Monte-Carlo 19/11/2021

1) X_t pour décrire l'évolution du marché : $X_t = X(t, W_t)$

$$dX_t = A\,dt + \beta\,dW_t \iff dX_t = \psi(t, W_t)\,dt + \varphi(t, W_t)\,dW_t$$

Il peut arriver que W_t dans ψ et φ se regroupent :

$$dX_t = \tilde{\psi}(t, X_t)\,dt + \tilde{\varphi}(t, X_t)\,dW_t$$ - équa diff stochastique

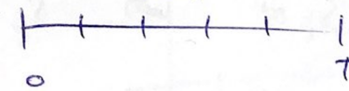
Exemple Le postulat de Black & Scholes

$$dS_t = \underbrace{r \cdot S_t}_{\psi}\,dt + \underbrace{\sigma S_t}_{\varphi}\,dW_t$$

On cherche une solution $S_t = F(S_0, W_t, r, \sigma)$

Situation 1 : Vous ne savez pas la solution analytique de l'EDS.

$$dS_t = r S_t\,dt + \sigma S_t\,dW_t$$



$\Delta t = \frac{T}{N}$

$t_i = \Delta t \cdot i$

On approxime dS_t par différences finies.

$$dS_t \sim S(t_{i+1}) - S(t_i)$$

approx. d'ordre Δt

$$S(t_{i+1}) - S(t_i) = r \cdot S(t_i) \cdot \Delta t + \sigma \cdot S(t_i) \cdot \left(W(t_{i+1}) - W(t_i)\right)$$

Mouv. Brownien et stationnaire $\Rightarrow$ $W_t - W_s$ et W_{t-s} suivent la même loi

$\Rightarrow W(t_{i+1}) - W(t_i)$ et $W(t_{i+1} - t_i) = W(\Delta t) \sim \mathcal{N}(0, \Delta t)$

Pour coder : $W_{\Delta t} = \sqrt{\Delta t} \cdot N(0,1)$ ← random sur Matlab

$\Rightarrow S(t_{i+1}) - S(t_i) = r S(t_i) \Delta t + \sigma S(t_i) \sqrt{\Delta t} \cdot N(0,1)$

$\Rightarrow S(t_{i+1}) = S(t_i) \left[1 + r \cdot \Delta t + \sigma \cdot \sqrt{\Delta t} \cdot N(0,1) \right]$

$r = 0{,}1 \quad \sigma = 0{,}5$

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S(1) = S0
Δt = T/N
for i = 1 : N
    S(i+1) = S(i) · (1 + r·Δt + σ √Δt · N(0,1))
    t(i+1) = t(i) + Δt
END.
plot(t, s)   ← une trajectoire
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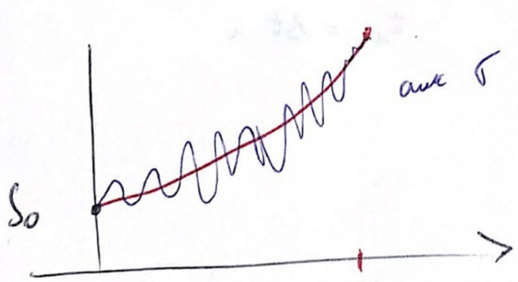
Si $dS_t = r S_t \, dt \Rightarrow S_t = S_0 \cdot e^{rt}$ — la richesse augmente de façon exponentielle.

σ : la mesure du risque, la volatilité



$$dX_t = \tilde{\varphi}(t, X_t)\,dt + \bar{\varphi}(t, X_t)\,dW_t$$

$X(1) = X_0$

for $k = 1 : NMC$

    for $i = 1 : N$

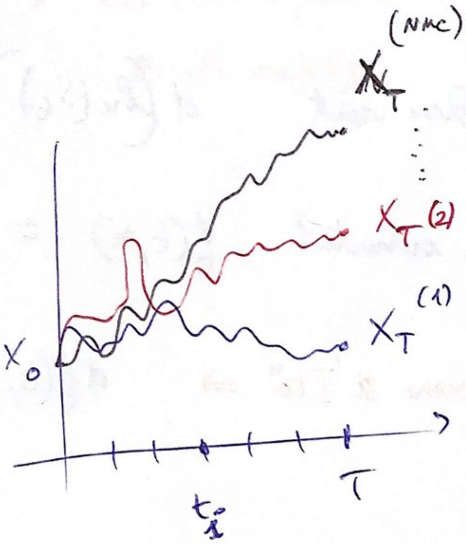
        $X(i+1) = X(i) + \tilde{\varphi}(t(i), X(i)) + \Delta t + \bar{\varphi}(t(i), X(i)) \cdot \sqrt{\Delta t} \cdot N(0,1)$

