MATHEMATICAL FOUNDATION FOR COMPUTER SCIENCE

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LOGIC, INDUCTION AND REASONING

- Proposition and Truth function
- Propositional Logic
- Expressing statements in Logic Propositional Logic
- Rules of Inference
- The predicate Logic
- Validity
- Informal Deduction in Predicate Logic
- Proofs(Informal Proofs & Formal Proofs)
- Elementary Induction
- Complete Induction
- Methods of Tableaux
- Consistency and Completeness of the System

Proposition:

- Declarative statement that is either TRUE or FALSE.

- Symbol 'T' for TRUE and 'F' for FALSE.

  Examples:

  i) Paris is in France(T).
  ii) Delhi is in Nepal(F).
  iii) 2<4(T).
  iv) 4=7(F).

  Example of statement that are not propositions:

  i) What is your name? (This is a Question)
  ii) Do your Homework (This is a command)
  iii) "x" is even number (It depends on the value of x)

- Small alphabets like 'p', 'q' , 'r' are used to represent prepositions.

  p: Paris is in France.

  q: We live on Earth

Proposition Logic:

- ➢ Deals with proposition also known as Propositional Calculus.

- ➢ First developed by Aristotle.

I) Atomic Proposition:

❖ Which cannot be further broken down.

Example:

“Today is Friday”

II) Compound Proposition:

❖ Which can further be broken down.

❖ Logical operators are used.

Example:

“Ram is intelligent and diligent.”

p: “Ram is intelligent”

q: “Ram is diligent”

1.Logical operators/connectives:

- Used to construct compound propositions.
- Some common logical connectives are:

1. NEGATION(NOT) ¬
2. CONJUCTION(AND) ^
3. DISJUNCTION(OR) ∨
4. EXCLUSIVE OR(XOR) ⊕
5. IMPLICATION(IF-THEN) →
   (Inverse, Converse and Contrapositive)

6. BICONDITIONAL(IF AND ONLY IF) ←→

1.Negation(not):

- If ‘p’ is the proposition , then the negation of ‘p’ is denoted by ‘¬p’.
- ‘¬p’ means “it is not case that p” or simply “not p”.

Examples:

1) p: “Today is Friday”
   ¬p: “It is not the case that today is Friday”
   ¬p: “Today is not Friday”

2) p: “London is in Denmark”
   ¬p: “It is not the case that London is in Denmark”
   ¬p: “London is not in Denmark”

TRUTH TABLE

p	¬p
T	F
F	T

2.conjunction(and):

- If ‘p’ and ‘q’ are two proposition , then the conjunction of ‘p’ and ‘q’ is denoted by ‘p^q’.

- p^q is TRUE only when both ‘p’ and ‘q’ are TRUE, otherwise FALSE.

  Examples:

  1) p: “Today is Friday”
     q: “It is raining Today”
     p^q: “Today is Friday and it is raining Today”

Truth Table

p	q	p^q
T	T	T
T	F	F
F	T	F
F	F	F

<u>3.disjunction(or):</u>

- If ‘p’ and ‘q’ are two proposition , then the disjunction of ‘p’ and ‘q’ is denoted by ‘pVq’.

- pVq is FALSE when both ‘p’ and ‘q’ are FALSE, otherwise TRUE.

  <u>Examples:</u>

  1) p: “Today is Friday”

     q: “It is raining Today”

  pVq: “Today is Friday or it is raining Today”

Truth Table

p	q	pVq
T	T	T
T	F	T
F	T	T
F	F	F

4.Exclusive or (xor):

- If ‘p’ and ‘q’ are two proposition , then the Exclusive or of ‘p’ and ‘q’ is denoted by ‘$p \oplus q$’ which means “Either p or q but not both”

- $p \oplus q$ is TRUE when either ‘p’ or ‘q’ is TRUE and FALSE when both are TRUE or both are FALSE.

Examples:

1) p: “Today is Friday”
   q: “It is raining Today”
   $p \oplus q$: “Either today is Friday or it is raining today”

Truth Table

p	q	$p \oplus q$
T	T	F
T	F	T
F	T	T
F	F	F

5.implication (if→then):

- If ‘p’ and ‘q’ are two proposition then the statement “if p then q” is called an implication and denoted by p→q.
- p→q is also called a conditional statement.
- ‘p’ is called ***hypothesis*** or ***antecedent*** or ***premise***.
- ‘q’ is called the ***conclusion*** or ***consequence***.

Some other terminologies used to express p→q are:

- ✓ If p , then q.
- ✓ p is sufficient for q
- ✓ q when p
- ✓ A necessary condition for p is q
- ✓ p only if q
- ✓ q unless ¬p
- ✓ q follows from p

5.implication (if→then):

Example:

p: “Today is holiday”

q: “The college is closed”

p→q: ”If today is holiday, then the college is closed”

TRUTH TABLE

p	q	p→q
T	T	T
T	F	F
F	T	T
F	F	T

inverse:

$p \rightarrow q$ → $\neg p \rightarrow \neg q$

"if p, then q" *"if not p, then not q"*

p: "Today is holiday"
q: "The college is closed"

¬p: "Today is not holiday"
¬q: "The college is not closed"

p→q: "If today is holiday, then the college is closed"
¬p→¬q: "If today is not holiday, then the college is not closed"

converse:

$p \rightarrow q$ $\longrightarrow$ $q \rightarrow p$

"if p, then q" *"if q, then p"*

p: "Today is holiday"

q: "The college is closed"

$p \rightarrow q$: "If today is holiday, then the college is closed"

$q \rightarrow p$: "if the college is closed, then today is holiday"

Contra-positive:

$p \rightarrow q$ → $\neg q \rightarrow \neg p$

"if p, then q" *"if not q, then not p"*

p: "Today is holiday" ¬p: "Today is not holiday"

q: "The college is closed" ¬q: "The college is not closed"

$p \rightarrow q$: "If today is holiday, then the college is closed"

$\neg q \rightarrow \neg p$: "If the college is not closed, today is not holiday"

6.Biconditional(if and only if):

- If ‘p’ and ‘q’ are two proposition , then the biconditional statement p←→q is the proposition “p if and only if q”
- (p→q)^(q→p)==p←→q
- These are also called bi-implications.

Examples:

p: “I am breathing”

q: “I am alive”

p←→q : “I am breathing if and only if I am alive”

Truth Table

p	q	p←→q
T	T	T
T	F	F
F	T	F
F	F	T

Operator precedence:

Operator	Precedence (higher the number higher the precedence)
¬	1
^	2
v	3
→	4
↔	5

Examples:

1) $\neg p \wedge q$ {Given p =True and q=False}

=F^F

=F

2) $p \wedge q \vee r$ {Given p=True, q =False, r=True}

=F v T

=T

3) p →q ^ ¬p{Given p=True, q=False}

=T →F^F

=T →F

=F

4)(p^q)→((¬p)V q) {Given p=True, q=False}

=(T^F) →(FvF)

=F →F

=T

Truth table of compound preposition:

- Construct the truth table of compound proposition

$(P \vee \neg Q) \rightarrow (P \wedge Q)$

P	Q	¬Q	PV¬Q	P^Q	(P∨¬Q)→(P∧Q)
T	T	F	T	T	T
T	F	T	T	F	F
F	T	F	F	F	T
F	F	T	T	F	F

Truth table of compound preposition:

- Construct the truth table of compound proposition 1. $(P \vee \neg Q) \rightarrow (P \wedge Q)$

P	Q	¬Q	A=(PV¬Q)	B=(P^Q)	A→B
T	T	F	T	T	T
T	F	T	T	F	F
F	T	F	F	F	T
F	F	T	T	F	F

Truth table of compound preposition:

- Construct the truth table of compound proposition 2. (P→Q)^(Q→R)

P	Q	R	(P→Q)	(Q→R)	(P→Q)^(Q→R)
T	T	T	T	T	T
T	T	F	T	F	F
T	F	T	F	T	F
T	F	F	F	T	F
F	T	T	T	T	T
F	T	F	T	F	F
F	F	T	T	T	T
F	F	F	T	T	T

TRANSLATING ENGLISH SENTENCES:

Examples:

1."You can access the internet from NCIT only if you are a masters student or you are a new student"

Let ,

p: You access the internet from NCIT

q: You are a masters student

r: You are a new student

$$\mathbf{p \rightarrow (q \wedge r)}$$

2."The automated reply can not be sent when file system is full"

Let,

p:The automated reply can be sent

q:File system is full

$$\mathbf{q \rightarrow \neg p}$$

Assignment 1 :

1.What are logical connectives explain each with example and truth table.

2. Construct truth table for

- **$\neg(p \land q) \lor (r \land \neg p)$**
- **$(p \lor \neg r) \land \neg((q \lor r) \lor \neg(r \lor p))$**
- **$((p \leftrightarrow q) \oplus (\neg p \rightarrow q)) \lor (q \rightarrow \neg r)$**

3.Let p, q ,r be:

p="You have flu"

q="You miss the final exam"

r="You pass the course"

Express each proposition as an English sentence and construct truth table:

- **$p \rightarrow q$**
- **$q \rightarrow \neg r$**
- **$(p \rightarrow \neg r) \lor (q \rightarrow \neg r)$**

4. Translate into mathematical expression

- You can't have voting right if you are mentally unfit and you are not over 18 years.
- Leaders will make correct decision only if you choose a good leader or you raise your voice against incorrect decision

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- ❑ TAUTOLOGY
- ❑ CONTRADICTION
- ❑ CONTINGENCY
- ❑ PROPOSITIONAL SATISFIABILITY
- ❑ LOGICAL EQUIVALENCE

TAUTOLOGY:

- Compound proposition that is always TRUE , not matter what the truth values of the propositional variables that occur in it, is called TAUTOLOGY.

Examples:

a) p V ¬p

p	¬p	p ^ ¬p
T	F	T
F	T	T

b) (p→q) v (q→p)

p	q	p→q	q→p	(p→q)V(q→p)
T	T	T	T	T
T	F	F	T	T
F	T	T	F	T
F	F	T	T	T

CONTRADICTION:

- Compound proposition that is always FALSE, not matter what the truth values of the propositional variables that occur in it, is called CONTRADICTION.

Examples:

a) p ^ ¬p

p	¬p	p ^ ¬p
T	F	F
F	T	F

b) ¬(p^q) ←→ (q^p)

p	q	(p ^ q)	¬(p^q)	(q^p)	¬(p^q) ←→ (q^p)
T	T	T	F	T	F
T	F	F	T	F	F
F	T	F	T	F	F
F	F	F	T	F	F

CONTINGENGY:

- Compound proposition that is neither a TAUTOLOGY or a CONTRADICTION

Examples:

a) $(p \rightarrow q) \wedge (q \rightarrow p)$

p	q	p→q	q→p	(p→q) ^ (q→p)
T	T	T	T	T
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

SATISFIABILITY:

- Compound proposition is satisfiable is there is at least one true value in its truth table.
- TAUTOLOGY is always satisfiable but satisfiable is not always TAUTOLOGY.

(p→q) ^ (q→p)

p	q	p→q	q→p	(p→q) ^ (q→p)
T	T	T	T	T
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

SATISFIABLE but not TAUTOLOGY

(p→q)V(q→p)

p	q	p→q	q→p	(p→q)V(q→p)
T	T	T	T	T
T	F	F	T	T
F	T	T	F	T
F	F	T	T	T

SATISFIABLE and alsoTAUTOLOGY

UNSATISFIABILITY:

- Compound proposition is unsatisfiable is there is no true value in its truth table.
- CONTRADICTION is always satisfiable.

¬(p^q) ←→ (q^p)

p	q	(p ^ q)	¬(p^q)	(q^p)	¬(p^q) ←→ (q^p)
T	T	T	F	T	F
T	F	F	T	F	F
F	T	F	T	F	F
F	F	F	T	F	F

VALID & INVALID:

VALID: Compound proposition always VALID when it is a TAUTOLOGY.

INVALID: Compound proposition always INVALID when it is either CONTRADICTION or CONTINGENCY.

SUMMARY

TAUTOLOGY
Always TRUE
Satisfiable
VALID

CONTRADICTION
Always FALSE
unsatisfiable
INVALID

CONTINGENCY
Sometimes TRUE or FALSE
Satisfiable
INVALID

LOGICAL EQUIVALENCES:

- Compound proposition ‘p’ and ‘q’ are logically equivalent is they have same Truth Values in all possible cases.
- Notation: $p \equiv q$ or $p \Leftrightarrow q$

Examples:

a) $\neg(p \vee q)$ and $(\neg p \wedge \neg q)$

p	q	(pVq)	¬(pVq)	¬p	¬q	(¬p^¬q)
T	T	T	F	F	F	F
T	F	T	F	F	T	F
F	T	T	F	T	F	F
F	F	F	T	T	T	T

Hence, $\neg(p \vee q) \Leftrightarrow (\neg p \wedge \neg q)$

Examples:

b) $(p \rightarrow q)$ and $(\neg p \lor q)$

p	q	$(p \rightarrow q)$	$\neg p$	$(\neg p \lor q)$
T	T	T	F	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

Hence, $(p \rightarrow q) \Leftrightarrow (\neg p \lor q)$

IMPORTANT EQUIVALENCE:

Equivalences	Laws
$p \wedge \mathbf{T} \equiv p$ $p \vee \mathbf{F} \equiv p$	Identity laws
$p \vee \mathbf{T} \equiv \mathbf{T}$ $p \wedge \mathbf{F} \equiv \mathbf{F}$	Domination laws
$p \vee p \equiv p$ $p \wedge p \equiv p$	Idempotent laws
$\neg(\neg p) \equiv p$	Double negation law
$p \vee q \equiv q \vee p$ $p \wedge q \equiv q \wedge p$	Commutative laws
$(p \vee q) \vee r \equiv p \vee (q \vee r)$ $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$	Associative laws
$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$ $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$	Distributive laws
$\neg(p \wedge q) \equiv \neg p \vee \neg q$ $\neg(p \vee q) \equiv \neg p \wedge \neg q$	De Morgan's laws
$p \vee (p \wedge q) \equiv p$ $p \wedge (p \vee q) \equiv p$	Absorption laws
$p \vee \neg p \equiv \mathbf{T}$ $p \wedge \neg p \equiv \mathbf{F}$	Negation laws

p	T	p ^ T
T	T	T
F	T	F

p	F	p V T
T	F	T
F	F	F

1. Identity Laws

p	T	p V T
T	T	T
F	T	T

p	F	p ^ F
T	F	F
F	F	F

2. Domination Laws

EQUIVALENCE INVOLVING CONDITION:

TABLE 7 Logical Equivalences Involving Conditional Statements.

