① X is $m \times n$ matrix. So, showing that rank $(X^T X) = n$ is the same as saying that it is invertible.

By rank-nullity theorem, rank(X) + nullity $(X) = n$. -(i)

$X^T X$ is a $n \times n$ matrix. So,

rank $(X^T X)$ + nullity $(X^T X) = n$ -(ii).

So,

$$\text{rank}(X) + \text{nullity}(X) = \text{rank}(X^T X) + \text{nullity}(X^T X).$$

If we show that nullity (X) = nullity $(X^T X)$, we can say that rank (X) = rank $(X^T X)$.

- If $Xy = 0$, then $X^T(Xy) = X^T 0 = 0$ i.e. $(X^T X) y = 0$. So, null-space $(X) \subseteq$ null-space $(X^T X)$.
- If $X^T X y = 0$ then $y^T X^T X y = y^T 0 = 0$
  $\therefore (Xy)^T (Xy) = 0$, i.e. $\langle Xy, Xy \rangle = 0$

But $X \neq 0$, so $y = 0$, i.e. null-space $(X^T X) \subseteq$ null-space (X).

Thus from these two, nullspace (X) = nullspace $(X^T X)$.
Thus, rank (X) = rank $(X^T X)$.

So, if columns of X are linearly independent, then rank $(X) = n =$ rank $(X^T X)$. Thus, $X^T X$ is invertible.

③. $X = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 3 \\ 1 & 4 \end{bmatrix}$, $Y = \begin{bmatrix} 2 \\ 1 \\ 2 \\ 3 \\ 2 \end{bmatrix}$, $X^T = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 3 & 4 \end{bmatrix} =$

$\theta = (X^TX)^{-1} XY$

$X^TX = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 3 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} 5 & 13 \\ 13 & 39 \end{bmatrix}$

$(X^TX)^{-1} = \frac{1}{|X^TX|} \cdot \begin{bmatrix} 39 & -13 \\ -13 & 5 \end{bmatrix} = \begin{bmatrix} 3/2 & -1/2 \\ -1/2 & 5/26 \end{bmatrix}$

$\therefore \theta = \begin{bmatrix} 3/2 & -1/2 \\ -1/2 & 5/26 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 3 \\ 1 & 4 \end{bmatrix}^T \begin{bmatrix} 2 \\ 1 \\ 2 \\ 3 \\ 2 \end{bmatrix}$

$= \begin{bmatrix} 3/2 & -1/2 \\ -1/2 & 5/26 \end{bmatrix} \begin{bmatrix} 10 \\ 27 \end{bmatrix} = \cancel{\pm} \begin{bmatrix} 3/2 \\ 5/26 \end{bmatrix}$

$\therefore \; a_1 = 3/2, \quad a_2 = 5/26$

So, $y(x) = \frac{5}{26}x + \frac{3}{2} = \frac{1}{2}\left(\frac{5}{13}x + 3\right)$