# Innovation, Technological Interdependence, and Economic Growth[*]

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**Abstract**

There is substantial heterogeneity across industries in the level of interdependence between new and old technologies. I propose a measure of this interdependence–an *index of sequentiality* in innovation–which is the transfer rate of patents in a particular industry. I find that highly sequential industries have higher profitability, higher variance of firm growth, lower exit rates, and lower rates of patent expiry. To better understand these trends, I construct a model of firm dynamics where the productivity of firms evolves endogenously through innovations. New innovators either replace existing technologies or must purchase the rights to existing technologies from incumbents in order to produce, depending on the level of sequentiality in the industry. Estimating the model using data on US firms and recent data on US patent transfers, I can account for a large fraction of the cross-industry trends described above. Because innovation results in larger monopoly distortions in more sequential industries, there is an overinvestment of research inputs into these industries. This misallocation, which amounts to 2.5% in consumption equivalent terms, can be partially remedied using a patent policy featuring weaker protection in more sequential industries, producing welfare gains of 1.7%.

**JEL classification:** L11, O31, O33, O34, O38
**Keywords:** Innovation, firm dynamics, technological change, optimal policy.

# 1   Introduction

The innovation process is known to be highly cumulative. New ideas are created because inventors "stand on the shoulders of giants" that preceded them. However, the extent to which new technology is dependent upon old technology varies substantially from field to field. In some areas, such as pharmaceuticals, new technologies often replace existing ones, rendering them obsolete. Here creative destruction is a natural byproduct of innovation. In other areas, such as computer software, new technologies complement existing ones and are integrated with them into a final product. In this setting, technologies are generated incrementally, potentially across multiple firms and over long periods of time, necessitating some form of technology transfer between firms.

Motivated by this last consideration, I proxy the level of technological interdependence in an industry by the rate of patent transfer between firms, which, following the literature, I refer to as sequentiality. Using this *index of sequentiality*, I find that more sequential industries have higher profitability, higher variance of firm growth, lower exit rates, and lower rates of patent expiry[1]. These trends may at first seem puzzling, but as I will show, they in fact arise naturally from a model of firm-driven technological progress featuring heterogeneity across industries in the level of sequentiality. By studying such a model and constraining it with data, we can address a number of important questions. For instance, how does the appropriability of the returns to innovation vary with sequentiality? Does cross-industry heterogeneity in sequentiality produce substantial research misallocation? And finally, what role can patent policy play in this setting?

Estimating this model using firm-level data on patenting and balance sheet information, each of the trends noted above is matched qualitatively, and a large fraction of the variation is accounted for quantitatively. I show theoretically that the larger the sequentiality in a particular industry, the more severe the monopoly distortions induced by a particular level of innovation. This leads to an overallocation of research inputs into more sequential industries. In line with this result, I find that implementing an optimal industry dependent patent policy, which features weaker patent protection in more sequential industries, can remedy a substantial fraction of this misallocation, over and above an optimal uniform patent policy.

This paper contributes to the existing body of literature along both empirical and theoretical dimensions. First, regarding theory, I construct a parsimonious, micro-founded model of

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[1]Patent holders must pay maintenance fees at 4, 8, and 12 years after granting or face permanent expiry.

sequential innovation and endogenous technological change that formalizes the process by which new ideas are generated, built upon, and subsequently transferred between firms or rendered obsolete. Sequential innovation has already been given treatment in the literature on innovation and endogenous growth, notably in Green and Scotchmer (1995) and Bessen and Maskin (2009), as well as Hopenhayn, Llobet, and Mitchell (2006), who analyze the inherent trade-off present between rewarding incumbents and subsequent innovators that will replace them. This model captures the same trade-off while incorporating features of more empirically focused models of firm dynamics such as those of Klette and Kortum (2004) and Lentz and Mortensen (2008).[2] I characterize the innovation decisions of firms in a manner that provides intuition for the various economic forces at play and solve for a variety of observable quantities.

In the model, new innovations have differing degrees of dependence on existing technology. High levels of dependence (sequential innovations) necessitate some form of patent sales agreement between the owners of existing technology and new innovators.[3] Conversely, low levels of dependence (independent innovations) necessitate little adjudication of rights between firms as the new innovation simply renders the old one obsolete, leading to the expiration of the original patent. Laitner and Stolyarov (2013) entertain a similar distinction in a model of exogenous innovation. In my model, innovation is endogenously determined and the frequency of sequential innovation varies from industry to industry.

As has been done in Akcigit and Kerr (2010) and Atkeson and Burstein (2010), firms can engage in two types of innovation: external, where they innovate on product lines owned by other firms, and internal, where they innovate on their own product lines. The effect of sequentiality on the rate of external innovation is *ex ante* ambiguous due to the presence of two countervailing forces. First, the value of owning an existing product line is larger in more sequential industries as they feature lower rates of creative destruction from competitors and larger streams of payments from subsequent sequential innovators who buy their patents. However, because of the increased probability of sequential innovation, which necessitates a payout to the existing incumbent, the net effect on innovators will be ambiguous. This stands

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[2]These in turn build upon foundational works such as Romer (1990), Aghion and Howitt (1992), and Grossman and Helpman (1991), as well as numerous other works produced in the interim. See Aghion et al. (2013) for a very recent survey.

[3]I assume that firms always sell their patents rather than licensing them. In the model, this will always be the optimal type of agreement due to monopolistic distortions.

in contrast to the model presented in Akcigit, Celik, and Greenwood (2013), which features the positive effect of revenues from patent sales, but not the inhibition of follow on innovation due to continuing patent protection. In the case of internal innovation, the picture becomes clearer, as only the positive effect described above remains.

Broadly speaking, the sequentiality dimension introduced here fills a gap between two classes of models commonly studied in the endogenous growth literature. That is, most models either feature firms that face no threat of replacement from other innovators at the product line level, as in the expanding variety model of Romer (1990), or just the opposite, that firms innovate solely on other firms' product lines and can take over production at will upon a successful innovation, as in Aghion and Howitt (1992), Grossman and Helpman (1991), or Klette and Kortum (2004). The model presented in this paper will act as a bridge between the two extremes presented above. In the extreme of full sequentiality, much of the gains from innovation will be internalized through repeated selling of patent rights down the quality ladder, though distortions from bargaining will complicate this process slightly. In contrast to the Romer model, however, this will come at the cost of a buildup of monopoly power. In the extreme of no sequentiality, we find ourselves with a standard model of creative destruction.

The second contribution of this paper is to enrich our understanding of the data on patenting and innovation by firms. To study cross-industry differences in the sequentiality of innovation, I propose a method of classifying technology classes–an index of sequentiality–based upon the fraction of patents that are transferred in their lifetime. Looking back to the introductory examples, patents in the major pharmaceutical patent classes are transferred 15% of the time, while the same figure for telecommunications is twice as large at 30%. Using this ordering, I document a variety of trends in both the patent data and in linked firm-level data. One would naturally expect the level of sequentiality in a particular industry to have an effect upon innovations dynamics in that industry. For instance, highly sequential industries should feature lower rates of patent obsolescence, as patents are more likely to be built upon and integrated into a larger portfolio rather than being replaced by a new type of technology. This buildup of larger patent portfolios should in turn cause profits to rise, as leading firms will have a larger technological lead over their nearest competitor.

In the data description section below, I document that these trends are in fact present in the data, and I enumerate other trends observed in the cross-industry data, namely that more sequential industries feature lower exit rates and higher variance in firm growth. The former is a natural implication of the reduction in the rate of creative destruction in more sequential

industries. The latter effect arises from interaction with firm heterogeneity. Higher sequentiality and the concomitant patent transfers result in the agglomeration of more productive control into the hands of high quality firms. This magnifies the persistent growth differences between firms of differing quality, resulting in more volatile firm growth overall, particularly over longer time periods.

In addition to cross-industry statistics, there are notable trends occurring at the firm and patent level. There is a strong tendency for patents to flow from older and larger firms to younger and smaller firms, with the age dimension showing a distinctly stronger trend than the size dimension. This echoes the finding of Figueroa and Serrano (2013) that small firms receive a disproportionate amount of patent transfers.[4] These facts, in conjunction with the cross-industry trend in firm growth volatility lend support to the notion that patent transfers reflect an underlying process of reallocation amongst firms. This is particularly compelling given the strong evidence that younger, smaller firms excel in many measures of firm performance such as growth and profitability (both in the data presented here and in other works such as Akcigit and Kerr (2010) and Acemoglu et al. (2013)).

The third contribution of this paper is to estimate the proposed model using data on public firms and patents in the US, provide a detailed quantitative analysis of the results, and study the implications for optimal patent policy. I use a Simulated Method of Moments (SMM) estimator to match various features of the US data on patent grants, transfers, and expiry and on firm level growth rates and profitability. The patent data comes from the USPTO/Google database and includes data on filings, grants, expiry, and transfers. Data on patent transfers in particular has not been utilized extensively in the literature, especially in a structural setting. The patent data is aggregated to the firm level and matched to Compustat balance sheet data using sophisticated name matching routines.

The estimated model is able to match the targeted moments quite well. In addition, the model can match various non-targeted features of the data, including some of the major trends noted above. The resulting eight-year patent expiration rate over all industries is 39%, compared to 34% in the data, while the standard deviation is 15% compared to 13% in the data. Thus the model captures the proportional variation in the data, while slightly overshooting the magnitude. Other cross-industry trends, such as the relationship between transfer rates and profitability, firm growth variance, and exit rate are qualitatively captured, and the model is able to quantitatively account for a large fraction of these tends.

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[4]They use firm size information (greater or less than 500 employees) contained in patent renewal applications.

As predicted by theory, the level of internal innovation rises with sequentiality, with a 52% increase moving from the least to most sequential industry. The level of external innovation, whose dependence on sequentiality was theoretically ambiguous, falls modestly along this dimension, largely due to bargaining distortions that limit the appropriability of back-loaded profit streams. To assess potential misallocation of production and research labor, both within and between industries, I consider a constrained social planner who can choose innovation rates but is still subject to monopoly distortions induced by patenting. I show theoretically that the larger the sequentiality in a particular industry, the more severe the monopoly distortions induced by an increase in innovation rates. Thus for either type of innovation, the planner optimally choses a profile that falls sharply with sequentiality–in the case of external innovation, much more so than in the equilibrium–so as to limit the monopoly distortions caused by the buildup of large, protected technological leads by firms. The equilibrium yields a consumption equivalent welfare 2.5% lower than that of the constrained social planner.

Finally, I investigate the implications of the model for patent policy. I consider both a uniform patent policy and one that depends upon the sequentiality of the industry in question. For the purposes of implementation, the sequentiality of a particular industry can be inferred using the monotonic relationship between sequentiality and the patent transfer rate in the equilibrium of the estimated model. The above discussion of the social planner's optimum leads one to suspect that the optimal patent policy would feature weaker protection in more sequential industries. Indeed, I find that for certain very low levels of sequentiality, an infinite patent is called for. The optimal patent length then decreases from infinity to a minimal value of 6 years in the most sequential industry. This policy results in welfare gains of 1.7% in consumption equivalent terms. For comparison, the optimal constant patent policy calls for a mean patent length of 12 years and delivers welfare gains of only 0.9%.

The remainder of the paper is laid out as follows: in Section 2, I describe the data set used and enumerate the notable trends in the data; in Section 3, I construct a model capable of matching these facts and describe its equilibrium properties; in Section 4, I describe the estimation procedure and results; in Section 5, I provide a detailed quantitative analysis of the estimated model with accompanying decompositions and policy experiments; and finally in Section 6, I conclude the analysis.

# 2   Empirical Findings

Data on patent grants, expirations, and transfers was acquired from the USPTO Bulk Download site (hosted by Google). Firm names are matched and aggregated into persistent entities based on a name matching algorithm described in Appendix A.[5]

For each patent transfer, the following information is provided: (1) the name of the origin firm and destination firm (assignor and assignee), (2) the date that the patent was legally transferred, (3) the date that the transfer was recorded by the patent office, and (4) the purpose of the transfer, amongst other things. In particular, the information on the purpose of the transfer (known as the conveyance text) is used to filter out mergers, licensing agreements, and collateralizations, leaving only simple patent sales, which account for about 85% of the original data points.

The names of the origin and destination firm were matched to the set of entities produced from the patent grant data using the same name matching algorithms. In order to focus on innovating firms and not firms that are simply acquiring patents for other reasons (such as resale), I keep only transfers to firms that have already acquired patents through filing and granting. This eliminates firms that act solely as patent brokers. Furthermore, to exclude instances where conglomerates are transferring patents amongst their constituent units, I eliminate transfers where the origin and destination firm names are sufficiently close, using a more aggressive version of the original name matching algorithm (this is also described in Appendix A).

To enrich the data on patent grants, I also use data on the payment of patent maintenance fees. Firms must pay fees to the US patent office after 4, 8, and 12 years from the time of granting. If these fees are not paid, the patent expires permanently. If a patent is maintained through the initial 12 year period, it remains valid until 20 years from its filing date.[6] This data thus gives discretized information on the active lifespan of a patent. In total, 35% of patents expire after 8 years, while 48% make it to the natural expiry date. Using the data on patent expiration gives us fairly direct information on the rate of patent obsolescence and hence a window into the level of product market competition faced by firms. Using this in conjunction

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[5]Python code to parse, match, and aggregate the USPTO patent data (along with Compustat data) can be found at https://github.com/iamlemec/patents.

[6]Traditionally, the maximum patent length was 17 years from the grant date. The 1994 Uruguay Round Agreements Act changed this to the above criterion. See Graham and Vishnubhakat (2013) for a review of the relevant statutes.

with the data on patent transfers helps us understand the importance of sequential innovation and its impact on firm dynamics and the incentives to innovate.

