Лабораторная работа 8

Двумерные начально-краевые задачи для дифференциального уравнения параболического типа

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Задание

Используя схемы переменных направлений и дробных шагов, решить двумерную начально-краевую задачу для дифференциального уравнения параболического типа. В различные моменты времени вычислить погрешность численного решения путем сравнения результатов с приведенным в задании аналитическим решением $U(x, y, t)$.

Вариант 1.

$$\begin{cases} \frac{\partial u}{\partial t} = a\frac{\partial^2 u}{\partial x^2} + a\frac{\partial^2 u}{\partial y^2}, \quad a > 0 \\ u(0, y, t) = cos(\mu_2 y)exp(-(\mu_1^2 + \mu_2^2)at) \\ u(\frac{\pi}{2}\mu_1, y, t) = 0 \\ u(x, 0, t) = cos(\mu_1 x)exp(-(\mu_1^2 + \mu_2^2)at) \\ u(x, \frac{\pi}{2}\mu_2, t) = 0 \\ u(x, y, 0) = cos(\mu_1 x)cos(\mu_2 y) \end{cases}$$

Аналитическое решение:

$$U(x, y, t) = cos(\mu_1 x)cos(\mu_2 y)exp(-(\mu_1^2 + \mu_2^2)at)$$

In [1]:
```
import sys
sys.path

sys.path.insert(0, r"c:\Users\никита\Desktop\учеба\чм\modules")
```

In [2]:
```
import numpy as np
import matplotlib.pyplot as plt
import seaborn as sns
import time
sns.set()

%matplotlib widget

plt.rcParams["figure.figsize"] = (10, 6)

import LinearAlgebra
```

In [3]:
```
x_begin = 0
x_end = lambda mu_1: np.pi * mu_1 / 2
```

```
y_begin = 0
y_end = lambda mu_2: np.pi * mu_2 / 2

t_begin = 0
t_end = 2

a = 1

mu = [(1, 1), (2, 1), (1, 2)]

h_x = 0.05
h_y = 0.05
tau = 0.01
```

Начальные и граничные условия:

In [4]:
```
def phi_3(y, t, mu_1, mu_2, a=1):
    return np.cos(mu_2 * y) * np.exp(-(mu_1**2 + mu_2**2) * a * t)

def phi_4(y, t, mu_1, mu_2, a=1):
    return 0

def phi_1(x, t, mu_1, mu_2, a=1):
    return np.cos(mu_1 * x) * np.exp(-(mu_1**2 + mu_2**2) * a * t)

def phi_2(x, t, mu_1, mu_2, a=1):
    return 0

def psi(x, y, mu_1, mu_2, a=1):
    return np.cos(mu_1 * x) * np.cos(mu_2 * y)

def solution(x, y, t, mu_1, mu_2, a=1):
    return np.cos(mu_1 * x) * np.cos(mu_2 * y) * np.exp(-(mu_1**2 + mu_2**2) * a *
```

Точное решение

In [5]:
```
def get_analytical_solution(x_begin, x_end, y_begin, y_end, t_begin, t_end, h_x, h_

    x = np.arange(x_begin, x_end(mu_1) + h_x, h_x)
    y = np.arange(y_begin, y_end(mu_2) + h_y, h_y)
    t = np.arange(t_begin, t_end + tau, tau)

    u = np.zeros((x.size, y.size, t.size))
    for i in range(x.size):
        for j in range(y.size):
            for k in range(t.size):
                u[i, j, k] = solution(x[i], y[j], t[k], mu_1, mu_2)

    return u
```

In [6]:
```
mu_1, mu_2 = mu[0]

u_exact = get_analytical_solution(x_begin, x_end, y_begin, y_end, t_begin, t_end, h
```

Метод переменных направлений

Для момента времени $k + 1/2$ производная по x апроксимруется неявно, а по y - явно. Для момента времени $k + 1$ наоборот.

Шаг 1.

