

$$= \frac{(y+x\cos t)+\sin t(x+y\sin t)}{\sqrt{(y+x\cos t)^2+(x+y\sin t)^2}}$$

$$\left.\frac{\partial u}{\partial y}\right|_{x=1,y=2,t=0} = \frac{(2+1)+0}{\sqrt{(2+1)^2+(1)^2}} = \frac{3}{\sqrt{10}}$$

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial t}\left(\sqrt{(y+x\cos t)^2+(x+y\sin t)^2}\right)$$

$$= \frac{1}{2}\left[(y+x\cos t)^2+(x+y\sin t)^2\right]^{-1/2}\{2(y+x\cos t)\cdot -x\sin t + 2(x+y\sin t)\cdot y\cos t\}$$

$$= \frac{2\{(y+x\cos t)(-x)\sin t+(x+y\sin t)\,y\cos t\}}{2\sqrt{(y+x\cos t)^2+(x+y\sin t)^2}}$$

$$= \frac{-x(y+x\cos t)\sin t+y(x+y\sin t)\cos t}{\sqrt{(y+x\cos t)^2+(x+y\sin t)^2}}$$

$$\left.\frac{\partial u}{\partial t}\right|_{x=1,y=2,t=0}$$

$$= \frac{1(2+1)\sin 0^\circ + 2(1+2\sin 0^\circ)\cos 0^\circ}{\sqrt{(2+1)^2+(1+0)^2}}$$

$$= \frac{2}{\sqrt{10}}$$

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(33) The temperature at a point (x,y) is $T(x,y)$ measured in degree Celscious. A bug crawls so that it's position after t seconds is given by $x = \sqrt{1+t}$, $y = 2 + \frac{1}{3}t$, where x and y are measured in centimeters. The temperature function satisfies $T_x(2,3) = 4$ and $T_y(2,3) = 3$. How fast is the temperature rising on the bug's path after 3 second?

Soln Here given temperature at a point (x,y) is $T(x,y)$. And,

$$x = \sqrt{1+t}, \quad y = 2 + \frac{1}{3}t$$

$$T_x(2,3) = 4 \qquad T_y(2,3) = 3$$

$$\frac{dT}{dt} = \frac{\partial T}{\partial x}\cdot\frac{\partial x}{\partial t} + \frac{\partial T}{\partial y}\cdot\frac{\partial y}{\partial t} \quad \text{— (i)}$$

$$T_x = \frac{\partial T}{\partial x}, \quad T_y = \frac{\partial T}{\partial y}$$

$$\frac{\partial x}{\partial t} = \frac{1}{2}(1+t)^{-1/2} \qquad \frac{\partial y}{\partial t} = \frac{1}{3}\cdot 1$$

From (i)

$$\frac{dT}{dt} = 4\cdot\frac{1}{2\sqrt{1+t}} + 3\cdot\frac{1}{3}$$

$$\frac{dT}{dt} = \frac{4}{2\sqrt{1+t}} + 1 = \frac{2}{\sqrt{1+t}} + 1$$

when $t = 3$ sec

$$\left.\frac{dT}{dt}\right|_{@t=3} = \frac{2}{\sqrt{1+3}} + 1 = \frac{2}{2} + 1 = 1 + 1 = 2\,^{\circ}C$$

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[34] Wheat production w in a given year depends on the average temperature T and the annual rainfall R. Scientists estimate that the average temperature is rising at a rate of $0.15°C$/year and rainfall is decreasing at a rate of 0.1 cm/year. They also estimate that, at current production levels, $\partial w/\partial T = -2$ and $\partial w/\partial R = 8$.

(a) What is the significance of the signs of these partial derivatives?

(b) Estimate the current rate of change of wheat production dw/dt.

