Optimization 2020 - Fourth compulsory assignment

Quadratic programming is the optimization problem where a quadratic function is to be minimized subject to linear constaints. We thus consider problems of the following form:

$$\begin{aligned} min \quad & c^\top x + \frac{1}{2} x^\top Q x \\ s.t. \quad & Ax \geq b \\ & x \geq 0 \end{aligned} \tag{1}$$

where Q is a symmetric $n \times n$ matrix, A is a $m \times n$ matrix, $c \in \mathbb{R}^n$, and $b \in \mathbb{R}^m$.

When the objective function is *convex*, which is precisely the case of the matrix Q being *positive definite*, we have discussed in lectures that the optimization problem can be solved in polynomial time using interior point methods.

In this assignment we consider the general case. It is now possible to model binary decision variables by minimizing the objective function $\sum_{i=1}^{n} x_i(1-x_i)$, which has objective value 0 if and only if $x_i \in \{0,1\}$ for all $i = 1,\ldots,n$. This can then form the basis of a reduction from e.g. SAT by building suitable linear constraints, and thereby showing that the problem is NP-hard.

Our goal is to prove the stronger statement that the problem is NP-hard even in the special case of when the quadratic part of the objective function is *bilinear* (i.e. where all entries q_{ii} on the diagonal of Q are zero for all i) and the set of feasible solutions is the unit hypercube $[0,1]^n$ (i.e. where A be the negative of the $n \times n$ identity matrix I and let $b \in \mathbb{R}^n$ be the all-(-1) vector)!

1. Assuming that the quadratic part of the objective function is bilinear and the set of feasible solutions is the unit hypercube, show that there exist an optimal solution to the quadratic program in the set $\{0,1\}^n$ of corner points of the unit hypercube.

2. Let $G = (V,E)$ be an undirected graph with $n = |V|$ vertices. Define the function $f : \{0,1\}^n \to \mathbb{R}$ by
$$f(x) = -\sum_{i \in V} x_i + \sum_{(i,j) \in E} n \cdot x_i x_j \quad .$$
Let $S \subseteq V$ be a set of vertices and define $x_i = 1$ if $i \in S$ and $x_i = 0$ if $i \notin S$. Show that (i) $f(x) \geq 0$ when S is not an independent set and that (ii) $f(x) = -|S|$ when S is an independent set.

3. Show that quadratic programming is NP-hard even when the quadratic part of the objective function is bilinear and the set of feasible solutions is the unit hypercube $[0,1]^n$.