LECTURE 8

Simple Harmonic Motion

Introduction

I open this lecture with a demonstration of what I claimed in Lecture 4 to be one of the most important forms of motion; the smooth oscillation. You can seen that it does not take much effort on my part to have myself oscillating up and down in front on the class while hanging by some springs from the ceiling. I want you to concentrate on this motion and try to figure out how I am causing it. Also how I finally killed it. You will find that it is not so easy to figure out.

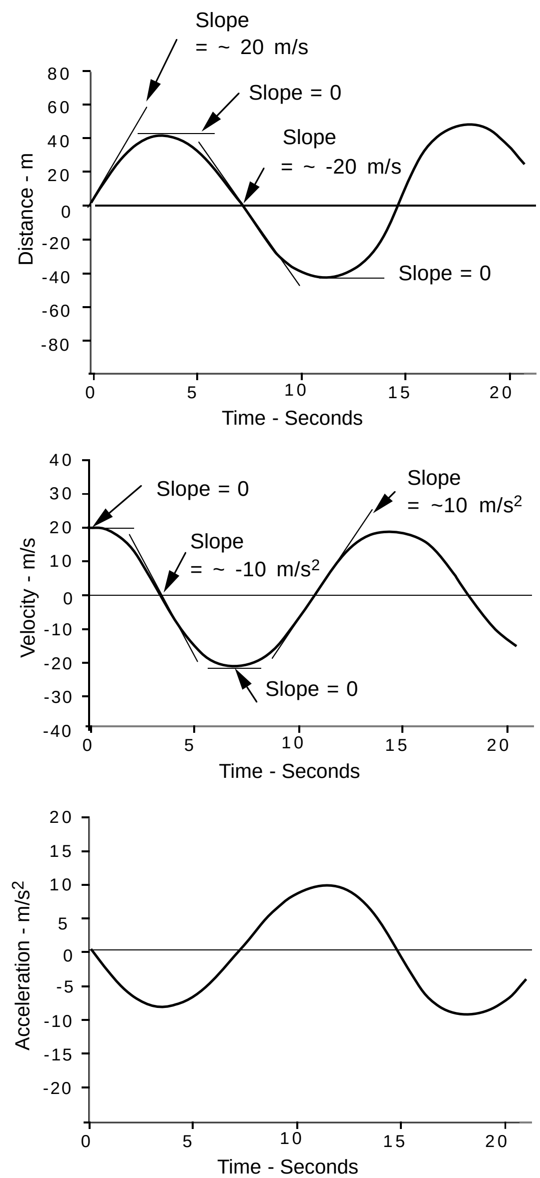
At the beginning of this course I claimed that physics replaced our ordinary intuitive feelings of the world and how it works with mathematical descriptions. Logic can then be used to deduce what relationships there might be between the various patterns we observe in our perceptions. In this way we can cope much better with the World and what it brings to us. It particular it allows us to cope with illusions.

Again, in Lecture 4 I pointed out that the smooth oscillation was one of the most important motions we have to deal with as humans. Not only does it occur as vibrations of things, such as of a building in a strong wind, which may be annoying if not destructive. Many problems with people's bones, the ligaments connecting them, and with the lubrication of the joints by the cartilage, can be traced to stresses caused by the oscillation of the body mass due to improper techniques for walking or running. This is particularly true for joggers, who are disdained for their technique by ultra-marathon runners. Another form of oscillatory motion that can cause real hardship to humans are the cycles in the economy.

But not all oscillations are undesirable. The Ancient Greek Pythagoras discovered that sound was the result of vibrations, the purest musical tone being caused by a smooth oscillation of some component of some object, such as the string of a lute. Light is also an oscillation, in this case of an electromagnetic field in space. Your vision is the result of such an oscillating electromagnetic field in your retina shaking the optic neurons causing them to fire into the optic neural network leading ultimately to the brain.

Indeed, as I have already pointed out, in modern physics all is vibrations, modern “string” theory of the basis of matter in this universe having to deal with vibrations in space that is up to 11 dimensions.

I have already applied some simple mathematics to the smooth oscillation as an example of motion in general. That mathematics was the hand-drawn graph of a smooth oscillatory motion produced in Lecture 4, and shown again below.



In that lecture I used the graph to get the velocities and accelerations of the motion. This was a form of mathematics, of the most basic kind, and it did reveal some secrets to the motion. To repeat Lecture 4, one secret was that, while the velocity and the acceleration do oscillate with, of course, the same frequency as the displacement, they do not crest at the same time. When the displacement is at a maximum the velocity is, in fact, zero. Similarly when the velocity is at a maximum, the displacement is zero. The secret of greatest importance that was revealed by the graphical method is that the acceleration is allways opposite to the displacement, a fact that is not so obvious from intuitive feelings about oscillations.

The graphical approach also gave some approximate numbers. These numbers were, for a motion that moved to extremes to both sides of about 40 meters from the center of the motion and with a cycle time of about 14 seconds, a peak velocity of about 20 m/s and a peak acceleration of about 10 m/s^2.

What are the mathematical relationships between these numbers?

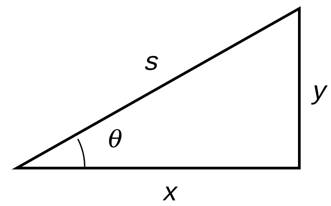
One that can be perhaps easily spotted is that the ratio of maximum displacement to maximum velocity is the same as the ratio of maximum velocity to maximum acceleration, i.e.about 2 s/m.

But where did this number of 2 s/m come from? For that it is necessary to go into some deeper mathematics dealing with oscillations.

