Exercice 1 : Risque non homogènes

$n = 250\,000 \qquad C = 50.000\,€ \qquad q = 1\% \qquad \alpha = 6\%$

$\mathcal{R} = 2M\,€$

1) l'espérance du résultat $\mathbb{E}(R_n)$

On a $R_n = (1+\alpha)\cdot\mathbb{E}(X_n) - X_n$ où $X_n = \sum_{i=1}^{n} Y_i$

avec $Y_i = \begin{cases} C & \text{avec proba } q = 1\% \\ 0 & \text{avec proba } 1-q = 99\% \end{cases}$

X_n représente la v.a. désignant la valeur du montant cumulé des sinistres

$Y_i \sim C\cdot\mathcal{B}(1,q)$ et les Y_i sont i.i.d pour $i \in \{1,..,n\}$

$X_n \sim C\cdot\mathcal{B}(n,q)$

$$\Rightarrow \mathbb{E}(R_n) = \mathbb{E}\left((1+\alpha)\,\mathbb{E}(X_n) - X_n\right)$$

$$= (1+\alpha)\cdot\mathbb{E}(X_n) - \mathbb{E}(X_n)$$

$$= \alpha\cdot\mathbb{E}(X_n)$$

$$= \alpha\cdot C\cdot n\cdot q$$

$$= 7{,}5\,M\,€$$

2) l'écart-type $\sigma(R_n)$

$$Var(R_n) = Var\left((1+\alpha)\cdot \mathbb{E}(X_n) - X_n\right) = Var(-X_n) = (-1)^2 \cdot Var(X_n)$$

$$= Var(X_n) = C^2 \cdot n \cdot q \cdot (1-q)$$

$$\rightarrow \sigma(R_n) = \sqrt{Var(R_n)} = C \cdot \sqrt{nq(1-q)} = 2{,}48 M €$$

3) Coefficient de sécurité β

On a $$\beta = \frac{R + \alpha \mathbb{E}(X_n)}{\sigma(X_n)} = 3{,}83$$

4) Probabilité de ruine avec approximation normale

On a $$\lim_{n \to +\infty} \frac{X_n - \mathbb{E}(X_n)}{\sigma(X_n)} \sim N(0,1)$$

Soit P la probabilité de ruine

$$\mathbb{P}(\{ruine\}) = \mathbb{P}(R_n + R < 0)$$

$$= \mathbb{P}\left((1+\alpha)\mathbb{E}(X_n) - X_n + R < 0\right)$$

$$\mathbb{P}(\text{" ruine "}) = \mathbb{P}\left((1+\alpha)\,\mathbb{E}(X_n) - X_n + R < 0\right)$$

$$= \mathbb{P}\left(-X_n < -R - (1+\alpha)\,\mathbb{E}(X_n)\right)$$

$$= \mathbb{P}\left(X_n > R + (1+\alpha)\,\mathbb{E}(X_n)\right)$$

$$= \mathbb{P}\left(\frac{X_n - \mathbb{E}(X_n)}{\sigma(X_n)} > \frac{R + (1+\alpha)\,\mathbb{E}(X_n) - \mathbb{E}(X_n)}{\sigma(X_n)}\right)$$

$$= \mathbb{P}\left(Z > \frac{R + \alpha\,\mathbb{E}(X_n)}{\sigma(X_n)}\right)$$

$$= \mathbb{P}(Z > \beta)$$

$$= 1 - \mathbb{P}(Z \leq \beta)$$

$$= 1 - \Phi(\beta)$$

$$= 1 - \Phi(3{,}83)$$

$$= 1 - 0{,}99$$

$$= 1{,}00 - 0{,}99$$

$$= 0{,}001$$

$$= 0{,}1\,\%$$

5) Contrat supplémentaire $C' = 50M €$ à n assuré

a) Nouvelle espérance du résultat et nouveaux écart-type et nouveau coef de sécurité

Soit $n' = n+1 = 250.001$

$$X'_n = X_n + Y_{n'}$$

où $Y_{n'} = \begin{cases} C' & \text{avec proba } q = 1\% \\ 0 & \text{avec proba } 1-q = 99\% \end{cases}$

$$Y_{n'} \sim C' \cdot \mathcal{B}(1, q)$$

Soit $R'_n = (1+\alpha)\,\mathbb{E}(X'_n) - X'_n$

$$= (1+\alpha)\,\mathbb{E}(X_n + Y'_n) - X_n - Y_{n'}$$

$$= (1+\alpha)\,\mathbb{E}(X_n) - X_n + \mathbb{E}(Y_{n'}) - Y_{n'}$$

$$= R_n + \mathbb{E}(Y_{n'}) - Y_{n'}$$

$$\Rightarrow \mathbb{E}(R'_n) = \mathbb{E}\big(R_n + \mathbb{E}(Y_{n'}) - (Y_{n'})\big) = \mathbb{E}(R_n)$$

$$= \mathbb{E}\big((1+\alpha)\,\mathbb{E}(X'_n) - X'_n\big)$$

$$= \alpha \cdot \mathbb{E}(X'_n) = \alpha\,\mathbb{E}(X_n + Y'_n) = 7\,530.000 €$$

$$= \alpha\big(\mathbb{E}(X_n) + \mathbb{E}(Y_{n'})\big) = \alpha(Cnq + C'q) = \alpha q(Cn + C')$$

$$\mathrm{Var}(R_n') = \mathrm{Var}\left((1+\alpha)\,\mathbb{E}(X_n') - X_n'\right) = (-1)^2\,\mathrm{Var}(X_n')$$

$$= \mathrm{Var}\left(X_n + Y_{n'}\right)$$

car $Y_{n'}$ est indépendante de $X_n = \sum_{i=1}^{n} Y_i$

$$= \mathrm{Var}(X_n) + \mathrm{Var}(Y_{n'})$$

$$\rightarrow \mathrm{Var}(R_n') = c^2 n\, q(1-q) + (c')^2 q(1-q)$$

$$\Rightarrow \sigma(R_n') = \sqrt{q(1-q)\left(n c^2 + c'^2\right)} = 558.776 \text{ €}$$

