Домашняя работа

Производные (часть 3)

№7.3.45

$$\lim_{x \to 1} \frac{x^{10} - 2x + 1}{x^{20} - 4x + 3} = \left[\frac{0}{0}\right] =$$

$$= \lim_{x \to 1} \frac{10x^9 - 2}{20x^{19} - 4} = \frac{8}{16} = \frac{1}{2}$$

№7.3.46

$$\lim_{x \to 0} \frac{\sin 5x}{\sqrt{x+1} - 1} = \left[\frac{0}{0}\right] = \lim_{x \to 0} \frac{5\cos 5x}{\frac{1}{2\sqrt{x+1}}} =$$

$$= \lim_{x \to 0} \left(5\cos 5x \cdot 2\sqrt{x+1}\right) =$$

$$= 10 \lim_{x \to 0} \left(\cos 5x \sqrt{x+1}\right) = 10 \cdot 1 = 10$$

№7.3.47

$$\lim_{x \to 0} \frac{e^x - e^{-x}}{\sin 2x} = \left[\frac{0}{0}\right] = \lim_{x \to 0} \frac{e^x + e^x}{2\cos 2x} =$$

$$= \lim_{x \to 0} \frac{e^x}{\cos 2x} = \frac{1}{1} = 1$$

w7.3.48

$$\lim_{x\to 0}\frac{\operatorname{arctg} x - x}{x^3} = \left[\frac{0}{0}\right] =$$

$$= \lim_{x\to 0}\left(\left(\frac{1}{x^2+1} - 1\right)\cdot\frac{1}{3x^2}\right) =$$

$$\cancel{= \lim_{x\to 0}\frac{1}{3x^2(x^2+1)} - \lim_{x\to 0}\frac{1}{3x^2}}$$

$$= \lim_{x\to 0}\left(-\frac{x^2}{1+x^2}\cdot\frac{1}{3x^2}\right) =$$

$$= \lim_{x\to 0}\left(-\frac{1}{3(1+x^2)}\right) = -\frac{1}{3}$$

w7.3.49

$$\lim_{x\to 0}\frac{e^{x^3}-1}{\sin^3 x} = \left[\frac{0}{0}\right] = \lim_{x\to 0}\frac{3x^2e^{x^3}}{3\cos 3x} =$$

$$= \frac{0}{1} = 0$$

w7.3.50

$$\lim_{x\to 2}\frac{\operatorname{ctg}\left(\frac{\pi x}{2}\right)}{\ln(x-2)} = \left[\frac{\infty}{\infty}\right] =$$

$$= \lim_{x\to 2}\frac{-\frac{1}{\sin^2\left(\frac{\pi}{2}x\right)}\cdot\frac{\pi}{2}}{\frac{1}{x-2}} =$$

$$= \lim_{x\to 2} \frac{\pi(x-2)}{2\sin^2(\frac{\pi}{2}x)} = \left[\frac{0}{0}\right] =$$

$$= \lim_{x\to 2} \frac{\pi}{-\cos(\frac{\pi}{2}x)\cdot 2\sin(\frac{\pi}{2}x)\cdot \cancel{\frac{\pi}{2}}} =$$

$$= \lim_{x\to 2} \frac{1}{\cos(\frac{\pi}{2}x)\cdot 2\cdot \sin(\frac{\pi}{2}x)} = \frac{1}{-1\cdot 2\cdot 0} =$$

$$= \infty$$

w7.3.51

$$\lim_{x\to +\infty} \frac{\log_2 x}{2^x} = \left[\frac{+\infty}{+\infty}\right] =$$

$$= \cancel{\lim_{x\to +\infty} \left(\frac{\log_2 \frac{1}{x}}{2^{\frac{1}{x}}}\right)} \lim_{x\to +\infty} \frac{\frac{1}{x\ln 2}}{2^x \ln 2} =$$

$$= \lim_{x\to +\infty} \frac{1}{2^x x \ln^2 2} = \frac{1}{+\infty} = +0$$

w7.3.52

$$\lim_{x\to \infty} \frac{x^3-x}{5x^3+x^2-7x+3} = \left[\frac{\infty}{\infty}\right] =$$

