1. *Assuming that the quadratic part of the objective function is bilinear and the set of feasible solutions is the unit hypercube, show that there exist an optimal solution to the quadratic program in the set {0,1}ⁿ of corner points of the unit hypercube.*

Ans:
Let Q be the quadratic part of $f(x)$. When Q is bilinear, that is, all entries q_{ii} along the diagonal of Q are zeros,

$$Q = \begin{bmatrix} 0 & a_{12} & a_{13} & \dots \\ a_{21} & 0 & a_{23} & \dots \\ a_{31} & a_{32} & 0 & \dots \\ \dots & \dots & \dots & \dots \end{bmatrix}$$

Thus the quadratic part of the objective function, $x^{\top}Qx$, is
$(a_{21} + a_{12})x_1x_2 + (a_{31} + a_{13})x_1x_3 + \dots + (a_{m1} + a_{1m})x_1x_m$
$+ \dots$
$\dots$
$+ \dots$
$+ (a_{2n} + a_{n2})x_nx_2 + (a_{3n} + a_{n3})x_nx_3 + \dots + (a_{mn} + a_{nm})x_nx_m$
for Q of m rows and n columns.

This shows that in the case of bilinear Q, $x^{\top}Qx$ does not contain any squared variables, that is, it does not contain any non-zero x_i^2 for all $i \in |V|$, where V is set of all variables in the function.

Since $c^{\top}x$ is linear, the objective function $f(x) = c^{\top}x + \frac{1}{2}x^{\top}Qx$ does not contain any non-zero x_i^2.

For any variable x_k, assuming all other variables x_i are constant, the objective function becomes $f(x_k) = ax_k + b$, where a and b are constants. In other words, the objective function is linear with respect to each variable x_i.

To minimize $f(x_k) = ax_k + b$, we want to make x_k as small as possible, that is $-\infty$, assuming a is positive. If a is negative, we want to make x_k as large as possible, that is ∞. The converse is true for maximization.
Since the set of feasible solutions is the unit hypercube $[0, 1]^n$, The optimal value for x_k is either the smallest possible value or largest possible value, that is 0 or 1. This means there exists an optimal solution to $f(x) = c^{\top}x + \frac{1}{2}x^{\top}Qx$ in the set $\{0, 1\}^n$ of corner points of the unit hypercube.

2. *Let $G = (V, E)$ be an undirected graph with $n = |V|$ vertices. Define the function $f : \{0, 1\}\ n \to R$ by*

   $f(x) = -\sum_{i \in V} x_i + \sum_{i,j \in E} n \cdot x_i x_j$

   Let $S \subseteq V$ be a set of vertices and define $x_i = 1$ if $i \in S$ and $x_i = 0$ if $i \notin S$. Show that (i) $f(x) \geq 0$ when S is not an independent set and that (ii) $f(x) = -|S|$ when S is an independent set.

   Ans:
   An independent set is a set of vertices such that there is no edge between any pair of vertices. In other words, for any vertices i and j in the independent set, $(i,j) \notin E$.

   Let's start by breaking $f(x)$ into two parts, $-\sum_{i \in V} x_i$ and $\sum_{i,j \in E} n \cdot x_i x_j$. (1)

   Case 1: S is an independent set

   If S is an independent set, the value for $-\sum_{i \in V} x_i$ from (1) is $-|S|$, due to the fact that $x_i = 1$ if $i \in S$ and $x_i = 0$ if $i \notin S$. For the second part of (1), $\sum_{i,j \in E} n \cdot x_i x_j$, $x_i \cdot x_j = 1$ if and only if $i,j \in S$. However, since S is independent, $(i,j) \notin E$. Thus either $x_i = 0$ or $x_j = 0$ for all $i,j \in S$. This follows that $x_i \cdot x_j = 0$, and hence $\sum_{i,j \in E} n \cdot x_i x_j = 0$. Therefore, if S is an independent set, $f(x) =- |S|$.

   Case 2: S is not an independent set

   If S is not an independent set, then there exists at least a pair of vertices i, j such $(i,j) \in E$. At the same time, $x_i = 1$ and $x_j = 1$ since both $i \in S$ and $j \in S$. The second part of (1), $\sum_{i,j \in E} n \cdot x_i x_j$, must be at least n, since there exists at least one pair of i,j such that $x_i \cdot x_j = 1$.

   The largest possible size for S is n, where all vertices in the original graph are also in S. This means the first part of (1), $-\sum_{i \in V} x_i$, have smallest possible value of $-n$. Therefore the objective function $f(x)$, which is the sum of the two parts, has value at least 0, that is, $f(x) \geq 0$.

3. *Show that quadratic programming is NP-hard even when the quadratic part of the objective function is bilinear and the set of feasible solutions is the unit hypercube* $[0,1]^n$

Ans:
This can be done by showing that a maximum independent set problem can be reduced to optimizing a quadratic programming problem, that is, every solution to a maximum independent set problem corresponds to optimal solution in some quadratic programming problem.

The first step is to reduce the Independent Problem to a Zero-One Integer Program.

Maximize

$$f(x) = \sum_{i \in V} x_i \qquad (1)$$

s.t.

$$\begin{array}{ll} x_i + x_j \le 1, & (i,j) \in E \\ x_i \ge 0, & i \in V \\ x_i \in \{0,1\} & i \in V \end{array}$$

For an independent set S, $x_i = 1$ if $i \in S$ and $x_i = 0$ if $i \notin S$. Maximizing $|S|$ thus corresponds to maximizing $f(x)$ subject to the contraints above.

Let us add a term $\sum_{i,j \in E} x_i x_j$ to the equation in (1), such that

$$f(x) = \sum_{i \in V} x_i + \sum_{i,j \in E} x_i x_j \qquad (2)$$

Since S is an independent set, either $x_i = 0$ or $x_j = 0$ for all $i,j \in S$. This follows that $\sum_{i,j \in E} x_i x_j = 0$ necessarily. Thus the equations in (1) and (2) are equivalent.

The equation in (2) is a quadratic function where the quadratic part is bilinear and the set of optimal solutions lie in the corners of unit hypercube $[0,1]^n$. Since we know that finding a maximum independent set is NP-hard, by using reduction of maximum independent set problem to optimal quadratic programming, finding an optimal solution for quadratic programming is also NP-hard.