Интегралы, часть 5
Практика

№8.5.2

$$\int \frac{dx}{\sin x} = [\text{табличный}] = \ln\left|tg\frac{x}{2}\right| + C$$

№8.5.3

$$\int \frac{dx}{5\cos x+3} = \left[t = tg\frac{x}{2},\ dx = \frac{2dt}{1+t^2};\ \cos x = \frac{1-t^2}{1+t^2} \right] = \int \frac{2dt}{(1+t^2)\cdot 5\cdot\left(\frac{1-t^2}{1+t^2}\right)+3} =$$

$$= \int \frac{2dt}{8-5t^2} = -2\int\frac{dt}{5t^2-8} =$$

$$= -\frac{2}{\sqrt{5}}\int\frac{\sqrt{5}\,dt}{5t^2-8} = -\frac{2}{\sqrt{5}}\cdot\frac{1}{4\sqrt{2}}\ln\left|\frac{\sqrt{5}t-2\sqrt{2}}{\sqrt{5}t+2\sqrt{2}}\right| + C =$$

$$= -\frac{2}{4\sqrt{10}}\ln\left|\frac{\sqrt{5}t-2\sqrt{2}}{\sqrt{5}t+2\sqrt{2}}\right| + C =$$

$$= \left[t = tg\frac{x}{2}\right] = -\frac{2}{4\sqrt{10}}\ln\left|\frac{\sqrt{5}\,tg\frac{x}{2}-2\sqrt{2}}{\sqrt{5}\,tg\frac{x}{2}+2\sqrt{2}}\right| + C$$

№8.5.4

$$\int\frac{dx}{3\sin^2x+5\cos^2x} = \left[tg x = t;\ x = arctg t;\ dx = \frac{dt}{1+t^2};\ \sin^2 x = \frac{1}{t^2\left(\frac{1}{t^2}+1\right)};\ \cos^2 x = \frac{1}{t^2+1} \right]$$

$$= \int \frac{\frac{dt}{1+t^2}}{3\left(\frac{1}{t^2\left(\frac{1}{t^2}+1\right)}\right)+5\left(\frac{1}{t^2+1}\right)} =$$

$$= \int \frac{\frac{dt}{t^2+1}}{\frac{8}{t^2+1}} = \frac{1}{8}\int dt = \frac{1}{8}t + C =$$

$$= \frac{1}{8}\operatorname{tg} x + C$$

w 8.5.6

$$\int \frac{dx}{\sin^5 x \cdot \cos x} = \left[t = \cos x ;\ \sin x = \sqrt{1-\cos^2 x} \right.$$

$$\left. = \sqrt{1-t^2} ;\ dx = -\frac{dt}{\sqrt{1-t^2}} \right] =$$

$$= -\int \frac{dt}{\sqrt{1-t^2}\left(\sqrt{1-t^2}\right)^5 \cdot t} = -\int \frac{dt}{t(1-t^2)^3} =$$

$$= \int \frac{dt}{t(t^2-1)^3} \quad \cancel{= \int \frac{dt}{t(t^6-3t^4+3t^2-1)}}$$

$$\cancel{= \int \frac{dt}{t^7-3t^5+3t^3-t}}$$

$$\frac{1}{t(t^2-1)^3} = \frac{1}{t(t-1)^3(t+1)^3} = \frac{A}{t} +$$

$$+ \frac{B}{(t-1)^3} + \frac{C}{(t-1)^2} + \frac{D}{t-1} + \frac{E}{(t+1)^3} + \frac{F}{(t+1)^2} + \frac{G}{t+1}$$

$$1 = A(t^2-1)^3 + B(t+1)^3 t + C(t+1)^3(t-1)t + D(t-1)^2(t+1)^3 t + E(t-1)^3 t + F(t+1)(t-1)^3 t + G(t+1)^2(t-1)^3 t =$$

$$= At^6 - 3At^4 + 3At^2 - A + Bt^4 + 3Bt^3 + 3Bt^2 + Bt + Ct^5 + 2Ct^4 - 2Ct^2 - Ct + Dt^6 + Dt^5 - 2Dt^4 - 2Dt^3 + Dt^2 + Dt + Et^4 - 3Et^3 + 3Et^2 - Et + Ft^5 - 2Ft^4 + 2Ft^2 - Ft + Gt^6 - Gt^5 - 2Gt^4 + 2Gt^3 + Gt^2 - Gt = t^6(A+D+G) + t^5(C+D+F-G) + t^4(-3A+B+2C-2D+E-2F-2G) + t^3(3B-2D-3E+2G) + t^2(3A+3B-2C+D+3E+2F+G) + t(B-C+D-E-F-G) - A;$$

$$\begin{cases} A+D+G=0 \\ C+D+F-G=0 \\ -3A+B+2C-2D+E-2F-2G=0 \\ 3B-2D-3E+2G=0 \\ 3A+3B-2C+D+3E+2F+G=0 \\ B-C+D-E-F-G=0 \\ A=-1 \end{cases} \Rightarrow$$