To register a patent reassignment with the USPTO, a firm must pay a one-time $40 flat fee. The bulk of the cost is likely to be found in simply filling out the paperwork. Firms already incur legal fees to arrange the contracts for the transfer deals, so going the extra step to register with the patent office is probably not a huge effort. Patent maintenance fees are slightly higher but still not large compared to the common estimates of patent value in the literature. Some studies, such as Pakes (1986), Pakes and Schankerman (1984), and Bessen and Meurer (2008), have used patent renewal patterns to estimate the distribution of patent valuations. A survey by Griliches (1998) reports that various studies found a highly skewed distribution of patent valuations, with mean valuation estimates in the hundreds of thousands of current US dollars, and an obsolescence rate of between 10% and 20% per year. The fees required for renewal at 4, 8, and 12 years are $1600, $3600, and $7200, respectively. These fees are cut in half for small entities (less than 500 employees), and halved again for "micro entities" (targeted towards individual inventors). This self-reported size information provides useful data on the actual size of particular patenting firms. Figueroa and Serrano (2013) utilize this to study the relationship between firm size and patent transferring activity.

An important consideration is the possibility that firms license the patents of other firms rather than buying them outright. Firms are required to register patent sales or transfers in order to retain patent rights for a particular technology. However, there is no such requirement for patent licensing, which is regulated by state law in the US.[7] Approximately 1% of patent transfer entries list licensing as the documented activity, though this cannot be guaranteed to be a complete record. Looking across industries, there is no systematic variation in this fraction of reported licensing activity.

## 2.1   Major Trends

Testing various cross-industry predictions necessitates dividing the sample of patents and firms into particular industry level categories. For this exercise, I employ the level one technology class utilized by the US patent office. There are 714 such classes represented in the full patent grant dataset, with a median size of around one thousand patents. Using the first-level classification provides sufficient granularity to capture the specific features of various tech-

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[7]See Dykeman and Kopko (2004) for an overview of the relevant statutes and case law.

nological fields while being large enough to avoid excessive noise in aggregate statistics due to small within-industry samples.

It is also possible to extend patent classification information to the firm level. By assigning to a firm the modal patent class amongst its portfolio of patents, we can look at how various firm characteristics vary with technological field. Though most firms have patents in multiple patent classes, the modal patent class accounts for an average of 50% of a firm's patents. This extension to firm characteristics will be important for analyzing trends in balance sheet data from Compustat. For patent data, much of the analysis can be done purely on the patent level. However, I do analyze the patent data using firm-level class assignment for robustness and find similar results.

Industry level regressions are done using weighted least squares. The weighting used is simply the size of the particular technology class in terms of total patents granted. Data on patenting is available from 1976 to the present. For the facts below, I look at the five-year period from 1995 to 2000. This allows sufficient lead time to have realistic values for firm patent stocks, which is the count of patents that are unexpired at any given time. In a steady state world, a lead in time equal to or greater than the patent length suffices. Additionally, it allows sufficient lag time to analyze future transfer, maintenance, and citation activity. The correlations presented below are also weighted by patent class size. In each of the figures accompanying the following facts, the point size represents the total number of patents granted (the weight) and the color represents the the numerical patent class, which ranges from 1 (lightest) to 800 (darkest). Because patent classes have been added incrementally over time, the patent class (color) also provides a very good proxy for how recently the patent class was created.

**Fact 1.** *There is a negative correlation between patent transfer rates and patent expiry rates across industries.*

The transfer rate is the fraction of patents granted in the data window that are transferred in their lifetime, while the expiry rate is the fraction of patents granted in the data window that are not renewed after the first eight-year window and hence expire. The extent of the negative relationship is portrayed in Figure 1. This trend highlights a central feature of the model proposed herein, namely that industries with innovation that is more sequential in nature will see higher transfer rates due to higher levels of technological interdependence and lower levels of creative destruction for the same reason.

FIGURE 1: TRANSFER–EXPIRY RELATIONSHIP
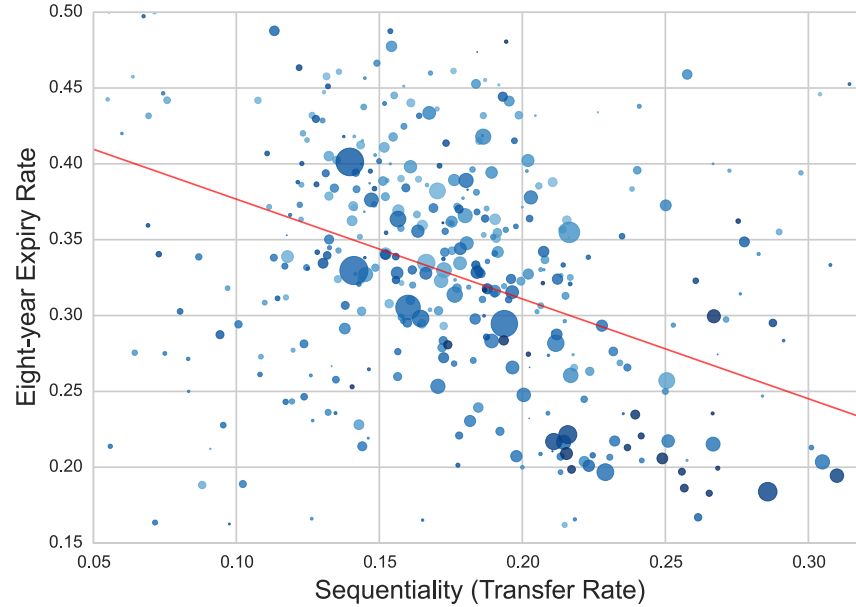


Relationship between patent transfer and expiry. Red line: WLS regression $\beta = -0.33$.
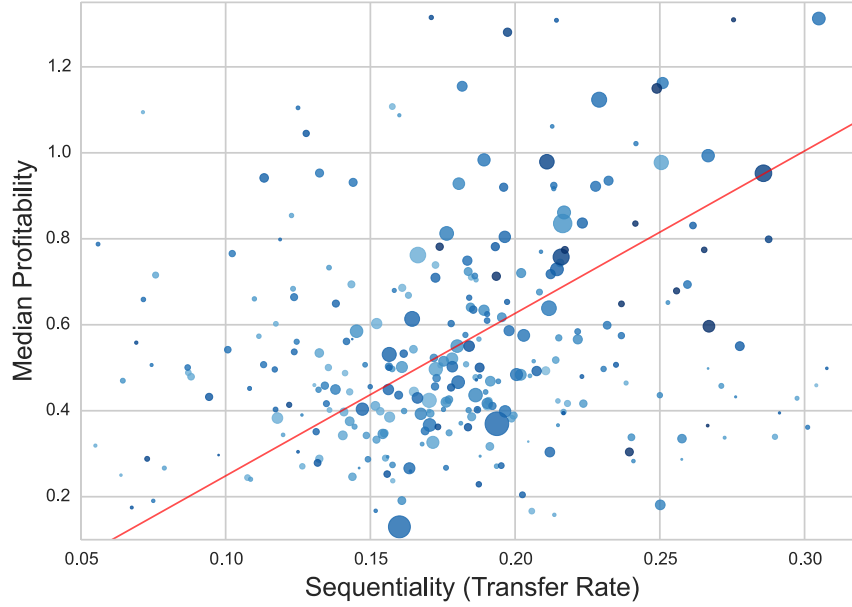
Correlation is $\rho = -0.34$.

**Fact 2.** *There is a positive correlation between patent transfer rates and firm profitability across industries.*

Here I calculate the firm-level profitability as the ratio of revenue to the cost of goods sold, as given in the Compustat data. This excludes operating costs and allows us to look purely at variable cost relationships, which are a primary variable of interest in quality ladder models. I then look at the median value within each industry. Intuitively speaking, we would expect that industries with high transfer rates see more aggregation of monopoly power and less reduction through creative destruction (from Fact Fact 1). This then leads to higher markups over cost being charged and higher profitability. This trend is portrayed in Figure 2. Looking at the relationship between log return on sales and transfer rate yields a similar trend, though with more noise on account of sales being in the denominator.

**Fact 3.** *The variance of firm growth rates is positively correlated with patent transfer rates across industries.*

Here the variance of firm growth is calculated using a number of common firm size statis-

FIGURE 2: TRANSFER–PROFITABILITY RELATIONSHIP



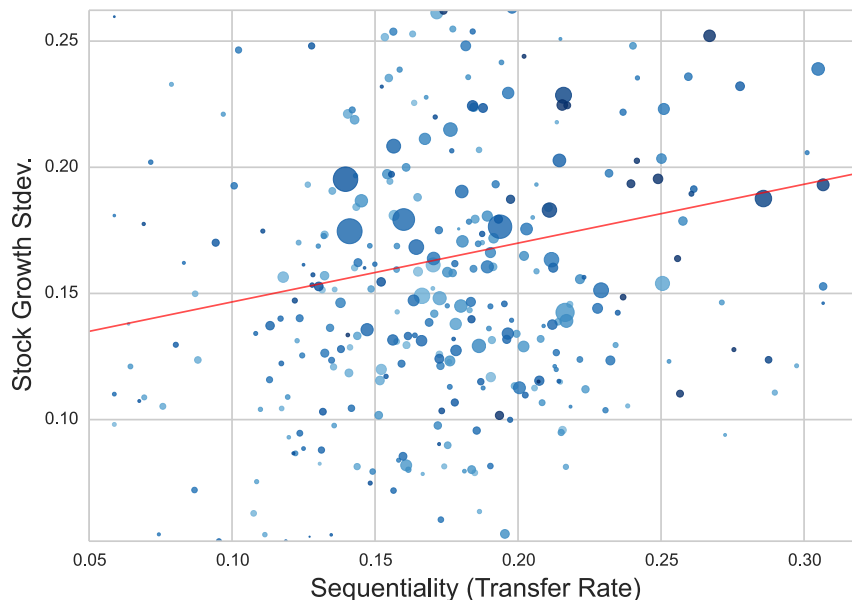Relationship between patent transfer and median profitability. Red line: WLS regression $\beta = 3.85$. Correlation is $\rho = 0.40$.

tics, including patent stock, employment level, and earnings. Regardless of the metric used, industries with higher transfer rates display higher variance in firm growth rates. One mechanism that could generate this trend is that patent transfers allow higher quality firms to grow faster at the expense of lower quality firms. Furthermore, because of lower levels of creative destruction present in industries with higher transfer rates, larger firms are also more able to protect their market share, thus widening the gap further between the performance of high and low quality firms. The fact that this gap is larger in industries with higher transfer rates leads to a higher variance of firm growth in these industries. In order to account for this potentially being driven by level differences, due to certain sectors growing or shrinking in the aggregate, I look at the variance of log growth rates within industry. Using this measure, changes in industry size will come out as a common additive factor for each firm and thus will not affect the computed variance.

**Fact 4.** *There is a negative correlation between firm exit rates and patent transfer rates across industries.*

In line with Fact Fact 1, interpreting lower expiry rates as indicating lower rates of creative

FIGURE 3: TRANSFER–GROWTH VOLATILITY RELATIONSHIP

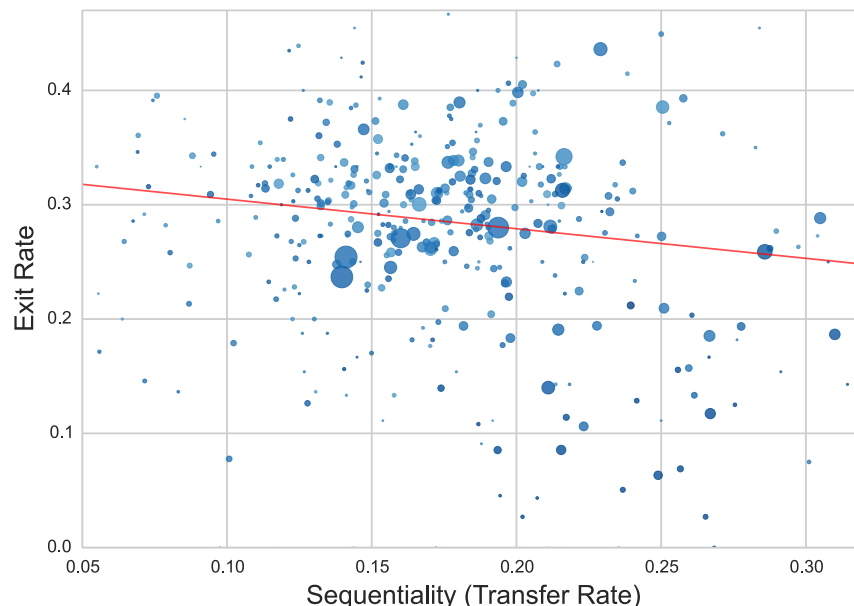

Relationship between patent transfer rate and growth volatility. Red line: WLS regression $\beta = 0.21$. Correlation is $\rho = 0.19$.

destruction, one would also expect lower exit rates in industries with high transfer rates. This is indeed born out in the data, where exiting from the sample of patenting firms is taken to occur when a firm no longer displays any patenting activity.

**Fact 5.** *Patent transfers are directed primarily toward young and small firms. Firms aged less than* 10 *years account for only* 14% *of the patent stock, while receiving approximately* 57% *of patent transfers.*

Small and young firms account for a disproportionate share of patent transfer receipts relative to their size. The size distribution of firms is highly skewed. Therefore small and young firms will invariably constitute a relatively small fraction of the patent stock. The analogous numbers to those presented in the fact above for firms below the 80th size percentile are that they account for 11% of the patent stock and 36% of patent receipts. This figure is still disproportionate in terms of firm size but not nearly as much as that for firm age. Looking at patent filings and patent transfer origination, we see similar but less extreme trends. Young firms account for 30% of filings and 33% of originations. The analogous figures for small firms are 19% of filings and 29% of originations.