Решаем систему уравнений для всех j, чтобы получить значения в момент времени $k + 1/2$:

$$\begin{cases} bu_{1j}^{k+1/2} + cu_{2j}^{k+1/2} = d_1, \\ au_{i-1j}^{k+1/2} + bu_{ij}^{k+1/2} + cu_{i+1j}^{k+1/2} = d_i\,, \quad i = 2 \dots N-2, \\ au_{N-2j}^{k+1/2} + bu_{N-1j}^{k+1/2} = d_{N-1}, \end{cases}$$

$$a = c = -a\tau h_y^2 \tag{1}$$
$$b_j = 2h_x^2 h_y^2 + 2a\tau h_y^2 \tag{2}$$
$$d_i = a\tau h_x^2 u_{ij-1}^k + (2h_x^2 h_y^2 - 2a\tau h_x^2) u_{ij}^k + a\tau h_x^2 u_{ij+1}^k \tag{3}$$
$$d_1 = d - au_{0j}^{k+1/2} \tag{4}$$
$$d_{N-1} = d - cu_{Nj}^{k+1/2} \tag{5}$$

Шаг 2.

Решаем систему уравнений для всех i, чтобы получить значения в момент времени $k + 1$:

$$\begin{cases} bu_{1j}^{k+1} + cu_{2j}^{k+1} = d_1, \\ au_{i-1j}^{k+1} + bu_{ij}^{k+1} + cu_{i+1j}^{k+1} = d_i\,, \quad i = 2 \dots N-2, \\ au_{N-2j}^{k+1} + bu_{N-1j}^{k+1} = d_{N-1}, \end{cases}$$

$$a = c = -a\tau h_x^2 \tag{6}$$
$$b_j = 2h_x^2 h_y^2 + 2a\tau h_x^2 \tag{7}$$
$$d_i = a\tau h_y^2 u_{i-1j}^{k+1/2} + (2h_x^2 h_y^2 - 2a\tau h_y^2) u_{ij}^{k+1/2} + a\tau h_y^2 u_{i+1j}^{k+1/2} \tag{8}$$
$$d_1 = d - au_{i0}^{k+1} \tag{9}$$
$$d_{N-1} = d - cu_{iN}^{k+1} \tag{10}$$

In [9]:
```
def alternating_directions_scheme(x_begin, x_end, y_begin, y_end, t_begin, t_end,
                                  h_x, h_y, tau, mu_1, mu_2, a=1):

    x = np.arange(x_begin, x_end(mu_1) + h_x, h_x)
    y = np.arange(y_begin, y_end(mu_2) + h_y, h_y)
    t = np.arange(t_begin, t_end + tau, tau)

    u = np.zeros((x.size, y.size, t.size))

    for i in range(x.size):
```

```
        for j in range(y.size):
            u[i, j, 0] = psi(x[i], y[j], mu_1, mu_2)

    for k in range(1, t.size):

        u[:, 0, k] = phi_1(x, t[k], mu_1, mu_2)
        u[:, -1, k] = phi_2(x, t[k], mu_1, mu_2)
        u[0, :, k] = phi_3(y, t[k], mu_1, mu_2)
        u[-1, :, k] = phi_4(y, t[k], mu_1, mu_2)

        u_half = np.zeros((x.size, y.size))

        u_half[:, 0] = phi_1(x, t[k] - tau / 2, mu_1, mu_2)
        u_half[:, -1] = phi_2(x, t[k] - tau / 2, mu_1, mu_2)
        u_half[0, :] = phi_3(y, t[k] - tau / 2, mu_1, mu_2)
        u_half[-1, :] = phi_4(y, t[k] - tau / 2, mu_1, mu_2)

        for j in range(1, y.size - 1):
            A = np.zeros((x.size - 2, x.size - 2))
            b = np.zeros((x.size - 2))

            A[0, 0] = 2 * h_x**2 * h_y**2 + 2 * a * tau * h_y**2
            A[0, 1] = -a * tau * h_y**2
            for ind in range(1, x.size - 3):
                A[ind, ind - 1] = -a * tau * h_y**2
                A[ind, ind] = 2 * h_x**2 * h_y**2 + 2 * a * tau * h_y**2
                A[ind, ind + 1] = -a * tau * h_y**2
            A[-1, -2] = -a * tau * h_y**2
            A[-1, -1] = 2 * h_x**2 * h_y**2 + 2 * a * tau * h_y**2

            for i in range(1, x.size - 1):
                b[i - 1] = (
                    u[i, j - 1, k - 1] * a * tau * h_x**2
                    + u[i, j, k - 1] * (2 * h_x**2 * h_y**2 - 2 * a * tau * h_x**2)
                    + u[i, j + 1, k - 1] * a * tau * h_x**2
                )
            b[0] -= (-a * tau * h_y**2) * phi_3(y[j], t[k] - tau / 2, mu_1, mu_2)
            b[-1] -= (-a * tau * h_y**2) * phi_4(y[j], t[k] - tau / 2, mu_1, mu_2)

            u_half[1:-1, j] = LinearAlgebra.sweep_method(A, b)

        for i in range(1, x.size - 1):
            A = np.zeros((y.size - 2, y.size - 2))
            b = np.zeros((y.size - 2))

            A[0, 0] = 2 * h_x**2 * h_y**2 + 2 * a * tau * h_x**2
            A[0, 1] = -a * tau * h_x**2
            for ind in range(1, y.size - 3):
                A[ind, ind - 1] = -a * tau * h_x**2
                A[ind, ind] = 2 * h_x**2 * h_y**2 + 2 * a * tau * h_x**2
                A[ind, ind + 1] = -a * tau * h_x**2
            A[-1, -2] = -a * tau * h_x**2
            A[-1, -1] = 2 * h_x**2 * h_y**2 + 2 * a * tau * h_x**2

            for j in range(1, y.size - 1):
                b[j - 1] = (
```

```
                    u_half[i - 1, j] * a * tau * h_y**2
                    + u_half[i, j] * (2 * h_x**2 * h_y**2 - 2 * a * tau * h_y**2)
                    + u_half[i + 1, j] * a * tau * h_y**2
                )
            b[0] -= (-a * tau * h_x**2) * phi_1(x[i], t[k], mu_1, mu_2)
            b[-1] -= (-a * tau * h_x**2) * phi_2(x[i], t[k], mu_1, mu_2)

            u[i, 1:-1, k] = LinearAlgebra.sweep_method(A, b)

    for k in range(t.size):
        u[:, 0, k] = phi_1(x, t[k], mu_1, mu_2)
        u[:, -1, k] = phi_2(x, t[k], mu_1, mu_2)
        u[0, :, k] = phi_3(y, t[k], mu_1, mu_2)
        u[-1, :, k] = phi_4(y, t[k], mu_1, mu_2)

    return u
```

