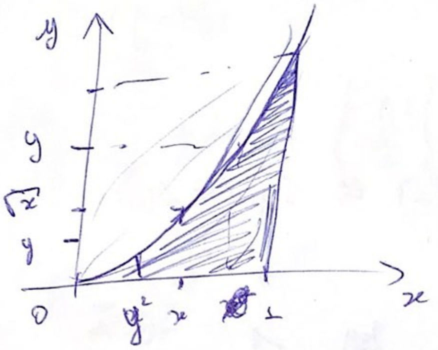
Exemple : Soit (X,Y) un couple de densité

$$f(x,y) = \frac{4y}{x^3} \mathbb{1}_{(0<x<1 \text{ et } 0<y<x^2)}$$

1) Calculer $\mathbb{E}(X|Y)$

Domaine :



a) $\mathbb{E}[X|Y] = \left(\mathbb{E}[X|Y=\cdot]\right)\circ Y = g(Y)$ $\qquad g : \mathbb{E}[X|Y=\cdot]$

$g(y) = \mathbb{E}(X|Y=y)$ est l'espérance de la loi $\mathcal{L}\, X|Y=y$ qui a pour densité

$$f_{X|Y=y}(x) = \frac{f(x,y)}{f_Y(y)} \mathbb{1}_{(f_Y(y)>0)}(x)$$

$$f_Y(y) = \int_{-\infty}^{+\infty} f(x,y)\,dx = \int_{-\infty}^{+\infty} \frac{4y}{x^3} \mathbb{1}_{\{0<x<1 \text{ et } 0<y<x^2\}}$$

On a $\begin{cases} 0<x<1 \\ 0<y<x^2 \end{cases} \Rightarrow \begin{cases} 0<y<x^2<1 \\ \sqrt{y}<x<1 \\ 0<y<1 \end{cases}$

$$f_Y(y) = \int_{\sqrt{y}}^{1} \frac{4y}{x^3} \mathbb{1}_{(0<y<1)}(y)\,dx = 4y \cdot \mathbb{1}_{(0<y<1)} \int_{\sqrt{y}}^{1} \frac{dx}{x^3}$$

$$= 4 \cdot y\, \mathbb{1}_{]0,1[}(y) \left[\frac{x^{-3+1}}{-3+1}\right]_{\sqrt{y}}^{1} = 4y\, \mathbb{1}_{]0,1[}(y) \left[\frac{-1}{2x^2}\right]_{\sqrt{y}}^{1} = 4y\left(\frac{1}{2y} - \frac{1}{2}\right)\mathbb{1}_{]0,1[}(y)$$

$$= \frac{4y(1-y)}{2y} \mathbb{1}_{]0,1[}(y) = 2(1-y)\, \mathbb{1}_{]0,1[}(y)$$

On a alors $f_{X|Y=y}(x) = \begin{cases} \dfrac{f_{(X,Y)}(x,y)}{f_Y(y)} & \text{si } f_Y(y) > 0 \Leftrightarrow y \in]0,1[\\ 0 & \text{sinon} \end{cases}$

$$f_{X|Y=y}(x) = \frac{f_{(X,Y)}(x,y)}{f_Y(y)} \mathbb{1}_{]0,1[}(y) = \frac{\frac{4y}{x^3} \mathbb{1}_D(x,y)}{2(1-y)}.$$

$$= \frac{2y}{x^3(1-y)} \mathbb{1}_{(0<y<1 \text{ et } \sqrt{y}<x<1)}$$

$$\Rightarrow \forall y \in]0,1[\quad f_{X|Y}(x) = \frac{2y}{x^3(1-y)} \mathbb{1}_{]\sqrt{y},1[}(x)$$

$$\Rightarrow g(y) = \mathbb{E}(X|Y=y) = \int_{-\infty}^{+\infty} x f_{X|Y=y}(x)\,dx = \int_{\sqrt{y}}^{1} x \cdot \frac{2y}{x^3(1-y)}\,dx$$

$$= \frac{2y}{1-y}\left[x^{-2}\right]_{\sqrt{y}}^{1} = \frac{2y}{1-y}\left[\frac{x^{-2+1}}{-2+1}\right]_{\sqrt{y}}^{1} = \frac{2y}{1-y}\left[\frac{1}{x}\right]_{1}^{\sqrt{y}}$$

$$= \frac{2y}{1-y}\left[\frac{1}{\sqrt{y}} - 1\right] = \frac{2y}{1-y}\,\frac{1-\sqrt{y}}{\sqrt{y}} = \frac{2\sqrt{y}(1-\sqrt{y})}{1-y}$$

$$= \frac{2\sqrt{y}(1-\sqrt{y})}{(1-\sqrt{y})(1+\sqrt{y})} = \frac{2\sqrt{y}}{1+\sqrt{y}}$$

$$\Rightarrow \mathbb{E}[X|Y] = g(Y) = \frac{2\sqrt{Y}}{1+\sqrt{Y}} \quad \mathbb{P}\text{-p.s}$$