Soit β' le nouveau coeff de sécurité

$$\beta' = \frac{\alpha\,\mathbb{E}(X_n') + R}{\sigma(X_n')} = 0{,}44$$

b) Nouvelle probabilité de ruine. avec approximation normale

$$p' = \mathbb{P}(\text{ruine}) = \mathbb{P}(R_n' + R < 0) = \mathbb{P}\left((1+\alpha)\mathbb{E}(X_n') + X_n' + R < 0\right)$$

$$= \mathbb{P}\left(-X_n' < -R - (1+\alpha)\mathbb{E}(X_n')\right) = \mathbb{P}\left(X_n > R + (1+\alpha)\,\mathbb{E}(X_n')\right)$$

$$= \mathbb{P}\left(\frac{X_n' - \mathbb{E}(X_n')}{\sigma(X_n')} > \frac{R + (1+\alpha)\mathbb{E}(X_n') - \mathbb{E}(X_n')}{\sigma(X_n')}\right)$$

$$\mathbb{P}\left(Z' > \frac{R + \alpha\,\mathbb{E}(X_n')}{\sigma(X_n')}\right) = 1 - P(Z' \leq \beta')$$

$$= 1 - \Phi(\beta')$$

$$= 1 - \Phi(0{,}44)$$

où $Z' \rightsquigarrow \mathcal{N}(0,1)$

$$Z' = \frac{X_n' - \mathbb{E}(X_n')}{\sigma(X_n')}$$

Exercice 2 : le nb de sinistre $N \sim \mathcal{PM}(\Lambda)$

$\mathbb{E}(N) = 10.000$ et $\sigma(N) = 1.000$

Calculer $\mathbb{E}(\Lambda)$ et $Var(\Lambda)$

Par définition $N \sim \mathcal{PM}(\Lambda)$

$\Rightarrow \exists \Lambda > 0$ v.a tq $\forall \lambda > 0$, $N | \{\Lambda = \lambda\} \sim \mathcal{P}(\lambda)$

$\forall n \in \mathbb{N}$, $\mathbb{P}(N = n) = \mathbb{E}\left(1\!\!1_{(N=n)}\right)$

$$= \mathbb{E}\left(\mathbb{E}\left(1\!\!1_{(N=n)} \mid \Lambda\right)\right)$$

$$= \mathbb{E}\left(\mathbb{P}\left(N = n \mid \Lambda\right)\right)$$

$$\Rightarrow \mathbb{P}(N=n) = \mathbb{E}\left(\frac{e^{-\Lambda}\Lambda^n}{n!}\right) = \int_{\mathbb{R}} \frac{e^{-\lambda}\lambda^n}{n!} f_\Lambda(\lambda)\, d\lambda$$

$$= \sum_{i=1}^{n} \mathbb{P}(\Lambda = \lambda_i) \cdot \frac{e^{-\lambda_i}\lambda_i^n}{n!}$$

$\Rightarrow \mathbb{E}(N) = \mathbb{E}(\mathbb{E}(N \mid \Lambda)) = \mathbb{E}(\Lambda)$ car $N | (\Lambda = \lambda) \sim \mathcal{P}(\lambda)$

$\mathbb{E}(N | (\Lambda = \lambda)) = \lambda$

$\Rightarrow \mathbb{E}(N) = \mathbb{E}(\Lambda) = 10.000$

$$\mathrm{Var}(\Lambda) = \mathbb{E}(\Lambda^2) - (\mathbb{E}(\Lambda))^2$$

On utilise l'esperance conditionnelle

$$\mathbb{E}\left(N^2 \mid (\Lambda = \lambda)\right) = \sum_{m=0}^{+\infty} m^2 \, \mathbb{P}(N = m \mid \Lambda = \lambda)$$

$$= \sum_{m=0}^{+\infty} m^2 \frac{e^{-\lambda} \lambda^m}{m!} \quad \text{car } N \mid (\Lambda = \lambda) \sim \mathcal{P}(\lambda)$$

$$= \sum_{m=0}^{+\infty} m \cdot \frac{e^{-\lambda} \lambda^m}{(m-1)!} = \sum_{m=1}^{+\infty} \frac{m \cdot e^{-\lambda} \lambda^m}{(m-1)!}$$

$$= \sum_{m=0}^{+\infty} (m+1) \frac{e^{-\lambda} \lambda^{m+1}}{m!} = \lambda e^{-\lambda} \sum_{m=0}^{+\infty} \frac{(m+1) \cdot \lambda^m}{m!}$$

