Fuzzy Logic Theory

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Outline

fuzzy set theory allows for partial membership in a set

- Fuzzy set theory

deals with the representation and manipulation of uncertainty and vagueness in data and decision-making.

  - Basic concepts in fuzzy sets

The membership function maps each element in the universe of discourse to a value between 0 and 1, indicating the degree to which the element belongs to the fuzzy set.

  - Operations on fuzzy sets

The universe of discourse (also known as the domain) is the set of all possible values that an element can take.

Fuzzy set theory also includes operations such as union, intersection, and complement,

- Fuzzy rules and reasoning

Fuzzy rules are typically expressed in the form of "if-then" statements, where the "if" part specifies the conditions or inputs, and the "then" part specifies the output or action. and the rules are defined by expert knowledge or heuristics.

- Fuzzy inference systems
  - Mamdani fuzzy systems

A fuzzy inference system (FIS) is a type of artificial intelligence system that uses fuzzy logic to model and control complex systems.

  - Sugeno fuzzy systems

Classical crisp sets are sets where each element belongs to the set or does not belong to the set, and there is no ambiguity
support: it is the set of elements that are at least partially in the fuzzy set.
The fuzzy complement is defined by the membership function, which assigns a degree of non-membership to each element.

Primer of Fuzzy Sets

Fuzzy union is defined as the maximum degree of membership of each element in the universe of discourse, while fuzzy intersection is defined as the minimum degree of membership of each element.

- From classical crisp sets to fuzzy sets
  - Change of range of membership functions
- Notions of fuzzy sets
  - Representation, support, α cut, convexity ...
- Fuzzy set operations
  - Intersection, union, complement, etc.
  - Share *almost* the same mathematics foundamentals with those of classical sets.

Basics of Fuzzy Sets

- Classical sets
  - *two-value* membership functions
  - An element either 'belongs to' or 'does not belong to' a given classical set (sharp boundary).
- Fuzzy sets
  - Extend the degrees of membership: $\{0,1\} \rightarrow [0,1]$
  - An element *partially* 'belongs to' and *partially* 'does not belong to' a given fuzzy set.
  - Smooth and Gradual boundary
- Membership functions
  - Assignment of membership functions is *subjective*.

Father of Fuzzy Logic: Lotfi A. Zadeh

Lotfi Asker Zadeh was a Professor in the Graduate School, Computer Science Division, Department of EECS, University of California, Berkeley. In addition, he was the Director of BISC (Berkeley Initiative in Soft Computing).

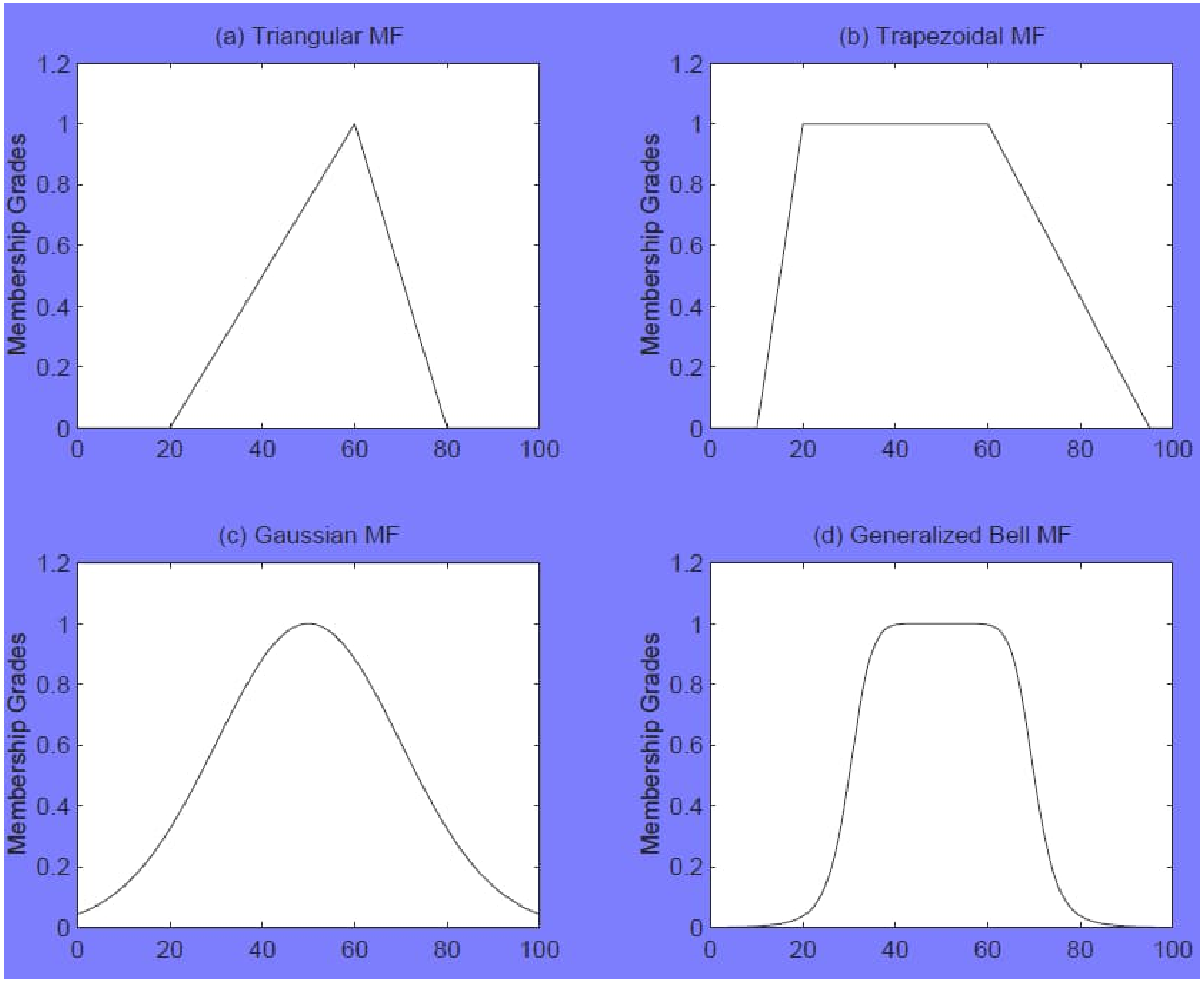
http://en.wikipedia.org/wiki/Lotfi_Asker_Zadeh
http://www-bisc.cs.berkeley.edu/

Lotfi Zadeh's Work

- "Fuzzy sets". *Information and Control*. 1965; 8: 338–353.
- "A fuzzy-set-theoretical interpretation of linguistic hedges". *Journal of Cybernetics* 1972; 2: 4–34.
- "Outline of a new approach to the analysis of complex systems and decision processes". *IEEE Trans. Systems, Man and Cybernetics*, 1973; 3: 28–44.
- "The concept of a linguistic variable and its application to approximate reasoning", I-III, *Information Sciences* 8 (1975) 199–251, 301–357; 9 (1976) 43–80.
- "From computing with numbers to computing with words — from manipulation of measurements to manipulation of perceptions". *International Journal of Applied Math and Computer Science*, pp. 307–324, vol. 12, no. 3, 2002.

Four Typical Membership Functions

Triangular is typically used to model fuzzy concepts that have a well-defined central value, such as temperature or speed.



Trapezoidal is typically used to model fuzzy concepts that have a range of values that are equally acceptable, such as age or income.

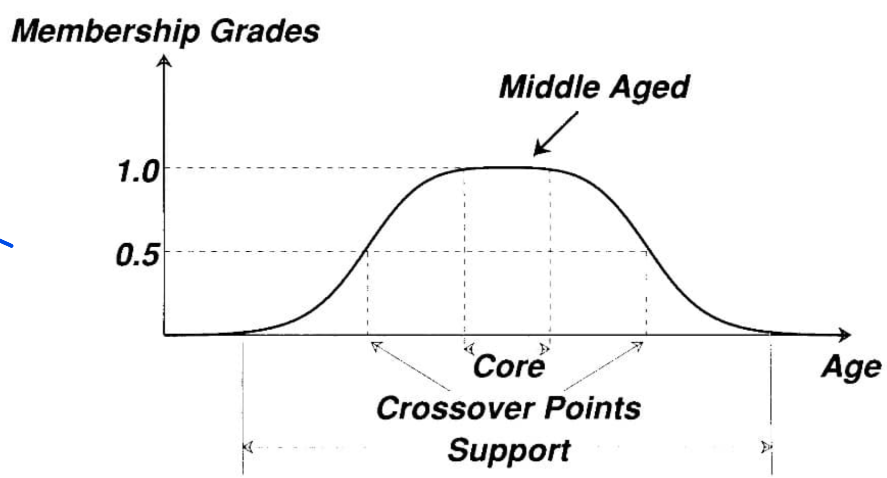
It is typically used to model fuzzy concepts that have a normal distribution, such as height or weight.

It is typically used to model fuzzy concepts that have a threshold or tipping point,

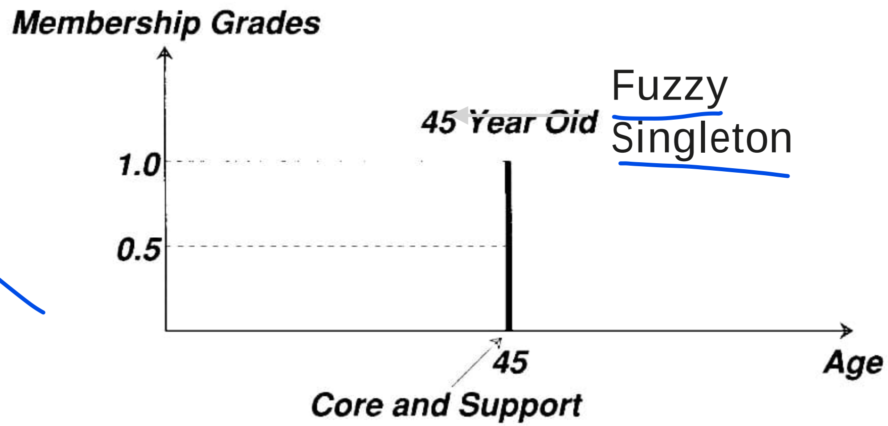
Definitions in Fuzzy Sets

- Supports $\mu_A(x) > 0$ The support of a membership function is the range of values for which the degree of membership is non-zero
- Core $\mu_A(x) = 1$ The core of a membership function is the range of values for which the degree of membership is maximum
- Crossover points $\mu_A(x) = 0.5$ The crossover points of a membership function are the points where the degree of membership is exactly 0.5.
- Height $\max\left[\mu_A(x)\right]$ the maximum degree of membership
- Fuzzy singleton A fuzzy singleton is a membership function that has maximum membership for a single point, and zero membership for all other points
  - A fuzzy singleton is a *crisp* set
  - A fuzzy set that has only one element x_0
  - $\mu_A(x_0) = 1$

Definitions in Fuzzy Sets



(a)

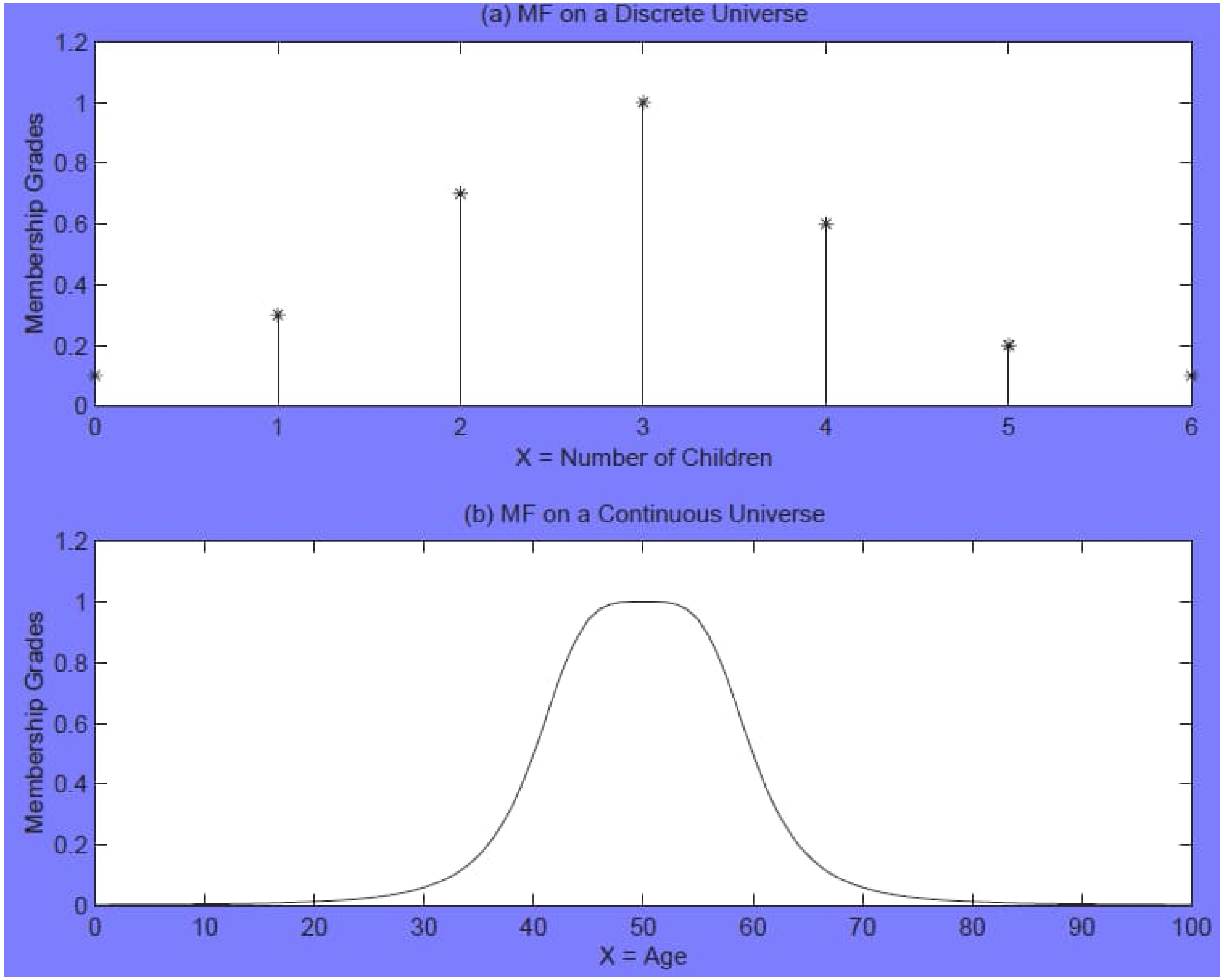


(b)

Representations of Membership Functions

- Discrete Fuzzy Sets
  - Universe of discourse is *discrete*
- Continuous Fuzzy Sets
  - Universe of discourse is *continuous*

Discrete and Continuous Fuzzy Sets



In fuzzy set theory, an alpha cut is a way to convert a fuzzy set into a crisp set. It involves setting a threshold value (called the alpha level) and selecting all the elements in the universe of discourse that have a degree of membership in the fuzzy set that is greater than or equal to this threshold.