FIGURE 4: TRANSFER–EXIT RATE RELATIONSHIP



Relationship between patent transfer rate and five-year exit rate. Red line: WLS regression $\beta = -0.26$. Correlation is $\rho = -0.16$.

Conventional wisdom dictates that small firms sell technologies to larger firms who are in a better position to bring products to market or integrate them into existing production processes (for instance, see Phillips and Zhdanov (2013) for a theoretical discussion of this dynamic). However, the data indicate a bulk flow towards small and, to a greater extent, young firms. This is consistent with a model where firms are imbued with persistent (though mutable) types and patent reassignment is a mechanism by which production control is transferred to higher quality firms.

## 2.2   Mechanism Evidence

In addition to the cross-industry trends presented above, I also provide more detailed evidence on the proposed mechanism. Of central importance is documenting that sequential innovation, in the sense of direct technological dependence, is the primary driving force behind the observed transfers of patent ownership. One might naturally expect citation patterns to shed light on this issue.

FIGURE 5: AGE/SIZE CUMULATIVE FRACTIONS
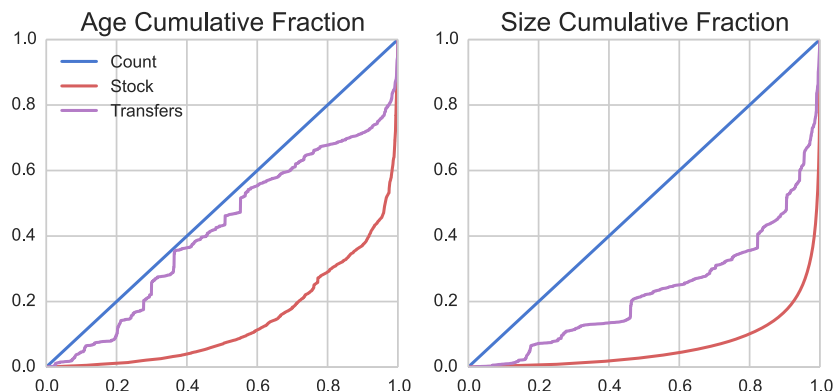


Cumulative fractions of patenting activity by firm age and size (patent stock).

As described above, I classify firms into various technological categories by using the modal patent class in their portfolio. The average firm cites only 3% of the other firms in its patent class. Relating this to the data on patent transfers, 52% of firm pairs that have transfers between them also cite each other. Breaking this down by the direction of transfer, the destination firm cites the origin firm in 50% of cases, while the reverse happens in only 23% of cases. Thus it is quite rare that the origin firm cites the destination firm without the reverse also happening. Higher citation rates between firms that transfer patents, in of itself, may merely indicate that these firms are closer together in a technological sense and that such firms are more likely to cite one another. However, the asymmetry in citation rates between the different directions of transfer lends further support to the notion that transfer are acting as a mechanism for reallocation of production and research towards higher quality firms.

**Fact 6.** *The internal citation rate is uncorrelated with transfer rates, while the external citation rate is highly correlated with transfer rates across industries.*

The data on citations show that the average number of citations per patent is higher in more sequential fields. Breaking these citations down into those that cite within firm and those that cite other firms, we see that internal citations are not related to industry sequentiality, while external citations are strongly related. This is consistent with the notion that technological dependency is the primary determinant of whether a new innovator must purchase the rights to existing ideas in the field. I use two measures of internal/external citations

FIGURE 6: TRANSFER–EXTERNAL CITATION RELATIONSHIP
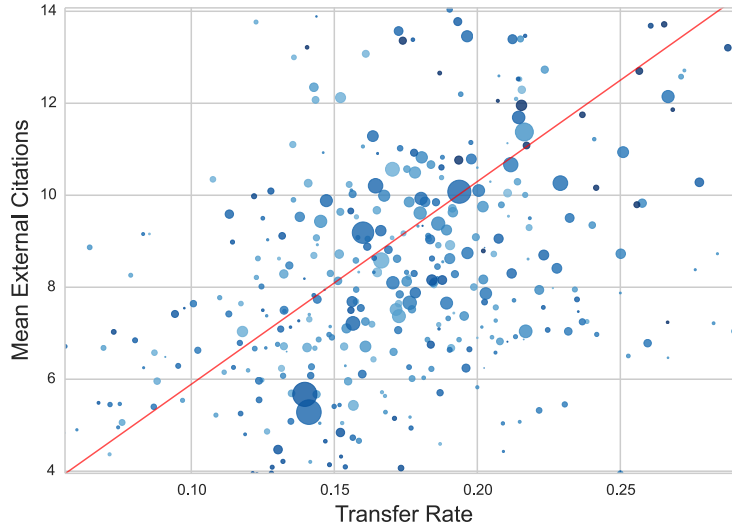


Relationship between patent transfer and external citations. Red line: WLS regression $\beta = 44.1$. Correlation is $\rho = 0.49$.

classification in this instance, both of which display the same trends across industries. First, I simply look at the aggregate number of internal and external citations per patent by industry. Second, I follow Akcigit and Kerr (2010) in classifying a patent as internally oriented if more than 50% of its citations are self-directed. Regardless of the measure used, I observe the trends noted above, as portrayed in Figure 6.

**Fact 7.** *There is a positive correlation across industries between the fraction of patents acquired by transfer and acquisition expenditures.*

The mechanism presented in this paper operates at the patent and product line level and firms are largely just collections of various product lines. However, it may be the case that resolution of patenting rights occurs not simply through the direct buying and selling of patents but at higher levels of aggregation such as the subsidiary of a conglomerate or entire firms of varying size. Particularly in the case of small firms or entrants, whose value is often encompassed in a single product line, this may be an important dynamic. And indeed, looking at the data on acquisition activity, we see a positive relationship between that and patent transfer rates. While it is certainly the case that there are other forces that can drive M&A activity, this

trend indicates that the technological landscape plays an important role. Discussion of how this data may be mapped into the model is deferred until the section on estimation.

# 3   Model

In this section, I present a continuous-time model of firm dynamics and endogenous techno-logical growth. After specifying the various elements of the model, I characterize the dynamic equilibrium. I then focus on the case of the steady state, with the objective of producing pre-dictions that map into the trends described in the previous section.

## 3.1   Consumers

There is a unit mass of immortal consumer-workers in the economy. Each has one unit of labor that they supply inelastically. Their utility is a function on the infinite flow stream of consumption starting at time $t = 0$. In particular, they discount the future at rate $\delta$ and have an instantaneous utility function of $u(c)$ with constant relative risk aversion parameter $\sigma$. Thus their utility function can be expressed as

$$U(c) = \int_0^\infty \left[ \frac{c(t)^{1-\sigma} - 1}{1 - \sigma} \right] \exp(-\delta t) dt$$

where $c$ is a consumption profile that specifies the level of consumption at each point in time. All agents earn a wage $w$ from employment. They also have access to a risk-free bond paying interest $r$ and having zero net supply in the aggregate. Let their bond holding profile be the function $a$. Their budget equation is then given by

$$c + \dot{a} = w + r a$$

where time dependence is suppressed for notational convenience. There is a single final good $Y$ for consumption, which is normalized to have a unit price at each point in time. Because all costs are purely in terms of labor, the final good resource constraint for the economy is simply $c = Y$. The associated Euler equation for this problem the delivers the result

$$g \equiv \frac{\dot{Y}}{Y} = \frac{\dot{c}}{c} = \frac{r - \delta}{\sigma}$$

Letting the common growth rate of $Y$ and $c$ be denoted by $g$, we arrive at $r = \delta + \sigma g$.

Finally, each worker can choose to be a production worker or a research worker. Let the respective masses of each type be $L_P$ and $L_R$. The labor market clearing condition is then

$$L_P + L_R = 1$$

Given the costless choice between being employed as a production worker and a research worker, any equilibrium of the model will feature a common wage $w$ for these two occupations.

## 3.2   Production

The final good is produced by combining a unit continuum of intermediate goods with the well-known Dixit-Stiglitz aggregator with unit elasticity

$$Y = \exp\left[\int_0^1 \ln(y_j)\, dj\right]$$

This technology is operated competitively by a continuum of firms. Each buys up certain quantities intermediate goods for respective prices $p_j$, combines them into a final good, and sells that for the normalized unit price. The objective of one such firm is then

$$\Pi = \max_{y_j}\left\{\exp\left[\int_0^1 \ln(y_j)\, dj\right] - \int_0^1 p_j y_j\, dj\right\}$$

Optimality dictates that $Y = p_j y_j$ for all $j$. Constant returns to scale ensure that these firms make zero profit in equilibrium.

Each intermediate good is in turn produced using a linear technology of the form $y_{jf} = q_{jf}\ell_{jf}$, where the $f$ subscript allows for the fact that different firms have different know-how in producing each particular good. Now consider the firm with the most advanced production technology and simply let $q_j = \max_f\{q_{jf}\}$. Furthermore, let the next best producer be $q_{-j} = q_j/\lambda_j$, where $\lambda_j \geqslant 1$. The leading firm can then price the runner up out of the market by charging a price $p_j = w/q_{-j} = w\lambda_j/q_j$, thus selling $y_j = Yq_j/(w\lambda_j)$ and employing $\ell_j = Y/(w\lambda_j)$ labor. This leads to profits of

$$\pi_j = p_j y_j - w\ell_j = (1 - \lambda_j^{-1})Y$$

Thus the labor utilization and profit of each product line are purely a function of the technological lead $\lambda_j$ and the not the absolute productivity value $q_j$. Using tilde to denote values

normalized by $Y$, the labor utilization and profit for a product line with technological lead $\lambda$ are given by

$$\ell(\lambda) = \frac{\lambda^{-1}}{\widetilde{w}} \qquad \text{and} \qquad \widetilde{\pi}(\lambda) = 1 - \lambda^{-1}$$

Having fully characterized the production decisions of intermediate goods producing firms, we can now use these production values to address the innovations decisions of firms.

## 3.3   Innovation

It was shown above that the only firm relevant variable for a particular product line is the technological lead $\lambda_j$. Each firm in the economy can thus be characterized simply as a portfolio of technological lead values for product lines in which it is the leading producer. For a firm with $n$ product lines, denote such a vector by

$$\vec{\lambda} = (\lambda_1, \ldots, \lambda_n)$$

Following the model presented in Klette and Kortum (2004), the external innovation production technology specified here uses only labor as an input and scales up linearly with firm size. In particular, a firm with $n$ product lines can achieve a Poisson flow rate of innovation $X$ by employing

$$C(n, X) = n c(X/n)$$

researchers. In other words, a firm must use $c(x)$ researchers per product line to achieve an innovation rate of $x$ per product line, where $x = X/n$. Firms can also undertake internal innovation on one of their existing product lines. Here I allow the cost of internally oriented innovation to scale with a firm's technological lead ($\lambda$). This is motivated partly by tractability and partly through existing empirical evidence. Akcigit and Kerr (2010) find that the intensity of internally oriented innovation does not scale strongly with firm size. In order to generate such a result, we can use the form

$$d(\lambda, z) = \lambda^{-1} d(z)$$

where $z$ is the flow rate of internal innovation. This ensures that the internal innovation rate will actually be constant across firms, regardless of their technological lead for a given product line.

The functional forms given allow one to treat each firm simply as a collection of research labs, each associated with a particular product line. Denote the value of a research lab with technological lead $\lambda$ by $V(\lambda)$. A firm with portfolio $\vec{\lambda}$ will then have value

$$V(\vec{\lambda}) = \sum_{i=1}^{\infty} V(\lambda_i)$$

A research lab will accrue profits from production and generate innovations. Successful innovations will garner new research labs with their associated production and innovation capabilities. Now we can characterize all firm decisions by addressing the problem at a product line level.

When an external innovation occurs, the state-of-the-art productivity of a random product line is incremented by a random factor $\beta$. For internal innovations, the productivity in the target product line is incremented by a factor drawn from the same distribution. Measuring and constructing systematic data on innovation sizes is difficult. However, in a limited sample, Scherer (1965) finds evidence that a Pareto distribution is appropriate. Meanwhile, Pakes and Schankerman (1984) are able to fit data on patent expiry in multiple countries using a Pareto distributed innovation size distribution, while Kortum (1997) find a Pareto distribution to be consistent with aggregate trends in research, growth, and patenting. Thus I assume that, $\beta$ is drawn from a Pareto distribution $F(\cdot)$ with tail index $1/\kappa$ and having cumulative density

$$F(\beta) = 1 - \beta^{-1/\kappa}$$

The inverted tail index is used as parameter so as to facilitate analogy to the step size parameter typically present in endogenous growth models. It will be of use later to know that the expected value of $\log(\beta)$ is simply $\kappa$, meaning a variable receiving such increments at Poisson rate $x$ will have expected growth rate $\kappa x$.

Upon the arrival of an internal innovation, with probability $\alpha$ the innovation is sequential and is dependent upon previous innovations. In this case, the innovating firm and the incumbent firm initiate a bargaining process by which either the existing patents of the incumbent are sold to the new innovator or the incumbent buys the new innovation and incorporates it into its portfolio. Conversely, with probability $1 - \alpha$, an innovation is independent. In this case, the new innovator assumes production responsibilities and the incumbent is summarily displaced.