$$= \lim_{x\to 0} \frac{\left(\frac{1}{x}\right)^3 - \frac{1}{x}}{5\left(\frac{1}{x}\right)^3 + \left(\frac{1}{x}\right)^2 - 7\frac{1}{x} + 3} =$$

$$= \lim_{x\to 0} \frac{1-x^2}{3x^3-7x^2+x+5} = \frac{1}{5}$$

№7.3.53

$$\lim_{x\to\infty} x\cdot\sin\frac{1}{x} = [\infty\cdot 0] =$$

$$= \lim_{x\to 0}\frac{\sin x}{x} = 1$$ (замечательный предел)

№7.3.54

$$\lim_{t\to\frac{\pi}{2}} \left(t-\frac{\pi}{2}\right)\operatorname{tg} t = [0\cdot\infty] =$$

$$= \lim_{t\to\frac{\pi}{2}}\frac{\operatorname{tg} t}{\frac{1}{\left(t-\frac{\pi}{2}\right)}} = \left[\frac{\infty}{\infty}\right] = \lim_{t\to\frac{\pi}{2}}\frac{\frac{1}{\cos^2 t}}{-\frac{1}{\left(t-\frac{\pi}{2}\right)^2}} =$$

$$= \lim_{t\to\frac{\pi}{2}}\frac{-\left(t-\frac{\pi}{2}\right)^2}{\cos^2 t} = \lim_{t\to\frac{\pi}{2}}\frac{2\left(t-\frac{\pi}{2}\right)}{2\cos t\sin t} =$$

$$= \lim_{t\to\frac{\pi}{2}}\frac{t-\frac{\pi}{2}}{\cos t\sin t} = \lim_{t\to\frac{\pi}{2}}\frac{1}{\cos^2 t-\sin^2 t} =$$

$$= \lim_{t\to\frac{\pi}{2}}\frac{1}{\cos 2t} = \frac{1}{1} = 1$$

№7.3.55

$$\lim_{x\to 0} x\ln\operatorname{ctg} x = [0\cdot\infty] =$$

$$= \lim_{x\to 0}\frac{\ln\operatorname{ctg} x}{\frac{1}{x}} = \left[\frac{\infty}{\infty}\right] =$$

$$= \lim_{x\to 0} \frac{\frac{1}{ctg x} \cdot \left(-\frac{1}{\sin^2 x}\right)}{-\frac{1}{x^2}} = \frac{0 \cdot 0}{\infty} = 0$$

№7.3.56

$$\lim_{x\to \frac{\pi}{2}} \left(\frac{1}{\cos x} - \frac{1}{\pi - 2x}\right) = [\infty - \infty] =$$

$$= \lim_{x\to \frac{\pi}{2}} \frac{\pi - 2x - \cos x}{\cos x(\pi - 2x)} = \left[\frac{0}{0}\right] =$$

$$= \lim_{x\to \frac{\pi}{2}} \frac{-2 + \sin x}{-2\cos x - \sin x(\pi - 2x)} = \frac{-2+1}{0-0} = \frac{-1}{0} =$$

$$= \infty$$

№7.3.58

$$\lim_{x\to 0} \left(\frac{1}{x} - \frac{1}{arctg x}\right) = [\infty - \infty] =$$

$$= \lim_{x\to 0} \frac{arctg x - x}{x\, arctg x} = \left[\frac{0}{0}\right] =$$

$$= \lim_{x\to 0} \frac{-\frac{x^2}{x^2+1}}{\frac{x}{x^2+1} + arctg x} = \left[\frac{0}{0}\right] =$$

$$= \lim_{x\to 0} \frac{-2x}{1 + \frac{1}{x^2+1} \cdot (x^2+1) + 2x\, arctg x} = \frac{0}{2} = 0$$

w7.3.59

$$\lim_{x\to 0}(ctg\,x)^{\sin x} = [\infty^0] =$$

$$= \lim_{x\to 0} \ln(ctg\,x)^{\sin x} = \lim_{x\to 0} \sin x \ln ctg\,x =$$

$$= [0\cdot\infty] = \lim_{x\to 0} \frac{\ln ctg\,x}{\frac{1}{\sin x}} = \left[\frac{\infty}{\infty}\right] =$$

