Exercice 3 : Soient X_1 et X_2 deux v.a. suivant resp. $P(\lambda_1)$ et $P(\lambda_2)$

a) Déterminer la loi de X_1 sachant $X_1 + X_2 = m$

On cherche $\mathbb{P}(X_1 = k \mid X_1 + X_2 = m)$ pour m fixé et $k \in \mathbb{N}$

On a
$$\mathbb{P}(X_1 = k \mid X_1 + X_2 = m) = \frac{\mathbb{P}(X_1 = k \text{ et } X_1 + X_2 = m)}{\mathbb{P}(X_1 + X_2 = m)}$$

$$= \frac{\mathbb{P}(X_1 = k \text{ et } X_2 = m-k)}{\mathbb{P}(X_1 + X_2 = m)}$$

par indépendance de X_1 et X_2

$$= \frac{\mathbb{P}(X_1 = k) \cdot \mathbb{P}(X_2 = m-k)}{\mathbb{P}(X_1 + X_2 = m)}$$

Lorsque $k > m \Rightarrow m - k < 0 \Rightarrow \mathbb{P}(X_2 = m-k < 0) = 0$

car $X_2 \sim P(\lambda_2) \Rightarrow X_2(\Omega) = \mathbb{N}$.

Lorsque $k \leq m \Rightarrow m - k \geq 0$, on a

$$\frac{\mathbb{P}(X_1 = k) \cdot \mathbb{P}(X_2 = m-k)}{\mathbb{P}(X_1 + X_2 = m)} = \frac{\dfrac{\lambda_1^k e^{-\lambda_1}}{k!} \quad \dfrac{\lambda_2^{m-k} e^{-\lambda_2}}{(m-k)!}}{\dfrac{(\lambda_1 + \lambda_2)^m e^{-(\lambda_1+\lambda_2)}}{m!}} = \frac{m!}{k!(m-k)!}$$

$$\frac{\mathbb{P}(X_1 = k) \times \mathbb{P}(X_2 = m-k)}{\mathbb{P}(X_1 + X_2 = m)} = \frac{\dfrac{\lambda_1^k e^{-\lambda_1}}{k!} \quad \dfrac{\lambda_2^{m-k} e^{-\lambda_2}}{(m-k)!}}{\dfrac{(\lambda_1+\lambda_2)^m e^{-(\lambda_1+\lambda_2)}}{m!}}$$

$$= \frac{m!}{k!\,(m-k)!} \left(\frac{\lambda_2}{\lambda_1+\lambda_2}\right)^m \cdot \left(\frac{\lambda_1}{\lambda_2}\right)^k$$

$$= \binom{m}{k} \cdot \left(\frac{\lambda_2}{\lambda_1+\lambda_2}\right)^{m-k} \left(\frac{\lambda_2}{\lambda_1+\lambda_2} \cdot \frac{\lambda_1}{\lambda_2}\right)^k$$

$$= \binom{m}{k} \left(\frac{\lambda_1}{\lambda_1+\lambda_2}\right)^k \cdot \left(\frac{\lambda_2}{\lambda_1+\lambda_2}\right)^{m-k}$$

$$\Rightarrow \mathbb{P}\left(X_1 = k \;\middle|\; X_1 + X_2 = m\right) = \binom{m}{k} \left(\frac{\lambda_1}{\lambda_1+\lambda_2}\right)^k \left(\frac{\lambda_2}{\lambda_1+\lambda_2}\right)^{m-k}$$

pour $k \in [0, m]$

$$\Longrightarrow \mathcal{L}_{X_1 \mid X_1+X_2=m} = \mathcal{B}\left(m, \frac{\lambda_1}{\lambda_1+\lambda_2}\right)$$

b) Calculer $\mathbb{E}(X_1 \mid X_1+X_2)$

$$\mathbb{E}(X_1 \mid X_1+X_2) = g(X_1+X_2)$$

où $g(x) = \mathbb{E}(X_1 \mid X_1+X_2=x)$

$$= \sum_k k\,\mathbb{P}(X_1=k \mid X_1+X_2=x)$$

$$= \sum_{k=0}^{m} k \binom{m}{k}\left(\frac{\lambda_1}{\lambda_1+\lambda_2}\right)^k\left(\frac{\lambda_2}{\lambda_1+\lambda_2}\right)^{x-k}$$

Puisque $Z = X_1 \mid X_1+X_2=x \rightsquigarrow \mathcal{B}\left(x, \frac{\lambda_1}{\lambda_1+\lambda_2}\right)$

$$\rightarrow \mathbb{E}(Z) = x \cdot \frac{\lambda_1}{\lambda_1+\lambda_2}$$

$$\Rightarrow g(x) = x \cdot \frac{\lambda_1}{\lambda_1+\lambda_2}$$

$$\Rightarrow \mathbb{E}(X_1 \mid X_1+X_2) = g(X_1+X_2) = (X_1+X_2) \cdot \frac{\lambda_1}{\lambda_1+\lambda_2}$$