Potential Flow Theory

Let us consider an inviscid, irrotational, incompressible and 2D flow.

So both ψ (streamfunction) and ϕ (velocity potential) exists.

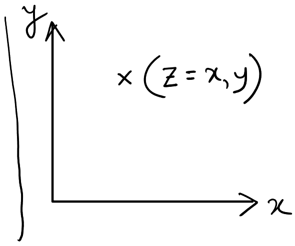
Let us define a function, f, such that:-

(f: Complex potential) $\qquad f = \phi + i\psi$, where $i = \sqrt{-1}$ and $f \equiv f(z)$

where $z = x + iy$, and (x, y) are real numbers.

In the Argand diagram, z represents a point in the x-y plane

Now, $\dfrac{df}{dz} = \dfrac{\partial f}{\partial x} \cdot \dfrac{\partial x}{\partial z} + \dfrac{\partial f}{\partial y} \cdot \dfrac{\partial y}{\partial z}$ $\qquad$ $\dfrac{\partial x}{\partial z} = 1$, $\quad \dfrac{\partial y}{\partial z} = \dfrac{1}{i} = \dfrac{i}{i^2} = -i$



$$\text{or} \quad \frac{df}{dz} = \frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}$$

$$= \frac{\partial}{\partial x}(\phi + i\psi) - i\frac{\partial}{\partial y}(\phi + i\psi)$$

$$= \left(\frac{\partial \phi}{\partial x} + i\frac{\partial \psi}{\partial x}\right) + \left(\frac{\partial \psi}{\partial y} - i\frac{\partial \phi}{\partial y}\right)$$

So differentiating in a direction parallel to y-axis ie

$$\left.\frac{df}{dz}\right|_{x=\text{Const}} = \frac{\partial \psi}{\partial y} - i\frac{\partial \phi}{\partial y} \qquad \text{——} \; (1)$$

Also differentiating in a direction parallel to x-axis ie

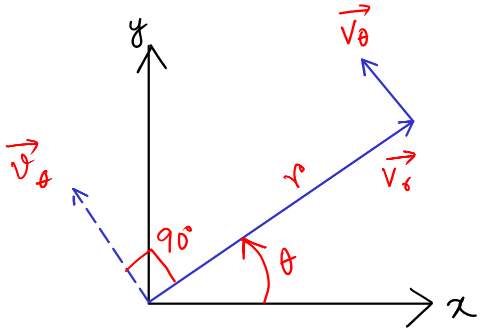
$$\left.\frac{df}{dz}\right|_{y=\text{Const}} = \frac{\partial \phi}{\partial x} + i\frac{\partial \psi}{\partial x} \qquad \text{——} \; (2)$$

As per the definition of ψ and ϕ, we have: $u = \dfrac{\partial \psi}{\partial y} = \dfrac{\partial \phi}{\partial x}$

$$v = -\frac{\partial \psi}{\partial x} = \frac{\partial \phi}{\partial y}$$

So from both ① and ②, we get: $\boxed{\frac{df}{dz} = u - iv}$ (irrespective of the path of differentiation)

In polar coordinate system:



So $\frac{df}{dz} = u - iv$

Here, $u = v_r \cos\theta - v_\theta \sin\theta$
$v = v_r \sin\theta + v_\theta \cos\theta$

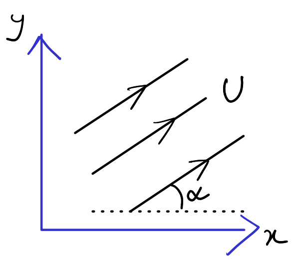
So $$\frac{df}{dz} = (v_r \cos\theta - v_\theta \sin\theta) - i(v_r \sin\theta + v_\theta \cos\theta)$$

or $$\frac{df}{dz} = v_r(\cos\theta - i\sin\theta) - iv_\theta(\cos\theta - i\sin\theta) = (v_r - iv_\theta)\underbrace{(\cos\theta - i\sin\theta)}_{e^{-i\theta}}$$

So $$\boxed{\frac{df}{dz} = (v_r - iv_\theta)e^{-i\theta} = u - iv}$$

Some elementary flows

① **Uniform flow**: Let U be a free-stream uniform flow inclined at an angle 'α' with x-axis