2) Calcul de $\mathbb{E}[Y|X]$ $\quad f_{(X,Y)}(x,y) = \frac{4y}{x^3} \mathbb{1}_{\{0<x<1 \text{ et } 0<y<x^2\}}$

$$f_{Y|X=x}(y) = \frac{f_{X,Y}(x,y)}{f_X(x)}$$ si x est tq $f_X(x) > 0$ c-à-d pour $x \in]0,1[$

$$f_X(x) = \int_{\mathbb{R}} f_{(X,Y)}(x,y)dy = \int_{\mathbb{R}} \frac{4y}{x^3} \mathbb{1}_{(0<x<1 \text{ et } 0<y<x^2)}$$

$$= \frac{4}{x^3} \mathbb{1}_{[0,1]}(x) \int_0^{x^2} y\,dy = \left[\frac{y^2}{2}\right]_0^{x^2} = \frac{x^4}{2}$$

$$= \frac{4x^4}{2x^3} \mathbb{1}_{]0,1[}(x)$$

$$\rightarrow f_X(x) = 2x \mathbb{1}_{[0,1]}(x)$$

On a $$f_{Y|X=x}(y) = \begin{cases} f_{X,Y}(x,y) & \text{si } f_X(x) > 0 \\ 0 & \text{sinon} \end{cases}$$

$$= \frac{\frac{4y}{x^3} \mathbb{1}_D(x,y)}{2x} = 2 \cdot \frac{y}{x^4} \mathbb{1}_D(x,y)$$

$x \in]0,1[$

$$f_{Y|X=x}(y) = \frac{2y}{x^4} \mathbb{1}_{]0,x^2[}(y)$$

$$\Rightarrow \mathbb{E}[Y|X=x] = \int_{\mathbb{R}} y \cdot f_{Y|X=x}(y)\,dy = \int_0^{x^2} y \cdot \frac{2y}{x^4}\,dy$$

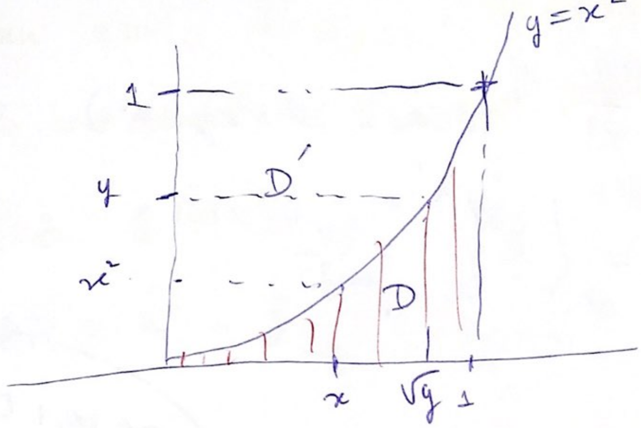
$$= \frac{1}{x^4} \int_0^{x^4} 2y^2 dy = \frac{1}{x^4} \left[\frac{2y^3}{3}\right]_0^{x^2}$$

$$= \frac{2x^6}{3x^4} = \frac{2}{3}x^2$$

Conclusion : $\mathbb{E}[Y|X] = \mathbb{E}(Y|X=\cdot)(X) = \frac{2}{3}X^2$

<u>Remarque :</u> À propos du domaine $D = \{(x,y) \in \mathbb{R}^2 \text{ tq } 0 < x < 1 \text{ et } 0 < y < x^2\}$

$D \neq D' = \{(x,y) \in \mathbb{R}^2 \text{ tq } 0 < x < 1 \text{ et } 0 < y < 1\}$



$D \subsetneq D'$

D peut être écrit d'1 deuxième façon (selon le besoin)

$D = \{(x,y) \in \mathbb{R}^2 \text{ tq } 0 < y < 1 \text{ et } \sqrt{y} < x < 1\}$

utile lorsqu'on calcule $\displaystyle\iint_D f(x,y)\,dx\,dy \underset{\text{Fubini}}{=} \int_0^1 \left[\int_{\sqrt{y}}^1 f(x,y)\,dx \right] dy$

alors que la 1ère écriture est utile si j'écris

$$\iint_D f(x,y)\,dx\,dy \overset{\text{Fubini}}{=} \int_0^1 \left[\int_0^{x^2} f(x,y)\,dy \right] dx$$

ce qui est vrai car $\quad 0 < y < x^2$

$\iff 0 < \sqrt{y} < |x|$

$\iff 0 < \sqrt{y} < x$

__Exemple :__ Soit (X,Y) un vecteur aléatoire tel que :

X a pour densité $f_X(x) = xe^{-x} \mathbb{1}_{]0,+\infty[}(x)$ ((densité de la loi gamma $\gamma(2,1)$)

et pour $x>0$, $\mathcal{L}_{Y|X=x} = \mathcal{U}([0,x])$

La loi conditionnelle de Y sachant $(X=x)$ est la loi uniforme $[0,x]$.