The first step in any mathematical intrepretaion of a phenomenon is to try to find a mathematical model that exhibits the same sort of phenomenon. In this case, we need a mathematical function that, when graphed in the same manner as the experimental observation, gives the same form. Those of you who have studied functions may well recognize the particular function that graphs like a smooth oscillation. For those that don't see it I will just tell you that it is the "sine" function.

The Sine Function

For those of you who don't quite see the sine of an angle as a function, consider the elementary definition of the sine of an angle. This in in terms of the famous right-angled triangle of Pythagoras.



The definitions of the sine, the cosine and the tangent of θ are

$$\sin\theta = \frac{y}{s} \quad ; \quad \cos\theta = \frac{x}{s} \quad ; \quad \tan\theta = \frac{y}{x}$$

Thus, when sines, cosines and tangents were first introduced to you they probably did not appear as "functions" but merely as handy tools for doing calculations on right-angled triangles. From these, if we knew any two quantities, such as the angle and the hypotenuse *s*, we could calculate the two other quantities, such as *x* and *y*. Likewise, if we knew *x* and *y*, we could calculate the hypotenuse from the relations

$$\theta = \tan^{-1}\left(\frac{y}{x}\right)$$

$$s = \frac{x}{\cos\theta}$$

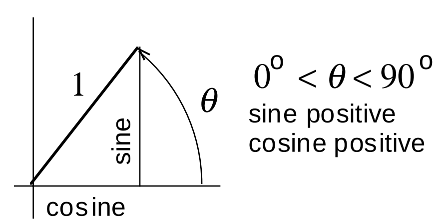
rather than going through the more cumbersome

$$s = \sqrt{x^2 + y^2}$$

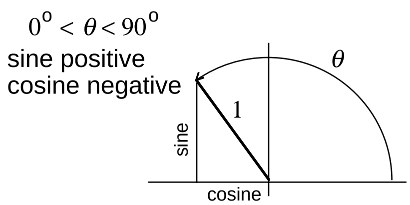
(Try these two alternatives on your calculator and see which is faster.)

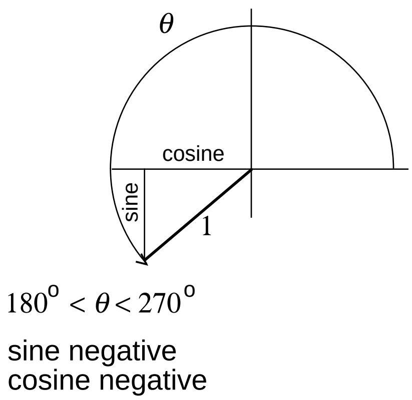
However, by now you should have gone through, or be going through, the painful realization that sine and cosines go beyond a simple right-angle triangle. In other words, you can have sines and cosines of angles that are greater than 90 degrees and where the numbers can be negative, as they cannot be for a right-angled triangle.

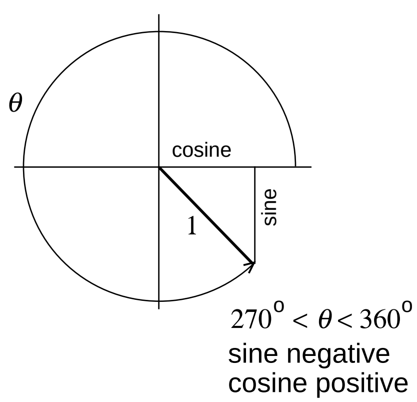
To see this, start with a right-angled triangle of which the hypotenuse is unity. The sine and the cosine are then just the vertical and horizontal sides of that triangle.



Extending the angle to beyond 90 degrees gives

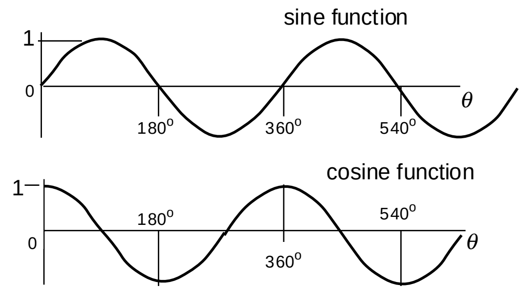






Thus sines and cosines can be calculated for any angle, even those beyond 360 degrees. They are thus *functions* of the angle and any modern "trig-function" calculator can calculate this function for any angle. (Try calculating the sine of 12,500 degrees.)

The nature of these functions should also now be familiar to you. They are shown on top right.



These are the two most important of the "trigonometric functions". For many reasons, when sines and cosines are treated as functions rather than properties of a right-angled triangle, the angle is expressed in radians. Already in this course I have shown the usefulness of radians in dealing with circular motion, and the velocities and accelerations that are involved. By now you may be encountering in your calculus course the rules for differentiating the sine and cosine functions:

$$\frac{d}{dx}(\sin x) = \cos x$$

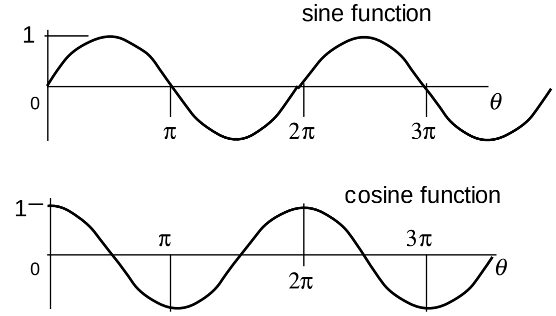
$$\frac{d}{dx}(\cos x) = -\sin x$$

Because in physics we often need both variables *x* and *y* for other purposes it is usual, as already in this course, to use the symbol θ for the angle.

$$\frac{d}{d\theta}(\sin\theta) = \cos\theta$$

$$\frac{d}{d\theta}(\cos\theta) = -\sin\theta$$

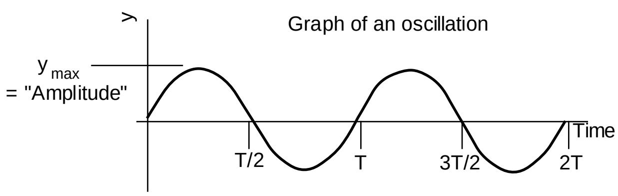
It is very important to remember that the calculus rules only have this simple form if the angle is expressed in radians. These functions are therefore usually graphed in radians instead of degrees. Since there are π radians in 180^0 the graphs appear as shown below.