So $u = U\cos\alpha$, $v = U\sin\alpha$

So $$\frac{df}{dz} = u - iv = U(\cos\alpha - i\sin\alpha)$$

or $$\frac{df}{dz} = Ue^{-i\alpha}$$

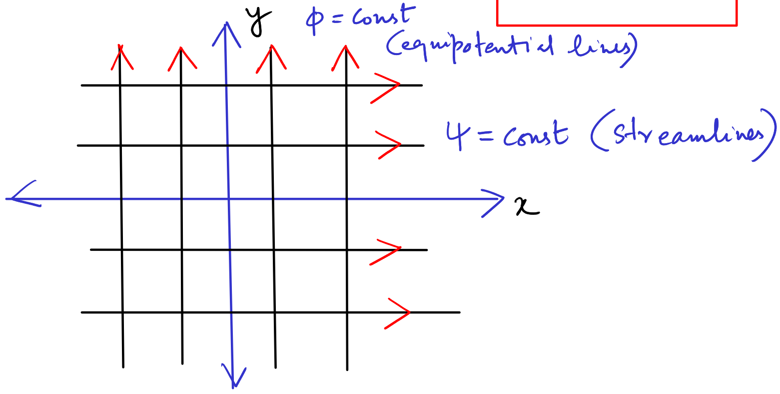
or $f = Ue^{-i\alpha}z + \text{Constant}$ → Arbiterary, Not important as $\frac{df}{dz}$ is more important than f

So choose the constant $= 0 \Rightarrow f = U e^{-i\alpha} z$

Now if $\alpha = 0$, the uniform flow is along the x-direction, $U \equiv U_\infty$

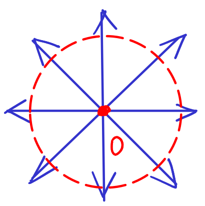
So $f = U_\infty z = \phi + i\psi$

$$\Rightarrow U_\infty (x + iy) = \phi + i\psi \Rightarrow \boxed{\begin{aligned}\phi &= U_\infty x \\ \psi &= U_\infty y\end{aligned}}$$



② Source: flow emerging from a point such that $v_r \propto \frac{1}{r}$, $v_\theta = 0$

or $v_r = \frac{c}{r}$ —(i)



Flow rate accross the circle, $q = v_r (2\pi r)$

$$v_r = \frac{q}{2\pi r} \quad \text{—(ii)}$$

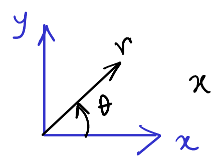
Comparing (i) & (ii), $c = \frac{q}{2\pi}$, q is also known as the strength of the source.

Now $\frac{df}{dz} = (v_r - i v_\theta) e^{-i\theta}$

$$= \frac{q}{2\pi r} \cdot e^{-i\theta} = \frac{q}{2\pi (r e^{i\theta})}$$

or $\frac{df}{dz} = \frac{q}{2\pi z}$

$r e^{i\theta} = r (\cos\theta + i \sin\theta)$



$x = r\cos\theta$

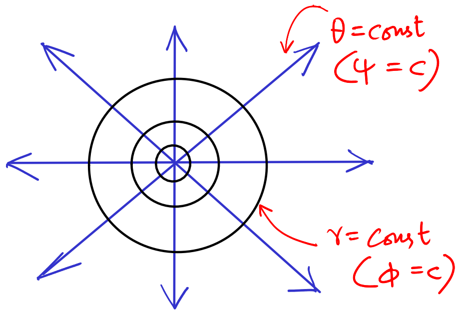
$y = r\sin\theta$

So $r e^{i\theta} = x + iy = z$

Integrating $\rightarrow$ $f = \dfrac{q}{2\pi} \ln(z) = \dfrac{q}{2\pi} \ln\left(re^{i\theta}\right)$

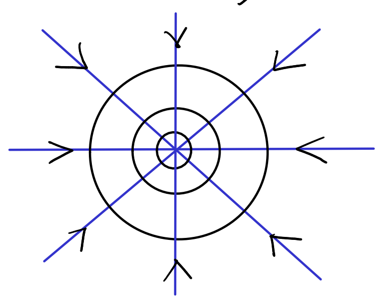
or $f = \phi + i\psi = \left(\dfrac{q}{2\pi} \ln r\right) + i\left(\dfrac{q\theta}{2\pi}\right)$

or $\phi = \dfrac{q}{2\pi} \ln r, \quad \psi = \dfrac{q\theta}{2\pi}$ } $\psi = $ Const. lines or streamlines signify $\theta = $ const. lines and $\phi = $ Const. lines or equipotential lines indicate $r = $ Const. lines.



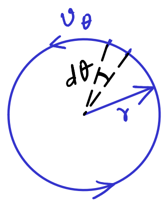
③ **Sink:** flow radially converging to a point, $v_r \propto \frac{1}{r}$, $v_\theta = 0$

Here $q < 0$ (rest same as the source)



④ **Vortex:** Free vortex flow, $v_r = 0$, $v_\theta \propto \dfrac{1}{r}$

or $v_\theta = \dfrac{c}{r}$;