Soln $w = w(T, R)$

(a) $\frac{\partial w}{\partial T} = -2$ Since it is $(-)$ sign which indicate decreasing production with with increasing temperature

and

$\frac{\partial w}{\partial T} = 8$ $(+)$ sign indicates increasing production with increasing rainfall

(b) $\frac{dw}{dt} = \frac{\partial w}{\partial T}\frac{dT}{dt} + \frac{\partial w}{\partial R}\frac{dR}{dt}$ and it is given that

$\frac{\partial w}{\partial t} = -2$, $\frac{\partial w}{\partial R} = 8$, $\frac{dT}{dt} = 0.15$ $\frac{dR}{dt} = -0.1$

$$\frac{dw}{dt} = \frac{\partial w}{\partial t}\frac{\partial t}{\partial T} + \frac{\partial w}{\partial R}\cdot\frac{dR}{\partial t}$$

$$= (-2)\cdot 0.15 + 8 \times 0.1$$

$$= -0.30 + 0.80$$

$$= 0.50$$

Hence current rate of change of wheat production

$\frac{dw}{dt} = 0.50$

(36) The radius of a right circular cone is increasing at a rate of 1.8 in/s while its height is decreasing at a rate of 2.5 in/s. At what rate is the volume of the cone changing when the radius is 120 in. and the height is 140 in?

Soln The volume is given by

$$V = \frac{1}{3}\pi r^2 h$$

Where r is the radius and h the height. These are functions of the time t since they are changing over time we want to find $\frac{dv}{dt}$. The chain rules gives,

$$\frac{dv}{dt} = \frac{\partial v}{\partial r}\frac{dr}{dt} + \frac{\partial v}{\partial h}\frac{dh}{dt}$$

$$\frac{dv}{dt} = \frac{2}{3}\pi r h \frac{dr}{dt} + \frac{1}{3}\pi r^2 \frac{dh}{dt}$$

We are given $\frac{dr}{dt} = 1.8$ and $\frac{dh}{dt} = -2.5$, so

$$\frac{dv}{dt} = \frac{2}{3} \times 1.8\pi r h + \frac{1}{3} \times (-2.5)\pi r^2$$

When $r = 120$ and $h = 140$ we find $\frac{dv}{dt} = 8160\pi \, in^3/s$.

[40] A manufacturer has modeled its yearly production function P (the value of its entire production in million of dollars) as a Cobb-Douglas function

$$P(L, K) = 1.47 L^{0.65} K^{0.35}$$

where L is the number of labour hours (in thousand) k is the invested capital (in millions of dollars) Suppose that when L = 30 and k = 8, the labour force is decreasing at a rate of 2000 labour hours per year and capital is increasing at a rate of \$ 500,000 per year. Find the rate of change of production.

Soln Here given function is,

$$P(L,k) = 1.47\, L^{0.65}\, k^{0.35} \quad \text{---} (*)$$

Where L is the number of labour hours & k is the invested capital.

Also, $\frac{\partial L}{\partial t} = -2000$

$$\frac{\partial k}{\partial t} = \frac{500,000}{12\times 60\times \cancel{365}} = 694.44$$

From (*)

$$\frac{\partial p}{\partial L} = \cancel{0.4}\ 1.47\times 0.65\, L^{0.65-1} k^{0.3}$$

$$= 0.955\, L^{-0.35} \cdot k^{0.35}$$

$$\frac{\partial P}{\partial L}\Big|_{30,8} = 0.955(30)^{-0.35} \cdot (8)^{0.35}$$

$$= 0.955\times 0.30 = 0.290\times 2.07$$

$$= 0.60045$$

$$\frac{\partial P}{\partial k} = \cancel{1.47}$$

$$= 1.47\, L^{0.65}(0.35)\, k^{-0.65}$$

$$\frac{\partial p}{\partial k}\Big|_{30,8} = (1.47)\times(0.35)\,(30)^{0.65}\,(8)^{-0.65}$$

$$= 0.5145\times 9.1228\times 0.2588$$

$$= 1.2147$$

Now

$$\frac{dp}{dt} = \frac{\partial p}{\partial L}\cdot\frac{\partial L}{\partial t} + \frac{\partial p}{\partial K}\cdot\frac{\partial K}{\partial t}$$

$$= (0.60045)\cdot(-2000) + (1.2147)\times 694.44$$

$$= -1200.90 + 843.53$$

$$= -357.36 \text{ ft.}$$

[41] One side of a traingle is increasing at a rate of 3 cm/s and a second side ~~of~~ is decreasing at a rate of 2cm/s. If the area of the traingle remains constant, at what rate does the angle between the side changes when the first side is 20cm long, the second side is 30cm, and the angle is $\pi/6$?

