Exercice 5 : Soit $X = \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix}$ un vect. gaussien de matrice de covariance $\Gamma_X = \begin{pmatrix} 3 & -1 & 0 \\ -1 & 3 & 0 \\ 0 & 0 & 2 \end{pmatrix}$ et $\mathbb{E}(Y) = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$

Trouver un vecteur $b = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \in \mathbb{R}^3$ et une Matrice $A \in M_3(\mathbb{R})$ tels que les coordonnées du vecteur $Y = AX + b$ soient indépendants et de la loi $N(0,1)$

On cherche A et b tq. $Y = AX + b \sim N_3(0, I_3)$

c-à-d
$$\begin{cases} \mathbb{E}(Y) = \mathbb{E}(AX+b) = A\,\mathbb{E}(X) + b = 0 \\ \Gamma_Y = A \cdot \Gamma_X A^t = I_3 \end{cases}$$

$$\Leftrightarrow \begin{cases} b = -A \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \\ A \Gamma_X A^t = I_3 \end{cases}$$

d'après l'exercice 4) on sait que

$$\Gamma_X = Q D Q^t \quad \text{où} \quad Q = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \end{pmatrix} \text{ et } D = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{pmatrix}$$

et $Q^{-1} = Q^t$

$$\Rightarrow Q^{-1} \Gamma_X (Q^t)^{-1} = D \quad \Rightarrow \quad Q^t \Gamma_X Q = D$$

On a donc : $\quad Q^t \Gamma_X Q = D$

$$\Leftrightarrow \quad Q^t \Gamma_X Q = \sqrt{D} \cdot \sqrt{D}$$

$$\Leftrightarrow \quad (\sqrt{D})^{-1} Q^t \Gamma_X Q (\sqrt{D})^{-1} = I_3$$

$$\Leftrightarrow \quad A \cdot \Gamma_X \cdot A^t = I_3$$

où $\quad A = (\sqrt{D})^{-1} Q^t$

$$A^t = \left((\sqrt{D})^{-1} Q^t\right)^t = (Q^t)^t \left((\sqrt{D})^{-1}\right)^t = Q \cdot (\sqrt{D})^{-1}$$

car $\quad D = \mathrm{diag}(2,2,4) \Rightarrow \sqrt{D} = \mathrm{diag}(\sqrt{2}, \sqrt{2}, 2)$

$$\Rightarrow (\sqrt{D})^{-1} = \mathrm{diag}\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, \frac{1}{2}\right)$$

$$\Rightarrow \left((\sqrt{D})^{-1}\right)^t = \mathrm{diag}\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, \frac{1}{2}\right)$$

Et $\quad Q^{-1} = Q^t$

Finalement,
$$A = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & \frac{1}{2} \end{pmatrix} \cdot \xcancel{\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \end{pmatrix}} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \end{pmatrix}$$

$$A = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{2\sqrt{2}} & -\frac{1}{2\sqrt{2}} & 0 \end{pmatrix}$$

Et $b = -A \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = -\begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{2\sqrt{2}} & \frac{-1}{2\sqrt{2}} & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = -\begin{pmatrix} 3/2 \\ \frac{3}{\sqrt{2}} \\ -\frac{1}{2\sqrt{2}} \end{pmatrix}$

<u>Ainsi</u> $Y = AX + b = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{2\sqrt{2}} & \frac{-1}{2\sqrt{2}} & 0 \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} + \begin{pmatrix} 3/2 \\ \frac{3}{\sqrt{2}} \\ \frac{-1}{2\sqrt{2}} \end{pmatrix}$

$$Y = \begin{pmatrix} \frac{1}{2}(X_1 + X_2 + 3) \\ \frac{1}{\sqrt{2}}(X_3 + 3) \\ \frac{1}{2\sqrt{2}}(X_1 - X_2 + 1) \end{pmatrix} \sim N_3(0, I_3)$$