Exo 9 : X et N des variables aléatoires à valeurs dans $\mathbb{N}$.

$$\begin{cases} \mathbb{E}(N) = m \\ Var(N) = \sigma^2 \end{cases} \qquad m, \sigma \in \mathbb{R}_+$$

$$X \mid N = m \geq 0 , \quad \mathbb{P}(X = k \mid N = m) = \begin{cases} \frac{1}{1+m} & 0 \leq k \leq m \\ 0 & \end{cases}$$

1) $\mathbb{E}(X \mid N) = \cancel{\mathbb{E}}(g(N))$

où $g(m) = \mathbb{E}(X \mid N = m)$

$$= \sum_{k \in \cancel{\mathbb{N}}}^{m} k \, \mathbb{P}(X = k \mid N = m)$$

$$= \sum_{k=0}^{\cancel{m}} \frac{k}{1+m} = \frac{1}{1+m} \sum_{k=0}^{m} k = \frac{m(m+1)}{2(1+m)} = \frac{m}{2}$$

Ainsi $\mathbb{E}(X \mid N) = \cancel{\mathbb{E}}(g(N)) = \cancel{\mathbb{E}}\left(\frac{N}{2}\right) = \frac{1}{2} \cancel{\mathbb{E}}(N) =$

$$\mathbb{E}(X^2|N) = (g(N))$$

où $g(m) = \mathbb{E}(X^2 | \{N=m\})$

$$= \sum_{k=0}^{m} k^2 \, \mathbb{P}(X=k \mid N=m)$$

$$= \sum_{k=0}^{m} \frac{k^2}{1+m} = \frac{m(m+1)(2m+1)}{6(1+m)} = \frac{m(2m+1)}{6}$$

Ainsi $\mathbb{E}(X^2|N) = (g(N)) = \left(\frac{N(2N+1)}{6}\right)$

$$\mathbb{E}(X) = \mathbb{E}\left(\mathbb{E}(X|N)\right)$$

$$= \mathbb{E}\left(\frac{N}{2}\right) = \frac{1}{2}\mathbb{E}(N) = \frac{m}{2}$$

$$\mathrm{Var}(X) = \mathbb{E}(X^2) - \mathbb{E}(X)^2 = \mathbb{E}(X^2) - \left(\frac{m}{2}\right)^2$$

On calcule $\mathbb{E}(X^2) = \mathbb{E}\left(\mathbb{E}(X^2|N)\right)$

$$= \mathbb{E}\left(\frac{N(2N+1)}{6}\right)$$

$$= \frac{2}{6}\mathbb{E}(N^2) + \frac{1}{6}\mathbb{E}(N)$$

$$= \frac{1}{3}\left[\mathrm{Var}(N) + \mathbb{E}(N)^2\right] + \frac{1}{6}m$$

$$= \frac{1}{3}\left(\sigma^2 + m^2\right) + \frac{1}{6}m$$

$$= \frac{1}{6}\left[2(\sigma^2+m^2) + m\right]$$

donc $\mathrm{Var}(X) = \frac{1}{6}\left[2(\sigma^2+m^2)+m\right] - \frac{m^2}{4}$

$$= \frac{8\sigma^2 + 8m^2 + 4m - 6m^2}{24} = \frac{2m^2 + 4m + 8\sigma^2}{24}$$

$$= \frac{1}{12}\left(m^2 + 2m + 4\sigma^2\right) = \frac{1}{12}m^2 + \frac{1}{6}m + \frac{1}{3}\sigma^2$$

2) On suppose que $Y = N - X$ et X sont indépendantes

$$\mathbb{E}(Y) = \mathbb{E}(N-X) = \mathbb{E}(N) - \mathbb{E}(X) = m - \frac{m}{2} = \frac{m}{2} = \mathbb{E}(X)$$

$$\begin{aligned}
Var(Y) &= \mathbb{E}(Y^2) - \mathbb{E}(Y)^2 \\
&= \mathbb{E}((N-X)^2) - \frac{m^2}{4} \\
&= \mathbb{E}(N^2) - 2\,\mathbb{E}(NX) + \mathbb{E}(X^2) - \frac{m^2}{4} \\
&= Var(N) + \mathbb{E}(N)^2 - 2\,\mathbb{E}(N\cdot X) + \mathbb{E}(X^2) - \frac{m^2}{4} \\
&= \sigma^2 + m^2 - 2\,\mathbb{E}(N\cdot X) + \frac{1}{3}(\sigma^2 + m^2) + \frac{m}{6} - \frac{m^2}{4} \\
&= \sigma^2 + m^2 - \frac{1}{6}(7m^2 + 2m + 4\sigma^2) + \frac{1}{3}(\sigma^2 + m^2) + \frac{m}{6} - \frac{m^2}{4}
\end{aligned}$$

Comme $Y \perp\!\!\!\perp X \Rightarrow cov(Y, X) = 0$

$$\begin{aligned}
&\Rightarrow Cov(N - X, X) = 0 \\
&\Rightarrow cov(N, X) - cov(X, X) = 0 \\
&\Rightarrow cov(N, X) = cov(X, X) \\
&\Rightarrow \mathbb{E}(NX) - \mathbb{E}(N)\mathbb{E}(X) = Var(X) \\
&\Rightarrow \mathbb{E}(NX) = Var(X) + \mathbb{E}(N)\,\mathbb{E}(X) \\
&\Rightarrow \mathbb{E}(NX) = \frac{1}{12}(m^2 + 2m + 4\sigma^2) + m\frac{m}{2}
\end{aligned}$$

$$\Rightarrow \mathbb{E}(NX) = \frac{1}{12}(7m^2 + 2m + 4\sigma^2)$$

$$\mathbb{E}(N \mid X) = \mathbb{E}(N - X + X \mid X)$$

$$= \mathbb{E}(Y + X \mid X)$$

$$= \mathbb{E}(Y \mid X) + \mathbb{E}(X \mid X)$$

$$= \mathbb{E}(Y) + X$$

$$= X + \frac{m}{2}$$

$$\mathbb{E}(N / Y) = \mathbb{E}(N - X + X \mid Y)$$

$$= \mathbb{E}(X + Y \mid Y)$$

$$= \mathbb{E}(X \mid Y) + \mathbb{E}(Y \mid Y)$$

$$= \mathbb{E}(X) + Y$$

$$= Y + \frac{m}{2}$$