In [10]:
```
mu_1, mu_2 = mu[0]

u_alternating = alternating_directions_scheme(x_begin, x_end, y_begin, y_end, t_beg
                                              h_x, h_y, tau, mu_1, mu_2, a=1)
```

Метод дробных шагов

Метод дробных шагов использует только неявные схемы

Шаг 1.

Решаем систему уравнений для всех j, чтобы получить значения в момент времени $k + 1/2$:

$$\begin{cases} bu_{1j}^{k+1/2} + cu_{2j}^{k+1/2} = d_1, \\ au_{i-1j}^{k+1/2} + bu_{ij}^{k+1/2} + cu_{i+1j}^{k+1/2} = d, \quad i = 2 \dots N-2, \\ au_{N-2j}^{k+1/2} + bu_{N-1j}^{k+1/2} = d_{N-1}, \end{cases}$$

$$a = c = -a\tau \tag{11}$$
$$b_j = h_x^2 + 2a\tau \tag{12}$$
$$d_j = h_x^2 u_{ij}^k \tag{13}$$
$$d_1 = d_j - au_{0j}^{k+1/2} \tag{14}$$
$$d_{N-1} = d_j - cu_{Nj}^{k+1/2} \tag{15}$$

Шаг 2.

Решаем систему уравнений для всех i, чтобы получить значения в момент времени $k + 1$:

$$\begin{cases} bu_{1j}^{k+1} + cu_{2j}^{k+1} = d_1, \\ au_{i-1j}^{k+1} + bu_{ij}^{k+1} + cu_{i+1j}^{k+1} = d, \quad i = 2 \ldots N-2, \\ au_{N-2j}^{k+1} + bu_{N-1j}^{k+1} = d_{N-1}, \end{cases}$$

$$a = c = -a\tau h_x^2 \tag{16}$$

$$b_j = 2h_x^2 h_y^2 + 2a\tau h_x^2 \tag{17}$$

$$d_i = a\tau h_y^2 u_{i-1j}^{k+1/2} + (2h_x^2 h_y^2 - 2a\tau h_y^2) u_{ij}^{k+1/2} + a\tau h_y^2 u_{i+1j}^{k+1/2} \tag{18}$$