Soln Given,

One side of traingle is increasing at a rate of 3cm/s.

Second side ~~of~~ is decreasing at a rate of 2cm/s

Length of Fist side is 20cm

length of second side is 30cm

Angle is $\pi/6$

area is constant

If x and y are the side of traingle and θ ~~be~~ is the angle between them, then area of traingle is

$$A = \frac{1}{2}xy\sin\theta$$

Area A is function of sides and angle (x, y, θ) and sides and angle are functions of time t.

$$\frac{dx}{dt} = 3 \quad \& \quad \frac{dy}{dt} = -2$$

Since Area is constand

$$\frac{dA}{dt} = 0$$

The rate of change of angle between two sides of triangle is $\frac{dA}{dt}$

By using chain rule

$$\frac{dA}{dt} = \frac{\partial A}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial A}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial A}{\partial \theta}\frac{\partial \theta}{\partial t}$$

$$0 = \frac{\partial}{\partial x}\left(\frac{1}{2}xy\sin\theta\right)3 + \frac{\partial}{\partial y}\left(\frac{1}{2}xy\sin\theta\right)(-2) + \frac{\partial}{\partial \theta}\left(\frac{1}{2}x \ldots\right)$$

By differenciating,

$$0 = \frac{3}{2}y\sin\theta - 2\,\tfrac{1}{2}x\sin\theta + \tfrac{1}{2}xy\cos\theta\cdot\frac{\partial\theta}{\partial t}$$

For $x = 20,\ y = 30,\ \theta = \pi/6$

$$0 = \frac{3}{2}\cdot 30\sin\pi/6 - 20\cdot\sin\pi/6 + \tfrac{1}{2}\,30\cdot 30\cos\pi/6\cdot\frac{d\theta}{dt}$$

Therefore, by calculating

$$0 = \frac{45}{2} - 10 + 150\sqrt{3}\cdot\frac{d\theta}{dt}$$

$$\frac{d\theta}{dt} = -\frac{\frac{25}{2}}{150\sqrt{3}} = -\frac{1}{12\sqrt{3}}$$

$\therefore$ rate of change of angle between two sides $-\frac{1}{12\sqrt{3}}$ rad/s.

[44] If $u = f(x,y)$, where $x = e^s\cos t$ and $y = e^s\sin t$ show that

$$\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 = e^{-2s}\left[\left(\frac{\partial u}{\partial s}\right)^2 + \left(\frac{\partial u}{\partial t}\right)^2\right]$$

Soln Here $u = f(x,y)$

$x = e^s\cos t \quad y = e^s\sin t.$

$$\cancel{\frac{\partial u}{\partial x} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial s}}$$

$$\frac{\partial u}{\partial s} = e^s \sin t \frac{\partial u}{\partial x} + e^s \cos t \frac{\partial u}{\partial y}$$

$$\frac{\partial u}{\partial t} = e^s \cos t \frac{\partial u}{\partial x} + - e^s \sin t \frac{\partial u}{\partial y}$$

$$\frac{\partial u}{\partial s} = e^s\left(\sin t \frac{\partial u}{\partial x} + \cos t \frac{\partial u}{\partial y}\right)$$

$$\frac{\partial u}{\partial t} = e^s\left(\cos t \frac{\partial u}{\partial x} - \sin t \frac{\partial u}{\partial y}\right)$$

