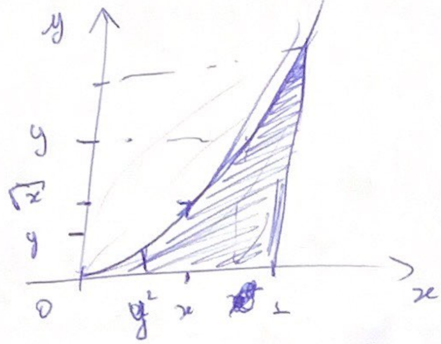
Exemple : Soit (X,Y) un couple de densité

$$f(x,y) = \frac{4y}{x^3} \mathbb{1}_{(0<x<1 \text{ et } 0<y<x^2)}$$

1) Calculer $\mathbb{E}(X|Y)$

Domaine :



a) $\mathbb{E}[X|Y] = \left(\mathbb{E}[X|Y=\cdot]\right)\circ Y = g(Y)$ $\qquad g : \mathbb{E}[X|Y=\cdot]$

$g(y) = \mathbb{E}(X|Y=y)$ est l'espérance de la loi $\mathcal{L}_{X|Y=y}$ qui a pour densité

$$f_{X|Y=y}(x) = \frac{f(x,y)}{f_Y(y)} \mathbb{1}_{(f_Y(y)>0)}$$

$$f_Y(y) = \int_{-\infty}^{+\infty} f(x,y)\,dx = \int_{-\infty}^{+\infty} \frac{4y}{x^3} \mathbb{1}_{\{0<x<1 \text{ et } 0<y<x^2\}}$$

On a $\begin{cases} 0<x<1 \\ 0<y<x^2 \end{cases} \Rightarrow \begin{cases} 0<y<x^2<1 \\ \sqrt{y}<x<1 \\ 0<y<1 \end{cases}$

$$f_Y(y) = \int_{\sqrt{y}}^{1} \frac{4y}{x^3} \mathbb{1}_{(0<y<1)}(y)\,dx = 4y \cdot \mathbb{1}_{(0<y<1)} \int_{\sqrt{y}}^{1} \frac{dx}{x^3}$$

$$= 4 \cdot y\, \mathbb{1}_{]0,1[}(y) \left[\frac{x^{-3+1}}{-3+1}\right]_{\sqrt{y}}^{1} = 4y\,\mathbb{1}_{]0,1[}(y)\left[-\frac{1}{2x^2}\right]_{\sqrt{y}}^{1} = 4y\left(\frac{1}{2y} - \frac{1}{2}\right)\mathbb{1}_{]0,1[}(y)$$

$$= \frac{4y(1-y)}{2y} \mathbb{1}_{]0,1[}(y) = 2(1-y)\,\mathbb{1}_{]0,1[}(y)$$

On a alors $f_{X|Y=y}(x) = \begin{cases} \dfrac{f_{(X,Y)}(x,y)}{f_Y(y)} & \text{si } f_Y(y) > 0 \Leftrightarrow y \in]0,1[\\ 0 & \text{sinon} \end{cases}$

$$f_{X|Y=y}(x) = \frac{f_{(X,Y)}(x,y)}{f_Y(y)} \mathbb{1}_{]0,1[}(y) = \frac{\frac{4y}{x^3} \mathbb{1}_D(x,y)}{2(1-y)}.$$

$$= \frac{2y}{x^3(1-y)} \mathbb{1}_{(0<y<1 \text{ et } \sqrt{y}<x<1)}$$

$$\Rightarrow \forall y \in]0,1[\quad f_{X|Y}(x) = \frac{2y}{x^3(1-y)} \mathbb{1}_{]\sqrt{y},1[}(x)$$

$$\Rightarrow g(y) = \mathbb{E}(X|Y=y) = \int_{-\infty}^{+\infty} x f_{X|Y=y}(x)\,dx = \int_{\sqrt{y}}^{1} x \cdot \frac{2y}{x^3(1-y)}\,dx$$

$$= \frac{2y}{1-y}\left[x^{-2}\right]_{\sqrt{y}}^{1} = \frac{2y}{1-y}\left[\frac{x^{-2+1}}{-2+1}\right]_{\sqrt{y}}^{1} = \frac{2y}{1-y}\left[\frac{1}{x}\right]_{1}^{\sqrt{y}}$$

$$= \frac{2y}{1-y}\left[\frac{1}{\sqrt{y}} - 1\right] = \frac{2y}{1-y}\,\frac{1-\sqrt{y}}{\sqrt{y}} = \frac{2\sqrt{y}(1-\sqrt{y})}{1-y}$$

$$= \frac{2\sqrt{y}(1-\sqrt{y})}{(1-\sqrt{y})(1+\sqrt{y})} = \frac{2\sqrt{y}}{1+\sqrt{y}}$$

$$\Rightarrow \mathbb{E}[X|Y] = g(Y) = \frac{2\sqrt{Y}}{1+\sqrt{Y}} \quad \mathbb{P}\text{-p.s}$$

c) $\mathbb{E}[Y|X] = g(X)$ où $g(x) = \mathbb{E}[Y|X=x]$

$$f_{Y|X=x}(y) = \frac{f_{(X,Y)}(x,y)}{f_X(x)} \mathbb{1}_{(f_X(x)>0)}$$

$$f_X(x) = \int_{\mathbb{R}} f(x,y)\,dy = \mathbb{1}_{]0,1[}(x) \frac{4}{x^3} \int_0^{x^2} y\,dy = \mathbb{1}_{]0,1[}(x) \frac{4}{x^3} \left[\frac{y^2}{2}\right]_0^{x^2} = \frac{4x^4}{x^3 2} \mathbb{1}_{]0,1[}(x)$$

$$= 2x\, \mathbb{1}_{]0,1[}(x)$$

$$\Rightarrow f_{Y|X=x}(y) = \frac{\frac{4y}{x^3} \mathbb{1}_D(x,y)}{2x} = \frac{2y}{x^4} \mathbb{1}_D(x,y)$$

$$\Rightarrow \forall x \in]0,1[\quad f_{Y|X=x}(y) = \frac{2y}{x^4} \mathbb{1}_{]0,x^2[}(y)$$

$$\Rightarrow g(x) = \mathbb{E}[Y|X=x] = \int_{\mathbb{R}} y\, f_{Y|X=x}(y)\,dy$$

$$= \int_{-\infty}^{+\infty} y \cdot \frac{2y}{x^4} \mathbb{1}_{]0,x^2[}(y)\,dy = \frac{1}{x^4} \int_0^{x^2} 2y^2\,dy$$

$$= \frac{1}{x^4} \left[\frac{2y^3}{3}\right]_0^{x^2} = \frac{2x^6}{3x^4} = \frac{2}{3} \cdot x^2$$

$$\Rightarrow \mathbb{E}[Y|X] = g(X) = \frac{2}{3} X^2$$

$$\begin{cases} 0 < x < 1 \\ 0 < y < x^2 \end{cases} \Rightarrow 0 < y < x^2 < 1 \Rightarrow \begin{cases} 0 < y < 1 \\ y < x^2 < 1 \end{cases} \Leftrightarrow \begin{cases} 0 < y < 1 \\ \sqrt{y} < x < 1 \end{cases}$$