$$= \lambda e^{-\lambda} \left(\sum_{m=0}^{+\infty} \frac{m \lambda^m}{m!} + \sum_{m=0}^{+\infty} \frac{\lambda^m}{m!} \right)$$

$$= \lambda e^{-\lambda} \left(\sum_{m=1}^{+\infty} \frac{m \lambda^m}{m!} + e^{\lambda} \right)$$

$$= \lambda e^{-\lambda} \sum_{m=1}^{+\infty} \frac{\lambda^m}{(m-1)!} + \lambda = \lambda e^{-\lambda} \sum_{m=0}^{+\infty} \frac{\lambda^{m+1}}{m!} + \lambda$$

$$= \lambda \cdot e^{-\lambda} \lambda \cdot \sum_{m=0} \frac{\lambda^m}{m!} + \lambda$$

$$= \lambda^2 e^{-\lambda} e^{\lambda} + \lambda = \lambda^2 + \lambda$$

$$\rightarrow \mathbb{E}(N^2 \mid \Lambda) = \Lambda^2 + \Lambda$$

$$\Rightarrow \mathbb{E}(N^2) = \mathbb{E}\left(\mathbb{E}(N^2 \mid \Lambda)\right) = \mathbb{E}(\Lambda^2 + \Lambda)$$

$$\Rightarrow \mathbb{E}(\Lambda^2) = \mathbb{E}(N^2) - \mathbb{E}(\Lambda)$$

$$\Rightarrow \mathrm{Var}(\Lambda) = \mathbb{E}(\Lambda^2) - \mathbb{E}(\Lambda)^2$$

$$= \mathbb{E}(N^2) - \mathbb{E}(\Lambda) - \mathbb{E}(\Lambda)^2$$

$$= \mathbb{E}(N^2) - \mathbb{E}(N)\left(1 + \mathbb{E}(N)\right)$$

$$= \mathbb{E}(N^2) - \mathbb{E}(N)^2 - \mathbb{E}(N)$$

$$= \mathrm{Var}(N) - \mathbb{E}(N)$$

Exercice 3 $\quad N \rightsquigarrow \mathcal{P}(\Lambda)$

$$\mu_3(N) = \mu_3(\Lambda) + 3\,\mathrm{Var}(\Lambda) + \mathbb{E}(N)$$

On a $\quad \mu_3(N) = \mathbb{E}\left(\left(N - \mathbb{E}(N)\right)^3\right)$

$$= \mathbb{E}\left(N^3 - 3N^2\mathbb{E}(N) + 3N\,\mathbb{E}(N)^2 - (\mathbb{E}(N))^3\right)$$

$$= \mathbb{E}(N^3) - 3\,\mathbb{E}(N^2)\cdot\mathbb{E}(N) + 3\,\mathbb{E}(N)\cdot\mathbb{E}(N)^2 - \mathbb{E}(N)^3$$

$$= \mathbb{E}(N^3) - 3\,\mathbb{E}(N)\cdot\left(\mathbb{E}(N^2) - \mathbb{E}(N)^2\right) - \mathbb{E}(N)^3$$

$$= \mathbb{E}(N^3) - 3\,\mathbb{E}(N)\cdot\mathrm{Var}(N) - \mathbb{E}(N)^3$$

$$\mathbb{E}(N^3) = \mathbb{E}\left[\mathbb{E}\left(N^3 \mid \{\Lambda = \lambda\}\right)\right]$$

$$\mathbb{E}\left(N^3 \mid \{\Lambda = \lambda\}\right) = \sum_{n=0}^{+\infty} n^3\,\mathbb{P}(N = n \mid \Lambda = \lambda)$$

$$= \sum_{n=0}^{+\infty} n^3 \frac{e^{-\lambda}\lambda^n}{n!}$$

$$= \sum_{n=1}^{+\infty} n^3 \frac{e^{-\lambda}\lambda^n}{n!} = \sum_{n=1}^{+\infty} \frac{n^2 e^{-\lambda}\lambda^n}{(n-1)!}$$

$$= \sum_{n=0}^{+\infty} \frac{(n+1)^2 e^{-\lambda}\lambda^{n+1}}{n!}$$

$$\mathbb{E}(N^3 \mid \Lambda = \lambda) = \sum_{n=0}^{+\infty} (n+1)^2 \frac{e^{-\lambda}\lambda^{n+1}}{n!} = \sum_{n=0}^{+\infty} (n^2+2n+1)\frac{e^{-\lambda}\lambda^{n+1}}{n!}$$

$$= \sum_{n=0}^{+\infty} n^2 \frac{e^{-\lambda}\lambda^{n+1}}{n!} + \sum_{n=0}^{+\infty} 2n \frac{e^{-\lambda}\lambda^{n+1}}{n!} + \sum_{n=0}^{+\infty}\frac{e^{-\lambda}\lambda^{n+1}}{n!}$$

$$= \sum_{n=1}^{+\infty} n\frac{e^{-\lambda}\lambda^{n+1}}{(n-1)!} + \sum_{n=1}^{+\infty} 2\frac{e^{-\lambda}\lambda^{n+1}}{(n-1)!} + \lambda \cdot e^{-\lambda}e^{\lambda}$$

$$= \sum_{n=0}^{+\infty}(n+1)\frac{e^{-\lambda}\lambda^{n+2}}{n!} + \sum_{n=0}^{+\infty} 2\frac{e^{-\lambda}\lambda^{n+2}}{n!} + \lambda$$

