MTH 203_ 20150826_WED

Review of Taylor's Formula

The significance of Taylor's Formula is :-
if f is a "well-behaved" function, and we know "everything" about f at a point, then we can determine the value of f "close to" the point.

The specifics of this are as follows:

A If f is a fn. of a single variable, say $f(x)$, and $f(x)$ and its derivatives are continuous around a point $x = a$, then the ~~values~~ value at a neighbouring pt. $a+h$ is given in terms of the differential operator $hD = h\frac{d}{dx}$, along with a numerical coefficient, specifically :-

$$f(a+h) = f(a) + \frac{hD}{1!}f(a) + \frac{h^2}{2!}D^2 f(a) + \cdots + \frac{h^n}{n!}D^n f(a) + R_n, \quad (1)$$

where $R_n = \dfrac{h^{n+1}}{(n+1)!} D^{n+1} f(a+ch), \quad 0 < c < 1.$ (2)

(B) Suppose though that f is a fn. of two variables, say $f(x,y)$. Then a pt. would be ~~(a,b)~~ $(x,y) = (a,b)$. The conditions for "well-behaved" become f and its partial derivatives are continuous in a neighbourhood of (a,b). It would be nice if the actual formula would be almost the same as (1). It turns out the answer is YES (though we have omitted the proof) but obviously the operator can't be the same. Actually the operator is

$$T = h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y},$$

and thus we get the following formula:

[effect of repeted applic. of operator is as if we expand (by Binomial Thm.)]

$$f(a+h, b+k) = f(a,b) + \frac{1}{1!}\,T f(a,b) + \frac{1}{2!}\,T^2 f(a,b) + \cdots + \frac{1}{n!}\,T^n f(a,b) + R_n \qquad (1')$$

where

$$R_n = \frac{1}{(n+1)!}\,T^{n+1} f(a+ch, b+ck) \qquad (2')$$

Usually, the remainder term R_n is small close to the given point, and can be ignored [by taking n sufficiently large] if we are satisfied with an approximate answer.

Prop. 16: If f has continuous 1st and 2nd order partial derivatives throughout an open region containing a rectangle R around (x_0, y_0), then the error $E(x, y)$ in using the linearization ~~satisfies~~ $L(x, y)$ satisfies:

$$|E(x,y)| \leq \frac{1}{2} M \left(|x - x_0| + |y - y_0| \right)^2$$

where M is an upper bound for $|f_{xx}|$, $|f_{yy}|$ and $|f_{xy}|$ on R.

Proof: The condition allow us to apply Taylor's formula upto the first term with the remainder for the second term:

i.e. $f(x_0+h, y_0+k) = f(x_0, y_0) + T(h,k) + \frac{1}{2!} T^2(h, k)$

where the 2nd order term is evaluated at some pt. $(x_0 + ch, y_0 + ck)$, $0 < c < 1$ $\uparrow R_2$

The error is nothing but

$$|E(x,y)| = \left|\frac{1}{2!} T^2(h,k)\right| \text{ which is expanded as}$$

$$= \frac{1}{2!}\left| h^2 \frac{\partial^2}{\partial x^2}(x_0+ch, y_0+ck) + 2hk \frac{\partial^2}{\partial x \partial y}(x_0+ch, y_0+ck) + k^2 \frac{\partial^2}{\partial y^2}(x_0+ch, y_0+ck) \right| \quad (1)$$

Noting that $h = x - x_0$, $k = y - y_0$, we get.

each of these is $\leq M$

$$|E(x,y)| \leq \frac{1}{2}\left(|x-x_0|^2 \, |f_{xx}(x', y')| + 2|x-x_0||y-y_0| \, |f_{xy}(x', y')| + |y-y_0|^2 \, |f_{yy}(x', y')|\right)$$

$$\leq \frac{1}{2} M \left(|x - x_0| + |y - y_0|\right)^2$$

as desired.