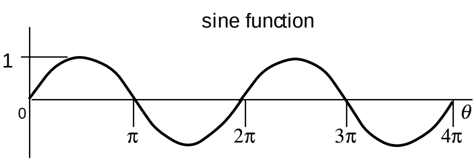


Already it should be apparent that the sine and cosine functions are ideal candidates for describing oscillations.

Using the Sine and Cosine Functions to Describe an Oscillation

To see how the sine and cosine functions may be used to describe oscillations look first at the sine function and an oscillation. Putting the two graphs side by side and lining them up gives





If the oscillation is as shown above, where the displacement starts at zero and immediately climbs to a maximum, then the connection should be easy to make. To turn the value 1 of the sine maximum into the y_{max}, or A for amplitude of the motion, you just multiply the sine function by A.

$$y = A\sin\theta$$

To convert the angles of the sine function to a time scale you just have to note that the angle 2π is equivalent to the period of the oscillation T. Thus when the variable t is T it must give the angle as 2π. The equation for the angle in terms of t is therefore

$$\theta = 2\pi\frac{t}{T}$$

which gives

$$y = A\sin\left(2\pi\frac{t}{T}\right)$$

As with describing circular motion, the collection of symbols

$$2\pi\frac{t}{T}$$

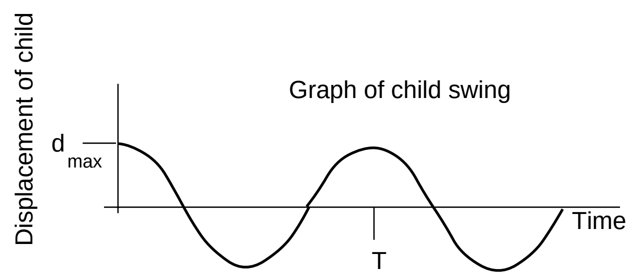
is cumbersome. As in circular motion, it is sometimes made a little simpler by using frequency f instead of the period T, which renders the equation into the form

$$y = A\sin(2\pi ft)$$

However, again as in circular motion, it can be made even simpler by going directly to radians per second for the frequency. The symbol for angle per second in radians is, as in the case of circular motion, ω. The equation for oscillations then takes on the elegant form

$$y = A\sin(\omega t)$$

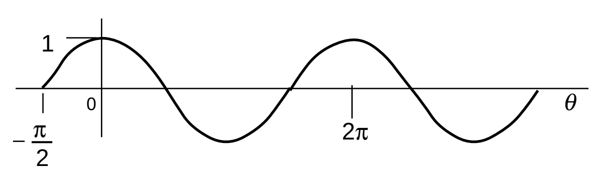
But we are not finished yet. The case where an oscillation starts at zero and immediately rises is a very special case. What about an oscillation that starts somewhere else in its cycle? To take a particular case, suppose it starts at its maximum. This would happen if you pulled a child forward in a swing and then let go of the swing. The result is shown below.



At first it might be tempting to go immediately to the cosine function, whereupon you would get:

$$d = d_{\max}\cos(\omega t).$$

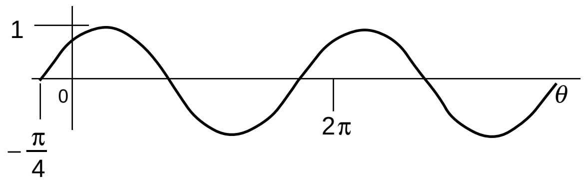
However, there is another way of expressing the form without abandoning the sine function. This is because the cosine function is just the sine function started a little earlier.



So the cosine function can be expressed as the sine function just by adding this starting angle. However, again you have to note that it should be expressed in radians, and therefore as π/2.

$$d = d_{\max} \sin\left(\omega t + \frac{\pi}{2}\right).$$

Although you may not favour this over the seemingly simpler cosine function, it is worth understanding this form because of the power it gives in expressing an oscillation that starts at any time in its cycle. Thus suppose that the oscillation starts at one eight of its cycle.



The angle here is 45^0, which of course must be expressed as π/4.

$$d = d_{\max} \sin\left(\omega t + \frac{\pi}{4}\right).$$

This equation is telling us that at t = 0, i.e. the start of the observation, the angle θ is already π/4. An oscillation that starts at zero but immediately goes negative starts at θ equal to π and so

$$d = d_{\max} \sin(\omega t + \pi).$$

The angle at which an oscillation starts is referred to as the "phase angle", or simply the "phase" of an oscillation. It is common to give this phase angle the symbol ϕ. The completely general form of the mathematical description of an oscillation is therefore

$$y = y_{\max} \sin(\omega t + \phi).$$

where the variable that is oscillating is y and $y_{\max}$ is its maximum value.

(The meaning of phase here derives from its ancient meaning as “face” or ”appearance”, as in the phases of the Moon, or as in “he is going though a phase”.)