$$= \sum_{n=0}^{+\infty} n\frac{e^{-\lambda}\lambda^{n+2}}{n!} + \sum_{n=0}^{+\infty}\frac{e^{-\lambda}\lambda^{n+2}}{n!} + 2\cdot\lambda^2 e^{-\lambda}\sum_{n=0}^{+\infty}\frac{\lambda^n}{n!} + \lambda$$

$$= \sum_{n=1}^{+\infty}\frac{e^{-\lambda}\lambda^{n+2}}{(n-1)!} + \lambda^2 e^{-\lambda}\sum_{n=0}^{+\infty}\frac{\lambda^n}{n!} + 2\lambda^2+\lambda$$

$$= \sum_{n=0}^{+\infty}\frac{e^{-\lambda}\lambda^{n+3}}{n!} + 3\lambda^2+\lambda$$

$$= \lambda^3 \cdot e^{-\lambda}\sum_{n=0}^{+\infty}\frac{\lambda^n}{n!} + 3\lambda^2+\lambda$$

$$= \lambda^3+3\lambda^2+\lambda$$

$$\Rightarrow \mathbb{E}(N^3 \mid \Lambda) = \Lambda^3+3\Lambda^2+\Lambda$$

$$\rightarrow \mathbb{E}(N^3) = \mathbb{E}(\Lambda^3 + 3\Lambda^2 + \Lambda)$$

<u>Ainsi</u> $$\mu_3(N) = \mathbb{E}(\Lambda^3 + 3\Lambda^2 + \Lambda) - 3\,\mathbb{E}(N)\cdot Var(N) - \mathbb{E}(N)^3$$

$$= \mathbb{E}(\Lambda^3) + 3\,\mathbb{E}(\Lambda^2) + \mathbb{E}(\Lambda) - 3\,\mathbb{E}(N)\cdot(Var(\Lambda) + \mathbb{E}(N)) - \mathbb{E}(N)^3$$

$$= \mathbb{E}(\Lambda^3) + 3\,\mathbb{E}(\Lambda^2) + \mathbb{E}(\Lambda) - 3\,\mathbb{E}(\Lambda)\,Var(\Lambda) - 3\,\mathbb{E}(\Lambda)^2 - \mathbb{E}(\Lambda)^3$$

$$= \mathbb{E}(\Lambda^3) - 3\,\mathbb{E}(\Lambda)\left(\mathbb{E}(\Lambda^2) - \mathbb{E}(\Lambda)^2\right) + 3\,\mathbb{E}(\Lambda^2) - 3\,\mathbb{E}(\Lambda)^2 - \mathbb{E}(\Lambda)^3 + \mathbb{E}(\Lambda)$$

$$= \mathbb{E}(\Lambda^3) - 3\,\mathbb{E}(\Lambda^2)\,\mathbb{E}(\Lambda) + 3\,\mathbb{E}(\Lambda)\cdot\mathbb{E}(\Lambda)^2 - \mathbb{E}(\Lambda)^3$$
$$+ 3\left(\mathbb{E}(\Lambda^2) - \mathbb{E}(\Lambda)^2\right) + \mathbb{E}(\Lambda)$$

$$= \mu_3(\Lambda) + 3\,Var(\Lambda) + \mathbb{E}(\Lambda)$$

<u>Ainsi</u> $$\mu_3(N) = \mu_3(\Lambda) + 3\,Var(\Lambda) + \mathbb{E}(N)$$

Exercice 4 : $N \sim PM(\Lambda)$

$\Lambda \sim \gamma(r, \alpha)$ $\qquad f_\Lambda(\lambda) = \dfrac{\alpha^r}{\Gamma(r)} \lambda^{r-1} e^{-\alpha\lambda} \mathbb{1}_{]0,+\infty[}(\lambda)$

Montrer que $N \sim BN\left(r, \dfrac{\alpha}{1+\alpha}\right)$

c-à-d $\mathbb{P}(N = k) = \dfrac{\Gamma(k+r)}{\Gamma(r)\, k!} \times \left(\dfrac{\alpha}{1+\alpha}\right)^r \left(1 - \dfrac{\alpha}{1+\alpha}\right)^k$

On a $\mathbb{P}(N = k) = \mathbb{E}\left(\mathbb{1}_{N=k}\right)$

$$= \mathbb{E}\left(\mathbb{E}\left(\mathbb{1}_{N=k} \mid \Lambda\right)\right)$$

$$= \mathbb{E}\left(\mathbb{P}(N=k \mid \Lambda)\right)$$

$$= \mathbb{E}\left(\frac{e^{-\Lambda} \Lambda^k}{k!}\right)$$

$$= \int_{\mathbb{R}} \frac{e^{-\lambda} \lambda^k}{k!} f_\Lambda(\lambda)\, d\lambda$$

$$= \int_0^{+\infty} \frac{e^{-\lambda} \lambda^k}{k!} \cdot \frac{\alpha^r}{\Gamma(r)} \lambda^{r-1} e^{-\alpha\lambda}\, d\lambda$$

$$\mathbb{P}(N=k) = \int_0^{+\infty} \frac{e^{-\lambda}\lambda^k}{k!} \cdot \frac{\alpha^r}{\Gamma(r)} \cdot \lambda^{r-1} e^{-\alpha\lambda}\, d\lambda$$

$$= \int_0^{+\infty} e^{-\lambda(\alpha+1)} \frac{1}{k!} \lambda^{k+r-1} \frac{\alpha^r}{\Gamma(r)}\, d\lambda$$