$\Rightarrow A = -1,\ B = \frac{1}{2},\ C = -\frac{1}{2},$
$D = \frac{1}{2},\ E = \frac{1}{2},\ F = \frac{1}{2},\ G = \frac{1}{2};$

$$\int\left(-\frac{1}{t} + \frac{1}{2(t-1)^3} - \frac{1}{2}\cdot\frac{1}{(t-1)^2} + \frac{1}{2(t-1)} + \frac{1}{2(t+1)^3} + \frac{1}{2(t+1)^2} + \frac{1}{2(t+1)}\right)dt =$$

$$= -\ln|t| - \frac{1}{2}\cdot\frac{1}{3(t-1)^2} + \frac{1}{2}\cdot\frac{1}{2(t-1)} - \frac{1}{2}\ln|t-1| - \frac{1}{2}\cdot\frac{1}{3(t+1)^2} - \frac{1}{2}\cdot\frac{1}{2(t+1)} +$$

$$+ \frac{1}{2}\ln|t+1| + C = -\ln|t| - \frac{1}{2}\ln|t-1| + \frac{1}{2}\ln|t+1| - \frac{1}{2}\left(\frac{2(t+1)^2 - 3(t-1)(t+1)^2 + 2(t-1)^2}{6(t^2-1)^2} + \frac{3(t+1)(t-1)^2}{6(t^2-1)^2}\right) + C = -\ln|t| - \frac{1}{2}\ln|t-1| +$$

$$+ \frac{1}{2}\ln|t+1| - \frac{1}{12}\left(\frac{2t^2+4t+2-3t^3-3t^2}{(t^2-1)^2} + \frac{3t+3+2t^2-4t+2+3t^3-3t^2-3t+3}{(t^2-1)^2}\right) + C =$$

$$= -\ln|t| - \frac{1}{2}\ln|t-1| + \frac{1}{2}\ln|t+1| - \frac{1}{12}\left(\frac{-2t^2+10}{(t^2-1)^2}\right) = -\ln|t| + \frac{1}{2}\ln\left|\frac{t+1}{t-1}\right| +$$

$$+ \frac{1}{6}\cdot\frac{t^2-5}{(t^2-1)^2} + C = -\ln|\cos x| +$$

$$+\frac{1}{2}\ln\left|\frac{\cos x+1}{\cos x-1}\right|+\frac{\cos^2 x-5}{6(\cos^2 x-1)^2}+C=$$

$$=-\ln|\cos x|+\frac{1}{2}\ln\left|\frac{\cos x+1}{\cos x-1}\right|+$$

$$+\frac{\cos^2 x-5}{6\sin^4 x}+C$$

№ 8.5.8

$$\int \sin^3 x\,dx=\Big[t=\cos x,\ \sin x=$$

$$=\sqrt{1-\cos^2 x}=\sqrt{1-t^2};\ dx=-\frac{dt}{\sqrt{1-t^2}}\Big]=$$

$$=-\int(1-t^2)\sqrt{1-t^2}\,\frac{dt}{\sqrt{1-t^2}}=-\int(1-t^2)\,dt=$$

$$=-\int dt+\int t^2\,dt=-t+\frac{t^3}{3}+C=$$

$$=-\cos x+\frac{\cos^3 x}{3}+C$$

№ 8.5.9

$$\int\frac{\cos^3 x\,dx}{\sin^4 x}=\Big[t=\sin x,\ \cos x=$$

$$=\sqrt{1-\sin^2 x}=\sqrt{1-t^2};\ dx=\frac{dt}{\sqrt{1-t^2}}\Big]=$$

$$=\int\frac{\sqrt{1-t^2}\,(1-t^2)\,dt}{t^4\sqrt{1-t^2}}=\int\frac{1-t^2}{t^4}\,dt=$$