α Cuts of Fuzzy Sets

- α (a constant) cut of a fuzzy set is a *crisp* set

$$A_a = \{x \mid \mu_A(x) \geq \alpha\}$$

- Strong α cuts

$$A'_a = \{x \mid \mu_A(x) > \alpha\}$$

$$\text{support}\ (A) = A'_0$$

$$\text{core}\ (A) = A_1$$

It involves breaking down a complex problem or proposition into simpler parts that can be more easily analyzed and evaluated.

Resolution Principle

- A fuzzy membership function can be expressed in terms of the characteristic functions of its α cuts.

$$\mu_A(x) = \max_{\alpha} \min[\alpha, A_\alpha]$$

- An example of decomposing a fuzzy set with discrete objects:

MF deerleri

$$A = \left\{ \frac{0.1}{1}, \frac{0.2}{2}, \frac{0.3}{3} \right\}$$

indeksler

Resolution Principle

$$A_{0.1} = \left\{ \frac{1}{1} + \frac{1}{2} + \frac{1}{3} \right\}$$

$$A_{0.2} = \left\{ \frac{1}{2} + \frac{1}{3} \right\}$$

$$A_{0.3} = \left\{ \frac{1}{3} \right\}$$

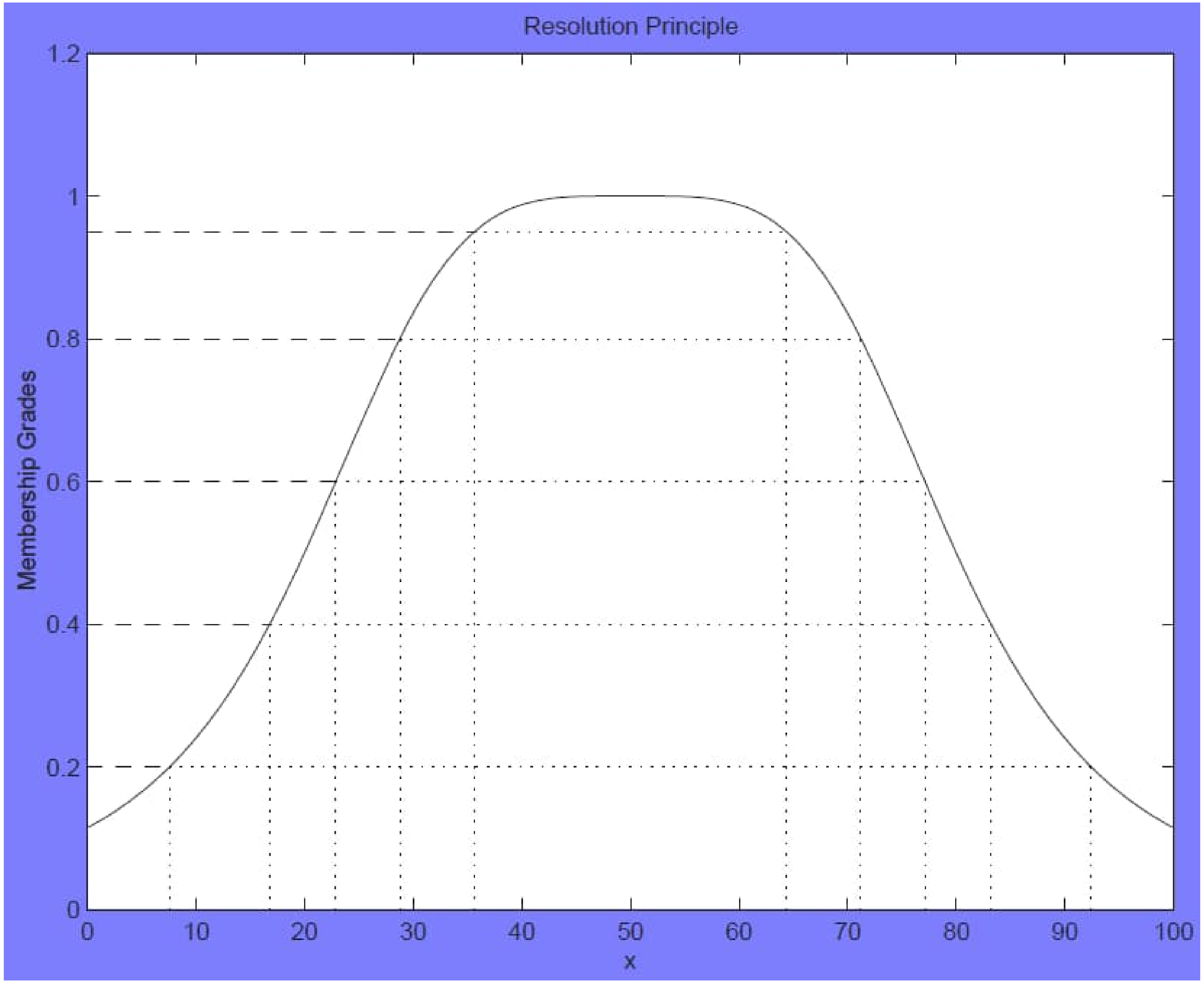
$$0.1 A_{0.1} = \left\{ \frac{0.1}{1} + \frac{0.1}{2} + \frac{0.1}{3} \right\}$$

$$0.2 A_{0.2} = \left\{ \frac{0.2}{2} + \frac{0.2}{3} \right\}$$

$$0.3 A_{0.3} = \left\{ \frac{0.3}{3} \right\}$$

$$A = \left\{ \frac{0.1}{1} + \frac{\max(0.1, 0.2)}{2} + \frac{\max(0.1, 0.2, 0.3)}{3} \right\} = \left\{ \frac{0.1}{1} + \frac{0.2}{2} + \frac{0.3}{3} \right\}$$

Resolution Principle



The extension principle is a fundamental concept in fuzzy set theory that allows us to extend a fuzzy set defined on a certain universe of discourse to a larger universe of discourse. The extension principle enables us to apply fuzzy set theory to situations where the universe of discourse is not completely known or is continuously changing.

Extension Principle

- Extension from *crisp* domains of mathematical expressions to *fuzzy* domains.
- Suppose *f* is a function mapping from X to Y, and A is a fuzzy set:

$$A = \frac{\mu_A(x_1)}{x_1} + \frac{\mu_A(x_2)}{x_2} + \cdots + \frac{\mu_A(x_n)}{x_n}$$

- We have image of A under *f*:

$$B = f(A) = \frac{\mu_A(x_1)}{f(x_1)} + \frac{\mu_A(x_2)}{f(x_2)} + \cdots + \frac{\mu_A(x_n)}{f(x_n)}$$

An Example of Extension Principle

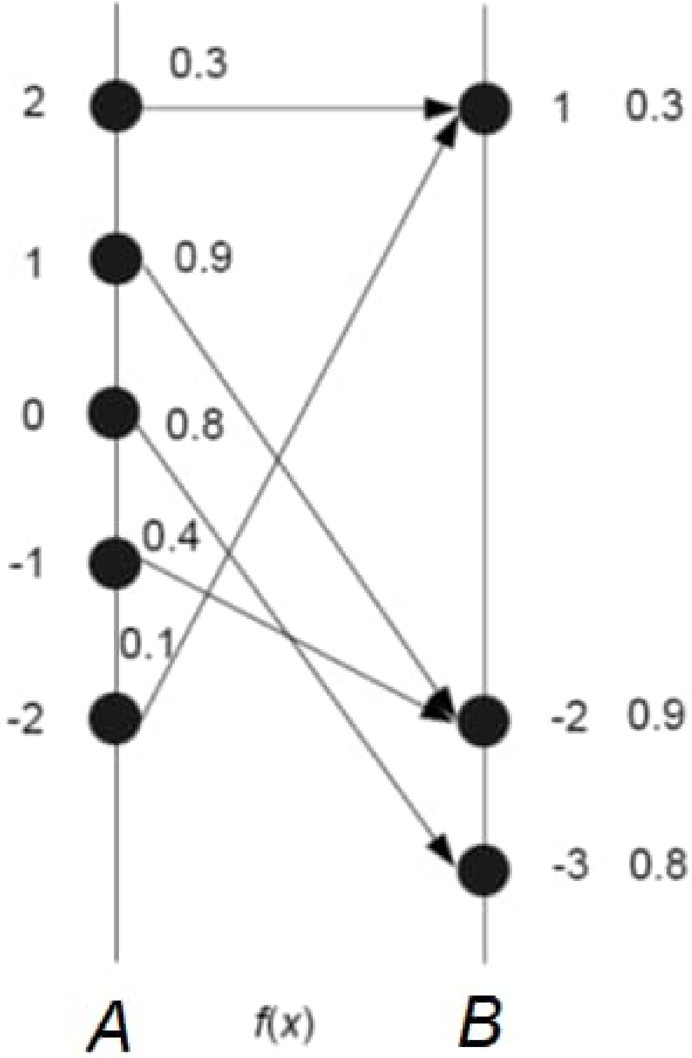
- An example of extension principle

$$A = \frac{0.1}{-2} + \frac{0.4}{-1} + \frac{0.8}{0} + \frac{0.9}{1} + \frac{0.3}{2}$$

$$f(x) = x^2 - 3$$

$$\begin{aligned} B &= \frac{0.1}{1} + \frac{0.4}{-2} + \frac{0.8}{-3} + \frac{0.9}{-2} + \frac{0.3}{1} \\ &= \frac{0.8}{-3} + \frac{(0.4 \vee 0.9)}{-2} + \frac{(0.1 \vee 0.3)}{1} \\ &= \frac{0.8}{-3} + \frac{0.9}{-2} + \frac{0.3}{1} \end{aligned}$$

An Example of Extension Principle



you take two points in the set and draw a straight line between them, then the degree of membership of any point on that line must be greater than or equal to the minimum of the degrees of membership of those two points.

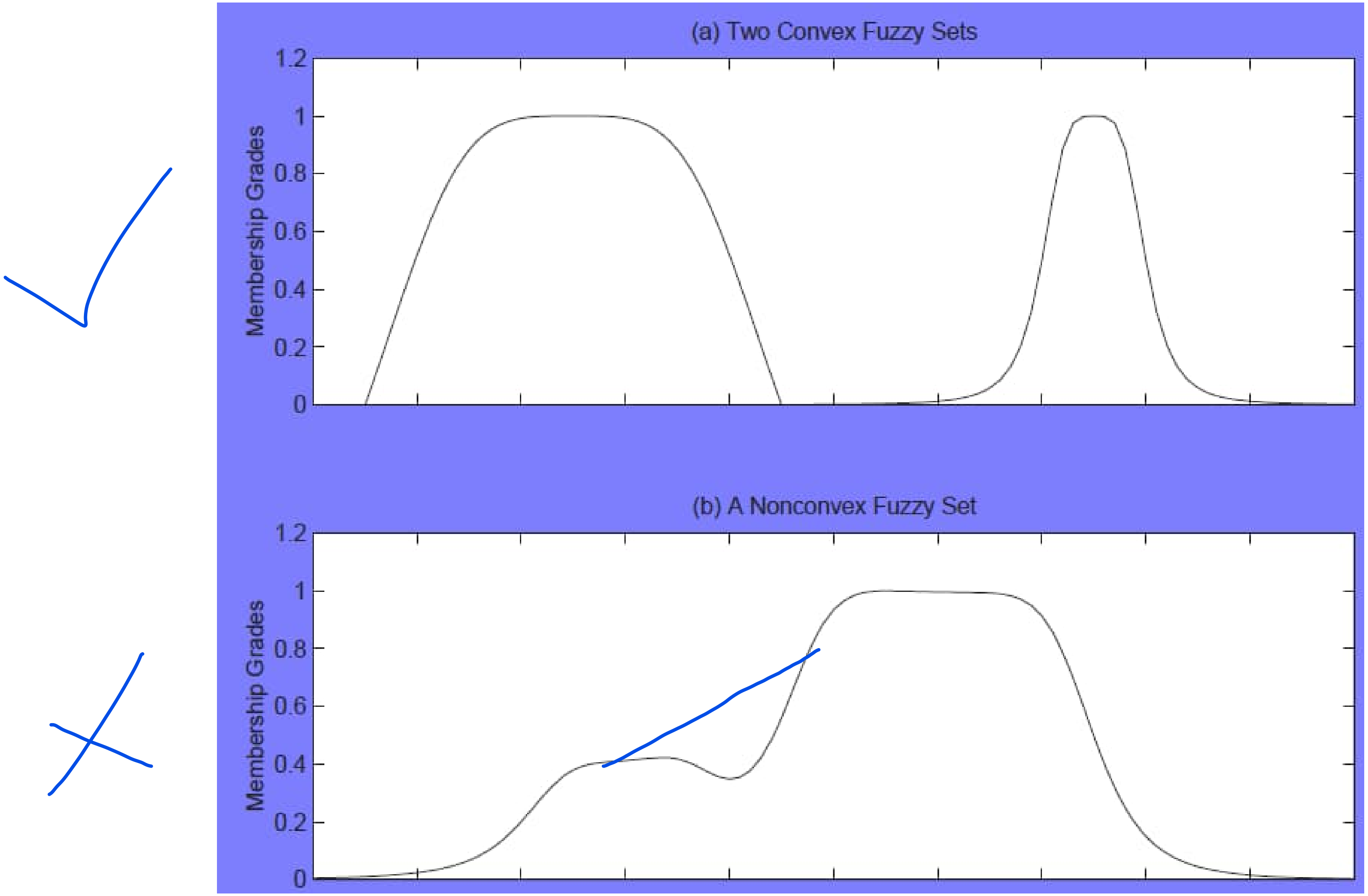
Convex Fuzzy Sets

- Convexity of fuzzy sets:

$$\mu_A(\lambda x_1 + (1-\lambda)x_2) \geq \min\{\mu_A(x_1), \mu_A(x_2)\}$$

where x_1 and x_2 are two arbitrary elements in A, and λ is an arbitrary constant.

