Практика

Интегрирование, часть 4

№8.4.2

$$\int \frac{\sqrt[3]{x}\,dx}{\sqrt[3]{x^2} - \sqrt{x}} = \left[\begin{matrix} n = 3 \\ q = 2 \end{matrix}\right| \Rightarrow k = \text{НОК}(3,2) = 6 \Rightarrow$$

$$\Rightarrow x = t^6 \Rightarrow dx = 6t^5\,dt \Big] =$$

$$= \int \frac{\sqrt[3]{t^6}\cdot 6t^5\,dt}{\sqrt[3]{(t^6)^2} - \sqrt{t^6}} = \int \frac{t^2 \cdot 6t^5\,dt}{t^4 - t^3} =$$

$$= \int \frac{6t^7\,dt}{t^3(t-1)} = 6\int \frac{t^4\,dt}{t-1} =$$

$$= 6\int \frac{t^4 - 1 + 1}{t-1}\,dt = 6\int \frac{t^4-1}{t-1}\,dt +$$

$$+ 6\int \frac{dt}{t-1} = 6\left(\int \frac{(t-1)(t+1)(t^2+1)}{t-1}\,dt +\right.$$

$$\left. + \int \frac{dt}{t-1}\right) = 6\left(\int (t+1)(t^2+1)\,dt +\right.$$

$$\left. + \int \frac{dt}{t-1}\right) = 6\int (t^3 + t^2 + t + 1)\,dt + 6\int \frac{dt}{t-1} =$$

$$= 6\int t^3\,dt + 6\int t^2\,dt + 6\int t\,dt + 6\int dt +$$

$$+ 6\int \frac{dt}{t-1} = \frac{6t^4}{4} + \frac{6t^3}{3} + \frac{6t^2}{2} + 6t +$$

$$+ 6\ln|t-1| + C = \frac{3t^4}{2} + 2t^3 + 3t^2 + 6t +$$

$+6\ln|t-1|+C = \cancel{\frac{3(x^6)^4}{2}+2(x^6)^3+}$

$\cancel{+3(x^6)^2+6x^6+6\ln|x^6-1|+C}$

$=\frac{3(\sqrt[6]{x})^4}{2}+2(\sqrt[6]{x})^3+3(\sqrt[6]{x})^2+$

$+6\sqrt[6]{x}+6\ln|\sqrt[6]{x}-1|+C=$

$=\frac{3}{2}\sqrt[3]{x^2}+2\sqrt{x}+3\sqrt[3]{x}+6\sqrt[6]{x}+$

$+6\ln|\sqrt[6]{x}-1|+C$

w 8.4.3

$\int\frac{dx}{\sqrt{x}+\sqrt[4]{x}}=\left[\begin{array}{l} n=2 \\ q=4 \end{array}\right| \Rightarrow k=\text{НОК}(2,4)=$

$=4\Rightarrow x=t^4\Rightarrow dx=4t^5dt\Big]=$

$=\int\frac{4t^3dt}{\sqrt{t^4}+\sqrt[4]{t^4}}=4\int\frac{t^3dt}{t^2+t}=$

$=4\int\frac{t^3dt}{t(t+1)}=4\int\frac{t^2dt}{t+1}=$

$=4\int\frac{t^2-1+1}{t+1}dt=4\int\frac{(t-1)(t+1)}{t+1}dt+$

$+4\int\frac{dt}{t+1}=4\int(t-1)dt+4\int\frac{dt}{t+1}=$

$=4\int t\,dt-4\int dt+4\int\frac{dt}{t+1}=$

$=\frac{4t^2}{2}-4t+4\ln|t+1|+C=$

$= 2t^2 - 4t + 4\ln|t+1| + C =$

$= 2(\sqrt[4]{x})^2 - 4\sqrt[4]{x} + 4\ln|\sqrt[4]{x}+1| + C =$

$= 2\sqrt[4]{x^2} - 4\sqrt[4]{x} + 4\ln|\sqrt[4]{x}+1| + C$

№ 8.4.5

$$\int \frac{dx}{\sqrt[3]{(2x+1)^2} - \sqrt{2x+1}} = \left[\begin{matrix} n=3 \\ q=2 \end{matrix} \right| \Rightarrow$$

$\Rightarrow k = \text{НОК}(2,3) = 6 \Rightarrow 2x+1 = t^6 \Rightarrow$

$\Rightarrow dx = \frac{1}{2}\cdot 6 \cdot t^5 dt =$

$= 3t^5 dt \Big] = 3\int \frac{t^5 dt}{\sqrt[3]{(t^6)^2} - \sqrt{t^6}} =$

$= 3\int \frac{t^5 dt}{t^4 - t^3} = 3\int \frac{t^5 dt}{t^3(t-1)} =$

$= 3\int \frac{t^2 dt}{t-1} = 3\int \frac{t^2 - 1 + 1}{t-1} dt =$

$= 3\int (t+1)dt + 3\int \frac{dt}{t-1} =$

$= 3\int t\,dt + 3\int dt + 3\int \frac{dt}{t-1} =$

$= 3\frac{t^2}{2} + 3t + 3\ln|t-1| + C$

$= \frac{3}{2}(2x+1)^{\frac{2}{6}} + 3(2x+1)^{\frac{1}{6}} + 3\ln|(2x+1)^{\frac{1}{6}} - 1| + C =$