    END

    last_value $(k) = X(N+1)$

END

$$\mathbb{E}[X_t] = \frac{1}{N_{mc}} \sum_{k=1}^{N_{mc}} X_t^{(k)}$$



Situation 2 : Vous connaissez la solution analytique de l'EDS. Vous simulez la trajectoire à partir de la solution analytique.

$$dS_t = S_t\, r\, dt + \sigma S_t\, dW_t$$

$$\Rightarrow \int_0^T \frac{dS_t}{S_t} = \int_0^T r\, dt + \int_0^T \sigma\, dW_t$$

⚠ $\int \frac{dS_t}{S_t} \neq \ln(S_t)$ car $d\ln(S_t) \neq \frac{dS_t}{S_t}$

Que vaut $d(\ln(S_t))$?

On introduit $f(t,x) = \ln(S_t)$

Lemme d'Itô $\Rightarrow$ $df(t, W_t) = \frac{\partial f}{\partial t} dt + \left.\frac{\partial f}{\partial x}\right|_{x=W_t} dW_t + \frac{1}{2} \left.\frac{\partial^2 f}{\partial x^2}\right|_{x=W_t} (dW_t)^2$

Or Nous avons montré que $\lim_{N \to +\infty} \sum_{i=0}^{N} (W_{i+1} - W_i)^2 = t$

$$\Rightarrow \int_0^T (dW_t)^2 = \int_0^T dt \Rightarrow dW_t^2 = dt$$

Et $(dW_t)^2 = d\langle W \rangle_t$

$$d f(t, S_t) = \frac{\partial f}{\partial t} dt + \frac{\partial f}{\partial x}\Big|_{x = S_t} dS_t + \frac{1}{2} \frac{\partial^2 f}{\partial x^2}\Big|_{x=S_t} (dS_t)^2$$

dans le cas Black & Schols :

Terme d'Itô

$$(dS_t)^2 = \left(S_t r dt + \sigma S_t dW_t\right)^2$$

$$= S_t^2 r^2 dt^2 + 2 \cdot \sigma r S_t^2 dt \cdot dW_t + \sigma^2 S_t^2 (dW_t)^2$$

$$= \underbrace{S_t^2 r dt \left(r dt + 2 \sigma dW_t\right)}_{\text{les termes petits on les néglige}} + \sigma^2 S_t^2 (dW_t)^2$$

$$dt \cdot dWt = dt \cdot (dt)^{1/2} = (dt)^{3/2}$$

$$(dWt)^2 = dt \iff dW_t = \sqrt{dt}$$

Ainsi $(dS_t)^2 = \sigma^2 S_t^2 dt$

Si $f = \ln x$

$$d(f(t,S_t)) = \frac{\partial f}{\partial x}\Big|_{x=S_t} dS_t + \frac{1}{2} \cdot \frac{\partial^2 f}{\partial x^2}\Big|_{x=S_t} \sigma^2 S_t^2 \, dt$$

$$d(\ln(S_t)) = \frac{1}{x}\Big|_{x=S_t} dS_t + \frac{1}{2}\left(\frac{-1}{x^2}\right)\Big|_{x=S_t} \sigma^2 S_t^2 \, dt$$

$$d(\ln(S_t)) = \frac{dS_t}{S_t} - \frac{1}{2} \cdot \frac{1}{S_t^2} \sigma^2 S_t^2 \, dt$$

$$= \frac{dS_t}{S_t} - \frac{1}{2}\sigma^2 dt$$

$$= r\,dt + \sigma dW_t - \frac{1}{2}\sigma^2 dt$$

$$\Rightarrow \int_0^T d(\ln(S_t)) = \int_0^T \left(r - \frac{\sigma^2}{2}\right) dt + \int_0^T \sigma \, dW_t$$

$$\Rightarrow \left[\ln(S_t)\right]_0^T = \left(r - \frac{\sigma^2}{2}\right)T + \left[\sigma . W_t\right]_0^T$$

