$$\mathbb{E}(\Lambda^3) = \int_{\mathbb{R}} x^3 f_{\Lambda}(x)\,dx = \int_0^1 \frac{x^{(a+3)-1}(1-x)^{b-1}}{B(a,b)}\,dx$$

$$= \frac{B(a+3,b)}{B(a,b)}$$

$$\mu_3(\Lambda) = c^3\left[\frac{B(a+3,b)}{B(a,b)} - 3\cdot\frac{B(a+2,b)}{B(a,b)}\cdot\frac{B(a+1,b)}{B(a,b)} + 3\left(\frac{B(a+1,b)}{B(a,b)}\right)^3 - \left(\frac{B(a+1,b)}{B(a,b)}\right)^3\right.$$

<u>donc</u>

$$\mu_3(N) = \mu_3(\Lambda) + 3\,\mathrm{Var}(\Lambda) + \mathbb{E}(N)$$

$$= c^3\frac{B(a+3,b)}{B(a,b)} - 3c^3\frac{B(a+2,b)B(a+1,b)}{B(a,b)^2} + 2c^3\frac{B(a+1,b)}{B(a,b)}$$

$$+ 3\left(\frac{B(a+2,b)}{B(a,b)} - \left(\frac{B(a+1,b)}{B(a,b)}\right)^2\right)$$

$$+ \frac{B(a+1,b)}{B(a,b)}$$

Exercice 6 :

$$N \sim \mathcal{P}(\Lambda) \qquad \begin{cases} 0 < \lambda_1 < \lambda_2 \\ 0 < p < 1 \end{cases}$$

$$\mathbb{P}(\Lambda = \lambda_1) = p$$

$$\mathbb{P}(\Lambda = \lambda_2) = q = 1-p$$

1) La loi de N par ses probabilités individuelles et par sa fonction génératrice des moments factoriels

$$\mathbb{P}(N=n) = \mathbb{E}\left(\mathbb{1}_{N=n}\right) = \mathbb{E}\left(\mathbb{E}[\mathbb{1}_{N=n} \mid \Lambda]\right) = \mathbb{E}[g(\Lambda)]$$

où
$$g(\lambda) = \mathbb{E}\left[\mathbb{1}_{N=n} \mid \{\Lambda = \lambda\}\right]$$
$$= \mathbb{P}\left(N=n / \{\Lambda = \lambda\}\right)$$
$$= \frac{\lambda^n e^{-\lambda}}{n!}$$

$$\rightarrow \mathbb{P}(N=n) = \mathbb{E}\left[\frac{e^{-\Lambda}\Lambda^n}{n!}\right] = \int_{\mathbb{R}} \frac{e^{-\lambda}\lambda^n}{n!} f_\Lambda(\lambda)\, d\lambda$$

$$= \sum_{n=0}^{+\infty} \frac{e^{-\lambda}\lambda^n}{n!} \mathbb{P}(\Lambda = \lambda n)$$

$$= \sum_{\lambda \in \{\lambda_1, \lambda_2\}} \frac{e^{-\lambda}\lambda^n}{n!} \mathbb{P}(\Lambda = \lambda)$$

$$= e^{-\lambda_1}\frac{(\lambda_1)^n}{n!} \mathbb{P}(\Lambda = \lambda_1) + e^{-\lambda_2}\frac{(\lambda_2)^n}{n!} \mathbb{P}(\Lambda = \lambda_2)$$

$$= p\, e^{-\lambda_1}\frac{(\lambda_1)^n}{n!} + (1-p)\, e^{-\lambda_2}\frac{(\lambda_2)^n}{n!}$$

$$g_N(t) = \mathbb{E}(t^N) = \sum_{n=0}^{+\infty} t^n \, \mathbb{P}(N=n)$$

$$= \sum_{n=0}^{+\infty} t^n \left(\frac{e^{-\lambda_1}(\lambda_1)^n}{n!} p + \frac{e^{-\lambda_2}(\lambda_2)^n}{n!} (1-p) \right)$$

$$= \sum_{n=0}^{+\infty} \frac{(t\lambda_1)^n}{n!} p \cdot e^{-\lambda_1} + \sum_{n=0}^{+\infty} \frac{(t\lambda_2)^n}{n!} e^{-\lambda_2} (1-p)$$

$$= p e^{-\lambda_1(1-t)} + (1-p) e^{-\lambda_2(1-t)}$$

$$= \sum_{i=1}^{2} p_i e^{-\lambda_i(1-t)}$$

où $\begin{cases} p_1 = p \\ p_2 = 1-p \end{cases}$

Utiliser $G_X^{(k)}(0) = k! \, \mathbb{P}(X=k)$, $\forall k \geq 0$

$$\mathbb{P}(X=1) = \frac{G_X'(0)}{1!} \quad \dots \quad \mathbb{P}(X=n) = \frac{G_X^{(n)}(0)}{n!}$$

$$\frac{d}{dt} g_N(t) = \frac{d}{dt} \left(p e^{-\lambda_1(1-t)} + (1-p) e^{-\lambda_2(1-t)} \right)$$

$$= \frac{d}{dt} \left(p \cdot e^{-\lambda_1} e^{\lambda_1 t} + (1-p) e^{-\lambda_2} e^{\lambda_2 t} \right)$$

$$= p \cdot e^{-\lambda_1} \lambda_1 e^{\lambda_1 t} + (1-p) e^{-\lambda_2} \lambda_2 e^{\lambda_2 t}$$

$$\frac{d^k}{dt^k}\left(g_N(t)\right) = \frac{d^k}{dt^k}\left(p e^{-\lambda_1} e^{\lambda_1 t} + (1-p)\, e^{-\lambda_2} e^{\lambda_2 t} \right)$$

$$= p \cdot e^{-\lambda_1} (\lambda_1)^k e^{\lambda_1 t} + (1-p)(\lambda_2)^k e^{-\lambda_2} e^{\lambda_2 t}$$

