COMP9417 : Homework 0 4/06/22

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Question 1. (Calculus Review)

a) $f(x,y) = a_1x^2y^2 + a_4xy + a_5x + a_7$

$$\frac{\partial f}{\partial x} = 2a_1xy^2 + a_4y + a_5$$

$$\frac{\partial f}{\partial x} = 2a_1x^2y + a_4x$$

$$\frac{\partial^2 f}{\partial x\partial y} = \frac{\partial^2 f}{\partial y\partial x} = 4a_1xy + a_4$$

$$\frac{\partial^2 f}{\partial x^2} = 2a_1y^2$$

$$\frac{\partial^2 f}{\partial y^2} = 2a_1x^2$$

b) $f(x,y) = a_1x^2y^2 + a_2x^2y + a_3xy^2 + a_4xy + a_5x + a_6y + a_7$

$$\frac{\partial f}{\partial x} = 2a_1xy^2 + 2a_2xy + a_3y^2 + a_4y + a_5$$

$$\frac{\partial f}{\partial y} = 2a_1x^2y + a_2x^2 + 2a_3xy + a_4x + a_6$$

$$\frac{\partial^2 f}{\partial x\partial y} = \frac{\partial^2 f}{\partial y\partial x} = 4a_1xy + 2a_2x + 2a_3y + a_4$$

$$\frac{\partial^2 f}{\partial x^2} = 2a_1y^2 + 2a_2y$$

$$\frac{\partial^2 f}{\partial y^2} = 2a_1x^2 + 2a_3x$$

c) $\sigma(x) = \frac{1}{1+e^{-x}} = (1+e^{-x})^{-1}$

$$\sigma'(x) = -1 \times -e^{-x} \times (1+e^{-x})^{-2}$$

$$= \frac{e^{-x}}{(1+e^{-x})^2}$$

$$= \frac{e^{-x}}{1+e^{-x}} \times \frac{1}{1+e^{-x}}$$

$$= (1-\sigma(x))\sigma(x)$$

d) • $y_1 = 4x^2 - 3x + 3$

$y_1' = 8x - 3$, $y_1'' = 8$

Therefore, $x = \frac{3}{8}$ when $y_1' = 0$ and $y_1'' = 8 > 0$. Hence, the local minimum points is $(\frac{3}{8}, \frac{39}{16})$ and there are no local maximum points.

• $y_2 = 3x^4 - 2x^3$

$y_2' = 12x^3 - 6x^2$

$y_2'' = 36x^2 - 12x$.

Therefore, $x=0$ or $\frac{1}{2}$ when $y_2' = 0$

When $x=0$ and $y_2'' = 0$ there are no local minimum or maximum points at $x=0$.

When $x = \frac{1}{2}$ and $y_2'' = 3 > 0$, hence the local minimum points at $(\frac{1}{2}, -\frac{1}{16})$. There are no local maximum points.

• $y_3 = 4x + \sqrt{1-x}$

$y_3' = 4 - \frac{1}{2} \times (1-x)^{-\frac{1}{2}}$, $y_3'' = -\frac{1}{4}(1-x)^{-\frac{3}{2}}$

Therefore, $x = \frac{63}{64}$ when $y' = 0$.

When $x = \frac{63}{64}$, $y_2'' < 0$, therefore, the local maximum points is $(\frac{63}{64}, \frac{65}{16})$ and there are no local minimum points.

• $y_4 = x + x^{-1}$

$y_4' = 1 - x^{-2}$, $y_4'' = 2x^{-3}$.