Convex and Nonconvex Fuzzy Membership Functions



Operations on Fuzzy Sets

- Intersection

$$\mu_C(x) = \min(\mu_A(x), \mu_B(x)) = \mu_A(x) \wedge \mu_B(x)$$

- Union

$$\mu_C(x) = \max(\mu_A(x), \mu_B(x)) = \mu_A(x) \vee \mu_B(x)$$

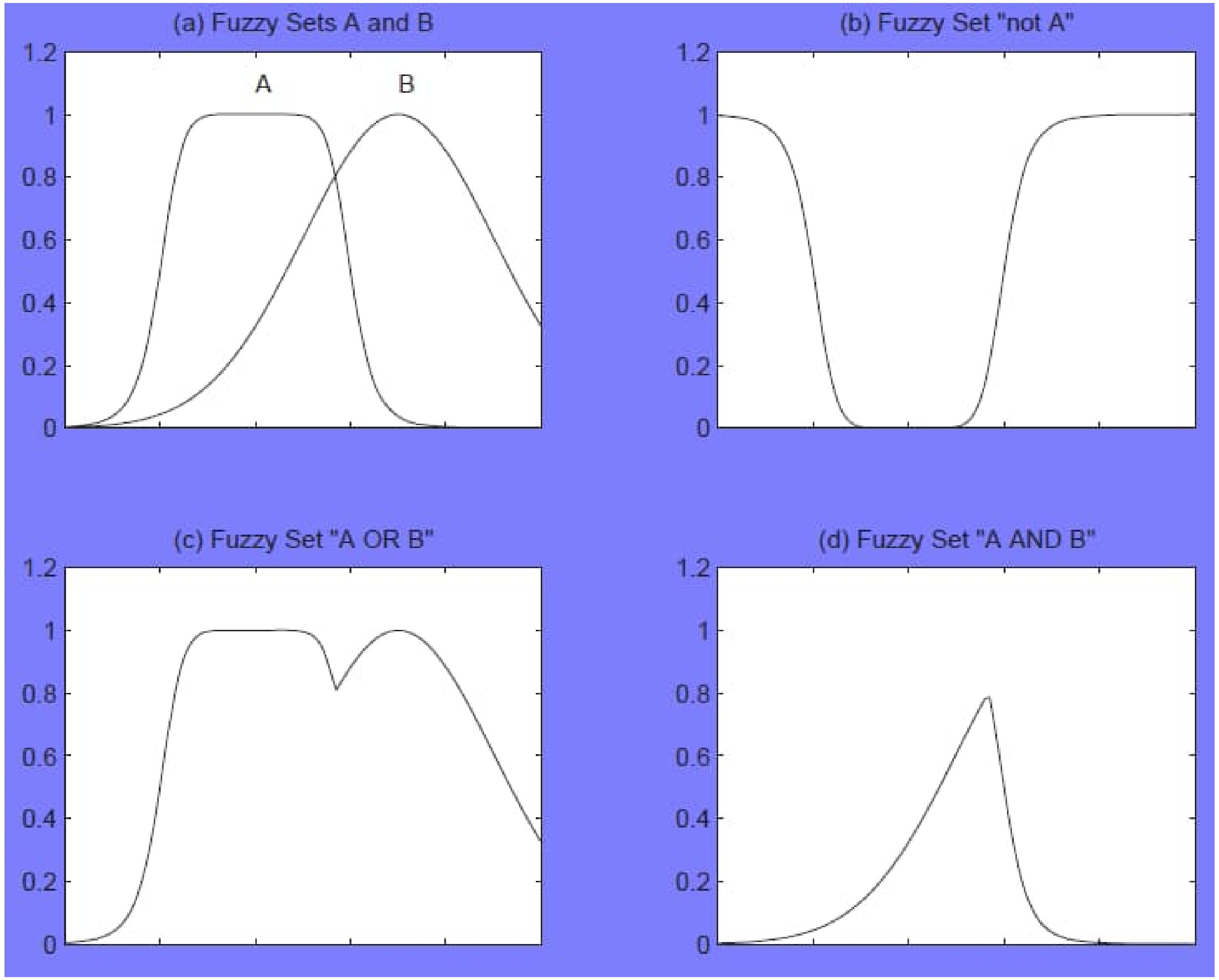
- Complement

$$\mu_{\overline{A}}(x) = 1 - \mu_A(x)$$

Operations on Fuzzy Sets

μ_A and μ_B

$\mu_{\overline{A}}$



$\mu_A \cup \mu_B$

$\mu_A \cap \mu_B$

Operations on Fuzzy Sets

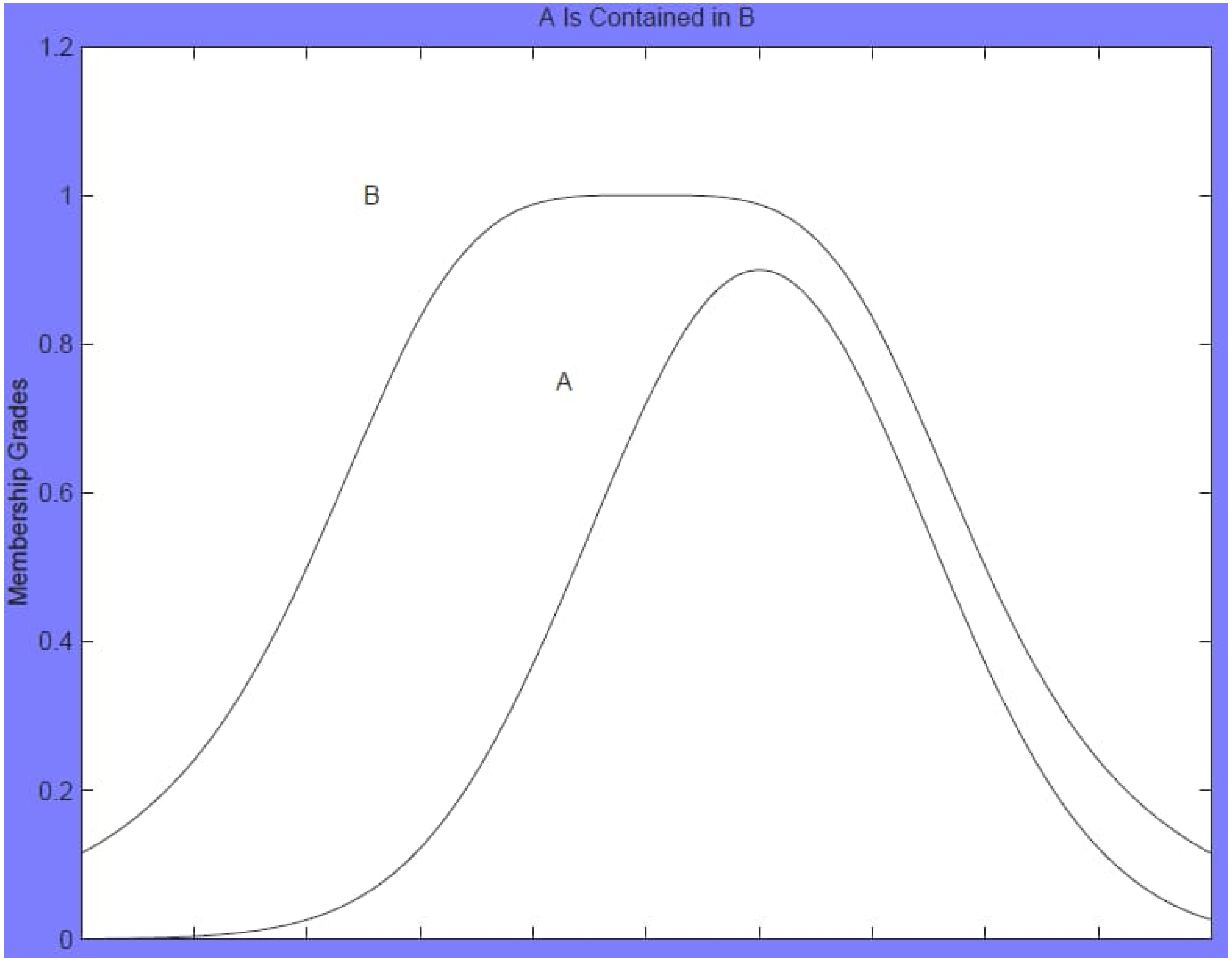
- Subset

$$A \subseteq B \Longleftrightarrow \mu_A(x) \leq \mu_B(x)$$

- Equality

$$A \subseteq B \quad \text{and} \quad B \subseteq A$$

Concept of $A \subseteq B$

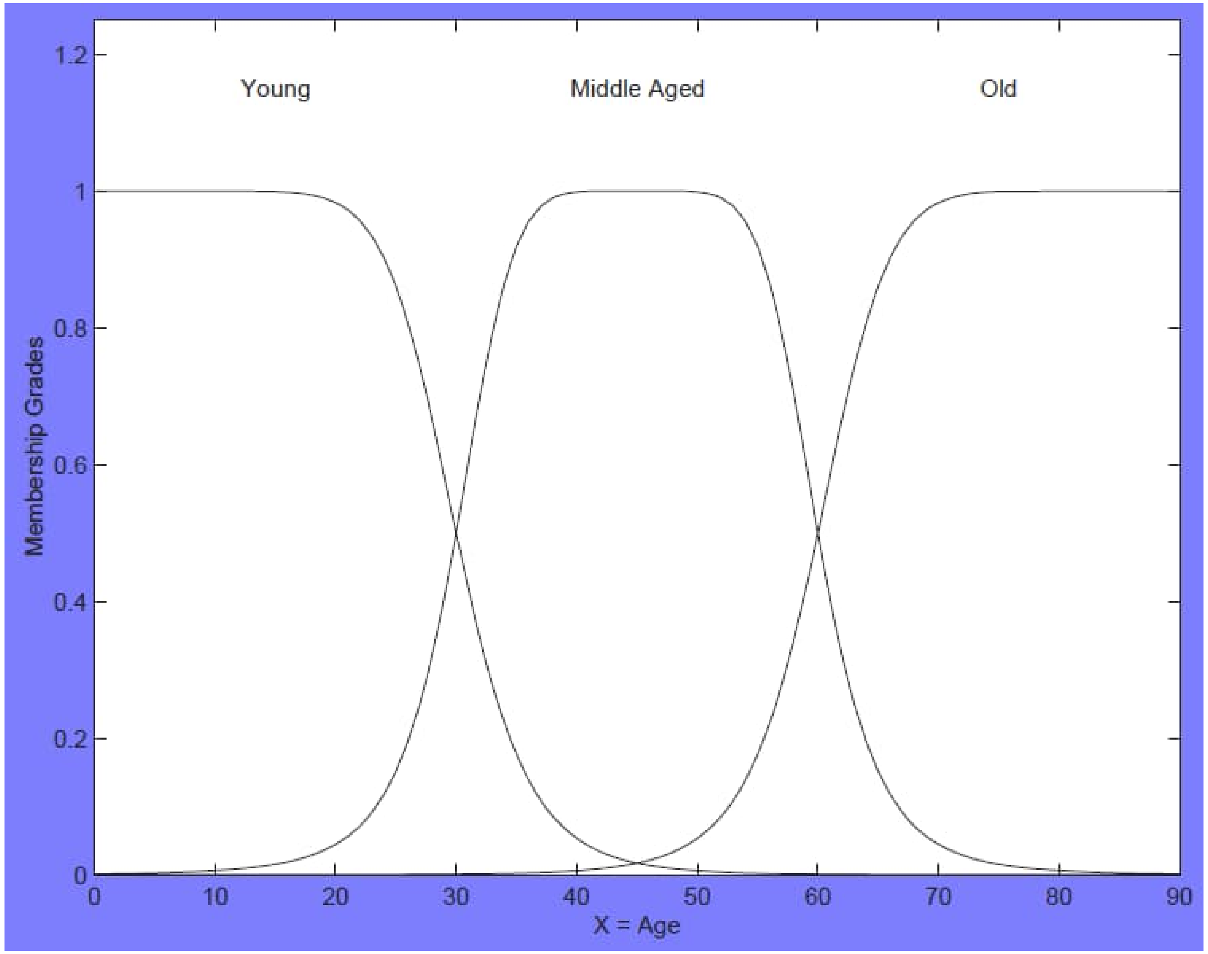


Linguistic Variables

- Modeling of human thinking
  - Numerical values are not sufficient.
  - Linguistic variables exist in real world.
    - e.g, chatting with a stranger on the phone:
      - Estimation of his/her age: (40? probability of 40? *About middle-aged*?)
- Linguistic variables are characterized by linguistic values (Age: [Young, Old, Very Old, etc.]).
- Linguistic values are described by their fuzzy membership functions.

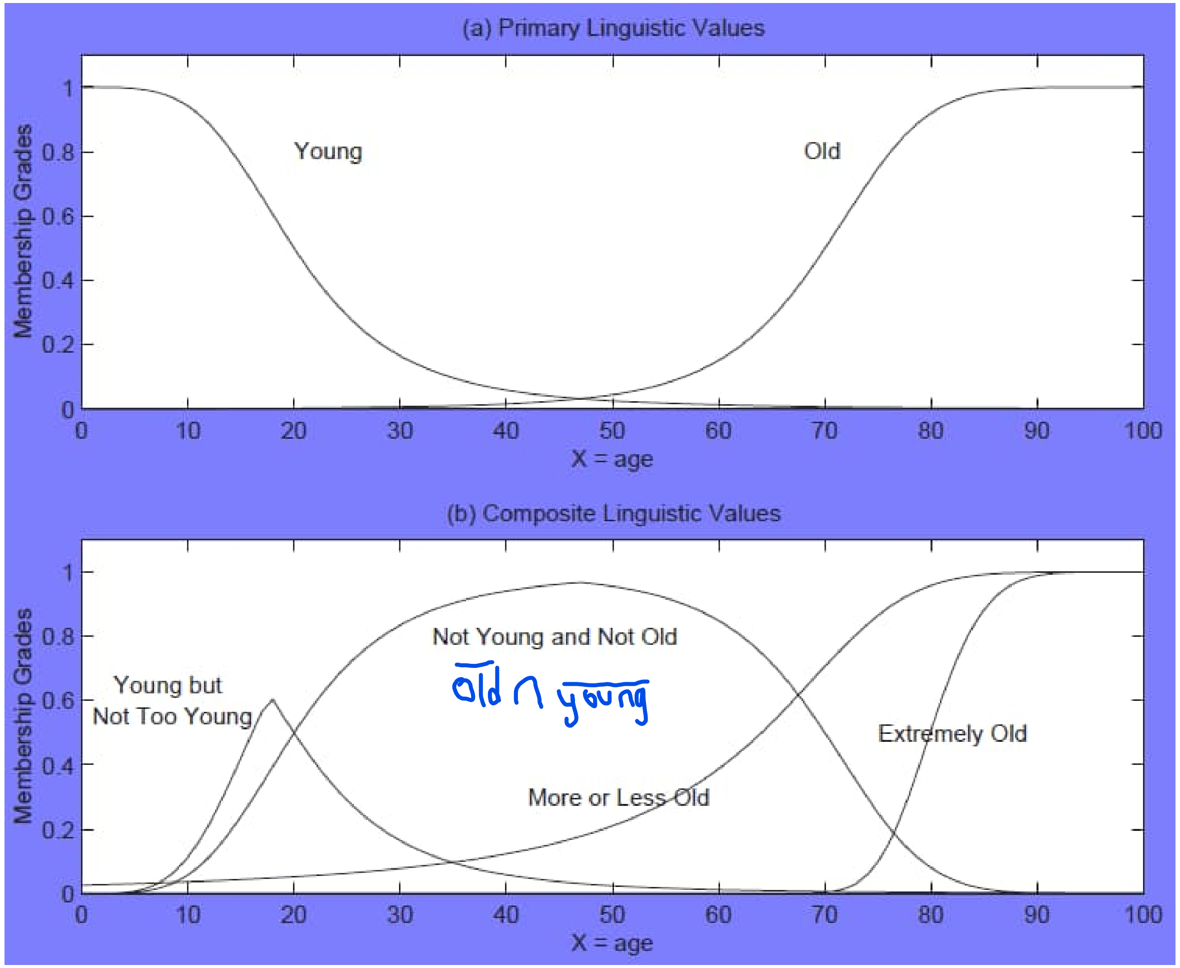
Membership Functions For Linguistic Values 'Young', 'Middle Aged', and 'Old'

Age: Linguistic Variable



Young, Middle Aged, and Old: Linguistic Values

Membership Functions For Primary and Composite Linguistic Values



Extremely Old = Old^4

Fuzzy IF-THEN Rules

- A fuzzy IF-THEN rule is expressed as:
  - IF *x* is A THEN *y* is B

Premise Part

Consequent Part

Both A and B are fuzzy membership functions.

- A fuzzy IF-THEN rule *associates* fuzzy input and output membership functions.

Fuzzy Reasoning

- Fuzzy reasoning derives conclusions based on *given* fuzzy IF-THEN rules and *known* facts.
  - An example:
    - Given a fuzzy rule: IF bath is very hot THEN add a lot of cold water.