Car $\frac{d^k}{dt^k}\left(e^{\lambda_1 t}\right) = \lambda_1^k e^{\lambda_1 t}$

$$\frac{d}{dt}\left(e^{\lambda_1 t}\right) = \lambda_1 e^{\lambda_1 t}$$

$$\frac{d}{dt}\left(\lambda_1 e^{\lambda_1 t}\right) = \lambda_1 \times \lambda_1 \, e^{\lambda_1 t} = \lambda_1^2 e^{\lambda_1 t}$$

Si $\frac{d^k}{dt^k} = \lambda_1^k e^{\lambda_1 t} \Rightarrow \frac{d}{dt}\left(\lambda_1^k e^{\lambda_1 t}\right) = \lambda_1^{k+1} \cdot e^{\lambda_1 t}$

donc $\mathbb{P}(N=k) = \frac{1}{k!} \frac{d^k}{dt^k}\left(g_N(t)\right)\Big|_{t=0}$

$$= \frac{1}{k!}\left[p \cdot e^{-\lambda_1} (\lambda_1)^k + (1-p)\, e^{-\lambda_2} (\lambda_2)^k \right]$$

2) $p = 0{,}9 \qquad \lambda_2 = 2\lambda_1 \qquad n = 500.$

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$$\mathbb{E}(\Lambda) = \lambda_1 p + (1-p)\lambda_2$$

$$= \lambda_1 p + (1-p)\, 2\lambda_1 = \lambda_1 (p + 2 - 2p)$$

$$\cancel{= 3\lambda_1 - p\lambda_1} \qquad = \lambda_1 (2 - p) = \lambda_1 (2 - 0{,}9)$$

$$= \lambda_1 \left(\frac{20-9}{10}\right) = \frac{11}{10}\lambda_1$$

$$\cancel{= \lambda_1 (3 - p)} \qquad = 1{,}1\,\lambda_1$$

$$\cancel{= \lambda_1 (3 - 0{,}9) = \lambda_1 \left(3 - \frac{9}{10}\right) = \lambda_1 \left(\frac{30-9}{10}\right)}$$

$$\cancel{= \lambda_1 \frac{21}{10} = \lambda_1 \left(\frac{20}{10} + \frac{1}{10}\right) = \lambda_1 \cdot 2{,}1}$$

$$\mathbb{E}(N) = \frac{500 - 460}{500} = \frac{40}{500} = \frac{4}{50} = \frac{2}{25}$$

$$\Rightarrow \quad \mathbb{E}(\Lambda) = \mathbb{E}(N)$$

$$\Leftrightarrow \quad 1{,}1\,\lambda_1 = \frac{2}{25}$$

$$\Leftrightarrow \quad \lambda_1 = \frac{10}{11} \cdot \frac{2}{25} = \frac{\cancel{5} \times 2 \times 2}{11 \times 5 \times \cancel{5}} = \frac{4}{55}$$

$$\lambda_2 = 2\lambda_1 = \frac{2 \cdot 4}{55} = \frac{8}{55}$$

Exercice 7 : $m = 400$. $q = 0{,}1\%$ $C = 1M€$

Cas 1 : les 400 voyagent indépendamment.

On a $Y_i \rightsquigarrow C \cdot B(1, q)$

$\Rightarrow X_m = \sum_{i=1}^{n} Y_i \rightsquigarrow C \cdot B(m, q)$ car Y_i indépendantes identiquement distribuées

$\mathbb{E}(X_m) = C \cdot m \cdot q = 1M \times 400 \times 0{,}1\ \%_0 = 400.000$

$Var(X_m) = Var\left(\sum_{i=1}^{n} Y_i\right) \overset{iid}{=} \sum_{i=1}^{n} Var(Y_i) = m \cdot C^2 q(1-q)$

$\Rightarrow \sigma(X_m) = \sqrt{m} \cdot C \cdot \times \sqrt{q(1-q)} = 632.139\ €$

Cas 2 : les 400 voyagent par couple, chaque couple voyage indépendamment les uns des autres

$\mathbb{E}(X_m) = C \cdot m \cdot q$

$Var(X_m) = Var\left(\sum_{i=1}^{m/2} Y_i'\right)$ où $Y_i' \rightsquigarrow \underset{\underset{2C}{\shortparallel}}{C'} B(1, q)$

$= \frac{m}{2} \times (C')^2 \times q(1-q) = \frac{m}{2}(2C)^2 q(1-q)$

$= 2mC^2 q(1-q)$

$\Rightarrow \sigma(X_m) = C\sqrt{2mq(1-q)}$

Cas 3 : les 400 voyagent ensemble, dans le même avion.

l'espérance $\mathbb{E}\left(\sum Y_i\right)$ et l'écart-type $\sigma\left(\sum Y_i\right)$