Therefore, when $y_4' = 0$, $x = \pm 1$.

when $x = 1$, $y_4'' > 0$, hence the local minimum point is $(1, 2)$,

$x = -1$, $y_4'' < 0$, hence the local maximum point is $(-1, -2)$

Question 2 (Probability Review)

a) $P(A) = 0.3, \quad P(B) = 0.4,$

$$P(A \cup B) = P(A) + P(B) - P(AB) = 0.3 + 0.4 - 0.1 = 0.6$$

$$P(\bar{A}\bar{B}) = P(\overline{A \cup B}) = 1 - P(A \cup B) = 0.4$$

b) i) Probabilities must add up to 1. Hence $r = \frac{1}{3}$

ii) $P(Y=2, Y=3) = \frac{1}{6}$ (from the table)

iii) $P(X=3) = P(X=3, y=1) + P(X=3, y=2) + P(X=3, y=3)$

$$= \frac{1}{3}$$

$$P(X=3 \mid Y=2) = \frac{P(Y=3, X=2)}{P(Y=2)}$$

$$= \frac{P(Y=3, X=2)}{P(X=1, Y=2) + P(X=2, Y=2) + P(X=3, Y=2)}$$

$$= \frac{\frac{1}{3}}{\frac{1}{12} + 0 + \frac{1}{3}}$$

$$= \frac{\frac{1}{3}}{\frac{5}{12}} = \frac{4}{5}$$

iv) $E(X) = 1 \times P(X=1) + 2 \times P(X=2) + 3 \times P(X=3)$

$$= 1 \times \frac{1}{3} + 2 \times \frac{1}{3} + 3 \times \frac{1}{3}$$

$$= 2.$$

$E(Y) = \frac{23}{12}$, $\quad E(XY) = 1 \times 1 \times P(X=1, Y=1) + 1 \times 2 \times P(X=1, Y=2)$
$+ 1 \times 3 \times P(X=1, Y=3) + 2 \times 1 \times P(X=2, Y=1)$
$+ 2 \times 2 \times P(X=2, Y=2) + 2 \times 3 \times P(X=3, Y=3)$
$+ 3 \times 1 \times P(X=3, Y=1) + 3 \times 2 \times P(X=3, Y=2)$
$+ 3 \times 3 \times P(X=3, Y=3)$

$$= \frac{47}{12}$$

v) $E(X^2) = 1^2 \times P(X=1) + 2^2 \times P(X=2) + 3^2 \times P(X=3)$

$= 1 \times \frac{1}{3} + 4 \times \frac{1}{3} + 9 \times \frac{1}{3} = \frac{14}{3}$

$E(Y^2) = (1 \times \frac{1}{3}) + (4 \times \frac{5}{12}) + (9 \times \frac{1}{4}) = \frac{1}{3} + \frac{5}{3} + \frac{9}{4} = \frac{17}{4}$

vi) $Cov(X,Y) = E(XY) - E(X)E(Y)$

$= \frac{47}{12} - 2 \times \frac{23}{12} = \frac{1}{12}$

vii) $Var(X) = E(X^2) - E(X)^2 = \frac{14}{3} - 2^2 = \frac{2}{3}$

$Var(Y) = E(Y^2) - E(Y)^2 = \frac{17}{4} - \left(\frac{23}{12}\right)^2 = \frac{83}{144}$

viii) $Corr(X,Y) = \frac{Cov(X,Y)}{\sqrt{Var(X)}\sqrt{Var(Y)}} = \frac{\frac{1}{12}}{\sqrt{\frac{2}{3}} \times \sqrt{\frac{83}{144}}} = 0.134$

ix) $E(X+Y) = E(X) + E(Y) = 2 + \frac{23}{12} = \frac{47}{12}$

$E(X+Y^2) = E(X) + E(Y^2) = 2 + \frac{17}{4} = \frac{25}{4}$

$Var(X+Y) = Var(X) + Var(Y) + 2Cov(X,Y) = \frac{2}{3} + \frac{83}{144} + 2 \times \frac{1}{12}$

$= \frac{323}{144}$

* $Var(X+Y^2) = Var(X) + Var(Y^2) + 2Cov(X,Y^2)$

$= Var(X) + E(Y)^2 \times Var(Y) + Var(Y) \times E(Y^2) + 2Cov(X,Y^2)$

Question 3 (Linear Algebra Review)

a) $dim(A) = 3 \times 5$, $dim(B) = 6 \times 1$, $dim(A^T) = 5 \times 3$

b) i) $A \in \mathbb{R}^{3\times3}$ and $B \in \mathbb{R}^{2\times2}$, hence AB and BA can't be computed.

ii) $AC = \begin{pmatrix} 21 & 14 & 14 \\ 20 & 10 & 10 \\ 56 & 28 & 28 \end{pmatrix}$ and $CA = \begin{pmatrix} 31 & 39 & 40 \\ 10 & 12 & 12 \\ 18 & 18 & 16 \end{pmatrix}$

iii) $AD = \begin{pmatrix} 20 & 32 \\ 17 & 19 \\ 43 & 45 \end{pmatrix}$, however DA is not defined hence it cannot be computed.

iv) DC is not defined, hence it can't be computed.