$$d_1 = d_i - au_{i0}^{k+1} \tag{19}$$

$$d_{N-1} = d_i - cu_{iN}^{k+1} \tag{20}$$

Системы будем решать методом прогонки

In [11]:
```
def fractional_steps_scheme(x_begin, x_end, y_begin, y_end, t_begin, t_end,
                            h_x, h_y, tau, mu_1, mu_2, a=1):

    x = np.arange(x_begin, x_end(mu_1) + h_x, h_x)
    y = np.arange(y_begin, y_end(mu_2) + h_y, h_y)
    t = np.arange(t_begin, t_end + tau, tau)

    u = np.zeros((x.size, y.size, t.size))

    for i in range(x.size):
        for j in range(y.size):
            u[i, j, 0] = psi(x[i], y[j], mu_1, mu_2)

    for k in range(1, t.size):

        u_half = np.zeros((x.size, y.size))

        u[:, 0, k] = phi_1(x, t[k], mu_1, mu_2)
        u[:, -1, k] = phi_2(x, t[k], mu_1, mu_2)
        u[0, :, k] = phi_3(y, t[k], mu_1, mu_2)
        u[-1, :, k] = phi_4(y, t[k], mu_1, mu_2)

        u_half[:, 0] = phi_1(x, t[k] - tau / 2, mu_1, mu_2)
        u_half[:, -1] = phi_2(x, t[k] - tau / 2, mu_1, mu_2)
        u_half[0, :] = phi_3(y, t[k] - tau / 2, mu_1, mu_2)
        u_half[-1, :] = phi_4(y, t[k] - tau / 2, mu_1, mu_2)

        for j in range(1, y.size - 1):
            A = np.zeros((x.size - 2, x.size - 2))
            b = np.zeros((x.size - 2))

            A[0, 0] = h_x**2 + 2 * a * tau
            A[0, 1] = -a * tau
            for ind in range(1, x.size - 3):
                A[ind, ind - 1] = -a * tau
                A[ind, ind] = h_x**2 + 2 * a * tau
                A[ind, ind + 1] = -a * tau
            A[-1, -2] = -a * tau
            A[-1, -1] = h_x**2 + 2 * a * tau
```

```
            for i in range(1, x.size - 1):
                b[i - 1] = u[i, j, k - 1] * h_x**2
            b[0] -= (-a * tau) * phi_3(y[j], t[k] - tau / 2, mu_1, mu_2)
            b[-1] -= (-a * tau) * phi_4(y[j], t[k] - tau / 2, mu_1, mu_2)

            u_half[1:-1, j] = LinearAlgebra.sweep_method(A, b)

        for i in range(1, x.size - 1):
            A = np.zeros((y.size - 2, y.size - 2))
            b = np.zeros((y.size - 2))

            A[0, 0] = h_y**2 + 2 * a * tau
            A[0, 1] = -a * tau
            for ind in range(1, y.size - 3):
                A[ind, ind - 1] = -a * tau
                A[ind, ind] = h_y**2 + 2 * a * tau
                A[ind, ind + 1] = -a * tau
            A[-1, -2] = -a * tau
            A[-1, -1] = h_y**2 + 2 * a * tau

            for j in range(1, y.size - 1):
                b[j - 1] = u_half[i, j] * h_y**2
            b[0] -= (-a * tau) * phi_1(x[i], t[k], mu_1, mu_2)
            b[-1] -= (-a * tau) * phi_2(x[i], t[k], mu_1, mu_2)

            u[i, 1:-1, k] = LinearAlgebra.sweep_method(A, b)

        u[:, 0, k] = phi_1(x, t[k], mu_1, mu_2)
        u[:, -1, k] = phi_2(x, t[k], mu_1, mu_2)
        u[0, :, k] = phi_3(y, t[k], mu_1, mu_2)
        u[-1, :, k] = phi_4(y, t[k], mu_1, mu_2)

    return u
```

In [12]:
```
mu_1, mu_2 = mu[0]

u_fractional = fractional_steps_scheme(x_begin, x_end, y_begin, y_end, t_begin, t_e
                                       h_x, h_y, tau, mu_1, mu_2, a=1)
```