Poson $\quad u = \lambda(\alpha+1) \Rightarrow du = (\alpha+1)\, d\lambda$

$$= \int_0^{+\infty} e^{-u} \frac{1}{k!} \left(\frac{u}{\alpha+1}\right)^{k+r-1} \frac{\alpha^r}{\Gamma(r)} \frac{du}{(\alpha+1)}$$

$$\Gamma(z) = \int_0^{+\infty} e^{-u} u^{z-1}\, du$$

$$= \frac{1}{k!} \frac{\alpha^r}{\Gamma(r)} \left(\frac{1}{\alpha+1}\right)^{k+r} \int_0^{+\infty} e^{-u} u^{k+r-1}\, du$$

$$= \frac{\Gamma(k+r)}{k!\,\Gamma(r)} \left(\frac{1}{\alpha+1}\right)^r \left(\frac{1}{\alpha+1}\right)^k \cdot \alpha^r = \left(\frac{\alpha}{\alpha+1}\right)^r \left(\frac{1}{\alpha+1}\right)^k$$

$$= \left(\frac{\alpha}{\alpha+1}\right)^r \left(\frac{1+\alpha-\alpha}{\alpha+1}\right)^k$$

$$\left(\frac{\alpha}{\alpha+1}\right)^r \left(1 - \frac{\alpha}{\alpha+1}\right)^k$$

Exercice 5 : $N \sim PM(\Lambda)$

$$f_\Lambda(\lambda) = \frac{1}{c} \cdot \mathbb{1}_{[0,c]}(\lambda) \qquad \Lambda \sim \mathcal{U}([0,c]) \quad c>0$$

1) $\mathbb{E}(N)$, $Var(N)$ $\mu_3(N)$ en fct de c

$$\begin{cases} \cdot \ \mathbb{E}(N) = \mathbb{E}(\Lambda) . \\ \cdot \ Var(N) = Var(\Lambda) + \mathbb{E}(N) . \\ \cdot \ \mu_3(N) = \mu_3(\Lambda) + 3\, Var(\Lambda) + \mathbb{E}(N) . \end{cases}$$

$$\cdot \ \mathbb{E}(\Lambda) = \int_{\mathbb{R}} \lambda \cdot f_\Lambda(\lambda)\, d\lambda = \int_{\mathbb{R}} \lambda \cdot \frac{1}{c}\, \mathbb{1}_{[0,c]}(\lambda)\, d\lambda$$

$$= \int_0^c \lambda \cdot \frac{d\lambda}{c} = \frac{1}{c} \cdot \left[\frac{\lambda^2}{2}\right]_0^c = \frac{c^2}{2c} = \frac{c}{2}$$

$$\cdot \ Var(\Lambda) = \mathbb{E}(\Lambda^2) - \mathbb{E}(\Lambda)^2$$

$$\mathbb{E}(\Lambda^2) = \int_{\mathbb{R}} \lambda^2 f_\Lambda(\lambda)\, d\lambda = \int_{\mathbb{R}} \frac{\lambda^2}{c}\, \mathbb{1}_{[0,c]}(\lambda)\, d\lambda$$

$$= \int_0^c \frac{\lambda^2}{c}\, d\lambda = \left[\frac{\lambda^3}{3c}\right]_0^c = \frac{c^3}{3c} = \frac{c^2}{3}$$

$$Var(\Lambda) = \frac{c^2}{3} - \frac{c^2}{4} = +\frac{c^2}{12}$$

$$\cdot \ \mu_3(\Lambda) = \mathbb{E}\left(\left(\Lambda - \mathbb{E}(\Lambda)\right)^3\right) = \mathbb{E}\left(\left(\Lambda - \frac{c}{2}\right)^3\right) = \int_{\mathbb{R}} \left(\lambda - \frac{c}{2}\right)^3 f_\Lambda(\lambda)\, d\lambda$$

$$\mu_3(\Lambda) = \mathbb{E}\left(\left(\Lambda - \mathbb{E}(\Lambda)\right)^3\right) = \mathbb{E}\left(\left(\Lambda - \frac{c}{2}\right)^3\right)$$

$$= \int_{\mathbb{R}} \left(\lambda - \frac{c}{2}\right)^3 f_{\Lambda}(\lambda)\, d\lambda = \int_0^c \left(\lambda - \frac{c}{2}\right)^2 \frac{1}{c}\, d\lambda$$

$$u = \lambda - \frac{c}{2} \Leftrightarrow du = d\lambda$$

$$= \int_{-c/2}^{c/2} u^3 \frac{1}{c}\, du = \frac{1}{c} \cdot \left[\frac{u^4}{4}\right]_{-c/2}^{c/2} = \frac{1}{c}\underbrace{\left(\frac{\left(\frac{c}{2}\right)^4}{4} - \frac{\left(-\frac{c}{2}\right)^4}{4}\right)}_{=0}$$

$$\Rightarrow \mu_3(\Lambda) = 0$$

<u>Ainsi</u>

$$\mathbb{E}(N) = \frac{c}{2}$$

$$\mathrm{Var}(N) = +\frac{c^2}{12} + \frac{c}{2} = \frac{c(c+6c)}{12}$$

$$\mu_3(N) = \mu_3(\Lambda) + 3\,\mathrm{Var}(\Lambda) + \mathbb{E}(N)$$

$$= 3 * \frac{c^2}{12} + \frac{c}{2} = \frac{c(3c+6)}{12} = \frac{3c(c+2)}{12}$$

$$= \frac{c(c+2)}{4}$$

$$\mathbb{E}(N) = \frac{c}{2} \quad \mathrm{Var}(N) = \frac{c(c+6c)}{12} \quad \mu_3(N) = \frac{c(c+2)}{4}$$