α

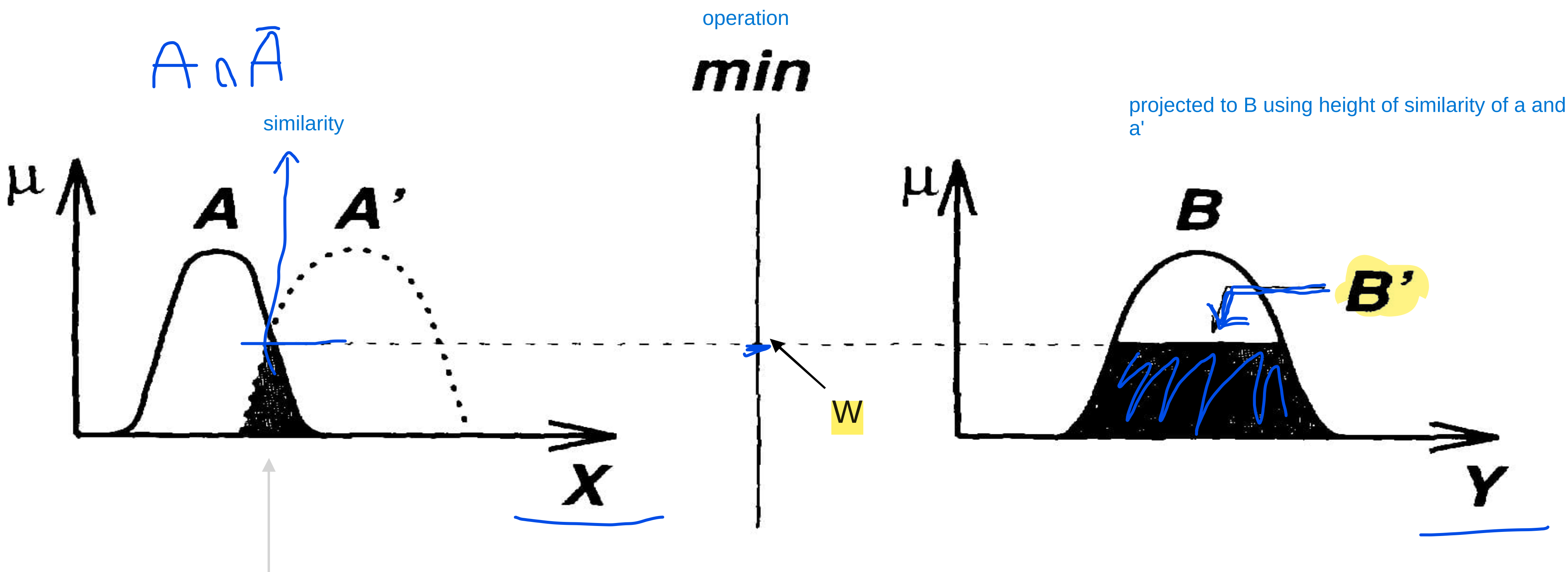
    - Known fact: bath is a little hot.
    - Conclusion: how much cold water should be added?

Fuzzy Reasoning

fuzzy membership functions

- Premise 1: IF X is A THEN Y is B
- Premise 2: IF X is A'
- Conclusion: THEN Y is B'
  - First, measure the *similarity* between A and A'.
  - Second, *project* this similarity to B.
- There are a few composition operations used in fuzzy reasoning procedure.
  - *Max-Min* is most widely employed.

Fuzzy Reasoning for Single Rule with Single Antecedent

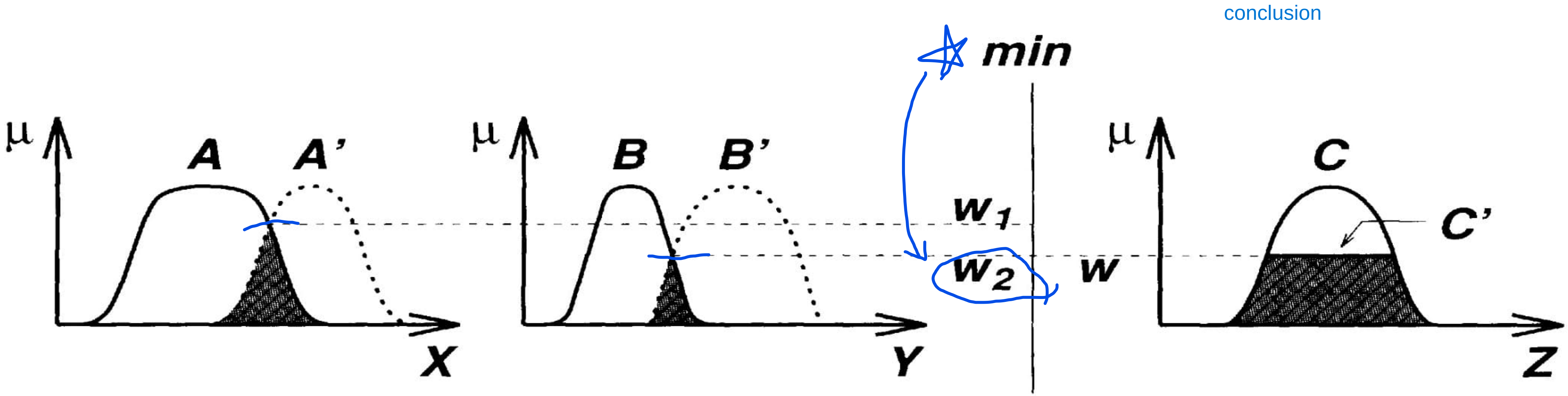


IF X is A THEN Y is B

fuzzy rule

Known Fact: X is A'

Fuzzy Reasoning for Single Rule with Multiple Antecedents

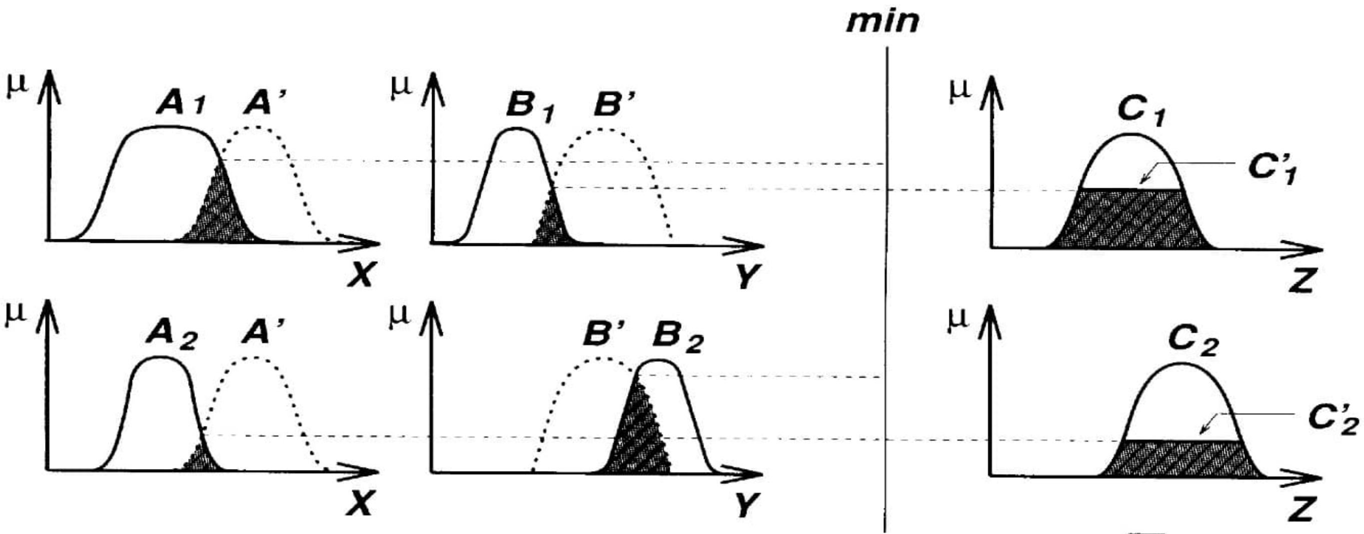


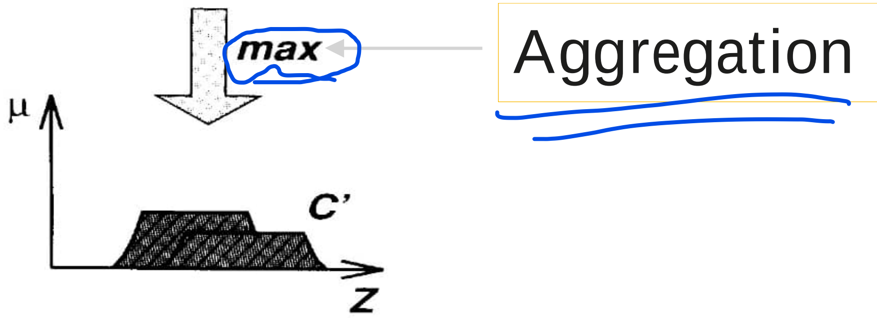
w=min(w1, w2)
or
w=w1*w2

IF X is A and Y is B THEN Z is C

Known Facts: X is A' and Y is B'

Fuzzy Reasoning For Multiple Rules with Multiple Antecedents

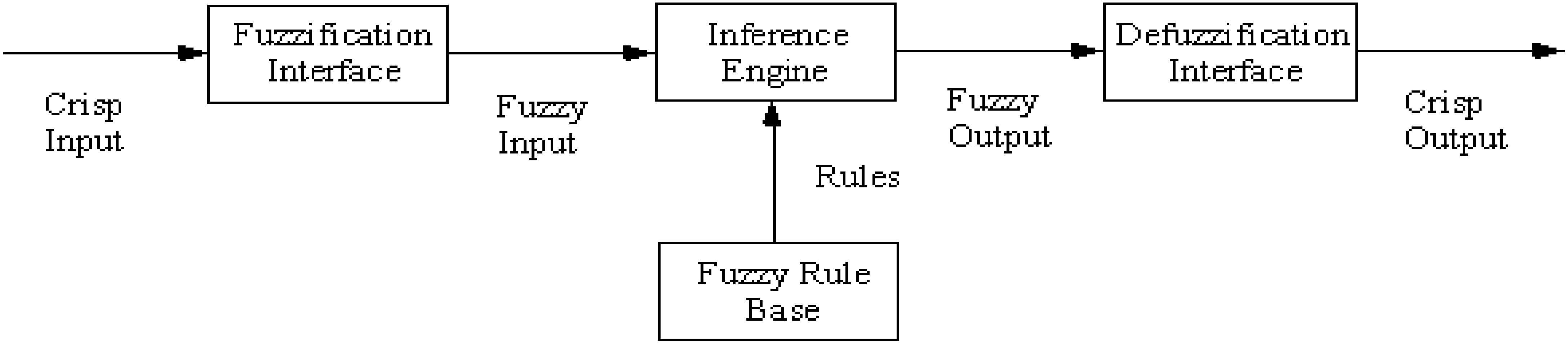




α If X is A1 and Y is B1 Then Z is C1
α If X is A2 and Y is B2 Then Z is C2

Known Facts: X is A' and Y is B'

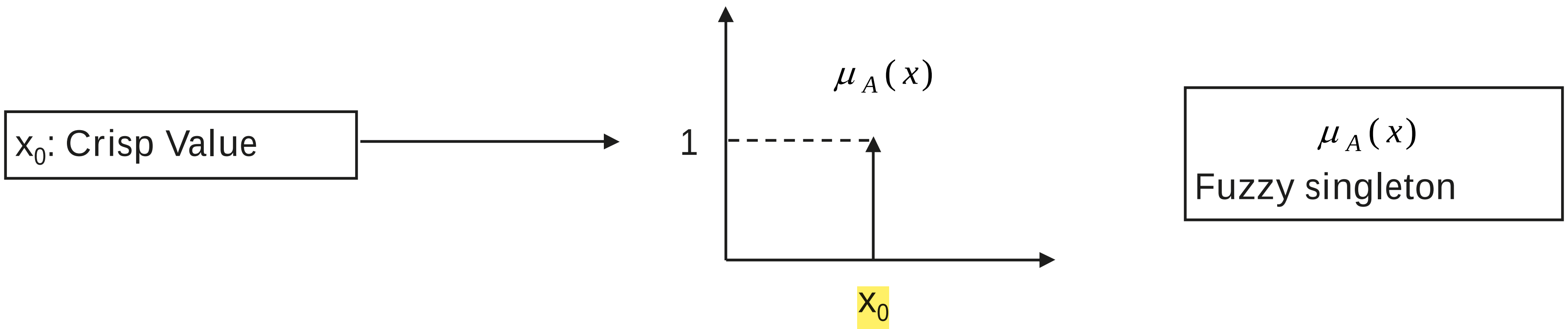
Fuzzy Logic Systems



Mamdani and Sugeno Fuzzy Inference Systems

Fuzzification

- Converts a *crisp* value (from sensors, devices, measurement meters, etc.) into a *fuzzy* value.
- A crisp value = A fuzzy singleton (*no* real fuzziness is introduced.)



Mamdani Fuzzy Inference Systems

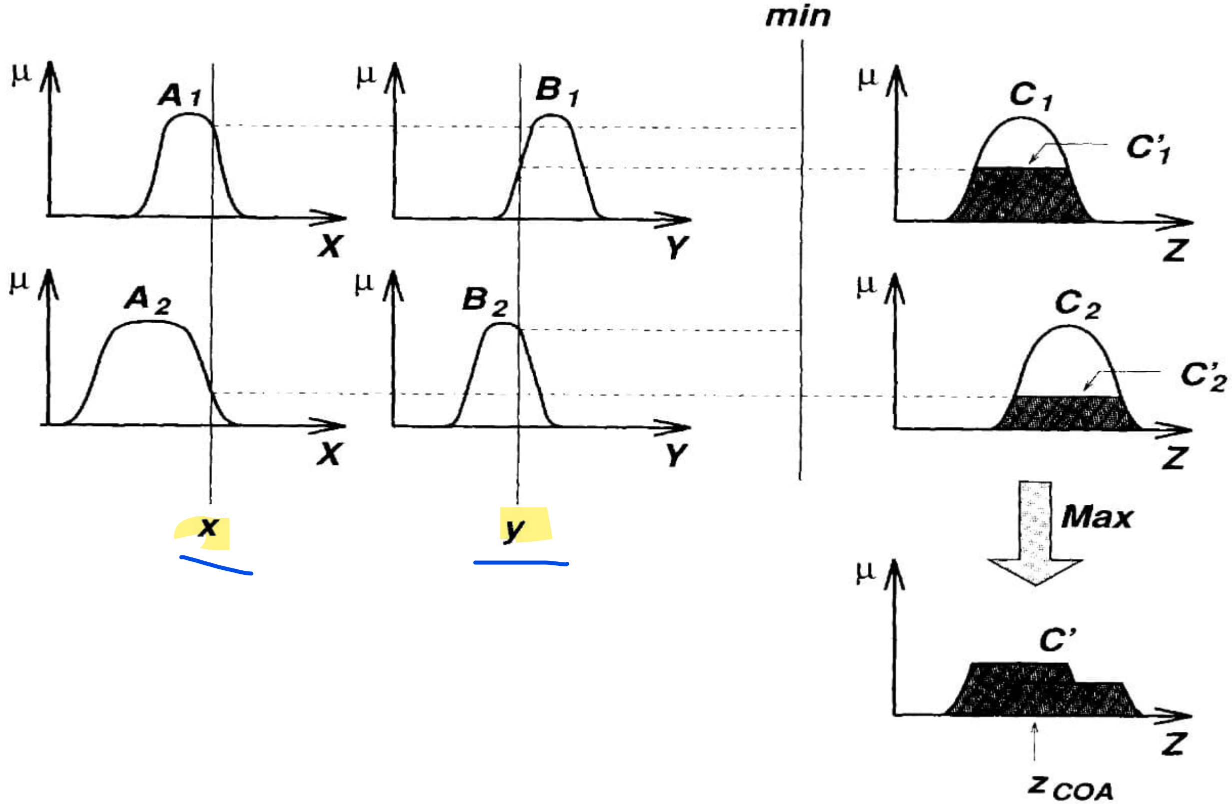
- Mamdani fuzzy inference systems
  - IF x is A THEN y is B
- Both premise and consequent parts consist of fuzzy membership functions.
- *Max-Min* composition can be applied in fuzzy reasoning procedure.