12:-

Prop. The Second Derivative Test for local extreme values.

Proof: Suppose the conditions of 1. hold, i.e.

$f_x(a,b) = f_y(a,b) = 0$ (1)

$f_{xx}(a,b) < 0$ (2)

$f_{xx}f_{yy} - f_{xy}^2 > 0$ at (a,b) (3)

We use Taylor's Theorem to write:

$$\Delta f = f(a+h, b+k) - f(a,b) = f_x(a,b) + f_y(a,b) + \frac{1}{2}\left[Ah^2 + 2Bhk + Ck^2\right]$$

$$* = \frac{1}{2}\left[Ah^2 + 2Bhk + Ck^2\right]$$ where

$A = f_{xx}(a+ch, b+ck)$, $B = f_{xy}(a+ch, b+ck)$, $C = f_{yy}(a+ch, b+ck)$ for some $0 < c < 1$. (4)

Now, by the continuity of f and its first and second partial derivatives at (a,b), conditions (2) and (3) will continue to hold in some disk of radius δ centred at (a,b). We take $h^2 + k^2 < \delta^2$ so that the pt. where A, B, C are evaluated is inside the ~~circle~~ disk.

$\therefore A < 0$, $AC - B^2 > 0$

Hence, $\Delta f = \frac{1}{2A}\left[(Ah + Bk)^2 + (AC - B^2)k^2\right] < 0$

Hence, f has a relative maximum at (a,b).

$\boxed{\text{The case 2, i.e. for relative minimum is similar}}$

* The idea of the proof is to show that $\Delta f < 0$ under the given conditions.

$\therefore f(a+h, b+k) < f(a) \rightarrow$ rel. maximum at (a,b)

We now consider the following conditions:

$$f_x = f_y = 0 \quad \text{at} \quad (a,b) \qquad (1)$$

and $$f_{xx}f_{yy} - f_{xy}^2 < 0 \quad \text{at} \quad (a,b) \qquad (5)$$

[i.e. statement 3 of the Proposition 12] – for saddle point.

We define A, B, C as before in (4)

Put $\alpha = f_{xx}(a,b)$, $\beta = f_{xy}(a,b)$, $\gamma = f_{yy}(a,b)$ (6),

so that A, B, C approach α, β, γ respectively as $h, k \to 0$ (by continuity).

Case I. $\alpha \neq 0$. Put $h = \lambda$, $k = 0$.

Then: $$\lim_{\lambda \to 0} \frac{\Delta f}{\lambda^2} = \lim_{\lambda \to 0} \frac{A}{2} = \frac{\alpha}{2} \qquad (7)$$

Now, set $h = -\lambda\beta$, $k = \lambda\alpha$.

Then, $$\lim_{\lambda \to 0} \frac{\Delta f}{\lambda^2} = \lim_{\lambda \to 0} \frac{1}{2}\left[A\beta^2 - 2B\alpha\beta + C\alpha^2\right]$$

$$= \frac{1}{2}\left(\alpha\beta^2 - 2\alpha\beta^2 + \gamma\alpha^2\right) = \frac{\alpha}{2}\left(\alpha\gamma - \beta^2\right). \qquad (8)$$

Since $\alpha\gamma - \beta^2 < 0$ by (5), the limits (7) and (8) have opposite signs; Hence Δf will have opposite signs for small λ (again by continuity) – hence, saddle point

Case II. $\gamma \neq 0$. This case is treated like Case I.

Case III. $\alpha = \gamma = 0$. Then $\beta \neq 0$ by (5).

First put $h = k = \lambda$, and then
$$\lim_{\lambda \to 0} \frac{\Delta f}{\lambda^2} = \beta$$
Then, put $h = -k = \lambda$, when
$$\lim_{\lambda \to 0} \frac{\Delta f}{\lambda^2} = -\beta$$

As in case I, we get that Δf will have opposite signs for small λ – hence saddle point