Again, as I have already done, it is usual to express the maximum value of an oscillating variable as its "amplitude" and to write this amplitude, if you want to retain its association with y, as A_y. The functional form of a smooth oscillation in y is then

$$y = A_y \sin(\omega t + \phi).$$

Again, it is sometimes useful to remember the special case of a phase angle such that the oscillation could be described by the cosine function. That is when the phase angle is π/2.

$$y = A_y \sin\left(\omega t + \frac{\pi}{2}\right)$$

$$y = A_y \cos(\omega t)$$

This equivalence arises because of the mathematical relationship between sines and cosines:

$$\cos\theta = \sin\left(\theta + \frac{\pi}{2}\right)$$

Because of this relationship it is possible to completely describe oscillations by a cosine formulation instead of the sine. Many text-books, such as your text-book by Hecht, do this. However, most scientific publications use the sine formulation and so I will use that form here.

The description of oscillations given here is completely general for all types of oscillations. The two most important examples in human perception are those of pressure oscillations in air, for sound and music, and of electromagnetic fields for sight (as already mentioned). Oscillating electromagnetic fields are also the basis of radio and TV, as well as of course the modern curse of cellular phones.

In your 102 course in electricity and magnetism you will also get to study oscillations of voltage and currents in electrical circuits. The most important such oscillation is that of the electrical voltage from electrical plugs. In North America this has an amplitude of 163 Volts and a frequency of 60 Hz (i.e. 377 radians per second). For energy considerations this voltage amplitude is divided by the square root of 2 to give what is called the "effective" or "root mean square (rms) voltage". This results in the normal specification of 115VAC, the "AC" meaning "Alternating Current" and refers to the fact that the voltage, and hence the current, is oscillating. In Europe, and most of the rest of the World outside North America, the effective voltage is 230 VAC and the frequency is 50 Hz.

Using the Sine and Cosine Functions to Analyze Oscillatory Motion

So far I have just used the sine function as a description of a simple oscillation. If it is to be a truly mathematical description of an oscillation then it should exhibit the general features of oscillatory motion, such as the relationships between displacement, velocity and acceleration. Can it do this?

To show that indeed it can, consider the mathematical description of velocity and acceleration. Recall that the velocity is the slope of the displacement-time graph and acceleration is the slope of the velocity-time graph. Again, in calculus terms the velocity is the derivative of the displacement-time function and acceleration is the derivative of the velocity-time function.

For the sine function these derivatives are simple;

$$y = A_y \sin(\omega t + \phi)$$

$$\begin{aligned} v_y &= \frac{dy}{dt} = \frac{d}{dt}\left(A_y \sin(\omega t + \phi)\right) \\ &= A_y \frac{d}{dt}\left(\sin(\omega t + \phi)\right) \\ &= A_y \cos(\omega t + \phi)\frac{d}{dt}(\omega t + \phi) \\ &= \omega A_y \cos(\omega t + \phi) \end{aligned}$$

Thus if the oscillation is expressed by the sine function the velocity is expressed by the cosine function. This means that it is advanced in phase by 90 degrees relative to the displacement and has amplitude ωA_y.

Let's see how this agrees with the displacement-time graph obtained in Lecture 4. That it agrees with the relative phasing is obvious; the velocity is a maximum when the displacement is zero. What about the amplitude of the velocity oscillation?

The amplitude of the motion was 40 m. The period of the motion was about 14 seconds, from which the angular frequency can be determined by the equation

$$\omega = \frac{2\pi}{T} \quad .$$

It is 2π/14 or 0.45. Multiplying by the amplitude of the displacement, 40 m, gives

$$A_{v_y} = 0.45 \times 40 \ = \ 18 \ \ m/s \quad .$$

The value obtained from the graph, i.e. 20 m/s, was close to this, the difference being no doubt due to the difficulty in measuring the slope of a graph (as many of you so painfully discovered in assignment number 2).

Continuing to the functional form of the acceleration we get

$$v_y = \omega A_y \cos(\omega t + \phi)$$

$$\begin{aligned} a_y &= \frac{dv_y}{dt} = \frac{d}{dt}\left(\omega A_y \cos(\omega t + \phi)\right) \\ &= \omega A_y \frac{d}{dt}\left(\cos(\omega t + \phi)\right) \\ &= -\omega A_y \sin(\omega t + \phi)\frac{d}{dt}(\omega t + \phi) \\ &= -\omega^2 A_y \sin(\omega t + \phi) \end{aligned}$$

Thus to get the amplitude of acceleration we again just multiply the amplitude of the velocity by the angular frequency;

$$A_{a_y} = \omega A_{v_y} = \omega^2 A_y$$

Checking this result with the graphical result we see that we get $0.45 \times 18 = 8.1$ m/s^2 instead of 10. Again, the difference can be explained by the difficulty of obtaining slopes from graphs, in this case compounded by requiring the determination of slopes from slopes.