Mamdani Fuzzy Inference Systems

If X is A1 and Y is B1 THEN Z is C1

If X is A2 and Y is B2 THEN Z is C2

x and y are two crisp input values

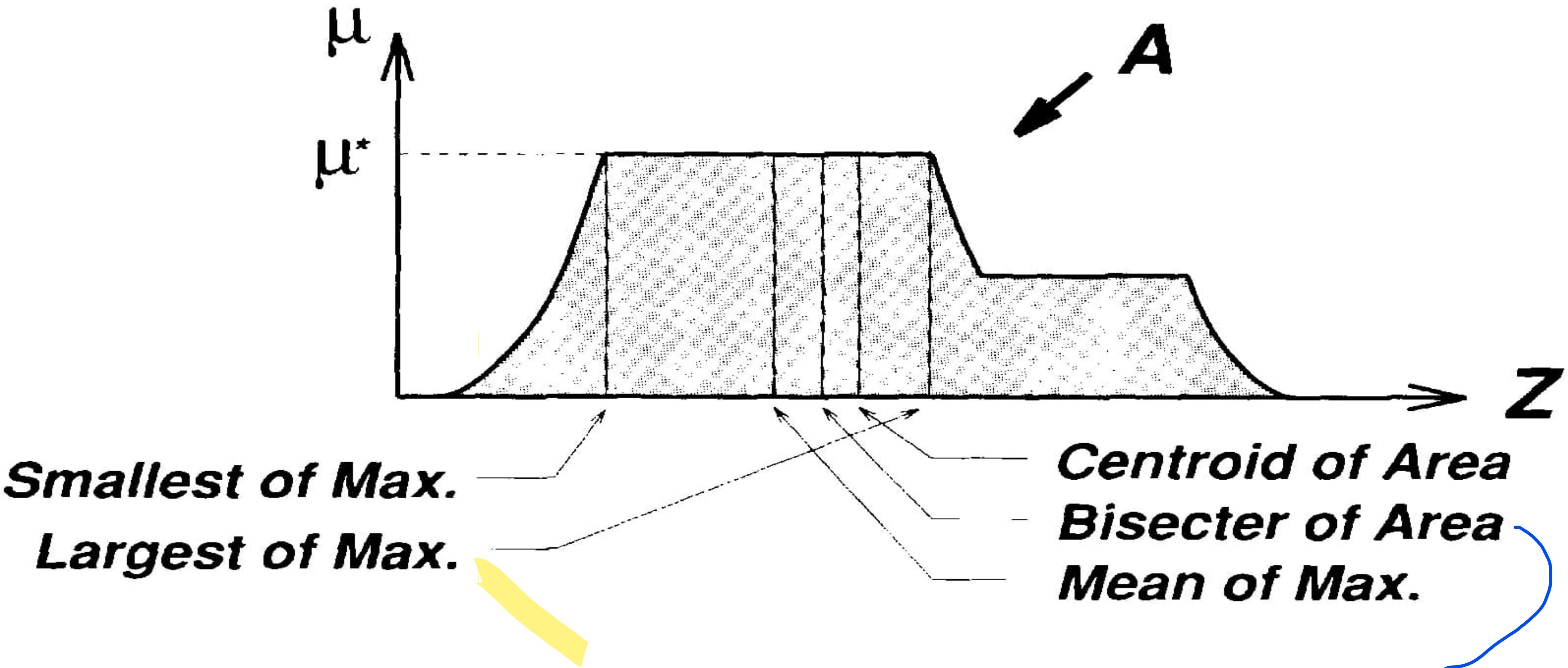


we need to convert this to a crisp value ->Defuzzification

Defuzzification Methods

we can use these metrics

loss of information



which divides the whole area into two parts

Defuzzification Methods

- Centroid of Area (COA)

$$Z^*_{COA} = \frac{\sum_{i=1}^{n} \mu_A(x_i) x_i}{\sum_{i=1}^{n} \mu_A(x_i)}$$

  - where z^*_{COA} is a crisp output.

- Bisector of Area (BOA)

$$\sum_{i=1}^{M} \mu_A(x_i) = \sum_{j=M+1}^{n} \mu_A(x_j)$$

  - where $Z^*_{Bi\sec t} = x_M$ is a crisp output.

Defuzzification Methods

- Mean of Maximum (MOM)

$$Z^*_{MOM} = \frac{\sum_{i}^{N} x_i^*}{N}$$

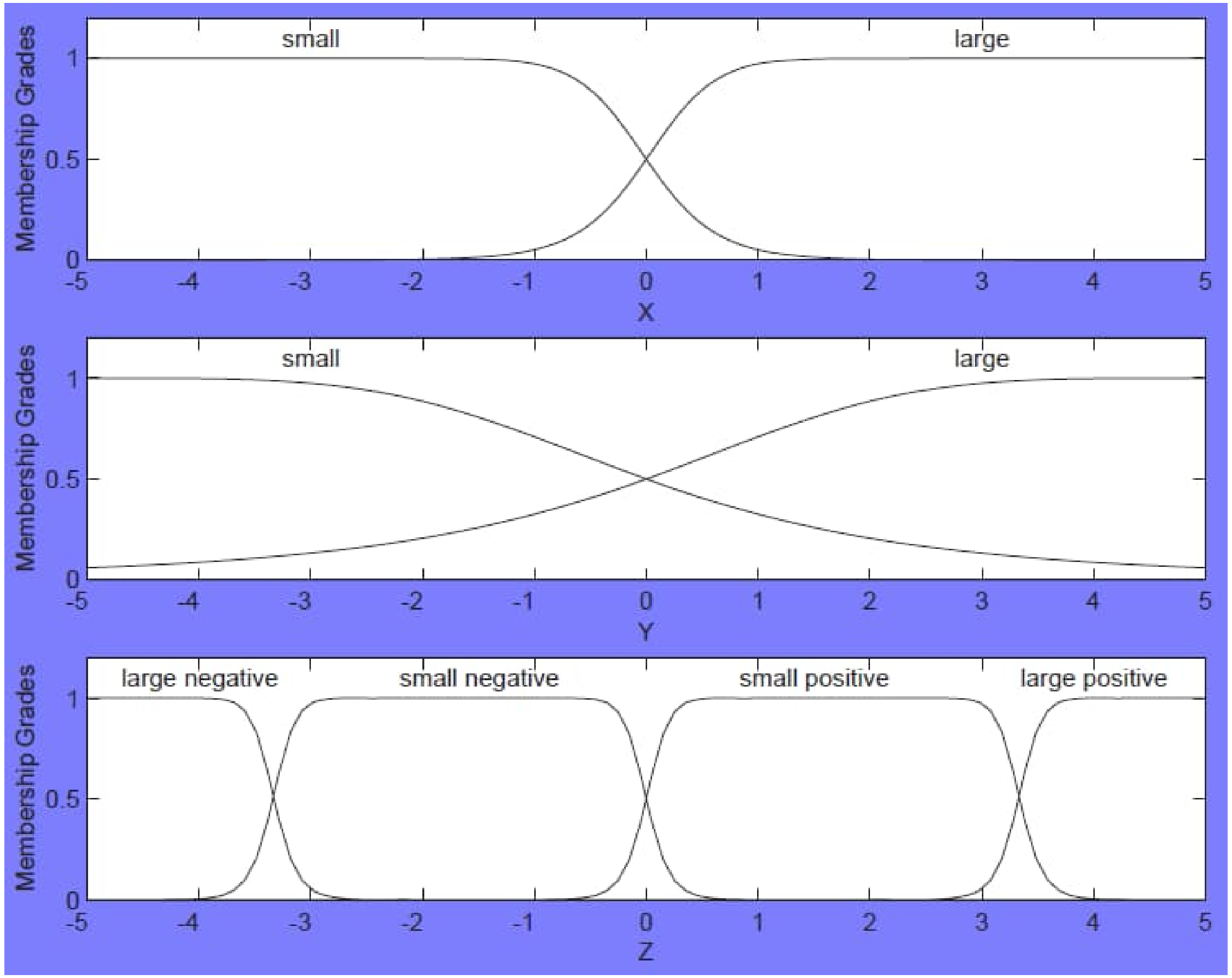
  - where $\mu_A(x_i^*)$ reaches maximal values of $\mu_A(x)$
- MOM may generatate very wrong discrete outputs in some certain cases.
- Different defuzzification methods may yield *similar* performances [Lee 88].

Mamdani Fuzzy Model: An Example

- A Two-Input and Single-Output Mamdani fuzzy logic system with four rules
  - IF X is small and Y is small THEN Z is large negative
  - IF X is small and Y is large THEN Z is small negative
  - IF X is large and Y is small THEN Z is small positive
  - IF X is large and Y is large THEN Z is large positive

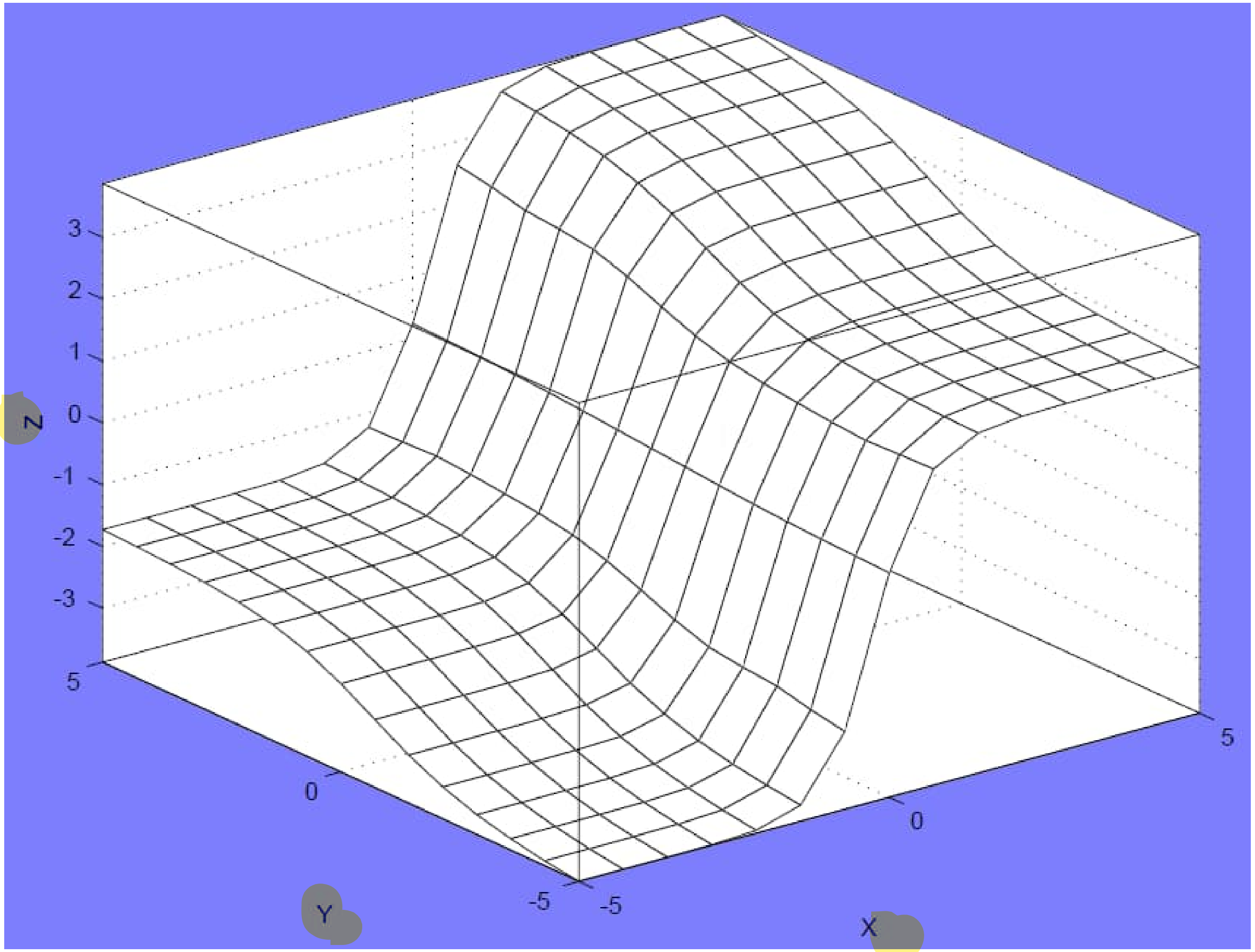
X and Y are different universe so their perception may differ from each other

Fuzzy Input and Output Membership Functions



Mamdani Input/Output Surface

Z = output



Nonlinear Input/Output Surface

Sugeno Fuzzy Inference Systems

- Sugeno fuzzy inference systems
  - IF x is A THEN y = f(x)
- Only *premise* part employs fuzzy membership functions.
- Consequent output is a *function* of input variables.
  - First order = linear consequent part
  - Higher orders = nonlinear consequent part

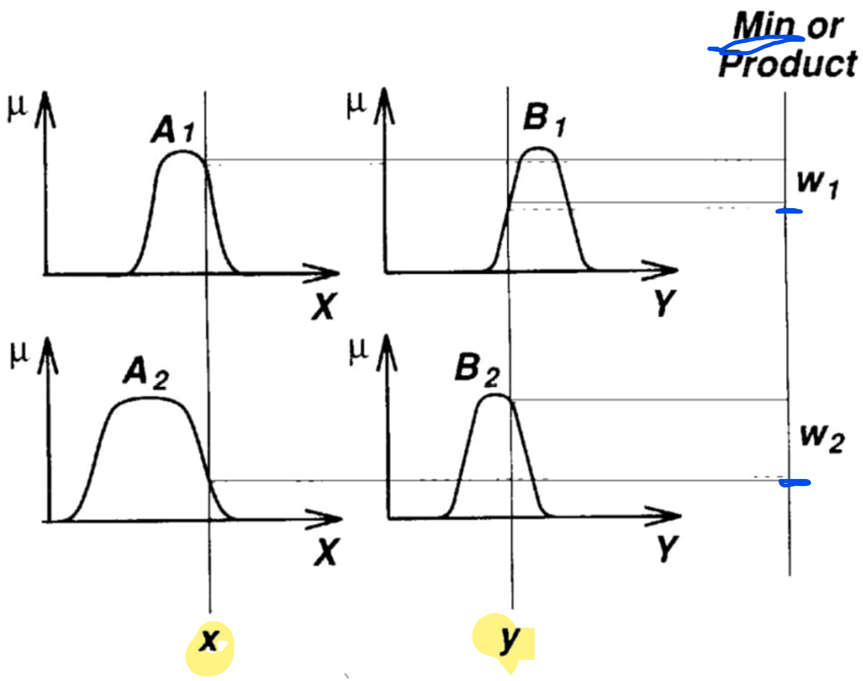
Sugeno Fuzzy Inference Systems

If X is A1 and Y is B1

THEN Z = p1x+q1y+r1

If X is A2 and Y is B2

THEN Z = p2x+q2y+r2



$$z_1 = p_1 x + q_1 y + r_1$$

$$z_2 = p_2 x + q_2 y + r_2$$

Weighted Average

$$z = \frac{w_1 z_1 + w_2 z_2}{w_1 + w_2}$$

z: Output

First Order Sugeno Inference System

Sugeno Fuzzy Inference Systems

- Advantages:
  - *No* defuzzification method is needed. z is already crisp, we do not need
  - Easy to analyze
    - *linear* with respect to consequent parameters (only in the first order consequent part case).