1) Calculer $\mathbb{E}(Y|X) = g(X)$ où $g(x) = \mathbb{E}[Y|X=x]$

$$g(x) = \mathbb{E}[Y|X=x] = \cancel{\mathbb{E}}\int_{\mathbb{R}} y \cdot f_{Y|X=x}(y)\,dy = \int_0^x y \frac{dy}{x} = \frac{1}{x}\left[\frac{y^2}{2}\right]_0^x = \frac{x}{2}$$

$$f_{Y|X=x}(y) = \frac{f(x,y)}{f_X(x)} \mathbb{1}_{(f_X(x)>0)}$$

$$f_{Y|X=x}(y) = \frac{f(x,y)}{xe^{-x}} \mathbb{1}_{(0<x<\infty \text{ et } 0<y<x)}$$

$$= \frac{1}{x-0} \mathbb{1}_{]0,x[}(y) \cdot \frac{1}{xe^{-x}} \mathbb{1}_{]0,+\infty[}(x)$$

$$\mathbb{E}(Y|X) = \frac{X}{2}$$

$$\cancel{f(x,y) = \frac{e^{-x}}{x} \mathbb{1}_{]0,+\infty[}(x)}$$

$$\frac{1}{x} \mathbb{1}_{]0,x[}(y) = \frac{f(x,y)}{xe^{-x}} \Leftrightarrow f(x,y) = \frac{xe^{-x}}{x} \mathbb{1}_{]0,x[}(y) \mathbb{1}_{\mathbb{R}_+^*}(x)$$

$$f_X(x) \cdot f_{Y|X=x}(y) = xe^{-x} \mathbb{1}_{]0,+\infty[}(x) \cdot \frac{1}{x} \mathbb{1}_{]0,x[}(y) = e^{-x} \mathbb{1}_{]0,+\infty[}(x) \cdot \mathbb{1}_{]0,x[}(y)$$

$$\Rightarrow f_Y(y) = \int_{\mathbb{R}} e^{-x} \mathbb{1}_{]0,+\infty[}(x) \cdot \mathbb{1}_{]0,x[}(y) =$$

2) $\mathbb{E}[X|Y] = g(Y)$ où $g(y) = \mathbb{E}[X|Y=y] = \int_{\mathbb{R}} x f_{X|Y=y}(x)\,dx$

• $f_{X,Y}(x,y) = \dfrac{f(x,y)}{f_X(x)} \cdot f_X(x)$

$= f_{Y|X=x}(y) \cdot f_X(x)$

$= \dfrac{1}{x} \mathbb{1}_{]0,x[}(y) \cdot x e^{-x} \mathbb{1}_{]0,+\infty[}(x)$

$= e^{-x} \mathbb{1}_{(0<x<+\infty \text{ et } 0<y<x)}$

$\cancel{g(y) = \mathbb{E}[X|Y=y] = \int_{\mathbb{R}} f^{x}}$

$f_Y(y) = \int_{\mathbb{R}} f(x,y)\,dx = \int_0^{+\infty} e^{-x} \mathbb{1}_{]0,x[}(y) = \left[-e^{-x}\right]_0^{+\infty} \mathbb{1}_{]0,x[}(y)$

$= \mathbb{1}_{]0,x[}(y)$

$\Rightarrow g(y) = \mathbb{E}[X|Y=y] = \int_{\mathbb{R}} x f_{X|Y=y}(x)\,dx = \int_{\mathbb{R}} x \cdot \dfrac{e^{-x} \mathbb{1}_D}{\ }\,dx$

$= \mathbb{1}_{]0,x[}(y) \cdot \int_0^{+\infty} x e^{-x}\,dx = \mathbb{1}_{]0,x[}(y) \left(\left[-xe^{-x}\right]_0^{+\infty} + \left[-e^{-x}\right]_0^{+\infty} \right)$

$= \mathbb{1}_{]0,x[}(y) = \begin{cases} 1 & \text{si } y \in]0,x[\\ 0 & \text{sinon} \end{cases}$

Soit (X,Y) vect. ~~gaussien~~ aléa.

