Exemple: Soit $X = \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix}$ un vecteur gaussien avec

$$\mathbb{E}[X] = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \quad \text{et} \quad \Gamma_X = \begin{array}{c|ccc|c} & X_1 & X_2 & X_3 & \\ X_1 & 1 & 1 & 1 & X_1 \\ X_2 & 1 & 2 & 2 & X_2 \\ X_3 & 1 & 2 & 3 & X_3 \end{array}$$

Cherchons $\mathbb{E}[X_3 \mid X_1, X_2] = \mathbb{E}[X_3 \mid \sigma(X_2, X_1)] = g(X_1, X_2)$

Il s'agit de la proj. $\perp$ de X_3 sur $L_2(\Omega, \sigma(X_1, X_2), \mathbb{P})$

$$\sigma(X_1, X_2) = \sigma(\mathcal{H}_1) = \sigma(\text{vect}\{X_1, X_2\}) = \sigma\left(\left\{ \sum_{i=1}^{2} \alpha_i X_i \;/\; \alpha_i \in \mathbb{R} \right\}\right)$$

$$\mathbb{E}[X_3 \mid X_1, X_2] = \alpha X_1 + \beta X_2 \qquad \alpha, \beta \in \mathbb{R}.$$

_Cherchons α et β tq $X_3 - (\alpha X_1 + \beta X_2) \perp \mathcal{H}_2 = \{\text{vect}\{X_1, X_2\}\}$

_c-à-d $\begin{cases} X_3 - (\alpha X_1 + \beta X_2) \perp X_1 \\ X_3 - (\alpha X_1 + \beta X_2) \perp X_2 \end{cases}$

$$\Leftrightarrow \begin{cases} \mathbb{E}[X_1(X_3 - \alpha X_1 - \beta X_2)] = 0 \\ \mathbb{E}[X_2(X_3 - \alpha X_1 + \beta X_2)] = 0 \end{cases} \Leftrightarrow \begin{cases} \mathbb{E}[X_1 X_3] = \alpha \mathbb{E}[X_1^2] + \beta \mathbb{E}[X_1 X_2] \\ \mathbb{E}[X_2 X_3] = \alpha \mathbb{E}[X_2 X_1] + \beta \mathbb{E}[X_2^2] \end{cases}$$

$$\begin{cases} 4 = \alpha \cdot 2 + \beta \cdot 3 \\ 8 = 3\alpha + 6\beta \end{cases}$$

On a $\mathbb{E}[X_1 X_3] = \text{cov}(X_1, X_3) + \mathbb{E}[X_1] \cdot \mathbb{E}[X_3]$
$= 1 + 1 \times 3 = 4$

$$\mathbb{E}[X_1^2] = \text{var}(X_1) + \mathbb{E}[X_1]^2 = 1 + 1^2 = 2$$

$$\mathbb{E}[X_1 X_2] = \text{cov}(X_1, X_2) + \mathbb{E}[X_1] \cdot \mathbb{E}[X_2] = 1 + 1 \cdot 2 = 3$$

Exo 5 :

Soit $X = \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix}$ un vect gaussien tq $\mathbb{E}(X) = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$

$\Gamma_X = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{pmatrix}$ On cherche $f \in \mathbb{R}^3$ et $A \in \mathcal{M}_3(\mathbb{R})$ tq

$\begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \end{pmatrix} = Y = AX + f$ soit un vecteur tq $\begin{cases} \mathbb{E}[Y] = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \\ \Gamma_Y = I_d = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix} \end{cases}$

$\begin{cases} \mathbb{E}[Y] = \mathbb{E}[AX + f] = A\mathbb{E}(X) + f = 0 \\ \Gamma_Y = A \cdot \Gamma_X A^t = I_d \end{cases} \iff \begin{cases} f = -A \cdot \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \\ A \Gamma A^t = I_d \end{cases}$

On a $A\Gamma A^t = Id$

$\Leftrightarrow \forall u \in \mathbb{R}^3 \quad \langle u, A\Gamma A^t u\rangle = \langle u, Id\, u\rangle$

$\Leftrightarrow \forall u \in \mathbb{R}^3 \quad \langle A^t u, \Gamma A^t u\rangle = \langle u, u\rangle$

$\Leftrightarrow \forall u \in \mathbb{R}^3 \quad \langle A^t u, \Gamma A^t u\rangle = \langle u, u\rangle$

Changement de variable naturelle :

$v = A^t u \Leftrightarrow (A^t)^{-1} v = u$

$\Leftrightarrow \forall v \in \mathbb{R}^3 \quad \langle u, u\rangle = \langle v, \Gamma v\rangle$ où $u = (A^t)^{-1}$

On a $\langle v, \Gamma v\rangle = v^t \Gamma v = (v_1, v_2, v_3)\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 2 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$

$= (v_1, v_2, v_3) \cdot \begin{pmatrix} v_1 + v_2 + v_3 \\ v_1 + 2v_2 + 2v_3 \\ v_1 + 2v_2 + 3v_3 \end{pmatrix} = v_1 [v_1 + v_2 + v_3] + v_2 [v_1 + 2v_2 + 2v_3] + v_3 [v_1 + 2v_2 + 3v_3]$

$= v_1^2 + v_1 v_2 + v_1 v_3 + v_2 v_1 + 2v_2^2 + 2 v_2 v_3 + v_3 v_1 + 2 v_2 v_3 + 3 v_3^2$

$= v_1^2 + 2v_1 v_2 + 2v_1 v_3 + 2 v_2 v_3 + v_2^2 + v_3^2 + v_2^2 + 2 v_2 v_3 + v_3^2 + v_3^2$

$= (v_1^2 + v_2^2 + v_3^2)^2 + (v_2 + v_3)^2 + v_3^2$

Et on a $\langle u, u\rangle = u^t u = (u_1, u_2, u_3)\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = u_1^2 + u_2^2 + u_3^2$

Il faut que $\langle v, \Gamma v\rangle = \langle u, u\rangle$

donc $(v_1 + v_2 + v_3)^2 + (v_2 + v_3)^2 + v_3^2 = u_1^2 + u_2^2 + u_3^2$

où $u = (A^t)^{-1} v$

on a $\begin{cases} u_1 = v_1 + v_2 + v_3 \\ u_2 = v_2 + v_3 \\ u_3 = v_3 \end{cases} \Leftrightarrow \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} v_1 + v_2 + v_3 \\ v_2 + v_3 \\ v_3 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$

Donc $({}^tA)^{-1} = \begin{array}{cc} \begin{array}{ccc} u & v & w \end{array} & \\ \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} & \begin{array}{c} e_1 \\ e_2 \\ e_3 \end{array} \end{array}$ ${}^tA = \begin{array}{cc} \begin{array}{ccc} e_1 & e_2 & e_3 \end{array} & \\ \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} & \begin{array}{c} u \\ v \\ w \end{array} \end{array}$

$\begin{cases} u = e_1 \\ v = e_1 + e_2 \\ w = e_1 + e_2 + e_3 \end{cases}$ $\begin{cases} e_1 = u \\ e_2 = -u + v \\ e_3 = -v + w \end{cases}$ $A = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}$

$f = -\begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix}$