So circulation, $\Gamma = \oint v_\theta \, dl$

$\Rightarrow \Gamma = c \displaystyle\int_0^{2\pi} \frac{1}{r} r \, d\theta$

Hence, $v_\theta = \dfrac{\Gamma}{2\pi r}$, $v_r = 0$

or $\Gamma = c \cdot 2\pi$ or $c = \dfrac{\Gamma}{2\pi}$

So $\dfrac{df}{dz} = \left(v_r - i v_\theta\right) e^{-i\theta} = -i \dfrac{\Gamma}{2\pi r} e^{-i\theta} = -i \dfrac{\Gamma}{2\pi\left(re^{i\theta}\right)} = -i \dfrac{\Gamma}{2\pi z}$

or $\quad \dfrac{df}{dz} = -i\dfrac{\Gamma}{2\pi z} \Rightarrow f = -i\dfrac{\Gamma}{2\pi}\ln(z) + \cancel{\text{Const}}^{\,0}$

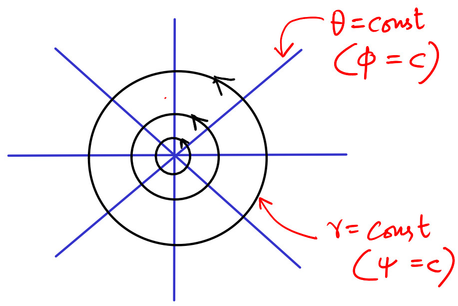
or $\quad f = \phi + i\psi = -i\dfrac{\Gamma}{2\pi}\ln\left(re^{i\theta}\right) = -i\dfrac{\Gamma}{2\pi}\ln(r) - \dfrac{\Gamma}{2\pi}i^2\theta$

or $\quad \phi + i\psi = \dfrac{\Gamma}{2\pi}\theta + i\left[-\dfrac{\Gamma}{2\pi}\ln(r)\right]$

$\Rightarrow \phi = \dfrac{\Gamma}{2\pi}\theta \;,\; \psi = -\dfrac{\Gamma}{2\pi}\ln(r)$

So ψ = const. or streamlines are r = constant lines.

ϕ = const. or equipotential lines are θ = const lines.



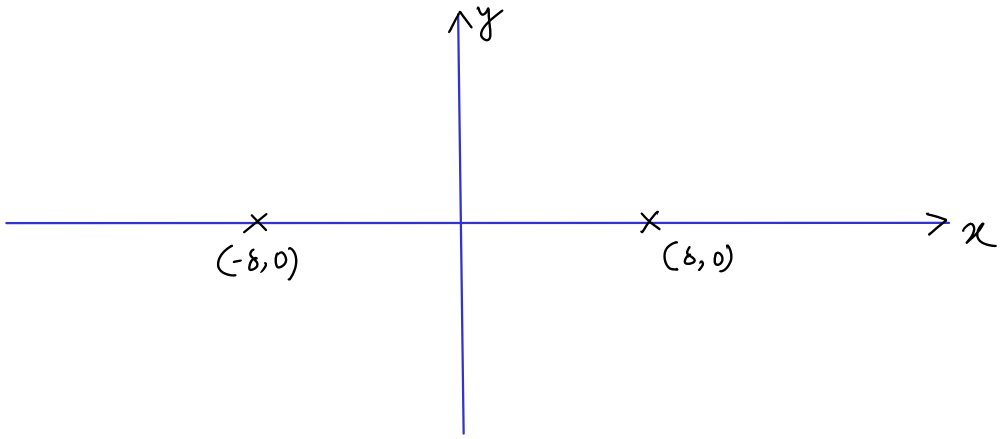
Combination of elementary flows :

① Combination of Source and Sink : Doublet

For an inviscid, incompressible, irrotational flow, we have $\nabla^2\phi = \nabla^2\psi = 0$

Hence due to linearity in the Laplace equation $\nabla^2(\phi + i\psi) = \nabla^2 f = 0$

Hence a linear superposition is possible for any elementary flows.



Let a sink be located at $(-\delta, 0)$ and a source at $(\delta, 0)$, both along the x-axis.

So $$f = \frac{q}{2\pi}\ln(z+\delta) - \frac{q}{2\pi}\ln(z-\delta)$$

$$= \frac{q}{2\pi}\left[\ln\left\{z\left(1+\frac{\delta}{z}\right)\right\} - \ln\left\{z\left(1-\frac{\delta}{z}\right)\right\}\right]$$

$$= \frac{q}{2\pi}\left[\ln\left(1+\frac{\delta}{z}\right) - \ln\left(1-\frac{\delta}{z}\right)\right]$$

Taking $\delta << z$,

$$f = \frac{q}{2\pi}\left[\left\{\left(\frac{\delta}{z}\right) - \frac{1}{2}\left(\frac{\delta}{z}\right)^2 + \cdots\right\} - \left\{\left(-\frac{\delta}{z}\right) - \frac{1}{2}\left(\frac{\delta}{z}\right)^2 - \cdots\right\}\right]$$

Neglecting higher order terms, $f \simeq \frac{q}{2\pi}\left(\frac{2\delta}{z}\right)$ or $f \simeq \frac{q\delta}{\pi z}$

or $f = \frac{m}{z}$, where $m = \frac{q\delta}{\pi}$ = strength of the doublet

when $\delta \to 0$, this source-sink combination is called a <u>doublet</u> but m = finite