Firms also face the rate of incoming external innovations by other firms. Let these events arrive at rate $\tau$. Finally, all patents in a particular product line expire at rate $b$, meaning the

technological lead goes to zero and production profits vanish. When this happens, the firm retains its research capacity in that product line but is displaced upon any subsequent innovation by another firm. Denote the present expected value of successful innovation by $\bar{V}$. Because both profits and labor costs scale up with output, I consider the output-normalized value of a patent protected product line with technological lead $\lambda$

$$
\begin{aligned}
\delta_F V(\lambda) - \dot{V}(\lambda) = {} & \tilde{\pi}(\lambda) \\
& + \max_x \left\{ -\tilde{w} c(x) + x\bar{V} \right\} + \max_z \left\{ -\tilde{w}\lambda^{-1} d(z) + z(\mathbb{E}V(\beta\lambda) - V(\lambda)) \right\} \\
& + \alpha\tau p(\mathbb{E}V(\beta\lambda) - V(\lambda)) + (1-\alpha)\tau(0 - V(\lambda)) + b(V_0 - V(\lambda))
\end{aligned}
$$

where $\delta_F = r - g$ is the effective discount rate used by the firm. The value of a product line without patent protection is simply

$$
\delta_F V_0 - \dot{V}_0 = \max_x \left\{ -\tilde{w} c(x) + x\bar{V} \right\} + \max_z \left\{ -\tilde{w}\lambda^{-1} d(z) + z(\mathbb{E}V(\beta\lambda) - V_0) \right\} + \tau(0 - V_0)
$$

Notice that $V(1) \neq V_0$, as expired product lines still retain their research capacity. To know the value of successful innovation, we must know the economy-wide distribution over $\lambda$. For now, denote the cumulative density for this variable by $\mu(\cdot)$. Furthermore, let $\mu_0$ be the mass of products whose patent has expired (meaning $\lambda = 1$) and $\mu_+(\cdot)$ be the cumulative density over those products whose patents are not expired. The value of successful external innovation is then given by

$$
\bar{V} = [(1-\alpha) + \alpha\mu_0)]\mathbb{E}V(\beta) + \alpha(1-p)\int_1^\infty (\mathbb{E}[V(\beta\lambda) - V(\lambda))d\mu_+(\lambda)
$$

As discussed earlier, each product line has a production value and a research value. The production value and internal research value will be a function of the technological lead, while the external research value will be independent of that variable since future innovations are undertaken on random external product lines. Thus it is useful to define the option values

$$
\begin{aligned}
\Omega_x &= \max_x \left\{ -\tilde{w} c(x) + x\bar{V} \right\} \\
\Omega_z(\lambda) &= \max_z \left\{ -\tilde{w}\lambda^{-1} d(z) + z(\mathbb{E}V(\beta\lambda) - V(\lambda)) \right\} \\
\Omega_0 &= \max_{z_0} \left\{ -\tilde{w} d(z_0) + z_0(\mathbb{E}V(\beta\lambda) - V_0) \right\}
\end{aligned}
$$

Notice that because successful internal innovation in an expired product line results in increased protection from external innovation, the incentive structure is slightly different. The

product line value function expressions can then be simplified to

$$(\delta_F + (1-\alpha)\tau)V(\lambda) - \dot{V}(\lambda) = \pi(\lambda) + \Omega_x + \Omega_z(\lambda) + \alpha\tau p(\mathbb{E}V(\beta\lambda) - V(\lambda)) + b(V_0 - V(\lambda))$$

$$(\delta_F + \tau)V_0 - \dot{V}_0 = \Omega_x + \Omega_0$$

Having characterized the firm value functions and their dynamics, we must also address the evolution of the state space, which in this case consists of the technological lead distributions. First, the respective masses of expired and unexpired product lines will satisfy the flow equations

$$\dot{\mu}_0 = b\mu_+ - (\tau + z_0)\mu_0 \qquad \text{and} \qquad \dot{\mu}_+ = (\tau + z_0)\mu_0 - b\mu_+$$

Focusing on unexpired product lines (where $\lambda > 0$), the distribution will satisfy the flow equation

$$\dot{\mu}_+(\lambda) = (b + (1-\alpha)\tau)\left[F(\lambda) - \mu_+(\lambda)\right] - (\alpha\tau + z)\int_1^\infty \left[1 - F(\lambda/\lambda')\right]d\mu_+(\lambda') \tag{1}$$

$$+ \left(\frac{\dot{\mu}_+}{\mu_+}\right)\left[F(\lambda) - \mu_+(\lambda)\right] \tag{2}$$

The first two terms are what we would expect in the case without patent expiry. Independent innovations arrive at rate $(1-\alpha)\tau$ and reset the technological lead to some random value $\beta$, and similarly for patent expiry $b$. Meanwhile, sequential and internal innovations arrive at rate $\alpha\tau + y$ and increment the technological lead by some random value $\beta$. The last term simply deals with the fact that there are also product lines flowing into and out of expiry.

### 3.4   Equilibrium

Having described the optimization problems faced by consumers and firms, we can now move on to characterizing their optimal behavior and setting forth conditions for aggregate consistency given the equilibrium variables we have introduced. The aggregate information needed by the firm to make decisions includes the rate of creative destruction $\tau$, the wage rate $w$, and the interest rate $r$. Finally, the firm needs to know the state space, namely the distribution over technological leads, which is fully described by the respective masses of expired and unexpired product lines $\mu_0$ and $\mu_+$ and the distribution of technological leads over unexpired product lines $\mu_+(\cdot)$. Eventually, it will be shown that the mean inverse over $\mu_+(\cdot)$

$$\Gamma_+ = \int_0^\infty \lambda^{-1}d\mu_+(\lambda)$$

will suffice for the purposes of the firm and for aggregate consistency. Now posit a linearly separable ansatz for the unexpired product line value function

$$V(\lambda) = A - B\lambda^{-1}$$

Recall that $\widetilde{\pi}(\lambda) = 1 - \lambda^{-1}$. Inserting the above into the product line value function and equating coefficients on the constant term and the $\lambda^{-1}$ terms yields the following characterization of the coefficients

$$(\delta_F + b + (1-\alpha)\tau)A - \dot{A} = 1 + \Omega_x + b V_0$$
$$(\delta_F + b + (1-\alpha)\tau)B - \dot{B} = 1 - \Omega_z - \alpha\tau p B/(1+\kappa^{-1})$$

Here $\Omega_x$ is the option value of external innovation. Because of the concavity of the profit function in the technological lead, the gross returns to internal innovation are decreasing. However, since the the cost also decreases by the same proportion, the net returns also scale down with the technological lead. Thus internal innovation shows up in the variable portion of the value function as

$$\Omega_z = \max_z \left\{ -\widetilde{w} d(z) + z B/(1+\kappa^{-1}) \right\}$$
$$\Omega_0 = \max_{z_0} \left\{ -\widetilde{w} d(z_0) + z_0(A - B/(1+\kappa) - V_0) \right\}$$

with $\Omega_z(\lambda) = \lambda^{-1}\Omega_z$. Using these expressions, the expected gain from innovation can be simplified to

$$\bar{V} = ((1-\alpha) + \alpha\mu_0)(A - B/(1+\kappa)) + \alpha(1-p)\mu_+\Gamma_+ B/(1+\kappa^{-1})$$

The labor market clearing condition will include contributions from production, external innovation (from incumbents and entrants) and internal innovation (on expired and unexpired product lines) as delineated below

$$1 = \frac{\Gamma}{\widetilde{w}} + (1+e)c(x) + \mu_0 d(z_0) + \mu_+\Gamma_+ d(z) \tag{3}$$

where $\Gamma$ is the average inverse technological lead over all product lines and satisfies $\Gamma = \mu_0 + \mu_+\Gamma_+$. The flow equation for $\Gamma_+$ is described in the next section and depends only on $\mu_0$, $\mu_+$, and $\Gamma_+$ itself. Therefore, with regards to solving the equilibrium by determining the evolution of the state space, $\Gamma_+$ is a sufficient statistic for $\mu_+(\cdot)$. In fact, just $\mu_0$ and $\Gamma$ would be a sufficient state space. However, the three variable specification proves to be notationally cleaner.

Aggregate consistency of the rate of external innovation requires that $\tau = (1+e)x$. Though it is not necessary for the equilibrium solution, the growth rate will naturally be of interest as an implication of this model. Each innovation, regardless of whether it is sequential or independent furthers the state of the art for a particular intermediate good by a random factor $\beta$ drawn from $F$. Because of the log-log aggregation in producing the final good, output can be decomposed into

$$Y = QL_P/\Delta$$

where $Q$ is the log aggregate productivity $\log(Q) = \int_0^1 \log(q_j)dj$ and $\Delta$ is a measure of labor misallocation given by

$$
\begin{aligned}
\log(\Delta) &= \log\left[\int_0^1 \ell_j\, dj\right] - \int_0^1 \log(\ell_j)dj \\
&= \log\left[\int_1^\infty \lambda^{-1} d\mu(\lambda)\right] - \int_1^\infty \log(\lambda^{-1})d\mu(\lambda) \geq 0
\end{aligned}
$$

where the inequality above follows from Jensen's inequality. It is straightforward to show that the growth rate of the aggregated productivity will simply be

$$g = \kappa(\tau + \bar{z}) \tag{4}$$

where $\bar{z} = \mu_0 z_0 + \mu_+ z$ is the aggregate rate of internal innovation. Outside of steady state the quantities $L_P$ and $\Delta$ can of course also change. So the overall growth rate of output will be composed of contributions from these three factors. In steady state, however, the growth rate of $Y$ will simply be $g$.

## 3.5   Steady State

The above section fully characterized the dynamic equilibrium of the model. In principle, this characterization could be used to describe the path of the economy starting from any given point in the state space. The usefulness of this capability is dampened by the inherent difficulty in simultaneously identifying the parameters of the model and the position in the state space. Therefore, I focus on the case of steady state.

In steady state, all normalized figures, such as those comprising the value function, will be constant. In addition, the position in the aggregate state space, as defined above, will be

invariant. Proceeding from this basis, the firm value function coefficients simplify to

$$A = \frac{1 + \Omega_x + b V_0}{\delta_F + b + (1-\alpha)\tau} \qquad \text{and} \qquad B = \frac{1 - \Omega_z}{\delta_F + b + ((1-\alpha) + \alpha p \kappa/(1+\kappa))\tau} \tag{5}$$

The value of a product line where patent protection has expired becomes simply

$$V_0 = \frac{\Omega_x + \Omega_0}{\delta_F + \tau} \tag{6}$$

These quantities, in conjunction with the state space position will determine the expected present value from successful innovation, which will in turn determine innovation rates, the growth rate, wages, and other observables of interest.

We now move on to the task of characterizing the steady distribution of technological leads. Imposing steady state on the flow equation for $\mu_+(\cdot)$ object given in (1), I find

$$(b + (1-\alpha)\tau)\left[F(\lambda) - \mu_+(\lambda)\right] = (\alpha\tau + z)\int_1^\infty \left[1 - F(\lambda/\lambda')\right] d\mu_+(\lambda')$$

For arbitrary $F$, the resulting distribution is intractable. However, given our assumption of Pareto distributed step sizes, one can show that the steady state distribution will in fact be Pareto as well.

**Proposition 1.** *The distribution of technological leads for patent protected product lines is Pareto with*

$$\mu_+(\lambda) = 1 - \lambda^{-1/m}$$

*where the tail index parameter satisfies*

$$m = \kappa\left[\frac{b + \tau + z}{b + (1-\alpha)\tau}\right] \tag{7}$$

*Proof.* Recall that the cumulative density for $\beta$ is simply $F(\beta) = 1 - \beta^{-1/\kappa}$. Now posit a similar form for the technological lead distribution with shape parameter $m$. Plugging this into the flow equation, one can verify that this shape parameter is given the above expression. $\square$

Thus the expected value of $\log(\lambda)$, conditional on being strictly positive is simply $m$. Here one can see that increasing either $\alpha$ and $\tau$ serves to attenuate the technological lead distribution while increasing the patent length $b$ draws it closer to unity. Furthermore, the mean inverse technological lead for unexpired product lines can then be expressed as

$$\Gamma_+ = \frac{1}{1+m}$$

**Note.** *As an aside, I will note that the assumption of Pareto distributed step sizes is not critical to the equilibrium solution, but is needed to ensure tractability of the technological lead distribution, which simplifies notation in various places. For arbitrarily distributed $\beta$, one can write the flow equation for the quantity $\Gamma_+$ as*

$$\dot{\Gamma}_+ = (b + (1-\alpha)\tau)\left(\mathbb{E}[\beta^{-1}] - \Gamma_+\right) - (\alpha\tau + z)\left(1 - \mathbb{E}[\beta^{-1}]\right)\Gamma_+ + \left(\mathbb{E}[\beta^{-1}] - \Gamma_+\right)\left(\frac{\dot{\mu}_+}{\mu_+}\right)$$

*Imposing $\dot{\Gamma}_+ = \dot{\mu}_+ = 0$ and solving then yields the equations*

$$\Gamma_+ = \frac{1}{1+m} \qquad where \qquad m = (1 - \mathbb{E}[\beta^{-1}])\left[\frac{b + \tau + z}{b + (1-\alpha)\tau}\right]$$

*though $m$ can no longer be interpreted as the tail index of the distribution $\mu_+$.*

The only remaining elements of the state space to be solved for are the aggregate shares of expired and unexpired product lines. Equating the flow equations for these quantities to zero yields the simple solution

$$\mu_0 = \frac{b}{b + \tau + z_0} \qquad \text{and} \qquad \mu_+ = \frac{\tau + z_0}{b + \tau + z_0}$$

Combining these with the results above, the inverse technological lead over all product lines, expired or unexpired, is then

$$\Gamma = \mu_0 + \mu_+\Gamma_+ = \frac{b + (\tau + z_0)/(1+m)}{b + \tau + z_0}$$

**Existence**    A balanced growth path equilibrium of this model is characterized by a vector $(\widetilde{w}, g, A, B, V_0)$ consisting of the wage rate $\widetilde{w}$ satisfying (3), the aggregate growth rate $g$ satisfying (4), the unexpired product line value coefficients $A$ and $B$ satisfying (5), and the unexpired product line value $V_0$ satisfying (6).