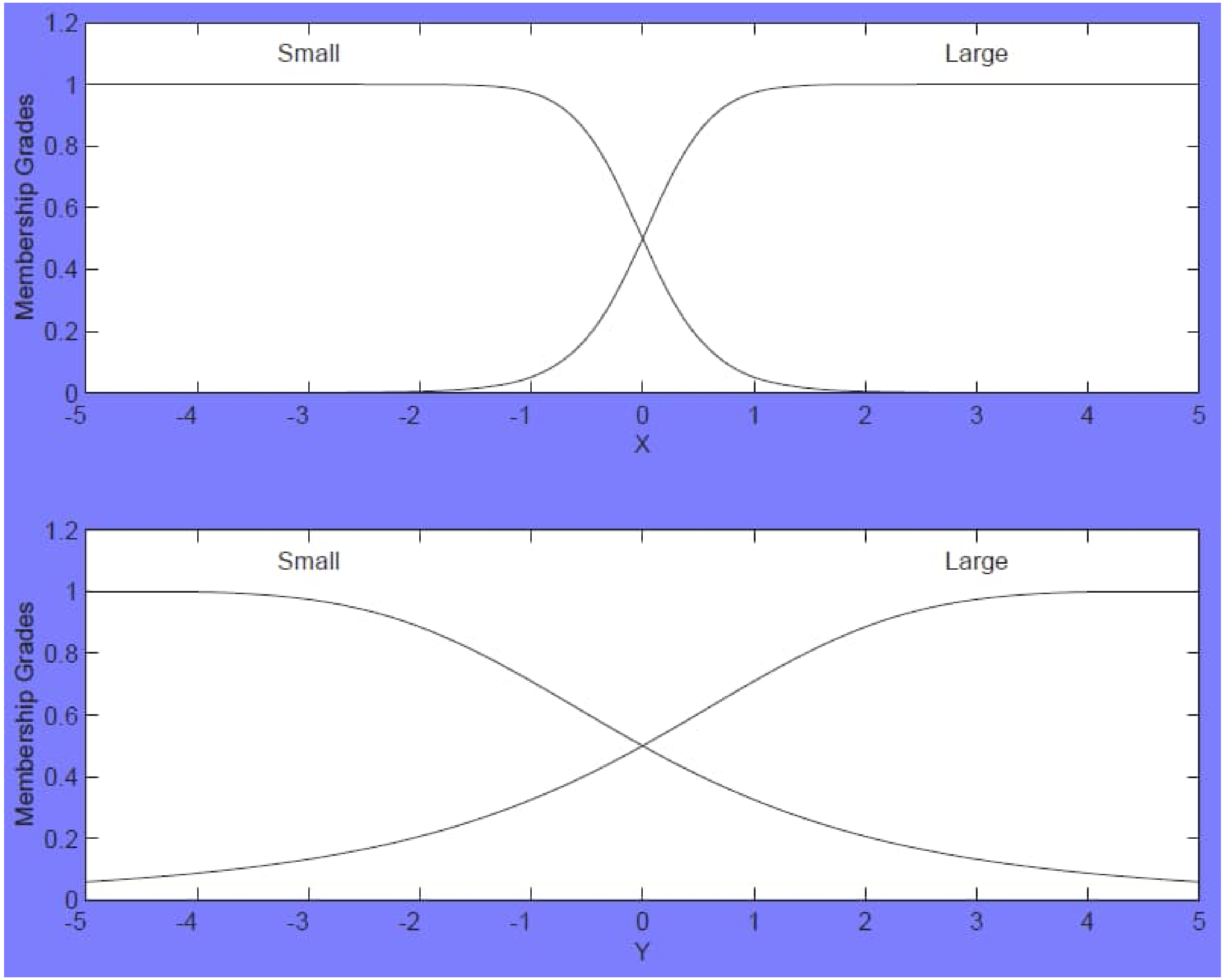
nonlinearity comes from if part

- Disadvantages
  - Very difficult to interpret practical meanings of consequent part.

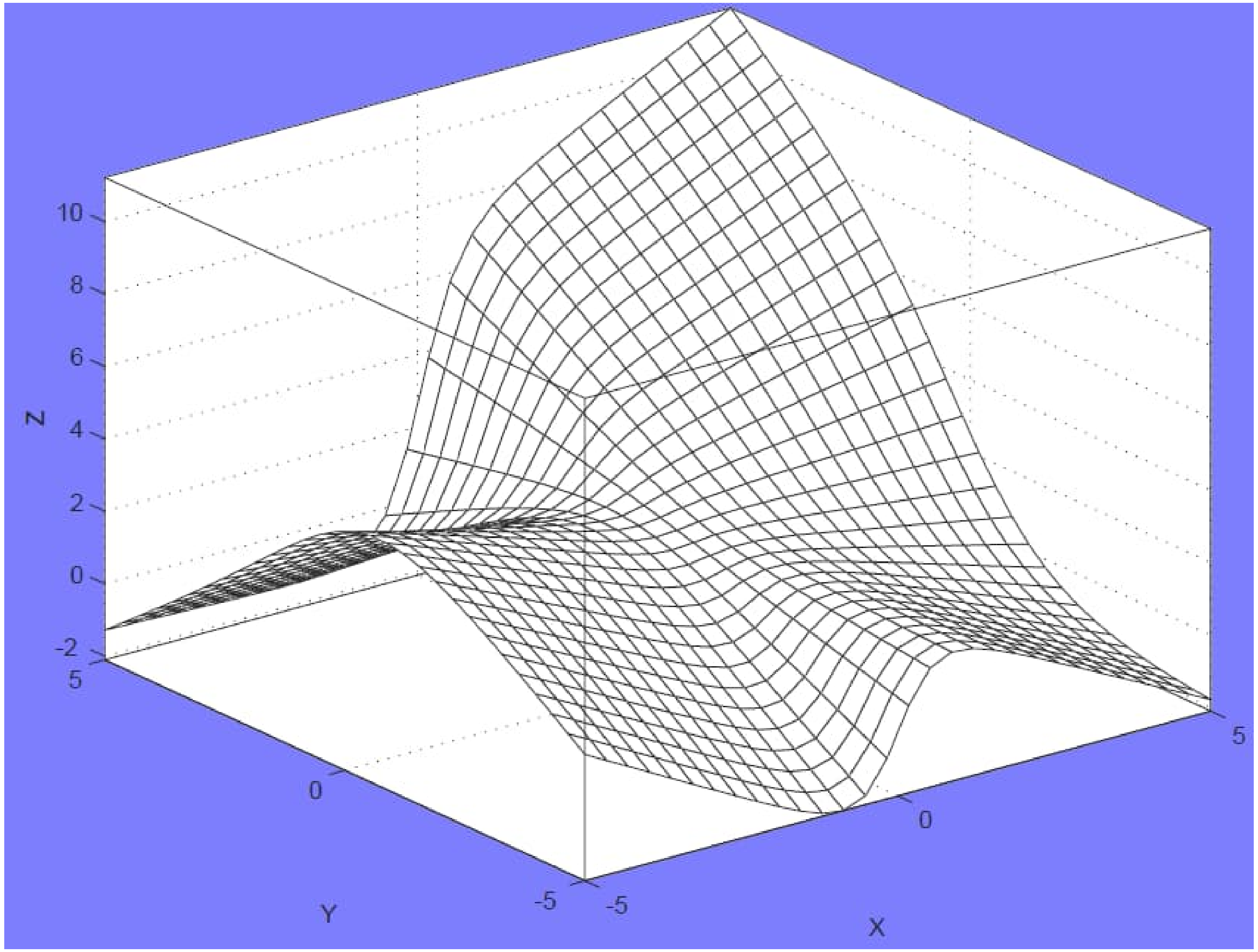
Sugeno Fuzzy Model: An Example

- A Two-Input and Single-Output Sugeno fuzzy logic system with four rules
  - IF X is small and Y is small THEN $z = -x+y+1$
  - IF X is small and Y is large THEN $z = -y+3$
  - IF X is large and Y is small THEN $z = -x+3$
  - IF X is large and Y is large THEN $z = -x+y+2$

Fuzzy Input Membership Functions

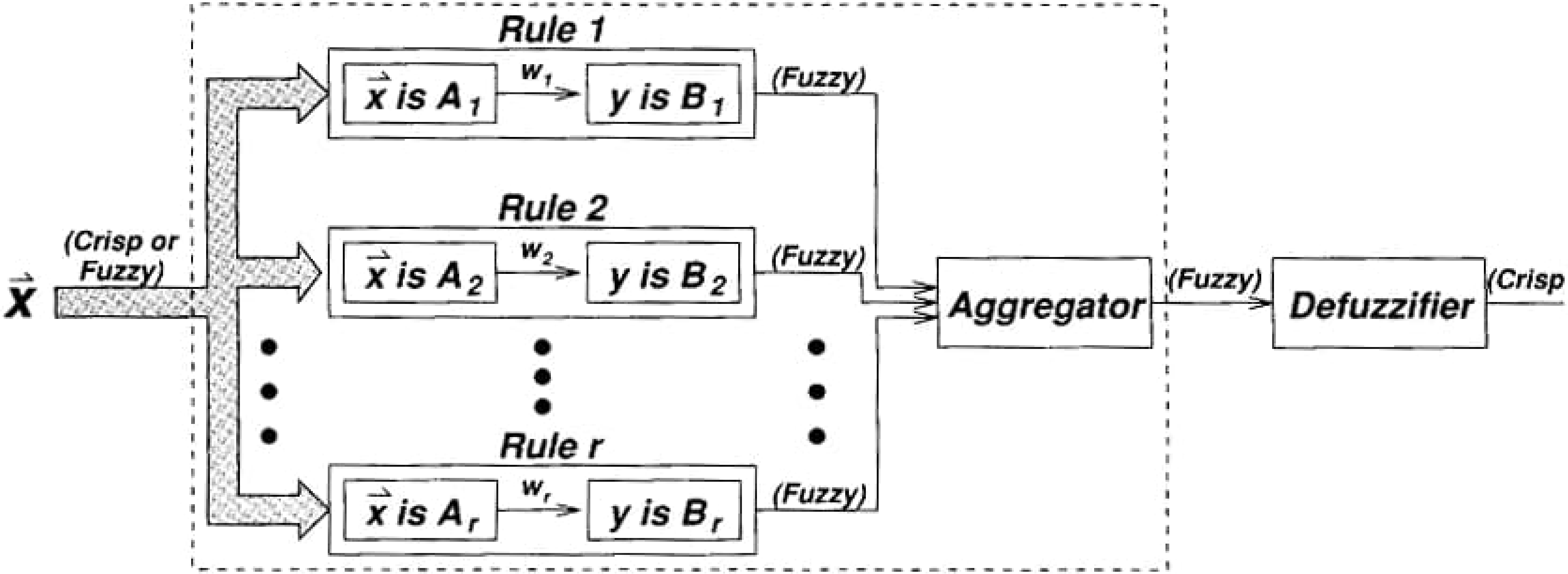


Sugeno Input/Output Surface



Nonlinear Input/Output Surface

Diagram of A Fuzzy Logic System



Applications of Fuzzy Logic

- Early applications include industrial systems in Europe, US, and China (1980s).
  - Steam engines, process control, etc.
- Further developed into consumer products in Japan (1990s).
  - Washing machines, vacuum cleaners, etc.
- Current applications cover *diverse* fields.
  - Control systems, signal processing, forecasting, etc.
- Pattern recognition is one of the *most* popular application areas for fuzzy logic.

Conclusions

- Background knowledge of fuzzy sets is briefly introduced.
- Concept of fuzzy sets is basically a *natural* generalization of classical sets.
- Fuzzy sets have some characteristics different from classical sets.
- Fuzzy inference systems are used to model human thinking.

Conclusions

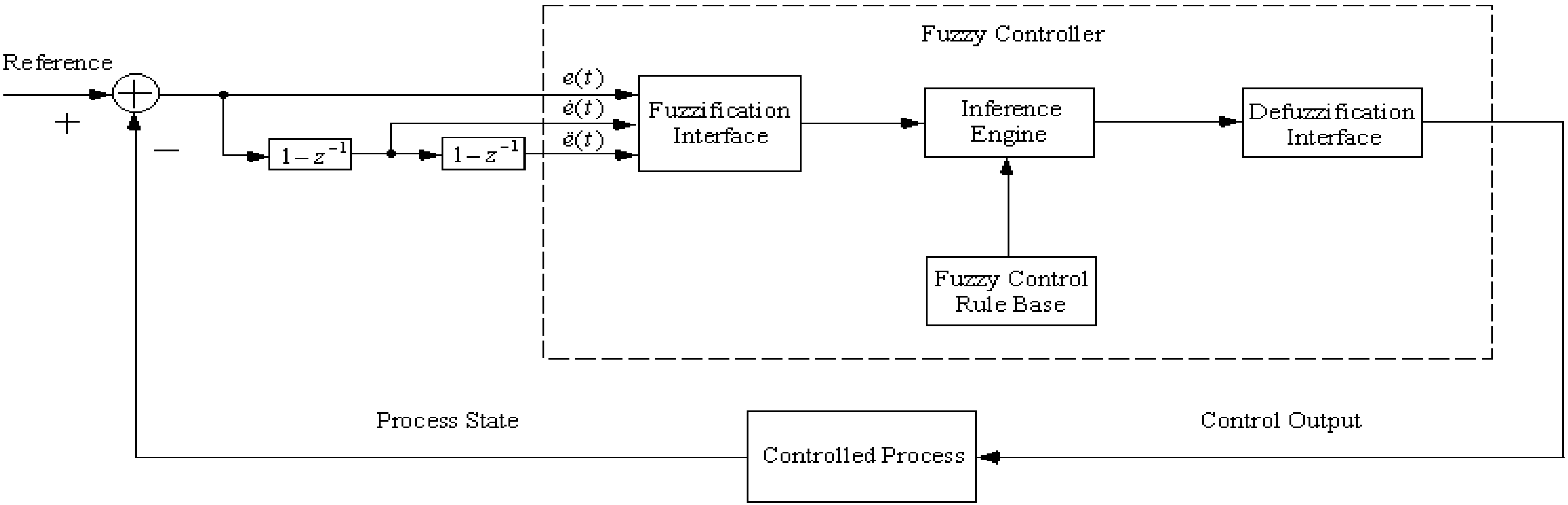
- Two fuzzy logic systems: Mamdani and Sugeno types.
- Engineering potential is more important than pure theoretical research.
- Applications of fuzzy logic in pattern recognition will be discussed later.
  - Fuzzy prediction
  - Fuzzy clustering
  - Fuzzy neural networks
  - etc.