So $$f = \phi + i\psi = \frac{m}{z} = \frac{m}{r}e^{-i\theta} = \frac{m}{r}(\cos\theta - i\sin\theta)$$

or $$\phi + i\psi = \frac{m}{r^2}(r\cos\theta + ir\sin\theta) = \frac{m(x-iy)}{(x^2+y^2)} \quad \text{—③}$$

So $\psi = \frac{-my}{x^2+y^2}$ or $x^2+y^2 = -\left(\frac{m}{\psi}\right)y$

$$\Rightarrow x^2 + \left\{y^2 + 2\left(\frac{m}{2\psi}\right)y + \frac{m^2}{4\psi^2}\right\} = \left(\frac{m}{2\psi}\right)^2$$

or $$x^2 + \left(y + \frac{m}{2\psi}\right)^2 = \left(\frac{m}{2\psi}\right)^2$$ → Signifies a family of circles with centre at $\left(0, -\frac{m}{2\psi}\right)$ and radius $\frac{m}{2\psi}$.

for ψ = const.

these family of circles represent streamlines.

Again from ③ $\quad \phi = \dfrac{mx}{x^2+y^2} \Rightarrow x^2+y^2 = \left(\dfrac{m}{\phi}\right)x$

$$\Rightarrow \underbrace{\left(x-\frac{m}{2\phi}\right)^2 + y^2 = \left(\frac{m}{2\phi}\right)^2}$$

↓

family of circles with centre as $\left(\frac{m}{2\phi}, 0\right)$ and radius as $\frac{m}{2\phi}$

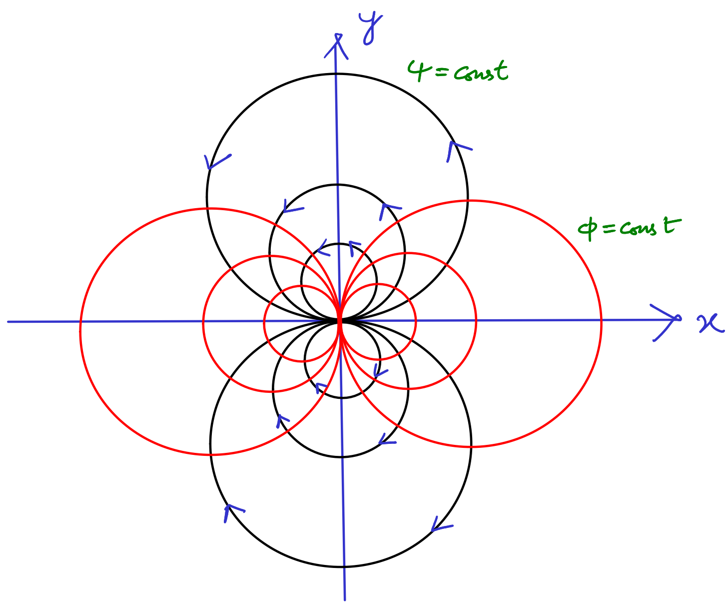
↓

for $\phi = \text{const}$, these represent equipotential lines

Lets try to draw it:-

As $\delta \to 0$

Doublet



[∵ m is positive for source and negative for sink]

② Combination of a uniform flow with doublet : uniform flow over a non-rotating cylinder

Let the uniform velocity be U_∞ along x-direction and let the strength of the doublet be m.

So $\displaystyle f(z) = U_\infty z + \frac{m}{z} = U_\infty r e^{i\theta} + \frac{m}{r} e^{-i\theta}$

$$= U_\infty r(\cos\theta + i\sin\theta) + \frac{m}{r}(\cos\theta - i\sin\theta)$$

$$\text{or } f(z) = \phi + i\psi = \left(U_\infty r + \frac{m}{r}\right)\cos\theta + i\left(U_\infty r - \frac{m}{r}\right)\sin\theta$$

So $\psi = \left(U_\infty r - \frac{m}{r}\right)\sin\theta$ So ψ = constant for a streamline.

Consider a streamline along the contour/surface of a body. For such case, by convention $\psi = \text{Constant} = 0$

$$\text{So } \psi = \left(U_\infty r - \frac{m}{r}\right)\sin\theta = 0 \quad \text{or } U_\infty r = \frac{m}{r}$$

$$\text{or } r^2 = x^2 + y^2 = \left(\frac{m}{U_\infty}\right) \rightarrow \text{At } r=R,\ \underline{m = U_\infty R^2}$$

This contour represents a circle in 2D. In a 3D scenario it represents a cylinder. Hence the streamlines indicate flow over a cylinder of radius $\sqrt{m/U_\infty}$.