But it can also be seen that there is a close relationship between the displacement and the acceleration at all times, not just at the peaks. It is

$$a_y = -\omega^2 A_y \sin(\omega t + \phi) = -\omega^2 y$$

Taking out the intermediate step:

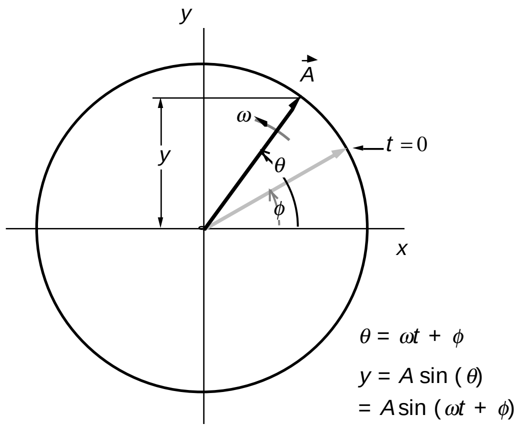
$$a_y = -\omega^2 y$$

The relationship between circular and oscillatory motion

The above relationship is very similar to the relationship between the acceleration and the displacement vectors in circular motion. There the acceleration vector points back along the displacement vector and has a magnitude which is the displacement magnitude multiplied by the square of the angular velocity of the motion. In algebraic form this relationship is

$$\vec{a} = -\omega^2 \vec{s}$$

This close similarity of rotational and simple harmonic motion should not be surprising since the sine and cosine functions are derived from circles. In fact the function used here for expressing simple harmonic motion is just the projection of a vector, of magnitude equal to the amplitude of the oscillation, on the *y*-axis as it rotates. This can be seen in the diagram below.



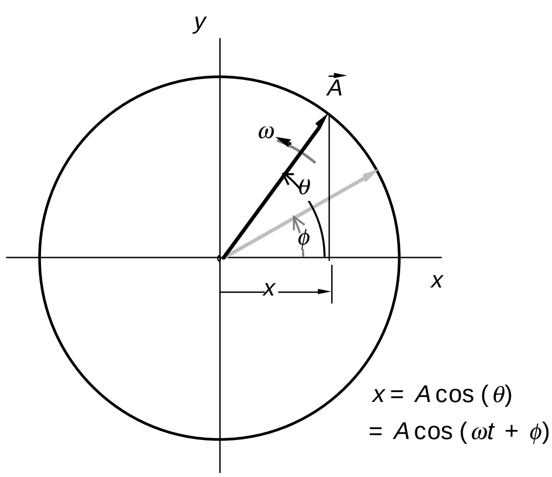
Here the amplitude vector $\vec{A}$ is imagined to rotate about the coordinate origin at angular velocity ω, starting at the angle ϕ when t = 0.The angle θ is therefore given by

$$\theta = \omega t + \phi$$

and so the oscillating variable y is given by

$$y = A\sin(\theta) = A\sin(\omega t + \phi)$$

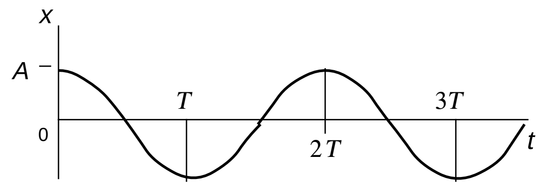
In this notation, there is no subscript y on the amplitude because it has no particular connection to y. It could just as well be used to describe the oscillation in x, where the amplitude would be the same as for y. This is shown below.

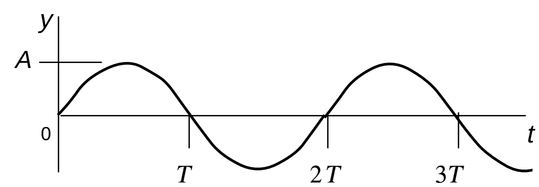


This is just the cosine form of the simple harmonic function, as used in Hecht.

To see further into the connection between simple harmonic motion and rotation consider a motion that is made up of two independent simple harmonic motions in x and y. Remember that, being orthogonal, motions in x and y can be independent.

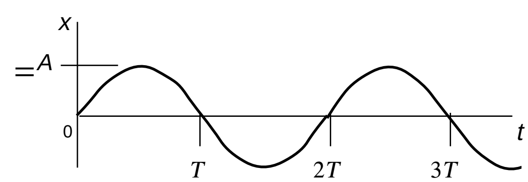
Further suppose that the simple harmonic motion in the x direction, leads the simple harmonic motion in the y direction by 90 degrees. This means that if at the start the x value is at its maximum, then the y value would be zero and going positive.

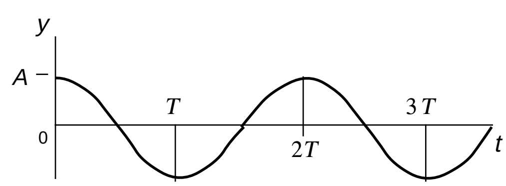




Consider now the result of plotting these x and y values for various times. The result will be a circle of radius A. (Try it for yourself for a few representative points.)

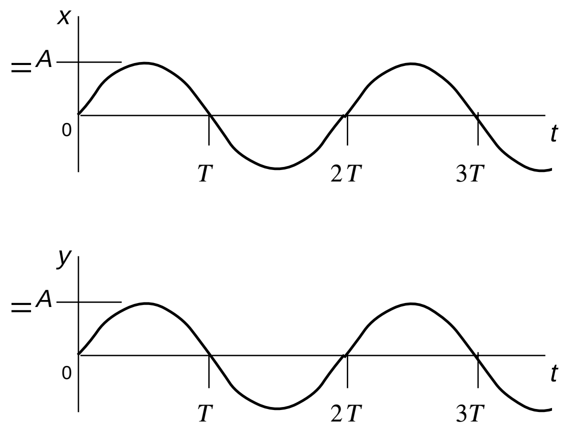
However, combining two orthogonal simple harmonic motions is more powerful than just reproducing counterclockwise circular motion. Consider first what happens when the order of the phases of the x and y motions are reversed.





