§6 Определённый интеграл. Приёмы вычисления

Основные понятия и св-ва

Пусть $y=f(x) \in [a;b]$ и $\exists x_0, x_1, \ldots, x_n \mid$
$a=x_0<x_1<\ldots<x_n=b$
$\exists c_i \in [x_{i-1}; x_i]$, $i=1,2,\ldots,n$

$\Rightarrow S_n=\sum_{i=1}^{n} f(c_i)\cdot\Delta x_i$,

$\Delta x_i = x_i - x_{i-1}$

- это интегральная сумма ф-ции на отрезке.

$\Rightarrow$ Опр. инт.:

$$\int_a^b f(x)dx = \lim_{\substack{n\to\infty \\ \max\Delta x_i\to 0}} \sum_{i=1}^{n} f(c_i)\Delta x_i$$

Теорема сущ-ния опр. инт.:
$f(x)$ - непр. на $[a;b] \Rightarrow \exists \lim$

Св-ва опр. инт.:

1. $\int_a^b f(x)dx = -\int_b^a f(x)dx$

2. $\int_a^a f(x)dx = 0$

3. $\int_a^b f(x)dx = \int_a^b f(t)dt$ - переменная интегрирования может быть любой.

4. $\int_a^b (f_1(x) \pm f_2(x))dx =$
$= \int_a^b f_1(x)dx \pm \int_a^b f_2(x)dx$

5. $\int_a^b cf(x)dx = c\int_a^b f(x)dx$

6. $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$,
$a < c < b$

7. $f(x) \geq 0$ на $[a;b]$ $\Rightarrow$
$\Rightarrow \int_a^b f(x)dx \geq 0$

$f(x) \leq 0 \ \forall x \in [a;b] \Rightarrow$
$\Rightarrow \int_a^b f(x)dx \leq 0$

8. $f(x) \le g(x)\ \forall x \in [a;b] \Rightarrow$

$\Rightarrow \int_a^b f(x)dx \le \int_a^b g(x)dx$

9. M - наиб. знач., m - наим. знач. $f(x)$ на $[a;b] \Rightarrow$

$\Rightarrow m(b-a) \le \int_a^b f(x)dx \le M(b-a)$

10. $\int_a^b f(x)dx = f(c)(b-a)$, $c \in [a;b]$

<u>(теорема о среднем)</u>

11. $\left| \int_a^b f(x)dx \right| \le \int_a^b |f(x)|\, dx$

12. $\left(\int_a^x f(t)dt \right)'_x = f(x)$

<u>Формула Ньютона - Лейбница</u>

$f(x)$ - непр. на $[a;b]$ и $\exists F(x) \Rightarrow$

$\Rightarrow \boxed{\int_a^b f(x)dx = F(x)\Big|_a^b = F(b) - F(a)}$

$$\int_{-a}^{a} f(x)dx = \begin{cases} 2\int_0^a f(x)dx, & f(x) \text{ - чёт. ф-ция} \\ 0, & f(x) \text{ - нечёт. ф-ция} \end{cases}$$

Интегрирование подстановкой

] $f(x)$ - непр. и для $\int_a^b f(x)dx$
$x = \varphi(t)$.
] $\varphi(t)$ и $\varphi'(t)$ - непр. на $[\alpha; \beta]$,
(*) причем $a = \varphi(\alpha)$ и $b = \varphi(\beta)$ =>
=>
$$\int_a^b f(x)dx = \int_\alpha^\beta f(\varphi(t))\varphi'(t)dt$$
формула замены переменной
в опр. инт.

1) $x = \varphi(t)$ нужно подобрать так, чтобы упростить выражение
2) новые пределы инт. находить из а и в (*)
3) возвращаться к старой переменной не требуется
4) также вместо $x = \varphi(t)$ — $t = \psi(x)$

Интегрирование по частям

$u = u(x)$, $v = v(x)$, ∃ u' и v' - непр.
на $[a; b]$ =>
$$\int_a^b u\,dv = uv\Big|_a^b - \int_a^b v\,du$$
формула интегрирования
по частям для опр. инт.