To find the velocity field, $\frac{df}{dz} = (v_r - i v_\theta)e^{-i\theta} = U_\infty - \frac{m}{z^2}$

$$\text{or } \frac{df}{dz} = U_\infty - \frac{m}{r^2}e^{-i(2\theta)} = \left(U_\infty e^{i\theta} - \frac{m}{r^2}e^{-i\theta}\right)e^{-i\theta}$$

$$= \left[U_\infty \cos\theta + iU_\infty \sin\theta - \frac{m}{r^2}\cos\theta + i\frac{m}{r^2}\sin\theta\right]e^{-i\theta}$$

$$= \left[\underbrace{\left(U_\infty - \frac{m}{r^2}\right)\cos\theta}_{v_r} - i\underbrace{\left(-U_\infty - \frac{m}{r^2}\right)\sin\theta}_{v_\theta}\right]e^{-i\theta}$$

$$\text{At } r=R,\ v_r\Big|_{r=R} = \left(U_\infty - \frac{m}{R^2}\right)\cos\theta = 0 \qquad (\because m = U_\infty R^2)$$

[No penetration condition at surface]

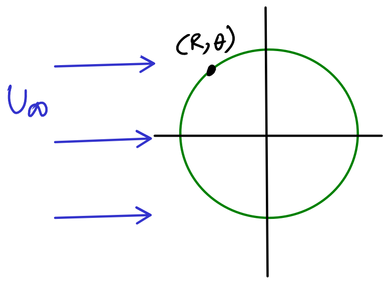
$$\text{At } r=R,\ v_\theta\Big|_{r=R} = \left(-U_\infty - \frac{m}{R^2}\right)\sin\theta = -2U_\infty \sin\theta$$

[No-slip condition does not hold for potential flows]

So for an inviscid uniform flow incident on a cylinder, B.E. can be applied between any two points in the flow field (since the flow is irrotational).

So between any point (R,θ) on the cylinder surface and far-field, let us apply B.E.,

$$p_\infty + \frac{1}{2}\rho U_\infty^2 = p_s + \frac{1}{2}\rho V^2$$



$$\Rightarrow \frac{p_s - p_\infty}{\frac{1}{2}\rho U_\infty^2} = 1 - \frac{V^2}{U_\infty^2} \quad \text{— (4)}$$

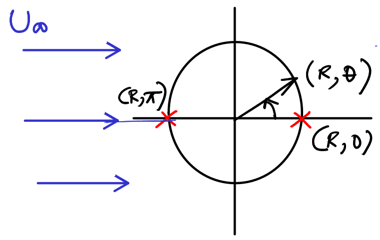
Now $V^2 = \left(v_r^2 + v_\theta^2\right)_{r=R} = 0 + 4U_\infty^2 \sin^2\theta$

So from (4),

$$\boxed{\frac{p_s - p_\infty}{\frac{1}{2}\rho U_\infty^2} = C_p = 1 - 4\sin^2\theta} = \underline{\text{Coefficient of pressure}}$$

→ This coefficient of pressure is the same within the boundary layer also till adverse pressure gradient or B.L. separation is encountered.

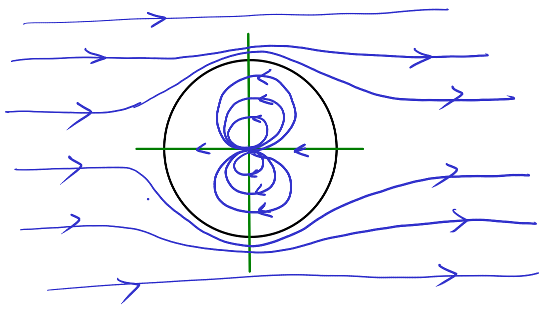
<u>Stagnation points</u>: $v_r = v_\theta = 0$ at $r = R$,



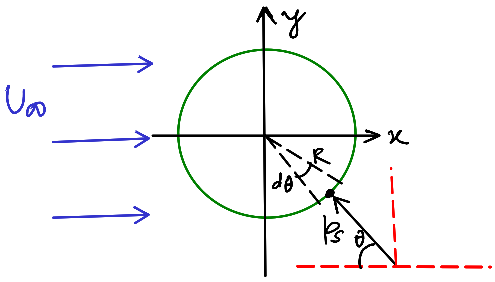
Now $v_\theta = 0$ at $r = R$, if $\theta = 0$ or π

$$\psi = \left(U_\infty r - \frac{m}{r}\right)\sin\theta$$

How does the streamlines look like?