**Proposition 2.** *A balance growth path equilibrium for this economy exists.*

*Proof.* See Appendix.                                                                                                      □

## 3.6    Welfare

As discussed in the previous section, aggregate output can be decomposed into contributions from three components

$$\log(Y) = \int_0^1 \log(q_j\ell_j)dj = \log(Q) + \int_1^\infty \mu(\lambda)\log(\ell(\lambda))d\lambda$$
$$= \log(Q) + \log(L_P) - \log(\Delta)$$

The term $\log(\Delta)$ is a measure of labor usage heterogeneity, which leads to productive misallocation. This implies that $Q$ is the maximum possible output of the economy and $QL_P$ is the maximal output given a certain amount of production labor $L_P$. In steady state, this takes on the value

$$\log(\Delta) = \log(\Gamma) + \mu_+ m = \log\left[\frac{b + (\tau + z_0)/(1 + m)}{b + \tau + z_0}\right] + \left(\frac{\tau + z_0}{b + \tau + z_0}\right) m \tag{8}$$

Notice that, holding innovation rates constant, this is decreasing in the patent length $b$ and increasing in sequentiality $\alpha$, since $m$ is also increasing in $\alpha$. Welfare is given according to

$$W = \int_0^\infty \left[\frac{Y(t)^{1-\sigma} - 1}{1 - \sigma}\right] \exp(-\delta t) dt$$

Without loss of generality, I can assume that $Q(0) = 1$. Furthermore, we know that $\dot{Q}/Q = g$. Plugging in for $Y$ and evaluating the integral then yields

$$\delta W = \left(\frac{\delta}{\delta + (\sigma - 1)g}\right)\left[\frac{g}{\delta} + \frac{(L_P/\Delta)^{1-\sigma} - 1}{1 - \sigma}\right] \tag{9}$$

Thus welfare can be easily expressed purely as a function of $\tau$, $z$, and $z_0$. One of the main implications of this model is that the welfare effects of monopoly distortions are more severe in more sequential industries. To see this, consider the effect of varying the aggregate innovation rates on the labor misallocation factor $\Delta$.

**Proposition 3.** *The effect of $\tau$, $z$, and $z_0$ on monopoly distortions is greater for more sequential industries. That is, $\frac{\partial \Delta}{\partial \tau}$, $\frac{\partial \Delta}{\partial z}$, and $\frac{\partial \Delta}{\partial z_0}$ are increasing in $\alpha$. Furthermore, the latter two derivatives are always positive, while the first is positive if $\alpha b > (1 - \alpha)z$.*

*Proof.* See Appendix. □

This is one of the major implications of the model presented here. Not only does more innovation induce higher production labor misallocation in most cases, this effects is larger for more sequential industries. Thus in considering patent policy, where the fundamental trade-off is between incentivizing innovation at the cost of monopoly distortions, the benefit is the same while the cost is larger in more sequential industries.

## 3.7  Social Optima

Before considering various policy interventions or changes, it is important to study this model from a social planner's prospective in order to gain insight into the types and levels of inefficiency present in the decentralized equilibrium. Two types of social planners will be considered. The first is a partially constrained social planner who can control the innovation decisions of firms but is still subject to the outcome of the static product market equilibrium with its associated monopoly distortions. In this case, the patent length is assumed to be the same as in the decentralized equilibrium. Choosing external innovation rate $\tau$, internal unexpired innovation rate $z$, and internal expired innovation rate $z_0$ allows one to determine the growth rate $g$, the production labor utilization $L_P$, and the labor misallocation factor $\Delta$. Using these, one can compute steady state welfare using (9).

The second type is an unconstrained planner who makes both innovation and production decisions for firms. This results in a simple closed form expressions for $L_P$ and $g$ as functions of $\tau$, $z$, and $z_0$. The unconstrained planner will optimally choose labor utilization to be equal across product lines, meaning $\ell_j = L_P$ for all $j$ and $\Delta = 1$.

## 3.8  Predictions

The only aggregate variables of interest in this setting are the wage $\widetilde{w}$ and the growth rate $g$. The following predictions will not be functions of these variables. They will depend only on the within industry variables, which are indexed by $\alpha$. In general, both $\tau$, $z$, and $z_0$ will be functions of $\alpha$, however their dependence is suppressed here for the sake of brevity.

**Transfer Rates**    The direction of transfer is indeterminate in this model. Let the probability that a transfer goes towards the innovator be $q$. Knowing this, what fraction of patents can one expect to be transferred in their lifetime? Patents are born at rate $\tau + \bar{z}$. They die at rate $b$ and an rendered obsolete at rate $(1 - \alpha)\tau$. Additionally, an external patent (fraction $\tau/(\tau + \bar{y})$) has a probability $\alpha(1 - q)$ of being transferred immediately upon birth, implying

$$P(0) = \left( \frac{\tau}{\tau + \bar{z}} \right) \alpha(1 - q)$$

Once born, patents of any type have a flow rate of transfer $\alpha q \tau$ for their entire lifetime. The probability of a particular patent surviving to age $t$ and being transferred for the first time is

then

$$P(t) = \left[1 - \left(\frac{\tau}{\tau + \bar{z}}\right) \alpha(1 - q)\right] \alpha q \tau \exp\left(-\left(b + (1 - \alpha)\tau + \alpha q \tau\right) t\right)$$

This exponential form is very close to what is seen in the data. A detailed evaluation of the match is given in the quantitative section. Finding the probability that a patent is never transferred is then a matter of evaluating the above expression in the limit as $t \to \infty$. This yields the expression

$$P = P(0) + (1 - P(0)) \left(\frac{\alpha q \tau}{b + (1 - \alpha)\tau + \alpha q \tau}\right) \in \left[\alpha\left(\frac{\tau}{\max\{\bar{z}, b\} + \tau}\right), \alpha\right]$$

Thus it is very closely related to the incidence of sequential innovation. Notice that the lower bound decreases with the fraction of innovations that are within-firm, since these innovations result in no transfers. Having only external innovation would result in implausibly high fractions of patents being transferred. Additionally, the lower bound decreases with the rate of patent expiry, since innovation on an expired product line induce no transfers as well.

**Patent Expiry**    Now consider the process of patent obsolescence. This occurs due to patent expiry at rate $b$ and due to technological replacement at the rate $(1 - \alpha)\tau$. Therefore, the distribution over the productive lifetime of a patent, $E(\cdot)$, is given by

$$E(t) = (b + (1 - \alpha)\tau) \exp(-(b + (1 - \alpha)\tau)t)$$

which arises from the properties of continuous time Poisson processes. Therefore, when looking across industries or patent classes, one would expect to see a negative relationship between the fraction of patents that are not renewed after a given period of time and the fraction of patents that are transferred at least once.

Another figure of interest the fraction of patents that become obsolete through creative destruction, rather than patent expiry. This value can be shown to be

$$E = \frac{(1 - \alpha)\tau}{b + (1 - \alpha)\tau}$$

Interpreting the loss of a patent due to failure to pay maintenance fees as creative destruction, this figure is roughly $1/2$ in the data.

**Markups and Profits**   We must also address the effect of $\alpha$ on markups over cost. Recall that for a given product line, total revenues are always $Y$, while production labor costs are to $\lambda^{-1}Y$. Thus the markup for any one product line is simply $\lambda$ and average markup at the product line level is equal to

$$\bar{\Lambda} = \mu_0 + \mu_+ \left( \frac{1}{1-m} \right) = \frac{b + (\tau + z_0)/(1-m)}{b + \tau + z_0}$$

Unfortunately, this is not always well defined since we cannot guarantee that $m < 1$. We can however give a well-defined expression for the median markup at the product line level. The complete cumulative density for markups is given by $\mu(\lambda) = \mu_0 + \mu_+(1 - \lambda^{-1/m})$. Equating this to one half yields

$$\widehat{\Lambda} = \max\left\{ 1, \left[ 2 \left( \frac{\tau + z_0}{b + \tau + z_0} \right) \right]^m \right\} \tag{10}$$

These cannot be directly observed in the Compustat data. However, one can look at total sales over variable cost at the firm level and target that using simulations. Meanwhile, the aggregate ratio of sales to cost over the entire economy ($\Lambda$) is simply the harmonic mean of individual product line markups

$$\Lambda = \left[ \mu_0 + \mu_+ \left( \frac{1}{1+m} \right) \right]^{-1} = \frac{b + \tau + z_0}{b + (\tau + z_0)/(1+m)}$$

This measure can be calculated directly from the Compustat data set or taken from existing estimates. It can also be shown to be increasing in $\alpha$. However, the endogenous response of $\tau$, $z$, and $z_0$ will determine the net effect.

**Patent Portfolios**   A closely related figure is the size of a firm's patent portfolio. In this case I will work with portfolios at the per-product line level, which can then be aggregated to the firm level. Let the mean portfolio size in a given industry be denoted by $\bar{n}$. The flow equation for this quantity is given by

$$\dot{\bar{n}} = \sum_{n=0}^{\infty} \dot{\mu}_n n = (1-\alpha)\tau + (\bar{n}+1)(\alpha\tau + \bar{z}) - (b + \tau + \bar{z})\bar{n}$$

which leads to the solution

$$\bar{n} = \frac{\tau + \bar{z}}{b + (1-\alpha)\tau}$$

This can be used to calculate a number of pertinent values. For instance, assuming independent innovations make no citations, the average number of external citations per patent will simply be $\left( \frac{\tau}{\tau + \bar{z}} \right) \alpha \bar{n}$.

## 3.9    Industry and Firm Heterogeneity

In order to match the cross-industry variation in quantities such as transfer rates, expiry rates, and profitability, I introduce the possibility of heterogeneity across industries in the sequentiality of innovation. Industries are segmented in terms of innovation but share a common pool of labor. Let there be $M$ equal-mass segments in total. Each will have a particular value for $\alpha$ and associated innovation rates $\tau(\alpha)$, $z(\alpha)$, and $z_0(\alpha)$ and average inverse technological lead $\Gamma(\alpha)$. The initial analysis then carries through unchanged at the industry level and we need only be concerned with aggregate labor and bond market clearing. The labor market clearing condition is given by

$$1 = \mathbb{E}_\alpha \left[ \frac{\Gamma(\alpha)}{\widetilde{w}} + (1+e)c(x(\alpha)) + \mu_0(\alpha)d(z_0(\alpha)) + \mu_+(\alpha)\Gamma_+(\alpha)d(z(\alpha)) \right]$$

In practice, one can deal in the limit where $M \to \infty$, using a continuous distribution over $\alpha$. Each industry is then treated as an infinitesimally small segment of the overall spectrum of products.

One potentially unsatisfying implication of the above model is that the direction of transfer is indeterminate and is effectively decided by a weighted coin flip $q$ between incumbent and challenger. By introducing firm level heterogeneity, either in production or research capability, we can not only break this indeterminacy, we can investigate the potential allocative implications of the transferring of patent rights. This modification of the model is motivated by the trends presented above as well. In particular, Fact Fact 5 regarding the predominance of patent flows towards younger and smaller firms leads one to believe such a dynamic is an important determinant of trends in the transfer of patent rights.

A two-type extension of the model presented herein is exposited in detail in Appendix B. This partially resolves the indeterminacy of patent transfer direction in the sense that interactions between firms of unlike types will result in final ownership being vested in the high type firm. However, interactions of like type firms will still require the introduction of *ad hoc* randomness. Firms differ in their cost of external innovation, while the cost of internal innovation and production capacity remain identical across firms. As a result, there will be differential firm-level external innovation rates $x^H$ and $x^L$ and expired internal innovation rates $z_0^H$ and $z_0^L$ but a common unexpired internal innovation rate $z$. Entering firms are high type with probability $\zeta$ and decay to low type at the Poisson flow rate $\nu$. The associated aggregate external innovation rate is $\tau = (\zeta e + \mu^H)x^H + ((1-\zeta)e + \mu^L)x^L$, where $\mu^i$ is the mass of products owned by a type $i$ firm, while the aggregate internal innovation rate is $\bar{z} = \mu_0^H z_0^H + \mu_0^L z_0^L + \mu_+ z$.

# 4    Estimation

In this section I bring the proposed model to the firm-level data. First, I provide a summary of the challenges associated with identifying the key aspects of the model quantitatively. Then I present the results of the estimation, assess the quality of the fit, and give some interpretation to the resulting parameters values.

## 4.1    Identification

In the basic model, there are seven parameters that need to be identified. First, those associated with the innovation production process, namely the step size distribution tail index $\kappa$ and the cost parameters $c$ and $\eta$. Additionally, there is the mass of entrants $e$ and the discount rate $\delta$, which is set to 0.05. Finally, there are two parameters specific to the phenomenon studied in this paper, the probability of sequential innovation $\alpha$ and the bargaining power of the incumbent $p$. Moving to the setting with own-product innovation will add an additional cost parameter $d$, and the introducing industry level heterogeneity in $\alpha$ will add in two distributional parameters. The specific form used will be the two parameter Beta distribution, a flexible choice for random variables on the unit interval.

First, the the innovation production parameters can be jointly constrained using aggregate moments. Naturally, their exact values will dependent upon the estimated values of other parameters in the model. However, this partitioning is still useful at a conceptual level. The tail index parameter $\kappa$ will be a strong determinant of profits in the economy. The expression given in (10) is calculated at the product line level. Aggregating this to the firm level has no apparent analytical form, so simulation must be used. Meanwhile, the R&D production function parameters can be effectively constrained using the aggregate growth rate and the share of R&D spending in the economy. Both these moments are readily available from BEA data. For the sake of consistency, analogues at the firm level, namely the incumbent innovation rate and R&D intensity, can be used as well.