$$= \lim_{x\to 0} \frac{\frac{1}{ctg\,x}\cdot\left(-\frac{1}{\sin^2 x}\right)}{-\frac{1}{\sin^2 x}} =$$

$$= \lim_{x\to 0} \frac{1}{ctg\,x} = \frac{1}{\infty} = 0 \Rightarrow$$

$$\Rightarrow \lim_{x\to 0} \ln\left((ctg\,x)^{\sin x}\right) = 0 \Rightarrow$$

$$\Rightarrow \lim_{x\to 0}(ctg\,x)^{\sin x} = 1$$

w7.3.60

$$\lim_{x\to 0}\left(\frac{2}{\pi}\arccos x\right)^{\frac{1}{x}} = [1^\infty] =$$

$$= \lim_{x\to 0} \ln\left(\left(\frac{2}{\pi}\arccos x\right)^{\frac{1}{x}}\right) =$$

$$= \lim_{x\to 0} \frac{1}{x} \ln\left(\frac{2}{\pi} \arccos x\right) = \left[\frac{0}{0}\right] \; (2)$$

$$\cancel{= \lim_{x\to 0} \frac{\pi}{2 \arccos x} = \frac{\pi}{2} \cdot \frac{2}{\pi} = 1 \Rightarrow}$$

$$\cancel{\Rightarrow \lim_{x\to 0} \ln\left(\left(\frac{2}{\pi} \arccos x\right)^{\frac{1}{x}}\right) = 1 \Rightarrow}$$

$$\cancel{\Rightarrow \lim_{x\to 0} \left(\frac{2}{\pi} \arccos x\right)^{\frac{1}{x}} = e}$$

$$(2) \; \lim_{x\to 0} \left(- \frac{1}{\sqrt{1-x^2} \arccos x}\right) = -\frac{2}{\pi} \Rightarrow$$

$$\Rightarrow \lim_{x\to 0} \ln\left(\left(\frac{2}{\pi} \arccos x\right)^{\frac{1}{x}}\right) = -\frac{2}{\pi} \Rightarrow$$

$$\Rightarrow \lim_{x\to 0} \left(\frac{2}{\pi} \arccos x\right)^{\frac{1}{x}} = e^{-\frac{2}{\pi}}$$

W7. 3.61

$$\lim_{x\to\infty} (1+2^x)^{\frac{1}{x}} = [\infty^0] =$$

$$= \lim_{x\to\infty} \frac{1}{x} \ln(1+2^x) = \left[\frac{\infty}{\infty}\right] =$$

$$= \lim_{x\to\infty} \frac{1}{1+2^x} \cdot 2^x \ln 2 = \left[\frac{\infty}{\infty}\right] =$$

$$= \lim_{x\to\infty} \frac{\cancel{2^x} \ln^2 2}{\cancel{2^x \ln 2}} = \lim_{x\to\infty} \ln 2 = \ln 2 \Rightarrow$$

$$\Rightarrow \lim_{x\to\infty} \ln\left(\left(1+2^x\right)^{\frac{1}{x}}\right) = \ln 2 \Rightarrow$$

$$\Rightarrow \lim_{x\to\infty}\left(1+2^x\right)^{\frac{1}{x}} = 2$$

№7.3.62

$$\lim_{x\to 0}\left(\frac{\sin x}{x}\right)^{\frac{3}{x}} = \left[1^{\infty}\right] =$$

$$= \lim_{x\to 0} \frac{3}{x} \ln\frac{\sin x}{x} = \left[\frac{3\cdot 0}{0}\right] =$$

$$= \lim_{x\to 0}\left(3\frac{x}{\sin x}\cdot\frac{x\cos x - \sin x}{x^2}\right) =$$

$$= \lim_{x\to 0}\frac{3x\cos x - 3\sin x}{x\sin x} = \left[\frac{0}{0}\right] =$$