Appendix: Fuzzy Logic in Control Systems

Fuzzy Logic Control

- Fuzzy control systems are oriented from Zadeh's 1973 paper [Zadeh 73].
- Mamdani designed the first practical fuzzy control system [Mamdani 75].
- Fuzzy logic control is now widely employed in industrial applications as well as household appliances.
  - From aircrafts to air conditioners

A Typical Fuzzy Control System



Design of A Fuzzy Controller

- 1. Choose input and output variables.
  - Feedback error, high order derivatives of feedback error, etc.)
- 2. Select membership functions for input and output variables.
  - Types of membership functions (Gaussian, Triangular, etc)
  - Number and parameters of membership functions (NB, NM, NS, ZE, PS, PM, PB)

Design of A Fuzzy Controller

- 3. Set up fuzzy control rules.
  - Mainly based on experts' experiences
    - Hard for humans to *simultaneously* handle more than seven fuzzy rules
  - Derived from practical data [Wang 92]
- 4. Design fuzzy inference mechanism.
  - Mamdani, Sugeno, etc.
  - Compositional operators
    - Max-Min, etc.

Design of A Fuzzy Controller

- 5. Choose defuzzification method.
  - COA, MOM, etc.
  - *No* apprarent performance difference
    - [Lee 88]
- 6. Evaluate and Fine-tune fuzzy controller.
  - Off-line tuning (rough)
  - On-line tuning (fine)

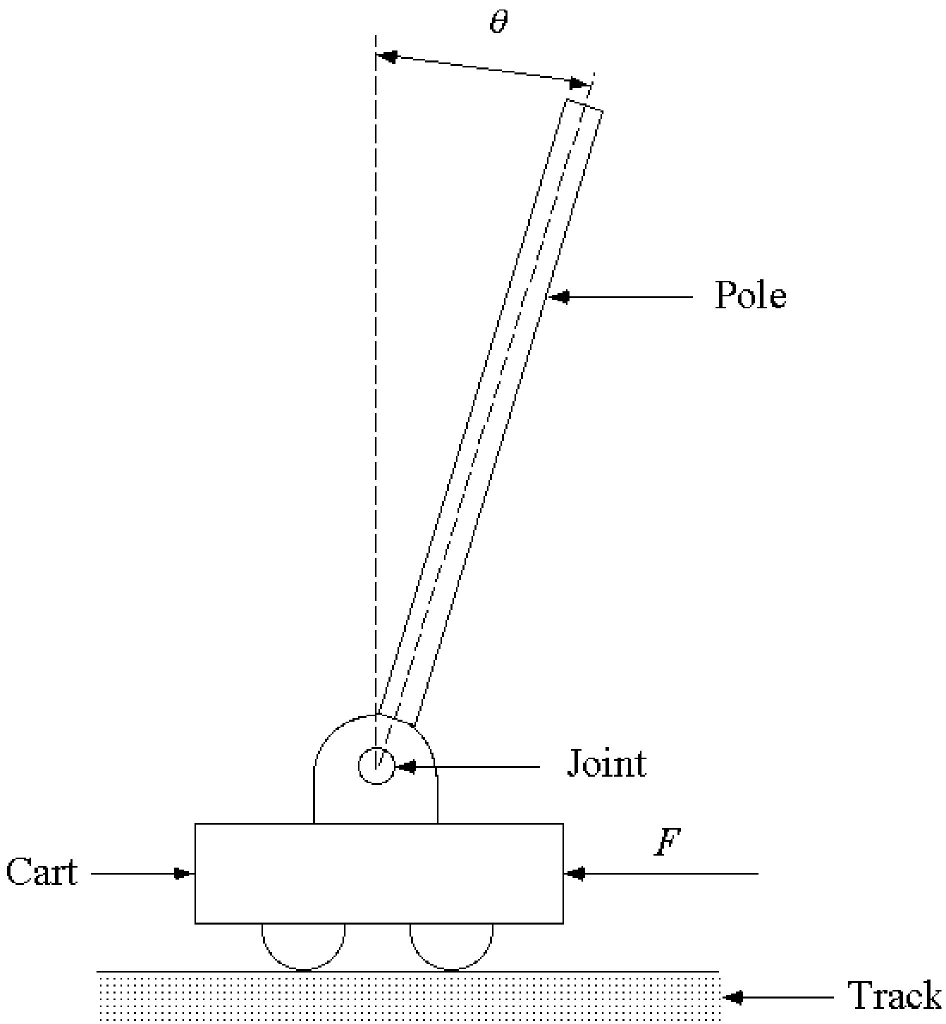
Two Fuzzy Logic Control Application Examples

- Fuzzy control of inverted pendulum
  - [Koczy 00]
- Fuzzy power PI control in mobile communications systems
  - [Chang 95]

Inverted Pendulum

- Inverted pendulum is a famous benchmark for control algorithms.
  - Very strong nonlinear characteristics
  - Multi-variable
  - Difficult to control using classical (hard computing) methods
- Inverted pendulum has been extensively investigated with soft computing methods.
  - Neural networks [Anderson 87] [Gao 97]
  - Fuzzy logic [Wang 96]
  - Fuzzy neural networks [Zhang 92]

Inverted Pendulum



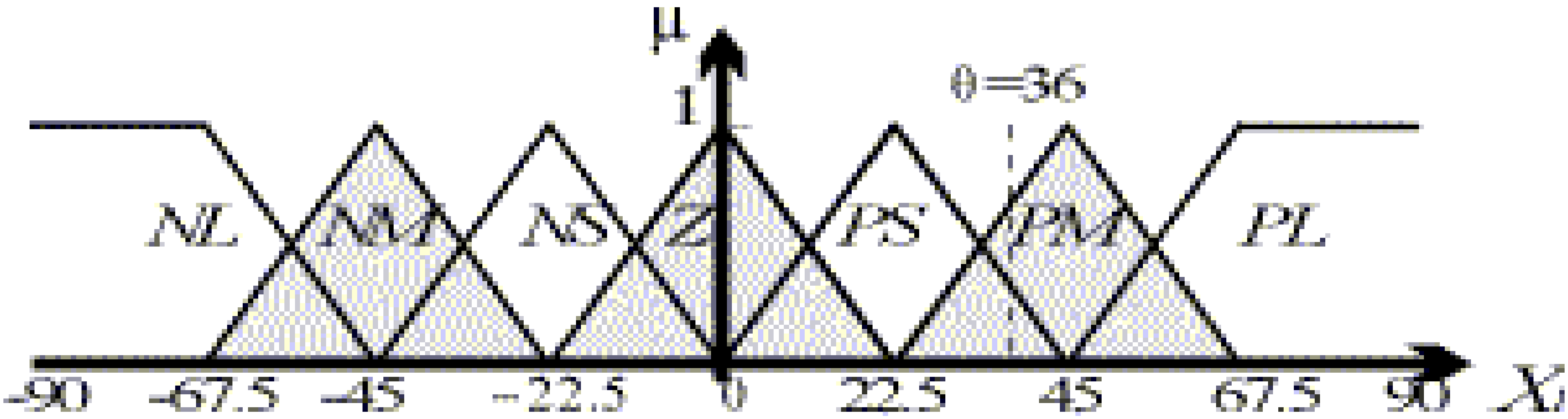
Mathematics Model of Pendulum

$$\ddot{\theta} = \frac{g \cdot \sin\theta + \cos\theta \cdot \left(\frac{-F - m_p \cdot l_p \cdot \dot{\theta}^2 \cdot \sin\theta + \mu_c \cdot \operatorname{sgn}(\dot{x})}{m_c + m_p} \right) - \frac{\mu_p \cdot \dot{\theta}}{m_p \cdot l_p}}{l_p \cdot \left(\frac{4}{3} - \frac{m_p \cdot \cos^2\theta}{m_c + m_p} \right)},$$

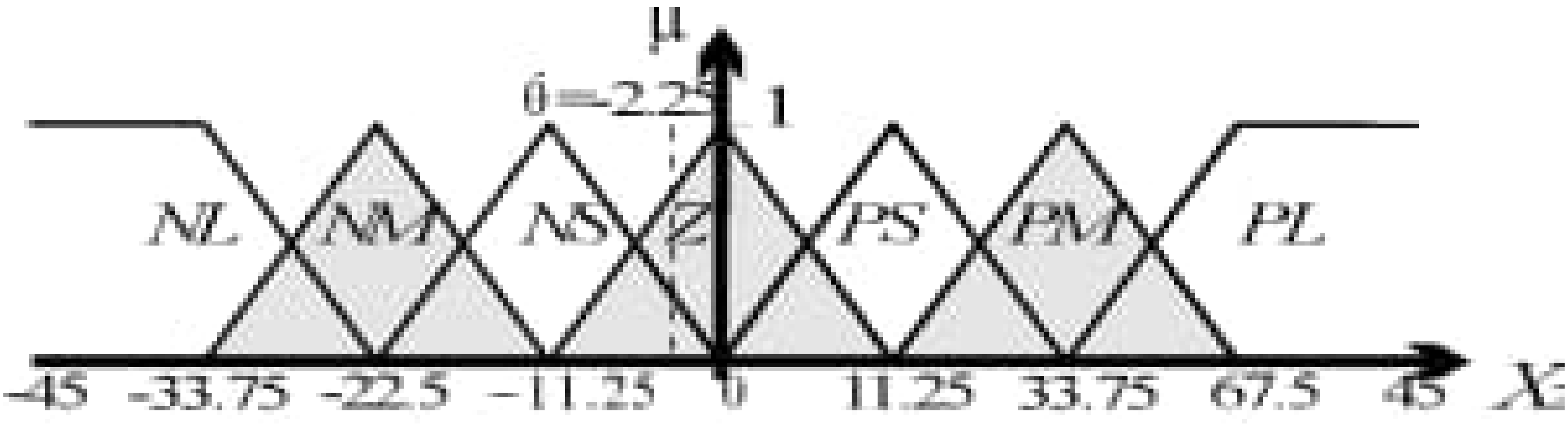
$$\ddot{x} = \frac{F + m_p \cdot l_p \cdot \left(\dot{\theta}^2 \cdot \sin\theta - \ddot{\theta} \cdot \cos\theta \right) - \mu_c \cdot \operatorname{sgn}(\dot{x})}{m_c + m_p}.$$

Fuzzy Inverted Pendulum Controller

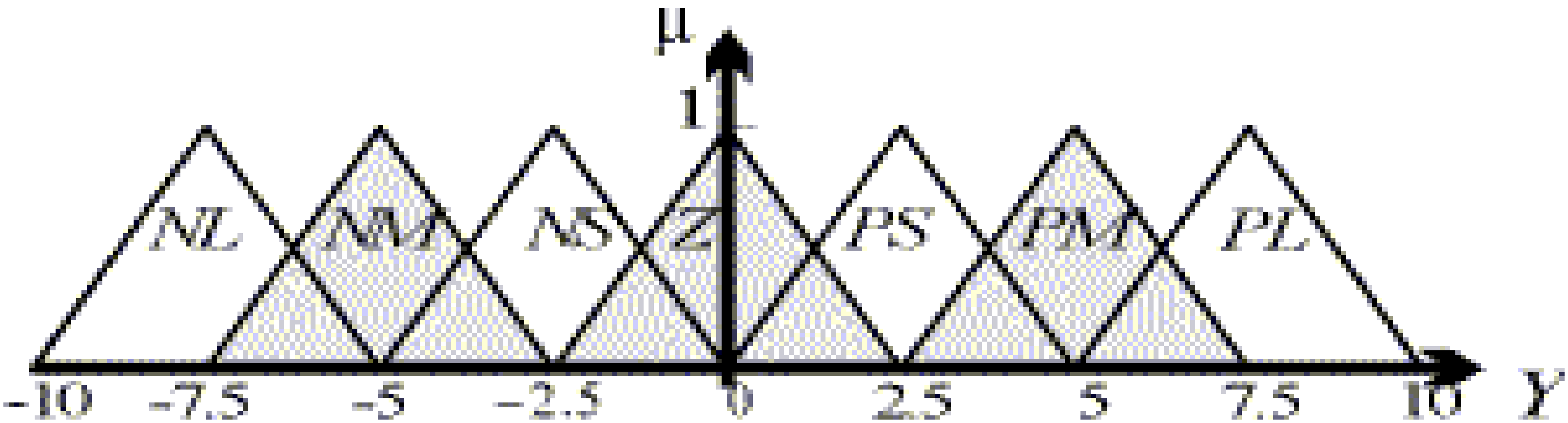
Linguistic Terms of Angle θ



Linguistic Terms of Angular Speed $\dot{\theta}$



Linguistic Terms of Force F



Fuzzy Rules for Pendulum Control

- Fuzzy inference rules for inverted pendulum control are derived *intuitively*:
  - IF angle θ is *zero* and angular velocity $\dot{\theta}$ is *zero*, THEN force *F* is *zero*
  - IF angle θ is *NL* and angular velocity $\dot{\theta}$ is *NS*, THEN force *F* is *NM*
- There are totally 19 fuzzy control rules.
- Performance is robust even with external random disturbances.