Примеры: $\int_a^b f(x)dx = F(b) - F(a)$

№ 9.1.1.

$$\int_1^4 x^2 dx = \frac{x^3}{3}\Big|_1^4 = \frac{4^3}{3} - \frac{1^3}{3} = \frac{64}{3} - \frac{1}{3} =$$

$$= \frac{63}{3} = 21$$

№ 9.1.2.

$$\int_{-4}^{-2} \frac{dx}{\sqrt{5-4x-x^2}} = \Big[\, 5-4x-x^2 =$$

$$= -(x^2+4x+5) = -(x^2+4x+5+9-9) =$$

$$= 9-(x^2+4x+16) = 9-(x+2)^2 = 3^2-(x+2)^2\Big]$$

$$= \int_{-4}^{-2} \frac{d(x+2)}{\sqrt{3^2-(x+2)^2}} = \arcsin\frac{x+2}{3}\Big|_{-4}^{-2} =$$

$$= \arcsin\frac{-2+2}{3} - \arcsin\frac{-4+2}{3} =$$

$$= \arcsin 0 - \arcsin\left(-\frac{2}{3}\right) = \arcsin\frac{2}{3}$$

№ 9.1.12

$$\int_0^{\frac{\pi}{2}} \cos^2\left(\frac{\pi}{6}-x\right)dx = \left[\cos^2 x = \frac{1+\cos 2x}{2}\right] =$$

$$= \int_0^{\frac{\pi}{2}} \frac{1}{2}\left(1+\cos\left(\frac{\pi}{3}-2x\right)\right)dx =$$

$$= \int_0^{\frac{\pi}{2}} \left(\frac{1}{2} + \frac{1}{2}\cos\left(\frac{\pi}{3}-2x\right)\right) dx =$$

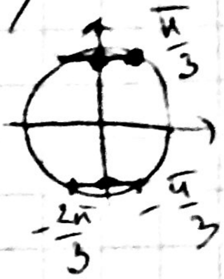
$$= \frac{1}{2}\int_0^{\frac{\pi}{2}} \cancel{\frac{1}{2}}\, dx + \frac{1}{2}\int_0^{\frac{\pi}{2}} \cos\left(\frac{\pi}{3}-2x\right) dx =$$

$$= \frac{1}{2}x\Big|_0^{\frac{\pi}{2}} + \frac{1}{2}\left(-\frac{1}{2}\right)\sin\left(\frac{\pi}{3}-2x\right)\Big|_0^{\frac{\pi}{2}} =$$

$$= \frac{1}{2}\cdot\frac{\pi}{2} - \frac{1}{2}\cdot 0 + \left(-\frac{1}{4}\right)\sin\left(\frac{\pi}{3}-2\cdot\frac{\pi}{2}\right) -$$

$$- \left(-\frac{1}{4}\right)\sin\left(\frac{\pi}{3}-2\cdot 0\right) = \frac{\pi}{4} - \frac{1}{4}\sin\left(-\frac{2\pi}{3}\right) +$$

$$+ \frac{1}{4}\sin\left(\frac{\pi}{3}\right) = \frac{1}{4}\left(\pi + \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2}\right) =$$



$$= \frac{1}{4}\left(\pi + \sqrt{3}\right)$$

W9.1.20

$$\int_1^2 \frac{x^4+1}{x^3(x^2+1)}\, dx = \left[\frac{x^4+1}{x^3(x^2+1)} = \frac{A}{x^3} + \frac{B}{x^2} + \frac{C}{x} + \frac{Dx+E}{(x^2+1)} \Rightarrow x^4+1 = A(x^2+1) +\right.$$

$$+ Bx(x^2+1) + Cx^2(x^2+1) + (Dx+E)x^3 \Rightarrow$$

$$\Rightarrow x^4+1 = Ax^2 + A + Bx^3 + Bx + Cx^4 + Cx^2 +$$

$$+ Dx^4 + Ex^3 \Rightarrow x^4+1 = x^4(C+D) +$$

$$+ x^3(B+E) + x^2(A+C) + x(B) + 1(A) \Rightarrow$$

$$\Rightarrow C+D=1;\quad B+E=0;\quad A+C=0;$$

$B=0;\ A=1 \Rightarrow 1+C=0 \Rightarrow C=-1;$

$0+E=0 \Rightarrow E=0;\quad -1+D=1 \Rightarrow D=2;$

$A=1,\ B=0,\ C=-1,\ D=2,\ E=0 \Rightarrow$

$$\frac{x^4+1}{x^3(x^2+1)} = \frac{1}{x^3} - \frac{1}{x} + \frac{2x}{x^2+1} \Bigg] =$$

$$= \int_1^2 \frac{1}{x^3}dx - \int_1^2 \frac{1}{x}dx + \int_1^2 \frac{2x\,dx}{x^2+1} \cancel{dx}$$

$$= \left[\cancel{2x}dx = d(x^2) = d(x^2+1) \right] =$$

$$= \int_1^2 \cancel{\frac{dx}{x^3}} x^{-3}dx - \int_1^2 \frac{dx}{x} + \int_1^2 \frac{d(x^2+1)}{x^2+1} =$$