2) Loi N par sa fct génératrice des moments et par ses probabilités individuelles

$$g_N(t) = \mathbb{E}(t^N) = \mathbb{E}\left(\mathbb{E}(t^N \mid \Lambda)\right)$$

où $\mathbb{E}(t^N \mid \Lambda) = g(\Lambda)$ où $g(\lambda) = \mathbb{E}(t^N \mid \Lambda = \lambda)$

$$g(\lambda) = \sum_{m=0}^{+\infty} t^m \frac{\lambda^m e^{-\lambda}}{m!} = e^{-\lambda} e^{t\lambda} = e^{-\lambda(1-t)}$$

$$g_N(t) = \mathbb{E}\left(\mathbb{E}(t^N \mid \Lambda)\right) = \mathbb{E}\left(g(\Lambda)\right)$$

$$= \mathbb{E}\left[e^{-\Lambda(1-t)}\right] = \int_{\mathbb{R}} e^{-\lambda(1-t)} f_\Lambda(\lambda)\, d\lambda$$

$$= \int_0^c e^{-\lambda(1-t)} \frac{1}{c}\, d\lambda = \left[\frac{1}{c} \cdot \frac{1}{-(1-t)} e^{-\lambda(1-t)}\right]_0^c$$

$$= \frac{1}{c(t-1)}\left(e^{-c(1-t)} - 1\right)$$

<u>donc</u> $$g_N(t) = \begin{cases} \dfrac{1}{c(t-1)} \times \left(e^{-c(1-t)} - 1\right) & \text{si } t \neq 1 \\ \displaystyle\int_{\mathbb{R}} f_\Lambda(\lambda)\, d\lambda = 1 & \text{si } t = 1 \end{cases}$$

$$\mathbb{E}(t^N) = \sum_{m=0}^{+\infty} t^m \times \mathbb{P}(N=m)$$

$$= \sum_{m=0}^{+\infty} t^m \times \int_{\mathbb{R}} e^{-\lambda} \frac{\lambda^m}{m!} f_\Lambda(\lambda)\, d\lambda$$

$$= \int_{\mathbb{R}} \sum_{m=0}^{+\infty} e^{-\lambda} t^m \frac{\lambda^m}{m!} f_\Lambda(\lambda)\, d\lambda$$

$$= \int_{\mathbb{R}} f_\Lambda(\lambda) \cdot \sum_{m=0}^{+\infty} \frac{(t\lambda)^m}{m!} e^{-\lambda}\, d\lambda$$

$$= \int_{\mathbb{R}} e^{-\lambda(1-t)} f_\Lambda(\lambda)\, d\lambda$$

$$= \int_0^c \frac{e^{-\lambda(1-t)}}{c}\, d\lambda = \left[\frac{1}{c(t-1)} e^{-\lambda(1-t)} \right]_{t=0}^{t=c}$$

$$= \frac{1}{c(t-1)} \left(e^{-c(1-t)} - 1 \right) \quad \text{si } t \neq 1$$

$$\mathbb{E}(1^N) = \sum_{m=0}^{+\infty} 1^m\, \mathbb{P}(N=m) = \sum_{m=0}^{+\infty} \int_{\mathbb{R}} e^{-\lambda} \frac{\lambda^m}{m!} f_\Lambda(\lambda)\, d\lambda$$

$$= \int_{\mathbb{R}} e^{-\lambda} \sum_{m=0}^{+\infty} \frac{\lambda^m}{m!} f_\Lambda(\lambda)\, d\lambda = \int_{\mathbb{R}} e^{-\lambda} e^{\lambda} f_\Lambda(\lambda)\, d\lambda$$

$$= \int_{\mathbb{R}} f_\Lambda(\lambda)\, d\lambda = 1$$

$$\mathbb{P}(N=n) = \mathbb{E}\left(\mathbb{1}_{(N=n)}\right) = \mathbb{E}\left(\mathbb{E}\left[\mathbb{1}_{N=n} \mid \Lambda \right] \right)$$

$$= \mathbb{E}\left(g(\Lambda) \right)$$

où $g(\lambda) = \mathbb{E}\left(\mathbb{1}_{N=n} \mid \{\Lambda = \lambda\} \right)$

$$= \mathbb{P}\left(N=n \mid \{\Lambda = \lambda\} \right)$$

$$= \frac{\lambda^n e^{-\lambda}}{n!}$$

$$\Rightarrow \quad \mathbb{P}(N=n) = \mathbb{E}\left(g(\Lambda)\right) = \mathbb{E}\left[\Lambda^n \frac{e^{-\Lambda}}{n!} \right]$$

$$= \int_{\mathbb{R}} \lambda^n \frac{e^{-\lambda}}{n!} f_\Lambda(\lambda)\, d\lambda$$