$= \frac{3}{2}(2x+1)^{\frac{1}{3}} + 3(2x+1)^{\frac{1}{6}} + 3\ln|(2x+1)^{\frac{1}{6}} - 1| + C$

№8.4.6

$$\int \frac{dx}{1+\sqrt[3]{x+1}} = \cancel{\int \frac{dx}{\sqrt[3]{(x+1)^2}}}$$

$$= [\, x+1 = t^3 \Rightarrow dx = 3t^2 dt \,] =$$

$$= 3\int \frac{t^2 dt}{1+\sqrt[3]{t^3}} = 3\int \frac{t^2 dt}{t+1} =$$

$$= 3\int \frac{t^2-1+1}{t+1} dt = 3\int (t-1) dt +$$

$$+ 3\int \frac{dt}{t+1} = 3\int t\,dt - 3\int dt + 3\int \frac{dt}{t+1} =$$

$$= 3\frac{t^2}{2} - 3t + 3\ln|t+1| + C =$$

$$= \frac{3}{2}(x+1)^{\frac{2}{3}} - 3(x+1)^{\frac{1}{3}} + 3\ln|(x+1)^{\frac{1}{3}}+1| + C$$

№8.4.8

$$\int \frac{\sqrt{x}}{x^2 \cdot \sqrt{x-1}} dx = \int \frac{1}{x^2} \cdot \frac{\sqrt{x}}{\sqrt{x-1}} dx =$$

$$= \int \frac{1}{x^2} \cdot \sqrt{\frac{x}{x-1}} dx = [\, \frac{x}{x-1} = t^2 \Rightarrow$$

$$\Rightarrow x = (x-1)t^2 = xt^2 - t^2;$$

$$x - xt^2 = -t^2; \quad xt^2 - x = t^2;$$

$$x(t^2-1) = t^2 \Rightarrow x = \frac{t^2}{t^2-1} \Rightarrow$$

$$dx = d\left(\frac{t^2}{t^2-1}\right) = \left(\frac{t^2}{t^2-1}\right)' dt =$$

$$= \frac{2t(t^2-1) - t^2 \cdot 2t}{(t^2-1)^2}\, dt =$$

$$= \frac{2t^3 - 2t - 2t^3}{(t^2-1)^2}\, dt = -\frac{2t}{(t^2-1)^2}\, dt \Big] =$$

$$= \int \frac{\cancel{(t^2-1)^2}}{t^{\cancel{4}\,2}} \cdot t \cdot \left(-\frac{2t}{\cancel{(t^2-1)^2}}\right) dt =$$

$$= -2 \int \frac{dt}{t^2} = -2 \cdot \left(-\frac{1}{t}\right) + C =$$

$$= \frac{2}{t} + C = 2 \cdot \frac{1}{\sqrt{\frac{x}{x-1}}} + C =$$

$$= 2 \cdot \frac{1}{\frac{\sqrt{x}}{\sqrt{x-1}}} + C = 2 \cdot \frac{\sqrt{x-1}}{\sqrt{x}} + C =$$

$$= 2\sqrt{\frac{x-1}{x}} + C$$

<u>№ 8.4.10</u>

$$\int \sqrt{x}\,(1+\sqrt[3]{x})^4\, dx = \int x^{\frac{1}{2}} (1 + x^{\frac{1}{3}})^4\, dx =$$

$$= \Big[\, m = \frac{1}{2},\ n = \frac{1}{3},\ p = 4 \Rightarrow 1)\ p - \text{целое} \Rightarrow$$

$$\Rightarrow x = t^k,\ k = \text{НОК}(2,3) = 6 \Rightarrow$$

$$\Rightarrow x = t^6 \Rightarrow dx = 6t^5\, dt \,\Big] =$$

$$= \int (t^6)^{\frac{1}{2}} \left(1 + (t^6)^{\frac{1}{3}}\right)^4 \cdot 6t^5\, dt =$$

$$= 6 \int t^3 (1+t^2)^4 \cdot t^5\, dt =$$

...

