Groups as Fundamental Groups

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Main Theorem

Let G be a group. There exists a space X such that the *fundamental group of* X is precisely G.

Introduction

A non-empty set G with a binary operation is a **group**. $X \subset G$ **generates** G if every element in G is the product of elements in $X \cup X^{-1}$, where $X^{-1} = \{x^{-1} | x \in X\}$. A group F is **free** over X, a set, if

$F = \{a_1 a_2 a_3 \ldots a_m | a_i \in X \cup X^{-1}, a_i a_{i+1} \neq x x^{-1}\}$

with binary operation of left-juxtaposition-and-reduction, denoted $*$, the *free product*. A map $\phi : G \to H$ between groups is a **homomorphism** if $\phi(ab) = \phi(a)\phi(b)$ for all $a, b \in G$. Every group G is the homomorphic image of a free group.

Let $\mathcal{A}$ be an indexing set and $n \geq 0$. Let $B^n(\mathcal{A}) = \amalg_{\alpha \in \mathcal{A}} B^n_\alpha$ where each $B^n_\alpha = B^n$. Let $S^n(\mathcal{A}) = \amalg_{\alpha \in \mathcal{A}} S^{n-1}_\alpha$ where $S^{n-1}_\alpha = S^{n-1}$. Then $B^n(\mathcal{A})$ is a topological collection of **n-balls** and $S^{n-1}(\mathcal{A})$ is a collection of their **boundaries**.
Let A be a topological space; let $f : S^{n-1}(\mathcal{A}) \to A$ be a map. Let $\sim$ be the minimum equivalence relation on $A \amalg B^n(\mathcal{A})$ such that $f(x) \sim x$ for $x \in S^{n-1}(\mathcal{A})$. Then $Y = (A \amalg B^n(\mathcal{A}))/\sim$ is the space made by **attaching** $B^n(\mathcal{A})$ **to** A **via f**.
Let $q : B^n(\mathcal{A}) \to Y$ be the quotient map above. Then for $e^n_\alpha = q(B^n_\alpha)$ an $n-cell$, $q_\alpha = q|_{B^n_\alpha}$ is the *characteristic map*, $f_\alpha = f|_{B^n_\alpha}$ is the *attaching map*, and f the *simultaneous attaching map*.
A **CW complex** is a topological space defined iteratively:
(1) $X^0 = \{x_0\}$;
(2) for $n \geq 1$, X^n, the *n-skeleton*, is obtained by attaching n-balls to X^{n-1} via f^n, a simultaneous attaching map.
The *dimension* of X is the index of the largest skeleton. Note that a one-dimensional CW complex is called a **graph**.

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Fundamental Group

Let X be a topological space and $x_0 \in X$.
A continuous map $f : [0,1] \to X$ is called a **loop at** x_0 if $f(0) = f(1) = x_0$. Two loops f, g at x_0 are **homotopic** if there exists a continuous function $F : [0,1] \times [0,1] \to X$ such that $F(0,t) = f(t)$ and $F(1,t) = g(t)$. Such an F is called a **homotopy** of f and g. The collection of all homotopy classes of loops at x_0 with the group operation of path concatenation is a group and is called the **fundamental group of** X **at** x_0 denoted $\pi(X, x_0)$. If space is path connected all such are isomorphic, and *the* fundamental group of $X, \pi(X)$.

(proof: Free Group is Fund. Group of Graphs) Let $f : [0,1] \to X$ be a loop with base point x_0 the center of X; let $\mathcal{O}$ be an open ball around x_0. Then $A_i = X^1_i \cup \mathcal{O}$ is open in X and $X = \cup^r_{j=1} A_j$, where X^1_i is the ith one-ball in the 1-skeleton. Note first that $f([0,1]) \cap A_i \subset X$ is open $\Rightarrow f^{-1}(f([0,1]) \cap A_i) \subset [0,1]$ is open by f continuous. By open sets of $\mathbb{R}$ a countable union of open intervals: $f^{-1}(f([0,1]) \cap A_i) = \cup^{n_i}_{k=1}(a_{ik}, b_{ik})$ and $\cup^n_{i=1}\{(a_{ik}, b_{ik})\}^{n_i}_{k=1}$ an open cover of $[0,1]$ whence there exists a finite subcover.