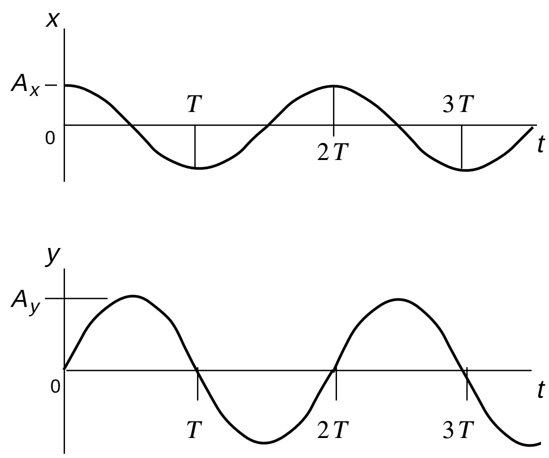
Try sketching out the result of these two motions. You will see that it is circular motion going clockwise.

Now look at what happens if the two motions are in phase.



Try sketching out the result of these two motions. You will see that it is just a line at 45 degrees to the *x* axis.

Finally consider what happens in the first example above if the amplitudes of the *x* and *y* motions are not the same. (This is the situation that will occur in your laboratory session on simple harmonic motion.)



Try sketching out the result of these two motions. You will see that it is an ellipse lined up with the *x* and *y* axis and with its major axis along the *y* axis.

This exercise could be carried on for many more examples but it should by now be clear that combining orthogonal simple harmonic motions can produce any sort of elliptical motion you wish, in either direction of rotation.

The importance of the relationship between displacement and acceleration

Now return to the question that was raised at the opening of the lecture. How do you get oscillations to start and how do you kill them?

The important point to start with is the relationship between the acceleration and the displacement. Whenever we have an oscillating body the acceleration of that body is always proportional but opposite to the displacement.

Inverting this logic, if you accelerate a body in a direction opposite to its displacement from some center but in proportion to its distance from that center, then that acceleration results in an oscillation.

To make this more concrete, suppose you have an object that is displaced from a point where you want it to be. You therefore try to move it toward that point. You must therefore give it some velocity, meaning that you must accelerate it.

However, as it accelerates the velocity increases. Suppose you therefore reduce this acceleration in proportion to the decreasing distance from where you want it to be, reducing your effort to zero when the object is where you want it, i.e. at the origin. What would be the result of this action?

You would find that the object would not come to rest at the center. Rather it would continue through to achieve a displacement on the other side of the center. If you then tried to correct this displacement by pulling it back you would find that you would be just oscillating the object back and forth.

Then how can you actually kill oscillations that you don't want? It may become clearer later in the course when we will return to the energy stored in oscillations but for now it is perhaps only clear that you can't kill them by a force that is opposite to the displacement. Rather, you have to apply the force when the object is at its maximum velocity. That is when its distance from the desired point is actually zero! With the child in a swing you have to apply the greatest retarding force when the child is at the bottom of the swing. Of course, you have to apply the force in the direction opposite to the motion.

This seems to be an extremely difficult principle for people to grasp. The error is particularly counterproductive in dealing with oscillations in the economy or in general social affairs. There is a tendency in inflationary times to put into place strong measures to reduce inflation, or in times of economic depression to put into place inflationary measures. However, strange as it may seem this just keeps the oscillation going. What is needed are anti-inflationary measures when the economy is normal but moving from depression to inflation, or inflationary measures when the economy is normal but moving from inflation to depression. In other words, you have to *foresee* the results of the motion and act accordingly.

Economic planners know this but because such measures are counterintuitive they are extremely difficult to put into place in a democracy. However, the human body has to take such measures in controlling oscillations within it. One dramatic example is the control of the orientation of the eye-ball in its socket so that an object can be seen while it is moving and while your head is turning. The forces of the muscles controlling the orientation of the eye in its socket are strong enough to produce very fast turning of the eyeball. However, they are also strong enough to easily produce oscillations.

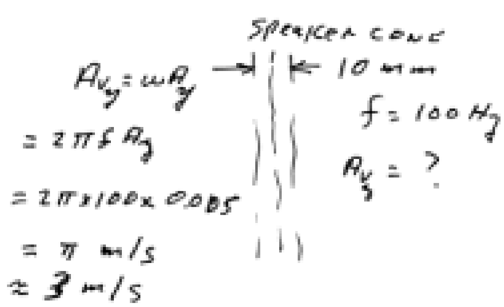
The control system for this, which is based on inputs from the vestibular organs, is highly sophisticated and works in the way that does produce the minimum in response time for the eye movement while still preventing intolerable oscillations. I will not go into the details here but it certainly isn't as simple as "Just shove it back to where it should be". One actually has to understand the dynamics that such a shoving action will produce.

Relevant material in Hecht

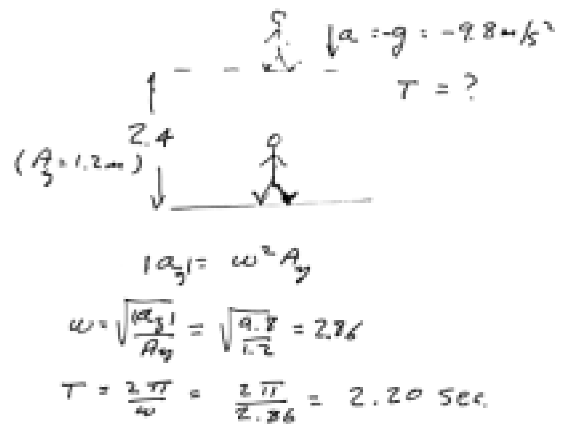
The relevant section of Hecht is Section 10.5

Worked Exercises in Class

1. You are observing the movement of the speaker cone on your bass system for your stereo. You see that at a 100 Hz tone it moves in and out to an extent of about 1 cm. What is the maximum velocity of the cone?