$$=\int\frac{dt}{t^4}-\int\frac{dt}{t^2}=-\frac{1}{3t^3}+\frac{1}{t}+C=$$

$$= \frac{1}{\sin x} - \frac{1}{3\sin^3 x} + C$$

№ 8.5.11

$$\int \cos^4 x\,dx = \int \left(\frac{1+\cos 2x}{2}\right)^2 dx =$$

$$= \frac{1}{4}\int (1 + 2\cos 2x + \cos^2 2x)\,dx =$$

$$= \frac{1}{4}\int dx + \cancel{\tfrac{1}{4}}\int \cancel{2}\cos 2x\,dx + \int \cos^2 2x\,dx =$$

$$= \frac{1}{4}\int dx + \int \cos 2x \cdot 2dx + \int \frac{1+\cos 4x}{2}\,dx =$$

$$= \frac{1}{4}\int dx + \int \cos 2x\,d(2x) + \frac{1}{2}\int (1+\cos 4x)\,dx =$$

$$= \frac{1}{4}\int dx + \int \cos 2x\,d(2x) + \frac{1}{2}\int dx + \frac{1}{2}\int \cos 4x\,dx =$$

$$= \frac{3}{4}\int dx + \int \cos 2x\,d(2x) + \frac{1}{8}\int \cos 4x\,d(4x) =$$

$$= \frac{3}{4}x + \sin 2x + \frac{1}{8}\sin 4x + C$$

№ 8.5.12

$$\int \sin^2 x \cos^2 x\,dx = \int (1-\cos^2 x)\cos^2 x\,dx =$$

$$= \int (\cos^2 x - \cos^4 x)\,dx =$$

$$= \int \frac{1+\cos 2x}{2}\,dx - \int \left(\frac{1+\cos 2x}{2}\right)^2 dx =$$

$= [$ 2) по № 8.5.11 $= \frac{3}{4}x + \sin 2x + \frac{1}{8}\sin 4x;$

1) $\frac{1}{2}\int(1+\cos 2x)dx = \frac{1}{2}\int dx + \frac{1}{2}\int\cos 2x dx =$

$= \frac{1}{2}x + \frac{1}{4}\sin 2x \,] = \frac{1}{2}x + \frac{1}{4}\sin 2x -$

$- \frac{3}{4}x - \sin 2x - \frac{1}{8}\sin 4x + C =$

$= -\frac{1}{4}x - \frac{3}{4}\sin 2x - \frac{1}{8}\sin 4x + C$

№ 8.5.14

$\int\cos 2x \sin 4x\, dx = \int\frac{1}{2}(\sin(4x-2x) +$

$+ \sin(4x+2x))dx = \frac{1}{2}\int\sin 2x dx +$

$+ \frac{1}{2}\int\sin 6x dx = \cancel{\int\sin x\cos x dx +}$

$\cancel{+\int\sin 3x\cos 3x dx} = \frac{1}{4}\int\sin 2x\, d(2x) +$

$+ \frac{1}{12}\int\sin 6x\, d(6x) = -\frac{1}{4}\cos 2x - \frac{1}{12}\cos 6x + C$

№ 8.5.15

$\int\sin\frac{x}{2}\sin\frac{3x}{2}dx = \int\frac{1}{2}\left(\cos\left(\frac{x}{2}-\frac{3x}{2}\right) - \cos\left(\frac{x}{2}+\frac{3x}{2}\right)\right)\cdot$

$\cdot dx = \int\frac{1}{2}(\cos x - \cos 2x)dx =$

$= \frac{1}{2}\int\cos x dx - \frac{1}{2}\int\cos 2x dx =$

$= \frac{1}{2}\sin x - \frac{1}{4}\sin 2x + C$

№8.5.16

$$\int ctg^3 x\,dx = \int\left(\frac{1}{\sin^2 x}-1\right)\frac{\cos x}{\sin x}\,dx =$$

$$= \int\left(\frac{\cos x}{\sin^3 x}-\frac{\cos x}{\sin x}\right)dx =$$

$$= \int\frac{\cos x\,dx}{\sin^3 x}-\int\frac{\cos x\,dx}{\sin x} = [t = \sin x,$$

$$dt = \cos x\,dx] = \int\frac{dt}{t^3}-\int\frac{dt}{t} =$$

$$= -\frac{1}{2t^2}-\ln|t|+C = -\frac{1}{2\sin^2 x}-\ln|\sin x|+C$$

№8.5.18

$$\int tg^2 x\,dx = \int\left(\frac{1}{\cos^2 x}-1\right)dx =$$

$$= \int\frac{dx}{\cos^2 x}-\int dx = tg x - x + C$$