Introducing the possibility of own-product innovation adds one additional cost scaling parameter $d$, while the elasticity $\eta$ is assumed to be common with the external innovation cost function. This can be constrained in a straightforward manner by using self-citation activity by firms. In particular, one can look at the fraction of citations that are internally directed versus externally directed. Alternatively, one can classify patents as primarily internal or external and look at the composition thereof. In the US patent data, approximately 23% of patents cite

other patents from their filing firm. Akcigit and Kerr (2010) study these patterns extensively, finding that approximately 20% of patents are self-citing. They also have access to analogous information at an R&D expenditure level (there termed product versus process innovation) and find a similar fraction.

The mass of entrants parameter can readily be determined by looking at the fraction of the patent stock owned by entrants over a five year period, for instance. Alternatively, one can undertake a growth decomposition and target the standard number given by Davis, Halti-wanger, and Schuh (1998) of $1/3$. As innovation and growth are one-to-one in this framework, we would then simply set $e = 1/2$. However, due to selective exit of low type firms early in the life cycle, targeting simulated numbers over a number of years is more accurate.

Capturing information about $\alpha$ and the distribution over various industries proves not to be too difficult. As shown above, the fraction of patents that are transferred in their lifetime varies with $\alpha$. This response is not provably one-to-one due to variability in the endogenous response of innovation rates, but in practice it proves to be robustly monotone. Thus we can effectively use the inverse of this function to relate data on patent transfer rates to sequential-ity measures $\alpha$ and the industry-level distribution thereof.

The most difficult parameter to estimate is the bargaining power parameter $p$. One possi-ble stance is that, there being no difference *ex ante* between firms in terms of bargaining po-sition, symmetry dictates the value be set to $1/2$. However, if one imagines a model in which bargaining does not happen instantaneously, the incumbent might be more patient if it thinks it can produce a workaround innovation. Alternatively, there may be asymmetries in patent protection for incumbents and new innovators that lead to asymmetries that can be captured by the bargaining power parameter. In the baseline case, I simply set $p = 1/2$, however, I also investigate the effects of changing this parameter on innovation rates and aggregate out-comes.

To match the facts regarding patent flows to younger and smaller firms, I also introduce firm-level heterogeneity in the cost of external innovation. This introduces three new param-eters. First, there are now two cost factors for external innovation ($c^H$ and $c^L$) rather than one. Additionally, there is the initial fraction of entrants that are high type ($\zeta$) and the rate at which high type firms decay into low type firms ($\nu$). By looking at growth rate differentials between young and old firms, as well as the fraction of the patents stock owned by young firms, we can constrain these various parameter values.

## 4.2   Results

As noted at the outset, the previous discussion is merely a conceptual overview of identification. Each of the parameters will affect each potential moment in varying degrees, as determined by the model. In order to match all of these simultaneously and intelligently trade off prediction errors in the case of a less-than-perfect match, I employ a simulated method of moments (SMM) estimator. This has the additional advantage of allowing for bootstrapping to determine parameter standard errors. See Bloom (2009) for further details on the usage of SMM. The details of the equilibrium solver and simulation algorithm are described in Appendix C.

The SMM objective function is simply quadratic form of the differences between the data and model predictions for a certain vector of moments. The requirements for identification discussed above will largely determine which moments are used. However, there are still some specific implementation details that should be explained.

First, the notion of entry employed here counts any previously unobserved firm that files for a patent. As the primary focus is on the innovation process, this would seem to be the most relevant statistic for our purposes. Additionally, it does not suffer from the selection issues that using Compustat would entail. It is susceptible to misclassification of firms in the sense that any novel name matching failure would show up as an entrant rather than simply a filing from an existing firm. Including a measure of the patent stock that recent entrants comprise could ease these concerns and potentially provide valuable information about the type distribution of entrants (the parameter $\zeta$).

To measure internal citations, one must take a stance on the exact mapping between the model and the data in this regard. Independent innovations of course cannot be entirely disparate from work that has come before. There is a sense in which general knowledge informs innovations within and across fields. The extent to which citations reflect general versus specific influence is not clear. Assuming that sequential and independent innovations have roughly similar amounts of external citations, the internal citation ratio would simply be $\bar{z}/(\tau + \bar{z})$. Note that here I assume that a new addition to a patent portfolio cites all of the patents below it, or at least a fixed fraction thereof.

The transfer rate mean and variance statistics are calculated as the probability that a patent in a particular industry filed for during the period in question is ever transferred (including beyond the end of the period). The three year transfer rate is simply the same probability but

restricting to the transfer occurring within three years of filing. The analog in the model is taken to be the probability of immediate transfer. The delay is chosen to allow for the fact that this process will not be instantaneous in a real-world setting.

TABLE 1: MOMENT VALUES

| Name | Data | Model |
|------|------|-------|
| Aggregate Growth | 0.034 | 0.034 |
| Entrant Stock Frac | 0.085 | 0.081 |
| Transfer Prob Mean | 0.180 | 0.187 |
| Transfer Prob Std | 0.105 | 0.104 |
| Internal Cite Frac | 0.229 | 0.268 |
| Median Profit | 0.164 | 0.175 |
| 3-year Transfer | 0.365 | 0.356 |
| Transfer To Younger | 0.699 | 0.621 |
| Transfer To Young | 0.573 | 0.546 |
| Young Stock Fraction | 0.139 | 0.143 |
| Young Filing Fraction | 0.197 | 0.198 |
| Average Filing Fraction | 0.166 | 0.182 |
| R&D Intensity | 0.092 | 0.051 |

The remaining aggregate moments are fairly straightforward. The aggregate growth figure is taken from the FRED data on output per person. The median profit is computed using the sample of Compustat firms. The moment values at the optimum are given in Table 1. It is evident that the fit of the model to the data is quite close in many dimensions, but misses the mark in some cases. The moments on aggregate growth, transfer statistics, citations, profits, and R&D levels are all very closely matched.

The parameter estimates themselves are summarized in the Table 2. Of prime importance are the sequentiality distribution parameters, which indicate that in the average industry, 35% of innovations are sequential as opposed to independent. Furthermore, looking across industries, this quantity has a standard deviation of 22%.

The R&D production function parameters are in line with those found in the existing literature. In particular, the curvature implies an elasticity of $53\%(= 1/1.880)$. This is similar to the

TABLE 2: ESTIMATED PARAMETER VALUES

| Name | Symbol | Value |
|------|--------|-------|
| Discount Rate | $\delta$ | 0.050 |
| Bargaining Power | $p$ | 0.500 |
| CRRA Parameter | $\sigma$ | 2.819 |
| Step Distribution | $\kappa$ | 0.339 |
| External R&D (High) | $c^H$ | 6.206 |
| External R&D (Low) | $c^L$ | 12.659 |
| Internal R&D Cost | $d$ | 20.257 |
| R&D Cost Curvature | $\eta$ | 1.880 |
| Mean Sequentiality | Mean($\alpha$) | 0.347 |
| Std Sequentiality | Std($\alpha$) | 0.216 |
| Entrant Mass | $e$ | 0.161 |
| Entry High Type | $\zeta$ | 0.580 |
| Type Decay Rate | $\nu$ | 0.126 |
| Transfer Direction | $q$ | 0.730 |

value of 0.61 found in Pakes and Griliches (1980). As documented by Kortum (1993), estimates of this parameter generally lie between 0.1 and 0.6. External innovation is more than twice as costly for low type firms compared to high type firms. Meanwhile, the common cost of internal innovation is nearly twice as costly again as low type innovation. The entrant mass of 0.19 is lower than what one might guess from directly calibrating to a growth decomposition attributing $1/3$ of growth to entrants. However, as noted in Foster, Haltiwanger, and Krizan (2001), the time horizon over which we measure entry contributions will affect even annualized figures due to selective exit of recent entrants. The firm selection dynamic present in the model induces a similar effect in simulated results.

The firm type dynamics parameters are consistent with previous structural studies. In particular, Acemoglu, Akcigit, Bloom, and Kerr (2013), on whose model of reallocation I build, estimate that the proportional variance in the cost faced by entrants is 1.21.[8] The analogous number using my estimates is 1.06. In their model, ignoring selection due to exit, the aver-

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[8]The fraction of entrants that are high type is not directly comparable, as the respective R&D cost parameters also differ.

age cost faced by incumbent firms rises by 5.5% over the course of one year, while I find that quantity to be 5.3%.

# 5   Quantitative Analysis

With these estimates in hand, we can evaluate the ability of the model to match the cross-industry and firm-level trends outlined in the data section. Figure 7 documents the model's qualitative success in reproducing four of the the major cross-industry trends seen in the data. In each pane, the variable of interest is plotted at the industry level against the sequentiality for that industry, which is simply the fraction of patents that are transferred in their lifetime. This measure is used so as to most closely match the facts presented in the data section and varies monotonically with the underlying theoretical sequentiality parameter $\alpha$. The median profitability, firm growth, and exit rate figures are the result of simulations and so have noise associated with each point, but the magnitude of this noise is not enough to obscure the evident trends.

To get an idea of the quantitative match of the model with regards to these trends, the average level of each variable over all industries in the data and in simulations is given in Table 3. The expiry rate and firm growth volatility are both matched well, while the profitability and exit rate figures are both not matched entirely. The profitability figure, though targeted at the economy-wide level, is grouped by industry and can thus be different due to aggregation effects. The exit rate may reflect asymmetries between entry and exit that are not present in the model, which features only entry of and exit by one-product firms. In the data, though entering firms are generally quite small, large firms may exit or cease to exist due to mergers. The framework used here implicitly categorizes innovations in to internal, sequential, and independent types. By targeting patent transfers and internal citation rates, the close alignment of the data here further supports the notion that sequential innovation is reflected in patent transfers, while independent innovation is reflected in patent expiry.

Having analyzed the levels of each of the four variables of interest here, I now study the variation across industries. To assess the amount of variation across industries in the various quantities, I perform a weighted least squares regression on sequentiality (the rate of patent transfer) and look at the predicted effect of moving between the mean plus or minus one standard deviation, as normalized by the mean value for that particular quantity. Comparing the values given by this metric for both the data and the simulated model, I can assess the pre-

FIGURE 7: CROSS-INDUSTRY PREDICTIONS
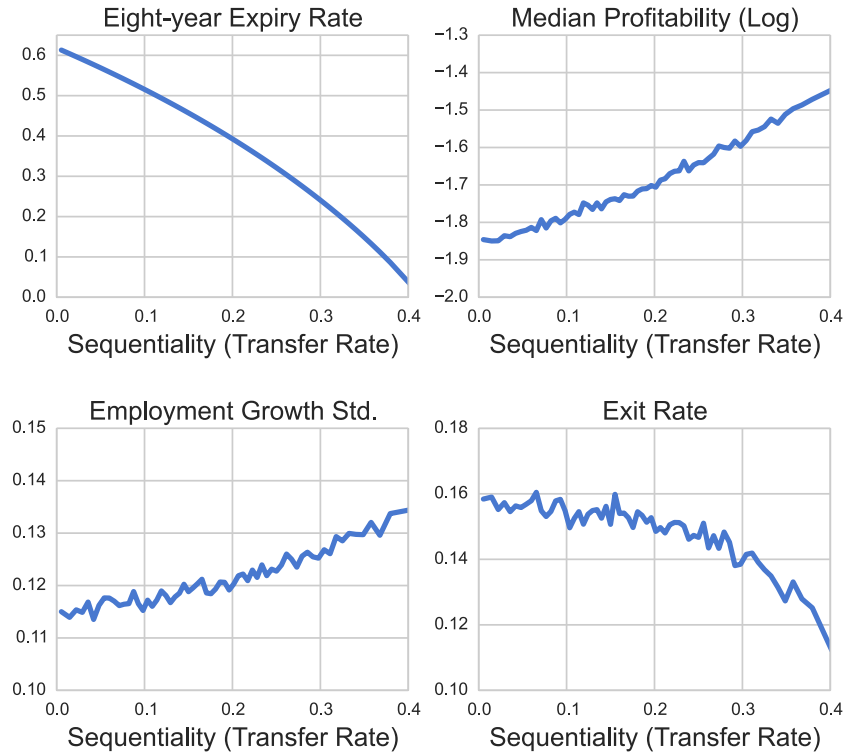


dictive power of the model. It is important to note that though the variation in sequentiality was targeted, variation in the other quantities has not been, meaning these implications come purely from the model structure. Using this metric, the model can account for approximately 65% of the variation in profitability (return on sales), 26% of the variation in firm growth volatility (employment), and 40% of the variation in exit rates. Meanwhile, the level of variation in expiry rates is overpredicted by the model. In Figure 8, I plot the proportional variation for each quantity. Both the data and model generated numbers are normalized by their respective mean values, as is the regression line for the data.

Figure 9 depicts a number of equilibrium variables of interest as they vary with industry sequentiality. Of particular interest in this framework is the effect of industry sequentiality on the incentives to innovate and the resulting innovation rates. The effect on external innovation is *ex ante* ambiguous due to the existence of two opposing forces on the expected product

TABLE 3: AVERAGE INDUSTRY-WIDE VALUES

| Variable Name | Data | Model |
|---:|:---:|:---:|
| Eight-year Expiry Rate | 34% | 39% |
| Median Profitability | 11% | 18% |
| Firm Growth Volatility | 13% | 12% |
| Exit Rate | 29% | 15% |

line valuation. The results of the estimate indicate a negative effect of sequentiality on the the rate of external innovation, with the effect being larger for high type firms. To understand this result, consider the time profile of returns delivered by an innovation. In non-sequential industries, the payoff is large early on but is cut off relatively soon due to creative destruction. Meanwhile, in highly sequential industries, the initial payoff is smaller, but there are ongoing payouts from future sequential innovations. Even if the total payout is similar in these two cases, a firm only captures a fraction of this surplus through bargaining, thus back-loaded incentives in highly sequential industries result in lower innovation rates.