Let X be a CW complex.
An **edge** is a non-degenerate one-cell e^1_γ with characteristic map $q_\gamma : B^1_\gamma \to e^1_\gamma$. An **edge loop** is a finite non-empty sequence of edges
$\tau = (\tau_1 \tau_2 \tau_3 \ldots \tau_n)$. The **inverse** of an edge τ^{-1}_γ is the same one-cell e^1_γ together with the characteristic map $q^{-1}_\gamma(x) = q_\gamma(-x)$. The *inverse* of an edge loop is $\tau^{-1} = (\tau^{-1}_n \ldots \tau^{-1}_1)$. An edge loop is **reduced** if it has no adjacent pairs $\tau_i \tau^{-1}_i$ or $\tau^{-1}_i \tau_i$.
Note that a two-cell e^2_α is attached to one-cells by the attaching map $f_\alpha : S^1_\alpha \to X^1$. If we start at $f_\alpha(1)$ and list the edges in clock-wise order we have a unique edge path called the **homotopical boundary** of e^2_α, denoted $\Delta e^2_\alpha = (u_1 u_2 \ldots u_m)$.

Theorem: Let X be a CW complex and let σ, ϕ be edge loops. Let $\sigma \sim \phi$ if either one is a reduction of the other, or if $\sigma\phi^{-1}$ is a reduction of $\tau \Delta e^2_\alpha \tau^{-1}$ for some two-cell e^2_α and any edge loop τ. Then $\sim$ is an equivalence relation on edge loops of X.
(proof) applications of definitions

The collection of all equivalence classes of edge loops forms a group together with the operation of edge product and is called the **fundamental group of** X denoted $\pi(X)$.

Free Groups are Fundamental Groups of Graphs

Let F_n be free over $\{x_1, x_2, \ldots x_n\}$. Then the graph X with $1-skeleton$ of n one-balls has fundamental group isomorphic to F_n.

Let $0 = s_0 < s_1 < \cdots < s_m = 1$ be a partition of $[0,1]$ with reordering and relabeling of the subcover interval's endpoints so that $(s_i, s_{i+1}) \subset (a_{ik}, b_{ik})$. Then $f(s_i, s_{i+1}) \subset A_i$ for some i.
Let f_n be the path obtained by restricting f to $[s_n, s_{n+1}]$. We note any loop in $\mathcal{O}$ is homotopic to the constant loop at x_0. It follows that f is the composition of $f_1, f_2, \ldots, f_p$ each f_i *homotopic* to $\hat{f}_i \subset X^1_i$ for some i.
It follows that f is *homotopic* to $\hat{f}_1 \hat{f}_2 \ldots \hat{f}_p$. Then every f has an expression as the composition of paths uniquely in some A_j.

This shows that $\pi(X) \leq *^r_{i=1} X^1_i$ the free product. Note that $\cap^r_i X^1_i = \{x_0\}$ the center of X. Whence f_i in A_i and f_{i+1} in A_{i+1} gives no alternative representation for $f_i f_{i+1}$ contained in other X_j unless $f_i f_{i+1}$ is homotopic to the constant loop. It follows that if f is homotopic to $\hat{f}_1 \hat{f}_2 \ldots \hat{f}_p$ each loop non-trivial and adjacent to loops in different A_k then $[\hat{f}_1][\hat{f}_2] \ldots [\hat{f}_p]$ is necessarily a reduced word $\Rightarrow *^r_{i=1} \pi(X^1_i) \leq \pi(X)$. Thus $\pi(X) \cong *^r_{i=1} \pi(X^1_i) \cong F_n$, the free group on n generators by $\pi(X^1_i) = \pi(S^1) \cong \mathbb{Z}$.

(proof: Free Groups... Graphs) Note that X is a CW-complex with no 2-skeleton $\Rightarrow \pi(X)$ is the group of reduced edge loops under edge concatenation. By reduced loops over n one-balls isomorphic to reduced words over n generators $\pi(X) \cong F_n$.

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