$$\left(\frac{\partial^2 u}{\partial s}\right)^2 + \left(\frac{\partial u}{\partial t}\right)^2 = e^{2s}\left[\left(\sin t \frac{\partial u}{\partial x} + \cos t \frac{\partial u}{\partial y}\right)^2 + \left(\cos t \frac{\partial u}{\partial x} - \sin t \frac{\partial u}{\partial y}\right)^2\right]$$

$$= e^{2s}\left[\sin^2 t\left(\frac{\partial u}{\partial x}\right)^2 + \cancel{2 \sin t \cos t \frac{\partial u}{\partial x}\cdot\frac{\partial u}{\partial y}} + \cos^2 t\left(\frac{\partial u}{\partial y}\right)^2 + \cos t^2\left(\frac{\partial u}{\partial x}\right)^2 - \cancel{2 \sin t \cos t \frac{\partial u}{\partial x}\frac{\partial u}{\partial y}} + \sin^2 t\left(\frac{\partial u}{\partial y}\right)^2\right]$$

$$= e^{2s}\left[(\sin^2 t + \cos^2 t)\left(\frac{\partial^2 u}{\partial x}\right)^2 + (\sin^2 t + \cos^2 t)\left(\frac{\partial u}{\partial y}\right)\right.$$

$$= e^{2s}\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2\right]$$

$$\Rightarrow \left(\frac{\partial^2 u}{\partial s}\right)^2 + \left(\frac{\partial u}{\partial t}\right)^2 = e^{2s}\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2\right]$$

$$\Rightarrow \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 = e^{-2s}\left[\left(\frac{\partial u}{\partial s}\right)^2 + \left(\frac{\partial u}{\partial t}\right)^2\right]$$ proved

[47] Show that any function of the form

$z = f(x+at) + g(x-at)$ is a solution of the wave eqn

$$\frac{\partial^2 z}{\partial t^2} = a^2 \frac{\partial^2 z}{\partial x^2}$$

[Hint: Let $u = x+at$, $v = x-at$]

Soln Here

$$z = f(x+at) + g(x-at)$$

$$u = x+at, \quad v = x-at$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial t} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial t} = af'(u) - ag'(v) = a(f'(u) - g'(v))$$

$$\frac{\partial^2 z}{\partial t^2} = a^2(f''(u) + g''(u))$$

Similarly

$$\frac{\partial^2 z}{\partial x^2} = f''(u) + g''(v)$$

then,

$$\frac{\partial^2 z}{\partial t^2} - a^2 \frac{\partial^2 z}{\partial x^2} = a^2(f''(u) + g''(v)) - a^2(f''(u) + g''(v)) = 0$$

$$\therefore \frac{\partial^2 z}{\partial t^2} = a^2 \frac{\partial^2 z}{\partial x^2}$$

[50] If $z = f(x,y)$ where $x = r\cos\theta$ and $y = r\sin\theta$

Find

(a) $\frac{\partial z}{\partial r}$, (b) $\frac{\partial z}{\partial \theta}$ (c) $\frac{\partial^2 z}{\partial r \partial \theta}$

Soln $z = f(x,y)$ $x = r\cos\theta$, $y = r\sin\theta$

(a) $$\frac{\partial z}{\partial r} = \cos\theta \frac{\partial z}{\partial x} + \sin\theta \frac{\partial z}{\partial y}$$

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(b)

$$\frac{\partial z}{\partial \theta} = -r\sin\theta \frac{\partial z}{\partial x} + r\cos\theta \frac{\partial z}{\partial y}$$

(c) $\frac{\partial^2 z}{\partial r \partial \theta}$

$$\frac{\partial z}{\partial \theta} = -r\sin\theta \frac{\partial z}{\partial x} + r\cos\theta \frac{\partial z}{\partial y}$$

$$\frac{\partial^2 z}{\partial r \partial \theta} = -\sin\theta \frac{\partial^2 z}{\partial x^2} + \cos\theta \frac{\partial^2 z}{\partial y^2}$$