$p \to q \equiv \neg p \vee q$

$p \to q \equiv \neg q \to \neg p$

$p \vee q \equiv \neg p \to q$

$p \wedge q \equiv \neg(p \to \neg q)$

$\neg(p \to q) \equiv p \wedge \neg q$

$(p \to q) \wedge (p \to r) \equiv p \to (q \wedge r)$

$(p \to r) \wedge (q \to r) \equiv (p \vee q) \to r$

$(p \to q) \vee (p \to r) \equiv p \to (q \vee r)$

$(p \to r) \vee (q \to r) \equiv (p \wedge q) \to r$

1. $(p \to q) \Leftrightarrow (\neg p \vee q)$

p	q	(p→q)	¬p	(¬pVq)
T	T	T	F	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

2. $(p \to q) \Leftrightarrow (\neg q \to \neg p)$

p	q	(p→q)	¬p	¬q	(¬q→ ¬p)
T	T	T	F	F	T
T	F	F	F	T	F
F	T	T	T	F	T
F	F	T	T	T	T

EQUIVALENCE INVOLVING BICONDITION:

TABLE 8 Logical Equivalences Involving Biconditional Statements.

$$p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$$

$$p \leftrightarrow q \equiv \neg p \leftrightarrow \neg q$$

$$p \leftrightarrow q \equiv (p \wedge q) \vee (\neg p \wedge \neg q)$$

$$\neg(p \leftrightarrow q) \equiv p \leftrightarrow \neg q$$

1. (p←→q) ⇔ (p→q)^(q→p)

p	q	(p←→q)	(p→q)	(q→p)	(p→q)^(q→p)
T	T	T	T	T	T
T	F	F	F	T	F
F	T	F	T	F	F
F	F	T	T	T	T

Prove the following are logically equivalent by developing a series of logical equivalence.

1. $\neg(p \to q) \equiv (p \wedge \neg q)$

<u>solution:</u>

Taking LHS,

$=\neg(p \to q)$

$=\neg(\neg p \vee q)$-----------{$p \to q \equiv \neg p \vee q$}

$= \neg(\neg p) \wedge (\neg q)$------{De- Morgan's Law}

$=p \wedge \neg q$---------------{Double Negation Law}

Prove the following are logically equivalent by developing a series of logical equivalence.

1. $\neg(p \vee (\neg p \wedge q)) \equiv (\neg p \wedge \neg q)$

   <u>solution:</u>

   Taking LHS,

   $= \neg(p \vee (\neg p \wedge q))$

   $= \neg p \wedge \neg(\neg p \wedge q)$ -------------------by the second De Morgan law

   $= \neg p \wedge [\neg(\neg p) \vee \neg q]$ ----------------------by the first De Morgan law

   $= \neg p \wedge (p \vee \neg q)$ -------------------------by the double negation law

   $= (\neg p \wedge p) \vee (\neg p \wedge \neg q)$ ------------by the second distributive law

   $= F \vee (\neg p \wedge \neg q)$ --because $\neg p \wedge p \equiv F$

   $= (\neg p \wedge \neg q) \vee F$---------- by the commutative law for disjunction

   $= \neg p \wedge \neg q$ ---by the identity law for F

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RULES OF INFERENCE

ARGUMENT:

- An argument is a sequence of proposition written as:

$$\begin{array}{c} P_1 \\ P_2 \\ . \\ . \\ . \\ . \\ . \\ . \\ P_n \\ \hline \therefore q_n \end{array}$$

$(P_1 \wedge P_2 \ldots\ldots P_n) \rightarrow Q$ is TAUTOLOGY

- P_1, P_2 …..are called the Hypothesis or premises and the proposition Q is called Conclusion.
- The argument is valid provided that P_1, P_2 ……and P_n all are TRUE ,then Q also must be TRUE.
- This process of Drawing a conclusion from a sequence of proposition is called Deductive Reasoning.

RULES OF INFERENCE:

- If an argument consists of 10 different proposition variable then $2^{10}=1024$ combination are needed for Truth Table which is a tedious approach.
- Instead we can first establish the validity of some relatively simple arguments forms, called Rules of Inference.
- These rules then can be used to construct more complicated valid arguments.

1. MODUS PONENS:

- It states that if P and P→Q is TRUE then, we can infer Q is true.
- That is, (P^(P→Q))→Q is TAUTOLOGY

$$\begin{array}{l} p \rightarrow q \\ p \\ \hline \therefore q \end{array}$$

Proof By Truth Table:

p	q	p→q	(p→q) ^p	((p→q) ^p)→q
T	T	T	T	T
T	F	F	F	T
F	T	T	F	T
F	F	T	F	T

2. MODUS TOLLENS:

- It states that if $P \rightarrow Q$ and $\neg Q$ is TRUE then, we can infer $\neg P$ is true.
- That is, $(\neg q \wedge (P \rightarrow Q)) \rightarrow \neg P$ is TAUTOLOGY

$$\begin{array}{l} p \rightarrow q \\ \neg q \\ \hline \therefore \neg p \end{array}$$

Proof By Truth Table:

p	q	p→q	¬p	¬q	(p→q) ^¬q	(p→q) ^¬q)→¬p
T	T	T	F	F	F	T
T	F	F	F	T	F	T
F	T	T	T	F	F	T
F	F	T	T	T	T	T

3. HYPOTHETICAL SYLLOGISM:

- It states that if P→Q and Q→R is TRUE then, we can infer P→R is true.
- That is, ((P→Q)^(Q→R))→(P→R) is TAUTOLOGY

$$\begin{array}{l} p \rightarrow q \\ q \rightarrow r \\ \hline \therefore p \rightarrow r \end{array}$$

P	q	r	p→q	q→r	p→r	(p→q)^(q→r)	((p→q)^(q→r))→(p→r)
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	T
T	F	T	F	T	T	F	T
T	F	F	F	T	F	F	T
F	T	T	T	T	T	T	T
F	T	F	T	F	T	F	T
F	F	T	T	T	T	T	T
F	F	F	T	T	T	T	T

4. DISJUNCTIVE SYLLOGISM:

- It states that if PvQ and ¬P is TRUE then, we can infer Q is true.
- That is, (¬P^(PvQ))→Q is TAUTOLOGY

$$\begin{array}{l} p \lor q \\ \neg p \\ \hline \therefore q \end{array}$$

Proof By Truth Table:

p	q	pVq	¬p	(pVq)^¬p	((pVq)^¬p)→q
T	T	T	F	F	T
T	F	T	F	F	T
F	T	T	T	T	T
F	F	F	T	F	T

5. ADDITION:

- It states that if P is TRUE then, PvQ will be TRUE.
- That is, P→(PvQ) is TAUTOLOGY

$$\frac{p}{\therefore\ p \vee q}$$

Proof By Truth Table:

p	q	pVq	P→(pVq)
T	T	T	T
T	F	T	T
F	T	T	T
F	F	F	T

6. SIMPLIFICATION:

- It states that if P^Q is TRUE then, P will be TRUE.
- That is, (P^Q)→P is TAUTOLOGY

$$\frac{p \wedge q}{\therefore p}$$

Proof By Truth Table:

p	q	p^q	(p^q)→p
T	T	T	T
T	F	F	T
F	T	F	T
F	F	F	T

7. CONJUNCTION:

- It states that if P is TRUE and Q is TRUE then, P^Q will be TRUE.
- That is, (P)^(Q)→(P^Q) is TAUTOLOGY

p
q

∴ p^q

Proof By Truth Table:

p	q	p^q	(p^q)→(p^q)
T	T	T	T
T	F	F	T
F	T	F	T
F	F	F	T

8. RESOLUTION:

- It states that if (PvQ) and (¬PvR) is TRUE then, we can infer (QvR) is true.
- That is,((PvQ)^(¬PvR))→(QvR) is TAUTOLOGY

$$\begin{array}{l} p \vee q \\ \neg p \vee r \\ \hline \therefore\ q \vee r \end{array}$$

p	q	r	¬p	pVq	¬pVr	qVr	((pVq)^(¬pVr))→(qVr)
T	T	T	F	T	T	T	T
T	T	F	F	T	F	F	T
T	F	T	F	T	T	T	T
T	F	F	F	T	F	F	T
F	T	T	T	T	T	T	T
F	T	F	T	T	T	T	T
F	F	T	T	F	T	T	T
F	F	F	T	F	T	T	T

RULES OF INFERENCE:

TABLE 1 Rules of Inference.

Rule of Inference	*Tautology*	*Name*
p $p \rightarrow q$ $\therefore q$	$(p \wedge (p \rightarrow q)) \rightarrow q$	Modus ponens
$\neg q$ $p \rightarrow q$ $\therefore \neg p$	$(\neg q \wedge (p \rightarrow q)) \rightarrow \neg p$	Modus tollens
$p \rightarrow q$ $q \rightarrow r$ $\therefore p \rightarrow r$	$((p \rightarrow q) \wedge (q \rightarrow r)) \rightarrow (p \rightarrow r)$	Hypothetical syllogism
$p \vee q$ $\neg p$ $\therefore q$	$((p \vee q) \wedge \neg p) \rightarrow q$	Disjunctive syllogism
p $\therefore p \vee q$	$p \rightarrow (p \vee q)$	Addition
$p \wedge q$ $\therefore p$	$(p \wedge q) \rightarrow p$	Simplification
p q $\therefore p \wedge q$	$((p) \wedge (q)) \rightarrow (p \wedge q)$	Conjunction
$p \vee q$ $\neg p \vee r$ $\therefore q \vee r$	$((p \vee q) \wedge (\neg p \vee r)) \rightarrow (q \vee r)$	Resolution

Q.1) State which rule of inference is the basis of the following argument:

"It is below freezing now. Therefore, it is either below freezing or raining now."

<u>Solution:</u>

Let, p : "*It is below freezing now*"

q : "*It is raining now*."

Then this argument is of the form:

$$\frac{p}{\therefore p \vee q}$$

This is an argument that uses the **addition rule.**

Q.2) State which rule of inference is the basis of the following argument:

"It is below freezing and raining now. Therefore, it is below freezing now."

<u>Solution:</u>

let, p: "*It is below freezing now*"

q : "*It is raining now*"

This argument is of the form:

$$\frac{p \wedge q}{\therefore p}$$

This argument uses the **simplification rule.**

Q.3) State which rule of inference is the basis of the following argument:

"If you have a current password, then you can log onto the network. You have a current password. Therefore, You can log onto the network."

<u>Solution:</u>

Let, p : "*you have a current password*"

q : "*you can log onto the network*"

Then this argument is of the form:

$$\begin{array}{l} p \rightarrow q \\ \underline{p \qquad\quad} \\ \therefore q \end{array}$$

This is an argument that uses the **Modus Ponens rule.**

Q.4) State which rule of inference is the basis of the following argument:

"If it rains today, then we will not have a barbecue today. If we do not have a barbecue today, then we will have a barbecue tomorrow. Therefore, if it rains today, then we will have a barbecue tomorrow"

<u>Solution:</u>

let, p: "*it rains today*"

q : "*We will have a barbecue today*"

r : "*we will have a barbecue tomorrow*"

This argument is of the form:

$$\begin{array}{l} p \rightarrow \neg q \\ \underline{\neg q \rightarrow r \qquad} \\ \therefore p \rightarrow r \end{array}$$

This argument uses the **Hypothetical rule.**

Q.5) Show that the premises: *"If I play football then I am tired the next day", "I will take rest if I am tired", "I did not take rest"* will lead to the conclusion *"I did not play football"*.

Solution:

Let, p: *"If I play football"*

q: *"I am tired"*

r: *"I will take rest"*

Hypothesis: i) $p \rightarrow q$

ii) $q \rightarrow r$

iii) $\neg r$

Conclusion: $\therefore \neg p$

s.n.	STEPS	REASONS
1.	$p \rightarrow q$	Given Hypothesis
2.	$q \rightarrow r$	Given Hypothesis
3.	$p \rightarrow r$	HYPOTHTICAL SYLLOGISM IN 1 & 2
4.	$\neg r$	Given Hypothesis
5.	$\neg p$	MODUS TOLLENSON 3 & 4

Q.6) Show that the premises "*It is not sunny this afternoon and it is colder than yesterday*". "*We will go swimming only if it is sunny.*" "*If we do not go swimming, then we will take a canoe trip.*" and "*If we take a canoe trip, then we will be home by sunset*" lead to the conclusion "*We will be home by sunset.*"

Solution:

Let, p: "*It is sunny this afternoon*"

q: "*It is colder than yesterday*"

r: "*We will go swimming*"

s: "*We will take canoe trip*"

t: "*We will be home by sunset*"

Hypothesis: i) $\neg p \wedge q$

ii) $r \rightarrow p$

iii) $\neg r \rightarrow s$

iv) $s \rightarrow t$

Conclusion: $\therefore t$

s.n.	STEPS	REASONS
1.	$\neg p \wedge q$	Given Hypothesis
2.	$\neg p$	SIMPLIFICATION ON 1
3.	$r \rightarrow p$	Given Hypothesis
4.	$\neg r$	MODUS TOLLENS ON 2 & 3
5.	$\neg r \rightarrow s$	Given Hypothesis
6.	s	MODUS PONENS ON 4 & 5
7.	$s \rightarrow t$	Given Hypothesis
8.	t	MODUS PONENS ON 6 & 7

Q.7) Show that the premises *"If you send me an e-mail message, then I will finish writing the program", "If you do not send me an e-mail message, then I will go to sleep early," "If I go to sleep early, then I will wake up feeling refreshed"* lead to the conclusion *"If I do not finish writing the program, then I will wake up feeling refreshed."*

Solution:

Let, p: "*you send me an e-mail message*"

q: "*I will finish writing the program*"

r: "*I will go to sleep early*"

s: "*I will wake up feeling refreshed*"

Hypothesis: i) $p \to q$

ii) $\neg p \to r$

iii) $r \to s$

Conclusion: $\therefore \neg q \to s$

s.n.	STEPS	REASONS
1.	$p \to q$	Given Hypothesis
2.	$\neg q \to \neg p$	CONTRAPOSITIVE ON 1
3.	$\neg p \to r$	Given Hypothesis
4.	$\neg q \to r$	Hypothetical syllogism using (2) and (3)
5.	$r \to s$	Given Hypothesis
6.	$\neg q \to s$	Hypothetical syllogism using (4) and (5)

Q.8) Show that the premises *"If it does not rain or if it is not foggy, then the sailing race will be held and the lifesaving demonstration will go on," "If the sailing race is held, then the trophy will be awarded,"* and *"The trophy was not awarded"* imply the conclusion *"It rained."*

Solution:

Let, p: *"It rains"*

q: *"It is foggy"*

r: *"The sailing race is held"*

s: *"Life saving demonstration is done"*

t: *"Trophy is awarded"*

Hypothesis: i) $(\neg p \vee \neg q) \rightarrow (r \wedge s)$

ii) $r \rightarrow t$

iii) $\neg t$

Conclusion: $\therefore p$

s.n.	STEPS	REASONS
1.	$r \rightarrow t$	Given Hypothesis
2.	$\neg t$	Given Hypothesis
3.	$\neg r$	MODUS TOLLENS ON 1 & 2
4.	$(\neg p \vee \neg q) \rightarrow (r \wedge s)$	Given Hypothesis
5.	$\neg r \vee \neg s$	Addition on 3
6.	$\neg(r \wedge s)$	DE-MORGAN'S LAW on 5
7	$\neg(\neg p \vee \neg q)$	MODUS TOLLENS ON 4 and 6
8	$p \wedge q$	DE-MORGAN'S LAW on 7
9	p	SIMPLIFICATION on 8

Q.9) Show that the premises *"If the interest rate drops , the housing market will improve"*, *"The federal discount rate will drop or the housing market will not improve"*, *"Interest rate will drop"* imply the conclusion *"The federal discount rate will drop"*

Solution:

Let, p: *"the interest rate drops "*
q: *"the housing market will improve"*
r: *"The federal discount rate will drop"*

Hypothesis: i) $p \rightarrow q$
ii) $r \vee \neg q$
iii) p

Conclusion: $\therefore r$

STEPS	REASONS
1. $p \rightarrow q$	Given Hypothesis
2. $\neg p \vee q$	Implication on 1
3. $r \vee \neg q$	Given Hypothesis
4. $\neg p \vee r$	Resolution From 2 and 3
5. p	Given hypothesis
6. r	From 4 and 5

Q.10) Show that the premises *"If my cheque book is in office, then I have paid my phone bill", "I was looking for phone bill at breakfast or I was looking for phone bill in my office", "If was looking for phone bill at breakfast then my cheque book is on breakfast table" , "If I was looking for phone bill in my office then my cheque book is in my office", "I have not paid my phone bill"* imply the conclusion *"My cheque book is on my breakfast table"*

Solution:

Let, p: *"my cheque book is in office"*
q: *"I have paid my phone bill"*
r: *"I was looking for phone bill at breakfast"*
s: *"I was looking for phone bill in my office"*
t: *"my cheque book is on breakfast table"*

Hypothesis: i) $p \rightarrow q$
ii) $r \vee s$
iii) $r \rightarrow t$
iv) $s \rightarrow p$
v) $\neg q$

Conclusion: $\therefore t$

Hypothesis: i) $p \rightarrow q$
ii) $r \vee s$
iii) $r \rightarrow t$
iv) $s \rightarrow p$
v) $\neg q$
Conclusion: $\therefore t$

STEPS	REASONS
1. $p \rightarrow q$	Given Hypothesis
2. $\neg q$	Given Hypothesis
3. $\neg p$	Modus Tollens on 1 and 2
4. $r \vee s$	Given Hypothesis
5. $r \rightarrow t$	Given Hypothesis
6. $\neg r \vee t$	Implication on 5
7. $s \vee t$	Resolution from 4 and 6
8. $s \rightarrow p$	Given Hypothesis
9. $\neg s \vee p$	Implication on 8
10. $t \vee p$	Resolution from 7 and 9
11. t	From 3 and 10

PROOF BY RESOLUTION:

- Propositional Resolution works only on expressions in *clausal form*. Before the rule can be applied, the premises and conclusions must be converted to this form.

A *literal* is either an atomic sentence or a negation of an atomic sentence. For example, if p is a logical constant, the following sentences are both literals.

$$p$$
$$\neg p$$

A *clausal sentence* is either a literal or a disjunction of literals. If p and q are logical constants, then the following are clausal sentences.

$$p$$
$$\neg p$$
$$\neg p \vee q$$

A *clause* is the set of literals in a clausal sentence. For example, the following sets are the clauses corresponding to the clausal sentences above.

$$\{p\}$$
$$\{\neg p\}$$
$$\{\neg p, q\}$$

CONVERTING TO CAUSAL FORM:

- Examples:
  1. p→q = ¬p V q
  2. p←→q = (¬p V q) ^ (¬q V p)--------{CNF}
  3. ¬(p ^ q) = ¬p V ¬q
  4. ¬(p V q) = ¬p ^ ¬q

CNF: CNF (Conjunctive normal form) if it is a ∧(Conjunction) of ∨(Disjunction s) of literals (variables or their negation.)

DNF: DNF (Disjunctive normal form) if it is a ∨(Disjunction s) of ∧(Conjunction) of literals (variables or their negation.)

Example:
1. ¬p V ¬q
2. (p^q) V (p^r)

Prove:

i) $P \rightarrow Q$

ii) $\neg P \rightarrow R$

iii) $R \rightarrow S$

$\therefore\ \neg Q \rightarrow S$

Solution:

The causal form of hypothesis and Conclusion are:

i) $\neg P \vee Q$

ii) $P \vee R$

iii) $\neg R \vee S$

$\therefore\ Q \vee S$

STEPS	REASONS
1. $\neg P \vee Q$	1. Given Hypothesis
2. $P \vee R$	2. Given Hypothesis
3. $Q \vee R = R \vee Q$	3. Using Resolution
4. $\neg R \vee S$	4. Given Hypothesis
5. $Q \vee S$	4. Using Resolution

Using Graphical Method:

Prove:

i) $P \rightarrow Q$

ii) $\neg P \rightarrow R$

iii) $R \rightarrow S$

$\therefore$ $\neg Q \rightarrow S$

STEP 1. The causal form of hypothesis and Conclusion are:

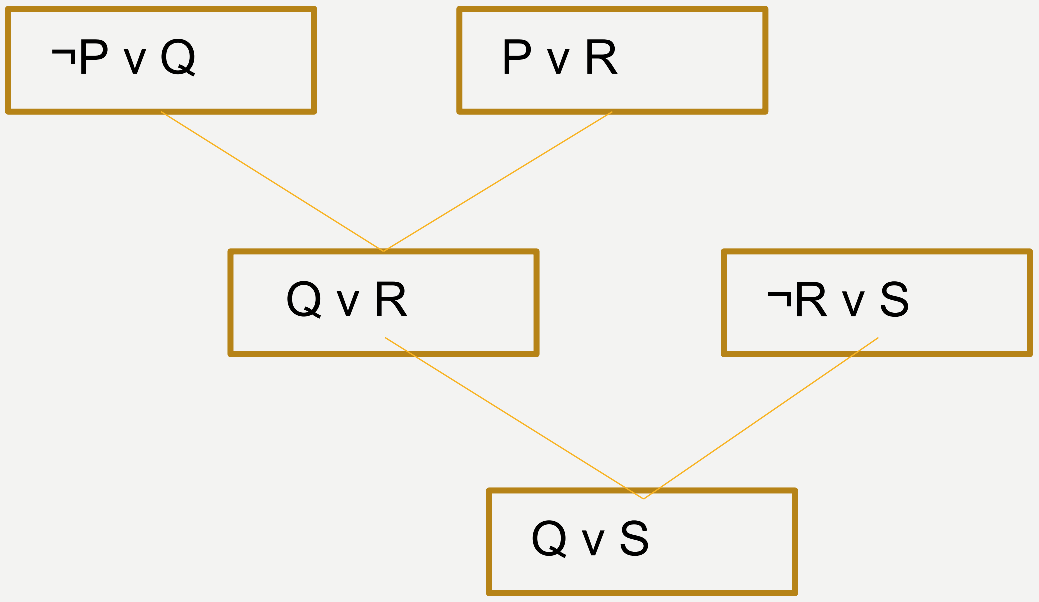
i) $\neg P \lor Q$

ii) $P \lor R$

iii) $\neg R \lor S$

$\therefore$ $Q \lor S$

STEP 2.



Using Graphical Method:

Prove:

i) P

ii) $P \rightarrow R$

iii) $R \rightarrow S$

$\therefore$ S

STEP 1. The causal form of hypothesis and Conclusion are:

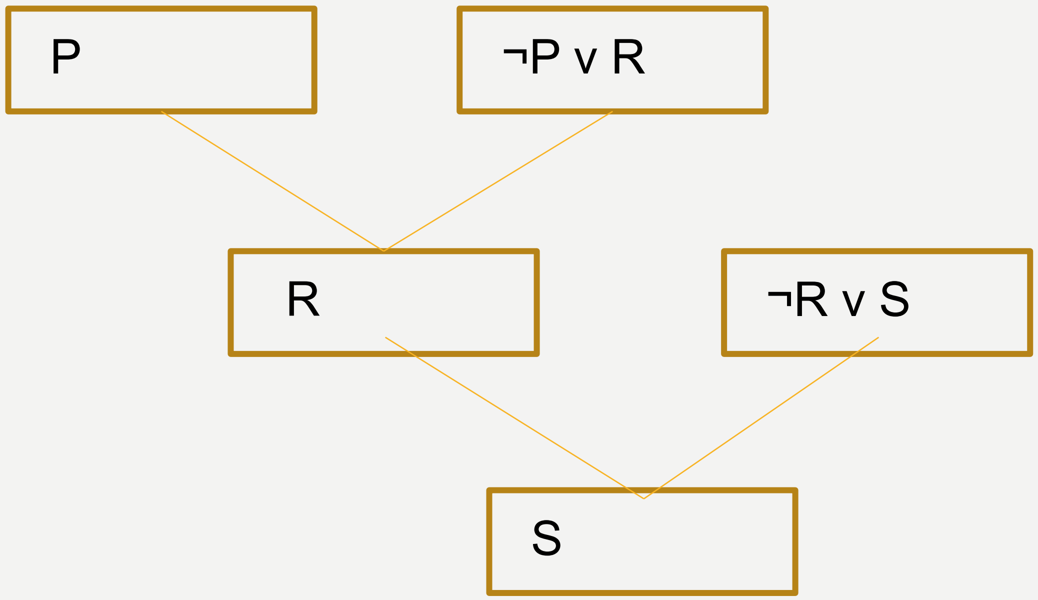
i) P

ii) $\neg P \vee R$

iii) $\neg R \vee S$

$\therefore$ S

STEP 2.



- Using resolution principle, prove that the hypotheses: "If today is Tuesday then I will have a test in Discreet Math or Microprocessor”. “If my Microprocessor teacher is sick then I will not have a test in Microprocessor." “Today is Tuesday and my Microprocessor teacher is sick." lead to the conclusion that “I will have a test in Discrete Math“

Solution:

Let, p : “Today is Tuesday”

q : “I will have test in Discrete Math”

r : “I will have test in Microprocessor”

s : “My Microprocessor teacher is sick”

Hypothesis: i) $p \rightarrow (q \vee r)$

ii) $s \rightarrow \neg r$

iii) $p \wedge s$

Conclusion: q

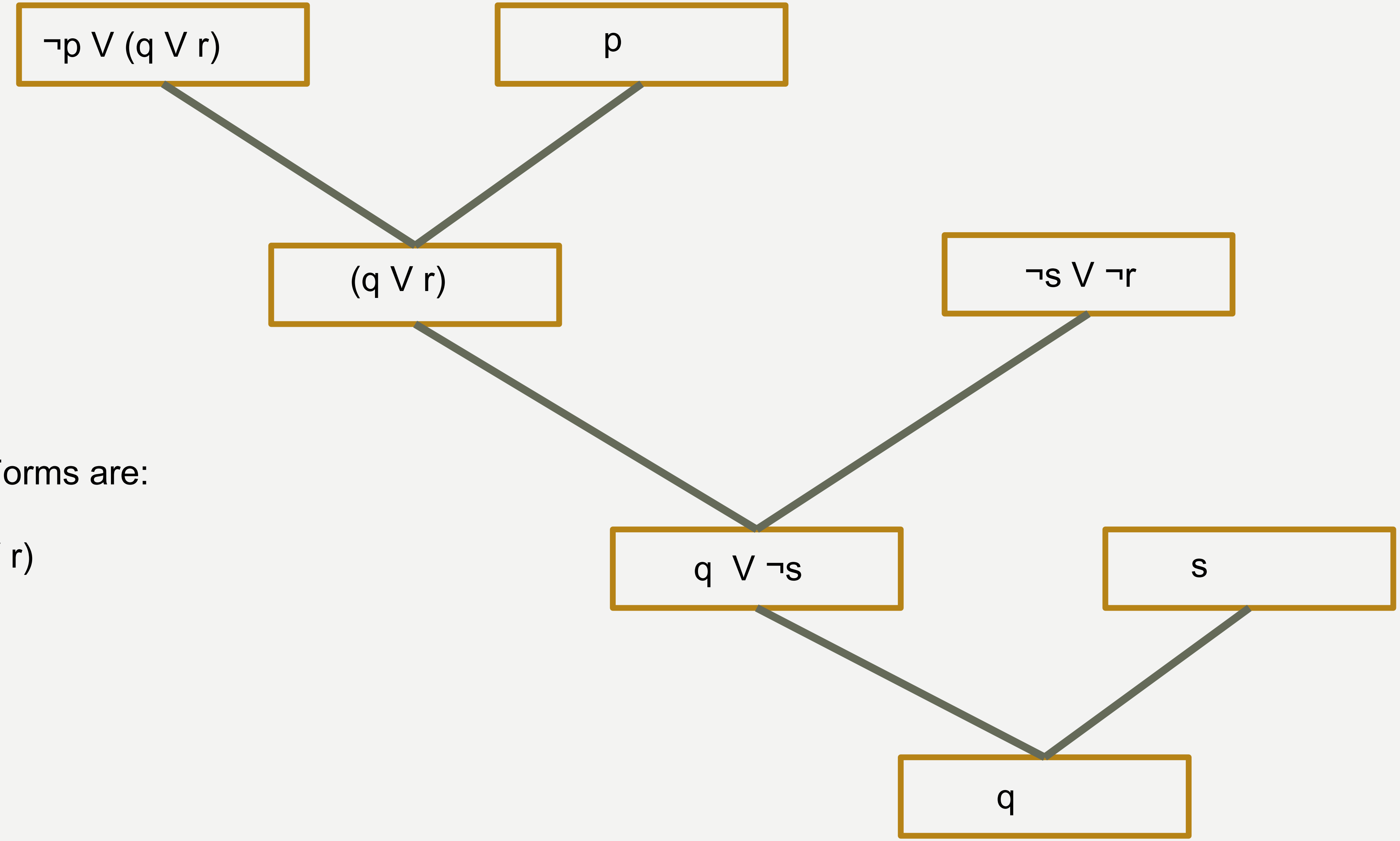
The Causal Forms are:
Hypothesis:
i) $\neg p \vee (q \vee r)$
ii) $\neg s \vee \neg r$
iii) p
iv) s

Conclusion:
$\therefore$ q

The Causal Forms are:
Hypothesis:
i) $\neg p \lor (q \lor r)$
ii) $\neg s \lor \neg r$
iii) p
iv) s

Conclusion:
$\therefore q$

STEPS	REASONS
1. $\neg p \lor (q \lor r)$	Given Hypothesis
2. p	Given Hypothesis
3. $q \lor r$	From 1 and 2
4. $\neg s \lor \neg r$	Given Hypothesis
5. $q \lor \neg s$	From 3 and 4
6. s	Given Hypothesis
7. q	From 5 and 6



The Causal Forms are:
Hypothesis:
i) ¬p V (q V r)
ii) ¬s V ¬r
iii) p
iv) s

Conclusion:
∴ q

SEMANTIC TABLEAUX:

- The formula is decomposed into its sub-formulas according to certain rules(α and β Rules), resulting a semantic tableau.

- Semantic Tableau is a binary tree constructed using semantic rules.

α Rule

a) P v Q
- P
- Q

b)P ^ Q
- P
  Q

c)P→Q
- ¬ P
- Q

d)P←→Q
- P
  Q
- ¬P
  ¬Q

β Rule

a) ¬(¬P)
- P

b) ¬(P v Q)
- ¬P
  ¬Q

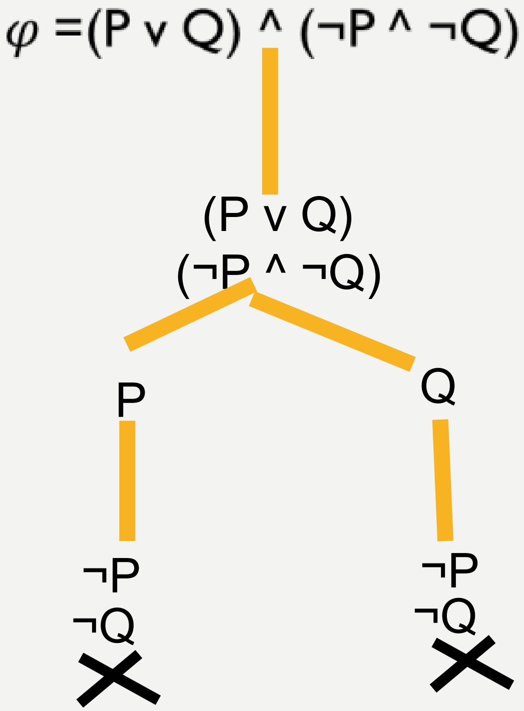
c) ¬(P ^ Q)
- ¬ P
- ¬ Q

d) ¬(P→Q)
- P
  ¬Q

e) ¬(P←→Q)
- P
  ¬ Q
- ¬P
  Q

SEMANTIC TABLEAUX:

- A finite set of formulas φ is satisfiable iff $T(\varphi)$ is open.
- As a corollary, φ is contradictory (not satisfiable) iff $T(\varphi)$ is closed

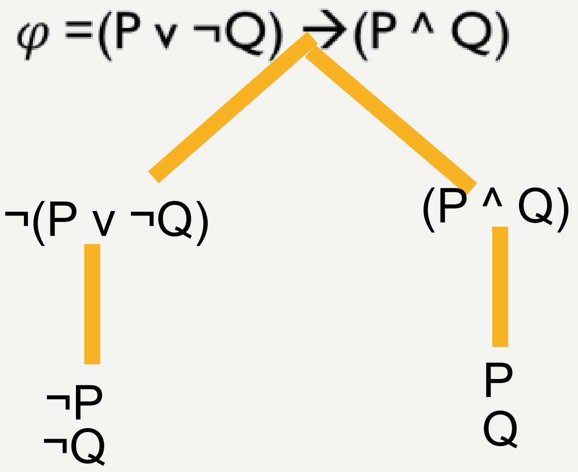


A=(P v Q)
B=(¬P ^ ¬Q)

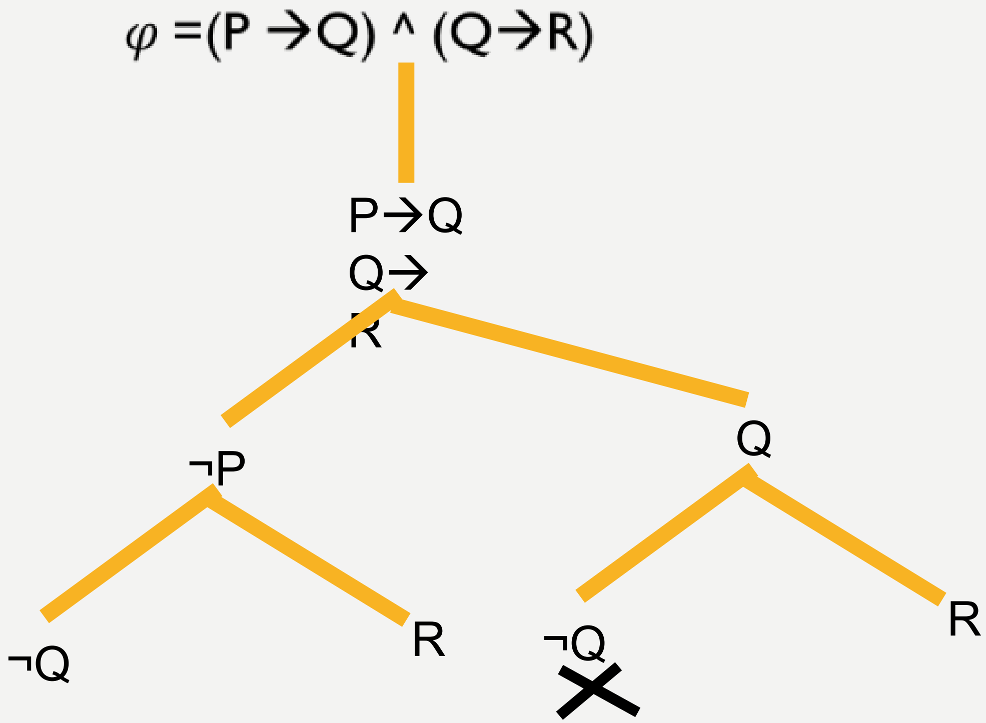
P	Q	¬P	¬Q	A	B	A^B
T	T	F	F	T	F	**F**
T	F	F	T	T	F	**F**
F	T	T	F	T	F	**F**
F	F	T	T	F	T	**F**

- When P and its negation ¬P appear on the same branch, a contradiction has been found and that branch is called closed.

SEMANTIC TABLEAUX:



SATISFIABLE
(There are open branches)

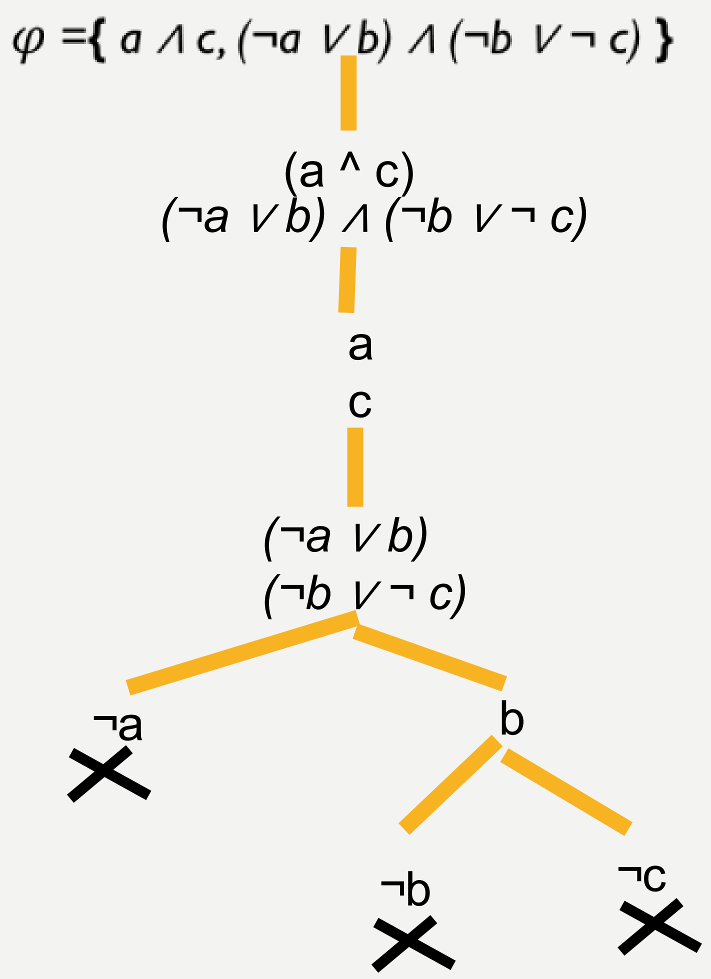


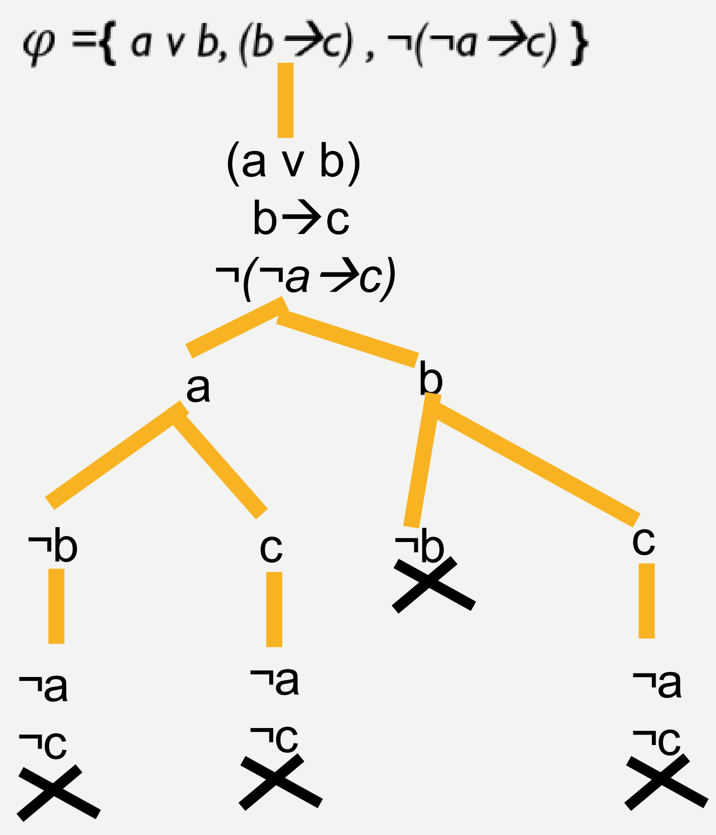
SATISFIABLE
(There are open branches)

P	Q	R	(P→Q)	(Q→R)	(P→Q)^(Q→R)
T	T	T	T	T	T
T	T	F	T	F	F
T	F	T	F	T	F
T	F	F	F	T	F
F	T	T	T	T	T
F	T	F	T	F	F
F	F	T	T	T	T
F	F	F	T	T	T

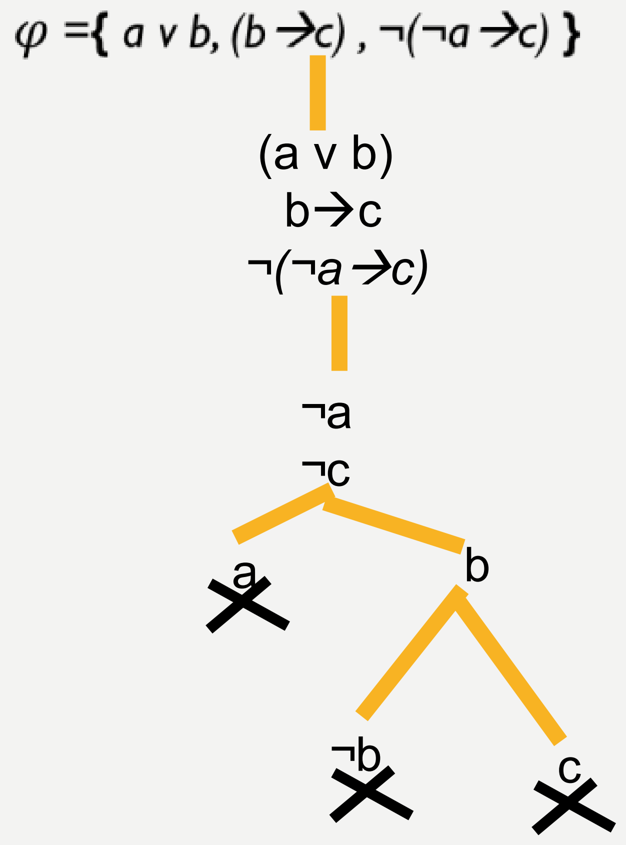
P	Q	¬Q	A=(PV¬Q)	B=(P^Q)	A→B
T	T	F	T	T	T
T	F	T	T	F	F
F	T	F	F	F	T
F	F	T	T	F	F

SEMANTIC TABLEAUX:





[Disjunction First Policy]



[Conjunction First Policy]

SEMANTIC TABLEAUX:

To prove the argument:

$$\begin{array}{l} P_1 \\ P_2 \\ P_3 \\ . \\ . \\ P_n \\ \hline \therefore Q \end{array}$$

$[(P_1 \wedge P_2 , \ldots\ldots\ \wedge P_n) \rightarrow Q]$ is Tautology.

- To prove the above argument we show that the set of premises along with negated conclusion is Unstisfiable i.e. we show semantic Tableau is Contradiction.

$$= \ [(P_1 \wedge P_2 , \ldots\ldots\ \wedge P_n) \rightarrow Q]$$
$$= \neg[(P_1 \wedge P_2 , \ldots\ldots\ \wedge P_n) \rightarrow Q]$$
$$= \ [P_1 \wedge P_2 , \ldots\ldots\ \wedge \ P_n \wedge \neg Q]$$

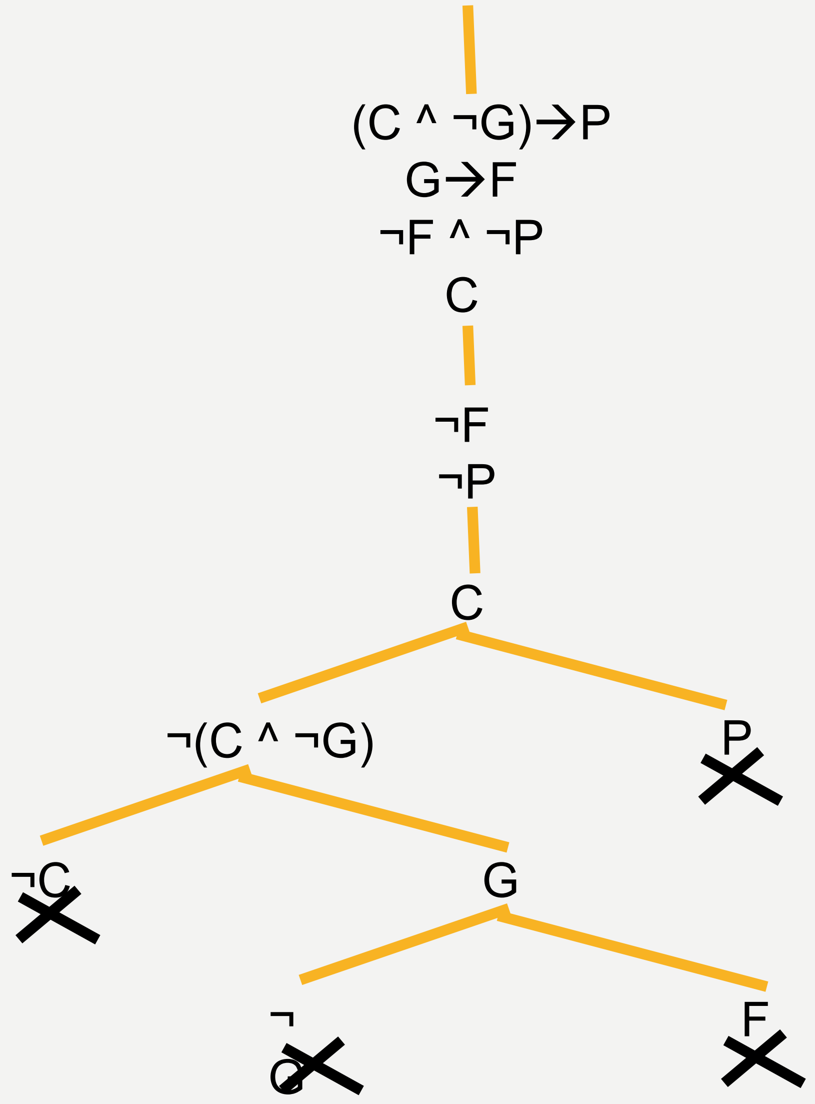
- $\varphi=\{P_1 , P_2 , P_3 , \ldots\ldots P_n , \neg Q \}$
- *If* $T(\varphi)$ is closed then the argument is valid else if $T(\varphi)$ is open the argument is invalid

Check if the following argument is valid or not.

(C ^ ¬G)→P
G→F
¬F ^ ¬P
∴ ¬C

φ= {(C ^ ¬G)→P , G→F , ¬F ^ ¬P , ¬(¬C) }

φ= {(C ^ ¬G)→P , G→F , ¬F ^ ¬P , ¬(¬C) }



The obtained Tableau is contradictory.Hence the argument is valid

i) $p \rightarrow q$
ii) $q \rightarrow r$
iii) $\neg r$
$\therefore \neg p$

Hypothesis: i) $(\neg p \vee \neg q) \rightarrow (r \wedge s)$
ii) $r \rightarrow t$
iii) $\neg t$
Conclusion: $\therefore p$

Hypothesis: i) $\neg p \wedge q$
ii) $r \rightarrow p$
iii) $\neg r \rightarrow s$
iv) $s \rightarrow t$
Conclusion: $\therefore t$

Hypothesis: i) $p \rightarrow q$
ii) $\neg p \rightarrow r$
iii) $r \rightarrow s$
Conclusion: $\therefore \neg q \rightarrow s$

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PREDICATE LOGIC

1. QUANTFIERS
2. TYPES OF QUANTIFIERS
3. NEGATING QUANTIFIERS
4. TRANSLATION FROM ENGLISH

LIMITATION OF PROPOSITIONAL LOGIC:

Consider the following :

p: "All men are mortal"
q: "Ram is man"
r: ∴ "Ram is mortal"

- No rule of propositional logic will allow us to conclude the truth of 'r'.
- Therefore, We need more powerful type of logic called First order Logic or PREDICATE LOGIC.

- To understand predicate Logic we need to understand:
  a) Subject
  b) Predicates
  c) Quantifiers
  d) Domain(Universe of Discourse0

Consider an statement:

“Jose is Tall”

Subject

Tall(Jose)

Predicate

- Subject: “The **subject** is what (or whom) the sentence is about”

- Predicate: “**Predicate** refers to a property that the subject of a statement can have”

Consider an statement:

“X is greater than 3” $(x>3)$

Subject: Variable “x”

Predicate: Greater than 3

We can denote “x>3” as: $P(x)$

The statement $P(x)$ becomes a proposition once the value has been assigned to the subject.

<u>Example:</u>

$P(5)$: “5 is greater than 3”(TRUE)

$P(2)$: “2 is greater than 3”(FALSE)

Q.1) Let Q(x) denotes the statement : “The word “x” contains the letter ‘a’ ”
What are the Truth value of Q(ankit), Q(Logic), Q(nothing) ?

Solution

Q(x): “ The word “x” contains the letter ‘a’ ”

Q(ankit): “”The word “ankit” contains the letter ‘a’ “ (TRUE)
Q(Logic): “”The word “Logic” contains the letter ‘a’ “ (FALSE)
Q(nothing): “”The word “nothing” contains the letter ‘a’ “ (FALSE)

Q.2) Let C(x, y) denotes the statement: “x is the capital of y”
What are the truth value of C(Kathmandu, Nepal), C(Texas, America) ?

Solution

C(x, y): “x is the capital of y”

C(Kathmandu, Nepal) : “Kathmandu is capital of Nepal” (TRUE)
C(Texas, America) : “Texas is capital of America” (FALSE)

Consider The Following:

P(x): “x is greater than 10”

Domain: All positive natural numbers.

Can we say the above statement is true for all values of x?

=No, because for x=1,2,3,4,5,6,7,8,9,10 above statement becomes FALSE.

So, We can say above statement as : **For some x, P(x) is TRUE.**

Consider The following:

Q(x): “x<x+1”

Domain : All positive natural numbers.

Can we say that Q(x) is TRUE For all values of x within our domain?

=Yes

So, we can say above statement as: **For all x, Q(x) is TRUE.**

QUANTIFIERS

In **predicate logic**, **predicates** are used alongside **quantifiers** to express the extent to which a **predicate** is true over a range of elements. Using **quantifiers** to create such propositions is called quantification.

1. UNIVERSAL QUANTIFICATION(∀):

"Every cat drinks milk"

CATS

X_1
X_2
X_3

Domain(Universe of Discourse)

Above statement is Equivalent to:

X_1 Drinks milk.
^
X_2 Drinks milk.
^
X_3 Drinks milk.

Milk(X_1)
^
Milk(X_2)
^
Milk(X_3)

$$\forall_X \text{Milk}(X)$$

=For all X, Milk(X)

or

=For ever X, Milk(X)

1. UNIVERSAL QUANTIFICATION(∀):

The Universal Quantification of P(x) is:

"P(x) for all value of x in the Domain"

$$= \forall_x P(x)$$

We can also read $\forall_x P(x)$ as:

"For all P(x) " or "for every x, P(x)"

<u>Example:</u>

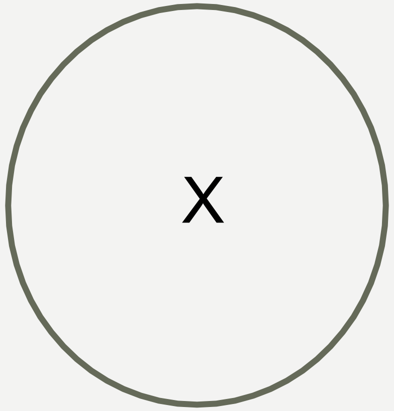
Q.1*) "All student of BEIT takes course on Discrete Mathematics"*

let,

D(x): "x takes course on Discrete Mathematics"

$$=\forall_x D(x)$$

<u>BEIT</u>



<u>Domain(Universe of Discourse)</u>

1. UNIVERSAL QUANTIFICATION(∀):

Example:

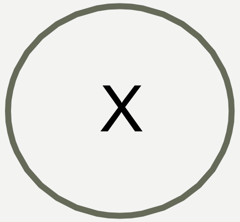
Q.2) ***"Every men are Mortal"***

let,

M(x): "x is mortal"

$=\forall_x M(x)$

MEN



Domain(Universe of Discourse)

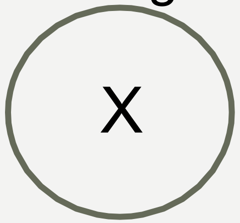
Q.3) "x+1>x"

let,

P(x): "x+1>x"

$=\forall_x P(x)$

+ve integers



Domain(Universe of Discourse)

1. UNIVERSAL QUANTIFICATION($\forall$):

Q.4) Let Q(x) be the statement “$x<5$”. What is the truth value of Quantification, $\forall_x Q(X)$, where domain of discourse is all real numbers.

Solution

Q(x) is not True for every real number.

for instance,

Q(6)=“$6<5$” is FALSE.

Thus , $\forall_x Q(x)$ is FALSE.

COUNTER EXAMPLE: An Element for which P(x) is False is called Counter Example of $\forall_x P(x)$.

Q.5) What is the truth value of $\forall_x P(x)$, where P(x) is the statement “$X^2 < 10$” and the domain consists of the positive integers not exceeding 4.

Solution: The statement $\forall_x P(x)$ is the same as the conjunction $P(1) \wedge P(2) \wedge P(3) \wedge P(4)$, because the domain consists of the integers 1, 2, 3, and 4.

Because P (4), which is the statement “42 < 10,” is false, it follows that $\forall_x P(x)$ is false.

2. EXISTENTIAN QUANTIFICATION(∃):

"some lion drinks milk"

LION

X_1
X_2
X_3

Domain(Universe of Discourse)

Above statement is Equivalent to:

X_1 Drinks milk.
V
X_2 Drinks milk.
V
X_3 Drinks milk.

Milk(X_1)
V
Milk(X_2)
V
Milk(X_3)

$$\exists_x \text{Milk}(X)$$

=There exist an x in the domain such that Milk(X)
=There is at least one x such that Milk(x)
=for some x, Milk(x)

2. EXISTENTIAN QUANTIFICATION(∃):

Example:

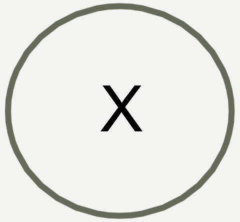
Q.1) *"Some people are Bad"*

let,

B(x): "x is Bad"

$=\exists_x B(x)$

People



Domain(Universe of Discourse)

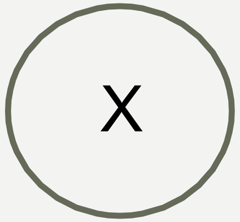
Q.2) *"Some dogs are big"*

let,

D(x): "x is Big"

$=\exists_x D(x)$

DOGS



Domain(Universe of Discourse)

2. EXISTENTIAN QUANTIFICATION(∃):

Q.1) Let P (x) denote the statement “x > 3.” What is the truth value of the quantification $\exists_x P(x)$, where the domain consists of all real numbers.

Solution:

Because “x > 3” is sometimes true—for instance, when x = 4—the existential quantification of P (x), which is $\exists_x P(x)$, is true.

Q.2) What is the truth value of $\exists_x P(x)$, where P(x) is the statement “$x^2 > 10$” and the universe of discourse consists of the positive integers not exceeding 4?

Solution:

Because the domain is {1, 2, 3, 4}, the proposition $\exists_x P(x)$ is the same as the disjunction P (1) ∨ P (2) ∨ P (3) ∨ P (4). Because P (4), which is the statement “42 > 10,” is true, it follows that $\exists_x P(x)$ is true.

Statement	When True?	When False?
$\forall_x P(x)$	P(x) is true for every x.	There is an x for which P(x) is false
$\exists_x P(x)$	There is an x for which P(x) is true	P(x) is false for every x.

FREE & BOUND VARIABLES:

- When the variable is assigned a value or it is quantified it is called bound variable. If the variable is not bounded then it is called free variable.

- An occurrence of a variable that is not bound by a quantifier or set equal to a particular value is said to be free.

- Example:
  1. $P(x, y)$ has two free variables x and y.
  2. $P(2, y)$ has one bound variable 2 and one free variable y.
  3. $\forall x\ P(x)$ has a bound variable x.
  4. $\forall x\ P(x, y)$ has one bound variable x and one free variable y.

- Expression with no free variable is a proposition.

- Expression with at least one free variable is a predicate only.

NEGATING QUANTIFICATIONS:

1. Negating Universal Quantification:

   $\neg[\forall_x P(x)]$

   a) Negate the Proposition Function$[\neg P(x)]$

   b) Change to Existential Quantification

$$\boxed{\neg[\forall_x P(x)] = \exists_x \neg[\ P(x)]}$$

2. Negating Existential Quantification:

   $\neg[\exists_x P(x)]$

   a) Negate the Proposition Function$[\neg P(x)]$

   b) Change to Universal Quantification

$$\boxed{\neg[\exists_x P(x)] = \forall_x[\neg P(x)]}$$

De-Morgan's Law For Quantifiers

Negate The Following :

1. ***"Every student in BEIT has Taken Data mining***" [Domain: All BEIT student]

Solution

let, p(x): "x has taken Data Mining"

$=\forall_x P(x)$

Negation:

$= \boldsymbol{\neg[\forall_x P(x)]}$

$\boldsymbol{=\exists_x \neg[\, P(x)]}$

"There is a student in BEIT who has not taken Data Mining"

2. "***There is a student in class who has long hair"*** [Domain: All BEIT student]

Solution

let, p(x): "x has long hair"

$= \boldsymbol{\exists_x} P(x)$

Negation:

$= \boldsymbol{\neg[\exists_x} P(x)\boldsymbol{]}$

$\boldsymbol{=\forall_x \neg[\, P(x)]}$

"All student in the class do not have long hair"

3. What are the negations of the statements:

a) $\forall_x(x^2 > x)$

Solution:

The negation of $\forall_x(x^2 > x)$ is,

$\neg\forall_x(x^2 > x)$, which is equivalent to

$\exists_x\neg(x^2 > x)$.

This can be rewritten as $\exists_x(x^2 \le x)$.

b) $\exists_x(x^2 = 2)$

Solution:

The negation of $\exists_x(x^2 = 2)$is,

$\neg\exists_x(x^2 = 2)$, which is equivalent to

$\forall_x\neg(x^2 = 2)$.

This can be rewritten as $\forall_x(x^2 != 2)$.

TRANSLATING FROM ENGLISH:

1. Express the statement *"Every student in BEIT class has studied calculus"* using predicates and quantifiers.

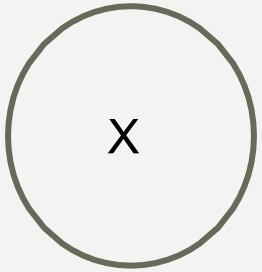
Solution:

First, we introduce a variable x so that our statement becomes "*For every student x in BEIT, x has studied calculus*."

Now, let C(x): "*x has studied calculus*."

Domain: BEIT

$=\forall_x C(x)$



DOMAIN: BEIT students

Domain: All people

Our statement becomes:

"For every person x, if person x is a student in BEIT, then x has studied calculus."

Now, let S(x) : "*x is a student in BEIT*."

C(x) : "*x has studied calculus*"

$=\forall_x[S(x) \rightarrow C(x)]$

all people

TRANSLATING FROM ENGLISH:

Domain: All people

Our statement becomes:

"For every person x, if person x is a student in BEIT, then x has studied calculus."

Now, let S(x) : "*x is a student in BEIT.*"

C(x) : "*x has studied calculus*"

$$=\forall_x[S(x) \rightarrow C(x)]$$

Q(x, Calculus) : "*Student x has studied Calculus*"

$$=\forall_x[S(x) \rightarrow Q(x, \text{Calculus})]$$

[Caution! Our statement cannot be expressed as $\forall_x[S(x) \wedge C(x)]$ because this statement says that all people are students in this class and have studied calculus!]

TRANSLATING FROM ENGLISH:

2. Express the statement *"Some student in this class has visited Jhapa"* using predicates and quantifiers.

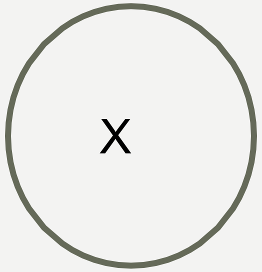
<u>Solution:</u>

First, we introduce a variable x so that our statement becomes "*There is a student x in this class that has visited Jhapa*"

Now, let p(x): "*x has visited Jhapa*"

Domain: BEIT

= $\exists_x p(x)$



DOMAIN: BEIT students

Domain: All people

Our statement becomes:

"There is a person x, if person x is in this class, then x has studied calculus."

Now, let S(x) : "*x is a student in BEIT.*"

p(x) : "*x has visited Jhapa*"

= $\exists_x$**[S(x) ^ p(x)]**

all people

Caution! Our statement cannot be expressed as $\exists_x(S(x) \rightarrow M(x))$, which is true when there is someone not in the class because, in that case, for such a person x, $S(x) \rightarrow M(x)$ becomes either $F \rightarrow T$ or $F \rightarrow F$, both of which are true.

TRANSLATING FROM ENGLISH:

3. Express the statement *"Every Student in this class has visited Jhapa or Kathmandu"* using predicates and quantifiers.

Solution:

Now, let k(x): "*x has visited Kathmandu*"

J(x): "*x has visited Jhapa*"

Domain: BEIT

$$= \forall_x [k(x) \vee j(x)]$$

Domain: All people

Our statement becomes:

"for all person x, if person x is in this class, then x has visited Jhapa or Kathmandu."

Now, let S(x) : "*x is a student in BEIT.*"

k(x): "*x has visited Kathmandu*"

J(x): "*x has visited Jhapa*"

$$= \forall_x [S(x) \rightarrow (k(x) \vee j(x))]$$

TRANSLATING FROM ENGLISH:

Consider these statement:

- *No professor are ignorant*
- *All ignorant people are vain*
- *Some professor are ignorant*

Let , P(x): *x is a Professor* , I(x): *x is ignorant* ,V(x): *x is vain*

Express above statement using quantifiers where domain consist of all people

a) No professor are ignorant

$\forall_x[p(x) \rightarrow \neg q(x)]$

b) All ignorant people are vain

$\forall_x[q(x) \rightarrow r(x)]$

c) Some professor are ignorant

$\exists_x[p(x) \wedge q(x)]$

TRANSLATING FROM ENGLISH:

Q. Let P (x) be the statement “x can speak Russian”

Q(x) be the statement “x knows the computer language C++.”

Express each of these sentences in terms of P (x), Q(x), quantifiers, and logical connectives. The domain for quantifiers consists of all students at your school.

a) There is a student at your school who can speak Russian and who knows C++.

$=\exists_x (P(x) \wedge Q(x))$.

a) There is a student at your school who can speak Russian but who doesn’t know C++.

$= \exists_x (P(x) \wedge \neg Q(x))$

a) Every student at your school either can speak Russian or knows C++.

$= \forall_x (P(x) \vee Q(x))$

a) No student at your school can speak Russian or knows C++.

$= \forall_x$ **$[\neg P(x) \wedge \neg Q(x)]$**

TRANSLATING FROM ENGLISH:

1. No one is sleeping.

   Negation of above: There is some who is sleeping

   $$=\exists_x[\mathbf{Person}(x) \wedge \mathbf{sleeping}(x)]$$

   Now , negate the predicate:

   $$=\neg\exists_x[\mathbf{Person}(x) \wedge \mathbf{sleeping}(x)]$$

2. Not everyone is sleeping.

   Negation of above: Everyone is sleeping.

   $$= \forall_x [\mathbf{Person}(x) \rightarrow \mathbf{Sleeping}(x)]$$

   Now , negate the predicate:

   $$=\neg\forall_x [\mathbf{Person}(x) \rightarrow \mathbf{Sleeping}(x)]$$

3. No one in this class is wearing glass and a cap.

   Negation of above statement: There is some one in this class who is wearing glass and a cap.

   $$=\exists_x[\mathbf{Glass}(x) \wedge \mathbf{Cap}(x)]$$

   Now, negate the predicate:

   $$=\neg\exists_x[\mathbf{Glass}(x) \wedge \mathbf{Cap}(x)]$$

MATHEMATICAL FOUNDATION FOR COMPUTER SCIENCE

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PREDICATE LOGIC

2.NESTED QUANTIFIERS

3.TRANSLATION FROM ENGLISH

1. NESTED QUANTIFIERS:

1. Consider the following :

"The sum of any two positive real number is positive"

This assertion can be restated as:

"for ever x and for every y , If x>0 and y>0, then x+y>0"

Let,

p(x, y) : "$(x>0)\wedge(y>0)\rightarrow(x+y)>0$

The given statement says that the sum of any two positive real number is positive, so we need two Universal quantifiers.

The Quantification is:

$$\boxed{\forall_x \forall_y [p(x , y)]}$$

1. NESTED QUANTIFIERS:

3. Consider the following :

"Some student in your class has taken some computer training course"

Restating above statement as:

"For some student x ,there exist a computer training course y such that x has taken y"

let, Q(x, y) : "Student x has taken training y"

The Quantification is:

$$\boxed{\exists_x \exists_y [Q(x , y)]}$$

1. NESTED QUANTIFIERS:

4. Consider the following :

"If a person is female and is a parent , then this person is someone's mother"

Restating above statement as:

"For every person x, if x is a female and person x is a parent , then there exist a person y such that person x is the mother of y.

let, $F(x)$: "x is female"

$P(x)$: " x is parent"

$M(x, y)$: "x is the mother of y"

The Quantification is:

$$= \forall_x [(F(x) \wedge P(x)) \rightarrow \exists_y M(x, y)]$$

$$= \forall_x \exists_y [(F(x) \wedge P(x)) \rightarrow M(x, y)]$$

1. NESTED QUANTIFIERS:

5. Consider the following :

"There is a man that has taken a flight on every airline in the world"

Restating above statement as:

"There is a man x, for all airlines a ,there exist a flight f such that x has taken flight f"

let, P(x, f) : "x has taken flight f"

Q(f, a) : " f is a flight on airline a"

The Quantification is:

$$=\exists_x \forall_a \exists_f[(P(x, f) \wedge Q(f, a)]$$

1. NESTED QUANTIFIERS:

Let,
L(x, y): "x loves y"

a) "Everyone Loves Somebody"
=For every person x, there exist person y such that x loves y
$=\forall_x \exists_y L(x, y)$

b) "Someone Loves Somebody"
=There exist some person x and some person y such that x loves y
$=\exists_x \exists_y L(x, y)$

c) "Someone is loved by everyone"
=There exist some person y for all x such that x loves y.
$=\exists_y \forall_x L(x, y)$

d) "Everybody Loves Everybody"
$= \forall_x \forall_y L(x, y)$

Q. Let Q(x, y) denote “x + y = 0.”
What are the truth values of the quantifications
a) $\exists_y \forall_x Q(x, y)$ and b)$\forall_x \exists_y Q(x, y)$, where the domain for all variables consists of all real numbers?

Solution:

a) The quantification $\exists_y \forall_x Q(x, y)$ denotes the proposition “There is a real number y such that for every real number x, Q(x, y).” No matter what value of y is chosen, there is only one value of x for which x + y = 0. Because there is no real number y such that x + y = 0 for all real numbers x, the statement $\exists_y \forall_x Q(x, y)$ is false.

b)The quantification $\forall_x \exists_y Q(x, y)$ denotes the proposition “For every real number x there is a real number y such that Q(x, y).” Given a real number x, there is a real number y such that x + y = 0; namely, y = −x. Hence, the statement $\forall_x \exists_y Q(x, y)$ is true.

Q. Let P (x, y) be the statement “x + y = y + x.”
What are the truth values of the quantifications a)$\forall_x \forall_y P(x, y)$ and b)$\forall_y \forall_x P(x, y)$ where the domain for all variables consists of all real numbers?

Solution:

a)The quantification $\forall_x \forall_y P(x, y)$ denotes the proposition “For all real numbers x, for all real numbers y,
x + y = y + x.” Because P (x, y) is true for all real numbers x and y the proposition $\forall_x \forall_y P(x, y)$ is true.

b)The quantification $\forall_y \forall_x P(x, y)$ says “For all real numbers y, for all real numbers x, x + y = y + x.”

Q. Let Q(x, y) denote “x + y = xy.”
What are the truth values of the quantifications
a) $\exists_x \exists_y Q(x, y)$ and b)$\exists_y \exists_x Q(x, y)$,domain for all variables consists of all positive real numbers?

Solution:

a) The quantification $\exists_x \exists_y Q(x, y)$ denotes the proposition “There is exist a number x such that for some number y, Q(x, y).” Q(x, y) is true for x=(0,2) and y= (0, 2). Hence, $\exists_x \exists_y Q(x, y)$ is TRUE

b) The quantification $\exists_y \exists_x Q(x, y)$ denotes the proposition “There is exist a number y such that for some number x, Q(x, y).” Q(x, y) is true for y=(0,2) and x= (0, 2). Hence, $\exists_y \exists_x Q(x, y)$ is TRUE

$\exists_y \forall_x Q(x, y)$!= $\forall_x \exists_y Q(x, y)$
$\forall_x \forall_y P(x, y) = \forall_y \forall_x P(x, y)$
$\exists_x \exists_y P(x, y) = \exists_y \exists_x P(x, y)$

2. NEGATING NESTED QUANTIFIERS:

- Statements involving nested quantifiers can be negated by successively applying the De-Morgan's rules for negating statements involving a single quantifier.

a) Express the negation of the statement $\forall_x \exists_y (xy = 1)$.

$=\neg[\forall_x \exists_y (xy = 1)]$

$=\exists_x \neg[\exists y(xy = 1)]$

$=\exists_x \forall_y \neg(xy = 1)$

$= \exists_x \forall_y (xy != 1)$

b) $\exists_x \exists_y P(x, y) \wedge \forall_x \forall_y Q(x, y)$

$= \neg[\exists_x \exists_y P(x, y) \wedge \forall_x \forall_y Q(x, y)]$

$=\neg[\exists_x \exists_y P(x, y)] \vee \neg[\forall_x \forall_y Q(x, y)]$

$= \forall_x \neg \exists_y P(x, y) \vee \exists_x \neg \forall_y Q(x, y)$

$= \forall_x \forall_y \neg P(x, y) \vee \exists_x \exists_y \neg Q(x, y)$

Statement	When True?	When False?
$\forall x \forall y P(x,y)$ $\forall y \forall x P(x,y)$	P(x,y) is true for every pair x,y	There is a pair x,y for which P(x,y) is false
$\forall x \exists y P(x,y)$	For every x there is a y for which P(x,y) is true	There is an x such that P(x,y) is false for every y
$\exists x \forall y P(x,y)$	There is an x for which P(x,y) is true for every y	For every x there is a y for which P(x,y) is false
$\exists x \exists y P(x,y)$ $\exists y \exists x P(x,y)$	There is a pair x,y for which P(x,y) is true	P(x,y) is false for every pair x,y

$\forall x \exists y P(x, y)$
$= \neg[\forall x \exists y P(x, y)]$
$= \exists_x \neg \exists y P(x, y)$
$\mathbf{= \exists_x \forall_y \neg P(x, y)}$

$\exists_x \forall_y P(x, y)$
$= \neg[\exists_x \forall_y P(x, y)]$
$= \forall_x \neg \forall_y P(x, y)$
$\mathbf{= \forall_x \exists_y \neg P(x, y)}$

3. TRANSLATING FROM NESTED QUANTIFIERS INTO ENGLISH:

Q. Translate the statement:
$\forall_x[C(x) \lor \exists_y(C(y) \land F(x, y))]$ into English, where
$C(x)$ is "x has a computer,"
$F(x, y)$ is "x and y are friends," and the domain for both x and y consists of all students in your school.

Solution:

The statement says that "for every student x in your school, x has a computer or there is a student y such that y has a computer and x and y are friends."

In other words, "*Every student in your school has a computer or has a friend who has a computer*"

3. TRANSLATING FROM NESTED QUANTIFIERS INTO ENGLISH:

Q. Translate the statement:
$\forall_x[S(x) \vee \exists_y(S(y) \wedge F(x, y))]$ into English, where
S(x) is “x uses snapchat”
F (x, y) is “x and y are friends,” and the domain for both x and y consists of all students in your school.

Solution:

The statement says that “for every student x in your school, x either uses snapchat or there is a student y such that y uses snapchat and y and x are friends.”

In other words, “*Every student in your school either uses snapchat or are friends with a student who uses snapchat*”

3. TRANSLATING FROM NESTED QUANTIFIERS INTO ENGLISH:

Q. Translate the statement

$\exists_x \forall_y \forall_z ((F(x, y) \wedge F(x, z) \wedge (y \mathrel{!=} z)) \rightarrow \neg F(y,z))$

into English, where F (a, b) means a and b are friends and the domain for x, y, and z consists of all students in your school.

Solution:

We first examine the expression $(F(x, y) \wedge F(x, z) \wedge (y = z)) \rightarrow \neg F(y, z)$. This expression says that if students x and y are friends, and students x and z are friends, and furthermore, if y and z are not the same student, then y and z are not friends.

It follows that the original statement, which is triply quantified, says that “there is a student x such that for all students y and all students z other than y, if x and y are friends and x and z are friends, then y and z are not friends. In other words,

“*There is a student none of whose friends are also friends with each other.*”

MATHEMATICAL FOUNDATION FOR COMPUTER SCIENCE

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RULES OF INTERFERENCE FOR QUANTIFIED STATEMENT

RULES OF INTERFERENCE FOR QUANTIFIED STATEMENT

1. UNIVERSAL INSTANTIATION:

- P(c) is true , Where c is a particular member of the Domain, given the Premise $\forall_x [p(x)]$

$$\frac{\forall_x [p(x)]}{\therefore p(c)}$$

Example: We can conclude from the statement “All women are wise” that “Lisa is wise” where Lisa is a member of the domain of all women.

2. UNIVERSAL GENERALIZATION:

- $\forall_x [p(x)]$ is True, given the premise that P(c) is True for all elements c in the domain.

$$\frac{p(c) \text{ for an arbitrary c}}{\therefore \forall_x [p(x)]}$$

Example: The Domain consist of the dogs Fido, Ruby,

Laika

"Fido is cute, Ruby is cute, Laika is cute".

Therefore, all dogs in the domain are cute.

3. EXISTENTIAL INSTANTIATION :

- There is an element c in the domain for which P (c) is true if we know that $\exists_x P(x)$ is true.

- We cannot select an arbitrary value of c here, but rather it must be a c for which P (c) is true. Usually we have no knowledge of what c is, only that it exists.

$$\frac{\exists_x [p(x)]}{\therefore p(c)\ c}$$

for some element

Example: "There is someone who got an A in the course."

"Let's call her c and say that c got an A"

4. EXISTENTIAL GENERALIZATION :

- That is, if we know one element c in the domain for which P (c) is true, then we know that $\exists_x P(x)$ is true.

$$\frac{p(c) \text{ for some element } c}{\therefore\ \exists_x [p(x)]}$$

Example: "Michelle got an A in the class."

"Therefore, there is someone who got an A in the class."

Q.1) “All King are men”

“All men are mortal”

∴ “All Kings are mortal

Solution:

Defining variables:

K(x): “x is king”

M(x): “x is man”

M_0(x): “x is mortal”

Hypothesis: i) $\forall_x[K(x) \rightarrow M(x)$

ii) $\forall_x[M(x) \rightarrow M_o(x)$

Conclusion: ∴ $\forall_x[K(x) \rightarrow M_o(x)]$

STEPS	REASONS
1. $\forall_x[K(x) \rightarrow M(x)]$	GIVEN HYPOTHESIS
2. $K(c) \rightarrow M(c)$	UNIVERSAL INSTANTIATION ON 1
3. $\forall_x[M(x) \rightarrow M_o(x)]$	GIVEN HYPOTHESIS
4. $M(c) \rightarrow M_o(c)$	UNIVERSAL INSTANTIATION ON 3
5. $K(c) \rightarrow M_o(c)$	HYPOTHETICAL SYLLOGISM ON 2 & 4
6. $\forall_x[K(x) \rightarrow M_o(x)]$	UNIVERSAL GENERALIZATION ON 5

Q.2) Show that the premises *"Everyone in this discrete mathematics class has taken a course in computer science"* and "*Sita is a student in this class*" imply the conclusion "*Marla has taken a course in computer science.*"

Solution:

Defining variables:

D(x): "*x is in this discrete mathematics class*"

C(x): "*x has taken a course in computer science.*"

Hypothesis: i) $\forall_x[D(x) \rightarrow C(x)]$

ii) D(sita)

Conclusion: $\therefore$ C(sita)

STEPS	REASONS
1. $\forall_x[D(x) \rightarrow C(x)]$	GIVEN HYPOTHESIS
2. D(sita)$\rightarrow$C(sita)	UNIVERSAL INSTANTIATION ON 1
3. D(sita)	GIVEN HYPOTHESIS
4. C(sita)	MODUS PONENS ON 2 & 3

Q.3) Show that the premises *"A student in this class has not read the book,"* and *"Everyone in this class passed the first exam"* imply the conclusion *"Someone who passed the first exam has not read the book."*

Solution:

Defining variables:

C(x): "x is in this class"

B(x): "x has read the book"

P(x): "x pas

Hypothesis: i) $\exists_x[C(x) \wedge \neg B(x)]$

ii) $\forall_x[C(x) \rightarrow P(x)]$

Conclusion: $\therefore \exists_x[P(x) \wedge \neg B(x)]$

STEPS	REASONS
1. $\exists_x[C(x) \wedge \neg B(x)]$	GIVEN HYPOTHESIS
2. $C(a) \wedge \neg B(a)$	EXISTENTIAL INSTANTIATION ON 1
3. $C(a)$	SIMPLIFICATION ON 2
4. $\forall_x[C(x) \rightarrow P(x)]$	GIVEN HYPOTHESIS
5. $C(a) \rightarrow P(a)$	UNIVERSAL INSTANTIATION FROM (4)
6. $P(a)$	MODUS PONENS FROM (3) AND (5)
7. $\neg B(a)$	SIMPLIFICATION ON 2
8. $P(a) \wedge \neg B(a)$	CONJUNCTION FROM (6) AND (7)
9. $\exists_x[P(x) \wedge \neg B(x)]$	EXISTENTIAL GENERALIZATION FROM (8)

Q.4) Show that the premises *"All rock music is loud", "Some rock music exist"*, imply the conclusion *"Some Loud music exists"*

<u>Solution:</u>

Defining variables:

R(x): "*x is in rock music*"

L(x): "*x is loud music*"

<u>Hypothesis:</u> i) $\forall_x[R(x) \rightarrow L(x)]$

ii) $\exists_x[R(x)]$

<u>Conclusion:</u> $\therefore \exists_x[L(x)]$

STEPS	REASONS
1. $\forall_x[R(x) \rightarrow L(x)]$	GIVEN HYPOTHESIS
2. $R(c) \rightarrow L(c)$	UNIVERSAL INSTANTIATION ON 1
3. $\exists_x[R(x)]$	GIVEN HYPOTHESIS
4. R (c)	EXISTENTIAL INSTANTIATION ON 3
5. L(c)	MODUS PONENS FROM (2) AND (4)
6. $\exists_x[L(x)]$	EXISTENTIAL GENERALIZATION FROM (5)

Q.4) Show that the premises *"Every computer science student works harder than somebody", "Everyone who works harder than any other person gets less sleep than that person", "Maria is a computer science student"* Implies the conclusion *"Maria gets less sleep than someone else"*

Solution:

Defining variables:

C(x) : "*x is Computer science student*"

W(x, y) : "*x works harder than y*"

S(x, y) : "x gets less sleep than y"

C(m) : "Maria is a computer science student"

Hypothesis: i) $\forall_x \exists_y [C(x) \rightarrow W(x, y)]$

ii) $[\forall_x \exists_y W(x, y)] \rightarrow S(x, y)$

iii) $C(m)$

Conclusion: $\therefore \exists_y [S(m, y)]$

Hypothesis: i) $\forall_x \exists_y [C(x) \rightarrow W(x, y)]$
ii) $[\forall_x \exists_y W(x, y)] \rightarrow S(x, y)$
iii) $C(m)$

Conclusion: $\therefore \exists_y [S(m, y)]$

STEPS	REASONS
1. $\forall_x \exists_y [C(x) \rightarrow W(x, y)]$	GIVEN HYPOTHESIS
2. $\forall_x [C(x) \rightarrow W(x, c)]$	EXISTENTIAL INSTANTIATION ON 1
3. $C(m) \rightarrow W(m, c)$	UNIVERSAL INSTANTIATION ON 2
4. $[\forall_x \exists_y W(x, y)] \rightarrow S(x, y)$	GIVEN HYPOTHESIS
5. $[\forall_x W(x, c)] \rightarrow S(x, c)$	EXISTENTIAL INSTANTIATION ON 4
6. $W(m\ c) \rightarrow S(m, c)$	UNIVERSAL INSTANTIATION ON 5
7. $C(m) \rightarrow S(m, c)$	FROM 3 AND 6
8. $C(m)$	GIVEN HYPOTHESIS
9. $S(m, c)$	MODUS TOLLENS ON 7 AND 8
10. $\exists_y [S(m, y)]$	EXISTENTIAL GENERALIZATION FROM 9

Students who pass the course either do the homework or attend lecture;” “Bob did not attend every lecture;” “Bob passed the course.” Therefore “ Bob must have done the homework.”

MATHEMATICAL FOUNDATION FOR COMPUTER SCIENCE

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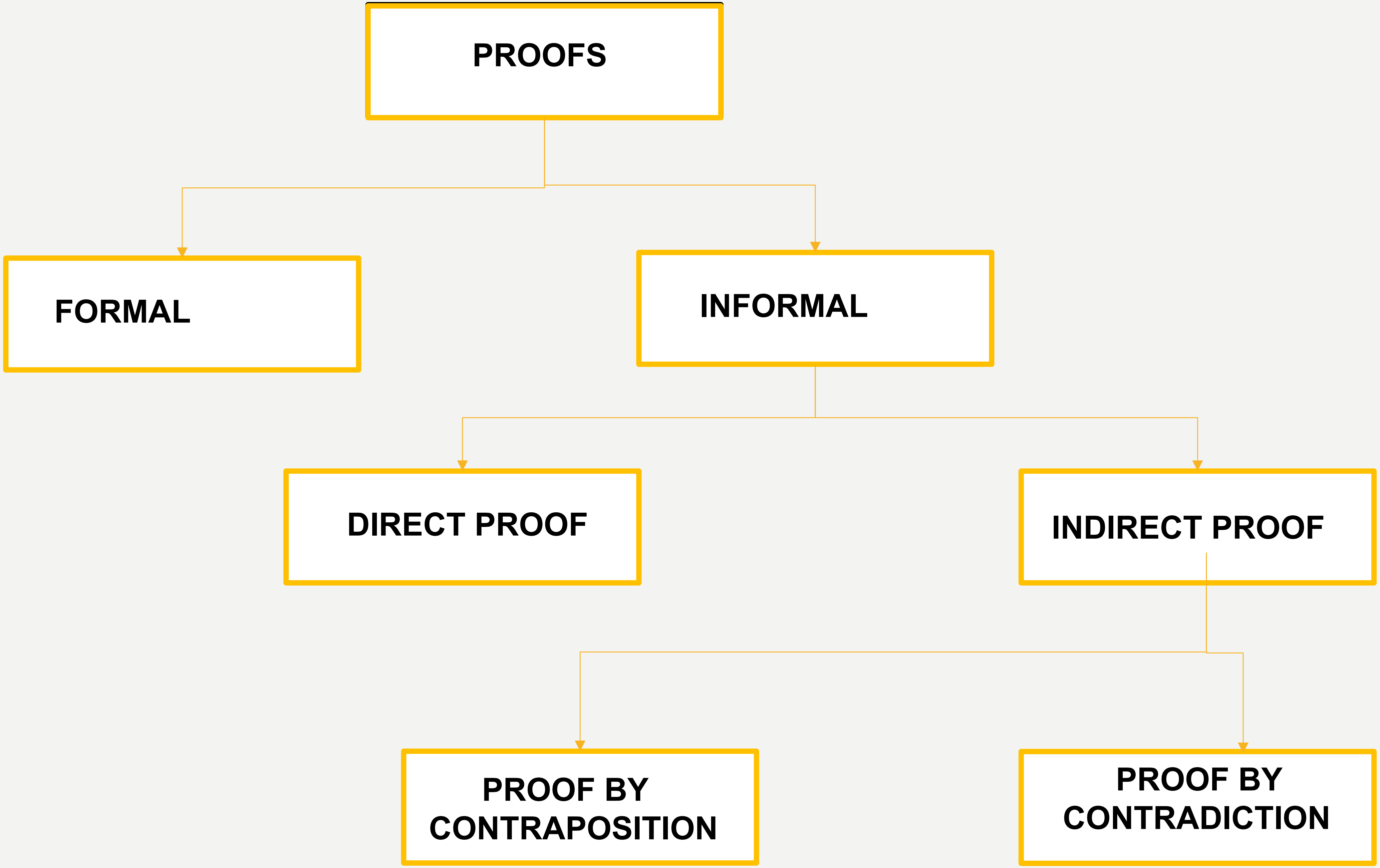
PROOFS
FORMAL
INFORMAL
DIRECT PROOF
INDIRECT PROOF
PROOF BY CONTRAPOSITION
PROOF BY CONTRADICTION

PROOF:

- An argument used to establish the truth of mathematical statement is called Proof.
- Mathematical proofs use **deductive reasoning**, where a conclusion is drawn from multiple premises.
- The premises in the proof are called statements.
- While establishing the truth, different rules and already proven facts are used.
- INDUCTIVE REASONONG : Drawing a general conclusion from what we see around us. For example, if all the sheep you have ever seen were white, you might conclude that all sheep are white.
- DEDUCTIVE REASONONG : You start from a general statement you know for sure is true and draw conclusions about a specific case. For example, if you know for a fact that all sheep like to eat grass, and you also know that the creature standing in front of you is a sheep, then you know with certainty that it likes grass.

SOME TERMINOLOGIES:

1. THEOREM: A mathematical statement that is proved using rigorous mathematical reasoning. The process of showing a theorem to be correct is called a proof.

2. AXIOM: An axiom is a statement, usually considered to be self-evident, that is assumed to be True without proof. It is used as starting point in mathematical proof.

Example: Parallel lines in same plane , do not meet one another in either direction when extended infinitely.

3. COROLLARY: A corollary is the theorem that can be proven to be a Logical consequence of another theorem.

Example: If $a+ b=c$ then an example of corollary is $c= b- a$.

SOME TERMINOLOGIES:

4. CONJECTURE : A conjecture is a mathematical statement that has not yet been rigorously proved. Conjectures arises when one notices a pattern that holds True for many cases.

Example: 2, 4, 6, ,8 ,10 ,12, ?

The next number is more likely to be 14.

5. AXIOM: It is generally minor, proven proposition which is used as a stepping stone to a larger result. It is also known as a “Helping Theorem” or “ Auxillary Theorem”

Example: For all real numbers r, |-r| = |r|

FORMAL & INFORMAL PROOF:

Definition: A formal proof of a conclusion q given hypotheses p1, p2, . . . , pn is a sequence of steps, each of which applies some inference rule to hypotheses or previously proven statements (antecedents) to yield a new true statement (the consequent).

Informal proof : where more than one rule of inference may be used in each step, where steps may be skipped, where the axioms being assumed and rule of inference used are not explicitly stated.

1. DIRECT PROOF:

- A direct proof of a conditional statement $p \to q$ is constructed when the first step is the assumption that p is true; subsequent steps are constructed using rules of inference, with the final step showing that q must also be true

- A direct proof shows that a conditional statement $p \to q$ is true by showing that if p is true, then q must also be true, so that the combination p true and q false never occurs.

- In a direct proof, we assume that p is true and use axioms, definitions, and previously proven theorems, together with rules of inference, to show that q must also be true

Example:

1. Prove that "If n is odd, then n^2 is odd."

Solution:

Let,

p: Hypothesis: " n is odd"

q: Conclusion: "n^2 is odd"

Now, we assume Hypothesis is TRUE. i.e.

n is odd(TRUE)

By the definition of odd integer, we can write,

$n = 2k+1$; for integer k

Squaring both sides,

$$
\begin{aligned}
n^2 &= (2k+1)^2 \\
&= 4k^2 + 4k + 1 \\
&= 2(2k^2 + 2k) + 1 \\
&= 2k_1 + 1; \text{ where } k_1 = (2k^2 + 2k) \text{ is an integer}
\end{aligned}
$$

Therefore, n2 is odd.

2. Prove that “If ‘m’ and ‘n’ are odd, then ‘m + n’ is even.” [The sum of two odd numbers is even.]

Solution:

Let,

p: Hypothesis: “ m and n are odd”

q: Conclusion: “m + n is even”

Now, we assume Hypothesis is TRUE. i.e.

m and n are odd(TRUE)

By the definition of odd integer, we can write,

$m = 2i+1$; for integer i

$n = 2j+1$; for integer j

Now,

$$\begin{aligned} m + n &= (2i+1) + (2j+1) \\ &= 2i+2j+2 \\ &= 2(i+j+1) \\ &= 2k \qquad \text{; where } k = (i+j+1) \text{ is an integer} \end{aligned}$$

Therefore, m + n is even.

3. If x is an even integer, then $x^2 - 6x + 5$ is odd.

Proof.

Suppose x is an even integer. Then $x = 2a$ for some $a \in Z$, by definition of an even integer. So

$$
\begin{aligned}
x^2 - 6x + 5 &= (2a)^2 - 6(2a) + 5 \\
&= 4a^2 - 12a + 5 \\
&= 4a^2 - 12a + 4 + 1 \\
&= 2(2a^2 - 6a + 2) + 1.
\end{aligned}
$$

Therefore we have $x^2 - 6x + 5 = 2b + 1$, where $b = 2a^2 - 6a + 2 \in Z$.

Consequently $x^2 - 6x + 5$ is odd, by definition of an odd number

4. If n is any even integer, then $(-1)^n = 1$.

Proof:

Suppose n is even integer. [We must show that $(-1)^n = 1$.].

Then by the definition of even numbers,

$n = 2k$ for some integer k

we have

$$
\begin{aligned}
(-1)^n &= (-1)^{2k} \\
&= ((-1)^2)^k \\
&= (1)^k \\
&= 1
\end{aligned}
$$

This is what was to be shown. And this completes the proof.

1. INDIRECT PROOF:

- Direct proofs lead from the premises of a theorem to the conclusion. They begin with the premises, continue with a sequence of deductions, and end with the conclusion.

- However, we will see that attempts at direct proofs often reach dead ends. We need other methods of proving theorems of the form $\forall x(P(x) \rightarrow Q(x))$. Proofs of theorems of this type that are not direct proofs, that is, that do not start with the premises and end with the conclusion, are called indirect proofs

i. Proof by Contraposition
ii. Proof by Contradiction

1.1 PROOF BY CONTRAPOSITION:

- Proofs by contraposition make use of the fact that the conditional statement $p \to q$ is equivalent to its contrapositive, $\neg q \to \neg p$. This means that the conditional statement $p \to q$ can be proved by showing that its contrapositive, $\neg q \to \neg p$, is true.

- In a proof by contraposition of $p \to q$, we take $\neg q$ as a premise, and using axioms, definitions, and previously proven theorems, together with rules of inference, we show that $\neg p$ must follow

Q.1 Prove that if n is an integer and 3n + 2 is odd, then n is odd.
Solution:
We first attempt a direct proof.
If 3n + 2 is odd, then n is odd.($p \rightarrow q$)
p: "3n + 2 is odd"
q: "n is odd"
To construct a direct proof, we first assume that 3n + 2 is an odd integer.
This means that 3n + 2 = 2k + 1 for some integer k.
$3n + 2 = 2k + 1$
$n = (2k-1)/3$
DEAD END
$p \rightarrow q = \neg q \rightarrow \neg p$[if n is even then 3n+2 is even]
$\neg q$="n is even"
$\neg p$= "3n + 2 is even"
Now , from the definition of an even integer
$n=2k$, for some integer k
Substituting 2k for n , We get,
$3(2k) + 2$
$=6k + 2$
$=2(3k + 1)$
i.e. 3n + 2 is even because it is a multiple of 2
Therefore, 3n+2 is even.

1.2 PROOF BY CONTRADICTION:

- The basic idea is to assume that the statement we want to prove is false, and then show that this assumption leads to nonsense. We are then led to conclude that we were wrong to assume the statement was false, so the statement must be true.

Prove there exist no integer a, b for which 5a + 15b =1.

Solution:

Step 1: Assume there exist integer a and b for which 5a + 15b =1.

Now,

$$5a + 15b = 1$$
$$5(a + 3b) = 1$$
$$a + 3b = 1/5$$

Because a and b are integers, a + 3b must also be an integer**[CONTRADICTION]**

Therefore, There exist no integer a, b for which 5a + 15b = 1.

Proposition: If a , $b \in Z$, then $a^2 - 4b != 2$.

Proof.

“If a and b are integers then, $a^2 - 4b != 2$”

Suppose this proposition is false. This conditional statement being false means:

There exist numbers a and b for which $a , b \in Z$ is true but $a^2 - 4b != 2$ is false.

“If a and b are integers then, $a^2 - 4b = 2$”

Thus there exist integers $a, b \in Z$ for which $a^2 - 4b = 2$.

From this equation we get

$a^2 = 4b + 2 = 2(2b + 1)$, so a2 is even. Since a^2 is even, it follows that a is even, so $a = 2c$ for some integer c.

Now plug $a = 2c$ back into the boxed equation $a^2 - 4b = 2$. We get

$(2c)^2 - 4b = 2$, so $4c^2 - 4b = 2$. Dividing by 2, we get $2c^2 - 2b = 1$.

Therefore $1 = 2(c^2 - b)$, and since $c^2 - b \in Z$, it follows that 1 is even. Since we know 1 is not even, something went wrong. But all the logic after the first line of the proof is correct, so it must be that the first line was incorrect. In other words, we were wrong to assume the proposition was false. Thus the proposition is true.

Prove that for all integer n, if $n^3 + 5$ is odd then n is even.

Solution:

Here,

Assume the conclusion, i.e. n is odd

Because n is odd , We can write,

$N = 2k + 1$

Putting value of n in $n^3 + 5$, We get

$=(2k + 1)^3 + 5$

$=8k^3 + 12k^2 + 6k +1 + 5$

$= 8k^3 + 12k^2 + 6k + 6$

$=2[8k^3 + 12k^2 + 6k + 6]$

=Even**[CONTRADICTION]**

Therefore, If $n^3 + 5$ is odd then n is even

Prove $\sqrt{2}$ is a irrational number using proof by contradiction.

Suppose $\sqrt{2}$ is rational. Then integers a and b exist so that $\sqrt{2} = a/b$.

Without loss of generality we can assume that a and b have no factors in common (i.e., the fraction is in simplest form).

Multiplying both sides by b and squaring,
we have $2b^2 = a^2$ so we see that a^2 is even.
This means that a is even so $a = 2m$ for some $m \in Z$.
Then $2b^2 = (2m)^2$
$= 4m^2$ which, after dividing by 2, gives $b^2 = 2m^2$ so b^2 is even. This means b is even.

We've seen that if $\sqrt{2} = a/b$ then both a and b must be even and so are both multiples of 2. This contradicts the fact that we know a and b can be chosen to have no common factors. Thus, $\sqrt{2}$ must not be rational, so $\sqrt{2}$ is irrational.

PRINCIPLE OF MATHEMATICAL INDUCTION:

- Let P(n) be a statement. Now, our concern is to show that P(n) is True using Mathematical Induction.

  a) First we show that P(n) is True for some initial value like n =0, 1 ….This is called the Basic Step.

  b) Then, we assume that P(n) is True for any arbitrary value k i.e. P(k) is True and show that P(n) is True for 'k + 1' i.e. P(k+1) is True. This step is called inductive step

Thus, Mathematical Induction can be defined as:

$$[P(1) \wedge (P(k) \rightarrow P(k+1))] \rightarrow P(n)$$

Q. Show that if n is positive integer then,

$$1 + 2 + \ldots\ldots\ldots + n = [n(n + 1)]/2$$

Solution:

Let, $P(n)$ be the proposition that the sum of first n positive integer , $1 + 2 + \ldots\ldots\ldots + n = [n(n + 1)]/2$

Basic Step: When $n=1$,

$$1= [1(1+1)]/2$$

$$1=1\text{(TRUE) i.e. } P(1) \text{ is True.}$$

Inductive Step: Assume $P(k)$ holds for arbitrary integer k.

i.e. $1+2+\ldots\ldots+k = [k(k+1)]/2$

Under this assumption , it must be shown that $P(k+1)$ is True

$$1 + 2 + \ldots.+ k + (k+1)= [(k+1)(k+2)]/2$$

L.H.S.

$1 + 2 + \ldots.+ k + (k+1)$

$=[k(k+1)]/2 + (k + 1)$

$=[k(k+1) + 2k + 2]/2$

$=[k(k+1) + 2(k+1)]/2$

$=[(k+1)(k+2)]/2$

=R.H.S

Q. Use Mathematical induction to show that:

$$2 + 2^2 + \ldots\ldots\ldots\ldots\ldots\ldots + 2^n = 2^{n+1} - 2$$

<u>Solution:</u>

Let, P(n) be the proposition that, $2 + 2^2 + \ldots\ldots\ldots\ldots\ldots\ldots + 2^n = 2^{n+1} - 2$

<u>Basic Step:</u> When n=1,

$2 = 2^2 - 2$

2=2(TRUE) i.e. P(1) is True.

<u>Inductive Step:</u> Assume P(k) holds for arbitrary integer k.

i.e. $2 + 2^2 + \ldots\ldots\ldots\ldots\ldots\ldots + 2^k = 2^{k+1} - 2$

Under this assumption , it must be shown that P(k+1) is True

$$2 + 2^2 + \ldots\ldots\ldots\ldots\ldots\ldots + 2^k + 2^{k+1} = 2^{(k+1)+1} - 2$$

L.H.S.

$2 + 2^2 + \ldots\ldots\ldots\ldots\ldots\ldots + 2^k + 2^{k+1}$

$= 2^{k+1} - 2 + 2^{k+1}$

$= 2 \cdot 2^{k+1} - 2$

$= 2^{(k+1)+1} - 2$

=R.H.S

Therefore, P(n) is true.

Q. Use Mathematical induction to show that:

$8^n – 3^n$ is divisible by 5.[n≥1]

Solution:

Let, P(n) be the proposition that, $8^n – 3^n$ is divisible by 5

Basic Step: When n=1,

$8^1 – 3^1$ is divisible by 5

5 is divisible by 5(TRUE) i.e. P(1) is True.

Inductive Step: Assume P(k) holds for arbitrary integer k.

i.e. $8^k – 3^k$ is divisible by 5

Under this assumption , it must be shown that P(k+1) is True

i.e. $8^{k+1} – 3^{k+1}$ is divisible by 5

Now,

$$
\begin{aligned}
& 8^{k+1} – 3^{k+1} \\
&= 8^k.8 – 3^k.3 \\
&= 8^k(5+3) - 3^k.3 \\
&= 8^k.5 + 8^k.3 - 3^k.3 \\
&= 8^k.5 + 3(8^k - 3^k)
\end{aligned}
$$

Here, $8^k.5$ is multiple of 5 and $(8^k - 3^k)$is divisible by 5.

Therefore, P(n) is true.