$$= 3\lim_{x\to 0}\frac{x\cos x - \sin x}{x\sin x} = \left[\frac{0}{0}\right] =$$

$$= 3\lim_{x\to\infty}\frac{\frac{1}{x}\cos\frac{1}{x} - \sin\frac{1}{x}}{\frac{1}{x}\sin\frac{1}{x}} = 3\lim_{x\to\infty}\frac{\cos\frac{1}{x} - x\sin\frac{1}{x}}{\sin\frac{1}{x}} =$$

$$= 3\lim_{x\to\infty}\frac{-\sin\frac{1}{x}\cdot\left(-\frac{1}{x^2}\right) - \sin\frac{1}{x} - x\cos\frac{1}{x}\cdot\left(-\frac{1}{x^2}\right)}{\cos\frac{1}{x}\cdot\left(-\frac{1}{x^2}\right)}$$

Понятно, что при дифф-нии дальше степень $\frac{1}{x}$ будет расти $\Rightarrow \lim = 0$

$$\Rightarrow \lim_{x\to 0} \ln\left(\left(\frac{\sin x}{x}\right)^{\frac{3}{x}}\right) = 0 \Rightarrow$$

$$\Rightarrow \lim_{x\to 0} \left(\frac{\sin x}{x}\right)^{\frac{3}{x}} = 1$$

№7.3.57

$$\lim_{\alpha\to 0}\left(ctg^2\alpha - \frac{1}{\alpha^2}\right) = [\infty - \infty] =$$

$$= \lim_{\alpha\to 0} \frac{\alpha^2 ctg^2\alpha - 1}{\alpha^2} = \left[\frac{0}{0}\right] =$$

$$= \lim_{\alpha\to\infty} \frac{\frac{1}{\alpha^2} ctg^2\frac{1}{\alpha} - 1}{\frac{1}{\alpha^2}} =$$

$$= \lim_{\alpha\to\infty} \frac{-\frac{2}{\alpha^3} ctg^2\frac{1}{\alpha} + \frac{1}{\alpha^2}\cdot 2\, ctg\frac{1}{\alpha}\cdot\left(-\frac{1}{\sin^2\frac{1}{\alpha}}\right)\cdot\left(-\frac{1}{\alpha^2}\right)}{-\frac{2}{\alpha^3}}$$

Понятно, что при диф-нии дальше, степень $\frac{1}{x}$ будет расти $\Rightarrow \lim = 0$

$$\Rightarrow \lim_{\alpha\to 0}\left(ctg^2\alpha - \frac{1}{\alpha^2}\right) = \infty$$

№7.3.63

$$\lim_{x\to 0}(1-x)^{\ln x} = [1^\infty] =$$

$$= \lim_{x\to 0} \ln x \ln(1-x) = [\infty\cdot 0] =$$

$$= \lim_{x\to 0} \frac{\ln(1-x)}{\frac{1}{\ln x}} = \left[\frac{0}{0}\right] =$$

$$= \lim_{x\to 0} \frac{-\frac{1}{1-x}}{\frac{1}{\frac{1}{x}}} = \lim_{x\to 0} \cancel{-\frac{1}{x(1-x)x}} =$$

$$= \cancel{\lim_{x\to\infty}} -\cancel{\frac{1}{\left(1-\frac{1}{x}\right)\frac{1}{x}}} = \cancel{\lim_{x\to\infty}} -\cancel{\frac{x}{\left(1-\frac{1}{x}\right)}}$$

$$= \lim_{x\to 0} \frac{\frac{1}{(1-x)^2}}{-\frac{1}{\frac{1}{x^2}}} = -\infty \Rightarrow$$

$$\Rightarrow \lim_{x\to 0} (1-x)^{\ln x} = 0$$

№ 7.3.64

$$\lim_{x\to\frac{\pi}{2}-0} (\pi-2x)^{\cos x} = \cancel{[0^0] = \frac{\pi}{4}}$$