As expected theoretically, internal innovation rates rise with sequentiality due to the reduced threat of creative destruction in more sequential industries. The n-shaped dependence of the average firm type on sequentiality is somewhat unexpected. One would anticipate that high type firms would face the largest drops in product loss rates moving to more sequential industries, which is indeed the case. However, this effect is eventually overwhelmed by the disproportionate drop in high type external innovation rates. Finally, the level of monopoly distortion ($\Delta$) on a per-industry basis is shown. As would be expected, this rises with sequentiality, as a result of the agglomeration of larger patent portfolios by firms.

Now consider the aggregate trends in the economy. The following table summarizes the growth contributions of entrants, external innovation, and internal innovation.

EQUILIBRIUM DECOMPOSITIONS (PERCENTAGES)

|  | Entrant | Inc. External | Inc. Internal | Total |
|---:|:---:|:---:|:---:|:---:|
| **Growth** | 0.49 ( 14.3%) | 2.02 ( 58.9%) | 0.92 ( 26.8%) | 3.43 (100.0%) |
| **Labor** | 1.37 ( 16.0%) | 5.38 ( 62.8%) | 1.81 ( 21.1%) | 8.56 (100.0%) |

External innovation by incumbents still plays a large role in innovation, followed by incum-

FIGURE 8: CROSS-INDUSTRY VARIATION
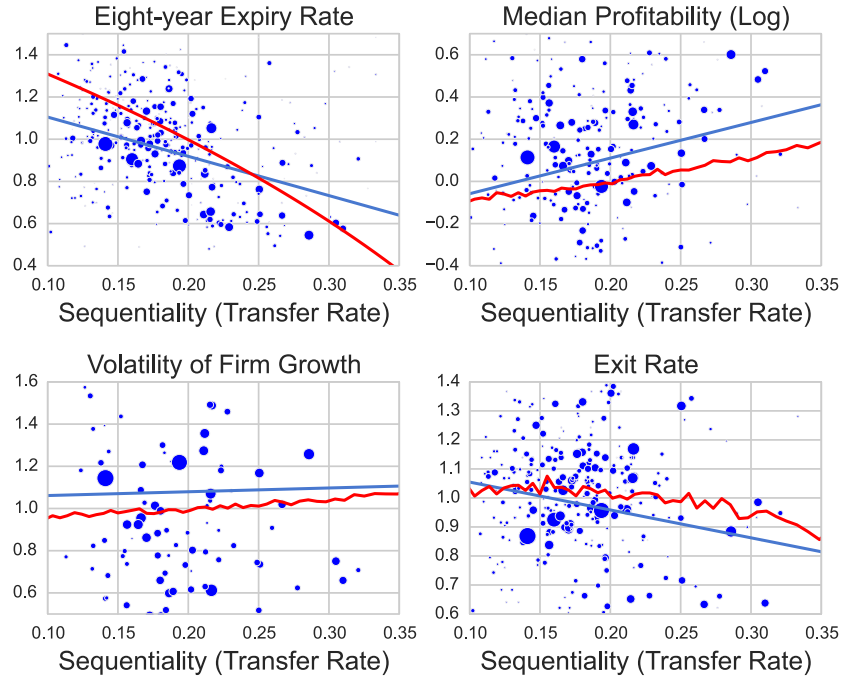


bent internal innovation, then entrants. Additionally, internal innovation achieves notable performance in terms of innovations per unit labor. Considering both the cost and capacity are similar to that of external innovation, this is largely a sign that it is simply utilized less.

As noted previously, introducing firm types into such a model naturally allows us to consider the potential effects of sequentiality on reallocation. Entering firms start out with a particular fraction of high-type firms ($\zeta = 58\%$). Over time the surviving firms decay at rate $\nu = 13\%$. High type firms have a higher survival probability, so the distribution over type by firm age exceeds the simple case of exponential decay. The transfer direction parameter dictates that when firms of like type transfer patents, the innovator becomes the eventual owner in 73% of cases. Using the respective shares high type and low type products, this implies that overall a high type innovator assumes final ownership in 96% of cases, while the same number for low types is 61%, and the overall number is 76%.

Looking at Figure 10, we can see that the match between the predicted and observed lag from patenting filing and time of first transfer is quite good. It should be noted, though, that the model delivers an exponential form for this function by construction, and relative mass at

FIGURE 9: CROSS-INDUSTRY EQUILIBRIUM VARIABLES
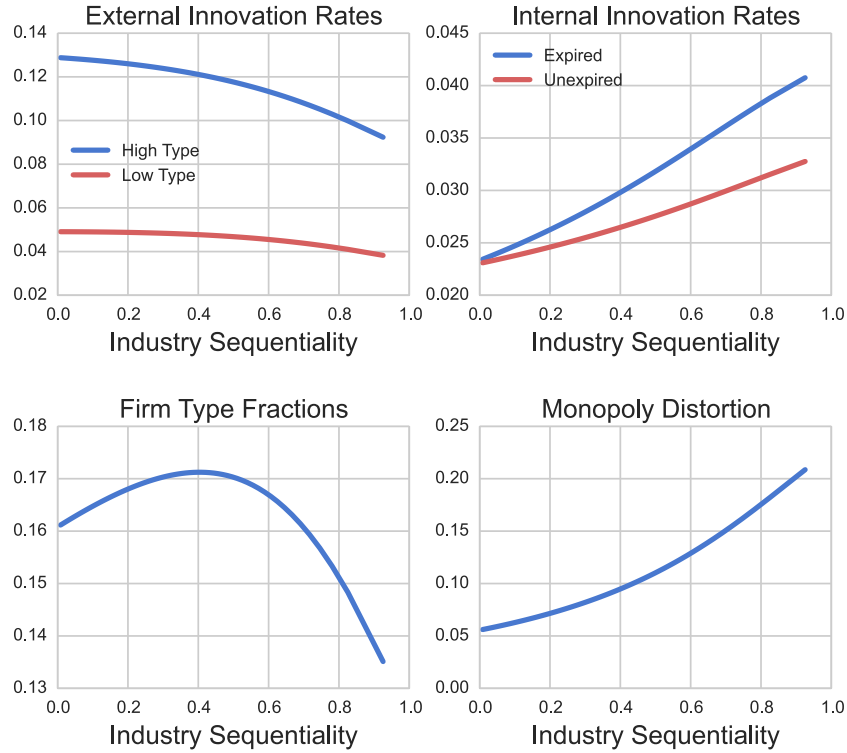


zero is indeed targeted for the purposes of estimating the direction of transfer parameter ($q$). As for the profitability distribution, the match between data and model is less exact. The median profit is targeted as a moment, however, the model generated clearly shows excess mass near zero. This could arise from compositional issues. Firm level profitability data is available only for Compustat firms, which are larger than the average firm, and hence the observed data could be "overaggregated", leading to greater weight in the middle of the distribution and less at the very bottom.

## 5.1    Mechanism Investigation

To better understand the means through which sequentiality affects the incentives for innovation, in this section I consider various modification to the estimated model. First, I investigate the effects of varying the bargaining power parameter, which was previously set to the sym-
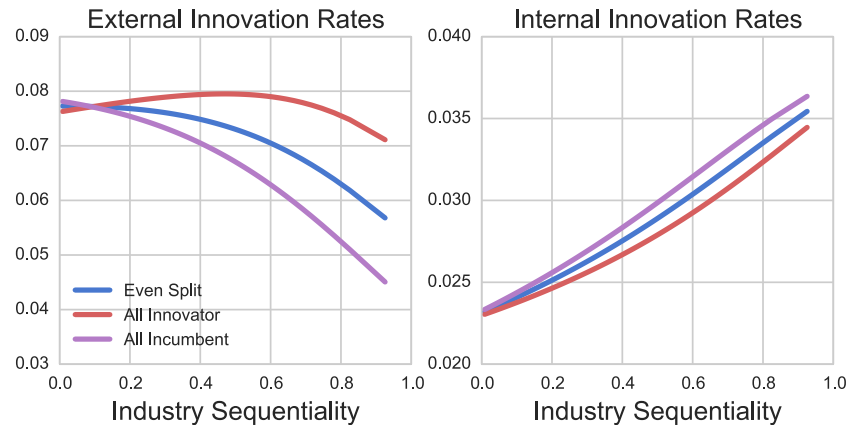
FIGURE 10: EQUILIBRIUM DISTRIBUTIONS
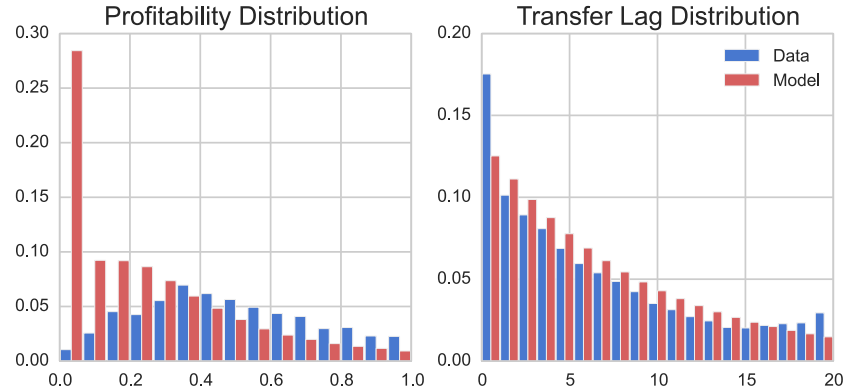


metric value of $1/2$. Because payoffs are back-loaded in more sequential industries and hence only partially delivered to the incumbent through the bargaining process, this parameter will be an important determinant of the incentives for external innovation.

FIGURE 11: BARGAINING PROCESS ALTERNATIVES



In Figure 11, the baseline case, as well as the two extremes of giving all the bargaining power to the innovator ($p = 0$) and to the incumbent ($p = 1$), are plotted. Here we see that though the rate of internal innovation is largely invariant to the bargaining parameter, the decrease in external innovation with sequentiality is larger the more bargaining power is vested with the incumbent. The intuition here is that when none of the incentives are transferred

intertemporally through bargaining, as is the case when the innovator has all the bargaining power, the backloading of payoffs that comes with increased sequentiality does not interact with the bargaining distortions, thus the profile is relatively flat.

A second modification I consider is simply eliminating sequential innovation across the economy, that is, setting $\alpha = 0$. This provides insight into the aggregate effects of sequentiality. Moving from the baseline case to the no sequential case, the growth rate falls from 3.42% to 3.27%. Most of this comes from changes in internal innovation rates, which fall from 2.7% to 2.2% annually. Distortion from production labor misallocation ($\Delta$) fall substantially from 9.4% to 5.5%. Recall that because of patent expiry dynamics and internal innovation, there will still be labor utilization heterogeneity, though it is much smaller here.

The final model modification performed is the elimination of firms type. Consider an economy where instead of having multiple firm types, there is a single firm type whose cost is equal to the expected cost of a new entrant, that is $\bar{C} = \zeta c^H + (1 - \zeta)c^L$. By studying the difference between our benchmark and this economy, we can illuminate the effects of firm types. The primary motivation for introducing firm types was to capture the disproportionate flow of patent transfers from older and larger firms to smaller and younger firms. Moving to the homogeneous settings, the fraction of patents transfers that are directed towards younger firms falls from 61% in the baseline to 39%. Thus firm types are critical for producing the disproportionate flows we see towards younger firms, which in either case constitute a small fraction of the overall patent stock (15% in the baseline and 9% in the homogeneous case).

There are also important effects on the firm size distribution. Moving from the homogeneous case to the heterogeneous case, we see mean firm size increase by 40%. However, this effect is concentrated mostly amongst the smaller firms. The skewness of the firm distribution is actually much larger in the heterogeneous case, at 20.5 compared to only 3.2 in the homogeneous case. For comparison, this value is 26.0 when looking at the patent stock data. Thus the baseline model captures nearly all the skewness in the data, and the introduction of firm types is partially responsible for this. In particular, the fact that certain firms (high type firms) undergo sustained periods of abnormally high growth is an important factor in generating realistic levels of variation in the firm size distribution.

## 5.2    Social Optima

As discussed previously, I consider both a constrained social planner, who can make innovation decisions but is still subject to patent policy and the resulting monopoly distortions, and an unconstrained planner who can make both innovation and production decisions at will. The constrained optimum yields innovations rates by type on a per-industry basis, while the unconstrained optimum will feature uniform innovation rates across industry. Note that the constrained planner is still also subject to firm type dynamics as in the decentralized case, while the unconstrained planner can reassign product lines to different firms at will but is still subject to exogenous heterogeneity in entrant types. In the constrained setting, the share of labor allocated to research rises from 8.6% to 15.3%, while the aggregate growth rate rises to 4.6%.