Decision Table (Rulebase)

R: θ

$\dot{\theta}$	NL	NM	NS	Z	PS	PM	PL
NL			PS	PL			
NM				PM			
NS	NM		NS	PS			
Z	NL	NM	NS	Z	PS	PM	PL
PS				NS	PS		PM
PM				NM			
PL				NL	NS		

[Koczy 00]

Helsinki Summer School

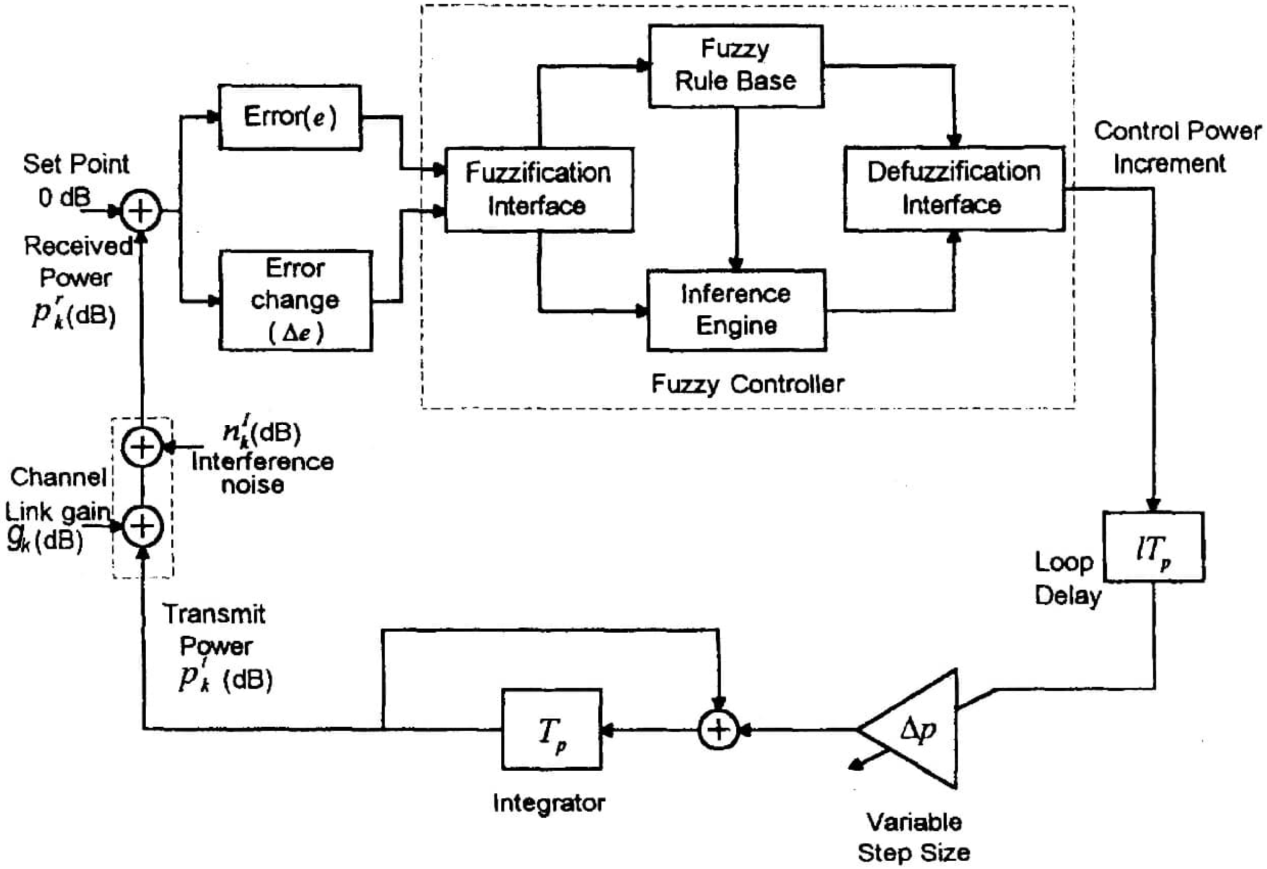
Satisfactory Control Results

Refer to demonstration in Fuzzy Logic Toolbox

Fuzzy Power Control in Mobile Communications Systems

- Conventional ’bang-bang’ power control always yields large overshoot, long rise time, and remaining steady state error.
- Fuzzy power control schemes utilize some *priori* knowledge of the dynamics of fading channels.
- Fuzzy power controllers provide a better regulation performance.

Fuzzy Power Control in Mobile Communications Systems



$$\Delta u^{FLC}$$

Fuzzy Power Control in Mobile Communications Systems

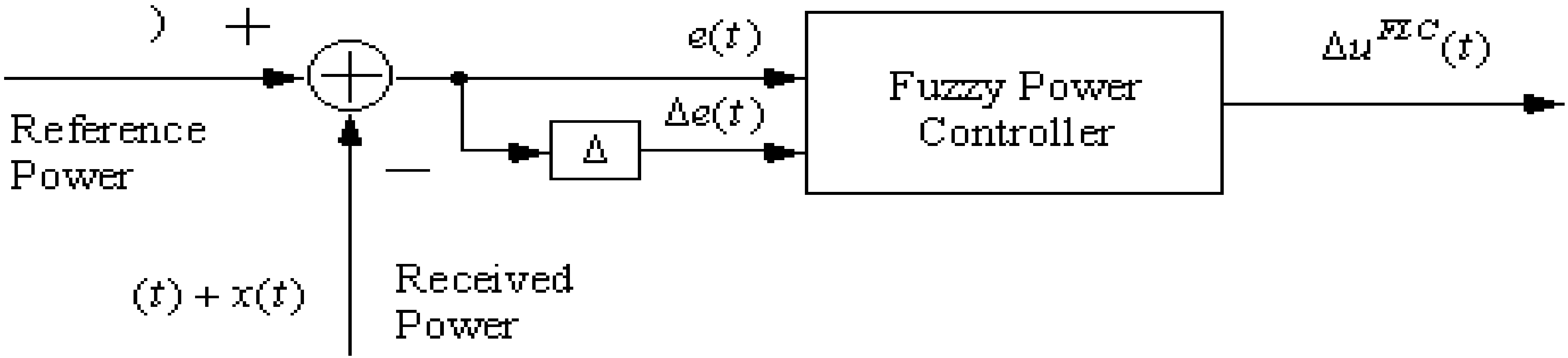
- Fuzzy PI power controller [Chang 95] has two inputs: $e(t)$ and $\Delta e(t)$
- Fuzzy PI control rules always have the following form:

$$R_I\text{: IF } e(t) \text{ is } A_i \text{ and } \Delta e(t) \text{ is } B_i\text{, THEN } \Delta u^{FLC} \text{ is } C_i$$

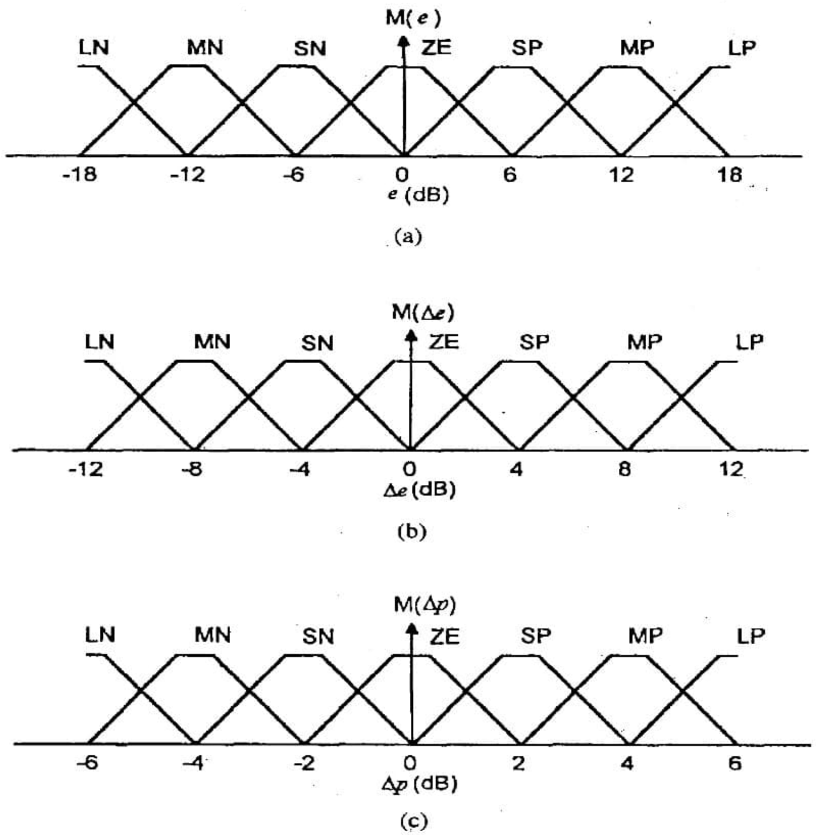
- An example is:

$$\text{IF } e(t) \text{ is ZE and } \Delta e(t) \text{ is ZE, THEN } \Delta u^{FLC} \text{ is ZE}$$

Fuzzy PI Power Controller



Membership Functions of Fuzzy PI Controller Variables



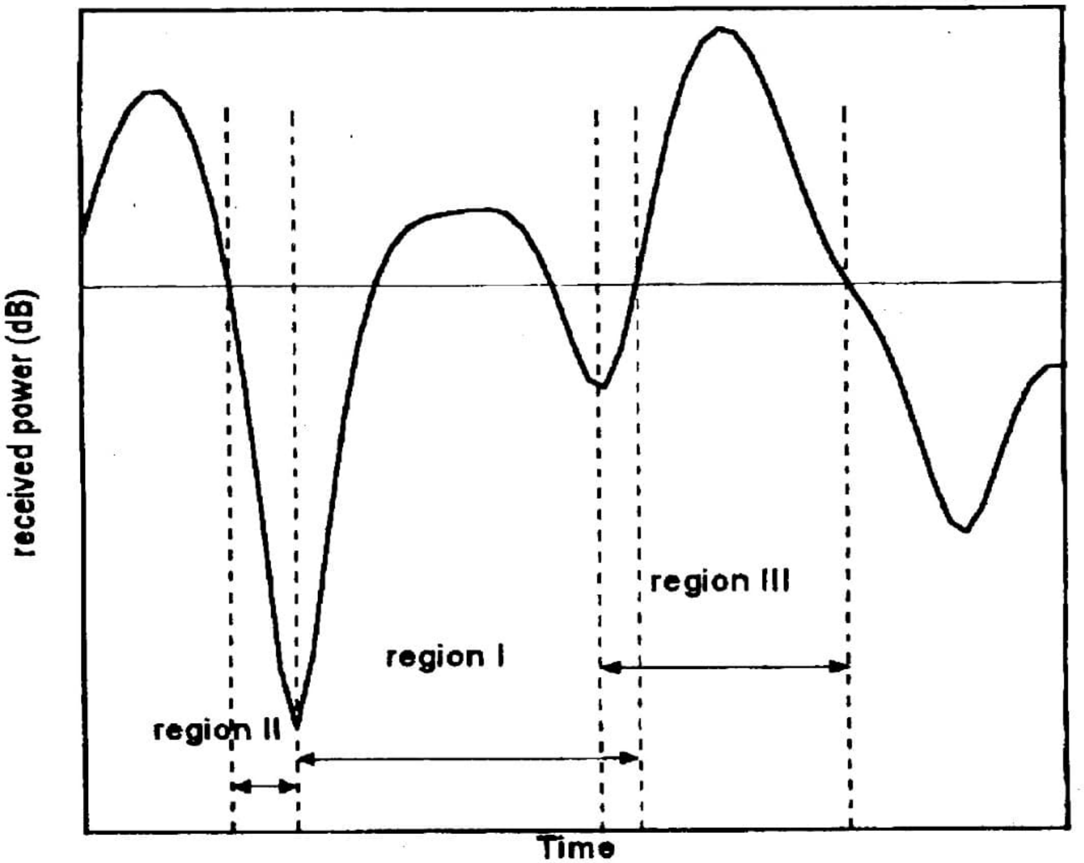
e

Δe

Δu^{FLC}

A Typical Fast Fading Signal

Region II:
Deep downward fading



Regions I and III:
Response of second order systems

Fuzzy Rules for Regions I and III

Δe

e	LN	MN	SN	ZE	SP	MP	LP
LN				LN (3)			
MN	LN (20)	LN (21)	LN (22)	MN (7)			
SN	LN (17)	LN (18)	MN (19)	SN (11)	ZE (23)		MP (24)
ZE	LN (2)	MN (6)	SN (10)	ZE (13)	SP (12)	MP (8)	LP (4)
SP	MN (15)		ZE (16)	SP (9)			
MP				MP (5)			
LP			LP (14)	LP (1)			

For Normal Fadings Only!

24 Rules

Fuzzy Rules for Region II

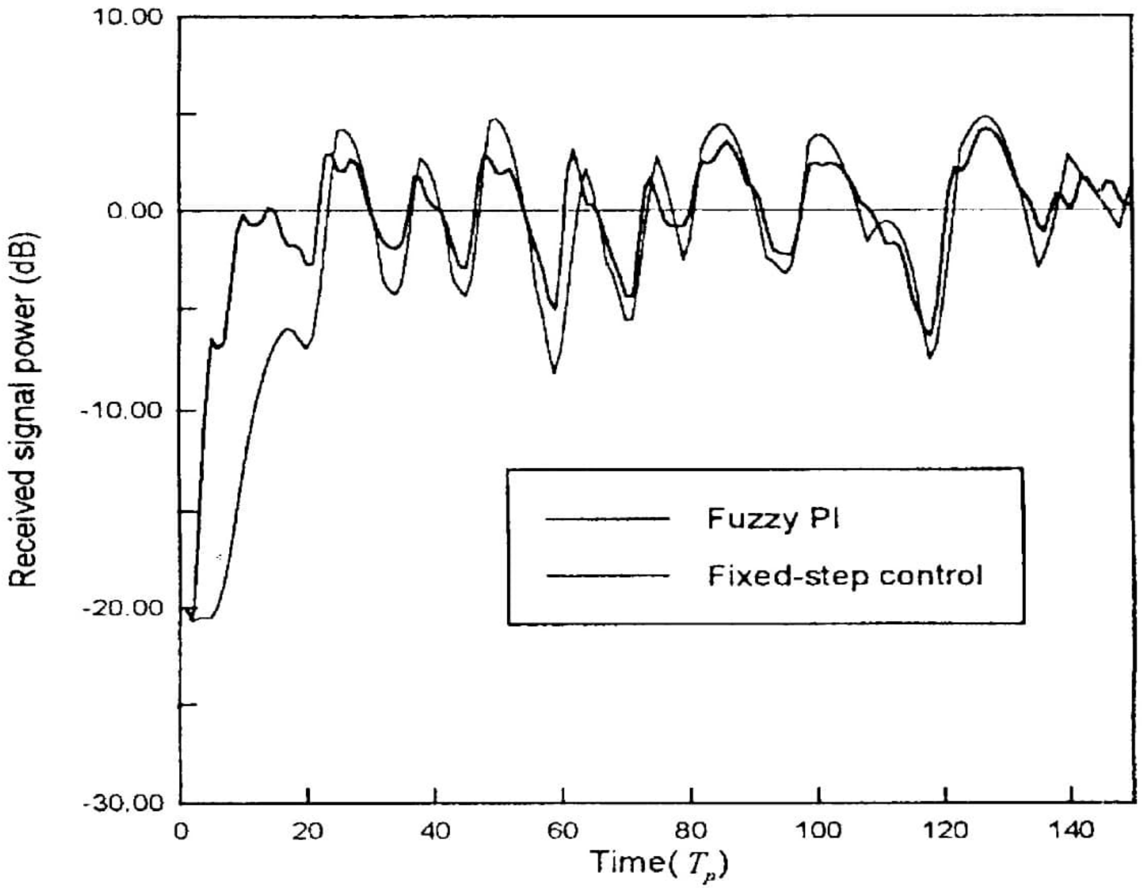
Δe

e		LN	MN	SN	ZE	SP	MP	LP
	LN	LN	LN	LN	LN	LN	LN	LN
	MN							
e	SN							
	ZE							
	SP					MP	MP	MP
	MP					MP	MP	MP
	LP	LP	LP	LP	LP	LP	LP	LP

For Deep Fadings Only!

20 Rules

Fuzzy PI Power Control Performance



Overshoot Reduced
&
Oscillation Elliminated

Comparison Between Fuzzy PI and Conventional Fixed-Step Power Control

	m = 2		m = 4	
	Fuzzy PI	Fixed-step	Fuzzy PI	Fixed-step
$f_D T_p = 0.05$	3.68	6.06	3.14	5.45
$f_D T_p = 0.0375$	3.34	5.75	3.10	5.38
$f_D T_p = 0.025$	3.13	5.57	3.07	5.34

RMS of Tracking Error

Remarks

- Two fuzzy control application examples are introduced.
- Fuzzy logic is effective in coping with complex, ill-defined, and nonlinear systems.
- Expert domain knowledge can be *explicitly* embedded into fuzzy controllers.
- Stability of fuzzy control systems is quite difficult to analyze.