Этот метод абсолютно устойчив

Полученные результаты

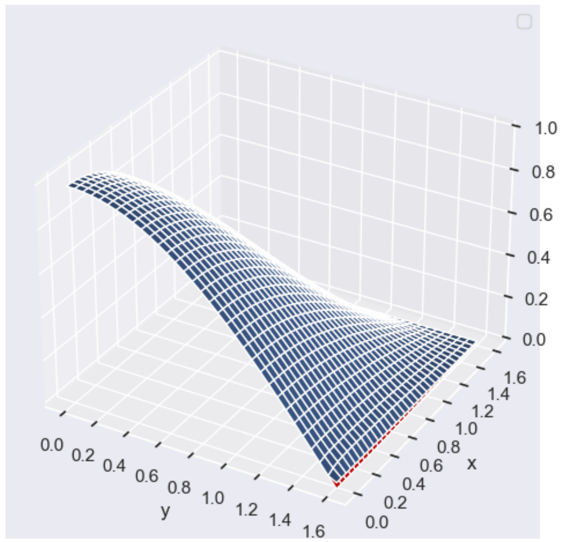
In [13]:
```
def plot_surface(u_exact, u):
    x = np.arange(x_begin, x_end(mu_1) + h_x, h_x)
    y = np.arange(y_begin, y_end(mu_2) + h_y, h_y)
    y, x = np.meshgrid(y, x)

    fig, ax = plt.subplots(subplot_kw={"projection": "3d"})
    ax.set_ylabel('x')
    ax.set_xlabel('y')
    ax.set_zlim(0, 1)
    ax.plot_surface(x, y, u_exact, color='red')
    ax.plot_surface(x, y, u)
```

```
    ax.legend()
    plt.show()
```

In [14]:
```
plot_surface(u_exact[:, :, 1], u_alternating[:, :, 1])
```

```
No artists with labels found to put in legend.  Note that artists whose label star
t with an underscore are ignored when legend() is called with no argument.
```

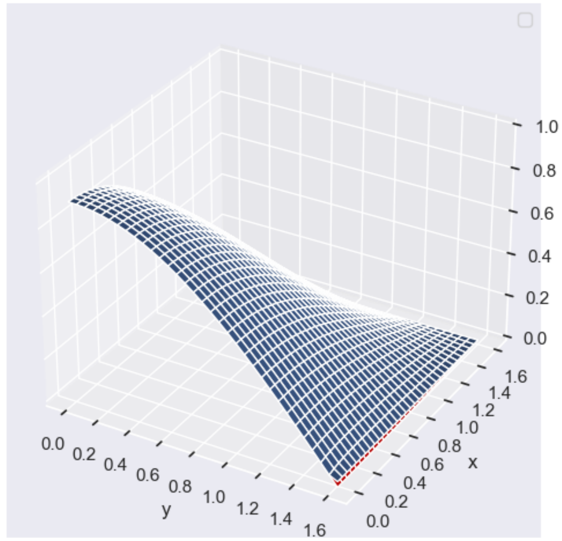
Figure



In [15]:
```
plot_surface(u_exact[:, :, 5], u_fractional[:, :, 5])
```

```
No artists with labels found to put in legend.  Note that artists whose label star
t with an underscore are ignored when legend() is called with no argument.
```

Figure



In [16]:
```
def print_errors():
    #mu_params = [(1, 1), (2, 1), (1, 2)]
    mu_params = [(1, 1)]
    for mu in mu_params:
        mu_1, mu_2 = mu[0], mu[1]
        print(f'mu1 = {mu_1}, mu2 = {mu_2} \n')
        u_exact = get_analytical_solution(x_begin, x_end, y_begin, y_end, t_begin,

        u_alternating = alternating_directions_scheme(x_begin, x_end, y_begin, y_en
                                                       h_x, h_y, tau, mu_1, mu_2, a=

        print(f'alternating directions, max abs error = {np.max(abs(u_alternating -
        print(f'alternating directions, mean abs error = {np.mean(abs(u_alternating

        u_fractional = fractional_steps_scheme(x_begin, x_end, y_begin, y_end, t_be
                                                h_x, h_y, tau, mu_1, mu_2, a=1)

        print(f'fractional steps, max abs error = {np.max(abs(u_fractional - u_exac
        print(f'fractional steps, mean abs error = {np.mean(abs(u_fractional - u_ex
```

In [17]:
```
print_errors()
```

```
mu1 = 1, mu2 = 1

alternating directions, max abs error = 0.029199522301288815
alternating directions, mean abs error = 0.0022884356346231237

fractional steps, max abs error = 0.02862133302091431
fractional steps, mean abs error = 0.0023677278088452356
```

Вывод

В данной лабораторной работе я научился решать двумерные начально-краевые задачи параболического типа.

Для решения я использовал два метода:

- метод переменных направлений
- метод дробных шагов

С помощью каждого метода мне удалось получить результат с хорошей точностью