$$\Rightarrow \ln(S_T) - \ln(S_0) = \left(r - \frac{\sigma^2}{2}\right)T + \sigma(W_T - \underset{0}{\overset{\|}{W_0}})$$

$$\Rightarrow \ln(S_T) = \ln(S_0) + \left(r - \frac{\sigma^2}{2}\right)T + \sigma W_T$$

$$\rightarrow S_t = e^{\ln(S_0) + \left(r - \frac{\sigma^2}{2}\right)T + \sigma W_T} = S_0 \cdot e^{\left(r - \frac{\sigma^2}{2}\right)T + \sigma W_t}$$

$$t_i = i\Delta t$$

$$\int_{t_i}^{t_{i+1}} d(\ln(S_t)) = \int_{t_i}^{t_{i+1}} \left(r-\frac{\sigma^2}{2}\right)dt + \int_{t_i}^{t_{i+1}} \sigma\, dW_t$$

$$\Rightarrow \ln(S_{i+1}) - \ln(S_i) = \left(r-\frac{\sigma^2}{2}\right)\Delta t + \sigma\left(W_{t_{i+1}} - W_{t_i}\right)$$

$$(i+1)\Delta t - i\Delta t = \Delta t = (t_{i+1} - t_i) \qquad \sigma(W_{i+1} - W_i)$$

$$\rightarrow S_{i+1} = S_i \exp\left(\left(r-\frac{\sigma^2}{2}\right)\Delta t + \sigma(W_{i+1} - W_i)\right)$$

t = linspace (0, T, N+1)

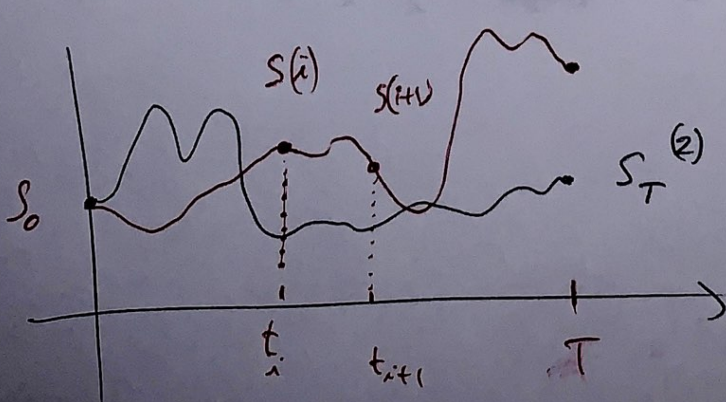
$S(1) = S_0$, $\Delta t = \frac{T}{N}$

for i = 1 : N

$S(i+1) = S(i) \cdot \exp\left(\left(r - \frac{\sigma^2}{2}\right)\Delta t + \sigma\sqrt{\Delta t} \cdot N(0,1)\right)$

END

Plot(t, S) ← une trajectoire



$$\mathbb{E}(S_T^2) = \mathbb{E}\left[\left(S_0 e^{\left(r-\frac{\sigma^2}{2}\right)T + \sigma W_T}\right)^2\right]$$

$$= \mathbb{E}\left(S_0^2 e^{(2r-\sigma^2)T + 2\sigma W_T}\right)$$

$$= S_0^2 e^{(2r-\sigma^2)T}\, \mathbb{E}\left(e^{2\sigma W_T}\right)$$ ← fonction Génératrice

$$= S_0^2 e^{(2r-\sigma^2)T} e^{\frac{(2\sigma)^2 T}{2}}$$

$$\mathrm{Var}(S_T)^{\text{théorique}} = \mathbb{E}(S_T^2) - \left(\mathbb{E}(S_T)\right)^2$$

$$= S_0^2 e^{(2r-\sigma^2)T} e^{\frac{(2\sigma)^2 T}{2}} - S_0 e^{2rT}$$

$$= S_0^2 e^{2rT} e^{-\sigma^2 T} e^{2\sigma^2 T} - S_0 e^{2rT}$$

$$= S_0^2 e^{2rT} e^{\sigma^2 T} - S_0 e^{2rT}$$

$$= S_0 e^{2rT} \left(S_0 e^{\sigma^2 T} - 1\right)$$