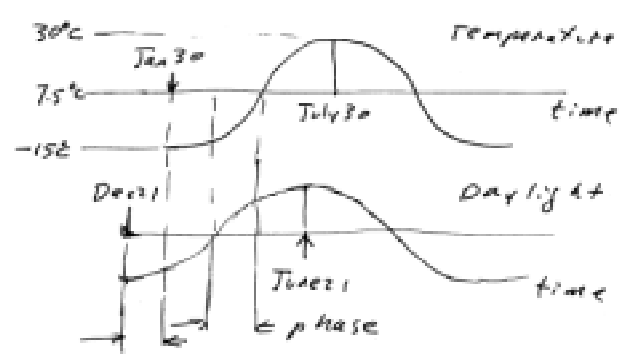


2 You are on a platform that is oscillating vertically to a total extent of 2.4 meters. At what oscillating period will you start to experience "zero gravity"?



3. The average daily temperature in Montreal follows a sinusoidal pattern over a year. It reaches a low of –15C at about Jan 30 and a high of 30C on July 31. (i.e 182 days later.) Sketch the graph of temperature vs time. What is the phase relationship of the temperature compared to the daylight hours? (The winter solstice comes on Dec 21.) What would be the expected temperature on Feb 18th?

This problem is intended to give some insight into phase relationships and possible cause and effect relationships. For this problem it is best to adopt the day as the unit of time. Graphs for the temperature, in degrees Celcius, and daylight hours with no vertical scale being necessary, vs time are shown below.



From these graphs, it should be clear that temperature is delayed in phase from daylight hours by the number of days from Dec. 21 to Jan. 30, i.e. 40 days. As an angle, where 360 degrees equals 365.26 days, this is 39.4 ^{O}, or 0.689 radians.

The reason for this delay between daylight hours (i.e. heat input from the Sun) and temperature is that it takes time for the heat input to warm up the Earth's surface. The temperature variation is caused by the variation in daylight hours.

In this case it is obvious that daylight hours cause temperature variations rather than the temperature variation causing the variation in daylight hours. (The motions of the Earth as it orbits around the Sun are not going to be affected by the fact that your feet are too cold in Montreal in the winter.)

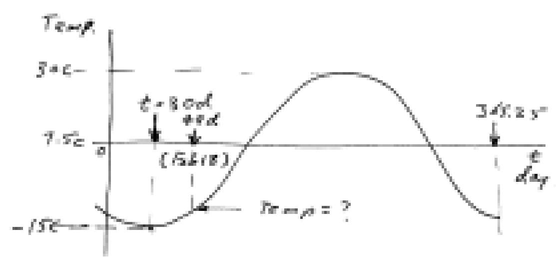
However, in many cases it is not so obvious what is the cause and what is the effect . For example, it is the common assumption that the content of CO_2 in the atmosphere will effect the global average temperature. Yet, if the global temperature, as measured by the recent ice-ages, is plotted on the same time scale as the measured CO_2 content of the atmosphere, as determined from geological studies, there is some (slight) evidence that the CO_2 cycle is *delayed* in phase relative to the temperature cycle. Is the CO_2 cycle therefore *caused* by the temperature cycle, which might itself be caused by something else? For example, it is known that a warming of the artic ocean will reduce its ability to absorb CO_2. Since this absorption by the artic ocean is the primary mechanism by which the CO_2 that we put into the atmosphere is permanently taken out of it this reduction will raise the CO_2 in the atmosphere. (The trees hold a minuscule amount of carbon compared to the oceans and even this is not held permanently but goes back into the atmosphere when the trees die. Despite all our attempts to compete with nature, almost 90% of the CO_2 put into the atmosphere stills comes from dead tress rotting in the forests.)

Thus a rise in global temperatures will itself cause a rise in CO_2 in the atmosphere. Which then causes which? This is like asking which came first, the chicken or the egg?

What is clearly happening here is a temperature cycle and the CO_2 cycle that are correlated in what is a fed-back system. That is to say excess CO_2 can cause heating, which in turn may well release more CO_2 from the soil and the oceans, thereby causing even more heating. This is the scenario that worries environmentalists. Indeed this cannot be expected to go on forever. What will eventually happen, according to history, is that the excess heating will cause so much humidity that there will be an almost permanent cloud cover that will reflect more of the sunlight back into space, thereby reducing the temperature and triggering the next ice-age.

In other words, there is a natural tendency for such a fed-back system to oscillate. The question of importance to human beings is at what phase are we in this oscillation, and are we changing this phase by our modern industrial activities?

As for the mundane question of the expected temperature on Feb 18th, what we require is the functional form of temperature oscillation so that we can plug in the value of *t* for Feb^{18th} (49 days) to get the temperature at that time. A graph of this variation is shown below.



This shows that the temperature oscillates about the yearly average of 7.5^{o} at an amplitude of 22.5^{o}. The basic form of the equation is then

$$Temp = 7.5 + A\sin(\omega t + \phi)$$

where A = 22.5.

The angular frequency is 2π radians per year, or $2\pi/365.25$ = 0.0172 radians per day. The equation then becomes

$$Temp = 7.5 + 22.5\sin(0.0172t + \phi)$$

where the only remaining uncertainty is the phase angle ϕ. To determine this we must take some time at which we know the value of the temperature. Take this to be January 30, when the temperature is –15^{o}. The equation then becomes

$$-15 = 7.5 + 22.5\sin(0.0172 \times 30 + \phi)$$

This means, of course, that

$$\sin(0.0172 \times 30 + \phi) = -1$$

For this to be true

$$0.0172 \times 30 + \phi = -\pi/2$$

Solving for ϕ gives

$$\phi = -\pi/2 - 0.0172 \times 30 = -2.09 \quad rad$$

The equation then becomes

$$Temp = 7.5 + 22.5\sin(0.0172t - 2.09)$$

Plugging in 49 for t gives a temperature of –13.8 degrees.