$X \sim \Gamma(2,1)$

$$X \sim \Gamma(\alpha,\beta) \Rightarrow f_X(x) = x^{\alpha-1} \frac{\beta^{\alpha} e^{-\beta x}}{\Gamma(a)} \mathbb{1}_{]0,+\infty[}(x)$$

$$\Rightarrow f_X(x) = x^{2-1} \frac{1^2 e^{-x}}{\Gamma(1)} \mathbb{1}_{]0,+\infty[}(x) = x e^{-x} \mathbb{1}_{]0,+\infty[}(x)$$

$$\frac{f(x,y)}{f_X(x)} = f_{Y|X=x}(y)$$

$$\mathcal{L}\ Y|X=x = \mathcal{U}([0,x]) \Rightarrow$$

$$\Rightarrow f(x,y) = f_{Y|X=x}(y) \cdot f_X(x)$$

$$f_{Y|X=x}(y) = \mathbb{1}_{[0,x)} \cdot \frac{1}{x-0} = \frac{1}{x} \mathbb{1}_{[0,x]}(y)$$

$$\mathbb{E}[Y|X=x] = \int_{\mathbb{R}} y \cdot f_{Y|X=x}(y)\, dy = \frac{1}{x} \int_0^x y\, dx$$

$$= \frac{1}{x}\left[\frac{y^2}{2}\right]_0^x = \frac{x}{2}$$

$$\frac{f(x,y)}{f_Y(y)} = f_{X|Y=y}(x)$$

$$\Rightarrow \mathbb{E}[Y|X] = \frac{X}{2}$$

$$\mathbb{E}[X|Y=y] = \int_{\mathbb{R}} x \cdot f_{X|Y=y}(x)\, dx = \int_{\mathbb{R}} x \cdot \frac{f(x,y)}{f_Y(y)}\, dx$$

$$f(x,y) = f_{Y|X=x}(y) \cdot f_X(x)\, dx$$

$$= \frac{1}{x} \cdot \mathbb{1}_{[0,x]}(y) \cdot x \cdot e^{-x} \mathbb{1}_{]0,\infty[}^{x}$$

$$= e^{-x} \mathbb{1}_{\{0 \leq y \leq x \cancel{\text{et}}\ x > 0\}}$$

$$f_Y(y) = \int_{\mathbb{R}} f(x,y)\, dx = \int_{\mathbb{R}} e^{-x} \mathbb{1}_{(0,x]}(y)\, \mathbb{1}_{]0,+\infty[}(x)\, dx$$

On a $\begin{cases} 0 \leq y \leq x \\ 0 \leq x \leq \infty \end{cases} \Rightarrow \begin{cases} 0 \leq y \leq \infty \\ y \leq x < \infty \end{cases}$

$$\Rightarrow \int_{\mathbb{R}} e^{-x} \mathbb{1}_{[y;\infty[}(x)\, \mathbb{1}_{]0,\infty}(y)\, dx$$

$$= \mathbb{1}_{\mathbb{R}_+}(y) \int_y^{\infty} e^{-x}\, dx = \left[-e^{-x}\right]_y^{\infty} \mathbb{1}_{\mathbb{R}_+}(y)$$

$$= \left[e^{-x}\right]_{\infty}^{y} \mathbb{1}_{\mathbb{R}_+}(y) = e^{-y} \mathbb{1}_{]0,+\infty[}(y)$$

$$\Rightarrow Y \sim \mathcal{E}xp(1)$$

$$\mathbb{E}[X \mid Y=y] = \int_{\mathbb{R}} x \cdot \frac{f(x,y)}{f_Y(y)}\,dx$$

	D	I
$+$	x	e^{-x}
$-$	1	$-e^{-x}$
$+$	0	e^{-x}
$-$		

$$= \int_{\mathbb{R}} x \cdot \frac{e^{-x}\, \mathbb{1}_{[0,x]}(y)\, \mathbb{1}_{]0,+\infty[}(x)}{e^{-y}\, \mathbb{1}_{]0,+\infty[}(y)}\,dx$$

$=$ on a $\begin{cases} 0 \leq y \leq x \\ 0 \leq x \leq \infty \end{cases} \Rightarrow \begin{cases} y \leq x \leq \infty \\ 0 \leq y \leq \infty \end{cases}$

$$= \mathbb{1}_{]0,+\infty[}(y)\, e^{y} \int_{y}^{+\infty} x e^{-x}\,dx$$

$$= \left[-(xe^{-x} + e^{-x})\right]_{y}^{+\infty} \mathbb{1}_{]0,+\infty[}(y)\, e^{y}$$

$$= \left[e^{-x}(x+1)\right]_{\infty}^{y} e^{y}\, \mathbb{1}_{]0,+\infty[}(y)$$

$$= e^{-y}(y+1)e^{y}\, \mathbb{1}_{]0,+\infty[}$$

$$\Rightarrow \mathbb{E}[X \mid Y=y] = y+1 \Rightarrow \mathbb{E}[X \mid Y] = Y+1$$

$\forall\, y \in]0,+\infty[$