$$= \cancel{\tfrac{1}{2x^2}} - \frac{1}{2x^2}\Big|_1^2 - \ln|x|\,\Big|_1^2 + \ln|x^2+1|\,\Big|_1^2 =$$

$$= -\frac{1}{2\cdot 2^2} + \frac{1}{2\cdot 1^2} - \ln 2 + \ln 1 + \ln(2^2+1) -$$

$$- \ln(1^2+1) = -\frac{1}{8} + \frac{1}{2} - 2\ln 2 + \ln 1 + \ln 5$$

$$= \ln\frac{1\cdot 5}{2^2} + \frac{3}{8} = \ln\frac{5}{4} + \frac{3}{8}$$

<u>w 9.1.26</u>

$$\int_0^2 f(x)\,dx,\quad f(x) = \begin{cases} e^x; & 0 \le x \le 1 \\ 2; & 1 \le x \le 2 \end{cases}$$

$\Downarrow$

$$\int_0^2 f(x)\,dx = \int_0^1 e^x dx + \int_1^2 2\,dx =$$

$$= e^x\Big|_0^1 + 2x\Big|_1^2 = e^1 - e^0 + 2\cdot 2 - 2\cdot 1 =$$

$$= e - 1 + 4 - 2 = e + 1$$

№ 9.1.46

$$\int_1^9 \frac{dx}{5+2\sqrt{x}} = \left[\begin{matrix} \sqrt{x} = t \\ x = t^2 \\ dx = 2t\,dt \end{matrix} ;\ \begin{array}{c|c|c} x & 1 & 9 \\ \hline t=\sqrt{x} & 1 & 3 \end{array} \right] =$$

$$= \int_1^3 \frac{2t\,dt}{5+2t} = \int_1^3 \frac{2t+5-5}{2t+5}\,dt =$$

$$= \int_1^3 \left(1 - \frac{5}{2t+5}\right) dt = \left[dt = \frac{1}{2} d(2t+5) \right]$$

$$= t\Big|_1^3 - \frac{5}{2} \ln|2t+5|\Big|_1^3 = 3 - 1 - \frac{5}{2}\ln|2\cdot 3+5| +$$

$$+ \frac{5}{2}\ln|2\cdot 1+5| = 2 - \frac{5}{2}(\ln 11 - \ln 7) =$$

$$= 2 - \frac{5}{2}\ln\frac{11}{7}$$

№ 9.1.51

$$\int_0^{\frac{\pi}{2}} \frac{dx}{3+2\cos x} = \left[\begin{matrix} t = tg\frac{x}{2} \\ x = 2\,arctg\,x, \end{matrix} \right.$$

$$dx = \frac{2}{1+t^2}dt;\ \cos x = \frac{1-t^2}{1+t^2};\ \begin{array}{c|c|c} x & 0 & \frac{\pi}{2} \\ \hline t & 0 & 1 \end{array}$$

$$= \int_0^1 \frac{\frac{2}{1+t^2}dt}{3+2\frac{1-t^2}{1+t^2}} = \int_0^1 \frac{2}{t^2+5}dt =$$

$$= \frac{2}{\sqrt5}\,\mathrm{arctg}\frac{t}{\sqrt5}\Big|_0^1 = \frac{2}{\sqrt5}\,\mathrm{arctg}\frac{1}{\sqrt5}$$

<u>w9. 1.52</u>

$$\int_2^3 x(3-x)^7dx = \left[\begin{matrix} t=3-x \\ x=3-t \end{matrix};\ dx=-dt;\ \begin{array}{c|c|c} x & 2 & 3 \\ \hline t & 1 & 0 \end{array}\right] = \int_1^0 (3-t)t^7(-dt) = \int_1^0 (t^8-3t^7)dt$$

$$= \frac{t^9}{9}\Big|_1^0 - \frac{3t^8}{8}\Big|_1^0 = \frac{0^9}{9} - \frac{1^9}{9} - \frac{3\cdot0^8}{8} + \frac{3\cdot1^8}{8}$$

$$= -\frac19 + \frac38 = \frac{19}{72}$$

<u>w9. 1.59</u>

$$\int_1^2 \frac{dx}{x\sqrt{x^2+x+1}} = \left[x=\frac1t,\ dx=-\frac{1}{t^2}dt,\ \begin{array}{c|c|c} x & 1 & 2 \\ \hline t & 1 & \frac12 \end{array}\right] = \int_1^{\frac12} \frac{-\frac{1}{t^2}dt}{\frac1t\cdot\sqrt{\frac{1}{t^2}+\frac1t+1}} =$$