Calculating the lift and drag force acting on the cylinder



Net drag force acting on the cylinder is due to the pressure distribution on its surface.

$$\text{So } F_D = -\int_0^{2\pi} (p_s \cos\theta)\, R\, d\theta = -R\int_0^{2\pi}\left[p_\infty + \frac{1}{2}\rho U_\infty^2 (1-4\sin^2\theta)\right]\cos\theta\, d\theta$$

$$\Rightarrow F_D = -R p_\infty \int_0^{2\pi} \cos\theta\, d\theta - \frac{1}{2}\rho U_\infty^2 R \int_0^{2\pi} \left(\cos\theta - 4\cos\theta \sin^2\theta\right) d\theta$$

$$\Rightarrow \boxed{F_D = 0}$$

Strange Result. No drag!! → D'Alembert's paradox

No viscous forces were considered, hence zero drag.

Total lift: $F_L = \int_0^{2\pi} (p_s \sin\theta)\, R\, d\theta = 0$ (So no lift forces acting)

Plot of C_p vs θ

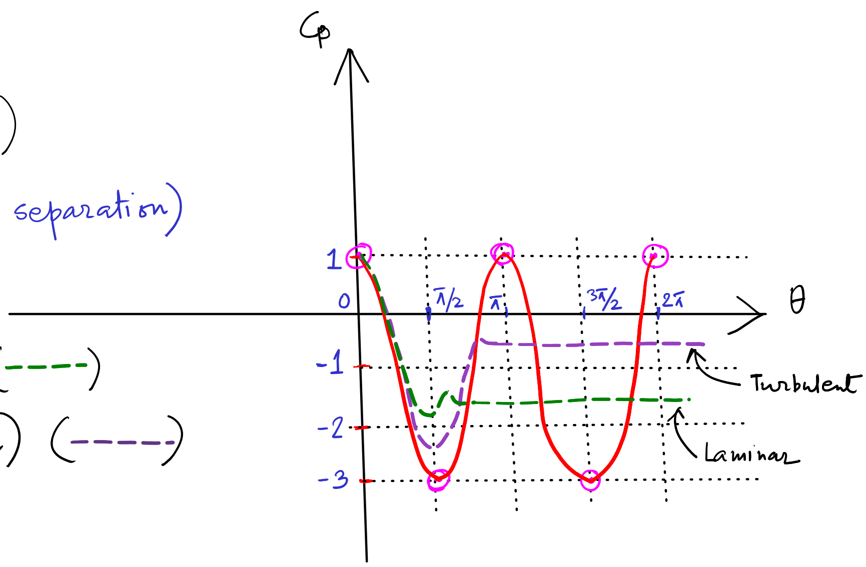
Potential flow solution (——)

$$C_p = \frac{p_s - p_\infty}{\frac{1}{2}\rho U_\infty^2} = 1 - 4\sin^2\theta$$ (No B.L. separation)

Considering B.L separation

Case I laminar flow ($Re < Re_{cr}$) (- - - -)

Case II Turbulent flow ($Re > Re_{cr}$) (- - - -)



③ Uniform flow + Doublet + Vortex : Uniform flow past a rotating cylinder

So, $f = U_\infty z + \dfrac{m}{z} + \dfrac{i\Gamma}{2\pi}\ln(z)$ $\qquad (m = U_\infty R^2)$

[Here the vortex is taken to be in clockwise direction. So $f_{vortex} = +i\frac{\Gamma}{2\pi}\ln(z)$]

$$\text{or } f = U_\infty r e^{i\theta} + \frac{m}{r}e^{-i\theta} + \frac{i\Gamma}{2\pi}\ln\left(re^{i\theta}\right)$$

$$\text{or } f = U_\infty(x+iy) + \frac{m}{r}(\cos\theta - i\sin\theta) + i\frac{\Gamma}{2\pi}\ln(r) - \frac{\Gamma\theta}{2\pi}$$

$$\text{or } f = \left(U_\infty x + \frac{m}{r}\cos\theta - \frac{\Gamma\theta}{2\pi}\right) + i\left(U_\infty y - \frac{m}{r}\sin\theta + \frac{\Gamma}{2\pi}\ln r\right)$$

$$\text{So } \phi = U_\infty x + \frac{m}{r}\cos\theta - \frac{\Gamma\theta}{2\pi}$$

$$\psi = U_\infty y - \frac{m}{r}\sin\theta + \frac{\Gamma}{2\pi}\ln r$$

$$\frac{df}{dz} = (v_r - iv_\theta)e^{-i\theta} = U_\infty - \frac{m}{z^2} + i\frac{\Gamma}{2\pi z}$$