$$= \int_{\mathbb{R}} e^{-\lambda} \frac{\lambda^n}{n!} \frac{1}{c} \cdot \mathbb{1}_{[0,c]}(\lambda)\, d\lambda$$

$$= \frac{}{n!\, c} \int_0^c e^{-\lambda} \lambda^n \, d\lambda =$$

$$= \frac{1}{c \cdot n!} \int_0^c e^{-\lambda} \lambda^n \, d\lambda$$

$$\int_0^c e^{-\lambda}\lambda^m d\lambda \qquad\qquad D \qquad\qquad I$$

$$= \left[-e^{-\lambda}\lambda^m\right]_0^c + \int_0^c m\lambda^{m-1}e^{-\lambda}d\lambda \qquad \begin{array}{ccc} + & \lambda^m & e^{-\lambda} \\ - & m\lambda^{m-1} & -e^{-\lambda} \end{array}$$

$$= -e^{-c}c^m + \frac{m}{\lambda}\int_0^c \lambda^m e^{-\lambda}d\lambda \qquad \pm$$

$$\Rightarrow I_m = -e^{-c}c^m + \frac{m}{\lambda} I_m$$

$$\Rightarrow I_m\left(1-\frac{m}{\lambda}\right) = -e^m e^{-c}$$

$$\Rightarrow I_m = \frac{-e^{-c}c^m \cdot \lambda}{\lambda - m} = \frac{\lambda c^m e^{-c}}{m-\lambda}$$

$$\Rightarrow \frac{1}{cm!} I_m = \frac{\lambda c^{m-1}e^{-c}}{m!\,(m-\lambda)}$$

$$\mathbb{P}(N=m) = \lambda e^{-c}\frac{c^{m-1}}{m!\,(m-\lambda)}$$

$$\mathbb{P}(N=m) = \frac{1}{c} \cdot \int_0^c \frac{e^{-\lambda}\lambda^m}{m!} d\lambda = \frac{1}{c} I_m$$

$$= \frac{1}{c} \cdot \left(1 - e^{-c} \sum_{k=0}^{m} \frac{c^k}{k!}\right)$$

	D	I
$+$	$\frac{\lambda^m}{m!}$	$e^{-\lambda}$
$-$	$\frac{m\lambda^{m-1}}{m!}$	$-e^{-\lambda}$

$$I_m = \left[-\frac{e^{-\lambda}\lambda^m}{m!}\right]_0^c = -\frac{e^{-c}c^m}{m!} + \int_0^c \frac{\lambda^{m-1}e^{-\lambda}}{(m-1)!} d\lambda$$

$$I_m = -\frac{c^m e^{-c}}{m!} + I_{m-1}$$

$$I_m - I_{m-1} = -\frac{c^m e^{-c}}{m!}$$

$$\sum_{k=1}^{m} (I_k - I_{k-1}) = -\sum_{k=1}^{m} \frac{c^k e^{-c}}{k!}$$

$$I_m - I_0 = -e^{-c} \sum_{k=1}^{m} \frac{c^k}{k!}$$

$$I_m = I_0 - e^{-c} \sum_{k=1}^{m} \frac{c^m}{m!}$$

$$= 1 - e^{-c} - e^{-c} \sum_{k=1}^{m} \frac{c^k}{k!}$$

$$= 1 - e^{-c}\left(1 + \sum_{k=1}^{m} \frac{c^k}{k!}\right) = \left(1 - e^{-c} \sum_{k=0}^{m} \frac{c^k}{k!}\right)$$

$$I_0 = \int_0^c \frac{e^{-\lambda}\lambda^0}{0!} d\lambda$$

$$= \int_0^c e^{-\lambda} d\lambda$$

$$= \left[-e^{-\lambda}\right]_0^c$$

$$= 1 - e^{-c}$$

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$$\sum_{n=0}^{+\infty} \mathbb{P}(N=n) = \sum_{n=0}^{+\infty} \frac{1}{c}\left(1-e^{-c}\sum_{k=0}^{n}\frac{c^k}{k!}\right)$$

$$\sum_{n=0}^{p}\frac{1}{c} - \sum_{n=0}^{\cancel{+\infty}}\frac{e^{-c}}{c}\sum_{k=0}^{n}\frac{c^k}{k!} = \sum_{n=0}^{p}\frac{1}{c}\left(1-e^{-c}\sum_{k=0}^{n}\frac{c^k}{k!}\right)$$

$$\frac{p+1}{c} - \frac{e^{-c}}{c}\sum_{n=0}^{p}\sum_{k=0}^{n}\frac{c^k}{k!} \qquad \begin{cases} 0\le k\le n \\ 0\le n\le p \end{cases}$$

$$\frac{p+1}{c} - \frac{e^{-c}}{c}\sum_{k=0}^{p}\sum_{n=k}^{p}\frac{c^k}{k!} \qquad \Leftrightarrow 0\le k\le n\le p \quad \Leftrightarrow \begin{cases} k\le n\le p \\ 0\le k\le p \end{cases}$$

$$\frac{p+1}{c} - \frac{e^{-c}}{c}\sum_{k=0}^{p}(p-k+1)\frac{c^k}{k!}$$

$$\frac{p+1}{c} - \frac{e^{-c}}{c}\left(\sum_{k=0}^{p}(p+1)\cdot\frac{c^k}{k!} - \sum_{k=0}^{p} k\frac{c^k}{k!}\right)$$