~~Экспериментально (не знаю как решить)~~

~~№ 7.3.65~~

$$= \lim_{x\to\frac{\pi}{2}-0} \cos x \ln(\pi-2x) = [0\cdot\infty] =$$

$$= \lim_{x\to\frac{\pi}{2}-0} \frac{\ln(\pi-2x)}{\frac{1}{\cos x}} = \left[\frac{\infty}{\infty}\right] =$$

$$= \lim_{x \to \frac{\pi}{2}-0} \frac{\frac{1}{\pi - 2x} \cdot 2}{-\frac{1}{\cos^2 x} \cdot (-\sin x)} =$$

$$= \lim_{x \to \frac{\pi}{2}-0} \frac{\frac{2}{\pi - 2x}}{\frac{\sin x}{\cos^2 x}} = \lim_{x \to \frac{\pi}{2}-0} \frac{2\cos^2 x}{\sin x(\pi - 2x)} =$$

$$= \left[\frac{0}{0}\right] = \lim_{x \to \frac{\pi}{2}-0} \frac{2 \cdot 2\cos x \cdot (-\sin x)}{\cos x(\pi - 2x) + \sin x \cdot (-2)} =$$

$$= \lim_{x \to \frac{\pi}{2}-0} \frac{-2\sin 2x}{(\pi - 2x)\cos x - 2\sin x} = \frac{0}{0-2} = 0$$

$$\Rightarrow \lim_{x \to \frac{\pi}{2}-0} \ln\left((\pi - 2x)^{\cos x}\right) = 0 \Rightarrow$$

$$\Rightarrow \lim_{x \to \frac{\pi}{2}-0} (\pi - 2x)^{\cos x} = 1.$$

№ 7.3.65

$P(x) = x^4 - 3x^2 + x - 1, \; x_0 = -2$

$n \leq 4$

$$P(x) = P(-2) + \frac{P'(-2)}{1!}(x+2) + \frac{P''(-2)}{2!}(x+2)^2 +$$

$$+ \frac{P'''(-2)}{3!}(x+2)^3 + \frac{P''''(-2)}{4!}(x+2)^4 + o((x+2)^4)$$

$$P(-2) = (-2)^4 - 3 \cdot (-2)^2 + (-2) - 1 =$$

$$= 16 - 12 - 2 - 1 = 1$$

$P'(x) = 4x^3 - 6x + 1$

$P'(-2) = 4\cdot(-2)^3 - 6\cdot(-2) + 1 = 45$

$P''(x) = 12x^2 - 6$

$P''(-2) = 12\cdot(-4)^2 - 6 = 186$

$P'''(x) = 24x$

$P'''(-2) = 24\cdot(-2) = -48$

$P^{(IV)}(x) = 24 = P^{(IV)}(-2)$

$$P(x) = 1 + \frac{45}{1!}(x+2) + \frac{186}{2!}(x+2)^2 - \frac{48}{3!}(x+2)^3 + \frac{24}{4!}(x+2)^4 + o((x+2)^4) =$$

$$= 1 + 45(x+2) + 93(x+2)^2 - 8(x+2)^3 + 6(x+2)^4 + o((x+2)^4)$$

№ 7.3.66

$P(x) = x^3 + 4x^2 + 8x + \frac{7}{8},\ x_0 = \frac{1}{2}$

$n \leq 3$

$$P(x) = P\left(\frac{1}{2}\right) + \frac{P'\left(\frac{1}{2}\right)}{1!}\left(x-\frac{1}{2}\right) + \frac{P''\left(\frac{1}{2}\right)}{2!}\left(x-\frac{1}{2}\right)^2 + \frac{P'''\left(\frac{1}{2}\right)}{3!}\left(x-\frac{1}{2}\right)^3 + o\left(\left(x-\frac{1}{2}\right)^3\right)$$

$$P'(x) = 3x^2 + 8x + 8$$

$$P''(x) = 6x + 8$$

$$P'''(x) = 6$$

$$P\left(\tfrac{1}{2}\right) = \frac{1}{8} + \frac{4}{4} + \frac{8}{2} + \frac{7}{8} = 6$$

$$P'\left(\tfrac{1}{2}\right) = \frac{3}{4} + \frac{8}{2} + 8 = \frac{51}{4}$$

$$P''\left(\tfrac{1}{2}\right) = \frac{6}{2} + 8 = 11$$

$$P'''\left(\tfrac{1}{2}\right) = 6$$

$$P(x) = 6 + \frac{51}{4}\left(x - \tfrac{1}{2}\right) + \frac{11}{2}\left(x - \tfrac{1}{2}\right)^2 +$$

$$+ 1\left(x - \tfrac{1}{2}\right)^3 + o\left(\left(x - \tfrac{1}{2}\right)^3\right)$$