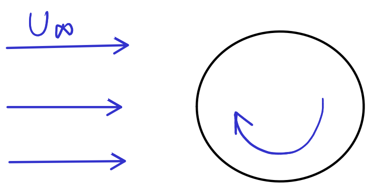
$$= U_\infty - \frac{m}{r^2}e^{-i2\theta} + i\frac{\Gamma}{2\pi r}e^{-i\theta}$$

$$= \left(U_\infty e^{i\theta} - \frac{m}{r^2}e^{-i\theta} + i\frac{\Gamma}{2\pi r}\right)e^{-i\theta}$$

$$= \left[U_\infty(\cos\theta + i\sin\theta) - \frac{m}{r^2}(\cos\theta - i\sin\theta) + i\frac{\Gamma}{2\pi r}\right]e^{-i\theta}$$

$$= e^{-i\theta}\left[\left(U_\infty - \frac{m}{r^2}\right)\cos\theta - i\left(-U_\infty\sin\theta - \frac{m}{r^2}\sin\theta - \frac{\Gamma}{2\pi r}\right)\right]$$

$$\text{So } v_r = \left(U_\infty - \frac{m}{r^2}\right)\cos\theta\ ; \quad v_\theta = -\left(U_\infty + \frac{m}{r^2}\right)\sin\theta - \frac{\Gamma}{2\pi r}$$



Stagnation points

At a stagnation point, $v_r = v_\theta = 0$

or $\left(U_\infty - \frac{m}{r^2}\right)\cos\theta = 0$ $\Rightarrow$ Zero radial velocity will occur along a circle of radius $r = \sqrt{\frac{m}{U_\infty}}$ or at $\theta = \pm \pi/2$

$\Downarrow$

$r^2 = \frac{m}{U_\infty}$ or $\theta = \pm \pi/2$

Also, $v_\theta = 0 \Rightarrow \left(U_\infty + \frac{m}{r^2}\right)\sin\theta = -\frac{\Gamma}{2\pi r}$ or $\theta = \sin^{-1}\left[\frac{-\Gamma/2\pi r}{\left(U_\infty + \frac{m}{r^2}\right)}\right]$

Since for a stagnation point, both $v_\theta = v_r = 0$

So, the stagnation point will be along the circle of radius $r = \sqrt{\frac{m}{U_\infty}}$ and

$$\theta = \sin^{-1}\left[\frac{-\Gamma/2\pi r}{\left(U_\infty + \frac{m}{r^2}\right)}\right]$$

$$\text{or} \quad \theta = \sin^{-1}\left[\frac{-\frac{\Gamma}{2\pi}\left(\frac{U_\infty}{m}\right)^{1/2}}{2\,U_\infty}\right] \quad \left(\text{Substituting } r^2 = \frac{m}{U_\infty}\right)$$

$$\text{or} \quad \boxed{\theta = \sin^{-1}\left(-\frac{\Gamma}{4\pi\sqrt{m U_\infty}}\right)} \rightarrow \text{Two values of } \theta \text{ except for } \sin^{-1}(\pm 1)$$

Substituting the stagnation coordinate (r, θ) in the expression for ψ will give us the equation for the streamline passing through the stagnation point.

$$\psi_{stag} = U_\infty r \sin\theta - \frac{m}{r}\sin\theta + \frac{\Gamma}{2\pi}\ln(r)$$

or $\psi_{stag} = \left(U_\infty r - \frac{m}{r}\right)\sin\theta + \frac{\Gamma}{2\pi}\ln(r)$

or $\psi_{stag} = \left[\cancel{U_\infty\left(\frac{m}{U_\infty}\right)^{1/2} - m\left(\frac{U_\infty}{m}\right)^{1/2}}\right]\left\{-\frac{\Gamma}{4\pi(mU_\infty)^{1/2}}\right\} + \frac{\Gamma}{2\pi}\ln\left(\frac{m}{U_\infty}\right)^{1/2}$ 0

or $\psi_{stag} = \frac{\Gamma}{2\pi}\ln\left(\frac{m}{U_\infty}\right)^{1/2}$

Now equating this with the general expression of a streamline:

$\therefore \psi = \psi_{stag}$

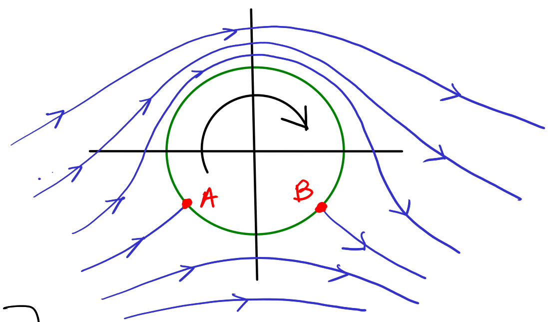
or $U_\infty r\sin\theta - \frac{m\sin\theta}{r} + \frac{\Gamma}{2\pi}\ln r = \frac{\Gamma}{2\pi}\ln\left(\frac{m}{U_\infty}\right)^{1/2}$

or $\left(U_\infty r - \frac{m}{r}\right)\sin\theta + \frac{\Gamma}{2\pi}\left\{\ln r - \ln\left(\frac{m}{U_\infty}\right)^{1/2}\right\} = 0$

So clearly the above equation is satisfied if $r = \left(\frac{m}{U_\infty}\right)^{1/2}$

Hence a streamline is present along the circle of radius, $r = \left(\frac{m}{U_\infty}\right)^{1/2}$ and two stagnation points also lie on this streamline. So this circle can be taken to be the contour of a solid cylinder.