$$\frac{p+1}{c} - \frac{e^{-c}}{c}\left((p+1)\sum_{k=0}^{p}\frac{c^k}{k!} - \sum_{k=1}^{p}\frac{c^k}{(k-1)!}\right)$$

$$\frac{p+1}{c} - \frac{e^{-c}}{c}\left((p+1)\left(e^{c}-\sum_{k=p+1}^{+\infty}\frac{c^k}{k!}\right) - \sum_{k=0}^{p-1}\frac{c^{k+1}}{k!}\right)$$

$$\frac{p+1}{c} - \frac{e^{-c}}{c}(p+1)e^{c} + \frac{e^{-c}}{c}\sum_{k=p+1}^{+\infty}(p+1)\frac{c^k}{k!} + \frac{e^{-c}}{c}\cdot c\sum_{k=0}^{p-1}\frac{c^k}{k!}$$

$$e^{-c}\left(\frac{1}{c}\sum_{k=p+1}^{+\infty}(p+1)\cdot\frac{c^k}{k!} + \sum_{k=0}^{p-1}\frac{c^k}{k!}\right)$$

$$= \frac{e^{-c}}{c}\sum_{k=0}^{+\infty}\frac{c^{p+k+1}}{(p+k+1)!}(p+1) + e^{-c}\sum_{k=0}^{p-1}\frac{c^k}{k!} \xrightarrow[p\to+\infty]{} 0 + e^{-c}e^{c} = 1$$

3) $\Lambda' = \Lambda / c \sim \beta(a,b)$

$$f_{\Lambda'}(x) = \frac{1}{B(a,b)} x^{a-1}(1-x)^{b-1} \mathbb{1}_{]0,1[}(x)$$

$$B(u,v) = \int_0^1 w^{u-1}(1-w)^{v-1}\, dw. \qquad u,v>0$$

$$B(u,v) = \frac{\Gamma(u)\cdot\Gamma(v)}{\Gamma(u+v)}$$

- $\mathbb{E}(N)$, $Var(N)$, $\mu_3(N)$

- $\mathbb{E}(N) = \mathbb{E}(\Lambda) = \mathbb{E}(c\Lambda') = c\,\mathbb{E}(\Lambda')$

$$= c \int_{\mathbb{R}} x \cdot f_{\Lambda'}(x)\,dx = c \int_{\mathbb{R}} x \cdot \frac{1}{B(a,b)} x^{a-1}(1-x)^{b-1} \mathbb{1}_{]0,1[}\,dx$$

$$= c \int_{\mathbb{R}} \frac{x^{(a+1)-1}(1-x)^{b-1}}{B(a,b)} \mathbb{1}_{]0,1[}(x)\,dx$$

$$= \frac{c}{B(a,b)} \int_0^1 x^{(a+1)-1}(1-x)^{b-1}\,dx$$

$$= c \frac{B(a+1,b)}{B(a,b)} = c \cdot \frac{\Gamma(a+1)\cancel{\Gamma(b)}}{\Gamma(a+b+1)} \cdot \frac{\Gamma(a+b)}{\Gamma(a)\cancel{\Gamma(b)}}$$

$$= c \cdot \frac{(a+1)\cdot\cancel{\Gamma(a)}\,\cancel{\Gamma(a+b)}}{(a+b+1)\,\cancel{\Gamma(a+b)}\,\cancel{\Gamma(a)}} = c \cdot \frac{a+1}{a+b+1}$$

$$\cdot \ Var(N) = Var(\Lambda) + \mathbb{E}(N)$$

$$= Var(c \cdot \Lambda') + \mathbb{E}(N)$$

$$= c^2 \cdot Var(\Lambda') + \mathbb{E}(N)$$

$$Var(\Lambda') = \mathbb{E}(\Lambda'^2) - \mathbb{E}(\Lambda')^2$$

$$\mathbb{E}(\Lambda'^2) = \int_{\mathbb{R}} x^2 f_{\Lambda'}(x)\,dx = \int_0^1 \frac{x^{(a+2)-1}(1-x)^{b-1}}{B(a,b)}\,dx = \frac{B(a+2,b)}{B(a,b)}$$

$$\Rightarrow Var(N) = \frac{B(a+2,b)}{B(a,b)} c^2 - \left(\frac{B(a+1,b)}{B(a,b)} c\right)^2 + c \cdot \frac{B(a+1,b)}{B(a,b)}$$

$$\mu_3(N) = \mu_3(\Lambda) + 3\,Var(\Lambda) + \mathbb{E}(N)$$

$$\mu_3(\Lambda) = \mathbb{E}\left(\left(\Lambda - \mathbb{E}(\Lambda)\right)^3\right) = \mathbb{E}\left[\left(c \cdot \Lambda' - \mathbb{E}(c \Lambda')\right)^3\right]$$

$$= \mathbb{E}\left(c^3\left(\Lambda' - \mathbb{E}(\Lambda')\right)^3\right) = c^3\, \mathbb{E}\left[\left(\Lambda' - \mathbb{E}(\Lambda')\right)^3\right]$$

$$= c^3\left[\mathbb{E}(\Lambda'^3) - 3\,\mathbb{E}(\Lambda'^2)\mathbb{E}(\Lambda') + 3\,\mathbb{E}(\Lambda')\,\mathbb{E}(\Lambda')^2 - 1\,\mathbb{E}(\Lambda')^3\right]$$

$=$