№ 7.3.67

$$f(x) = xe^x, \quad x_0 = -1$$

$$f(-1) = -e^{-1} = -\frac{1}{e}$$

$$f'(x) = e^x + xe^x$$

$$f''(x) = e^x + e^x + xe^x = 2e^x + xe^x$$

$$f'''(x) = 2e^x + e^x + xe^x = 3e^x + xe^x$$

$$\Rightarrow f^{(n)}(x) = ne^x + xe^x$$

$$\cancel{f(x) = -\frac{1}{e} +}$$

$$f'(-1) = e^{-1} + (-1)e^{-1} = \frac{1}{e} - \frac{1}{e} = 0$$

$$f''(-1) = 2e^{-1} + (-1)e^{-1} = \frac{2}{e} - \frac{1}{e} = \frac{1}{e}$$

$$f'''(-1) = 3e^{-1} + (-1)e^{-1} = \frac{3}{e} - \frac{1}{e} = \frac{2}{e}$$

$$f^{(n)}(-1) = ne^{-1} + (-1)e^{-1} = \frac{n}{e} - \frac{1}{e} = \frac{n-1}{e}$$

$$f(x) = -\frac{1}{e} + \frac{1}{2!e}(x+1)^2 + \frac{2}{3!e}(x+1)^3 + \dots +$$
$$+ \frac{n-1}{n!e}(x+1)^n + o((x+1)^n)$$

N 7.3.68

$$f(x) = \ln(2x-1), \ x_0 = 1$$

$$f'(x) = \frac{1}{2x-1} \cdot 2 = \frac{2}{2x-1}$$

$$f''(x) = -\frac{2}{(2x-1)^2} \cdot 2 = -\frac{4}{(2x-1)^2}$$

$$f'''(x) = \frac{4}{(2x-1)^3} \cdot 2 = \frac{8}{(2x-1)^3}$$

$$\Rightarrow f^{(n)}(x) = (-1)^{n-1} \frac{2^n}{(2x-1)^n}$$

$$f(1) = \ln(2 \cdot 1 - 1) = \ln 1 = 0$$

$$f'(1) = \frac{2}{2 \cdot 1 - 1} = 2$$

$$f''(1) = -\frac{4}{(2\cdot 1-1)^2} = -4$$

$$f'''(1) = \frac{8}{(2\cdot 1-1)^3} = 8$$

$$\Rightarrow f^{(n)}(1) = (-1)^{n-1} \cdot 2^n$$

$$f(x) = 2(x-1) - \frac{4}{2!}(x-1)^2 + \frac{8}{3!}(x-1)^3 + \ldots +$$
$$+ (-1)^{n-1} \cdot \frac{2^n}{n!}(x-1)^n + o((x-1)^n)$$

№ 7.3.69

$$f(x) = \sin^2 x, \quad k = 4$$

$$f(0) = 0$$

$$f'(0) = 2\sin 0 \cos 0 = 0$$

$$f''(0) = 2\cdot\cos 2\cdot 0 = 2$$

$$f'''(0) = -4\cdot\sin 2\cdot 0 = 0$$

$$f^{(IV)}(0) = -8\cdot\cos 2\cdot 0 = -8$$

$$f(x) = x^2 - \frac{x^4}{3} + o(x^4)$$

№ 7.3.70

$$f(x) = \operatorname{ch} x, \quad k = 5$$

$$f(0) = 1$$

$$f'(0) = 0$$

$f''(0) = 1$

$f'''(0) = 0$

$f^{(IV)}(0) = 1$

$f^{(V)}(0) = 0$

$$f(x) = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + o(x^5)$$