Figueratively



Stagnation points : A and B.

[$\because \sin\theta < 0$ only for $180 < \theta < 360$]

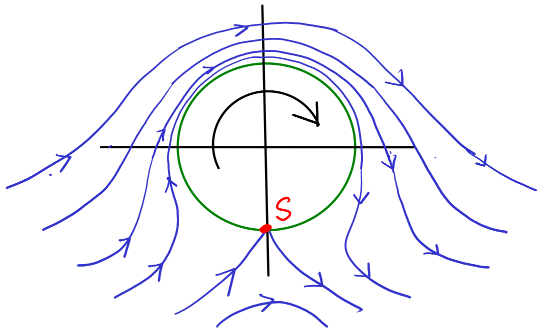
Special case

We have $\theta = \sin^{-1}\left[\frac{-\Gamma}{4\pi(mU_\infty)^{1/2}}\right]$

$= \sin^{-1}\left[-\frac{\Gamma}{4\pi U_\infty r}\right]$

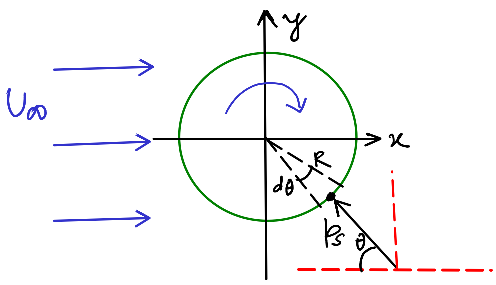
$\boxed{\text{If } \frac{\Gamma}{4\pi U_\infty r} = 1, \text{ then } \theta = -\frac{\pi}{2}}$

Here the two stagnation points now converge at a single point. $\theta = -\pi/2$. So:



S: Stagnation point.

Lift and Drag force for flow about a rotating cylinder



To calculate the lift force :- $F_L = \displaystyle\int_0^{2\pi} (p_s \sin\theta)\, R\, d\theta$, $R = \left(\frac{m}{U_\infty}\right)^{1/2}$

Applying B.E,
$$\frac{p_s - p_\infty}{\frac{1}{2}\rho U_\infty^2} = 1 - \frac{\left(\cancelto{0}{v_r^2} + v_\theta^2\right)\Big|_{r=R}}{U_\infty^2}$$

$$= 1 - \frac{1}{U_\infty^2}\left[-\left(U_\infty + \frac{m}{R^2}\right)\sin\theta - \frac{\Gamma}{2\pi R}\right]^2$$

$$= 1 - \frac{1}{U_\infty^2}\left[2U_\infty \sin\theta + \frac{\Gamma}{2\pi R}\right]^2$$

$$= 1 - \left[2\sin\theta + \frac{\Gamma}{2\pi R U_\infty}\right]^2$$

or $$p_s = p_\infty + \frac{1}{2}\rho U_\infty^2\left[1 - \left\{4\sin^2\theta + \frac{\Gamma^2}{4\pi^2 R^2 U_\infty^2} + \frac{2\Gamma\sin\theta}{\pi R U_\infty}\right\}\right]$$

$$\text{So} \quad F_L = \int_0^{2\pi} \left[\left[p_\infty R \sin\theta + \frac{R}{2} \rho U_\infty^2 \sin\theta \left\{ 1 - \left(4\sin^2\theta + \frac{\Gamma^2}{4\pi^2 R^2 U_\infty^2} + \frac{2\Gamma \sin\theta}{\pi R U_\infty} \right) \right\} \right] \right] d\theta$$

$$\text{or} \quad F_L = 0 + \frac{R\rho U_\infty^2}{2} \left(\frac{2\Gamma}{\pi R U_\infty} \right) \underbrace{\int_0^{2\pi} \sin^2\theta \, d\theta}_{\pi} = \frac{\cancel{2}\Gamma \cancel{R} \rho U_\infty^{\cancel{2}} \cancel{\pi}}{\cancel{2}\cancel{\pi} \cancel{R} \cancel{U_\infty}} = \rho \Gamma U_\infty$$

$$\left[\because \int_0^{2\pi} \sin\theta \, d\theta = \int_0^{2\pi} \sin^3\theta \, d\theta = 0 \right]$$

or

$$\boxed{F_L = \rho \Gamma U_\infty} \rightarrow \text{Kutta-Joukowski Theorem}$$

→ This lift force is independent of the body shape

→ Even though this is derived for a potential flow, it holds true for most real flows

→ This effect of generation of a lift force due to rotation of a body in a flow is called Magnus effect : Application :– swing in cricket ball.

Drag force : A similar calculation can be done for the drag force

$$F_D = -\int_0^{2\pi} (p_s \cos\theta) R \, d\theta = 0$$

Again the drag force is zero, due to the viscous forces not being considered

Next : Aerofoil Theory