It might be of interest to note when the temperature is expected to be midway between winter and summer, i.e. at the mythical period called spring. It is, of course, when

$$\sin(0.0172t - 2.09) = 0$$

$$0.0172t = 2.09$$

$$t = 121.5 \quad days$$

or on about the afternoon of May 1.

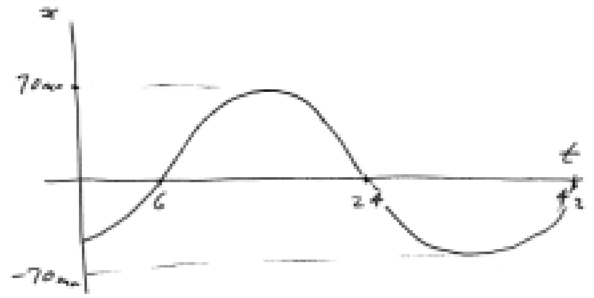
But in Montreal you have to be careful or you might miss it that afternoon.

Why is this such a time-worn Montreal joke? In fact, it applies to any Northern climate in which the range of temperature variation is so extreme. (Saskatoon is even worse.) Because of the large amplitude of the swing, the rate of change of the temperature from day to day, i.e. the “velocity” of the swing, is also great. This velocity is at its maximum when the temperature deviation itself is at zero. Thus the rate of change of temperature from day to day is at its greatest on about May 1, i.e. at “spring”. Thus while we huddle for many days at the end of January waiting for some sign that this dreadful cold is going to abate, we rocket from being too cold to being too hot before we have had a chance to change the clothes in our closets.

The following questions were not solved in class but are shown here as being representative of questions dealing with simple harmonic motion in general and are meant to prepare you for the third laboratory exercise.

4. An object undergoes simple harmonic motion in the x direction at an amplitude of 70 mm and a period of 36 seconds. You start your timer arbitrarily and observe that the object crosses the x axis, going in the positive x direction, at 6 seconds.

Graph the x motion of the particle and calculate its phase.



$\phi = ?$

$x = A\sin(\omega t + \phi)$

at $t = 6$ s $\quad x = 0$

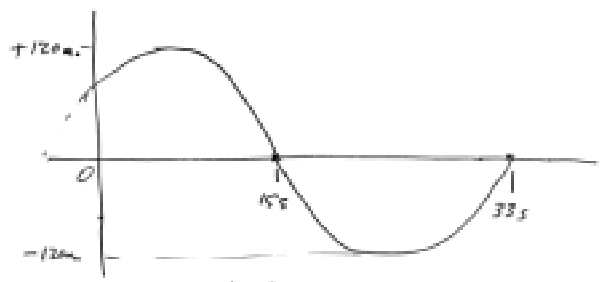
$\omega t + \phi = 0 \quad$ (not π)

$\phi = -\omega t$

$= -\frac{2\pi t}{T} = -2\pi \times \frac{6}{36} = -\frac{\pi}{3}$ rad/sec

$(= -60°)$

5. At the same time the object undergoes simple harmonic motion in the y direction at an amplitude of 120 mm and a period of 36 seconds. You observe that, after the start of your clock the object first crosses the y axis going in the negative y direction, at 15 seconds. Graph the y motion of the particle and calculate its phase.



$\phi = ?$

$y = 0$ at $t = 15$ s going negative

$y = A_y \sin(\omega t + \phi_y)$

$0 = A_y \sin(\omega \times 15 + \phi_y)$

$15\omega + \phi_y = \pi$

$\phi_y = \pi - \frac{2\pi}{36} \times 15 = \frac{6\pi}{36} = \frac{\pi}{6}$ rad

$(= 30°)$

6. Sketch the actual trajectory of the object.

   The equations and representative values for the *x* and *y* motion are given below.

$$x = 70 \sin\left(\frac{2\pi t}{T} - \frac{\pi}{3}\right)$$

$$y = 120 \sin\left(\frac{2\pi t}{T} + \frac{\pi}{6}\right)$$

t	x	y
0	$70 \sin(-\frac{\pi}{3}) = -60.6\,mm$	$120 \sin(\frac{\pi}{6}) = 60\,mm$
$\frac{T}{12}$	$70 \sin(-\frac{\pi}{6}) = -35\,mm$	$120 \sin(\frac{\pi}{3}) = 104\,mm$
$\frac{T}{6}$	$70 \sin(0°) = 0$	$120 \sin(\frac{\pi}{2}) = 120\,mm$
$\frac{T}{4}$	$70 \sin(\frac{\pi}{6}) = 35\,mm$	$120 \sin(\frac{2\pi}{3}) = 104\,mm$
$\frac{T}{3}$	$70 \sin(\frac{\pi}{3}) = 60.6\,mm$	$120 \sin(\frac{5}{6}\pi) = 60\,mm$
$\frac{5T}{12}$	$70 \sin(\frac{\pi}{2}) = 70\,mm$	$120 \sin(\pi) = 0\,mm$
$\frac{T}{2}$	$70 \sin(\frac{2\pi}{3}) = 60.6\,mm$	$120 \sin(\frac{7\pi}{6}) = -60\,mm$
⋮	⋮	⋮

The graph of this motion is: