Dynamics of Machines

- Force Analysis of Mechanics
- Flywheel
- Gear Dynamics
- Governors
- Mechanical Vibration 90% syllabus

× Kinematics & Dynamics of Machines – ghosh & Mallik.

Theory

✓ Mechanical Vibrations – S. S. Rao

✓ Theory of Vibrations with Applications – W.T. Thomson et. al

✓ Elements of Vibration Analysis – L. Meirovitech

Problems

Mechanical Vibrations → Seto

—— 11 —— → Kelly.

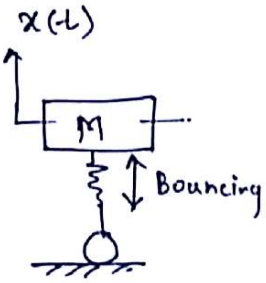
} Schaum's outline Series. MacGraw hill

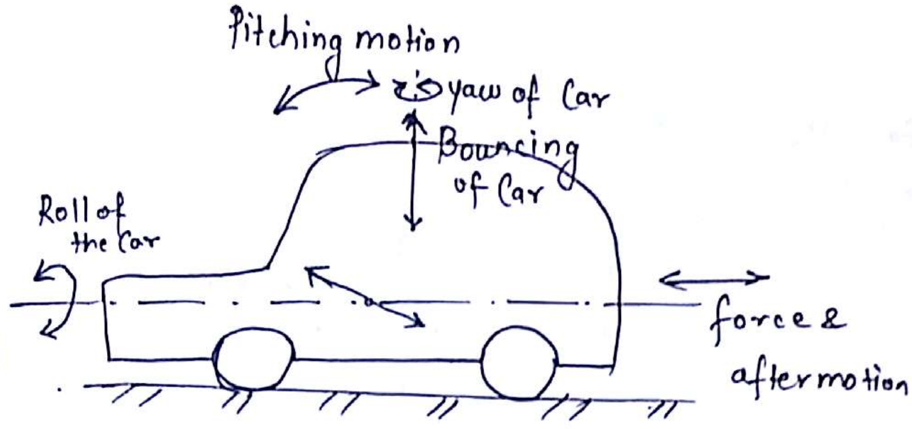
6 Simple Machines

1) Pulley
2) Lever
3) Inclined Planes
4) Double Inclined Planes
5) Wheel & Axle
6) Screw

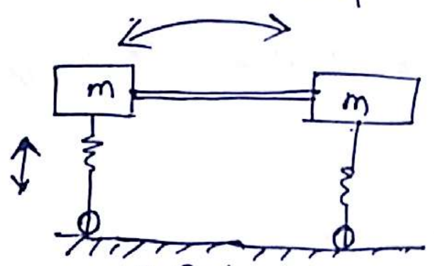
A Simple Definition of Machine:

A m/c is a device which transmits energy/power from One place or Another. A m/c mass also transfers Energy from one form to Another.



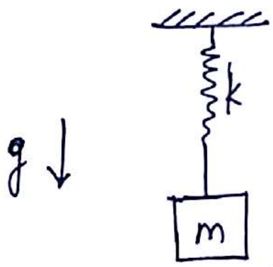


1 Dof



2 Dof

Free Vibration of an undamped 1 Dof Spring Mass System



A Discrete Dynamic System

Assumption:

(i) The Mass (block) undergoes translation only such that the C.M. Moves in Verticle Line. The block is arrested in other direction by frictionless walls.

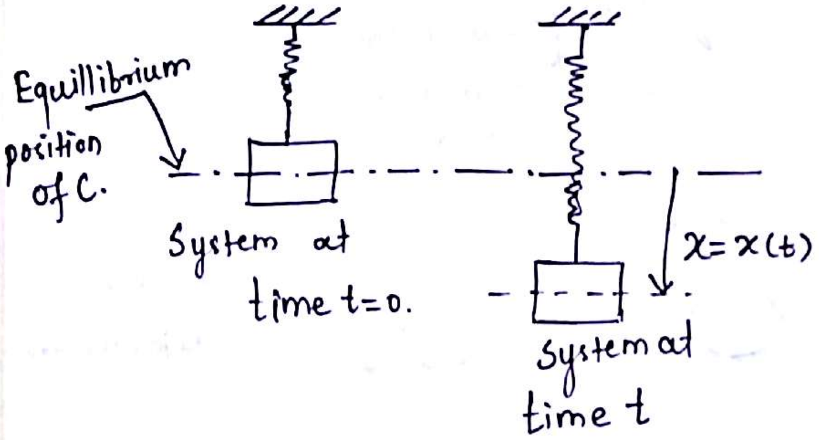
(ii) All Sorts of Damping Neglected.

(iii) The Mass is absolutely Rigid

(iv) The Spring is Linear (V) The Spring is Massless.

Free Vibration Means

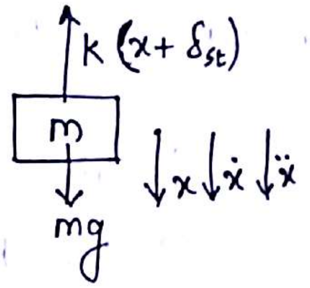
The Mass would be given an initial displacement &/or an initial Velocity & then the System will lift to itself. The Subsequent motion is <u>free Vibration</u>.



<u>Aim</u>: To Obtain the Differential Equation of Motion (DEOM) of our System.

<u>Method I</u>: (Using Newtons 2nd Laws of Motion).

FBD of Mass at time t.



$$mg = k\delta_{st}$$

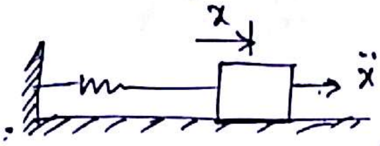
$m\ddot{x} = \sum$ interal Forces in +ve x direction

$$m\ddot{x} = \cancel{mg} - k(x+\delta_{st}) = -kx$$

$$\frac{d^2x}{dt^2} + \frac{k}{m}x = 0$$

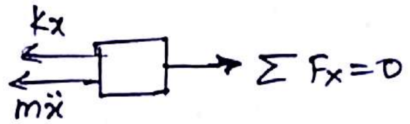
Method II: (D'Alembert's Method)

D'Alembert's said that If we Introduced Inertia terms a_{cm} a problem in Dynamics Can be reduced in a problem in Statics.



The inertia force will be $-m\ddot{x}$

Then the FBD of block is



$$m\ddot{x} + kx = 0$$

Method III (The Energy balance Method)

Ours ~~is~~ is Conservative System & Mechanical Energy (total) is Conserved.

$T = K.E. =$ Kinetic energy of our system

$$= \frac{1}{2} m\dot{x}^2$$

U (or V) $=$ P.E. $=$ Potential Energy (includes strain Energy) $=$?

Here $U =$ The change in P.E. Over and above the Equillibrium.

$$U = -mgx + \frac{1}{2} k (x+\delta_{st})^2 - \frac{1}{2} k\delta_{st}^2$$

$$= \frac{1}{2} kx^2$$

So

$$\frac{d}{dt}(T+U) = 0$$

$$m\dot{x}\ddot{x} + kx\dot{x} = 0 \qquad \dot{x} \neq 0$$

$$\underline{m\ddot{x} + kx = 0}$$

Method IV: The Use of Langrange's Equation (of 2^{nd} kind)

$$\cancel{\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} + \frac{\partial U}{\partial q_j}}$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} + \frac{\partial U}{\partial q_j} + \frac{\partial D}{\partial \dot{q}_j} = Q_j$$

$j = 1, 2, \ldots, n$

q_j = The j^{th} generalized Coordinate

Q_j = Generalised force associated with q_j

holonomic Differential System.

n = no. of D.O.F. of the Holonomic Systems.

$$\omega_n = \sqrt{\frac{K}{m + \frac{m_s}{3}}} \quad \text{(for Spring with Mass)}$$

for Our System

$j = 1$ & $q_1 = x$ $\dot{q}_1 = \dot{x}$

Also $D \equiv 0$ (no damper) $Q_j \equiv 0$ ($\because$ only free Vibration is Considered)

Here $T = KE = \frac{1}{2} m \dot{x}^2$

$U = PE = \frac{1}{2} K x^2$

* Remember that at this Stage of derivation of the DEOM, x & $\dot{x}$ are Independent Variable.

Langrange Equation for our System is

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}}\right) - \cancel{\frac{\partial T}{\partial x}}^{\,0} + \frac{\partial U}{\partial x} = 0$$

$$\frac{dT}{d\dot{x}} = \frac{d}{d\dot{x}}\left(\frac{1}{2} m \dot{x}^2\right) = m\dot{x} \qquad \frac{dU}{dx} = Kx$$

$$m\ddot{x} + kx = 0$$

Method V: Hamilton's Principle

$$\delta \int_{t_1}^{t_2} \underbrace{(T-U)}_{L} dt = 0$$

Feynmann Physics Lecture Volume 2
Chapter 21

$L \rightarrow$ The Langrangian of the system

$\delta x = dx$

$$\delta\left(\int_{t_1}^{t_2} L \, dt\right) = 0$$

$$\int_{t_1}^{t_2} \delta L \, dt = 0$$

$$L = T - U = -\frac{1}{2} kx^2 + \frac{1}{2} m\dot{x}^2$$

$$\delta L = \delta\left(\frac{1}{2} m\dot{x}^2\right) - \delta\left(\frac{1}{2} kx^2\right)$$

$$\delta L = m\dot{x}\,\delta\dot{x} - kx\,\delta x$$

$$0 = \int m\dot{x}\,\delta\dot{x}\,dt - \int kx\,\delta x\,dt$$

$$0 = \int m\dot{x}\,\frac{d}{dt}(\delta x)\,dt - \int kx\,\delta x\,dt$$

$$0 = m\dot{x}\,\delta x - \int m\ddot{x}\,\delta x\,dt - \int kx\,\delta x\,dt$$

B.C.

$\delta x = 0$ at $t = t_1$ & $t = t_2$ | $(m\ddot{x} + kx) = 0$ for $t \in (t_1, t_2)$

- Method VI - Use of Schrödinger Wave Equation

1 D wave Equation of our system Can be Shown to be

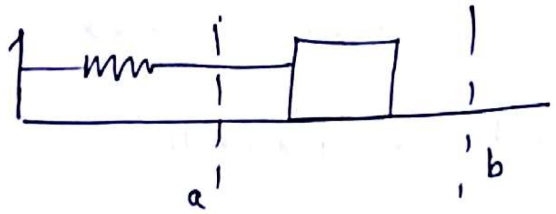
$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2 \psi}{\partial x^2} + V(\psi)$$

A potential

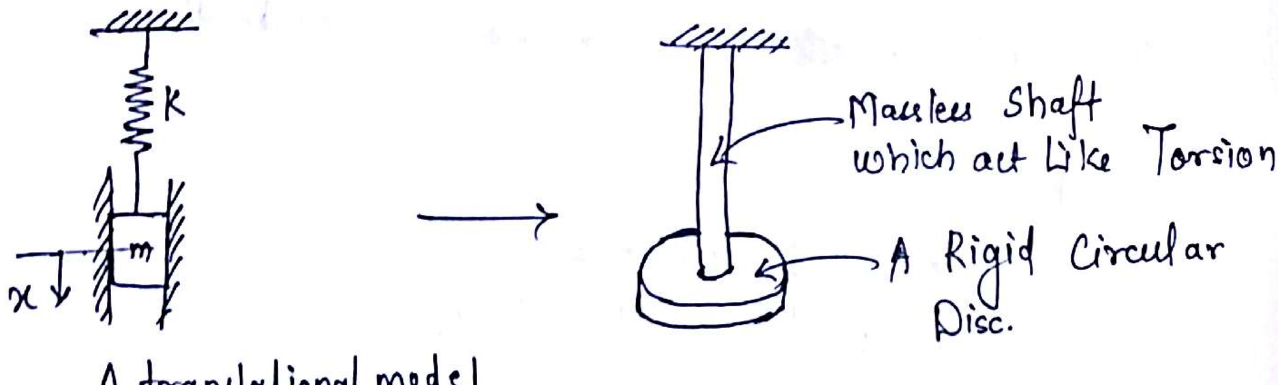
$i = \sqrt{-1}$ $\hbar = \frac{h}{2\pi}$

h = plank Constant

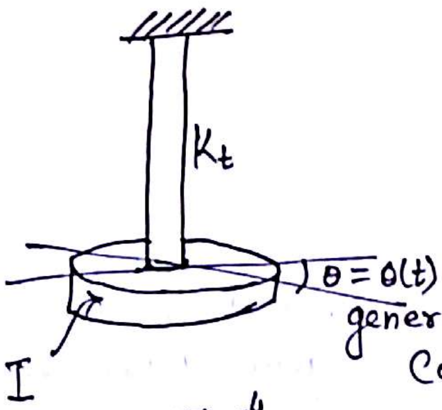
$\int_a^b |\psi|^2 dx$ gives probability of finding particle in $]a, b[$



The Rotational Counterpart of our Model.



A translational model



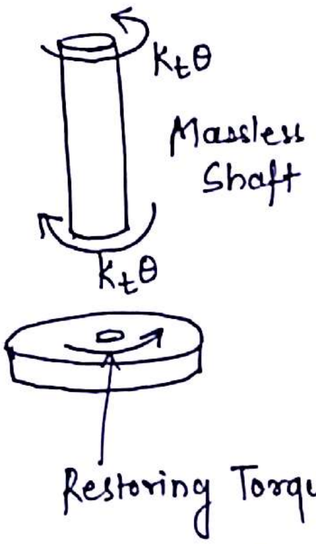
$$\frac{T}{J} = \frac{G\phi}{l}$$

$$\boxed{K_t = \frac{GJ}{l}}$$

$$J = \frac{\pi r^4}{2}$$

DEOM for Small, translational Oscillation

Method I



$$I\ddot{\theta} = -k_t\theta$$

Method 2 Energy Method

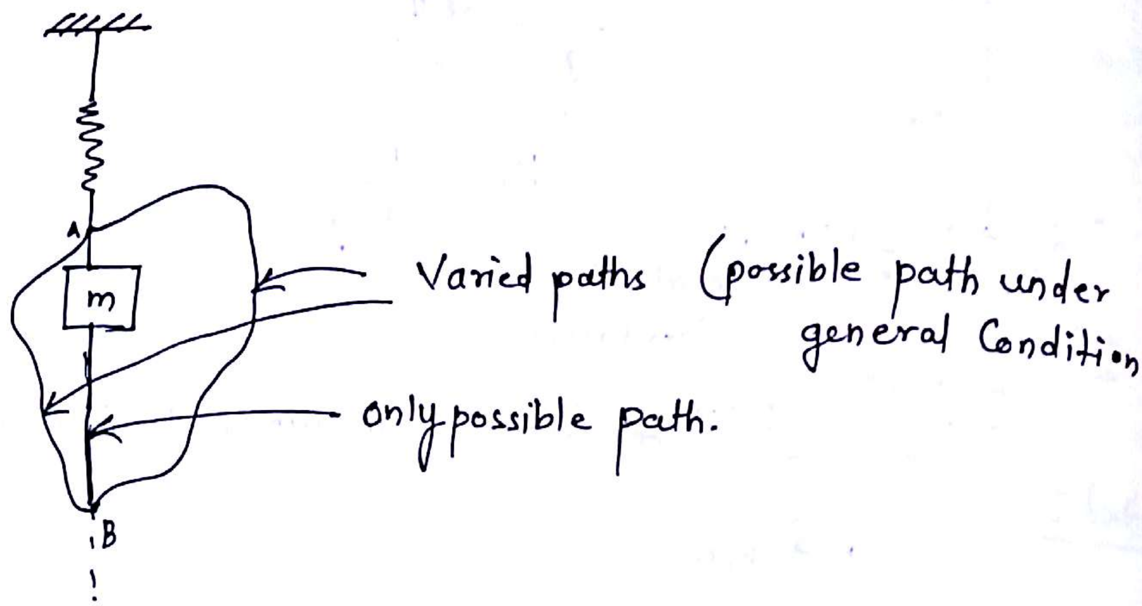
$$T = \frac{1}{2} I \dot{\theta}^2$$

$$U = \frac{1}{2} K_t \theta^2$$

$$\frac{d}{dt}(T+U) = 0 \qquad I\ddot{\theta} + k_t\theta = 0$$

$$\int_{t_1}^{t_2} L\,dt \longrightarrow \text{The Action Integral}$$

$$\delta \int_{t_1}^{t_2} L\,dt = 0 \quad \text{A principle of Least action}$$



$$m\ddot{x} + kx = 0$$

Let $x = c\, e^{st}$

$$ms^2 + k = 0$$

$$x = c_1 e^{s_1 t} + c_2 e^{s_2 t}$$

$$x = c_1 e^{-\sqrt{\frac{k}{m}}\, i t} + c_2 e^{j\sqrt{\frac{k}{m}}\, t}$$

$$x = A \sin \omega_n t + B \cos \omega_n t$$

$$x = X_0 \sin(\omega t + \phi)$$

$$\omega_n = \sqrt{\frac{K}{m}}$$

$$X_0 = \sqrt{A^2 + B^2}$$

$$\tan\phi = B/A$$

$$\cancel{x} = A \sin(\omega_n t) + B \cos(\omega_n t)$$

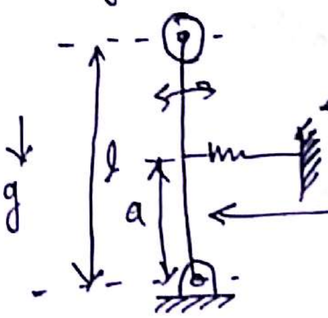
A, B to be deduced from $X(0)$ $\dot{X}(0)$

W_n = Natural frequency of given System

$$f_n = \frac{\omega_n}{2\pi}$$

Example 1

Obtain Condition for Stable Oscillation for the following system.

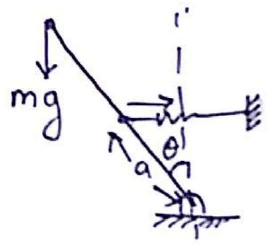


Step 1 : Obtain DEOM

N's method.

Let $\theta(t)$ be the generalised Co-ordionate

(Note that the System given only One DOF hence only one gen. coordinate)



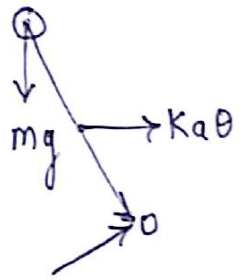
Net Torque Restoring $= Ka^2\theta - mgl\theta$

$$-I\alpha = (Ka^2 - mgl)\,\theta$$

$$I\omega^2 \cancel{\theta} = (ka^2 - mgl)\,\cancel{\theta}$$

$$\omega = \sqrt{\frac{Ka^2 - mgl}{ml^2}}$$

FBD



for stable oscillation

$$\left(\frac{Ka^2 - mgl}{ml^2}\right) > 0$$

Method 2 . Energy Method

Method 3 Langranges Equation

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot\theta}\right) - \frac{\partial T}{\partial \theta} + \frac{\partial U}{\partial \theta} = 0$$

$$T = \frac{1}{2} I_o \omega^2 = \frac{1}{2} m l^2 \dot\theta^2$$

$$U = -l(1-\cos\theta)\, mg + \frac{1}{2} k (a\theta)^2$$

→ Never Linearize before differentiation

$$\frac{\partial T}{\partial \theta} = 0 \qquad \frac{\partial T}{\partial \dot\theta} = \frac{1}{2} m l^2 \times 2\dot\theta = m l^2 \dot\theta$$

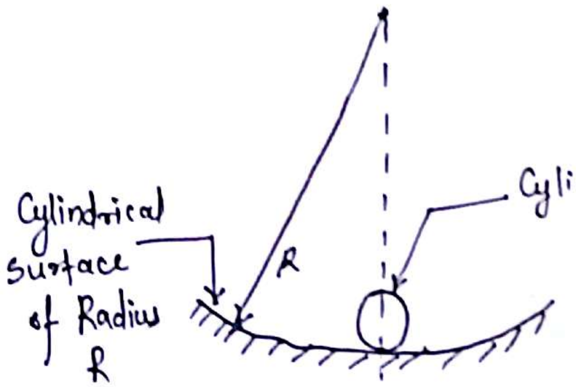
$$\frac{\partial}{\partial t}\left(\frac{\partial T}{\partial \dot\theta}\right) = m l^2 \ddot\theta$$

$$\frac{\partial U}{\partial \theta} = -mgl\sin\theta + ka^2\theta$$

$$\sim (ka^2 - mgl)\,\theta$$

$$m l^2 \ddot\theta + (ka^2 - mgl)\theta = 0$$

An Important Example



The Cylinder rolls w/o slipping & Oscillates back and forth obtain the DEOM & ω_n

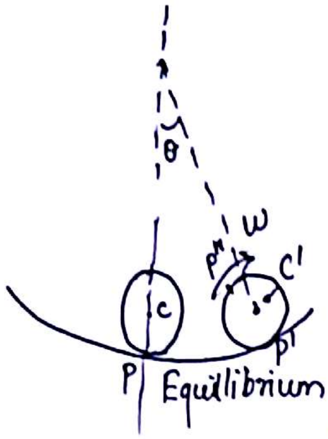
Note that for Rolling w/o slipping friction force act as a Workless/ Wattless Constraints. This System has only one DOF.

Let θ be the ~~geralised~~ generalized Coordinate choosen

Q. What is ω in terms of θ ?

Is $\omega = \dot{\theta}$ Ans. No

To Obtain ω, we may Consider the Velocity of c'



Equilibrium position $\omega = \dot{\phi} - \dot{\theta} = \left(\frac{R}{r} - 1\right)\dot{\theta}$

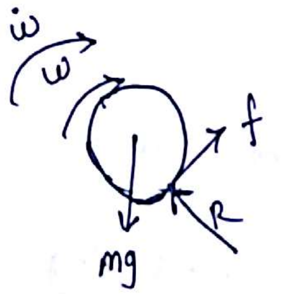
$$V_r = (R-r)\dot{\theta} = r\omega$$

$$\widehat{PP'} = \widehat{P'P''}$$

The Langrange Equation is

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}}\right) - \frac{\partial T}{\partial \theta} + \frac{\partial U}{\partial \theta} = 0$$

Newtons Method



$$+ mg\, r\sin\theta + I_p\dot{\omega} = 0$$

$$T = \text{K.E of System} = \frac{1}{2} m V_{c'}^2 + \frac{1}{2} I_{c'}\omega^2$$

$$= \frac{1}{2} m (r\omega)^2 + \frac{1}{2} I_{c'}\omega^2$$

$$= \frac{3}{4} m r^2 \omega^2$$

$$= \frac{3}{4} m r^2 \left(\frac{R}{r} - 1\right)^2 \dot{\theta}^2$$

$$U = mg(R-r)(1-\cos\theta)$$

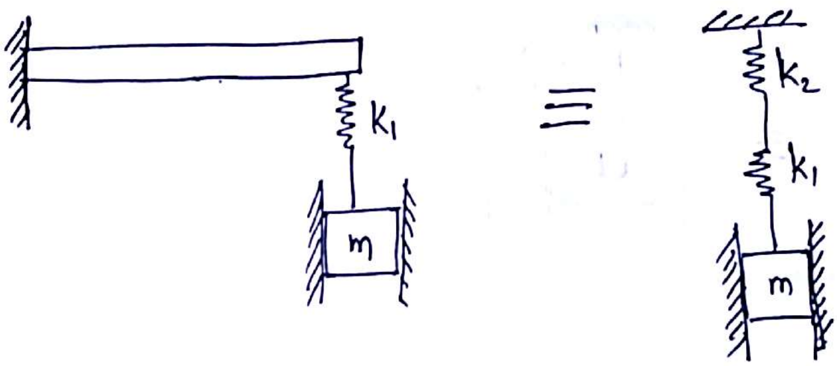
$$\frac{\partial T}{\partial \dot\theta} = \frac{3}{2} m r^2 \left(\frac{R}{r}-1\right)^2 \dot\theta$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot\theta}\right) = \frac{3}{2} m r^2 \left(\frac{R}{r}-1\right)^2 \ddot\theta$$

$$\frac{\partial U}{\partial \theta} = mg\,(R-r)\sin\theta \qquad \frac{\partial T}{\partial \theta} = 0$$

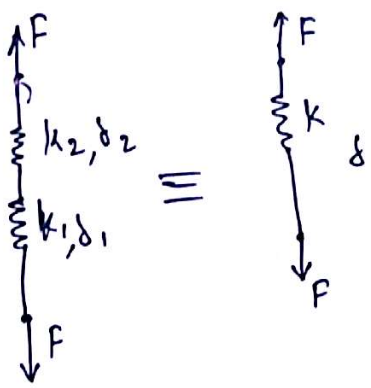
$$\frac{3}{2} \cancel{m} (R-r)^{\cancel{2}}\, \ddot\theta + \cancel{m} g \cancel{(R-r)}\, \cancel{\sin}\, \theta = 0$$

$$\omega_n = \sqrt{\frac{2\cancel{m}g}{3\cancel{m}(R-r)}} = \sqrt{\frac{2g}{3(R-r)}}$$



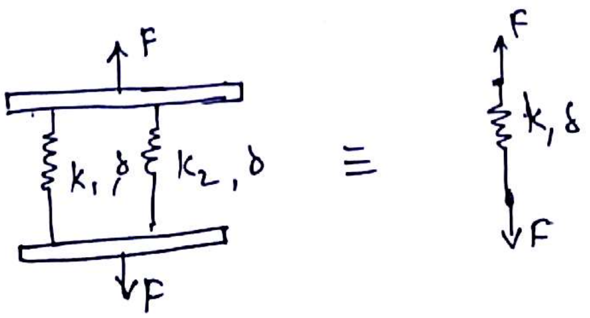
$$\omega_n = \sqrt{\frac{k}{m}}$$

$$\frac{P}{\delta} = \frac{3EI}{l^3} = k_2$$



$$k\delta = k_1\delta_1 + k_2\delta_2$$

$$\delta_1 + \delta_2 = \delta$$



$$k_1 + k_2 = k$$

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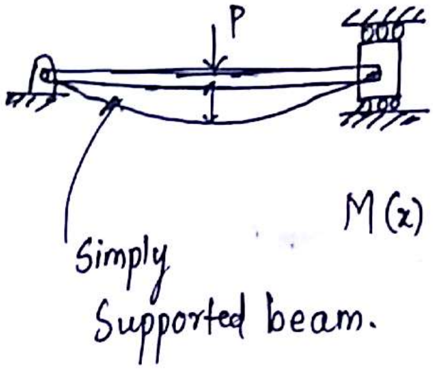
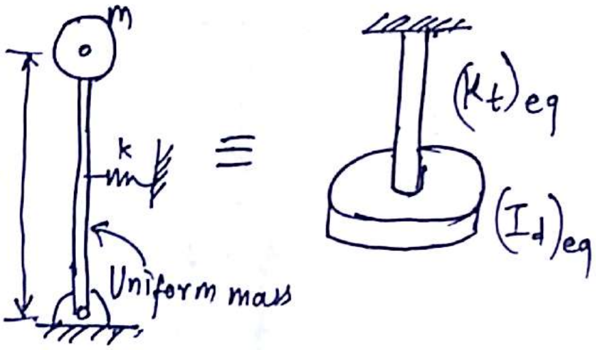
$$\omega_n = \sqrt{\frac{k_1 k_2}{(k_1+k_2)m}}$$



$$\delta_{max} = \frac{P l^3}{48EI}$$

$$M(x) = \pm EI \frac{d^2w}{dx^2}$$

$$I_d \ddot{\theta} + k_t \theta = 0$$



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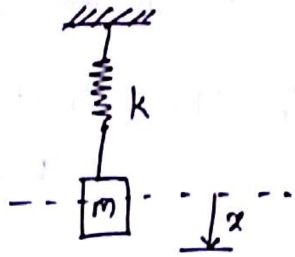
The Rayleigh Method

This Method enables us to Compute the fundamental (natural) frequency of a Vibrationg Mechanical Systems.

For a Conservative System,



$\omega_n = ?$

$$T_{max} = U_{max}$$

Assume

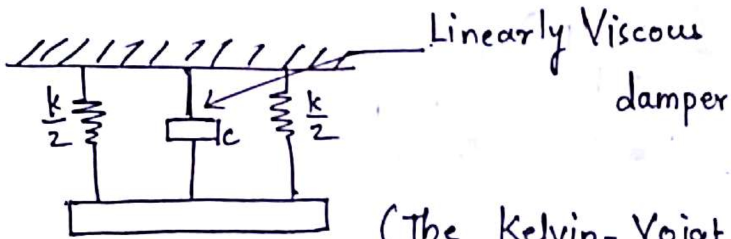
$$x = A\sin(\omega_n t)$$

$$\dot{x} = A\omega_n \cancel{\sin}\cos(\omega_n t)$$

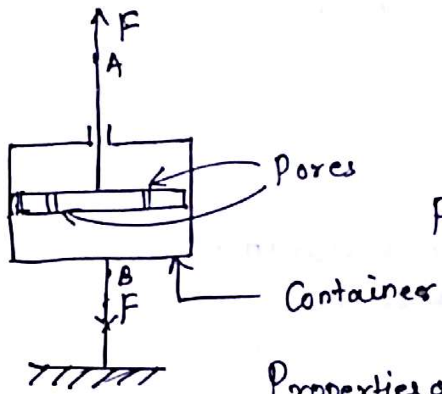
$$T_{max} = \frac{1}{2}m\dot{x}^2 = \frac{1}{2}mA^2\omega_n^2\cos^2\omega_n t$$

$$U_{max} = \frac{1}{2}kx^2 = \frac{1}{2}kA^2\sin^2\omega t$$

$$T_{max} = U_{max} = \frac{1}{2}kA^2$$



(The Kelvin-Voigt Model)



$$F = c\,v_{rel}$$

Properties of damper

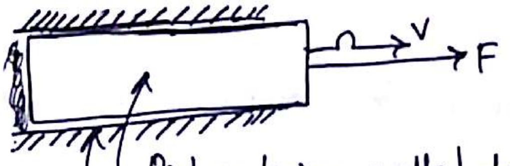
(i) It is Massless

(ii) It is Capable of restricting force only if there is a relative motion between its ends.

v_{rel} = relative Velocity of end A w.r.t End B.

(iii) Its Constitutive Law is $F = c\,v_{rel}$

c = damping Constant of damper



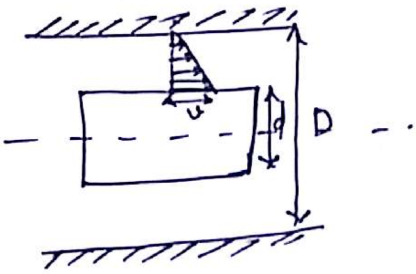
We shall Ignore the Force on Flat ~~plate~~ ~~of~~ End of Cylindrical piston.

Piston being pulled to the right

thin walled Cylinder

<u>Aim</u>: To Show that for laminar flow Conditions

$$F \propto v$$

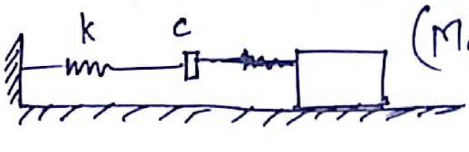


$$\tau = \mu \frac{\partial u}{\partial y} = \mu \frac{v}{\frac{(D-d)}{2}}$$

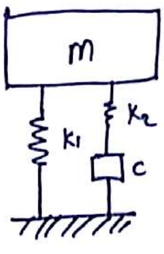
$$\boxed{F = F_{visc} = \pi D l \tau}$$

$$F = \left(\frac{2\pi D l \mu}{(D-d)}\right) v$$

$$\therefore c = \frac{2\pi D l \mu}{(D-d)}$$



(Maxwell Model)

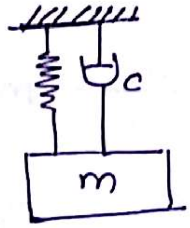


Harri's

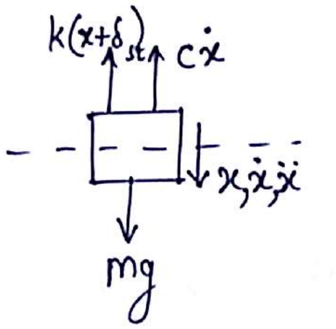
SHock & Vibrations Handbook

The Kelvin-Voigt Model

Aim: To Determine the DEOM of the K-V Model for free Vibrations.



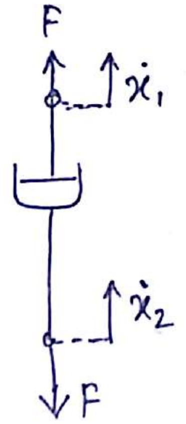
(i) Newton's Method:



$$m\ddot{x} = \Sigma F_x = mg - k(x+\delta_{st}) - c\dot{x}$$

Constitutive Equation of damper

$$c(\dot{x}_1 - \dot{x}_2) = F$$



$$m\ddot{x} + c\dot{x} + kx = 0$$

Method 2 Use of Langrange Equation

(Note that $\frac{d}{dt}(T+U) = 0$ won't work)

The Langrange Equation here will be

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}}\right) - \frac{\partial T}{\partial x} + \frac{\partial U}{\partial x} + \frac{\partial D}{\partial \dot{x}} = 0$$

Where $D = \frac{1}{2} c\dot{x}^2$ = Rayleigh dissipation function

$T = \frac{1}{2} m\dot{x}^2 \quad U = \frac{1}{2} kx^2$

Here $\frac{\partial T}{\partial \dot{x}} = m\dot{x}$ $\quad \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}}\right) = m\ddot{x}$

$$\frac{\partial T}{\partial x} = 0, \quad \frac{\partial U}{\partial x} = kx, \quad \frac{\partial D}{\partial \dot{x}} = c\dot{x}$$

$$m\ddot{x} + c\dot{x} + kx = 0 \quad \text{—— (1)}$$

To Solve (1) for $x(t)$

Let $x = Ce^{st}$ $\quad \dot{x} = Cs\,e^{st}$ $\quad \ddot{x} = Cs^2 e^{st}$

$$ms^2 + cs + k = 0 \quad \text{—(3)}$$

If s_1 & s_2 be Root of (3)

$$s_1 = \frac{-c - \sqrt{c^2 - 4mk}}{2m}$$

$$s_2 = \frac{-c + \sqrt{c^2 - 4mk}}{2m}$$

Here $x = C_1 e^{s_1 t} + C_2 e^{s_2 t}$ —— (4)

C_1, C_2 are arbitary Constant of Integration

The Advantage of Non- Dimensionalisation

Aim: To Non-Dimensionalize $m\ddot{x} + c\dot{x} + kx = 0$ —(1)

We Introduce Non-dimensional Displacement and non-dim time ~~and~~ as

$\bar{x}$ & $\bar{t}$

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Let $\bar{x} = \frac{x}{l}$ l being some characteristic length associated with our System. It Can be the free length.

$\bar{t}$ = non-dim time $= \frac{t}{t_0}$, t_0 being some Characteristic time associated with our System.

$$t_0 = \frac{1}{\omega_n}$$

$$x = \bar{x}\, l$$

$$\dot{x} = \frac{dx}{dt} = l\,\frac{d\bar{x}}{d\bar{t}}\,\frac{d\bar{t}}{dt} = l\omega_n \frac{d\bar{x}}{d\bar{t}}$$

$$\ddot{x} = l\omega_n^2 \frac{d^2\bar{x}}{d\bar{t}^2}$$

substitute in DEOM

$$m\cancel{l}\omega_n^2 \frac{d^2\bar{x}}{d\bar{t}^2} + c\cancel{l}\omega_n \frac{d\bar{x}}{d\bar{t}} + k\cancel{l}\bar{x} = 0$$

$$\cancel{\frac{d^2\bar{x}}{d\bar{t}^2}} \quad \frac{d^2\bar{x}}{d\bar{t}^2} + \frac{c}{m\omega_n}\frac{d\bar{x}}{d\bar{t}} + \frac{\cancel{k}\bar{x}}{\cancel{m\omega_n^2}} = 0 \qquad \text{— (1')}$$

(1') is required momentum DEOM.

Remember

$$\frac{c}{2\sqrt{km}} = \zeta = \text{damping factor} = \frac{c}{2m\omega_n}$$

$$\frac{c}{2m} = \zeta\omega_n$$

$$s_{1,2} = \frac{-c \pm \sqrt{c^2 - 4km}}{2m} = -\frac{c}{2m} \pm \sqrt{\left(\frac{c}{2m}\right)^2 - \omega_n^2}$$

$$s_{1,2} = -\zeta\omega_n \pm \left(\sqrt{\zeta^2 - 1}\right)\omega_n \qquad [\zeta \neq 1]$$

Case (i) $\zeta > 1$: The 'Overdamped' Case (Case of Overdamping)

$$x = A e^{(-\zeta - \sqrt{\zeta^2 - 1})\omega_n t} + B e^{(-\zeta + \sqrt{\zeta^2 - 1})\omega_n t}$$

non-oscillatory motion.

Case (ii) $\zeta = 1$: This is the Case of Critical damping

Here $s_1 = s_2 = -\zeta\omega_n$

& $x = (A + Bt) e^{-\zeta\omega_n t}$

Many Measuring Instruments are Critically damped so are the automatic door closing Mechanism.

Case (iii) $\zeta < 1$ (The Case of Underdamping)

$$s_{1,2} = -\zeta\omega_n \pm j\left(\sqrt{1-\zeta^2}\right)\omega_n \qquad j = \sqrt{-1}$$

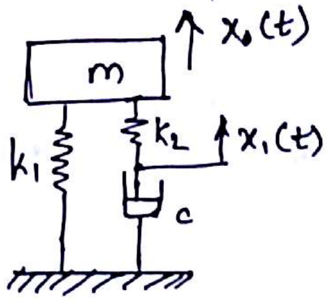
$$= -\zeta\omega_n \pm j\,\omega_d \; ; \quad \omega_d = \omega_n \sqrt{1-\zeta^2} = \text{The damped natural frequency.}$$

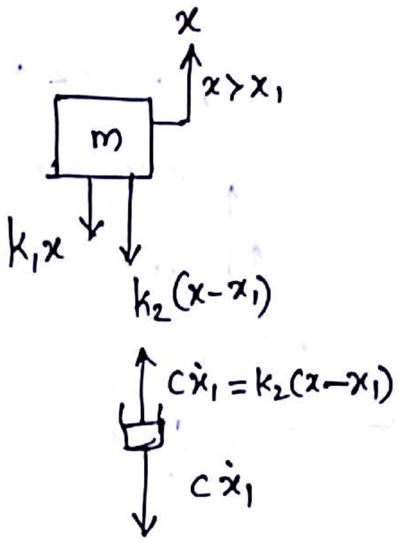
$$x = c_1 e^{s_1 t} + c_2 e^{s_2 t} = e^{-\zeta\omega_n t}\left(c_1 e^{j\omega_d t} + c_2 e^{-j\omega_d t}\right)$$

HW $\boxed{x = X_0 e^{-\zeta\omega_n t} \sin(\omega_d t + \phi)}$ Remember

X_0 & ϕ $\rightarrow$ to be obtained from given $x(0)$ & $\dot{x}(0)$

Ex 1 Obtain the DEOM of the following system.





Obtain 3rd Order DEOM

$$m\ddot{x} + k_1 x + k_2(x - x_1) = 0$$

$$c\dot{x}_1 = k_2(x - x_1) \qquad \therefore \ \frac{c}{k_2}\dot{x}_1 + x_1 = x$$

$$m\left(\frac{c}{k_2}\dddot{x}_1 + \ddot{x}_1\right) + k_1\left(\frac{c}{k_2}\dot{x}_1 + x_1\right) + k_2\left(\frac{c}{k_2}\dot{x}_1 + \cancel{x_1} - \cancel{x_1}\right) = 0$$

$$\frac{mc}{k_2}\dddot{x}_1 + m\ddot{x}_1 + \left(\frac{k_1}{k_2} + 1\right)c\dot{x}_1 + k_1 x_1 = 0$$

The DEOM of System

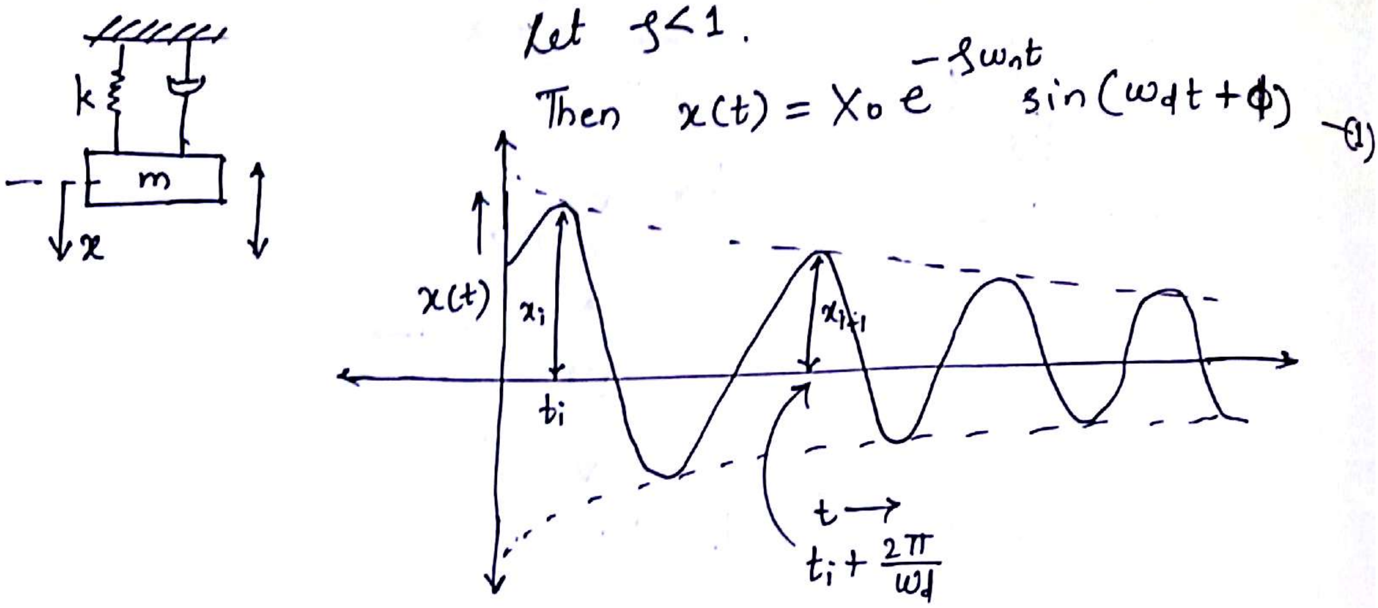
$$\cancel{\frac{mc}{k_2}} \quad \frac{m\ddot{x} + k_2 x + k_1 x}{k_2} = x_1$$

$$c\left(\frac{m\dddot{x} + k_2\dot{x} + k_1\dot{x}}{k_2}\right) = k_2\left(x - \left(\frac{m\ddot{x} + k_2 x + k_1 x}{k_2}\right)\right)$$

$$cm\dddot{x} + (k_2 + k_1)\dot{x} = \cancel{k_2 x} - m\ddot{x} - \cancel{k_2 x} - k_1 x$$

$$cm\dddot{x} + (k_1 + k_2)\dot{x} + m\ddot{x} + k_1 x = 0$$

The Logarithmic Decrement :



Let $\zeta < 1$.

Then $x(t) = X_0 e^{-\zeta\omega_n t} \sin(\omega_d t + \phi)$ —(1)

For Experimental determination of ζ, we use the logarithmic decrement δ

Where $\delta = \ln\left(\frac{x_i}{x_{i+1}}\right) = \ln\left[\frac{e^{-\zeta\omega_n t_i}\cancel{\sin(\omega_d t_i + \phi)}}{e^{-\zeta\omega_n (t_i + \frac{2\pi}{\omega_d})}\cancel{\sin(\omega_d(t_i + \frac{2\pi}{\omega_d}) + \phi)}}\right]$

$$\delta = \ln\left[e^{\frac{2\pi\omega_n\zeta}{\omega_d}}\right] = \frac{2\pi\omega_n\zeta}{\omega_d} = \frac{2\pi\cancel{\omega_n}\zeta}{\cancel{\omega_n}\sqrt{1-\zeta^2}}$$

$$\boxed{\delta = \frac{2\pi\zeta}{\sqrt{1-\zeta^2}}}$$ Will be Obtained Experimentally.

$$\zeta = \frac{\delta}{\sqrt{4\pi^2 + \delta^2}}$$

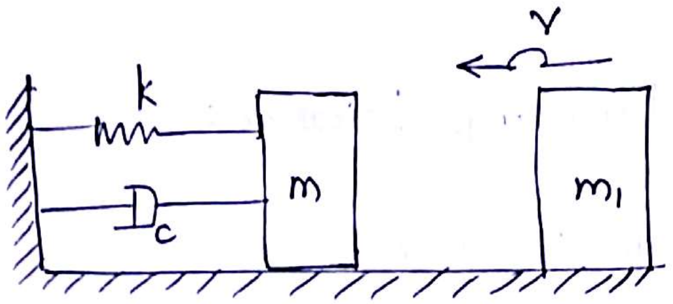
For $\zeta << 1$ (Which happen often)

x_i will be quite close to x_{i+1} & Experimental Error occurs. For Better Accuracy, We often measure $x_i(t)$ & $x_{i+n}(t_i + \frac{2\pi n}{\omega_d})$, where n, a +ve integer, could be as high as 10 or more.

In Such a Situation

$$\delta_n = \ln\left(\frac{x_i}{x_{i+n}}\right) = \frac{2\pi \zeta n}{\sqrt{1-\zeta^2}}$$

Ex1



Assumption: Completely Inelastic Collision
Neglect friction
Conservation of momentum (Not true in Real scenerio as Impulse will be there)

Step 1 : find ζ (Zeta)

Let $\zeta < 1$

$$(m+m_1)\,\dot{x}(0^+) = m_1 V$$

$$x(0) = \int_0^{0^+} \dot{x}\,dt \approx 0$$

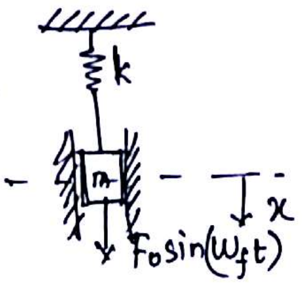
$$\zeta = \frac{c}{\sqrt{(m+m_1)k}}$$

$$\dot{x}(0^+) = \frac{m_1 V}{(m_1+m)}$$

(§) Forced Vibration of 1 DOF System

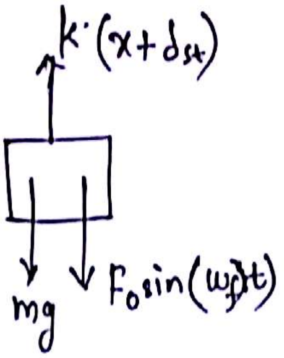
An Undamped 1 DOF System.

$\omega_d = \sqrt{\frac{k}{m}}$



Step 1: Obtain the DEOM.

(a) By Newton's Method.



$$m\ddot{x} = mg + F \sin(\omega_f t) - k(x+\delta_{st})$$

$$m\ddot{x} + k(x) = F_0 \sin(\omega_f t)$$

(b) <u>By Using the Langrange Equation:</u>

The Langrange Eq^n Here is

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}}\right) - \frac{\partial T}{\partial x} + \frac{\partial U}{\partial x} = Q_x$$

Where Q_x is the generalised force Corresponding to general Coordinate x.

$$T = \frac{1}{2} m\dot{x}^2 \qquad U = \frac{1}{2} k x^2$$

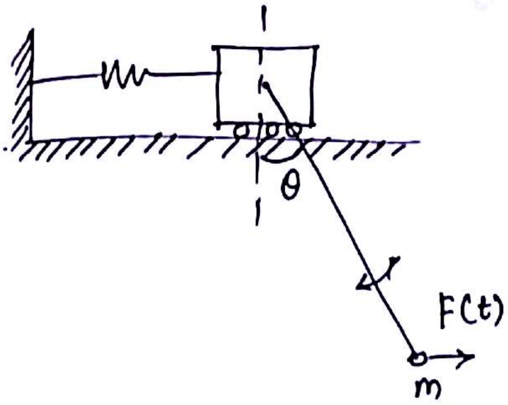
$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}}\right) = m\ddot{x} \qquad \frac{\partial T}{\partial x} = 0 \qquad \frac{\partial U}{\partial x} = kx$$

For Computing Q_i, obtain the Virtual Work done by the applied force Over a Virtual displacement δx of the mass. If δW is this Virtual Work, then

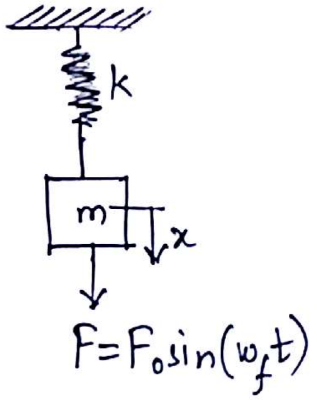
$$Q_1 = \frac{\partial W}{\partial x}$$

$$\delta W = F(t)\, \delta x$$

$$Q_1 = \frac{\delta W}{\delta x} = F(t)$$



Undamped Forced Vibration



DEOM is

$$m\ddot{x} + kx = F_0 \sin(\omega_f t) \quad \text{--- (1)}$$

Here

$$x(t) = \underset{\substack{\text{Complementary} \\ \text{free vibration}}}{x(t)} + \underset{\text{forced/steadystate}}{x(t)}$$

Let $\frac{d}{dt} = D$ Then (1) becomes

$$(D^2 + \omega_n^2)\, x = \frac{F_0}{m} \sin(\omega_f t)$$

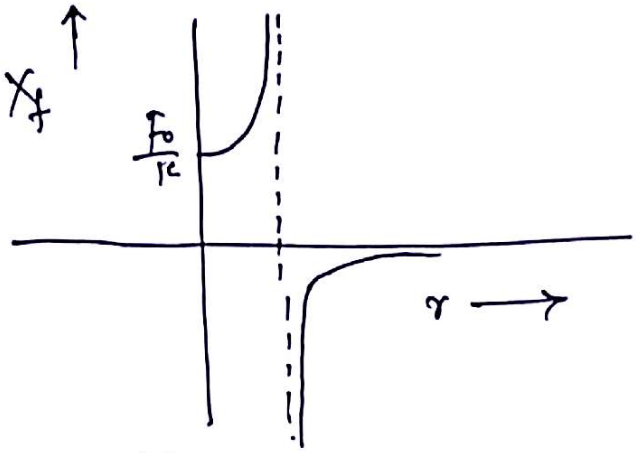
~~parabola~~ $x_{particular} = \frac{F_0}{m} \frac{1}{(D^2 + \omega_n^2)} \sin(\omega_f t)$

If $\omega_f \neq \omega_n$ then

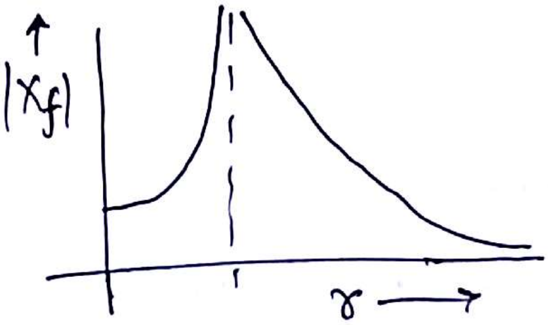
$$x_p = \frac{F_0}{m} \frac{1}{(\omega_n^2 - \omega_f^2)} \sin(\omega_f t) = \frac{F_0/k}{(1-r^2)} \sin(\omega_f t)$$

$$= X_f \sin(\omega_f t)$$

Where $\gamma = \frac{\omega_f}{\omega_n}$ = The frequency ratio.



or



Q What happens when $\omega_f = \omega_n$?

Now the DEOM becomes

$$\ddot{x} + \omega_n^2 x = \frac{F_0}{m} \sin(\omega_n t) \quad \text{—— (2)}$$

We start with a forcing from $\frac{F_0}{m}$ ~~sin~~ $e^{j\omega_n t}$ —(3)

So that are ~~$\frac{F_0}{2m\omega_n}$~~ derived steady state solution ~~with~~ will be the imaginary part of the particular Integral of $\ddot{x} + \omega_n^2 x = \frac{F_0}{m} e^{j\omega_n t}$ —(4)

$$x_p = \frac{F_0}{m} \frac{1}{[D^2+\omega_n^2]} e^{j\omega_n t}$$

$$= \frac{F_0}{m} \left[\frac{1}{(D+j\omega_n)(D-j\omega_n)}\right] e^{j\omega_n t}$$

$$= \frac{F_0}{2mj\omega_n} \frac{1}{(D-j\omega_n)} e^{j\omega_n t}$$

$$= \frac{F_0}{2mj\omega_n} e^{j\omega_n t} \int \underbrace{e^{-j\omega_n t} e^{j\omega_n t}}\, dt$$

$$= \frac{F_0 t}{2mj\omega_n} e^{j\omega_n t}$$

$$y = \frac{1}{(D+P)} Q \qquad \frac{dy}{dx} + P(x) y = Q(x)$$

IF $\quad e^{\int P dx}$

$$\int d\left[y\, e^{\int P dx}\right] = \int e^{\int P dx} Q(x)\, dx$$

$$y = e^{-\int P dx} \int e^{\int P dx} Q(x)\, dx$$

So at $\omega_n = \omega_f$ for the forced Response is given as

$$x_p = x_{forced} = x_{ss} = -\frac{F_0 t}{2m\omega_n} \cos(\omega_n t)$$

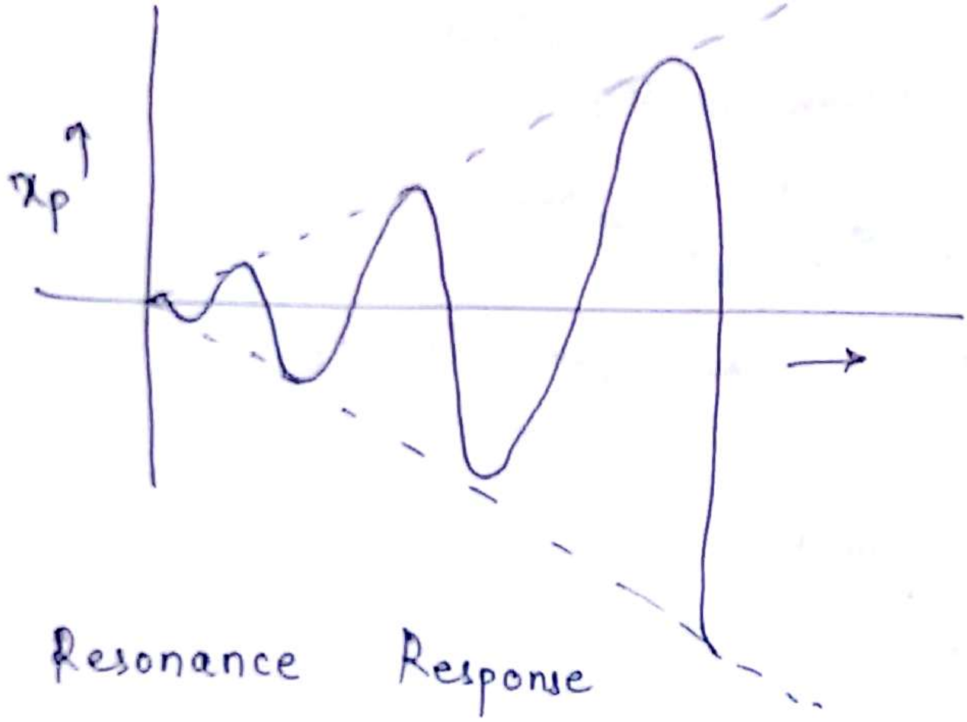
$$x_p = -\frac{jF_0 t}{2m\omega_n} (\cos\omega_n t + j\sin\omega_n t)$$

of DE4

So, required $x_p = I_m$ (x_p of DE 4)

~~So required $x_p =$ I~~

$$= -\frac{F_0 t}{2m\omega_n} \cos(\omega_n t)$$



Resonance Response

Hence total Response $x(t) = x_0 \sin(\omega_n t + \phi)$ + (S.S) Responce.

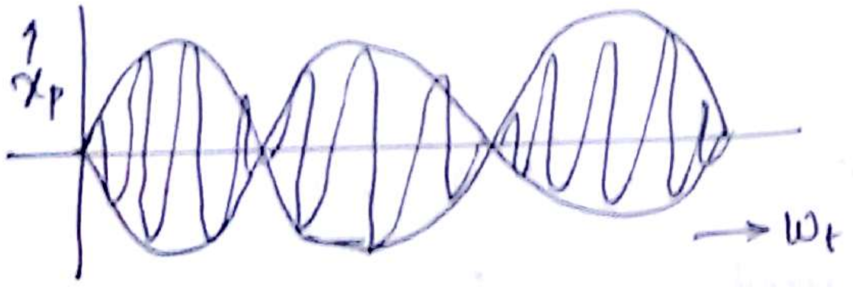
$$= x_0 (A \sin(\omega_n t) + B \sin(\omega_n t)) + \left(\frac{F_0/K}{1-r^2}\right) \sin(\omega_f t)$$

So, ~~what $\omega_f = \omega_n$ the forced~~ response is given as $x_p = x_{force} = x_{ss} = -\frac{F_0 t}{2m\omega_n} \cos \omega_n t$

So phenomenon of Beating Occures

History of Mechanics - Truesdell et al.

Q Obtain the forced Responce which gives beats formation



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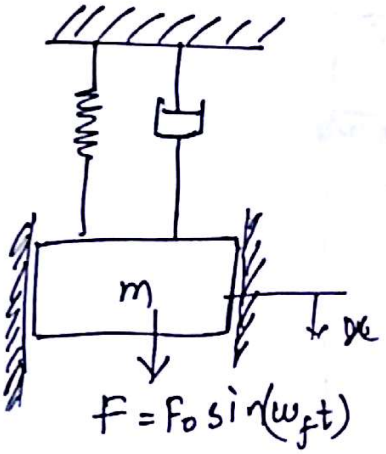
Forced Vibration of Damped System



The DEOM is

$$m\ddot{x} + c\dot{x} + kx = F_0 \sin(\omega_f t)$$

$$\ddot{x} + 2\zeta\omega_n \dot{x} + \omega_n^2 x = \frac{F_0}{m}\sin(\omega_f t)$$

$$x_c = X_0 e^{-\zeta\omega_n t} \sin(\omega_d t + \phi)$$

$$\boxed{x_p = \frac{F_0/k}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}} \sin(\omega_f t \pm \psi)}$$

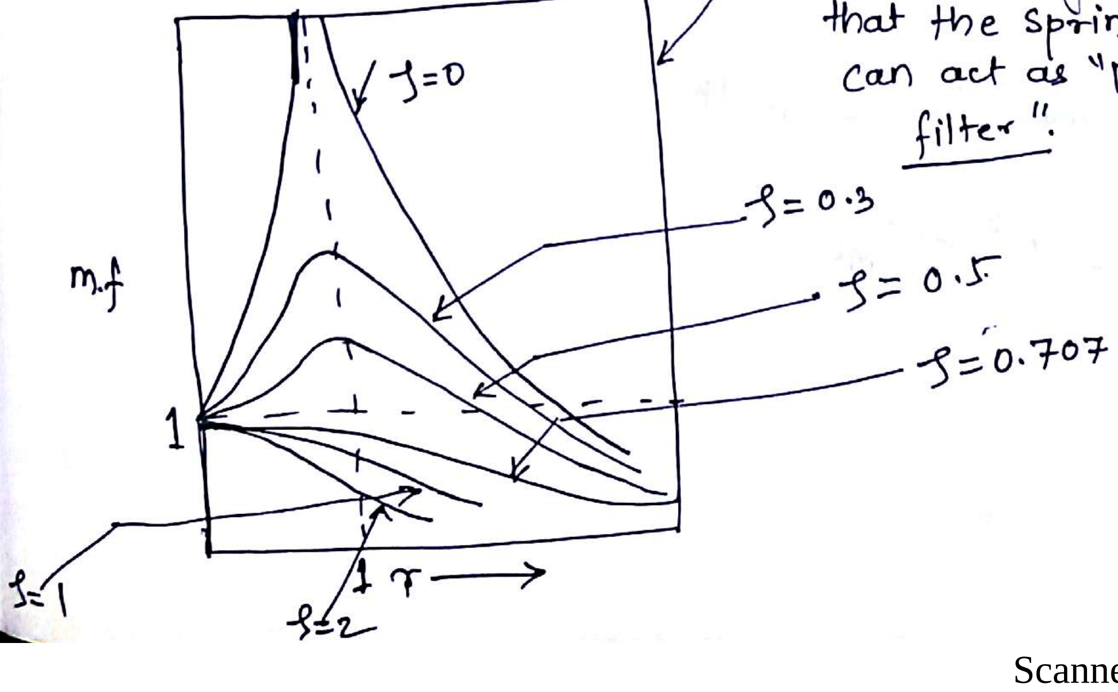
Or x_{ss}

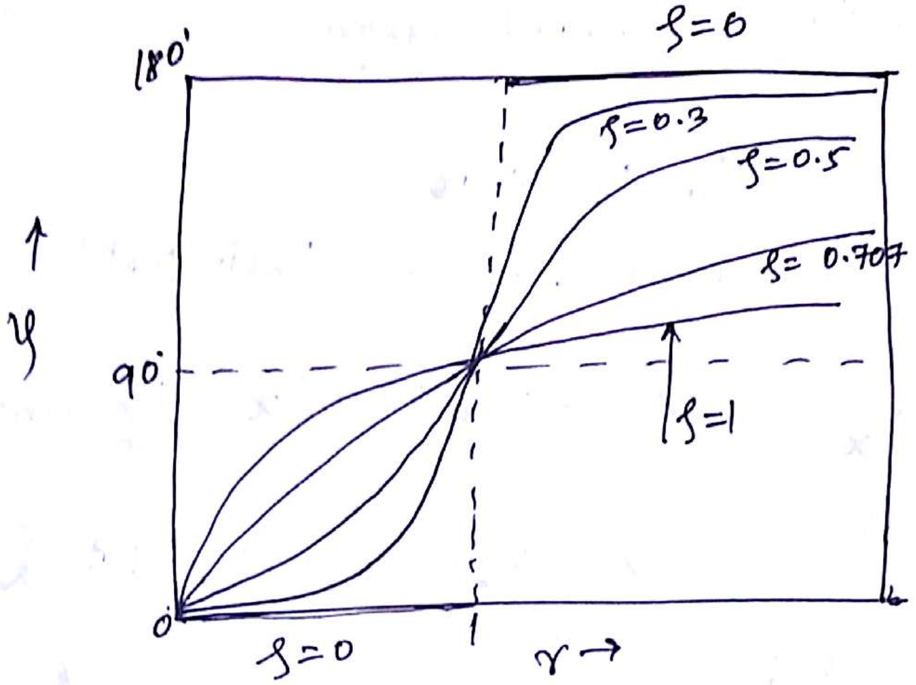
$$\psi = \tan^{-1}\left(\frac{2\zeta r}{1-r^2}\right)$$

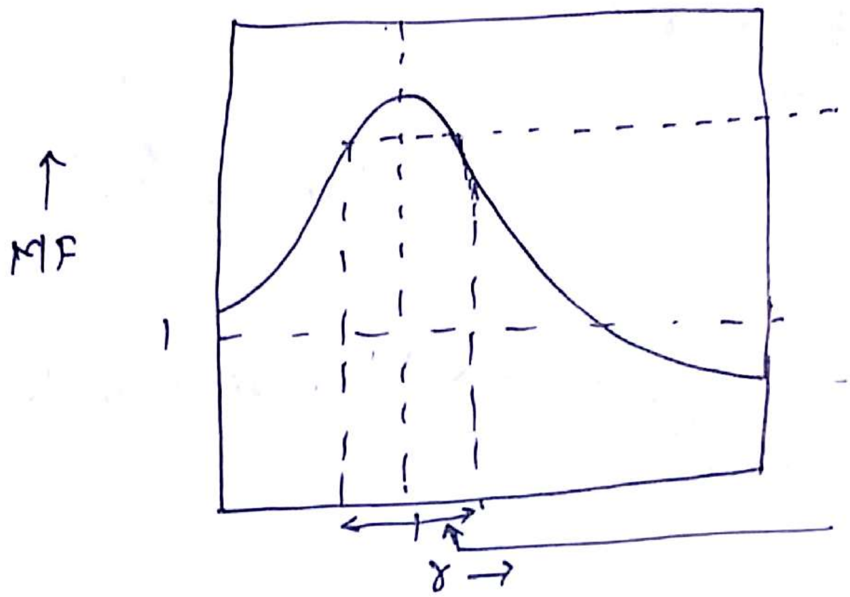
$$\frac{1}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}} \longrightarrow \text{Magnification factor}$$

$$\zeta = \frac{c}{2m\omega_n}$$

This figure shows that the spring mass system can act as "Mechanical filter".





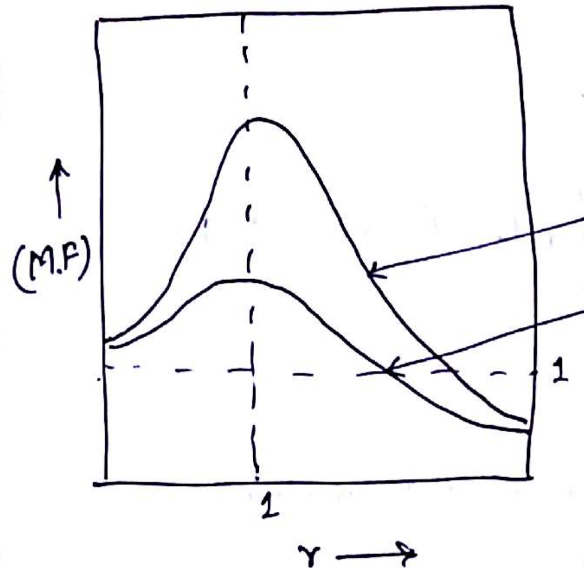


$(MF)_{max} \rightarrow$ Amplitude Resonance

$$20 \log \left(\frac{(MF)_{max}\, 0.707}{(MF)_{max}} \right) \simeq -3$$

$$10 \log \left(\frac{P}{P_0} \right) = (\text{in dB})$$

$$P_0 = 10^{-12}\ \text{W/cm}^2$$



H.W.

Set

$$\left.\frac{\partial (MF)}{\partial r}\right|_{\xi=\xi_1} = 0$$

find Where MF is a max or a min.

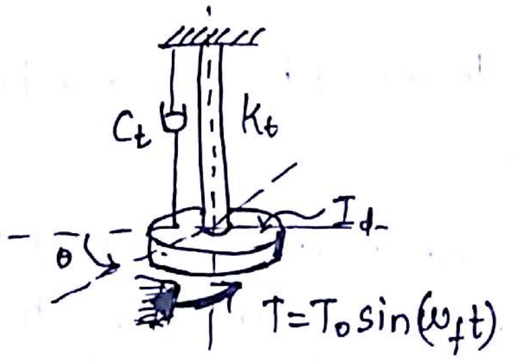
Amplitude Resonance Occurs at

$$r = \sqrt{1-2\xi^2} \qquad \left(\xi < \frac{1}{\sqrt{2}}\right)$$

If $\xi \ll 1$

$$(MF)_{max} \approx \left(\frac{1}{2\xi}\right) \longleftarrow \text{The } Q \text{ Factor}$$

- Phase Resonance Occurs at $r=1$ $(\phi = 90^\circ)$



DEOM is

$$I_d \ddot{\theta} + C_t \dot{\theta} + K_t \theta = T_0 \sin(\omega_f t)$$

$$\theta_c(t) = H_1 \sin(\omega_n t) + H_2 \cos(\omega_n t)$$

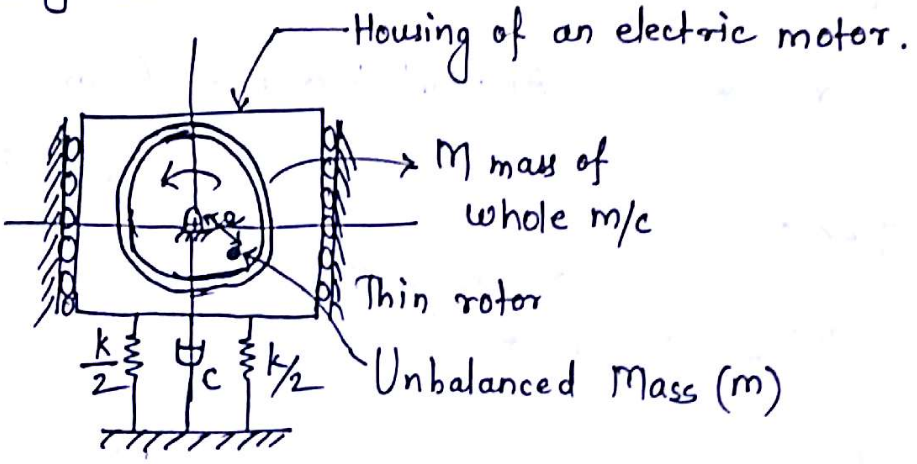
$$\theta_{ss} = \frac{T_0/K_t}{\sqrt{(1-r^2)^2 + (2\xi r)^2}} \sin(\omega_f t - \phi)$$

$$\theta = \theta_c + \theta_{ss} \qquad \xi = \frac{c}{c_c} = \frac{c}{2m\omega_n}$$

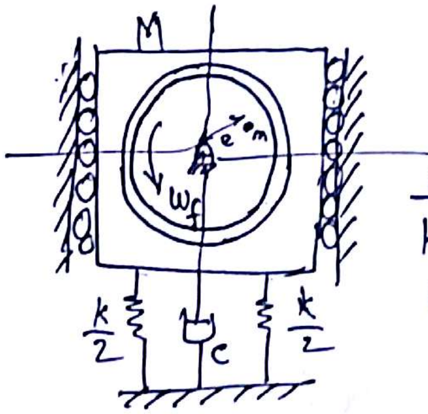
$$\omega_n = \sqrt{\frac{K_t}{I_d}} \qquad \zeta = \frac{C_t}{2 I_d \omega_n}$$

$C_c \rightarrow$ Critical damping factor (Corr. $\zeta = 1$)

⑥ Rotating Unbalance



Q



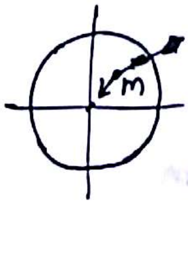
How do we get m for a Real thin Rotor ?

Ans. We never get m experimentally, However we Condition the product (e, m) which is Called the Unbalanced U.

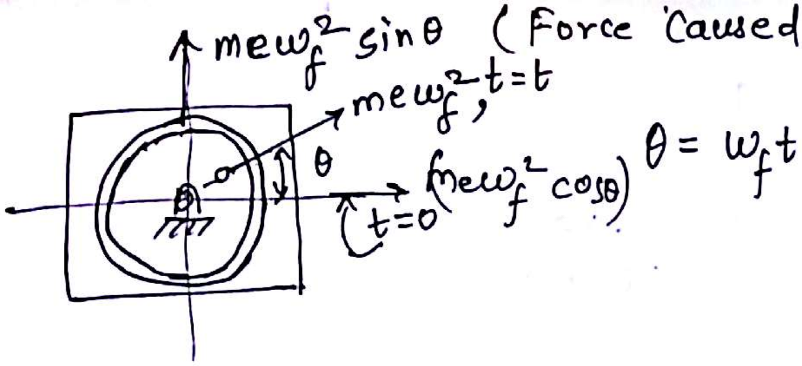
(U = me = Unbalancy)

$me = 10^{-3}$ gm cm

Let θ is measured from the right hand horizontle line through Center of Rotor, +ive CCW.

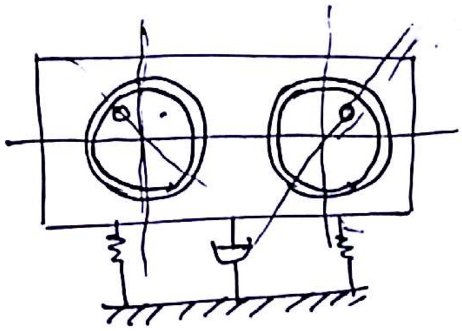


$me\omega_f^2$ = ~~mass~~ of Centri~~fugal~~petal force on m.

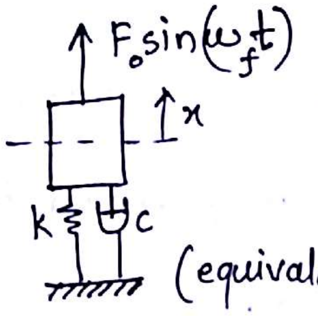


(Force Caused by Unbalanced mass on Housing)

$\theta = \omega_f t$



A shaker



$F_0 = me\omega_f^2$

$\omega_n = \sqrt{\dfrac{k}{M}}$

(equivalent system)

$$x = x_c + x_{ss}$$

$$x_{ss} = \frac{F_0/k}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}} \sin((\omega_f t) - \varphi)$$

$$x_{ss} = \frac{me\omega_f^2/k}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}} \sin(\omega_f t - \varphi)$$

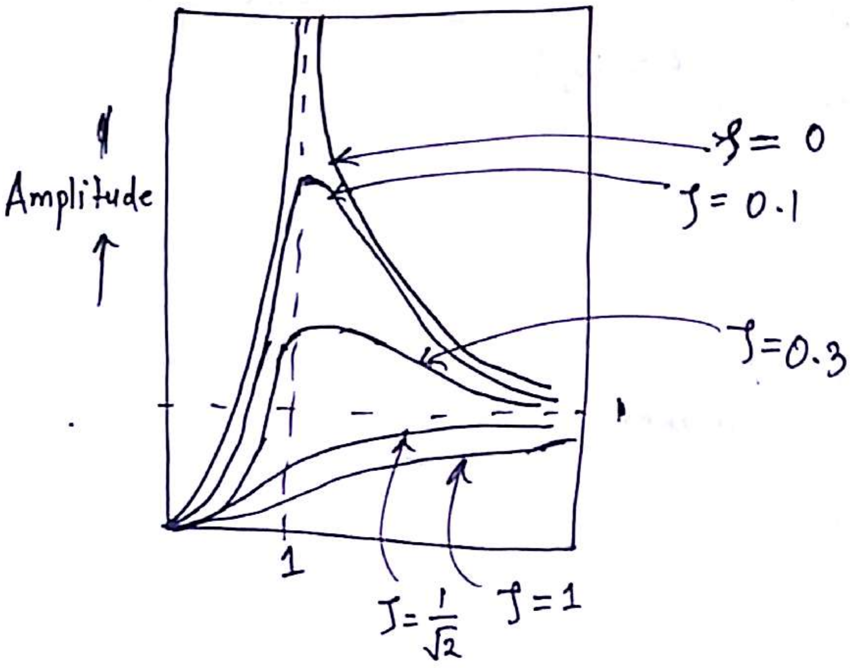
$$\varphi = \tan^{-1}\left(\frac{2\zeta r}{1-r^2}\right)$$

$r = \dfrac{\omega_f}{\omega_n}$ So, $\dfrac{me\,\omega_f^2/M}{k/M} = \dfrac{U\,\omega_f^2/M}{\omega_n^2}$

$= \dfrac{U}{M} r^2$

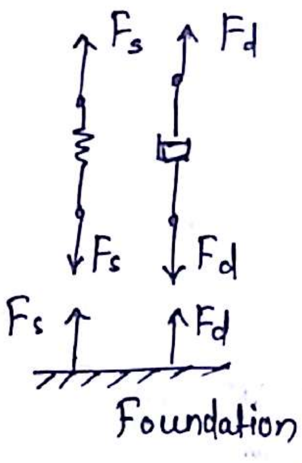
So

$$x_{ss} = \frac{\frac{U}{M} r^2}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}} \sin(\omega_f t - \varphi)$$



We are Interested in finding the Characteristic of the force transmitted to the m/c foundation by the Spring & damper Combinations.

$F_A = kx$ Hence F_T = force transmitted to foundation



$F_s = kx$

$F_d = c\dot{x}$

Hence Net transmitted force to foundation is

$F_s + F_d = kx + c\dot{x}$

$$x = \frac{F_0/k}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}} \sin(\omega_f t - \varphi)$$

$$\dot{x} = \frac{F_0/k \;\omega_f}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}} \cos(\omega_f t - \varphi)$$

$$F_{trans} = \frac{F_0/k}{\sqrt{(1-r^2)^2+(2\xi r)^2}} \left(\sin(\omega_f t-\psi)\, k + c\,\omega_f \cos(\omega_f t-\psi) \right)$$

$$F = f_T \sin(\omega_f t-\psi-\beta)$$

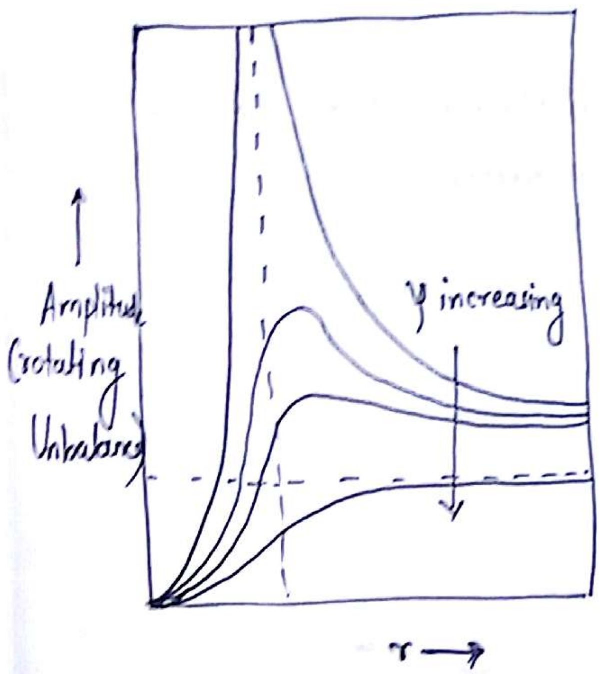
$$F_{trans} = \frac{F_0/k}{\sqrt{(1-r^2)^2+(2\xi r)^2}} \; \frac{\sqrt{k^2+c^2\omega_f^2}}{\cancel{\sqrt{k^2+c^2\omega_f^2}}} \sin(\omega_f t-\psi+\beta)$$

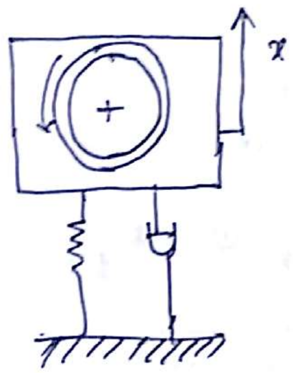
$$f_T = \frac{F_0/k \sqrt{k^2+c^2\omega_f^2}}{\sqrt{(1-r^2)^2+(2\xi r)^2}}$$

$$F_T = \frac{F_0\sqrt{1+(2\xi r)^2}}{\sqrt{(1-r^2)^2+(2\xi r)^2}} \sin(\omega_f t-\psi+\beta)$$ (Force transmitted to foundation)

$$\underline{\beta = \tan^{-1}(2\xi r)}$$

$$\frac{c\,\omega_f}{k} = 2\xi r$$

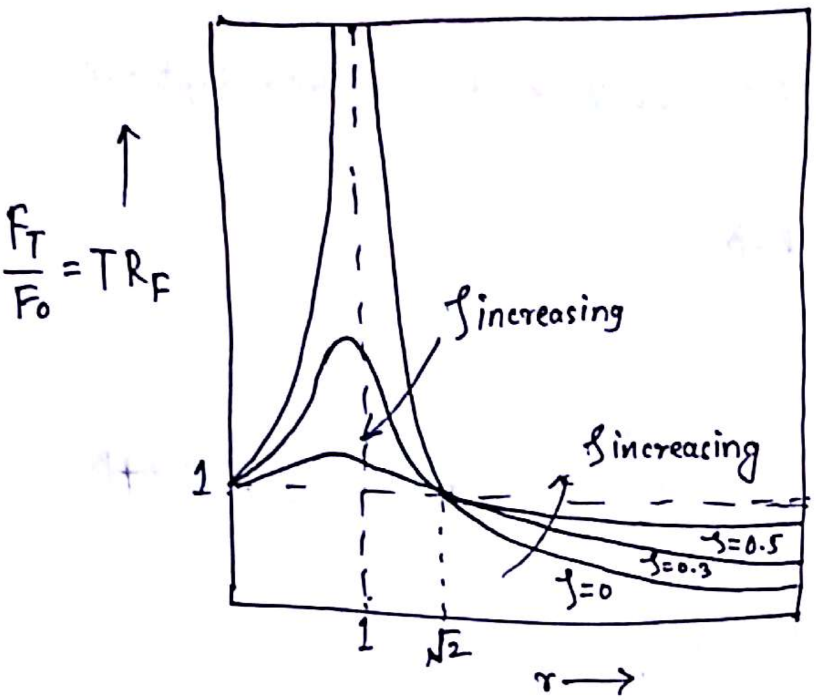




Definition:

Force Transmissibility TR_F = $\dfrac{\text{Amplitude of force Transmitted}}{\text{Amplitude of Applied force}}$

$$\frac{F_T}{F_0} = \frac{\sqrt{1+(2\xi r)^2}}{\sqrt{(1-r^2)^2+(2\xi r)^2}}$$

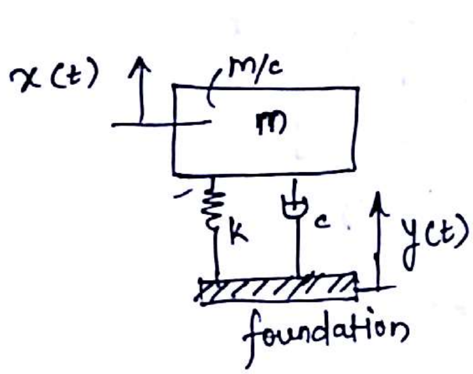


The Smaller the Value of ζ, the better the force isolation force Isolation (for $r > \sqrt{2}$)

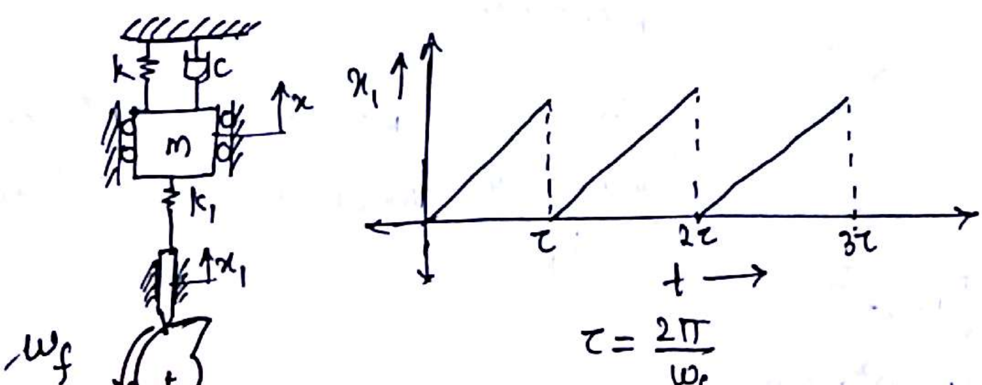
A larger ζ means Smaller amplitude of Vibration for the m/c ($r > \sqrt{2}$).

To keep the Vibration Amplitude of the m/c to a required minimum & also to obtain a better force isolation, a design has to make a Compromise.

(c) Base Excitation:

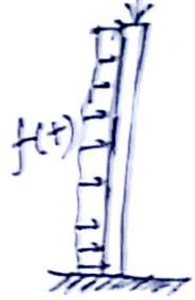


$y(t) \rightarrow$ displacement of the m/c foundation.



$$\tau = \frac{2\pi}{\omega_f}$$

$F(t) = F_0 + F_1 (\cos \omega_f t)$



$$\ddot{x} + (\alpha + \beta \cos \omega_f t) x = f(t)$$

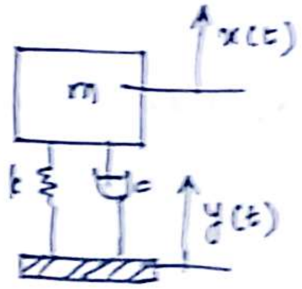
Mathieu Equations

Buckling of Column

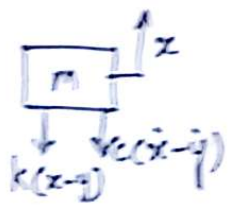
$n = 1$ principle Resonances

$n > 1$ Higher order Resonances

$$\omega_f = \frac{\omega_a \pm \omega_b}{n}$$ parametric Resonances ..



H.W. Let $x > y$, $\dot{x} > \dot{y}$,



$$m\ddot{x} = -k(x-y) - c(\dot{x}-\dot{y})$$

$$m\ddot{x} + kx + c\dot{x} = ky + c\dot{y} \quad \text{—— (1)}$$

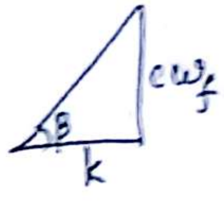
Let $y(t) = Y_0 \sin(\omega_f t)$

$$m\ddot{x} + c\dot{x} + kx = Y_0 \left(k \sin(\omega_f t) + c\omega_f \cos(\omega_f t) \right)$$

$$= Y_0 \left(\frac{k}{\sqrt{k^2 + (c\omega_f)^2}} \sin \omega_f t + \frac{c\omega_f}{\sqrt{k^2 + (c\omega_f)^2}} \cos(\omega_f t) \right)$$

$$= Y_0 k \sqrt{1 + (2\zeta r)^2} \sin(\omega_f t + \beta) \; ; \quad \beta = \tan^{-1}(2\zeta r)$$

$$\frac{c\omega_f}{k} = 2\zeta r \qquad \zeta = \frac{c}{c_c}$$



$c_c \rightarrow$ Critical Damping Constant.

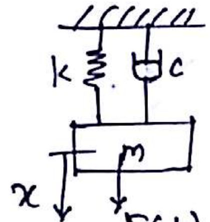
$$x(t) = x_{ss}(t) = \frac{Y_0 \sqrt{1+(2\zeta r)^2}}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}} \sin(\omega_f t - \psi + \beta)$$

Amplitude of ~~fourd~~ forced Vibration of Our System due to Base Excitation.

$$\psi = \tan^{-1}\left(\frac{2\zeta r}{1-r^2}\right)$$

$$TR_m = \text{Motion transmissibility} = \frac{\text{Amplitude of force Vibration}}{\text{Amplitude of base Excitation}} = \frac{\sqrt{1+(2\zeta r)^2}}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}}$$

(8) Periodic Excitation:



We Express F(t) as Fourier Series

$$F(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos(n\omega_f t) + b_n \sin(n\omega_f t)\right)$$

$$F(t) = F(t+\tau) \qquad \tau = \frac{2\pi}{\omega_f}$$

$$a_n = \frac{2}{\tau}\int_0^{\tau} F(t)\cos(n\omega_f t)\,dt \; ; \; n = 0, 1, \ldots$$

$$b_n = \frac{2}{\tau}\int_0^{\tau} F(t)\sin(n\omega_f t)\,dt \; ; \; n = 1, 2, \ldots$$

$$m\ddot{x} + c\dot{x} + kx = \frac{a_0}{2}$$

x_{ss} due to $\frac{a_0}{2}$ is $\frac{a_0}{2k}$

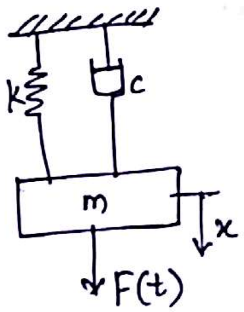
x_{ss} due to $a_n \cos(\omega_f n t)$ is $\dfrac{(a_n/k) \times \cos(n\omega_f t - \psi_n)}{\sqrt{(1-r_n^2)^2 + (2\zeta r_n)^2}}$

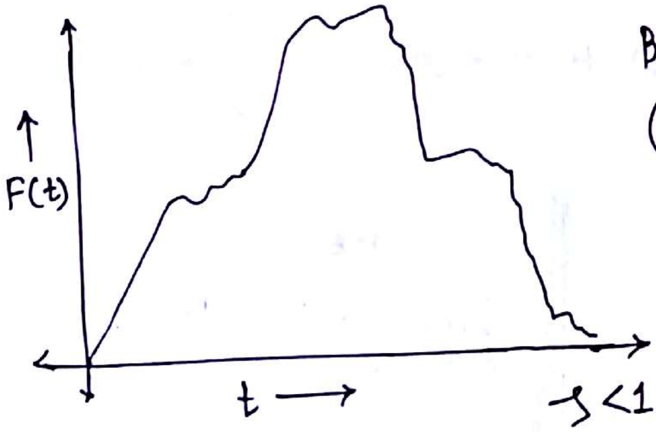
$$r_n = \frac{n\,\omega_f}{\omega_n} \qquad \psi_n = \tan^{-1}\left(\frac{2\zeta r_n}{1-r_n^2}\right)$$

x_{ss} due to $b_n \sin(n\omega_f t)$ is $\dfrac{(b_n/k) \times \sin(n\omega_f t - \psi_n)}{\sqrt{(1-r_n^2)^2 + (2\zeta r_n)^2}}$

By Principle of Superposition,

$$x_{ss} = \frac{a_0}{2k} + \sum_{n=1}^{n} \frac{1}{k\sqrt{(1-r_n^2)^2 + (2\zeta r_n)^2}} \left[a_n \cos(n\omega_f t) + b_n \sin(n\omega_f t)\right]$$





Brüel & Kjaer (Vibration Monitoring Instruments)

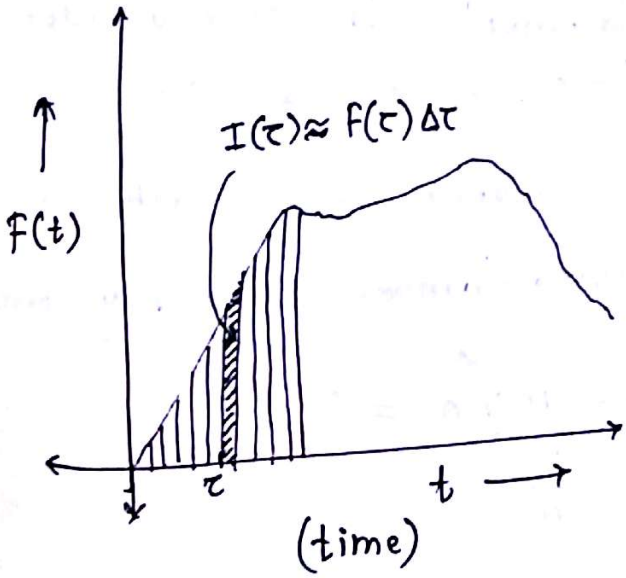
$\zeta < 1$

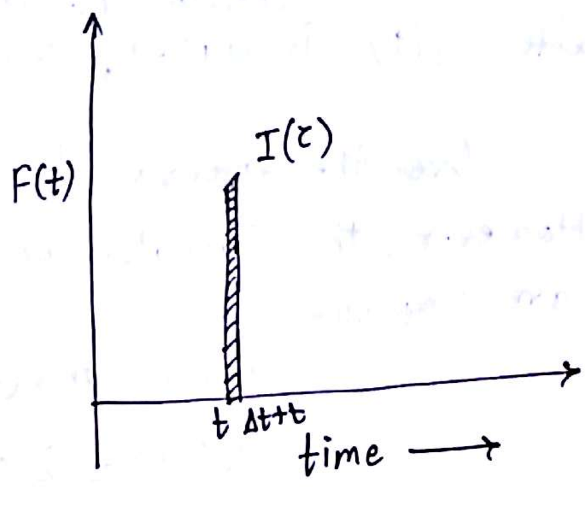
$$\boxed{x(t) = \int_0^t F(\tau)\, g(t-\tau)\, d\tau}$$ → The Duhamel Integral / Convolution Theorem.

Where $g(t)$ is the Impulse Response function.

$x(t)$ = forced Response of the K-V Model Under the Action of a force $F(t)$ Which can have any Complex form.

The force acts for $t \geq 0$ & we have



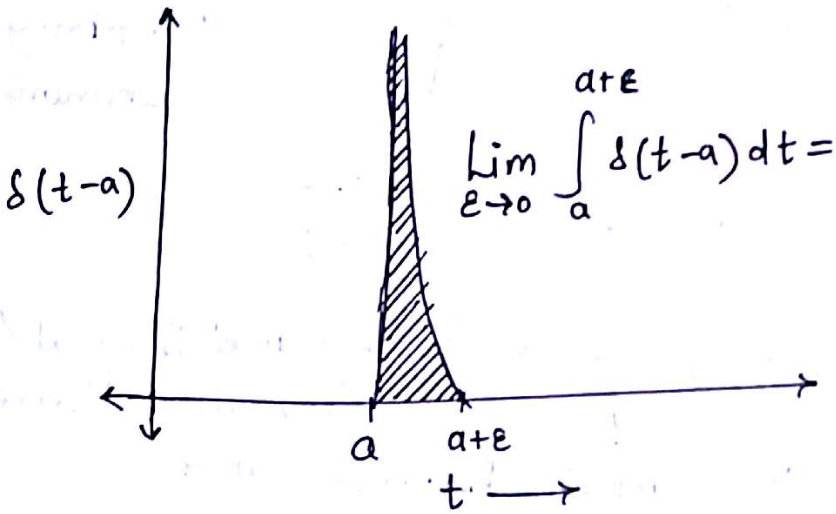


The Dirac – Delta Function (Unit Impulse Fuction)

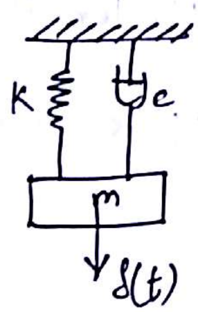
$$\int_0^\infty \delta(t)\,dt = 1 \qquad \delta(t) = 0 \text{ for } t \neq 0$$

Impulse of $\delta(t)$

$$\int_0^\infty f(t)\,\delta(t)\,dt = f(0) \qquad \int_0^\infty f(t)\,\delta(t-a)\,dt = f(a)$$



$$\lim_{\varepsilon \to 0} \int_a^{a+\varepsilon} \delta(t-a)\,dt = 1$$



$x(0) = 0 \qquad \dot{x}(0) = 0$

$\delta(t)$ will Produce, as $[0, \varepsilon]$ $(\varepsilon \to \overset{+}{0})$ a finite Velocity of the Mass but it won't Produce any Considerable displacement. In Other words, after $\delta(t)$ is applied, $x(0^+) = 0$, $\dot{x}(0^+) \to$ finite.

Over the Interval $[0, \varepsilon]$ Several forces acts on body However, the Impulses of Spring damper & gravity forces are negligible.

Hence, $mv(0^+) - m\cancelto{0}{y(0)} = 1$

$$V(0^+) = \dot{x}(0^+) = \frac{1}{m}$$

$$x(0^+) = \lim_{\epsilon \to 0^+} \int_0^{\epsilon} \dot{x}(t)\, dt = \lim_{\epsilon \to 0^+} \left[\dot{x}(\epsilon_1)\, \epsilon \right] = 0$$

$$\boxed{\begin{aligned} x(0^+) &= 0 \\ \dot{x}(0^+) &= \frac{1}{m} \end{aligned}}$$

$$m\ddot{x} + c\dot{x} + kx = F(t)$$

$$x_{ss} \qquad\qquad \downarrow$$

$$\alpha x_{ss} \qquad\qquad \alpha F(t)$$

Let $\zeta < 1$

Then $\quad x(t) = X_0 e^{-\zeta \omega_n t} \sin(\omega_d t + \phi)$

$$x(0^+) = 0 \Rightarrow \phi = 0$$

$$\dot{x}(t) = -X_0 \zeta \omega_n e^{-\zeta \omega_n t} \sin(\omega_d t) + X_0 \omega_d e^{-\zeta \omega_n t} \cos(\omega_d t)$$

$$\dot{x}(0^+) = \frac{1}{m} \Rightarrow X_0 \omega_d = \frac{1}{m}$$

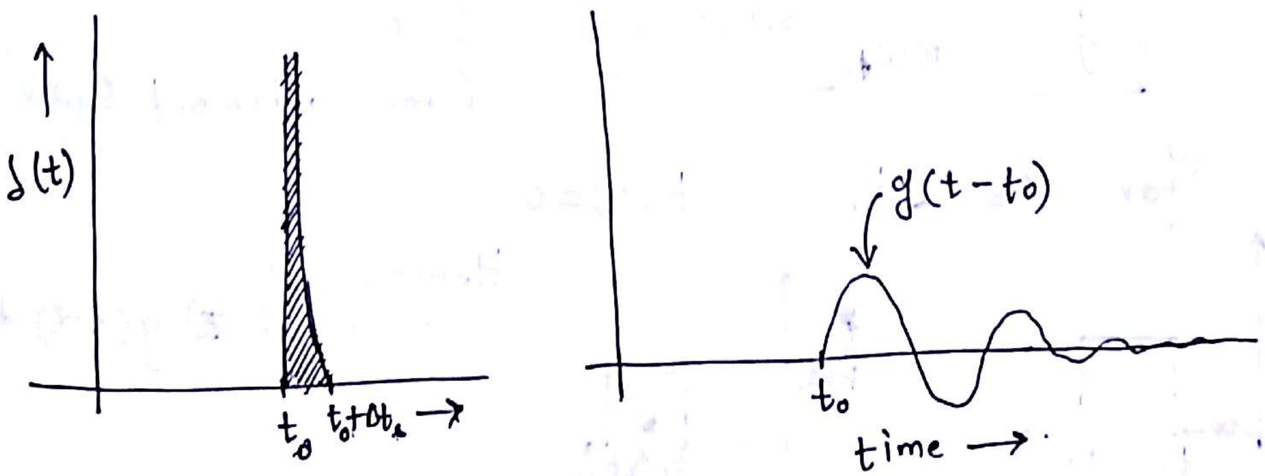
$$X_0 = \frac{1}{m\omega_d}$$

Hence

$$\boxed{x(t) = \frac{1}{m\omega_d} e^{-\zeta \omega_n t} \sin(\omega_d t)}$$

$\downarrow$

$g(t)$, the Impulse Response function.



$$\boxed{\begin{aligned} g(t-t_0) &= 0 \quad \text{for } t<t_0 \\ g(t-t_0) &= \frac{1}{m\omega_d} e^{-\xi\omega_n(t-t_0)} \sin(\omega_d(t-t_0)) \end{aligned}}$$

$g(t-t_0) \rightarrow$ The Response of the System ~~of~~ due to unit Impulse excitation applied at $t=t_0$

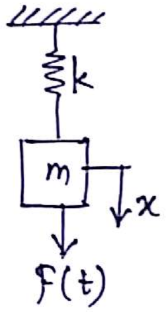
This $I(t_0)$ generates a Response of

$$I(t_0)\, g(t-t_0) = F(t_0)\, g(t-t_0)\, \Delta t_0$$

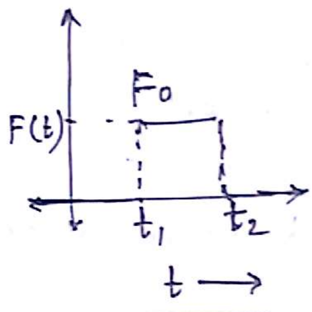
The Total Response at time t Can be Obtained by the Principle of Super position & will be obiviously given by (as $\Delta t_0 \rightarrow 0$)

$$\boxed{x(t) = \int_0^t F(\tau)\, g(t-\tau)\, d\tau}$$

$$= \int_0^t F(t-\tau)\, g(\tau)\, d\tau$$



$\xi = 0$

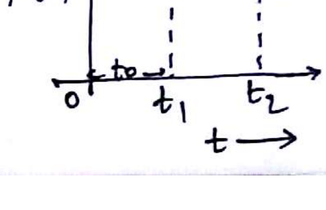


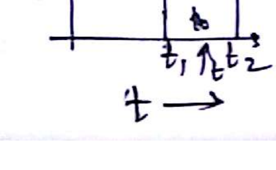
Find $x(t)$ for $t>0$ Using Duhamels Integral's

$$\boxed{g(t) = \frac{1}{m\omega_n} \sin(\omega_n t)}$$

$\xi = 0$
(for Undamped System)

For $0 \leq t \leq t_1$ $\quad F(t) = 0$





Hence.
$$x(t) = \int_0^t F(\tau)\, g(t-\tau)\, d\tau$$

For $t_1 \leq t \leq t_2$

$F(t) = F_0$

Hence

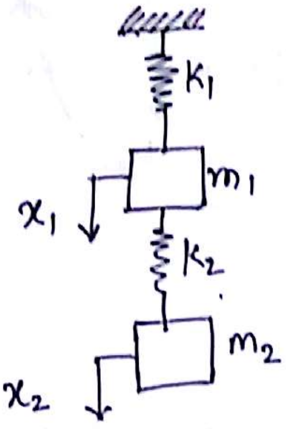
$$x(t) = \int_0^{t_1} \cancel{\cdots}^{\,0} + \int_{t_1}^{t_2} F(\tau)\, g(t-\tau)\, d\tau + \int_{t_2}^{t} \cancel{\cdots}^{\,0}$$

for $t > t_2$

$F(t) = 0$

$$x(t) = \int_0^{t_1} \cancel{\cdots}^{\,0} + \int F_0 \frac{1}{m\omega_n} \sin(\omega_n (t-\tau))\, d\tau$$

Two DOF Systems

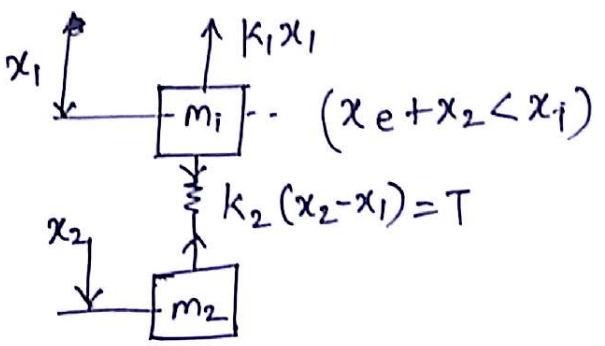


Undamped free Vibration of 2 DOF Systems

$x_1, x_2 \rightarrow$ A Set of generalised 'Coordinates, measured from Static equillibrium positions.

Derivation of DEOM:

1) Newton's Method:



① and ② Can be put in the matrix form:

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} K_1 + K_2 & -K_2 \\ -K_2 & K_2 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$[m]\{\ddot{x}\} + [K]\{x\} = \{0\} \quad — ③$$

So $m_1 \ddot{x}_1 = -k_1 x_1 + k_2(x_2 - x_1) \Rightarrow m_1 \ddot{x}_1 + (k_2 + k_1)x_1 - k_2 x_2 = 0$

$m_2 \ddot{x}_2 = -k_2(x_2 - x_1) \Rightarrow m_2 \ddot{x}_2 - k_2 x_1 + k_2 x_2 = 0$

$[m] = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}$ is the mass / Inertia matrix

$[K] = \begin{bmatrix} K_1+K_2 & -K_2 \\ -K_2 & K_2 \end{bmatrix}$ is the Stiffness or Elastic Matrix

$\{x\} = \begin{Bmatrix} x_1(t) \\ x_2(t) \end{Bmatrix}$ is the displacement Vector

$\{\ddot{x}\} = \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix}$ is the acceleration Vector.

2) Use of Langrange's Equation:

Here, we have 2 Langrange Equations.

$$\frac{\partial}{\partial t}\left(\frac{\partial T}{\partial \dot{x}_1}\right) - \frac{\partial T}{\partial x_1} + \frac{\partial U}{\partial x_1} = 0$$

$$\frac{\partial}{\partial t}\left(\frac{\partial T}{\partial \dot{x}_2}\right) - \frac{\partial T}{\partial x_2} + \frac{\partial U}{\partial x_2} = 0$$

$$T = \frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} \cdot m_2 \dot{x}_2^2$$

$$U = \frac{1}{2} K_1 x_1^2 + \frac{1}{2} K_2 (x_2 - x_1)^2$$

$$\frac{\partial T}{\partial \dot{x}_1} = m_1 \dot{x}_1 \qquad \frac{\partial}{\partial t}\left(\frac{\partial T}{\partial \dot{x}_1}\right) = m_1 \ddot{x}_1$$

$$\frac{\partial T}{\partial x_1} = 0 \qquad \frac{\partial U}{\partial x_1} = K_1 x_1 - K_2 (x_2 - x_1)$$

Similarly for x_2 and Substituting the above Values in Langrange Equation will give the required DEOM.

The DEOM are Coupled note that So we can't solve for $x_1(t)$ & $x_2(t)$ independently.

Here we take Heuristic approach. with our Experience for the Single DOF Spring-mass System we try the following Solution

$$x_1 = A_1 \sin(\omega t + \phi) \quad \text{--- (4)}$$

If (4) is true then x_2 Can Only take the form

$$x_2 = A_2 \sin(\omega t + \phi) \quad \text{--- (5)}$$

Substituting x_1 in DEOM we get x_2

$$-m_1 A_1 \omega^2 \sin(\omega t+\phi) = -A_1 \sin(\omega t+\phi)(K_1+K_2) + K_2 x_2.$$

$$x_2 = \left[\frac{A_1(K_1+K_2)}{K_2} - \frac{m_1 A_1 \omega^2}{K_2}\right] \sin(\omega t+\phi)$$

$$A_2 = \frac{A_1(K_1+K_2)}{K_2} - \frac{m_1 A_1 \omega^2}{K_2}$$

The DEOM are

$$m_1\ddot{x}_1 + (K_1+K_2)x_1 - K_2 x_2 = 0 \quad \text{--- (1)}$$

$$m_2\ddot{x}_2 + K_2 x_2 - K_2 x_1 = 0 \quad \text{--- (2)}$$

Let $x_1 = A_1 \sin(\omega t+\phi)$

$x_2 = A_2 \sin(\omega t+\phi)$

$A_1, A_2, \omega, \phi \rightarrow$ to be determined.

Substituting 4, 5, 6 & 7 in 1 & 2 we get

$$\left.\begin{aligned} &\left((K_1+K_2)-m_1\omega^2\right)A_1 - k_2 A_2 = 0 \quad — (8) \\ &\left[-K_2 A_1 + (K_2 - m_2\omega^2)A_2\right] = 0 \quad — (9) \end{aligned}\right\} \text{ since } \sin(\omega t+\phi)\neq 0 \text{ for all } t$$

Now we will Solve 8 & 9

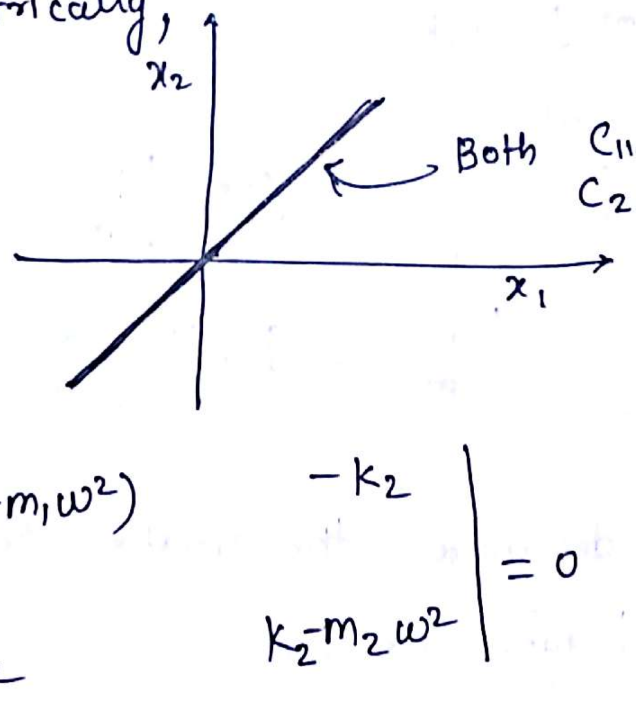
$$\left(K_1+K_2-m_1\omega^2\right)\frac{\left(K_2-m_2\omega^2\right)}{K_2}A_2 - K_2 A_2 = 0$$

$A_2 \neq 0$ So we have only one choice to have Infinitely many solutions.

$$\frac{K_1+K_2-m_1\omega^2}{-K_2} = \frac{-K_2}{(K_2-m_2\omega^2)}$$

{ for Non-trivial A_1 & A_2 we must have }

Geometrically,



$$\begin{vmatrix} (K_1+K_2-m_1\omega^2) & -K_2 \\ -K_2 & K_2-m_2\omega^2 \end{vmatrix} = 0 \quad — (10)$$

Which is Called Characteristic Equation or frequency equation of our system.

Let $m_1 = m_2 = m$

$K_1 = k_2 = k$

→ Obtain fre equation

→ Obtain the natural frequencies

→ obtain A_1 & A_2

$$\begin{bmatrix} 2k - m\omega^2 & -k \\ -k & k - m\omega^2 \end{bmatrix} = 0.$$

$$2k^2 - 2km\omega^2 - km\omega^2 + m^2\omega^4 - k^2 = 0$$

$$m^2\omega^4 - 3km\omega^2 + k^2 = 0$$

$$\omega^2 = \frac{3km \pm \sqrt{9k^2m^2 - 4m^2k^2}}{2 \cdot m^2}$$

$$\omega^2 = \frac{3km \pm \sqrt{5}\, mk}{2m^2}$$

$$\omega_2 = \sqrt{\frac{3+\sqrt{5}}{2}} \cdot \sqrt{\frac{k}{m}} = 1.618\sqrt{\frac{k}{m}}$$

$$\omega_1 = \sqrt{\frac{3-\sqrt{5}}{2}} \cdot \sqrt{\frac{k}{m}} = 0.618\sqrt{\frac{k}{m}}$$

Always designate the smaller of natural frequency ω_1 & bigger one ω_2.

The Amplitudes A_1 & A_2 are to be Obtained from

$(k_1+k_2-m_1\omega^2)A_1 - k_2A_2 = 0$

$-k_2A_1 + (k_2-m_2\omega^2)A_2 = 0$

$(2k-m\omega^2)A_1 - kA_2 = 0$

$-kA_1 + (k-m\omega^2)A_2 = 0$

For $\omega=\omega_1$ $\quad A_1 \rightarrow A_{11}$, $A_2 \rightarrow A_{21}$

Amplitude for m_2 for first Principle mode of Vibration.

$A_{11} \rightarrow$ Amplitude of m_1 for first principle mode of Vibration.

$x_1 = A_{11}\sin(\omega_1 t + \phi_1)$

$x_2 = A_{21}\sin(\omega_1 t + \phi_1)$

for $\omega = \omega_2$

$x_1 = A_{12}\sin(\omega_2 t + \phi_2)$

$x_2 = A_{22}\sin(\omega_2 t + \phi_2)$

$$\mu_1 = \frac{A_{21}}{A_{11}} = 1.618 = \frac{2k-(0.618)^2k}{k} = \frac{k}{k-(0.618)^2k}$$

$\mu_1 \rightarrow$ Amplitude Ratio for first principle mode of Vibration.

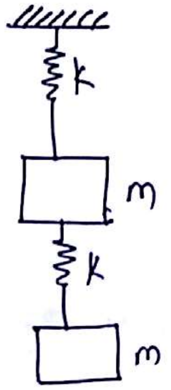
$x_1 = A_{11}\sin(\omega_1 t + \phi_1)$

$x_2 = A_{11}(1.618)\sin(\omega_1 t + \phi_1)$

$$\mu_2 = \frac{A_{22}}{A_{12}} = -0.618$$

Here the 2^{nd} Principle mode of Vibration given by

$$x_1(t) = A_{12} \sin(\omega_2 t + \phi_2)$$
$$x_2(t) = A_{12}\mu_2 \sin(\omega_2 t + \phi_2)$$



The Principal modes of Vibration are :

$$\left.\begin{aligned} x_1(t) &= A_{11} \sin(\omega_1 t + \phi_1) \\ x_2(t) &= A_{11}\mu_1 \sin(\omega_1 t + \phi_1) \end{aligned}\right\} \text{1}^{st} \text{ principle mode}$$

$$\left.\begin{aligned} x_1(t) &= A_{12} \sin(\omega_2 t + \phi_2) \\ x_2(t) &= A_{12}\mu_2 \sin(\omega_2 t + \phi_2) \end{aligned}\right\} \text{2}^{nd} \text{ principle mode.}$$

$$\mu_1 = \frac{A_{21}}{A_{11}} \qquad \mu_2 = \frac{A_{22}}{A_{12}}$$

Note that $x(t)$ for first principle mode is Linearly Independent of $x_1(t)$ for 2^{nd} Principle mode.

[This is so because
$$\frac{A_{11}\sin(\omega_1 t + \phi_1)}{A_{12}\sin(\omega_2 t + \phi_2)} \text{ isn't Constant.}$$

This means that the two Principal Modes represent two Linearly independent Solutions of Our DEOM.

By the theory of differential equations, the general Solutions (responses) of the masses would be the representation of the principal modes.

Hence the general free-Vibration responses of the Systems is given by

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$$x_1(t) = A_{11}\sin(\omega_1 t+\phi_1) + A_{12}\sin(\omega_2 t+\phi_2)$$
$$x_2(t) = A_{21}\sin(\omega_1 t+\phi_1) + A_{22}(\sin(\omega_2 t+\phi_2))$$
Remember

The arbitary Constants of Integration are $\phi_1, \phi_2, A_{11}, A_{12}$. These are to be determined for a given solution by using given Initial Conditions $x_1(0)$, $\dot{x}_1(0)$, $(x_2(0))$, $\dot{x}_2(0)$

Q. What are the necessary and Sufficient Conditions for a principle mode of Vibration?

Part-a: The Necessary Conditions

Let the System is executing the first principal mode of Vibrations.

Then

$$x_1(t) = A_{11}\sin(\omega_1 t+\phi_1) \qquad \dot{x}_1 = A_{11}\omega_1\cos(\omega_1 t+\phi_1)$$
$$x_2(t) = A_{11}\mu_1\sin(\omega_1 t+\phi_1) \qquad \dot{x}_2 = A_{11}\mu_1\omega_1\cos(\omega_1 t+\phi_1)$$

Hence

$$x_2 = \mu_1 x_1 \qquad \boxed{x_2(0) = \mu_1 x_1(0)} \text{ --- } (\alpha)$$

Also $$\dot{x}_2 = \mu\dot{x}_1 \qquad \boxed{\dot{x}_2(0) = \mu_1\dot{x}_1(0)} \text{ --- } (\beta)$$

α & β are necessary Condition for 1st principle mode of Vibration.

(H.W) Prove that Condition (α) & (β) are also Sufficient Condition for 1st Mode of Vibration.

Similarly we show that the necessary & sufficient Condition for 2nd Mode are

$$x_2(0) = \mu_2 x_1(0)$$
$$\dot{x}_2(0) = \mu_2 \dot{x}_1(0)$$

Hence for our Example Problem

to Excite the first principle mode, we could do the following (Here $\mu_1 = 1.618$ $\mu_2 = -0.618$)

give m_1 an Initial displacement of say 4mm

" m_2 an Initial displacement of 1.618×4 mm

The System will execute first Principal mode of Vibration.

* Linear Independence of a set of n functions :

$f_1(x), \dots, f_n(x)$

The Wronskian of the set of function is defined as

$$W(x) = \begin{vmatrix} f_1(x) & f_2(x) & \dots & f_n(x) \\ f_1'(x) & \dots & \dots & f_n'(x) \\ & & & \\ f^{(n-1)}(x) & \dots & \dots & f_n^{(n-1)}(x) \end{vmatrix}$$

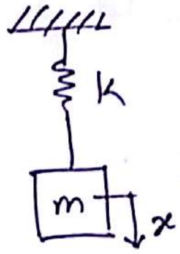
$$f_i^r = \frac{d^r f_i}{dx^r}$$

If $W(x)$ is identically zero over the interval of Interest, then the functions are Linearly dependent. otherwise they are Linearly Independent

$$W = \begin{vmatrix} x_1 & x_2 \\ \dot{x}_1 & \dot{x}_2 \end{vmatrix} = \begin{vmatrix} A\sin\omega_n t & B\cos\omega_n t \\ A\omega_n\cos\omega_n t & -B\omega_n\cos\omega_n t \end{vmatrix}$$

$x_1 = A\sin(\omega_n t)$

$x_2 = B\cos(\omega_n t)$



$W = -AB\omega_n \neq 0$

$$\{A_1\} = \begin{Bmatrix} A_{11} \\ \mu_1 A_{11} \end{Bmatrix} \text{ \& } \{A_2\} = \begin{Bmatrix} A_{12} \\ \mu_2 A_{12} \end{Bmatrix}$$ are the modal vectors.

for ~~the~~ 1^{st} & 2^{nd} principal Modes.

Note that A_{11} & A_{12} are arbitary.

However, for some Analytical advantage, we often normalize the Modal Vectors. There are several Standard ways to Normalize a modal Vector.

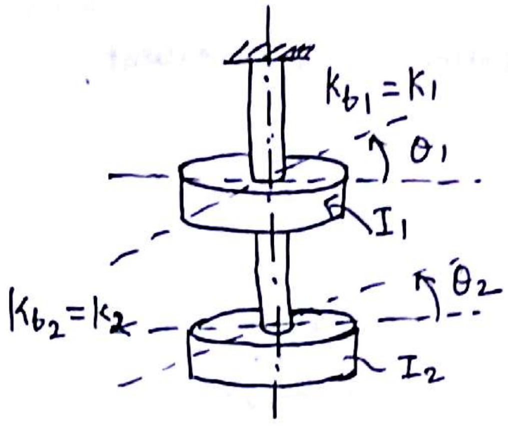
one way is to set $A_{11} = 1$, $A_{12} = 1$.

$$\{A_1\} = \begin{Bmatrix} 1 \\ \mu_1 \end{Bmatrix} \qquad \{A_2\} = \begin{Bmatrix} 1 \\ \mu_2 \end{Bmatrix}$$

Definition of a Modal Matrix: The matrix $[\mu] = [\{A_1\}\ \{A_2\}]$

$$= \begin{bmatrix} A_{11} & A_{12} \\ \mu_1 A_{11} & \mu_2 A_{12} \end{bmatrix}$$ is Called a modal Matrix of our System.

A Rotational 2 DOF systems:



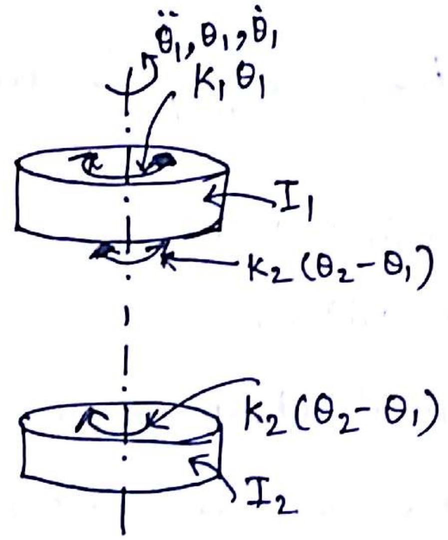
To Obtain DEOM

(I) by Newtons Method:

Let $\theta_2 > \theta_1$

$$\underbrace{\theta_1=\theta_1(t) \qquad \theta_2=\theta_2(t)}_{\text{set of generalised Coordinates,}}$$

from ~~Equation~~ Equillibrium in +ve in CCW as seen from above.



$$I_1\ddot{\theta}_1 + K_1\theta_1 - K_2(\theta_2-\theta_1)=0$$

$$I_2\ddot{\theta}_2 + K_2(\theta_2-\theta_1)=0$$

(II) By Langranges Equation

$$\frac{d}{dt}\left(\frac{dT}{d\dot{\theta}_1}\right)-\frac{dT}{d\theta_1}+\frac{dU}{d\theta_1}=0$$

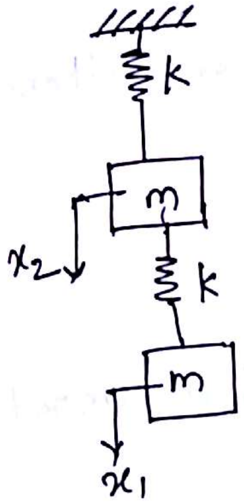
$$\frac{d}{dt}\left(\frac{dT}{d\dot{\theta}_2}\right)-\frac{dT}{d\theta_2}+\frac{dU}{d\theta_2}=0$$

$$T=\frac{1}{2}I_1\dot{\theta}_1^2+\frac{1}{2}I_2\dot{\theta}_2^2$$

$$U=\frac{1}{2}K_1\theta_1^2+\frac{1}{2}K_2(\theta_2-\theta_1)^2$$

Modal Analysis:

A Modal matrix can be used as a Coordinate transformation matrix ~~is~~ to uncouple the DEOM. This Leads to the Modal Analysis.



For this System

$$[\mu] = \begin{bmatrix} A_{11} & A_{12} \\ \mu_1 A_{11} & \mu_2 A_{12} \end{bmatrix}$$

$$[\mu] = \begin{bmatrix} 1 & 1 \\ \mu_1 & \mu_2 \end{bmatrix}$$ (a normalised modal Matrix)

Sometimes it proves Convenient to make each modal Vector on orthonormal Vector.

Let A_{11} be Such that that $|\{A_1\}| = 1$

i.e. $A_{11}^2 + \mu_1^2 A_{11}^2 = 1$

or $$A_{11} = \frac{1}{\sqrt{1+\mu_1^2}}$$

Similarly make $|\{A_2\}| = 1$

where $\{A_2\} = \begin{Bmatrix} A_{12} \\ \mu_2 A_{12} \end{Bmatrix}$

$$A_{12}^2 = \frac{1}{\mu_2^2+1} \qquad A_{12} = \frac{1}{\sqrt{\mu_2^2+1}}$$

Hence

$$\{\mu\} = \begin{bmatrix} \dfrac{1}{\sqrt{\mu_1^2+1}} & \dfrac{1}{\sqrt{\mu_2^2+1}} \\ \dfrac{\mu_1}{\sqrt{\mu_1^2+1}} & \dfrac{\mu_2}{\sqrt{\mu_2^2+1}} \end{bmatrix}$$

is also a modal Matrix.

Where the Modal Vectors $\{A_1\} = \begin{Bmatrix} \frac{1}{\sqrt{\mu_1^2+1}} \\ \frac{\mu_1}{\sqrt{\mu_1^2+1}} \end{Bmatrix}$

& $\{A_2\} = \begin{Bmatrix} \frac{1}{\sqrt{\mu_2^2+1}} \\ \frac{\mu_2}{\sqrt{\mu_2^2+1}} \end{Bmatrix}$, are set of Orthonormal

Vectors.

Let the DEOM of the System for ~~independ~~ undamped free Vibration be

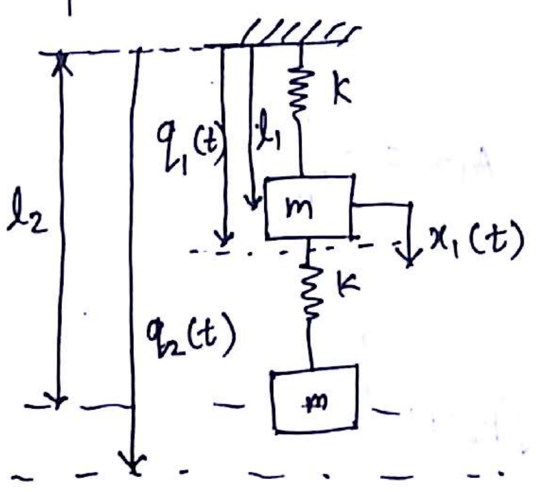
$$[m]\{\ddot{x}\} + [k]\{x\} = \{0\} \quad \text{—— (1)}$$

Let $\{x(t)\} = [\mu]\{p(t)\}$ —— (2)

defined a new set of Coordinates

$\{p(t)\}$ where $\{p(t)\} = [\mu]^{-1}\{x(t)\}$ —— (3)

$x_1(t)$ & $x_2(t)$ are measured from Static Equillibrium positions.



$q_1(t) = l_1 + x_1(t)$

$q_2(t) = l_2 + x_2(t)$

Another Set of generalised Coordinates.

$$\{A_1\} = \begin{Bmatrix} A_{11} \\ 1.618\,A_{11} \end{Bmatrix} \qquad \{A_2\} = \begin{Bmatrix} A_{12} \\ -0.618\,A_{12} \end{Bmatrix}$$

$$A_{11} = A_{12} = 1$$

$$\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{bmatrix} 1 & 1 \\ 1.618 & -0.618 \end{bmatrix} \begin{Bmatrix} p_1 \\ p_2 \end{Bmatrix}$$

Our DEOM are (for Example Problem)

$$[m]\{\ddot{x}\} + [K]\{x\} = \{0\} \quad \text{—— (i)}$$

Where

$$[m] = \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} \qquad [K] = \begin{bmatrix} 2K & -K \\ -K & K \end{bmatrix}$$

$$\{\ddot{x}\} = \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} \qquad \{x\} = \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix}$$

If $\{x\} = [\mu]\{p\}$ —— (ii)

Then substitution of (ii) in (i) leads to

$$[m][\mu]\{\ddot{p}\} + [K][\mu]\{p\} = \{0\} \quad \text{—— (iii)}$$

premultiply both sides of (iii) by $[\mu]^T$

$$[\mu]^T[m][\mu]\{\ddot{p}\} + [\mu]^T[K][\mu]\{p\} = \{0\} \quad \text{—— (iv)}$$

$$[\mu]^T[m][\mu] = \begin{bmatrix} 1 & 1.618 \\ 1 & -0.618 \end{bmatrix} \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1.618 & -0.618 \end{bmatrix}$$

$$= \begin{bmatrix} m\,(1+(1.618)^2) & m\,(1-1.618\times0.618) \\ m\,(1-1.618\times0.618) & m\,(1+(0.618)^2) \end{bmatrix}$$

$$= \begin{bmatrix} 3.617\,m & 7.6\times10^{-5}\,m \\ 7.6\times10^{-5}\,m & 1.381\,m \end{bmatrix} \approx \begin{bmatrix} 3.617\,m & 0 \\ 0 & 1.381\,m \end{bmatrix}$$

$$[\mu^T][k][\mu] = \begin{bmatrix} 1 & 1.618 \\ 1 & -0.618 \end{bmatrix} \begin{bmatrix} 2k & -k \\ -k & k \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1.618 & -0.618 \end{bmatrix}$$

$$= \begin{bmatrix} 0.382\,k & 0.618\,k \\ 2.618\,k & -1.618\,k \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1.618 & -0.618 \end{bmatrix}$$

$$= \begin{bmatrix} 1.382\,k & 7.6\times10^{-5}\,k \\ 7.6\times10^{-5}\,k & 3.618\,k \end{bmatrix}$$

$$\approx \begin{bmatrix} 1.382\,k & 0 \\ 0 & 3.618\,k \end{bmatrix}$$

So Eq^n (iv) becomes

$$\begin{bmatrix} 3.617\,m & 0 \\ 0 & 1.381\,m \end{bmatrix} \begin{Bmatrix} \ddot{p}_1 \\ \ddot{p}_2 \end{Bmatrix} + \begin{bmatrix} 1.381\,k & 0 \\ 0 & 3.617\,k \end{bmatrix} \begin{Bmatrix} p_1 \\ p_2 \end{Bmatrix} = \{0\} \qquad \text{— (v)}$$

Equation (vi) & (vii) are the required uncoupled DEOM in terms of the principle coordinates $p_1(t)$ & $p_2(t)$.

$$3.617\,m\ddot{p}_1 + 1.382\,k p_1 = 0$$

$$1.381\,m\ddot{p}_2 + 3.617\,k p_2 = 0$$

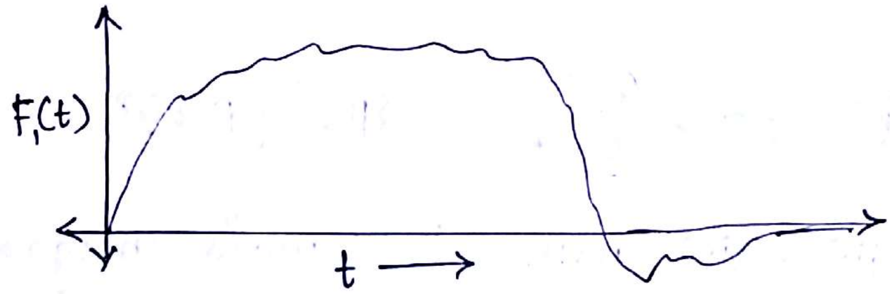
Note that there can be Infinitely many principle (sets) coordinates since A_{11} & A_{12} are arbitary.

Since we normalized our modal matrix ($Setting\ A_{11}=1,\ A_{12}=1$)

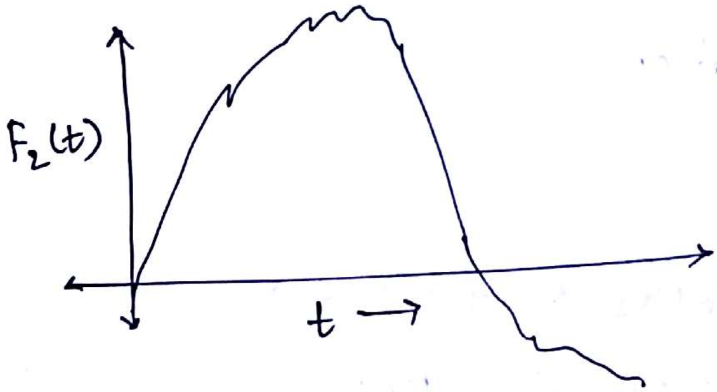
We are getting only one particular Set of Principal Coordinates.

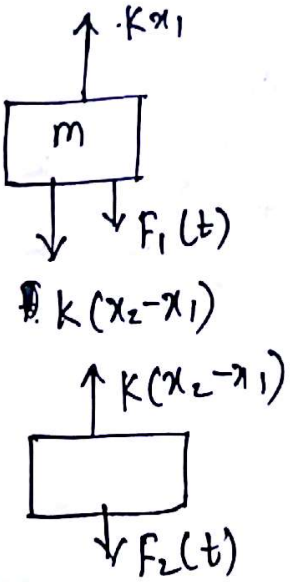
$$\omega_1 = \sqrt{\frac{1.382k}{3.617m}} = 0.618\sqrt{\frac{k}{m}}$$

$$\omega_2 = \sqrt{\frac{3.617k}{1.381m}} = 1.618\sqrt{\frac{k}{m}}$$



<u>Aim</u>: to obtain forced response of the System at any time t.





The DEOM will be

$$\left.\begin{aligned} m\ddot{x}_1 + 2kx_1 - kx_2 &= F_1(t) \\ m\ddot{x}_2 - kx_1 + kx_2 &= F_2(t) \end{aligned}\right\}$$

$$[m]\{\ddot{x}\} + [k]\{x\} = \{F(t)\} \quad \text{(I)}$$

$$\{F(t)\} = \begin{Bmatrix} F_1(t) \\ F_2(t) \end{Bmatrix}$$

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Let

$$\{x\} = [\mu]\{p\} \quad \text{— ②}$$

from ① & ②

$$[m]\{\mu\}\{\ddot{p}\} + [k]\{\mu\}\{p\} = \{F(t)\}$$

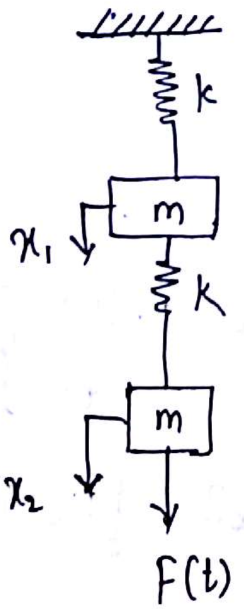
$$\{\mu^T\}[m]\{\mu\}\{\ddot{p}\} + \{\mu\}^T[k]\{\mu\}\{p\} = \{\mu\}^T\{F(t)\}$$

$$M_{11}\ddot{P}_1 + K_{11}P_1 = Q_1(t) \quad \text{— (iii)}$$

$$M_{22}\ddot{P}_2 + K_{22}P_2 = Q_2(t) \quad \text{— (iv)}$$

$$\{Q(t)\} = \begin{Bmatrix} Q_1(t) \\ Q_2(t) \end{Bmatrix} = \{\mu\}^T\{F(t)\}$$

solve (iii) & (iv) using Duhamel's integral for forced Vibration.



$$m\ddot{x}_1 - K(x_2 - x_1) + Kx_1 = 0$$

$$m\ddot{x}_2 + K(x_2 - x_1) = F(t)$$

<u>Step I</u> obtain DEOM

$$\begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} 2K & -K \\ -K & k \end{bmatrix}\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ F(t) \end{Bmatrix} \quad \text{— ①}$$

Step II: Obtain modal Vectors

$$\{A_1\} = \begin{Bmatrix} A_{11} \\ \mu_1 A_{11} \end{Bmatrix} \qquad \{A_2\} = \begin{Bmatrix} A_{12} \\ \mu_2 A_{12} \end{Bmatrix}$$

Step III: Use the Normalised modal Matrix $[\mu] = \begin{bmatrix} 1 & 1 \\ \mu_1 & \mu_2 \end{bmatrix}$

to uncouple the DEOM ①.

This Results in the following uncoupled DEOM in terms of principal Coordinates $\{p\} = \begin{Bmatrix} P_1(t) \\ P_2(t) \end{Bmatrix}$ where $\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = [\mu] \begin{Bmatrix} P_1 \\ P_2 \end{Bmatrix}$

Pre-Multiply $[\mu]^T$ to ①.

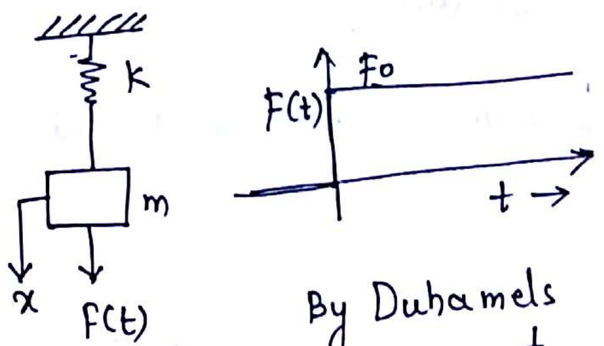
$$M_{11} \ddot{P}_1 + K_{11} P_1 = Q_1(t) \quad — ②$$

$$M_{22} \ddot{P}_2 + K_{22} P_2 = Q_2(t) \quad — ③$$

M_{11} & M_{22} called generalised masses.

K_{11} & K_{22} called generalised Stiffness.

$$\omega_1 = \sqrt{\frac{K_{11}}{M_{11}}} \qquad \omega_2 = \sqrt{\frac{K_{22}}{M_{22}}}$$



By Duhamels

$$x_{forced} = x(t) = \int_{0}^{t} F(\tau)\, g(t-\tau)\, d\tau$$

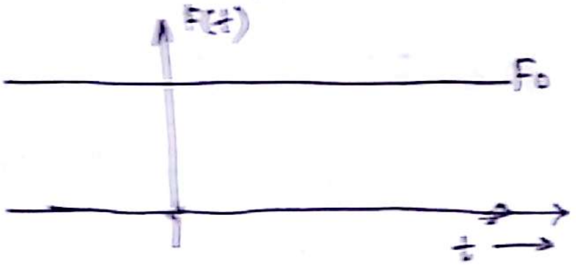
$$= \int_0 \frac{F_0}{m\omega_n} \sin \omega_n (t-\tau)\, d\tau$$

$$x_{forced} = x(t) = \frac{F_0}{k}\left[1 - \cos(\omega_n t)\right]$$

$x(0) = \dot{x}(0) = 0$

<u>Note</u> : $m\ddot{x} + kx = F_0$

$x_{particular} = F_0/k$ will only valid when force is



But for Impulsive force we must write $x_T = x_p + x_c = \frac{F_0}{k} + A\sin\omega t + B\cos\omega t$ and use $x(0) = \dot{x}(0) = 0$ will give Response. But the Resultant will be x_{forced}. If $x(0)$ & $\dot{x}(0)$ $\neq 0$ then $x_{Resp} = x(0) + x_{forced}$

From DEOM ② using Duhamel's Integral,

we get

$$P_1 = \frac{1}{M_{11}\omega_1}\int_0^t F(\tau)\sin\omega_1(t-\tau)\,d\tau$$

$$= \frac{\mu_1 F_0}{M_{11}\omega_1^2}\left[1 - \cos(\omega_1 t)\right]$$

Similary solving ③ by Duhamels Integral. We get

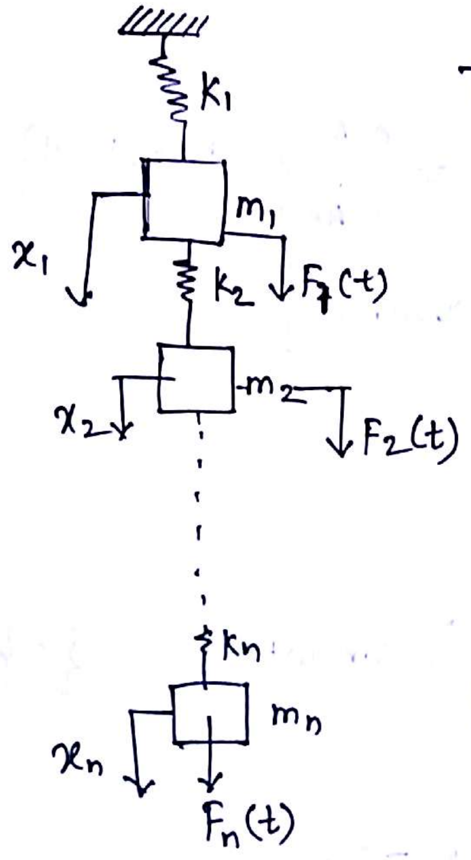
$$\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{bmatrix} 1 & 1 \\ \mu_1 & \mu_2 \end{bmatrix}\begin{Bmatrix} P_1 \\ P_2 \end{Bmatrix}$$

$$x_1(t) = P_1 + P_2 = \frac{\mu_1 F_0}{K_{11}}\left[1 - \cos\omega_1 t\right] + \frac{\mu_2 F_0}{K_{22}}\left[1 - \cos(\omega_2 t)\right]$$

$$x_2(t) = \mu_1 P_1 + \mu_2 P_2 = \frac{\mu_1^2 F_0}{K_{11}}\left[1 - \cos\omega_1 t\right] + \frac{\mu_2^2 F_0}{K_{11}}\left[1 - \cos\omega_2 t\right]$$

This is required forced Vibration Response

Generalization for an n-DOF System



The DEOM are

$$\underset{n\times n}{[m]}\underset{n\times 1}{\{\ddot{x}\}} + \underset{n\times n}{[k]}\underset{n\times 1}{\{x\}} = \underset{n\times 1}{\{F\}} \quad \text{—(1)}$$

Where,

$$\{\ddot{x}\} = \begin{Bmatrix} \ddot{x}_1 \\ \vdots \\ \ddot{x}_n \end{Bmatrix}$$

$$\{x\} = \begin{Bmatrix} x_1 \\ x_{\cdot} \\ \vdots \\ x_n \end{Bmatrix} \qquad \{F\} = \begin{Bmatrix} F_1(t) \\ \vdots \\ F_n(t) \end{Bmatrix}$$

$$[m] = \begin{bmatrix} m_{11} & m_{12} & \cdots & m_{1n} \\ \vdots & m_{22} & & \vdots \\ \vdots & & \ddots & \vdots \\ m_{n1} & - & \cdots & m_{nn} \end{bmatrix}$$

$$[K] = \begin{bmatrix} K_{11} & \cdots & \cdots & K_{1n} \\ \vdots & K_{22} & & \vdots \\ \vdots & & \ddots & \vdots \\ K_{n1} & - & \cdots & K_{nn} \end{bmatrix}$$

To Obtain the Natural frequencies $\omega_1, \omega_2, \ldots \omega_n$ & the Associated modal Vectors $\{A_1\}, \{A_2\}, \ldots \{A_n\}$

$\left(\text{Where } \{A_r\} = \begin{Bmatrix} A_{1r} \\ A_{2r} \\ \vdots \\ A_{nr} \end{Bmatrix} \quad r = 1, \ldots, n \right)$

We Consider Only the homogeneous part of (1) that is,

$$[m]\{\ddot{x}\} + [K]\{x\} = \{0\} \quad \text{— (2)}$$

We Assume,

$$\{x\} = \{A\} \sin(\omega t+\phi) = \begin{Bmatrix} A_1 \\ \vdots \\ A_n \end{Bmatrix} \sin(\omega t+\phi) \quad \text{— (3)}$$

[Just As in a 2DOF Systems]

After (3) is Substituted in (2)

We get

$$\left(-\omega^2[m]\{A\} + [K]\{A\}\right) \sin(\omega t+\phi) = \{0\}$$

& This Implies

$$[K]\{A\} - \omega^2[m]\{A\} = \{0\}$$

or

$$\left([K] - \omega^2[m]\right)\{A\} = 0 \quad \text{— (4)}$$

For Non-Trivial Solution

We must have

$$\left|[K] - \omega^2[m]\right| = 0 \quad \text{— (5)}$$

Characteristic frequency Equation

(5) gives an algebraic equation of the form.

$$a_1\omega^{2n} + a_2\omega^{2(n-1)} + \dots + a_n\omega^2 + a_{n+1} = 0$$

Which in term gives us $\omega_1, \omega_2 \dots \omega_n$

After $\omega_1 \dots \omega_n$ obtained each of these is to be Substituted in ④ to get Corresponding modal Vector.

Next, we can form a modal matrix.

$$[\mu] = \begin{bmatrix} A_{11} & \cdots & A_{1n} \\ A_{21} & \cdots & A_{2n} \\ \vdots & & \vdots \\ A_{n1} & \cdots & A_{nn} \end{bmatrix}$$

& use it in the Coordinate transformation

$$\{x\} = [\mu]\{p\} \qquad \{p\} = \begin{Bmatrix} p_1(t) \\ \vdots \\ p_n(t) \end{Bmatrix}$$

in DEOM ① [with force Vector $\{F\}$ on RHS]

Finally

$$[\mu]^T[m]\{\mu\} = \begin{bmatrix} M_{11} & 0 & \cdots & 0 \\ 0 & M_{22} & 0 & \vdots \\ \vdots & 0 & \ddots & \vdots \\ 0 & \cdots & \cdots & M_{nn} \end{bmatrix}$$

&

$$[\mu]^T[k]\{\mu\} = \begin{bmatrix} K_{11} & 0 & 0 & \cdots & 0 \\ 0 & \ddots & & 0 & \vdots \\ \vdots & 0 & & \ddots & \vdots \\ 0 & \cdots & & \cdots & K_{nn} \end{bmatrix}$$

& we shall get n uncouple DEOM

$$M_{11}\ddot{p}_1 + K_{11}p_1 = Q_1$$

$$\vdots$$

$$M_{nn}\ddot{p}_n + K_{nn}p_n = Q_n$$

Each of these Can be solved using Duhamel Integral.

$$\{Q\} = [\mu]^T\{F\} = \begin{Bmatrix} Q_1 \\ \vdots \\ Q_n \end{Bmatrix}$$ a Vector of generalised Forces.

Orthogonality of Modal Vectors

If $r \neq s$ & $\omega_r \neq \omega_s$

then

$$\{A_r\}^T [m] \{A_s\} = 0$$

$$\& \quad \{A_r\}^T [K] \{A_s\} = 0$$

that is $\{A_r\}$ & $\{A_s\}$ ar Orthogonal Vector w.r.t. weighing matrices $[m]$ & $[K]$.

$$[m]\{\ddot{x}\} + [K]\{x\} = \{0\}$$

$$\{x\} = \{A\} \sin(\omega t + \phi)$$

$$-\omega^2 [m]\{A\} + [K]\{A\} = \{0\}$$

$$\omega^2 [m] \{A\} = [K]\{A\} \quad \text{—— (1)}$$

When $\omega = \omega_r \qquad \{A\} = \{A_r\}$

& $\omega = \omega_s \qquad \{A\} = \{A_s\}$

Then (1) becomes

$$\omega_r^2 [m] \{A_r\} = [K] \{A_r\} \quad \text{—— (2)}$$

&

$$\omega_s^2 [m] \{A_s\} = [K] \{A_s\} \quad \text{—— (3)}$$

Premultiply ② by $\{A_s\}^T$ & ③ by $\{A_r\}^T$

This gives

$$w_r^2 \{A_s\}^T [m] \{A_r\} = \{A_s\}^T [K] \{A_r\} \quad — ④$$

$$w_s^2 \{A_r\}^T [m] \{A_s\} = \{A_r\}^T [K] \{A_s\} \quad — ⑤$$

$$w_r^2 \{A_r\}^T [m]^T \{A_s\} = \{A_r\}^T [K]^T \{A_s\}$$

Let $[m]^T = [m]$ & $[K]^T = [K]$

$[m]$ & $[K]$ both are symmetric

Then

$$w_r^2 [m] \{A_s\} = \{A_r\}^T [K] \{A_s\} \quad — ⑥$$

R.H.S of ⑤ & ⑥ are Identical.

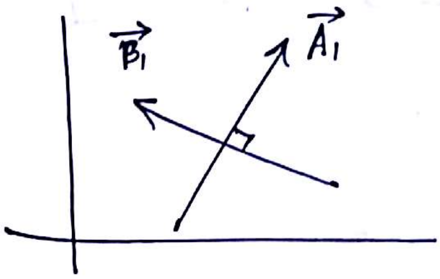
Hence, $(w_r^2 - w_s^2) \{A_r\}^T [m] \{A_s\} = \{0\}$

But $w_r \neq w_s$

Hence, $\{A_r\}^T [m] \{A_s\} = \{0\}$

From ⑥

$$\{A_r\}^T [K] \{A_s\} = \{0\}$$



$$\{A_1\} = \begin{Bmatrix} A_{11} \\ A_{21} \end{Bmatrix} \quad \{B_1\} = \begin{Bmatrix} B_{11} \\ B_{21} \end{Bmatrix}$$

If

$\{A_1\}^T \{B_1\} = \{0\}$ Hence

$\{A_1\}$ & $\{B_1\}$ are orthogonal in an ordinary sense.

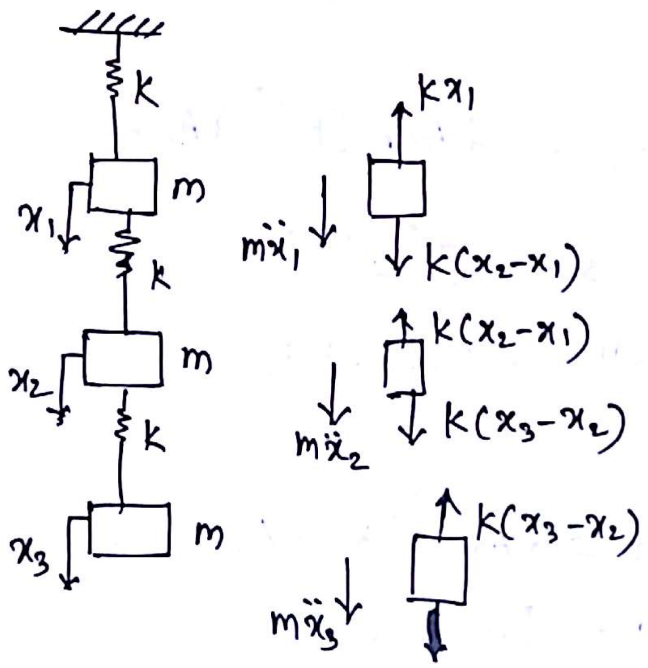
$$\{A_1\}^T [I] \{B_1\} = \{0\} \qquad [I] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

weighing matrix

$$\omega_r^2 = \frac{\{A_r\}^T [k] \{A_r\}}{\{A_r\}^T [m] \{A_r\}}$$

$$r = 1, 2, \dots n$$

Ex



$$m\ddot{x}_1 + 2kx_1 - kx_2 + (0)x_3 = 0$$

$$m\ddot{x}_2 - kx_1 + 2kx_2 - kx_3 = 0$$

$$m\ddot{x}_3 + (0)x_1 - kx_2 + kx_3 = 0$$

$$[m] = \begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & m \end{bmatrix} \qquad [k] = \begin{bmatrix} 2k & -k & 0 \\ -k & 2k & -k \\ 0 & -k & k \end{bmatrix}$$

$$\begin{vmatrix} 2k - m\omega^2 & -k & 0 \\ -k & 2k - m\omega^2 & -k \\ 0 & -k & k - m\omega^2 \end{vmatrix} = 0$$

$$(2K-m\omega^2)\left[(2K-m\omega^2)\cdot(K-m\omega^2) - K^2\right] + K\,(m\omega^2 K - K^2) = 0$$

$$(2K-m\omega^2)\left[2K^2 + m^2\omega^4 - 3mK\omega^2 - K^2\right] - K^3 + (m\omega^2 K^2) = 0$$

$$(2K^3 + 2Km^2\omega^4 - 6mK^2\omega^2 - K^2 m\omega^2 - m^3\omega^6 + 3m^2K\omega^4)$$
$$-K^3 + m\omega^2K^2 = 0$$

$$m^3\omega^6 - 5Km^2\omega^4 + 6mK^2\omega^2 - K^3 = 0$$

$$\omega_1^2 = 3.2469\left(\frac{K}{m}\right)$$

$$\omega_2^2 = 1.555\left(\frac{K}{m}\right)$$

$$\omega_3^2 = 0.198\left(\frac{K}{m}\right)$$

$$\omega_1 = 0.444\left(\frac{K}{m}\right)^{1/2}$$

$$\omega_2 = 1.2469\left(\frac{K}{m}\right)^{1/2}$$

$$\omega_3 = 1.8019\left(\frac{K}{m}\right)^{1/2}$$

The Rayleigh Method for Multi-DOF Systems

For an n DOF Systems,

$$\omega_r^2 = \frac{\{A_r\}^T [K] \{A_r\}}{\{A_r\}^T [m] \{A_r\}} \quad - (1)$$

So, we need to know $\{A_r\}$ to evaluate ω_r

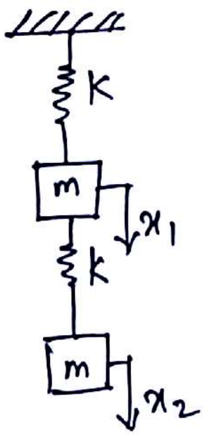
However, we first obtain ω_r and then $\{A_r\}$ by the Analytical

Method Hence, fourmula ① is apparantly useless Computing any ω_r.

However, this is not for $\omega = \omega_1$ fourmula ① Can be used to obtain a pretty good estimate for ω_1 by making a (Somewhat wild) guess for $\{A_1\}$ That is the Rayleigh Method for an n-DOF System.

It Can be Shown that a 100% error in the guess for $\{A_1\}$ results in only atmost 10% error in estimated ω_1.

To illustrate Consider the System



we already know that

$$\omega_1 = 0.618\sqrt{\frac{K}{m}}$$

& $\{A_1\} = \begin{Bmatrix} 1 \\ 1.618 \end{Bmatrix}$

To apply the Rayleigh Method, we assume a trial Vector $\begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$ for $\{A_1\}$

Then ω_R^2 = square of Rayleigh frequency ω_r

$$= \frac{\begin{Bmatrix} 1 \\ 1 \end{Bmatrix}^T [K] \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}}{\begin{Bmatrix} 1 \\ 1 \end{Bmatrix}^T [m] \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}}$$

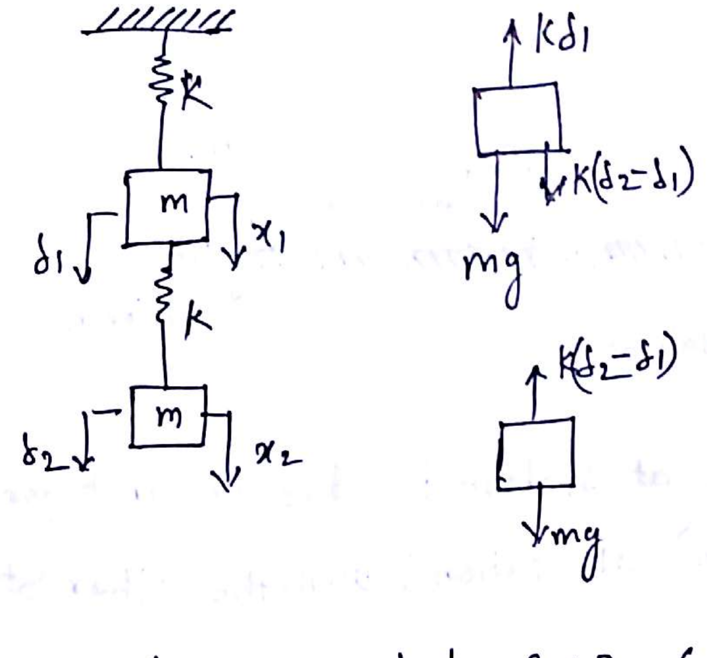
$$\omega_R^2 = \frac{\{1 \quad 1\}\begin{bmatrix} 2k & -k \\ -k & k \end{bmatrix}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix}}{\{1 \quad 1\}\begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix}}$$

$$= \frac{k}{2m}$$

$$\omega_R = 0.707\sqrt{\frac{k}{m}}$$

Hence % error $= \dfrac{0.707 - 0.618}{0.618} \times 100 = 14.4\%$.

A Better estimate for ω_1 Can be obtained by taking the Static deflection Vector for $\{A_1\}$.



$$mg = 2k\delta_1 - k\delta_2$$

$$\delta_2 = \delta_1 + \frac{mg}{k}$$

$$\delta_1 = \frac{2mg}{k}$$

Thus, we take $\{A_1\} \approx \{\delta_1\} = \begin{Bmatrix} \delta_1 \\ \delta_2 \end{Bmatrix} = \begin{Bmatrix} 2mg/k \\ 3mg/k \end{Bmatrix}$

noralized $\{\delta\} = \begin{Bmatrix} 2 \\ 3 \end{Bmatrix}$

$$\omega_R^2 = \frac{\{2 \quad 3\}\begin{bmatrix} 2k & -k \\ -k & k \end{bmatrix}\begin{Bmatrix} 2 \\ 3 \end{Bmatrix}}{\{2 \quad 3\}\begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix}\begin{Bmatrix} 2 \\ 3 \end{Bmatrix}} = \frac{5k}{13m}$$

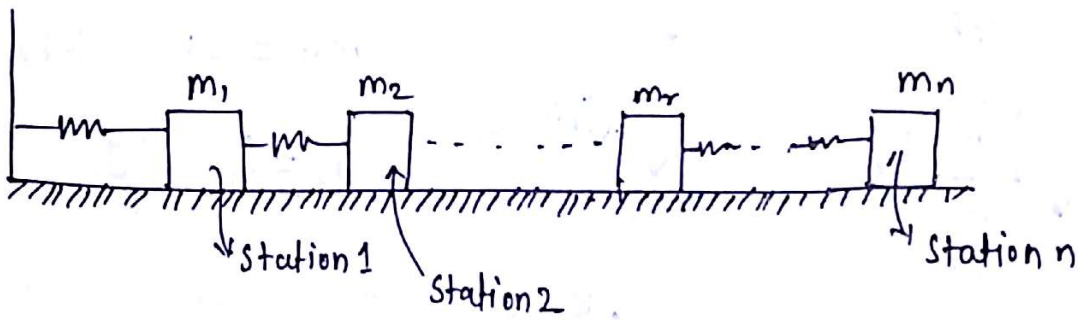
$$\omega_1 = \sqrt{\frac{5}{13}}\sqrt{\frac{K}{m}} = 0.62\sqrt{\frac{K}{m}}$$

$$\%\ \text{error} = \left(\frac{0.62 - 0.618}{0.618}\right) \times 100 = 0.35\ \%$$

(8) The flexibility Influence Coefficients a_{ij} & flexibility matrix $\underset{n\times n}{[a]} = \underset{n\times n}{[a_{ij}]}$ for an n DOF System.

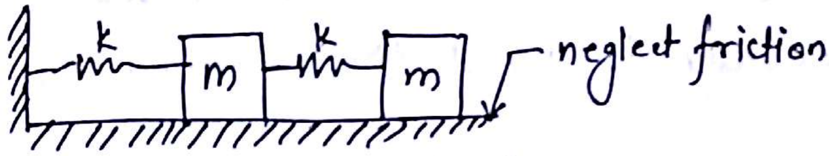
The flexibility influence Coefficients a_{ij} are Very useful for obtaining the Stiffness matrix $[K]$ of a Complex System by experimentation.

It Can be Shown that $[K] = [a]^{-1}$.

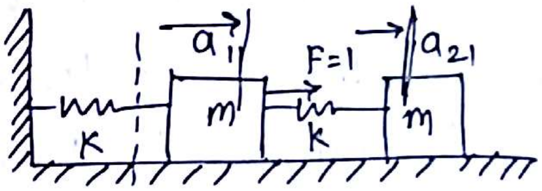


a_{ij} = deflection at Station i due to unit force (at an appropriate direction) at Station j with the other Stations free of Such forces.

Ex obtain $[a] = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$



<u>Step 1</u> : Apply force $F = 1$ unit at Station 1 & Compute/measure deflections at Stations 1 & 2. These will give a_{11} & a_{21}, i.e., the first Column of $[a]$.

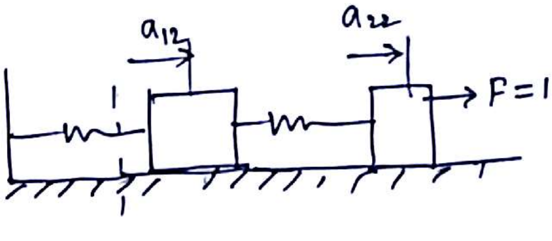


$$k a_{11} = 1 \qquad a_{11} = \frac{1}{K}$$

$$K(a_{21} - a_{11}) \qquad a_{21} = a_{11} = \frac{1}{K}$$

$$[a] = \begin{bmatrix} 1/K & 1/K \\ 1/K & 2/K \end{bmatrix}$$

for a_{12} & a_{22} we will apply $F = 1$ on Station 2



$$K a_{12} = F = 1$$

$$a_{12} = 1/K$$

$$K(a_{22} - a_{12}) = 1$$

$$a_{22} = \frac{2}{K}$$

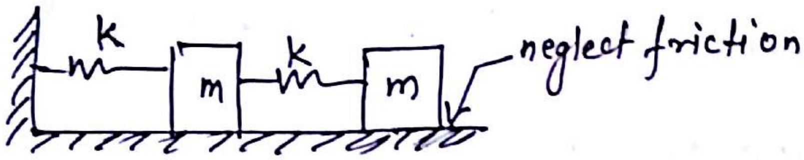
$$[a]^{-1} = \frac{adj(a)}{|a|} = \begin{bmatrix} 2K & -K \\ -K & K \end{bmatrix} = [K]$$

(8) The Stiffness Influence Coefficients K_{ij} & the Stiffness matrix $[K] = [K_{ij}]$.

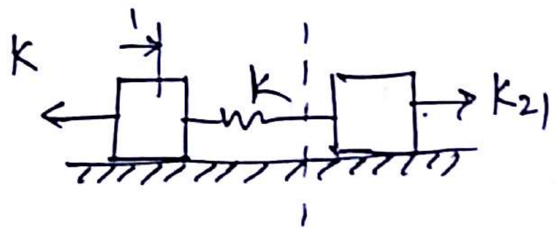
Scanned by CamScanner

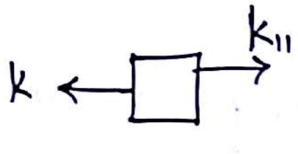
K_{ij} = Force required at Station i to Cause unit deflection at Station j, all other Stations being Subject to forces Simultaneously to arrest their movements.

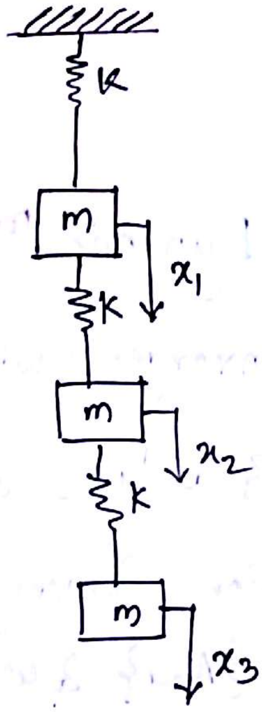
Ex. obtain $[K] = \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix}$ using definitions of k_{ij}



Step 1: To get k_{11}, we apply a force on Station 1 to Cause a unit deflection at Station 2. Clearly, we must apply a force $= k_{21}$ to Station 2 to Accomplish this.







$$\omega_1 = 0.44497\sqrt{\frac{K}{m}} \qquad \{A_1\} = \begin{Bmatrix} 1.0 \\ 1.802 \\ 2.247 \end{Bmatrix}$$

$$\omega_2 = 1.247\sqrt{\frac{K}{m}} \qquad \{A_2\} = \begin{Bmatrix} 1.0 \\ 0.445 \\ -0.802 \end{Bmatrix}$$

$$\omega_3 = 1.802\sqrt{\frac{K}{m}} \qquad \{A_3\} = \begin{Bmatrix} 1.0 \\ -1.247 \\ 0.555 \end{Bmatrix}$$

Matrix Iteration Method (MI)

(The power Method)

This is Numerical Method for Obtaining ω_i and $\{A_i\}$ for an undamped 'n' DOF System with $0 < \omega_1 < \omega_2 < \cdots < \omega_n$

Let the DEOM be

$$[m]\{\ddot{x}\} + [K]\{x\} = \{0\}$$

$$\{x\} = \{A\}\sin(\omega t + \phi)$$

$$\omega^2 [m]\{A\} = [K]\{A\}$$

$$\omega^2 [K]^{-1}[m]\{A\} = \{A\}$$

$$[K]^{-1}[m]\{A\} = \frac{1}{\omega^2}\{A\}$$

$$[D]\{A\} = \frac{1}{\omega^2}\{A\} \quad \text{—— (1)}$$

$[D] = [K]^{-1}[m]$ is Called a Dynamic Matrix.

We Also have $[m]^{-1}[K]\{A\} = \omega^2\{A\}$

$$[E]\{A\} = \omega^2\{A\} \quad -② $$

Where $[E] = [D]^{-1}$ is an other Dynamic Matrix.

If we ~~have~~ use ① for Iteration, Convergence will be to ω_1 and $\{A_1\}$ first, next to $\{A_2\}$ and ω_2 & so on. However, if ② is used for iteration, Convergence will be to ω_n and $\{A_n\}$ and then to $\{A_{n-1}\}$ & ω_{n-1}. etc.

[D] (or [E]) must be Correctly Obtained.

We shall use ① for iteration since the 1st few Natural frequency are normally the most Important ones.

$$\{u_1\} = \begin{Bmatrix} -10^3 \\ 1 \\ 10^3 \end{Bmatrix}, \quad \{u_1\} = \begin{Bmatrix} 1 \\ 2 \\ 3 \end{Bmatrix} \text{ etc}$$

Here $[K] = \begin{bmatrix} 2k & -k & 0 \\ -k & 2k & -k \\ 0 & -k & k \end{bmatrix}$ $\quad [m] = \begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & m \end{bmatrix}$

$[K]^{-1} = \frac{1}{k}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$ $\quad$ So $[D] = [K]^{-1}[m]$

$$[D] = \frac{m}{k}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$$

We start Iterating assuming a trial Vector $\{u\}$ for $\{A_1\}$.

If $\{u\} \equiv \{A_1\}$, then $[D]\{u\} = \frac{1}{\omega^2}\{u\}$

Let 1^{st} trial Vector be $\{u_1\}$

Let $\{u_1\} = \begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix}$

$$[D]\{u_1\} = \frac{m}{K}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}\begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix} = \frac{m}{K}\begin{Bmatrix} 3 \\ 5 \\ 6 \end{Bmatrix} = \frac{3m}{K}\begin{Bmatrix} 1 \\ 1.667 \\ 2 \end{Bmatrix}$$

Then

$\{u_2\} = \begin{Bmatrix} 1 \\ 1.667 \\ 2 \end{Bmatrix}$ where $\{u_2\} \rightarrow$ next Trial Vector

$$[D]\{u_2\} = \frac{4.6667\,m}{K}\begin{Bmatrix} 1 \\ 1.7857 \\ 2.1242 \end{Bmatrix} \rightarrow \{u_3\}$$

$$[D]\{u_3\} = \frac{4.9999\,m}{K}\begin{Bmatrix} 1 \\ 1.8000 \\ 2.2428 \end{Bmatrix} \rightarrow \{u_4\}$$

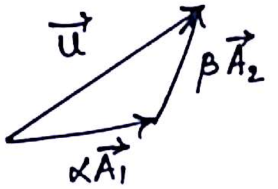
$$[D]\{u_4\} = \frac{5.0428\,m}{K}\begin{Bmatrix} 1 \\ 1.8017 \\ 2.2465 \end{Bmatrix} \leftarrow \{u_5\}$$

$$[D]\{u_5\} = \frac{5.0482\,m}{K}\begin{Bmatrix} 1 \\ 1.8019 \\ 2.2469 \end{Bmatrix} \leftarrow \{u_6\}$$

So $\{A_1\} \approx \{u_6\} = \begin{Bmatrix} 1 \\ 1.8019 \\ 2.2469 \end{Bmatrix}$

$$\frac{1}{\omega_1^2} = \frac{5.0482\,m}{K} \qquad \omega_1 = 0.4451\sqrt{\frac{k}{m}}$$

Irrespective of the Initial guess value, the Convergence will be to 'ω_1' and $\{A_1\}$ always. This is because,



$$\vec{u} = \alpha\vec{A_1} + \beta\vec{A_2}$$

$$\{u\} = C_1\{A_1\} + C_2\{A_2\} + C_3\{A_3\}$$

$\omega_1 < \omega_2 < \omega_3$

$$\underbrace{[D]\{u_1\}}_{\{u_2\}} = C_1[D]\{A_1\} + C_2[D]\{A_2\} + C_3[D]\{A_3\}$$

$$[D]\{u_2\} = \frac{C_1}{(\omega_1^2)^2}\{A_1\} + \frac{C_2}{(\omega_2^2)^2}\{A_2\} + \frac{C_3}{(\omega_3^2)^2}\{A_3\}$$

So after 'p' iterations

$$\frac{1}{(\omega_1^2)^p} >> \frac{1}{(\omega_2^2)^p} >> \frac{1}{(\omega_3^2)^p}$$

$\{A_1\}$ term is prominent

If we want $\{A_2\}$ Should be prominent then C_1 should be Zero.

For Convergence to ω_2 and $\{A_2\}$ the trial Vector $[v]$ must be orthogonal to $\{A_1\}$.

$$\{v\} = \alpha\{A_1\} + \beta\{A_2\} + \gamma\{A_3\}$$

$$\{A_1\}^T[m]\{v\} = \alpha\{A_1\}^T[m]\{A_1\} + \cancel{\beta\{A_1\}^T[m]\{A_2\}} + \cancel{\gamma\{A_1\}^T[m]\{A_3\}}$$

If $\{A_1\}^T[m]\{v\} = 0$ then $\alpha = 0$

So $\{A_{11} \;\; A_{21} \;\; A_{31}\}^T \begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & m \end{bmatrix} \begin{Bmatrix} v_1 \\ v_2 \\ v_3 \end{Bmatrix} = 0$

$$A_{11} v_1 + A_{21} v_2 + A_{31} v_3 = 0$$

$$v_1 = 0 \cdot v_1 - \frac{A_{21}}{A_{11}} v_2 - \frac{A_{31}}{A_{11}} v_3$$

$$v_2 = 0 v_1 + 1 \cdot v_2 + 0 \cdot v_3$$

$$v_3 = 0 \cdot v_1 + 0 \cdot v_2 + 1 \cdot v_3$$

$$\begin{Bmatrix} v_1 \\ v_2 \\ v_3 \end{Bmatrix} = \underbrace{\begin{bmatrix} 0 & -A_{21}/A_{11} & -A_{31}/A_{11} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\text{A Sweeping Matrix}} \begin{Bmatrix} v_1 \\ v_2 \\ v_3 \end{Bmatrix}$$

$\{v\} = [S^{(2)}]\{v\}$ for Convergence to ω_2 and $\{A_2\}$

where $\{v\}$ is an arbit - trial Vector.

Instead of pre-multiplying $\{v\}$ by $[S^{(2)}]$ at every stage, we can make a new dynamic Matrix $[D^{(2)}] = [D][S^{(2)}]$

Then Iteration Should Start with $[D^{(2)}]\{A_2\} = \frac{1}{\omega_2^2}\{A_2\}$

For Convergence to ω_3 & $\{A_3\}$ a trial Vector $\{w\}$ must be Orthogonal to both $\{A_1\}$ & $\{A_2\}$ & hence,

$$\{A_1\}^T [m] \{w\} = 0$$

$$\rightarrow w_1 A_{11} + w_2 A_{12} + w_3 A_{13} = 0$$

$$\{A_2\}^T [m] \{w\} = 0$$

$$w_1 A_{21} + w_2 A_{22} + w_3 A_{23} = 0$$

This leads to a new Sweeping Matrix $[S^{(3)}]$. The Dynamic matrix for ω_3 & $\{A_3\}$

$$[D^{(3)}] = [D][S^{(3)}]$$

Solving for W_1 & W_2 in terms of w_3, we get equation like

$$W_1 = 0.W_1 + 0.W_2 + \alpha W_3$$
$$W_2 = 0.W_1 + 0.W_2 + \beta W_3$$
$$W_3 = 0.W_1 + 0.W_2 + 1.W_3$$

$$\begin{Bmatrix} W_1 \\ W_2 \\ W_3 \end{Bmatrix} = \begin{bmatrix} 0 & 0 & \alpha \\ 0 & 0 & \beta \\ 0 & 0 & 1 \end{bmatrix} = [S^{(3)}]$$

Actally after obtaining $\{A_1\}$ & $\{A_2\}$ it is no longer necessary to Continue iteration since $\{A_3\}$ Can be obtained by invoking orthogonality viz, A good trial Vector $\{A_3\}$ should have 1 sign change $\begin{Bmatrix} 1 \\ 1 \\ -1 \end{Bmatrix}$

— " ——— $\{A_3\}$ —— " ~~2~~ ——— $\begin{Bmatrix} -1 \\ 1 \\ 1 \end{Bmatrix}$

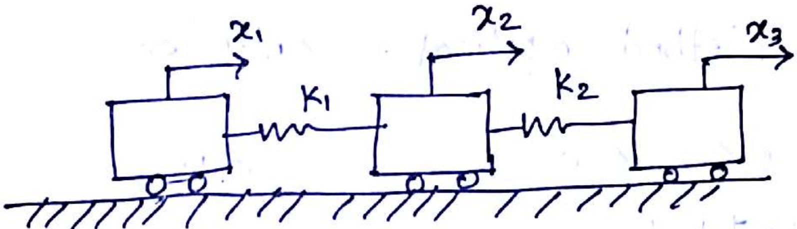
$\{A_1\}^T [m] \{A_3\} = 0$
&
$\{A_2\}^T [m] \{A_3\} = 0$

Solve these two equations for A_{32} & A_{33} in terms of A_{31}.
Let $A_{32} = \gamma A_{31}$ & $A_{33} = \delta A_{31}$
then $\{A_3\} = \begin{Bmatrix} 1 \\ \gamma \\ \delta \end{Bmatrix}$

Then

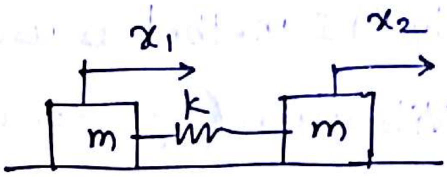
$$\omega_3^2 = \frac{\{A_3\}^T [k] \{A_3\}}{\{A_3\}^T [m] \{A_3\}}$$

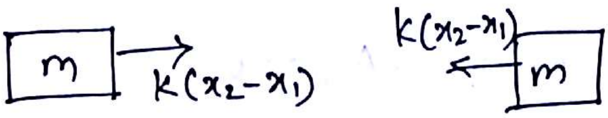
Semi- Definite Systems:



(A· free-free System)

Ex





$m\ddot{x}_1 = k(x_2 - x_1)$

$m\ddot{x}_2 = -k(x_2 - x_1)$

$$[m] = \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix}$$

$$[k] = \begin{bmatrix} k & -k \\ -k & k \end{bmatrix}$$

$$\begin{vmatrix} k-m\omega^2 & -k \\ -k & k-m\omega^2 \end{vmatrix} = 0$$

$$\cancel{k^2} + m^2\omega^4 - 2mk\omega^2 - \cancel{k^2} = 0$$

$$\omega^2(m^2\omega^2 - 2mk) = 0$$

$$\omega_1 = 0 \quad \omega_2 = \sqrt{\frac{2k}{m}}$$

$\omega_1 = 0 \rightarrow$ Rigid Body Mode of Motion is possible.

CM moves with uniform Velocity or remains Rest.

Matrix Method applicable only for

$$0 < \omega_1 < \omega_2 < \ldots < \omega_n$$

as $[K]^{-1}$ does not exist.

When $\omega_1 = 0$ we can reduce the DOF of the System by one of ~~the~~ & then apply MI method as usual

Analytical Methods in Vibration by Leonard Meirowitz

only 2 are independent

Solve A_{21} & A_{31} in terms of A_{11}

$$(2k - m\omega_1^2)A_{11} - kA_{21} + 0 \cdot A_{31} = 0$$

$$-k\,A_{11} + (2k - m\omega_1^2)A_{21} - k\,A_{31} = 0$$

$$0 \cdot A_{11} - kA_{21} + (2k - m\omega_1^2)A_{31} = 0$$

$$\{A_1\} = \begin{Bmatrix} A_{11} \\ \alpha A_{21} \\ \beta A_3 \end{Bmatrix}$$

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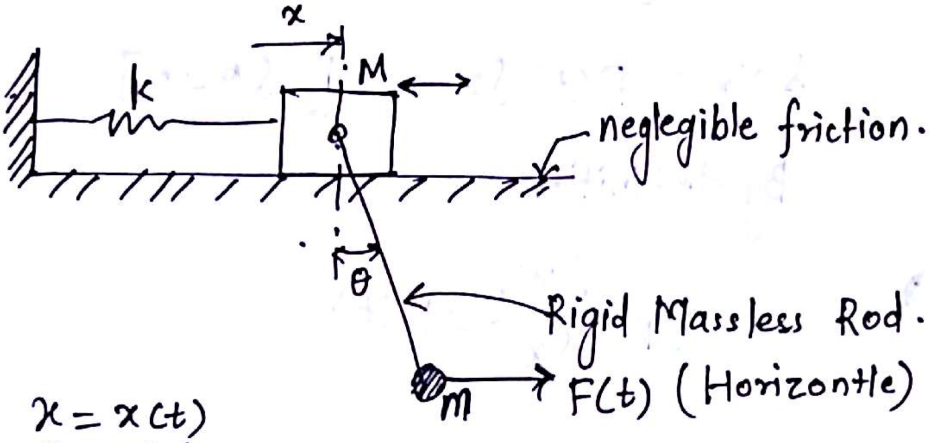
Lagrange's Equation for 2&3 DOF Systems

Ex1.



$x = x(t)$
$\theta = \theta(t)$

Let x, θ be a set of generalised Coordinates.

Obtain the DEOM using Lagrange Equations

Linearize the DEOM.

The Lagranges equations are

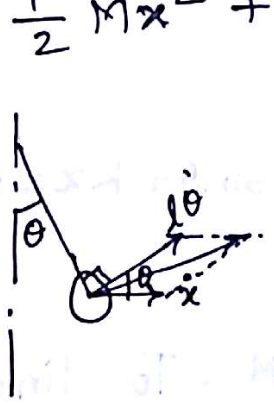
$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}}\right) - \frac{\partial T}{\partial x} + \frac{\partial U}{\partial x} = Q_1$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}}\right) - \frac{\partial T}{\partial \theta} + \frac{\partial U}{\partial \theta} = Q_2$$

generalised forces

$$\frac{\partial D}{\partial \dot{x}} = \frac{\partial D}{\partial \dot{\theta}} = 0 \quad \} \text{ No Damper}$$

$$T = \frac{1}{2} M\dot{x}^2 + \frac{1}{2} m v_{bob}^2$$



$$T = \frac{1}{2} M\dot{x}^2 + \frac{1}{2} m(\dot{x} + l\dot{\theta}\cos\theta)^2 + \frac{1}{2} m(l\dot{\theta}\sin\theta)^2$$

$$T = \frac{1}{2} M\dot{x}^2 + \frac{1}{2} m\left(\dot{x}^2 + l^2\dot{\theta}^2\cos^2\theta + 2l\dot{\theta}\dot{x}\cos\theta\right) + \frac{1}{2} m l^2\dot{\theta}^2\sin^2\theta$$

$$T = \frac{1}{2} M\dot{x}^2 + \frac{1}{2} m(\dot{x}^2 + l^2\dot{\theta}^2 + 2l\dot{\theta}\dot{x}\cos\theta)$$

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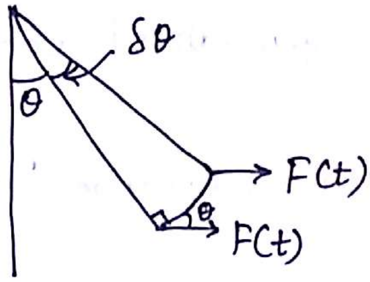
$$U = \frac{1}{2}kx^2 + mgl(1-\cos\theta)$$

$$\delta W_x = F(t)\,\delta x \qquad [\delta\theta = 0]$$

$l(1-\cos\theta)$

By definition

$$Q_1 = \frac{\delta W_x}{\delta x} = F(t)$$



$$Q_2 = \frac{\delta W_\theta}{\delta\theta}$$

$$\delta W_\theta = F(t).\, l\,\delta\theta \cos\theta$$

$$Q_2 = \frac{\delta W_\theta}{\delta\theta} = F(t)\, l \cos\theta$$

$$\left(\frac{\partial T}{\partial \dot{x}}\right) = (M+m)\dot{x} + ml\dot{\theta}\cos\theta$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}}\right) = (M+m)\ddot{x} + ml\ddot{\theta}\cos\theta - ml\dot{\theta}^2\sin\theta$$

$$\frac{\partial T}{\partial x} = 0 \qquad \frac{\partial U}{\partial x} = kx$$

$$(M+m)\ddot{x} + ml\ddot{\theta}\cos\theta - ml\dot{\theta}^2\sin\theta + kx = 0 \quad \text{— (A)}$$

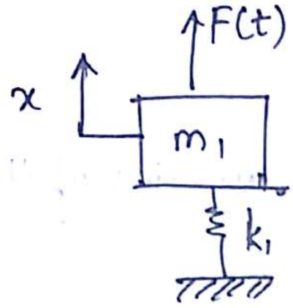
(A) is a non-linear DEOM. To linearize it. We Must assume that $\dot{\theta}^2\sin\theta \approx 0$ & $\cos\theta \approx 1$

The Undamped dynamic Vibration absorber

(The tuned damper)

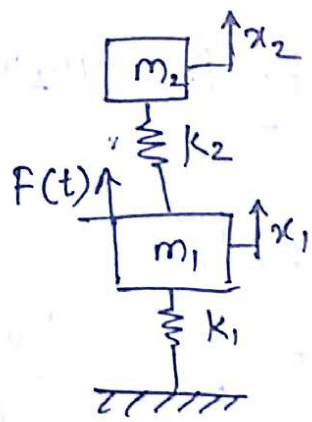
The basic Problem – We have a dynamic System approximately modeled as a Simple Spring-mass System. The mass is subjected to a force. $F(t) = F_0 \sin(\omega_f t)$

It is required to avoid Resonance by using another Spring mass System.



(original System)

The absorber



<u>Given</u>: $\omega_n = \sqrt{\frac{k_1}{m_1}} = \omega_f$

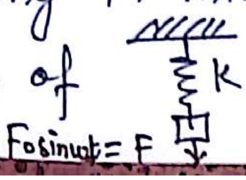
<u>Aim</u>:- To Change m_2 & k_2 Such that the Amplitude of m_1 (i.e. x_1) is acceptable.

The DEOM of the (main System + absorber) are:

$$m_1 \ddot{x}_1 + (k_1 + k_2) x_1 - k_2 x_2 = F_0 \sin(\omega_f t) \quad \text{—— (1)}$$

$$m_2 \ddot{x}_2 - K_2 x_1 + K_2 x_2 = 0 \quad \text{—— (2)}$$

Keeping in mind what happens to the force Vibration of



we Assume $(x_1)_{forced} = X_1 \sin(\omega_f t)$

& $(x_2)_{forced} = X_2 \sin(\omega_f t)$

$$\left.\begin{aligned} \ddot{x}_1 &= -X_1 \omega_f^2 \sin(\omega_f t) \\ \ddot{x}_2 &= -X_2 \omega_f^2 \sin(\omega_f t) \end{aligned}\right\} \text{ substitute in (1) \& (2)}$$

This gives

$$(K_1 + K_2 - m_1\omega_f^2)X_1 - K_2 X_2 = F_0$$

$$-K_2 X_1 + (K_2 - m_2\omega_f^2)X_2 = 0$$

So, if $\omega_f \neq \omega_1$ or ω_2

(ω_1, ω_2 are the natural frequencies of the 2DOF System)

So by Crammers Rule

$$X_1 = \frac{\begin{vmatrix} F_0 & -K_2 \\ 0 & K_2 - m_2\omega_f^2 \end{vmatrix}}{\Delta} = \frac{F_0(K_2 - m_2\omega_f^2)}{\Delta}$$

$$X_2 = \frac{\begin{vmatrix} K_1 + K_2 - m_1\omega_f^2 & F_0 \\ -K_2 & 0 \end{vmatrix}}{\Delta} = \frac{F_0 K_2}{\Delta}$$

Where,

$$\Delta = \begin{vmatrix} K_1 + K_2 - m_1\omega_f^2 & -K_2 \\ -K_2 & K_2 - m_2\omega_f^2 \end{vmatrix}$$

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Hence $X_1 = 0$

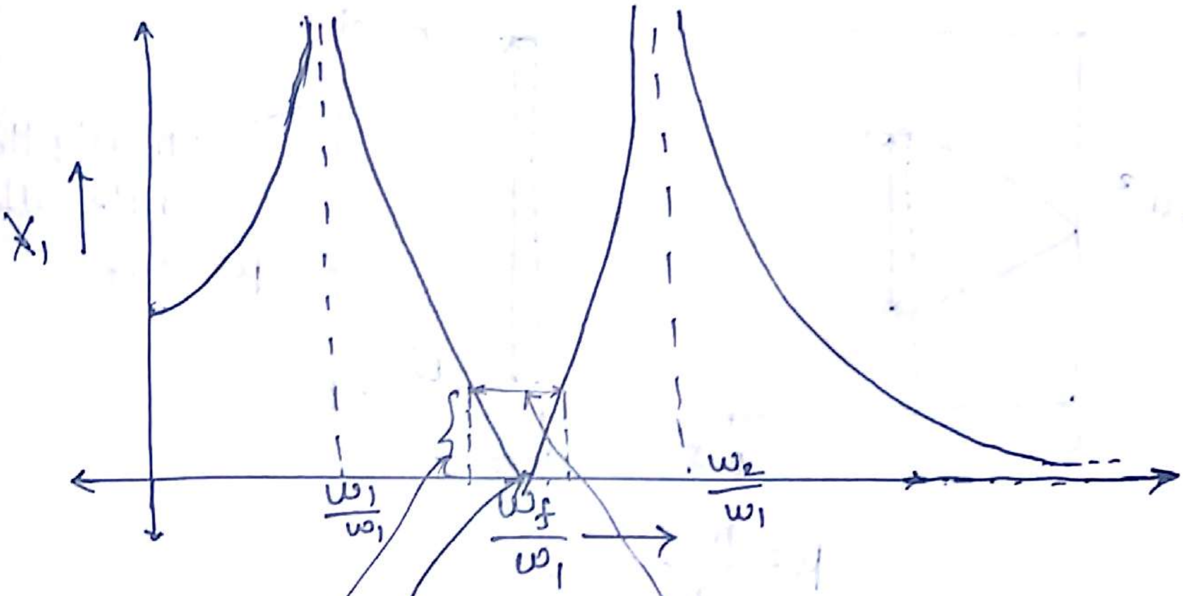
if $k_2 - m_2 \omega_f^2 = 0$

i.e. if $\omega_f^2 = \frac{k_2}{m_2}$

Since, $\omega_f = \omega_n$ This Implies

$$\boxed{\frac{k_2}{m_2} = \frac{k_1}{m_1}} \quad \text{——} \quad (3)$$

(3) gives us a way of Choosing k_2 & m_2. The Forced Response Amplitudes look like the following



Limit of Amplitude X_1 of m_1

$\omega_f = \omega_n$

Permissible fluctuation in ω_f / ω_n.

The frequency Equation

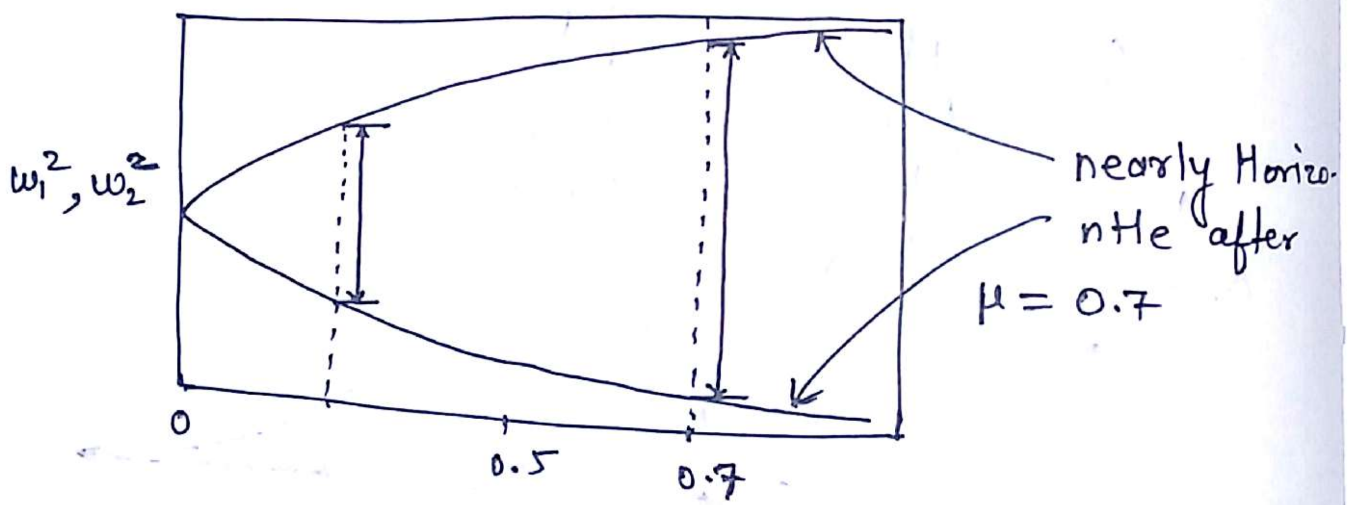
$$\begin{vmatrix} k_1 + k_2 - m_1\omega^2 & -k_2 \\ -k_2 & k_2 - m_2\omega^2 \end{vmatrix} = 0$$

$$(K_1+K_2)K_2 - \omega^2\left[K_2 m_1 + m_2(K_1+K_2)\right] + m_1 m_2 \omega^4 - K_2^2 = 0$$

$$m_1 m_2 \omega^4 - \left[(K_1+K_2)m_2 + K_2 m_1\right]\omega^2 + \left[(K_1+K_2)K_2 - K_2^2\right] = 0$$

$$\omega^4 - \left[\frac{(K_1+K_2)}{m_1} + \frac{K_2}{m_2}\right]\omega^2 + \frac{K_1 K_2}{m_1 m_2} = 0$$

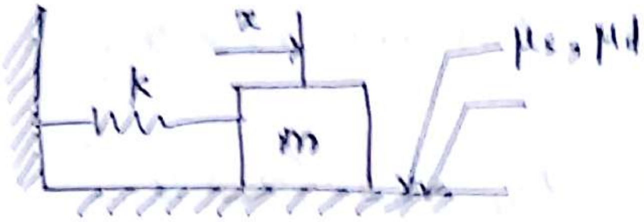
$$\omega^4 - \left[\frac{(K_1+K_2)}{m_1} + \frac{K_2}{m_2}\right]\omega^2 + \omega_n^2 = 0$$



$\mu = \frac{m_2}{m_1} \longrightarrow$

A damped Vibration absorber.

⑤ Coulomb Damping:



The DEOM is

$$m\ddot{x} + kx = \pm F_f \qquad \dot{x} > 0$$

Where F_f = force of friction = $\mu_d\, mg$, while m is in motion.

+ve sign → While mass moves to the right
−ve singn → While mass moves to the left.

While $\dot{x} > 0$

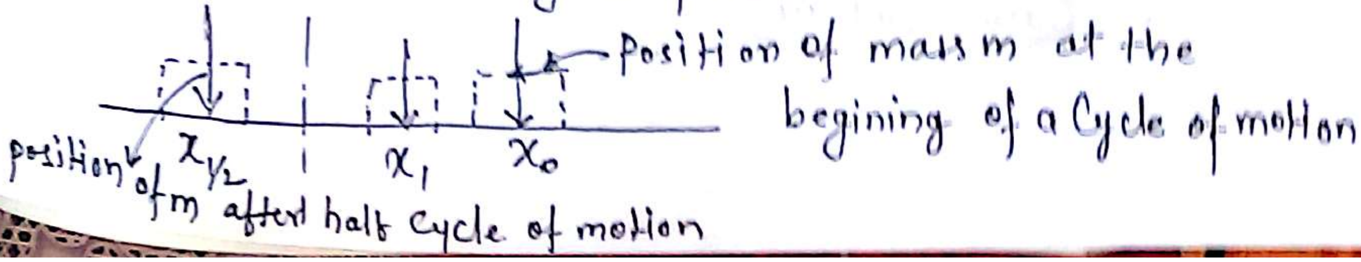
Then $x = A\sin(\omega_n t) + B\cos(\omega_n t) + \frac{F_f}{k}$ is the Complete Response.

This is Valid until t become t_1, such that $\dot{x}(0) = 0$

After $t = t_1$, the motion is governed by

$$\omega^4 - \omega^2\omega_n^2\left(2 + \frac{k_2}{k_1}\right) + \omega_n^4 = 0$$

$$m\ddot{x} + kx = F_f$$ provided $x(t_1)$ is sufficient to Overcome $\mu_s mg$. This clearly shows that the Oscillations occur at the undamped natural frequency ω_n with diminishing amplitude.



Aim : $x_0 - x_1 = ?$

For movement Between x_0 & $x_{1/2}$

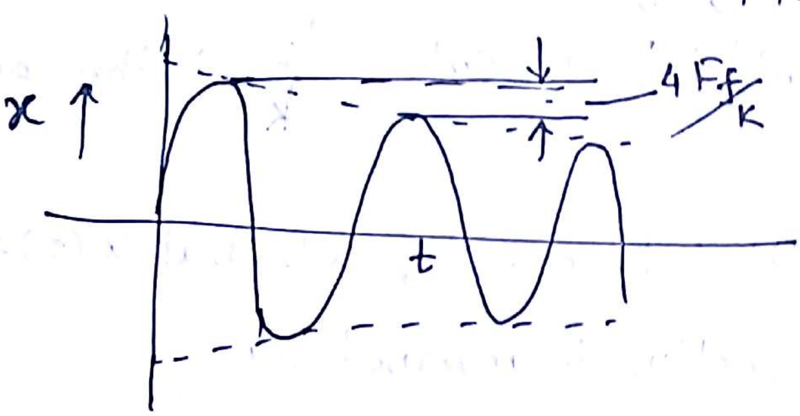
$$\frac{1}{2}k\left(x_0^2 - x_{1/2}^2\right) = F_f\left(x_0 + x_{1/2}\right)$$

$$x_0 - x_{1/2} = \frac{2F_f}{k}$$

Similarly,

$$x_{1/2} - x_1 = \frac{2F_f}{k}$$

$$x_0 - x_1 = \frac{4F_f}{k} =$$ reduction in Amplitude of free Vibration per Cycle of Motion



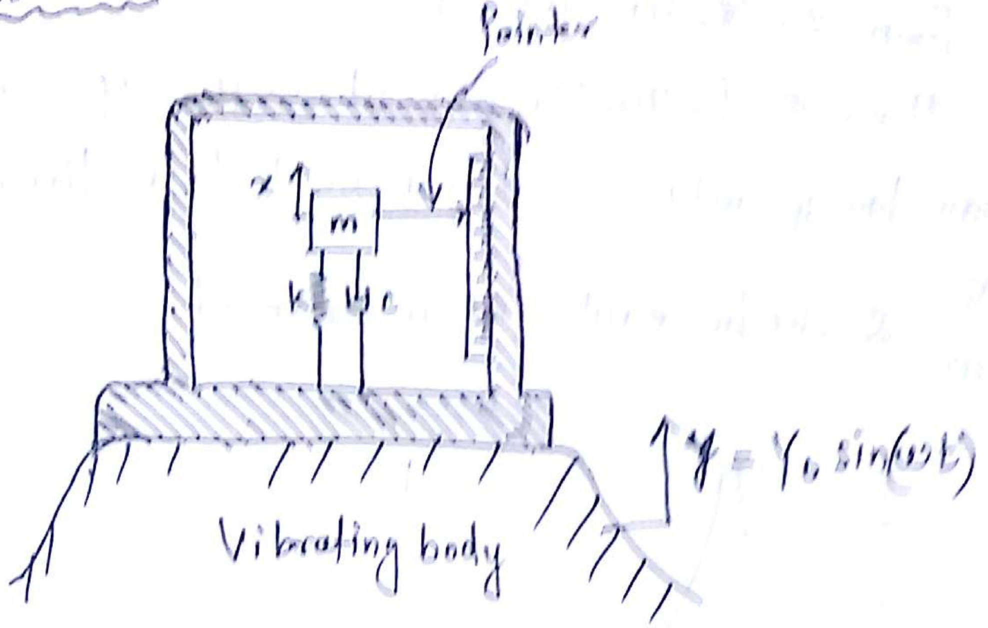
Vibration Measuring Instrument

(Ch 10. Harry's shock and Vibration Handbook

→ Shock & Vibration transducers)

The Basic Set-up

Vibrometer



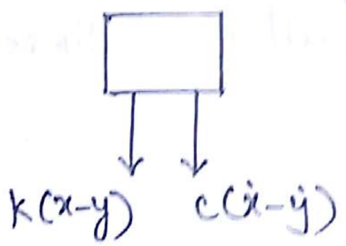
Aim: to measure y $(\dot{y}, \ddot{y})$

The DEOM of m is

$$m\ddot{x} + c(\dot{x} - \dot{y}) + k(x - y) = 0$$

$$m\ddot{z} + c\dot{z} + kz = -m\ddot{y} = mY_0\omega^2 \sin(\omega t)$$

where $z = x - y$



where $z = x - y$

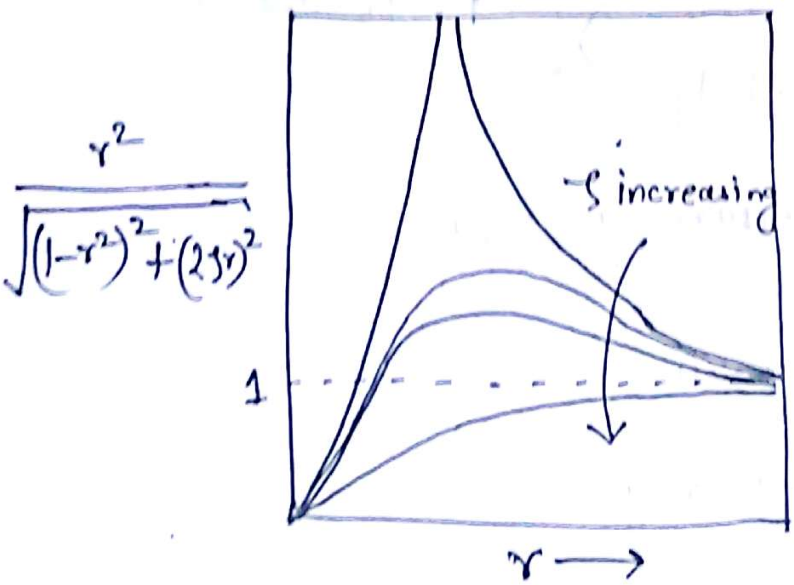
$$z = \frac{mY_0\omega^2/k}{\sqrt{(1-r^2)^2 + (2\xi r)^2}} \sin(\omega t - \psi) \quad \text{—} ②$$

$$\tan(\psi) = \frac{2\xi r}{1 - r^2} \quad \text{—}$$

Hence, if $\dfrac{r^2}{\sqrt{(1-r^2)^2 + (2\xi r)^2}} = 1$ (satisfied only if r is very large.

Then $z = Y_0 \sin(\omega t - \varphi)$

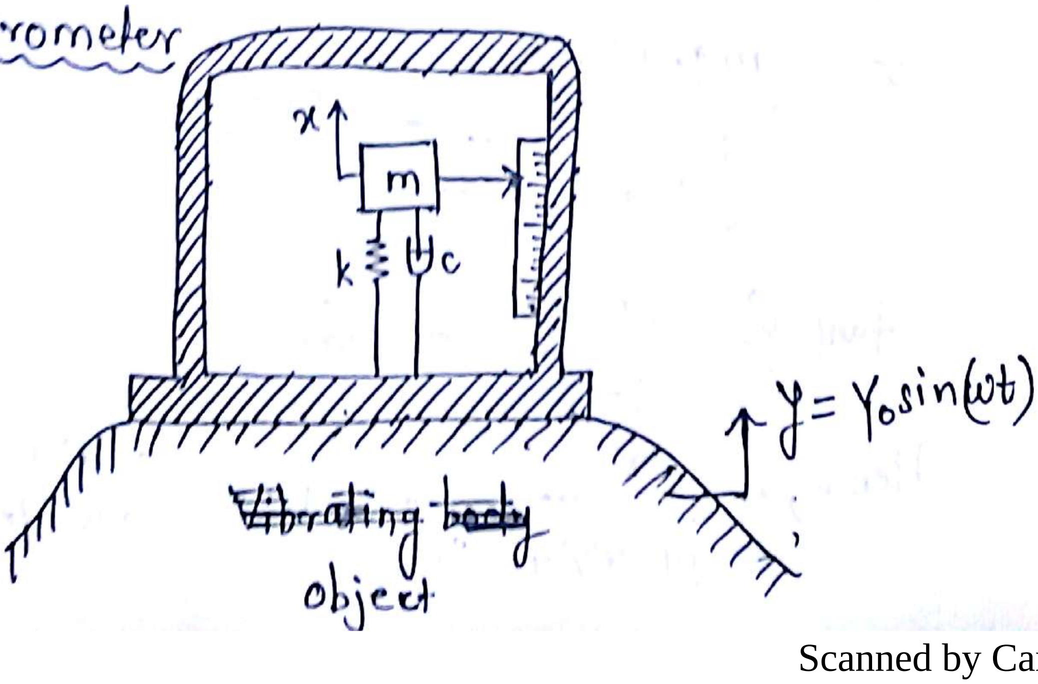
Then z faithfully reproduces y except for the phase lag φ which is Equivalent to a time lag $\frac{\varphi}{\omega}$ & Can be easily taken Care of.



$r = \frac{\omega}{\omega_n}$

Hence, Vibrometer which measure dispacement of a Vibrating body turn out to be bulky & these are not Commonly used now adays.

accelerometer



We have Seen that

$$z_{ss} = z = x - y = \frac{Y_0 r^2}{\sqrt{(1-r^2)^2 + (2\xi r)^2}} \sin(\omega t - \psi) .$$

$$r = \frac{\omega}{\omega_n}$$

$$\Rightarrow \quad -\omega_n^2 z = -\frac{Y_0 \, \omega_f^2 \sin(\omega_f t - \psi)}{\sqrt{(1-r^2)^2 + (2\xi r)^2}} .$$

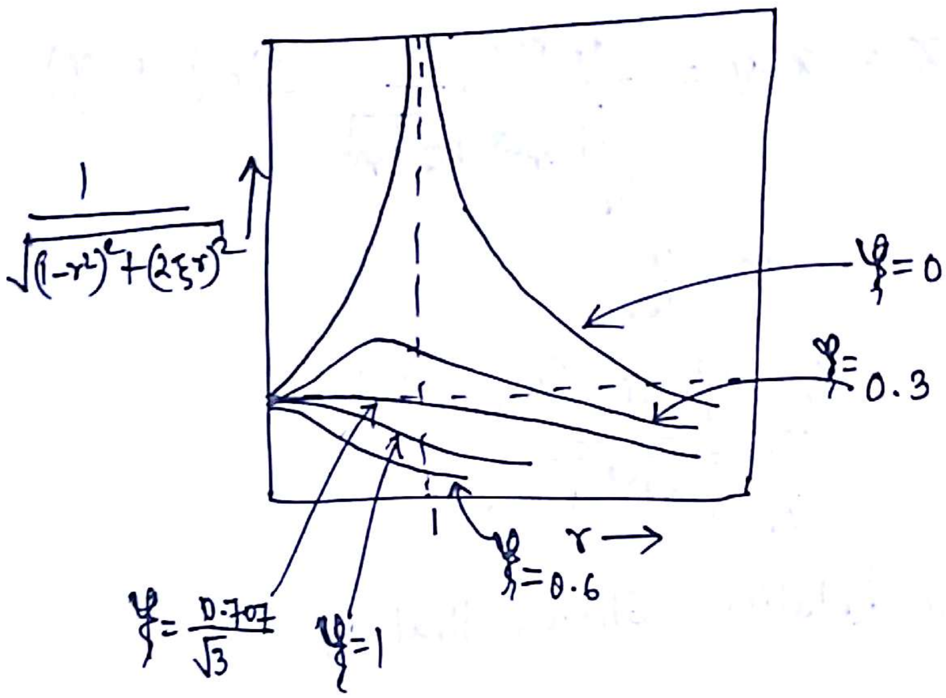
& this Relation Shows that if

$$\frac{1}{\sqrt{(1-r^2)^2 + (2\xi r)^2}} \approx 1$$

Then z reproduces the acceleration $\ddot{y} = -Y_0 \omega^2 \sin \omega t$ faithfully except for sign of a multiplying factor and phase all of which Can be easily taken care of by proper Calibration of Scale..

Hence, for our Spring-mass damper system to act as an accelerometer we must have

$$\frac{1}{\sqrt{(1-r^2)^2 + (2\xi r)^2}} \approx 1$$



So if $\psi = 0.707$
$\xi < 0.6$ then

Hence, an accelerometer should have a large ω_n i.e. a small m and large k both of which can be easily acheived.

Let $y = \sum_{n=1}^{\infty} Y_n \sin(n\omega t)$

(i.e. y is periodic with period $\frac{2\pi}{\omega}$)

Hence,

$$z = \sum \frac{Y_n r_n^2}{\sqrt{(1-r_n)^2+(2\xi r_n)^2}} \sin(n\omega t - \psi_n)$$

Where $\tan\psi_n = \frac{2\xi r_n}{1-r_n^2}$ $\quad r_n = \frac{n\omega}{\omega_n}$

Hence for our accelerometer to work properly we must have $\frac{1}{\sqrt{(1-r_n)^2+(2\xi r_n)^2}} \approx 1$

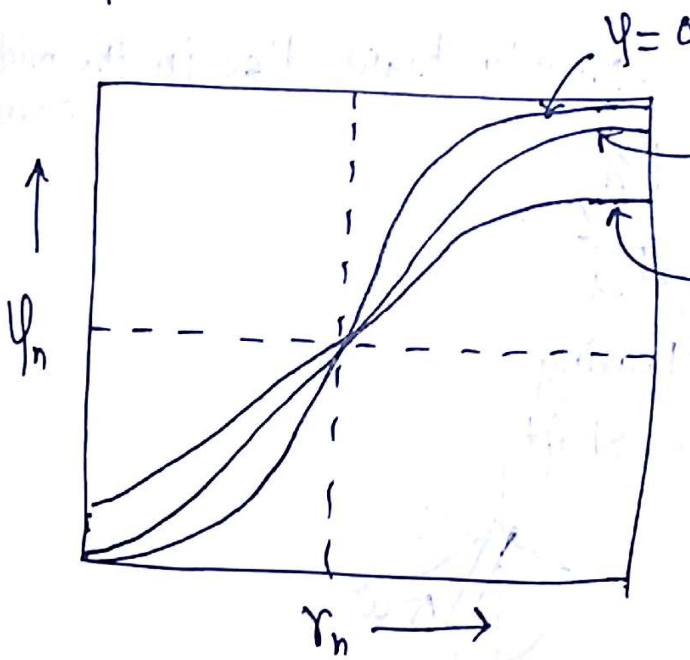
& also $\dfrac{\psi_n}{n\omega} = \text{Constant}$

$$-\omega_n^2 z = \sum_{n=1}^{\infty} \frac{-Y_n (n\omega)^2}{\sqrt{(1-r_n)^2 + (2\xi r_n)^2}} \sin(n\omega t - \psi_n)$$

$$\ddot{y} = \sum_{n=1}^{\infty} (-Y_n)(n\omega)^2 \sin(n\omega t)$$

Hence, by making ω_n sufficiently high, we can easily make $r_n < 0.6$ for values of n upto say 5

for $n > 5$, the Contribution from higher harmonic be quite neglegible & hence, the set up works



For $\psi = 0.707$

ψ_n vs r_n is STL (straight Line)

hence, $\psi_n = \beta\, r_n$

$\beta = \text{const.}$

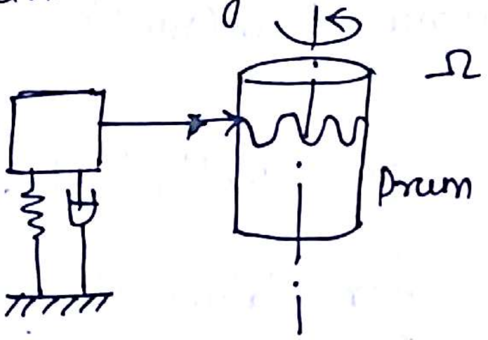
$$\psi_n = \beta \frac{n\omega}{\omega_n} \Rightarrow \frac{\psi_n}{n\omega} = \frac{\beta}{\omega_n} = \text{Const} = \alpha$$

hence, phase distortion error is taken Care of

$$-\omega_n^2 z \approx \sum_{n=1}^{\infty} \left(-Y_n (n\omega)^2\right) \sin(n\omega\tau)$$

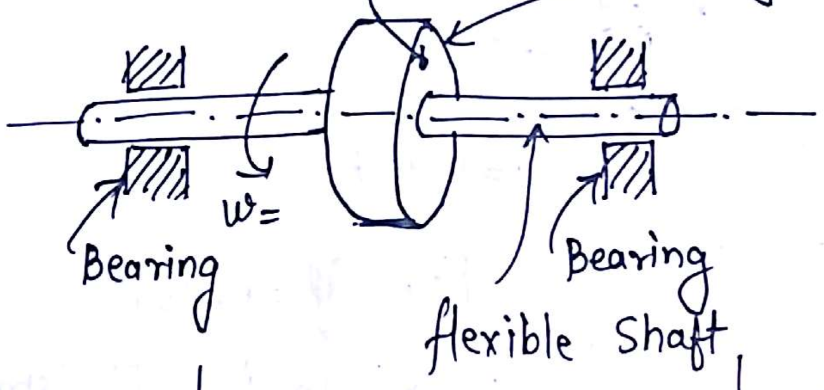
Where $\tau = (t-\alpha)$

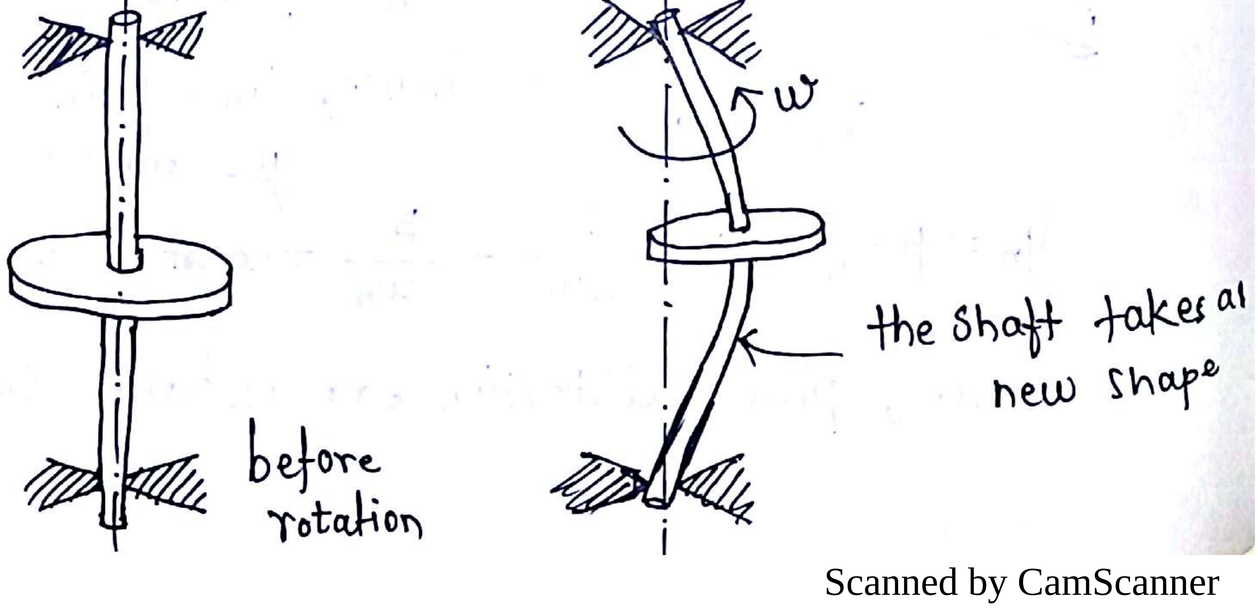
In earlier days



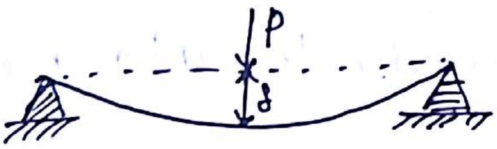
Rotor Dynamics

The Synchro~~chrous~~nous whirling of Shaft :

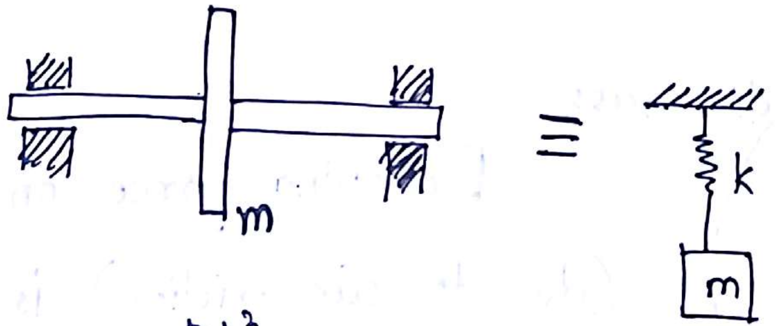




It can be shown that whatever ω = the natural frequency of transverse vibration resonance occurs & large amplitude vibrations can destroy the setup.

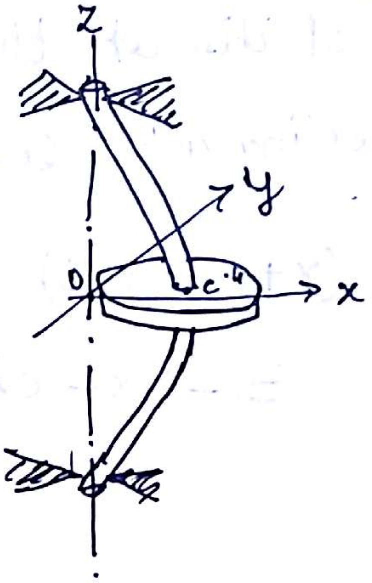


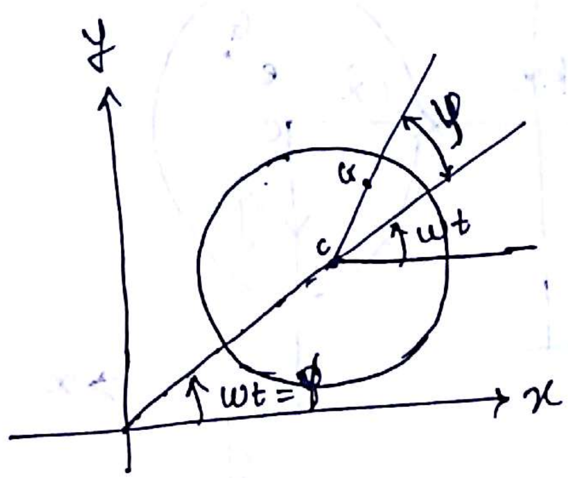
ω_n for beam or lateral or transverse vibration $\sqrt{\frac{k}{m}}$



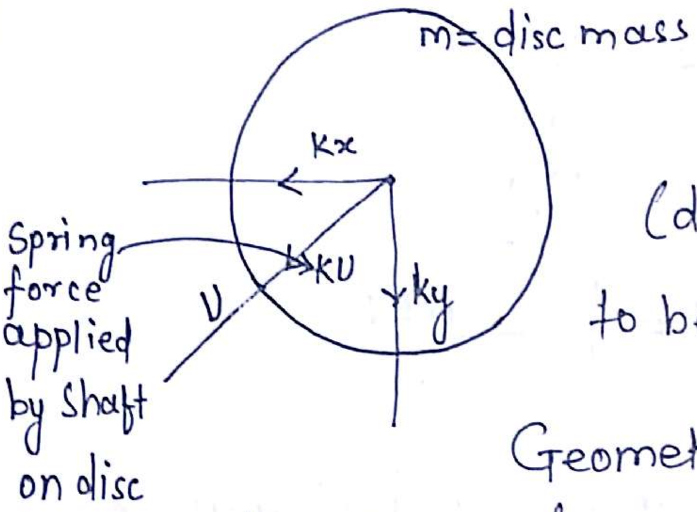
$$\delta = \frac{Pl^3}{48EI} \longrightarrow k = \frac{P}{\delta} = \frac{48EI}{l^3}$$

Hence the plane made by bent axis of the shaft & axis of bearing's rotate which is called whirling of shaft. In asychronous whirl, the shaft and disc have different angular velocities.

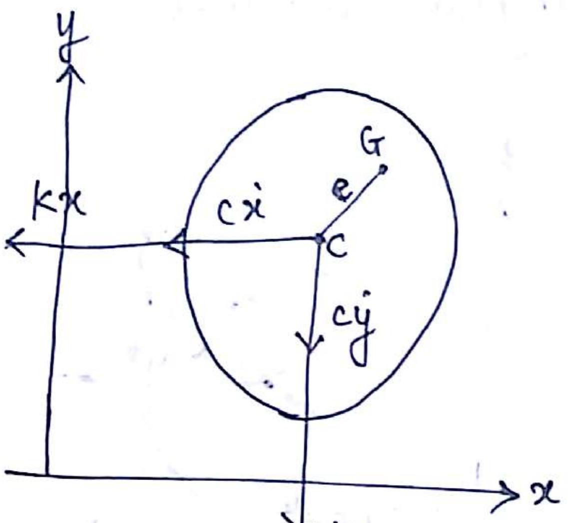




Let us Consider a Special Situation when the geometric Center C of the disc/rotor is different from its Center of gravity G. The Shaft is Connected at C. This Situation gives rise to unbalance & Whirling would occur.



Damping force on a rotor (due to air friction) is assumed to be proportional to the speed of Geometric Center. Hence, the ~~geometric~~ damping force Components would be $c\dot{x}$ and $c\dot{y}$ in $-$ve 'x' and $-$ve 'y' direction.



FBD of disc at time t (neglecting ωt) so,

$$m\frac{d^2}{dt^2}(x+e\cos\omega t) = -kx - c\dot{x}$$

$$\& \; m\frac{d^2}{dt^2}(y+e\sin\omega t) = -ky - c\dot{y}$$

$$m\ddot{x} + c\dot{x} + kx = me\omega^2 \cos(\omega t)$$

$$m\ddot{y} + c\dot{y} + ky = me\omega^2 \sin(\omega t)$$

$$x_{ss} = x = \frac{me\omega^2/k}{\sqrt{(1-r^2)^2 + (2\xi r)^2}} \times \cos(\omega t - \varphi)$$

$$y_{ss} = y = \frac{me\omega^2/k}{\sqrt{(1-r^2)^2 + (2\xi r)^2}} \times \sin(\omega t - \varphi)$$

$$r = \frac{\omega}{\omega_n} \qquad \omega_n = \sqrt{\frac{k}{m}}$$

$$\frac{me\omega^2}{k} = er^2$$

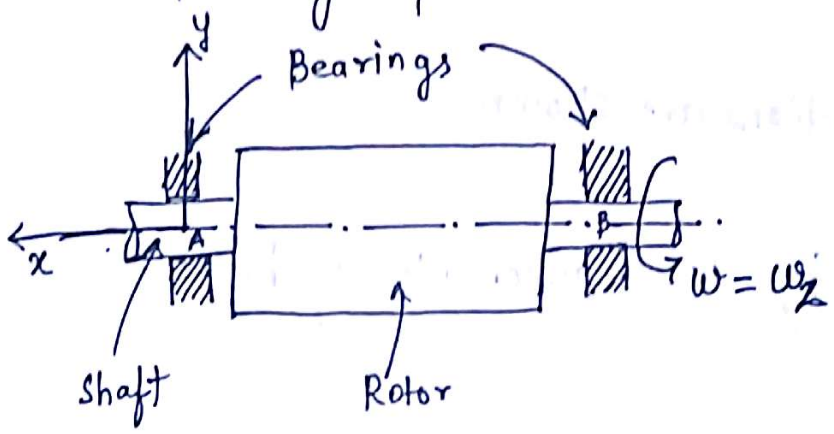
hence, $U = \sqrt{x^2+y^2} = \dfrac{er^2}{\sqrt{(1-r^2)^2 + (2\xi r)^2}}$

Balancing of M/c's

Effects of Unbalance :

1) Large (unacceptable) levels of Vibration.
2) Fatigue / premature failure of bearings

3) Noisy Operation.

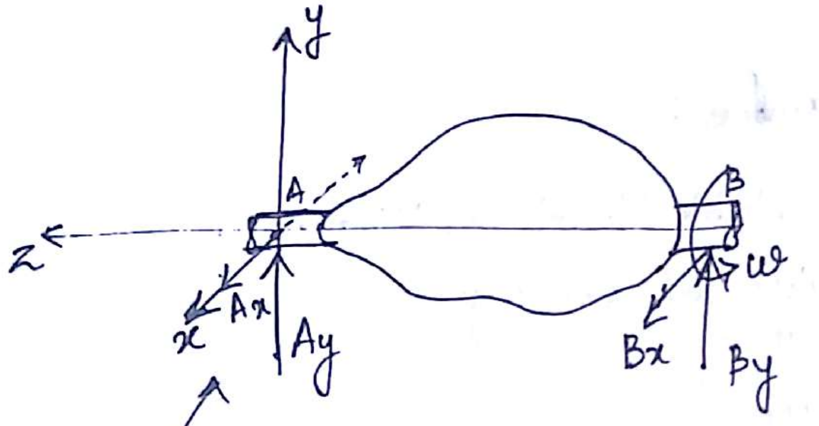


There are two types of unbalance.

(i) Static Unbalance
(ii) Dynamic Unbalance

The System is Said to be in Static balance if its Center of mass lies on axis of rotation. It is Said to be in dynamic balance if it is in Static balance & in addition, the axis of rotation is a principal axis at any point on the axis of rotation. If the rotor-Shaft System is in Static balance, it will remain in equillibrium at any angular position. If the System is in dynamic balance no unbalanced forces will act on the bearings & Hence Vibration

& failure of the Setup may be avoided.



$l = AB$ (Dist. bet^n mid pt of bearings)

Only Dynamic Reactions are shown

A_x, A_y, B_x, B_y are Components of bearing Reactions at A & B.

$z \rightarrow$ axis of Rotation

We have plane motion since Every particle of the System moves in a plane parallel to the xy plane. Hence, MOM Equations are.

$$M_x = B_y l = I_{xz}\dot{\omega}_{\cancel{y}z} - I_{yz}\,\omega_z^2$$

$$M_y = B_x l = I_{yz}\,\dot{\omega}_z - I_{xz}\,\omega_z^2$$

$$M_z = I_{zz}\,\dot{\omega}_z$$

where $[I] = \begin{bmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{xy} & I_{yy} & -I_{yz} \\ -I_{xz} & -I_{yz} & I_{zz} \end{bmatrix}$ is the

Inertia tensor for the Shaft-rotor System at A.

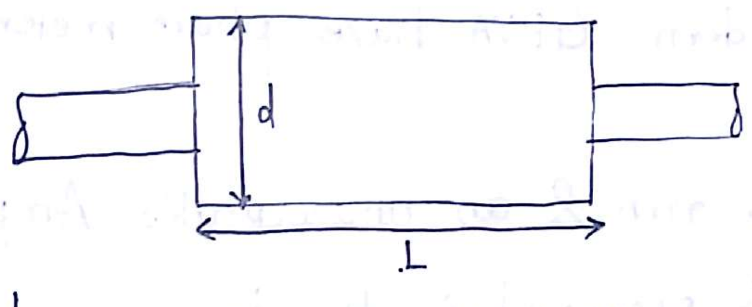
Hence, if $I_{xz} = I_{yz} = 0$

then $B_y = 0$ & $B_x = 0$
~~then~~ Hence, $A_x = 0$ & $A_y = 0$ } ~~Thies~~ no rotating fatigue Causing forces on bearings. They would Balance the weight of setup at all times.

Formally, the Necessary & Sufficient Conditions for the dynamic balance of a rotor System are:

(1) The CM Must lie on axis of Rotation

(2) This axis must be Principal axis at any point on axis of Rotation.



If $\frac{L}{d} \sim 1$ we Call it a long rotor which Requires 2-plane balancing.

If ~~L~~$\approx 0.5d$ or less we may Call it a thin rotor ~~which requires~~ where Single-Plane balancing will do.

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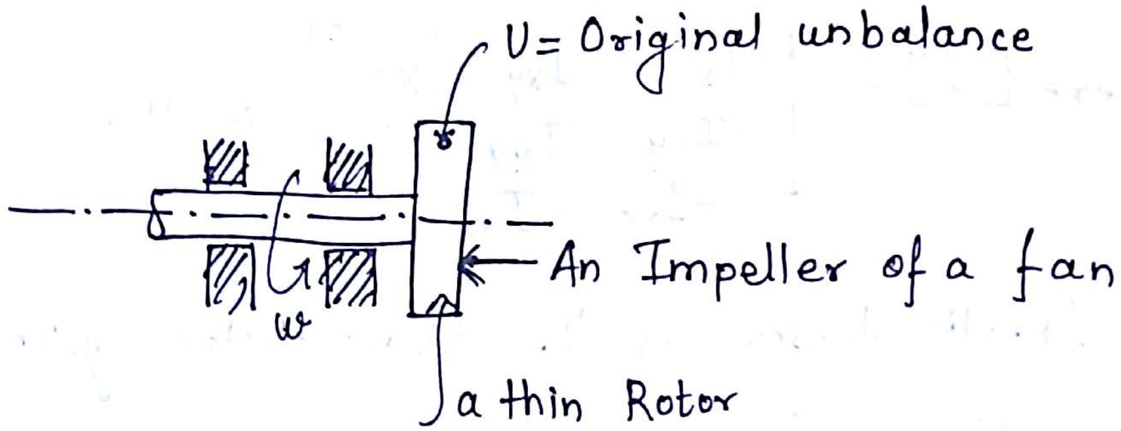


Hence Single - Plane balancing will do.

Single-Plane field balancing of Rotors.

This method is application to thin rotors only.

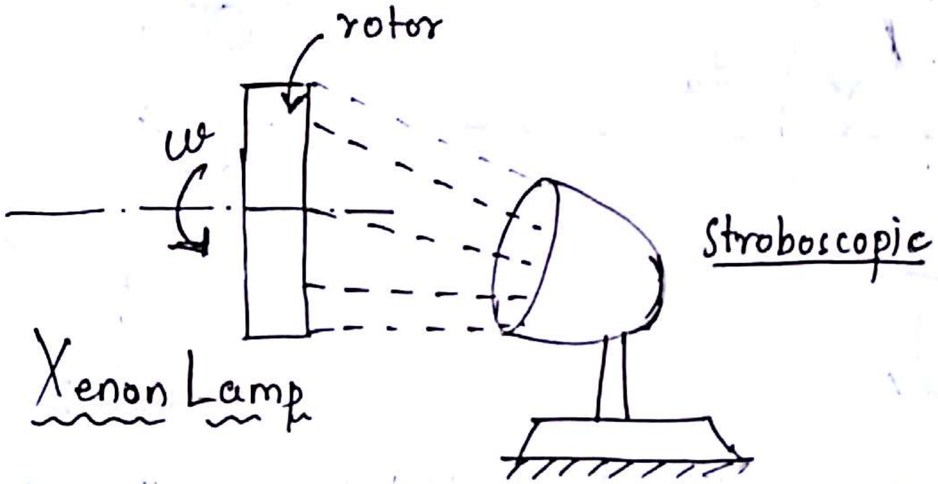
A General Purpose Vibration analyser Contains mechanical & electrical/electronic Components The electronic/electrical unit Converts the mechanical/electrical Output of the Vibration transducers (accelerometer) to Suitable form using frequency filters.

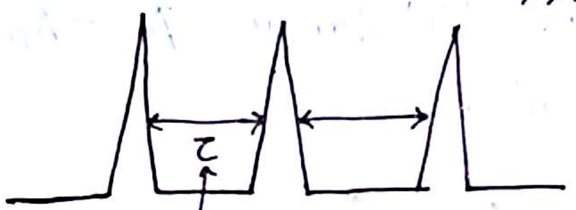
There is a Stroboscopic unit for measuring phase angles. (Modern GPVA have phase meters too.)

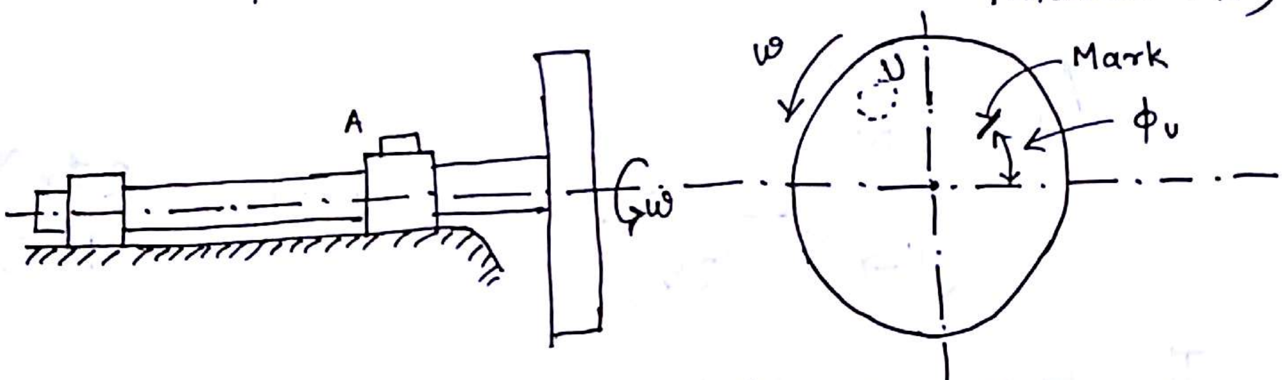
The Rotor is run & an unacceptable Amplitude of Vibrations is observed in the transverse direction at the Rotational frequency (ω) of the Shaft.