$$= 6\int t^8(1+t^2)^4\,dt =$$

$$= [(1+t^2)^4 = (1+t^2)^2(1+t^2)^2 =$$

$$= (1+2t^2+t^4)(1+2t^2+t^4) =$$

$$= 1+2t^2+t^4+2t^2+4t^4+2t^6+$$

$$+t^4+2t^6+t^8 = 1+4t^2+6t^4+$$

$$+4t^6+t^8] = 6\int t^8(1+4t^2+$$

$$+6t^4+4t^6+t^8)\,dt = 6\int t^8\,dt+$$

$$+24\int t^{10}\,dt+36\int t^{12}\,dt+$$

$$+24\int t^{14}\,dt+6\int t^{16}\,dt =$$

$$= 6\frac{t^9}{9}+24\frac{t^{11}}{11}+36\frac{t^{13}}{13}+$$

$$+24\frac{t^{15}}{15}+6\frac{t^{17}}{17}+C =$$

$$= \frac{2}{3}t^9+\frac{24}{11}t^{11}+\frac{36}{13}t^{13}+\frac{8}{5}t^{15}+$$

$$+\frac{6}{17}t^{17}+C = \frac{2}{3}(x^{\frac{1}{6}})^9+\frac{24}{11}(x^{\frac{1}{6}})^{11}+$$

$$+\frac{36}{13}(x^{\frac{1}{6}})^{13}+\frac{8}{5}(x^{\frac{1}{6}})^{15}+$$

$$+\frac{6}{17}(x^{\frac{1}{6}})^{17}+C = \frac{2}{3}x^{\frac{3}{2}}+\frac{24}{11}x^{\frac{11}{6}}+$$

$$+\frac{36}{13}x^{\frac{13}{6}}+\frac{8}{5}x^{\frac{5}{2}}+\frac{6}{17}x^{\frac{17}{6}}+C =$$

$= \frac{2}{3} x\sqrt{x} + \frac{24}{11} x \sqrt[6]{x^5} + \frac{36}{13} x^2 \sqrt[6]{x} +$

$+ \frac{8}{5} x^2 \sqrt{x} + \frac{6}{17} x^2 \sqrt[6]{x^5} + C$

<u>№ 8.4.11</u>

$\int \frac{dx}{x^4\sqrt{x^2+1}} = \int x^{-4}(x^2+1)^{-\frac{1}{2}} dx =$

$= [\, m = -4,\ n = 2,\ p = -\frac{1}{2} \Rightarrow$

$1)\ p \notin \mathbb{Z};\ 2)\ \frac{m+1}{n} = \frac{-4+1}{2} = \frac{-3}{2} \notin \mathbb{Z};$

$3)\ \frac{m+1}{2} + p = \frac{-4+1}{2} - \frac{1}{2} = -\frac{3}{2} - \frac{1}{2} =$

$= -\frac{4}{2} = -2 \in \mathbb{Z} \Rightarrow 1 \cdot x^{-2} + 1 = t^2;$

$x^{-2} + 1 = t^2;\ x^{-2} = t^2 - 1;$

$\frac{1}{x^2} = t^2 - 1;\ x^2 = \frac{1}{t^2-1};\ x = \frac{1}{\sqrt{t^2-1}} \Rightarrow$

$\Rightarrow dx = \left(\frac{1}{\sqrt{t^2-1}}\right)' dt = \left((t^2-1)^{-\frac{1}{2}}\right)' dt =$

$= -\frac{1}{2}(t^2-1)^{-\frac{3}{2}} \cdot 2t\, dt =$

$= -\frac{t\,dt}{(t^2-1)^{\frac{3}{2}}} \,] = \int \left((t^2-1)^{-\frac{1}{2}}\right)^{-4} \left(\left((t^2-1)^{-\frac{1}{2}}\right)^2 +\right.$

$\left. + 1\right)^{-\frac{1}{2}} \cdot \left(-\frac{t\,dt}{(t^2-1)^{\frac{3}{2}}}\right) = -\int (t^2-1)^2 \left((t^2-1)^{-1} + 1\right)^{-\frac{1}{2}} \cdot$

$\cdot \frac{t}{(t^2-1)^{\frac{3}{2}}} dt = -\int (t^2-1)^2 \sqrt{\frac{t^2-1+1}{t^2-1}} \cdot \frac{t\,dt}{(t^2-1)^{\frac{3}{2}}} =$

$$= -\int (t^2-1)^2 \frac{t}{\sqrt{t^2-1}} \cdot \frac{t\,dt}{(t^2-1)\sqrt{t^2-1}} =$$

$$= -\int t^2\,dt = -\frac{t^3}{3} + C =$$

$$= -\frac{1}{3}\left(\sqrt{x^{-2}-1}\right)^3 + C =$$

$$= -\frac{1}{3}\left(\sqrt{\frac{1}{x^2}-1}\right)^3 + C =$$

$$= -\frac{1}{3}\left(\sqrt{\frac{1-x^2}{x^2}}\right)^3 + C =$$

$$= -\frac{1}{3}\frac{\left(\sqrt{1-x^2}\right)^3}{x^3} + C =$$

$$= -\frac{(1+x)^{\frac{3}{2}}(1-x)^{\frac{3}{2}}}{3x^3} + C = -\frac{(1+x)(1-x)\sqrt{(1+x)(1-x)}}{3x^3}$$

$$+ C = -\frac{(1-x^2)\sqrt{1-x^2}}{3x^3} + C =$$

$$= \frac{(x^2-1)\sqrt{1-x^2}}{3x^3} + C$$