$CD = \begin{pmatrix} 43 & 41 \\ 13 & 13 \\ 18 & 22 \end{pmatrix}$, $D^TC = \begin{pmatrix} 38 & 18 & 18 \\ 32 & 18 & 18 \end{pmatrix}$

v) uB does not exist and $Bu = \begin{pmatrix} 14 \\ 4 \end{pmatrix}$

vi) Av does not exist and thus cannot be computed.

vii) $Av = \begin{pmatrix} 18 \\ 13 \\ 31 \end{pmatrix}$, vA is not defined so cannot be computed.

viii) Av + Bv cannot be computed due to Bv

c) i) $\|u\|_1 = |1| + |3| = 4$ $\quad \|u\|_2^2 = u^T u = [10]$

$\|u\|_2 = \sqrt{1^2 + 3^2} = \sqrt{10}$ $\quad \|u\|_\infty = 3$

ii) $\|v\|_1 = |2| + |4| + |1| = 7$ $\quad \|v\|_2^2 = v^T v = [21]$

$\|v\|_2 = \sqrt{2^2 + 4^2 + 1^2} = \sqrt{21}$ $\quad \|v\|_\infty = 4$

iii) $v + w = \begin{bmatrix} 3 \\ 2 \\ 3 \end{bmatrix}$ $\quad \|v+w\|_2 = \sqrt{3^2 + 2^2 + 3^2} = \sqrt{22}$

$\|v+w\|_1 = |3| + |2| + |3| = 8$ $\quad \|v+w\|_\infty = 3$

iv) $Av = \begin{bmatrix} 18 \\ 13 \\ 31 \end{bmatrix}$, $\|Av\|_2 = \sqrt{18^2 + 13^2 + 31^2} = 38{,}131$

$A(v-w) = \begin{bmatrix} 15 \\ 12 \\ 27 \end{bmatrix}$, $\|A(v-w)\|_\infty = 27$

d) $\langle u, v \rangle = 1 \times 1 + 2 \times 1 = 3$, $\langle u, w \rangle = 0$, $\langle v, w \rangle = -\frac{1}{2}$

$\langle x, y \rangle = \|x\|_2 \|y\|_2 \cos(\theta_{xy})$

$$\theta_{uv} = \cos^{-1}\left(\frac{3}{\sqrt{5} \times \sqrt{2}}\right) = 18.43^\circ$$

$\theta_{uw} = 90^\circ$, $\theta_{uw} = 108.43^\circ$

e) When the dot product is positive, the resulting angle is acute, when the dot product is zero, then the ~~acute~~ angle is a right angle, and when the dot-product is negative, the result is an obtuse angle.

When calculating the dot products are used to measure the strength of relationship between two objects (magnitude). A dot product between 2 vectors which is positive means that the object points in the same direction.

f) $A = \begin{pmatrix} 1 & 3 \\ 4 & 1 \end{pmatrix}$

$$A^{-1} = \frac{1}{1\times1-4\times3}\begin{pmatrix} 1 & -3 \\ -4 & 1 \end{pmatrix} = -\frac{1}{11}\begin{pmatrix} 1 & -3 \\ -4 & 1 \end{pmatrix} = \begin{pmatrix} -1/11 & 3/11 \\ 4/11 & 1/11 \end{pmatrix}$$

g) $A = \begin{pmatrix} 3 & 3 \\ 4 & 4 \end{pmatrix}$

$\det(A) = 0$, hence inverse does not exist.

h) A is symmetric if and only if $A = A^T$.

Thus for X^TX to be symmetric:

$(X^TX)^T = X^T(X^T)^T = X^TX$. Thus X^TX is always symmetric.