It is Very Likely caused by an unbalance U in the rotor.

Aim: To measure U & take Corrective action to eliminate unbalance.







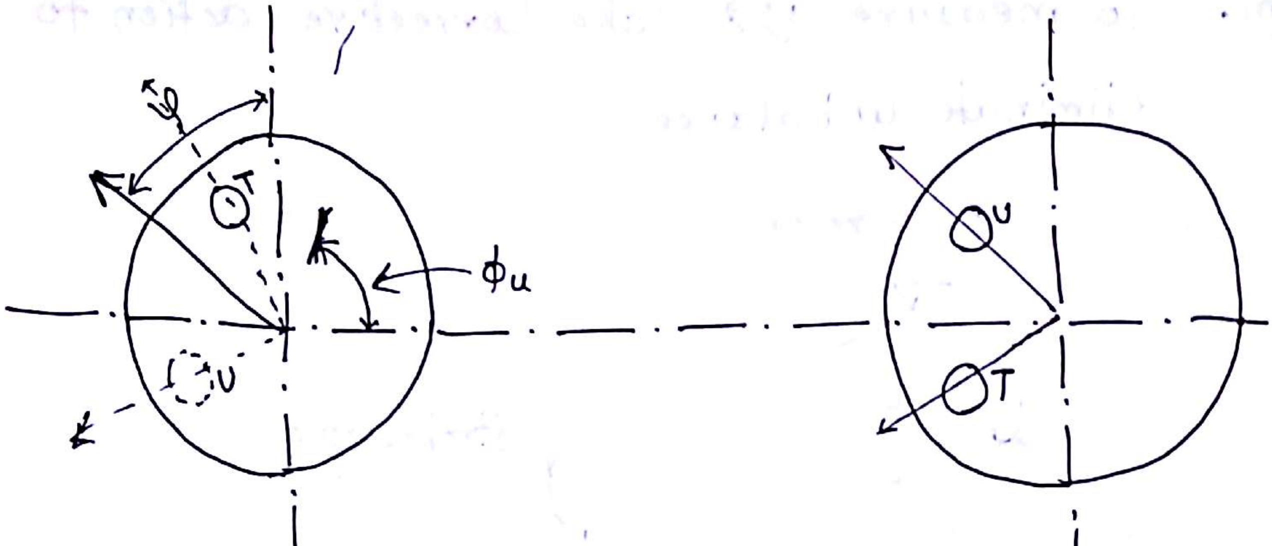
Procedure: 1) Make a radial (phase) mark on rotor.

2) Adjustust the GPVA so that it Locks on frequency ω & the Stroboscope flashes at moment when the Vibration pickup A undergoes maximum deflection in vertical direction.

3) The m/c is run with original Unbalacy U & the position of phase to be ϕ_u. Amplitude

A_u is also measured.

4) The m/c is stopped & trial Unbalancy T is put.



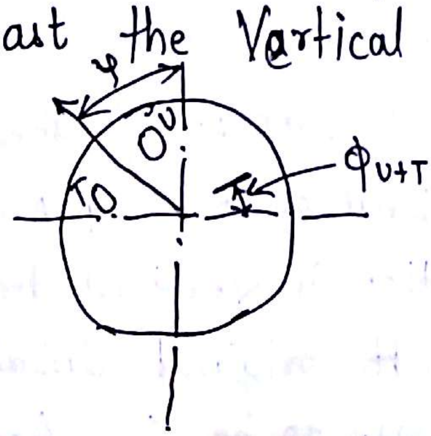
Case A   Case B

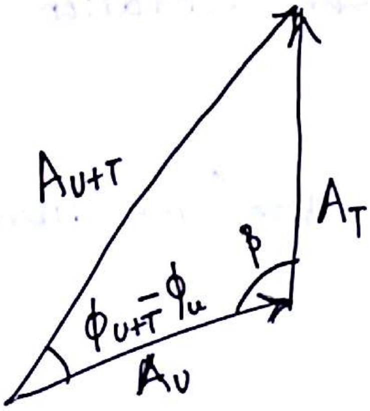
5) Run the M/c with trial Unbalancy A_T = Amplitude of Vibration with trial Unbalancy.

$$\frac{A_T}{A_u} = \frac{T}{U} \qquad A_u = \frac{U}{T} A_T$$

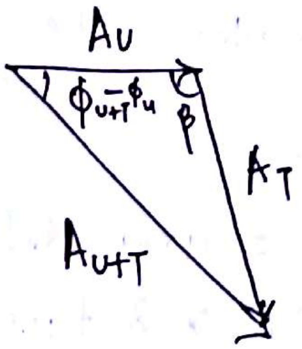
$$A_u = \frac{\frac{Ur^2}{M}}{\sqrt{(1-r^2)^2 + (2\xi r)^2}} \qquad \varphi = \tan^{-1}\left(\frac{2\xi r}{1-r^2}\right)$$

The Stroboscope flashes Once the Resultant U goes past the Vartical by an angle φ.





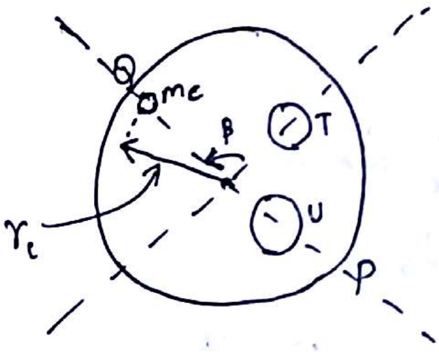
Case B



Case A

$$A_T^2 = A_{U+T}^2 + A_u^2 - 2 A_u A_{U+T} \cos(\phi_{U+T} - \phi_u)$$

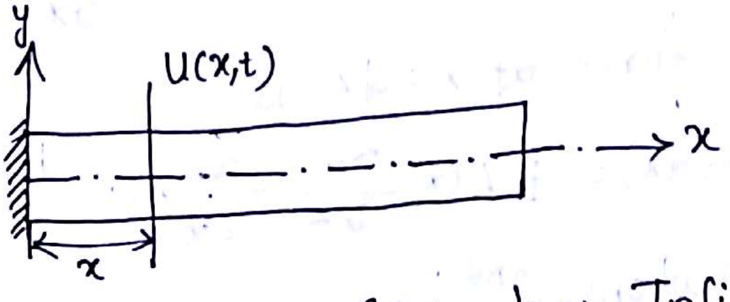
$$U = T \frac{A_u}{A_T}$$



$U = m_c r_c$

To balance the M/c either some mass is removed at Convinient location on Radius OP or a mass is added at OQ.

* Vibration of Continuous Systems



A Continuous System has Infinitely many-natural frequencies

$m(x) \rightarrow$ mass per unit length at x.

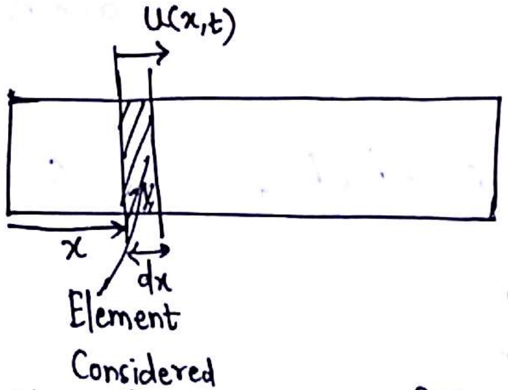
$A(x) \rightarrow$ c/s-section at ~~a~~ location x.

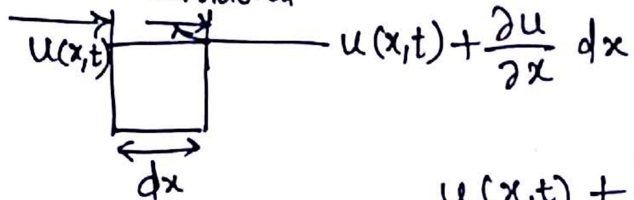
An Example

Let us Consider the Axial ~~force~~ free Vibration of damped - free bar.

Let $u(x,t)$ = axial deflection due to ~~the~~ free Vibration at Location x & time t.

Aim : To obtain DEOM.





$$\varepsilon_x = \frac{u(x,t) + \frac{\partial u}{\partial x}dx - u(x,t)}{dx}$$

$$\varepsilon_x = \frac{\partial u}{\partial x} \qquad \boxed{\sigma_x = E\frac{\partial u}{\partial x}}$$

axial force at x $= \sigma_x \times (A(x)) = EA(x)\frac{\partial u}{\partial x} = P(x)$

So, axial force at $x+dx$ is

$$P(x+dx) = EA(x)\frac{\partial u}{\partial x} + \frac{\partial}{\partial x}\left[EA(x)\frac{du}{dx}\right]dx$$

Hence by Newtons 2nd Law

$$\underbrace{m(x)\,dx}_{\text{mass of element}} \times \frac{\partial^2 u}{\partial t^2} = P(x+dx) - P(x)$$

Hence

$$m(x)\frac{\partial^2 u}{\partial t^2}=\frac{\partial}{\partial x}\left[EA(x)\frac{\partial u}{\partial x}\right]$$ is the Required DEOM.

The Boundary Conditions are -

$u(x,t)=0$ at $x=0$ at all times

$EA\frac{\partial u}{\partial x}\Big|_{x=l}=0$ at all times

The DEOM of bar in Axial Vibration was Seen to be

$$m(x)\frac{\partial^2 u}{\partial t^2}=\frac{\partial}{\partial x}\left[EA(x)\frac{\partial u}{\partial x}\right]$$

For a Uniform bar, it reduces to

$$m\frac{\partial^2 u}{\partial t^2}=EA\frac{\partial^2 u}{\partial x^2}$$

$$\frac{\partial^2 u}{\partial t^2}=c^2\frac{\partial^2 u}{\partial x^2} \quad — (2)$$

$$C=\sqrt{\frac{EA}{m}}=\text{wave Speed}$$

(2) is 1-D wave Equation

To Solve it we can use methods of Variable Separation

Let $u(x,t)=u(x)\,f(t)$ — (3)

Where $u(x)$ is an eigen function & $f(t)$ is generalised Coordinate.

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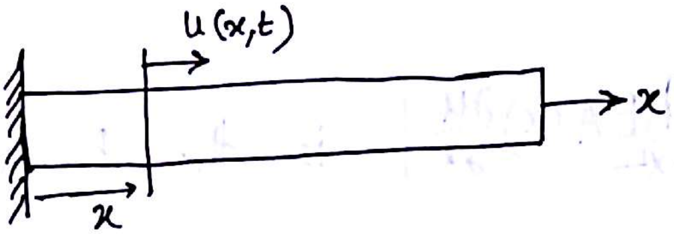


$u(x,t)$ is axial displacement of Location x & time t.

Substituting ③ into ② give

$$u(x)\ddot{f}(t) = c^2 u'' f$$

or

$$\frac{\ddot{f}(t)}{f(t)} = \frac{u'' c^2}{u} \quad \text{—} ④$$

Where $\ddot{f}(t) = \frac{\partial^2 f}{\partial t^2}$ & $u'' = \frac{\partial^2 u(x)}{\partial x^2}$

Each Ration in ④ must be Constant

Hence

$$\frac{\ddot{f}(t)}{f(t)} = \frac{c^2 u''}{u} = -\omega^2 \quad \text{a -ve constant}$$

(otherwise $f(t)$ would increase exponentially which isn't possible for our System executing Stable Oscillations)

So then,

$$\ddot{f}(t) + \omega^2 f(t) = 0 \quad \text{—} ⑤$$

& $$\frac{\partial^2 u(x)}{\partial x^2} + \left(\frac{\omega}{c}\right)^2 u = 0 \quad \text{—} ⑥$$

$$\frac{\omega}{c} = \beta$$

From ⑤ we get

$$f(t) = A\sin(\omega t) + B\cos(\omega t) \quad \text{— (7)}$$

From ⑥ we get

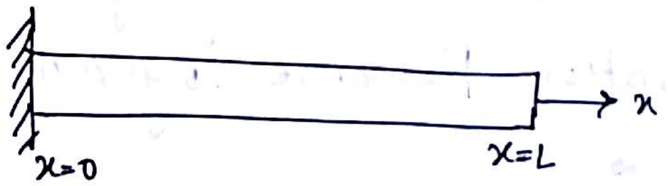
$$u(x) = C\sin(\beta x) + D\cos(\beta x) \quad \text{— (8)}$$

where $\beta = \frac{\omega}{c}$

A & B are to be obtained from given initial Condition $u(x,0)$ & $\left.\frac{\partial u}{\partial t}\right|_{t=0}$

to get C and D we need the boundary Conditions (BCs)

Let Our bar be damping free



The BC's

$$u(0,t) = 0$$

$$EA\left.\frac{\partial u}{\partial x}\right|_{x=l} = 0$$

From (8)

$$u(0) = 0$$

Then $u(x) = A\sin(\beta x)$

$$\left.\frac{\partial u}{\partial x}\right|_{x=l} = A\beta \left.\cos(\beta x)\right|_{x=l} = 0$$

$$\beta_n L = (2n-1)\frac{\pi}{2} \qquad n = 1, 2, 3 \ldots\ldots$$

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$$\omega_n = \frac{(2n-1)\pi c}{2l}$$

$$\omega_n = (2n-1)\frac{\pi}{2}\sqrt{\frac{EA}{ml^2}} \qquad n = 1, 2 \ldots$$

Hence the bar Has Infinitely many natural frequencies of which the first few matter in practice.

Also, $U_n(x) = C_n \sin\left(\frac{\omega_n x}{c}\right) + D_n \cos\left(\frac{\omega_n x}{c}\right)$

Finally, $f_n(t) = A_n \sin(\omega_n t) + B_n \cos(\omega_n t)$

So, $u_n(x,t) = U_n(x)\, f_n(t)$ is a Solution of equation (2) Thus by the principle of Super-position (Since ours is a Linear System the general free Vibration Response is given as)

$$u(x,t) = \sum_{n=1}^{\infty} U_n(x)\, f_n(t)$$

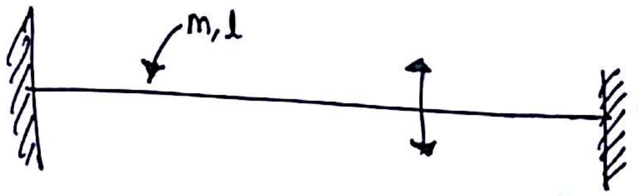
$$= \sum_{n=1}^{\infty} \left(A'_n \sin(\omega_n t) + B'_n \cos(\omega_n t)\right) \sin\left(\frac{\omega_n x}{2}\right)$$

A'_n & B'_n are evaluated using given initial Conditions.

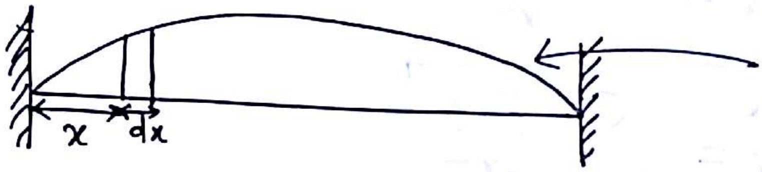
Transverse free Vibration of a Stretched Spring with a high Initial Tension T

(T does not change appreciably during Vibration)

m = mass per unit length / Linear density.

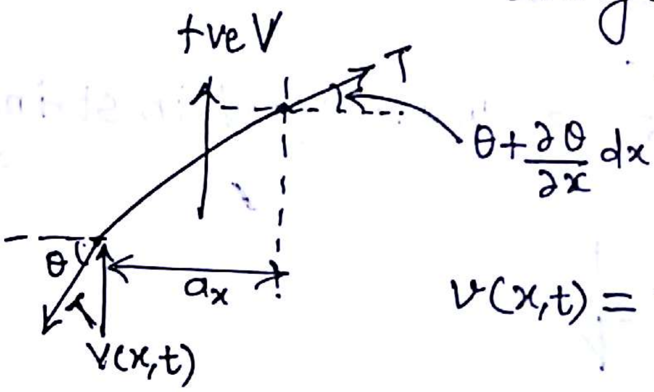


Initial - Configuration



String at time t

String Element Considered



$v(x,t)$ = transverse displacement of Spring at location x and time t

Transverse acceleration of element $\frac{\partial^2 v}{\partial t^2}$

So by Newtons ~~Method~~ 2nd Law

$$m\,dx \times \frac{\partial^2 v}{\partial t^2} = \text{net Component of } T \text{ in +ve direction}$$

$$= T\sin\left(\theta + \frac{\partial \theta}{\partial x} dx\right) - T\sin\theta$$

$$= T\left(\cancel{\theta} + \frac{\partial \theta}{\partial x} dx - \cancel{\theta}\right)$$

$$m\frac{\partial^2 v}{\partial t^2} = T\frac{\partial v}{\partial x}$$

Now $\theta \approx \frac{\partial v}{\partial x}$ = slope of the String/Cable at Location x & time t.

Thus we have

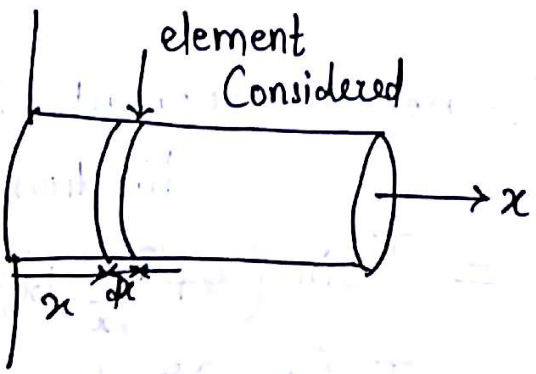
$$m\frac{\partial^2 v}{\partial t^2} = T\frac{\partial^2 v}{\partial x^2} \quad \text{or}$$

$$\boxed{\frac{\partial^2 v}{\partial t^2} = c^2\frac{\partial^2 v}{\partial x^2}}$$

$c = \sqrt{\frac{T}{m}}$ = wave speed in string

B.C $v(0,t) = 0$

Torsional free Oscillation of a Circular bar



$G, J, I, l, \rho(r)$

G = Shear modulus

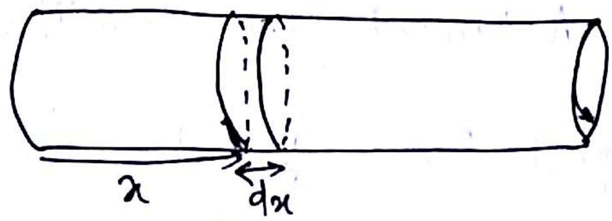
J = Polar area moment of C/s

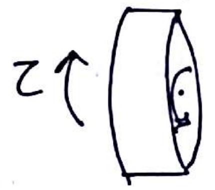
l = length of shaft $\qquad J = \frac{\pi r^4}{2}$

I (per ~~point~~ unit length if shaft is Uniform)

$$= \frac{I_{total}}{l} = \frac{1}{2}\frac{Mr^2}{l}$$

$$M = \rho \pi r^2 l$$





$$\left(\tau + \frac{\partial \tau}{\partial x}.dx\right)$$

$$\tau(x,t) =$$

Torque acting on section at location x at time t

$$\frac{\tau}{J} = G\frac{\partial \phi}{\partial x} = \frac{G\phi_{max}}{l}$$

MOM Equation for our System is

$$I\,dx\,\frac{\partial^2 \phi(x,t)}{\partial t^2} = \tau + \frac{\partial \tau}{\partial x}dx - \tau$$

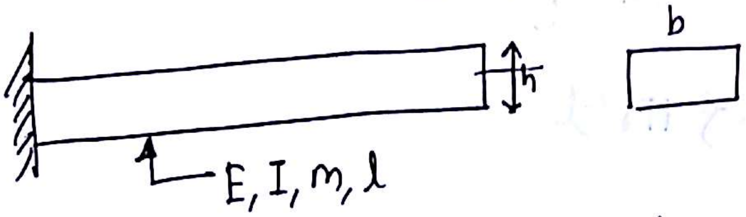
$$I\frac{\partial^2 \phi}{\partial t^2} = \frac{\partial}{\partial x}\left[GJ\frac{\partial \phi}{\partial x}\right]$$

$$I(x)\frac{\partial^2 \phi}{\partial t^2} = \frac{\partial}{\partial x}\left[GJ(x)\frac{\partial \phi}{\partial x}\right]$$

If the shaft is uniform then $\frac{\partial^2 \phi}{\partial x^2} = c^2\frac{\partial^2 \phi}{\partial x^2}$ is the DEOM where $c = \sqrt{\frac{GJ}{I}}$

C= speed of Shear wave in bar

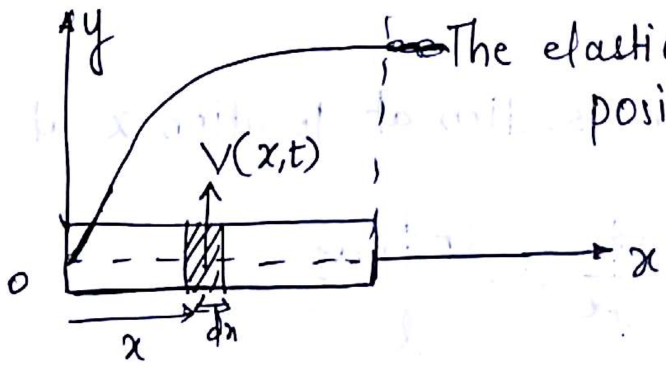
Transverse Vibration of an Euler-Bernoulli beam



· A damped free (cantilever EB Beam)

If the aspect Ratio $\frac{l}{h}$ of a beam is ≥ 10, we may call it as EB Beam.

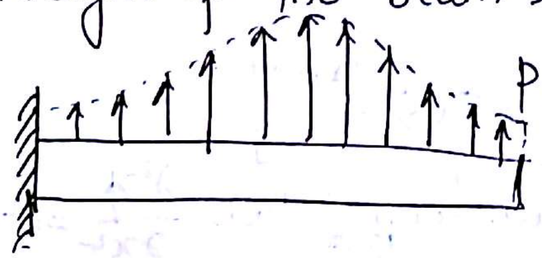
Also ~~b~~ b Should be Comparable with h.



The element Considered in equillibrium position

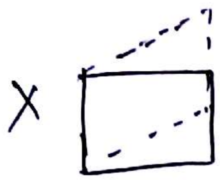
$v(x,t) \longrightarrow$ Lateral deflection of an element at Location x at time t.

Weight of the beam is neglected.



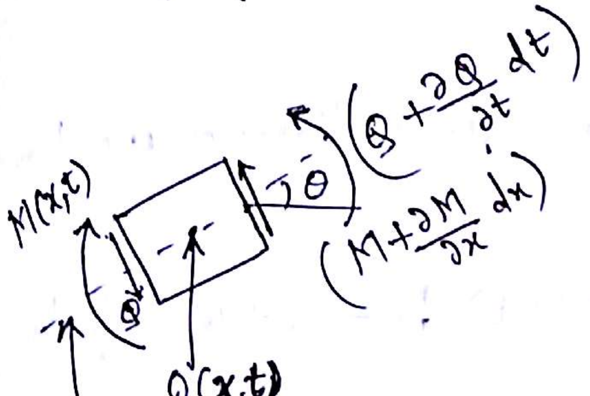
$p(x,t) \rightarrow$ transverse loading intensity

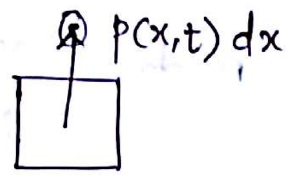
= load/unit length at x at time t



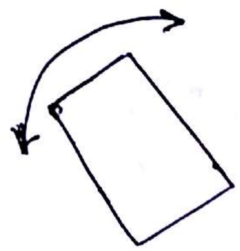
Still Remain Rectangle

If Shear deformation is Concidered, we get rayleigh beam.





Elastic Line at time t



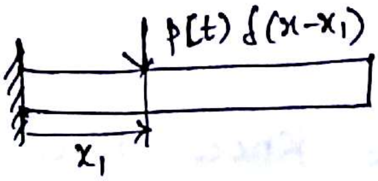
Rotary (Rotational) Inertia of beam is neglected

If it is taken into account, we get a Timoshenko beam. These effects are important for Short beam only.

$\sin\theta \approx \theta$ $\cos\theta \approx 1$ also, slope of beam $= \frac{\partial u}{\partial x} \approx \theta$

Assumptions: Deflection as well as Slope of beam Remains Small.

$\delta(x-x_1) \longrightarrow$ spatial dirac delta function $p(t)\,\delta(x-x_1)$



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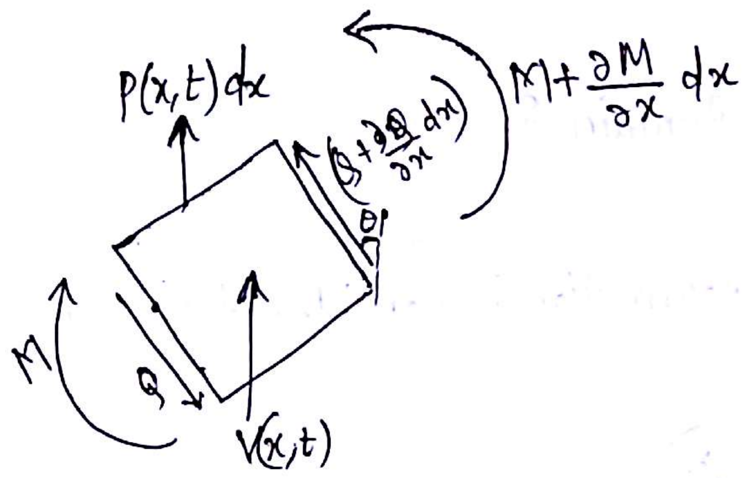

$\cos\theta \approx 1$

Using Newtons 2nd Law in y direction we have

$$m\,dx \frac{\partial^2 v(x,t)}{\partial t^2} = p(x,t)\,dx - Q + Q + \frac{\partial Q}{\partial x}dx$$

$$m\frac{\partial^2 v}{\partial t^2} = p(x,t) + \frac{\partial Q}{\partial x} \quad \text{—} ①$$

We Know that

$$M = \text{Bending Moment} = EI\frac{\partial^2 v}{\partial x^2}$$

Also, $Q = -\frac{\partial M}{\partial x}$

Hence ① Reduce to

$$m\frac{\partial^2 v}{\partial t^2} + \frac{\partial^2}{\partial x^2}\left[EI\frac{\partial^2 v}{\partial x^2}\right] = p \quad \text{—} ②$$

Which is Required DEOM

Taking Moment about A.

$$Q\,dx + \frac{\partial M}{\partial x}dx = 0$$

$Q = -\frac{\partial M}{\partial x}$ We know that

If $m \neq m(x)$

$I \neq I(x)$

i.e. If the beam is uniform (2) reduces to

$$m\frac{\partial^2 v}{\partial t^2} + EI\frac{\partial^4 v}{\partial x^4} = p(x,t) \quad \text{— (3)}$$

Which is the DEOM of an EB Beam for forced Vibration Under a Continuous loading.

(3) can be solved Using the method of separation of Variable for free Vibrations.

When $p(x,t) \equiv 0$

Hence for free Vibrations

$$m\frac{\partial^2 v}{\partial t^2} + EI\frac{\partial^4 v}{\partial x^4} = 0$$

$$v(x,t) = V(x)\ f(t)$$

generalised Coordinate

Eigen function

$$m\frac{\partial^2 v}{\partial t^2} + EI\frac{\partial^4 v}{\partial x^4} = 0 \quad \text{— (I)}$$

$$v(x,t) = V(x)\ f(t) \quad \text{— (II)}$$

$$\left.\begin{aligned} \frac{\partial^2 v}{\partial t^2} &= V(x)\ \ddot{f}(t) \\ \frac{\partial^4 v}{\partial x^4} &= V^{IV}(x)\ f(t) \end{aligned}\right\} \text{Substituting.}$$

$$m\,V(x)\ \ddot{f}(t) = EI\ V^{IV}(x)\ f(t)$$

$$\frac{\ddot{f}}{f} = -\frac{EI}{m}\frac{V^4(x)}{V(x)} = -\omega^2$$

$$\ddot{f} + \omega^2 f = 0 \quad \text{— (III)}$$

$$\frac{d^4V}{\partial x^4} - \left(\frac{m\omega^2}{EI}\right)V = 0 \quad \text{— (IV)}$$

(IV) Represents an eigen value Problems subject to proper BC's

$$V = Ce^{sx}$$

$$s^4 - \beta^4 = 0$$

$s_1 = \beta$ $\qquad$ $s_2 = -\beta$

$s_3 = j\beta$ $\qquad$ $s_4 = -j\beta$

$$\frac{\partial^4 V}{\partial x^4} - \beta^4 V = 0 \quad \text{— (V)}$$

Where $\boxed{\beta^4 = \frac{m\omega^2}{EI}}$

The general solution of (V) is

$$V(x) = A\sin(\beta x) + B\cos(\beta x) + C\sinh(\beta x) + D\cosh(\beta x)$$

$A, B, C, D \rightarrow$ to be determined using the 4 BC's

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① Clamped free (Cantilever)
(CF)

$$\left.\begin{aligned} v &= 0 \\ \frac{\partial v}{\partial x} &= \text{Slope} = 0 \end{aligned}\right\} \quad \text{at } x=0$$

$$EI\frac{\partial^2 v}{\partial x^2} = BM = 0$$

$$-EI\frac{\partial^3 v}{\partial x^2} = \text{Transverse force} = 0 \quad \text{at } x=L$$

② Simply Supported
(pinned-pinned)
(SS)

$v = 0$

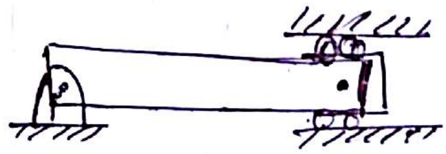
$$EI\frac{\partial^2 v}{\partial x^2} = 0.$$

at $x=0$

$v=0$

$$EI\frac{\partial^2 v}{\partial x^2} = 0$$

at $x=0$ at all times.



③ Clamped Pinned
(C-P)

④ Clamped-clamped
(C-C)

The Orthogonality of the Eigen functions :-

For a Uniform bar in axial, free Vibration the eigen function are given by.

$$U_r(x) = A_r \sin(\beta_r(x))$$

We Can easily Show that

$$\int_0^l U_r(x)\, U_s(x)\, dx = 0 \quad \text{if } r \neq s$$

We Can also Show that

$$\int_0^l U_r(x) \cdot \frac{\partial^2 U_s(x)}{\partial x^2}\, dx = 0 \quad \text{if } r \neq s$$

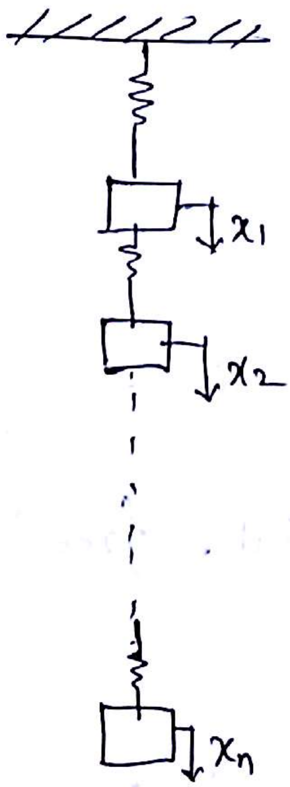
These 2 relations are the Orthogonality principal for the Uniform bar in axial Vibrations

$$\frac{\partial^2 u(x,t)}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

$$\beta_r = \frac{\omega_r}{c}$$

$$c = \sqrt{\frac{EA}{m}}$$

The Expansion theorem for a Continuous System



$\omega_1 < \omega_2 < \omega_3 \cdots < \omega_n$

$\{A_1\}\ \{A_2\} \cdots \{A_n\} \Rightarrow$ eigenfuncti

Any n-dimensional Vector

$$\{u\} = \begin{Bmatrix} u_1 \\ u_2 \\ \vdots \end{Bmatrix}$$ Can be

expressed

$$\{u\} = C_1 \{A_1\} + \cdots + C_n \{A_n\}$$

$C_i \rightarrow$ Constants not all zero

Any Continuous founded function of x can be expressed as a linear Combination of the eigen-function

$$f(n) = \sum_{l=1}^{\infty} C_i V_i(n)$$