$$= -\int_1^{\frac12}\frac{dt}{\sqrt{t^2+t+1}} = \int_{\frac12}^1 \frac{d(t+\frac12)}{\sqrt{(t+\frac12)^2+\left(\frac{\sqrt3}{2}\right)^2}} =$$

$$= \ln\left|t+\frac{1}{2}+\sqrt{t^2+t+1}\right|\Bigg|_{\frac{1}{2}}^{1} =$$

$$= \ln\left|1+\frac{1}{2}+\sqrt{1^2+1+1}\right| - \ln\left|\frac{1}{2}+\frac{1}{2}+\right.$$
$$\left.+\sqrt{\left(\frac{1}{2}\right)^2+\frac{1}{2}+1}\right| = \ln\frac{3+2\sqrt{3}}{2} - \ln\frac{2+\sqrt{7}}{2} =$$
$$= \ln\frac{3+2\sqrt{3}}{\sqrt{7}+2}$$

<u>№ 9.1.61</u>

$$\int_0^2 \frac{dx}{(4+x^2)^2} = \left[\begin{matrix} x = 2\operatorname{tg} t \\ dx = \frac{2}{\cos^2 t}dt \end{matrix},\ \begin{array}{c|c|c} x & 0 & 2 \\ t & 0 & \frac{\pi}{4} \end{array}\right] =$$

$$= \int_0^{\frac{\pi}{4}} \frac{2dt}{\cos^2 t\,(4+4\operatorname{tg}^2 t)^2} = \int_0^{\frac{\pi}{4}} \frac{2dt}{16\cos^2 t\left(\frac{1}{\cos^2 t}\right)} =$$

$$= \frac{1}{8}\int_0^{\frac{\pi}{4}} \cos^2 t\,dt = \frac{1}{16}\int_0^{\frac{\pi}{4}} (1+\cos 2t)\,dt =$$

$$= \frac{1}{16}\left(t+\frac{1}{2}\sin 2t\right)\Big|_0^{\frac{\pi}{4}} = \frac{1}{16}\left(\frac{\pi}{4}+\frac{1}{2}\right) =$$

$$= \frac{\pi+2}{64}$$

<u>№ 9.1.86</u>

$$\int_1^e (x+1)\ln x\,dx = [u = \ln x,\ dv = (x+1)dx,$$

$du = \frac{1}{x}dx,\ v = \frac{x^2}{2} + x \Big] = \left(\frac{x^2}{2} + x\right)\ln x\Big|_1^e -$

$- \int_1^e \left(\frac{x^2}{2} + x\right)\frac{dx}{x} = \frac{e^2}{2} + e - 0 - \left(\frac{x^2}{4} + x\right)\Big|_1^e =$

$= \frac{e^2}{2} + e - \frac{e^2}{4} - e + \frac{1}{4} + 1 = \frac{e^2+5}{4}$

N 91.91

$\int_0^1 \operatorname{arctg} x\,dx = \Big[u = \operatorname{arctg} x,\ dv = dx,$

$du = \frac{1}{1+x^2}dx,\ v = x \Big] = x \operatorname{arctg} x \Big|_0^1 -$

$- \int_0^1 \frac{x}{1+x^2}dx = \frac{\pi}{4} - \frac{1}{2}\ln(1+x^2)\Big|_0^1 =$

$= \frac{\pi}{4} - \frac{1}{2}\ln 2 = \frac{\pi - \ln 4}{4}$

N 91.95

$J = \int_0^{\frac{\pi}{4}} x^2 \sin 2x\,dx$

$\left[\begin{array}{ll} u = x^2 & du = 2x\,dx \\ dv = \sin 2x\,dx; & v = -\frac{1}{2}\cos \end{array}\right]$

$J = -\frac{1}{2}x^2\cos 2x\Big|_0^{\frac{\pi}{4}} - \int_0^{\frac{\pi}{4}}\left(-\frac{1}{2}\right)\cos 2x \cdot 2x\,dx =$

$= 0 + \int_0^{\frac{\pi}{4}} x\cos 2x\,dx$

$\left[\begin{array}{ll} u = x & du = dx \\ dv = \cos 2x\,dx; & v = \frac{1}{2}\sin 2x \end{array}\right]$

$J = \frac{1}{2}x\sin 2x\Big|_0^{\frac{\pi}{4}} - \int_0^{\frac{\pi}{4}}\frac{1}{2}\sin 2x\,dx = \frac{\pi}{8} + \frac{1}{2}\cdot\frac{1}{2}.$

$\cdot \cos 2x \Big|_0^{\frac{\pi}{4}} = \frac{\pi}{8} - \frac{1}{4}$