FIGURE 12: SOCIAL OPTIMUM INNOVATION RATES
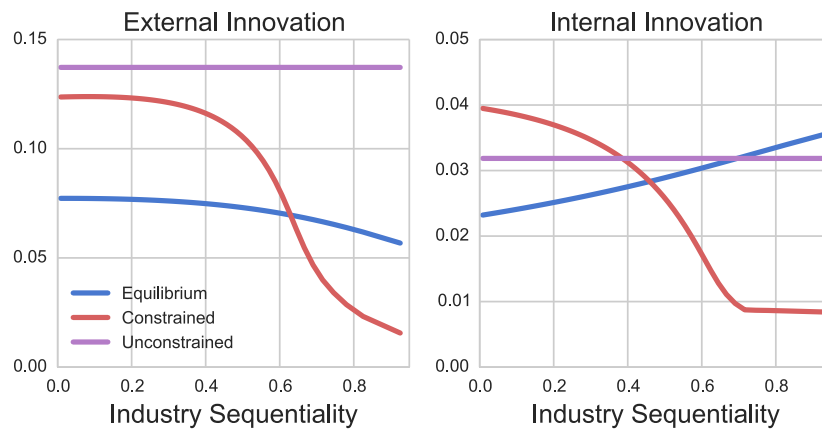


In Figure 12, both the internal and external innovation rates are plotted for the equilibrium and the constrained and unconstrained planner. Because monopoly distortions arising from innovation are much more severe in sequential industries, the planner reduces innovation of both types in these industries. As a result the level of monopoly distortion in the economy goes down by 0.50 percentage points. The net welfare gains from moving to the constrained planner's allocation are 2.5%. The aggregate growth and labor decompositions are show in the table below

CONSTRAINED OPTIMUM DECOMPOSITIONS

|  | Entrant | Inc. External | Inc. Internal | Total |
|---|---|---|---|---|
| **Growth** | 0.64 ( 13.9%) | 2.96 ( 64.0%) | 1.02 ( 22.1%) | 4.62 (100.0%) |
| **Labor** | 2.41 ( 15.8%) | 10.54 ( 68.9%) | 2.33 ( 15.3%) | 15.29 (100.0%) |

In the aggregate, in addition to the increase in the level of research labor overall, there is a shift from internal to external innovation. This is consistent with the intuition presented in Aghion and Howitt (1992) by which externally oriented innovation can be either under or over-invested in, dependent on the step size.

In the unconstrained optimum, we actually see a partial reversal of the trends that occurred when going from the equilibrium to the constrained optimum. The external innovation rate is uniformly higher than in the equilibrium case. However, due to the shift to all high type research, this is done using proportionately less labor. Meanwhile, the uniform innovation rate of the unconstrained planner is quite close to the equilibrium rate seen in highly sequential industries, where one would expect firms to internalize much of the social gains from innovation. The following table summarizes the overall allocation by research type

UNCONSTRAINED OPTIMUM DECOMPOSITIONS

|  | Entrant | Inc. External | Inc. Internal | Total |
|---|---|---|---|---|
| **Growth** | 0.51 (8.9%) | 4.14 (72.2%) | 1.08 (18.8%) | 5.72 (100.0%) |
| **Labor** | 1.47 (8.9%) | 11.92 (72.2%) | 3.11 (18.8%) | 16.49 (100.0%) |

Here we see a further shift towards external innovation, with accompanying gains due to the increase in the quality of incumbent firms. The total amount of research labor rises only slightly above the constrained planners case, while the growth rate rises considerably to 5.72%. It is interesting to note the alignment of the share of growth and labor from each source. This arises from the fact that from the perspective of the unconstrained planner, each type of innovation has the same effect, namely increasing the overall productivity level by a common factor in perpetuity, and there is a common cost elasticity across each type. The consumption equivalent welfare gains associated with moving to the unconstrained planner's allocation are 16%. Much of this, however, comes from the total elimination of monopoly distortions, while the rest is from growth related effects.

## 5.3   Policy Implications

The primary policy lever that will be considered here is the strength of patent protection, as embodied in the rate of patent expiry $b$. This will not be directly analogous to the real world notion of patent length, as patent expiry is stochastic in the model and fixed-length in reality. However, we will map policies between the two based in the mean patent length.

In highly sequential industries, a greater fraction of the benefits from a particular innovation are internalized by the original inventor, in the form of payments for patent sales to subsequent innovators. Additionally, these industries also feature a greater concentration of patent portfolios, leading to larger monopoly distortions. The fundamental trade-off of patent policy, as articulated by Arrow (1962), is between increasing the incentives to innovate so as to more closely align private and public returns to innovation and the deleterious effects of granting monopolies in production. Looked at through this lens, both of the above mentioned features of sequential industries lead us to expect that an optimal policy will include lower patent protection in more sequential industries. The first implies less of a need for the realignment of incentives, while the second implies that the inherent costs associated with granting patents are more severe in these industries.

Motivated by these considerations, I study both policies that are constant across industries and those that vary linearly with industry sequentiality. In terms of implementation, though sequentiality cannot be directly observed, one can infer its value for each industry using the equilibrium relationship between sequentiality and the probability of patent transfer (or any other observable that varies monotonically with sequentiality).

The optimal constant policy calls for a patent expiry rate of 8.6% annually, implying a mean patent length of 11.6 years. This causes the growth rate to fall to 2.94%. However, since the fraction of production labor rises to 93.4% and monopoly distortions fall to 6.0%, there is a net welfare increase of 0.9%. The optimal linear patent length decreases sharply with sequentiality. For industries with extremely low sequentiality, the policy calls for an infinite patent. The patent policy eventually reaches a minimal mean length of 6 years in the most sequential industries. The full path of the patent expiry rate is depicted in Figure 13. It is interesting to note that the optimal patent length in the median industry (sequentiality 17%) is about 18 years, almost exactly what current law prescribes. The resulting growth rate is 3.25% and the share of production labor is 91.9%, which are quite similar to the equilibrium values. However, the monopoly distortions fall noticeably, having been curbed in the most offending industries, to

FIGURE 13: OPTIMAL LINEAR PATENT POLICY
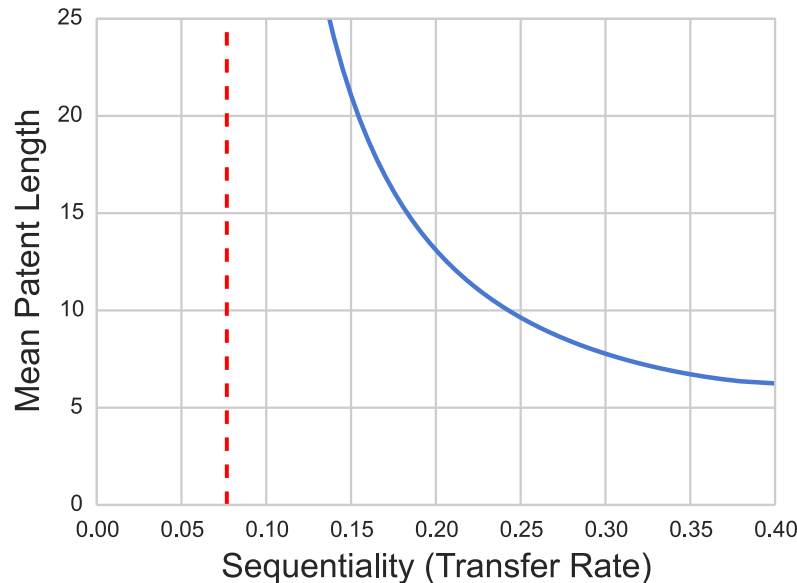


6.7%, resulting in a net welfare increase of 1.7%.

# 6   Conclusion

In this paper, I propose a notion of sequentiality in innovation and argue that it is an important determinant of a firm's incentives to innovate and of firm dynamics. Specifically, though externally oriented innovation has generally been assumed to result in creative destruction (or new products), I emphasize the notion that patent protection applies not just to contemporaneous imitators but to future cumulative innovators. The result is that innovating firms must in some cases come to an agreement with incumbent firms regarding ownership of the underlying portfolio. This ultimately has strong effects upon firms overall incentive to innovate. Not only that, these patent sales allows us to use data on the transfer of patent ownership as a window into the nature of the innovation process.

When looking at the cross-industry trends in patenting, a number of notable trends regarding transfer rates, expiry rates, profitability, and firm dynamics are apparent. Additionally, patent transfers flow disproportionately towards smaller and younger firms. To capture these trends, I take the basic model of sequential innovation described above and introduce

heterogeneity both across industries and across firms within industry. The resulting estimated model is able to match these trends qualitatively and account for a large fraction of the cross-industry variation.

The quantitative estimates point to an underallocation in the quantity of labor devoted to research. This result is not surprising given the existing theoretical and empirical literature. However, even fixing the quantity of labor devoted to research, there is a misallocation of research labor towards those industries with the highest sequentiality. I find that monopoly distortions are quantitatively important in this setting. In highly sequential industries, firms accumulate large patent portfolios, allowing them acquire a substantial technological lead over their nearest legal competitor. To remedy these misallocations in both research and production, I introduce both constant and industry dependent patent policies and evaluate their welfare effects.

I find that a patent policy featuring weaker patent protection in more sequential industries can generate large welfare gains over both current policy and the optimal uniform patent policy. The implications of this finding are not out of line with certain existing proposals by policymakers. In particular, there have been numerous calls by interest groups to either eliminate or severely restrict patenting of software, an industry which I find to be highly sequential. Other authors, such as Boldrin and Levine (2008), who advocate the elimination of patents frequently cite the software industry as an example of patenting gone wrong. Similarly, those concerned about "patent trolls" cite software as an industry which has been hit particularly hard by costs associated with intellectual property litigation.[9] Though I do not address these costs directly, reducing patent protection in these industries would limit the potential damage that patent trolls could cause.

Though this paper takes a very detailed approach to modeling patenting dynamics and the incentives to innovate, this setting is undoubtedly extremely complex, featuring a large quantity of heterogeneity both at the firm and industry level. There are many potential avenues of research that remain to explored. For instance, the notion of sequentiality may be related to patent breadth, meaning there are patent policy levers in addition to length that could be explored in this context. Additionally, not all innovation is necessarily protected with patenting. Incorporating an endogenous decision to patent (as opposed to using secrecy, for instance) could have interesting implications, though observability is naturally an issue.

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[9]http://www.schumer.senate.gov/record.cfm?id=341612&

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# Appendices

# A   Data Construction

**Name Matching**   The following procedure is used to match firm entities by name from both the US patent data and Compustat balance sheet information

1. Remove non-corporate entities

2. Drop corporate name identifiers and common English words

3. Group and standardizing suspected acronyms

4. Construct a similarity score basic on token and positional information for each pair of names

5.  Group names by a given cutoff similarity score.

   As noted by Hall, Jaffe, and Trajtenberg (2001), weighting tokens by their frequency of appearance would enhance match. The fact that certain uncommon words (such as "Samsung") appear in so many patents may skew this process, so an iterative is needed.

# B   Two Type Model

I now introduce heterogeneity amongst firms in the form of persistent types in order to explain trends regarding innovation, firm growth, and patent transfers between and amongst large/small and young/old firms. Much of the previous derivations carry through here. The production environment, in particular, is identical.

   One major difference that arises when introducing firm-level heterogeneity is that the indeterminacy in the direction of transfer is partially broken. When a high-type and low-type firm have a sequential interaction (in either direction), the patents are ultimately operated entirely by the high-type firm. However, it is still the case that when two firms of common type interact, the direction must be chosen arbitrarily. Moving to a model with a continuum of firm types would entirely eliminate this indeterminacy, but at the cost of tractability.

**Equilibrium**   There are two types of firms: high-intensity and low-intensity innovators, herein referred to as high and low type, that are differentiated by their R&D cost functions. High type firms transition to non-adopting firms at flow rate $\nu$. Being a low type firm is an absorbing state. Denote a generic firm type with $i \in \{H, L\}$. There is a type-specific innovation cost function $c^i(\cdot)$ for external innovation. The internal innovation technology $d(\cdot)$ is the same across types. An entrant firm starts as type $H$ with probability $\zeta^H$ and type $L$ with probability $\zeta^L = 1 - \zeta^H$.

Successful innovation yields a present value $\bar{V}^i$. The per-product value of a firm is then

$$
\begin{aligned}
\delta_F V^H(\lambda) - \dot{V}^H(\lambda) = {} & \tilde{\pi}(\lambda) + \Omega_x^H + \Omega_z^H \\
& + \alpha\tau p(\mathbb{E}V^H(\beta\lambda) - V^H(\lambda)) + (1-\alpha)\tau(0 - V^H(\lambda)) \\
& + b(V_0^H - V^H(\lambda)) + \nu(V^L(\lambda) - V^H(\lambda)) \\
\delta_F V^L(\lambda) - \dot{V}^L(\lambda) = {} & \tilde{\pi}(\lambda) + \Omega_x^L + \Omega_z^L \\
& + \alpha\tau^L p(\mathbb{E}V^L(\beta\lambda) - V^L(\lambda)) + \alpha\tau^H p(\mathbb{E}V^H(\beta\lambda) - V^L(\lambda)) \\
& + (1-\alpha)\tau(0 - V^L(\lambda)) + b(V_0^L - V^L(\lambda))
\end{aligned}
$$

where

$$
\begin{aligned}
(\delta_F + \tau)V_0^H - \dot{V}_0^H &= \Omega_x^H + \Omega_0^H + \nu(V_0^L - V_0^H) \\
(\delta_F + \tau)V_0^L - \dot{V}_0^L &= \Omega_x^L + \Omega_0^L
\end{aligned}
$$

and

$$
\begin{aligned}
\Omega_x^i &= \max_{x^i}\left\{-\tilde{w}c^i(x^i) + x^i\bar{V}^i\right\} \\
\Omega_z^i(\lambda) &= \max_{z^i}\left\{-\tilde{w}\lambda^{-1}d(z^i) + z^i(\mathbb{E}[V^i(\beta\lambda) - V^i(\lambda)))\right\} \\
\Omega_0^i &= \max_{z_0^i}\left\{-\tilde{w}d(z_0^i) + z_0^i(\mathbb{E}[V^i(\beta\lambda) - V_0^i))\right\}
\end{aligned}
$$

As before, posit a linearly separable form

$$
\begin{aligned}
V^H(\lambda) &= A^H - B^H\lambda^{-1} \\
V^L(\lambda) &= A^L - B^L\lambda^{-1}
\end{aligned}
$$

We then find for the high type

$$
\begin{aligned}
(\delta_F + b + (1-\alpha)\tau)A^H - \dot{A}^H &= 1 + \Omega_x^H + bV_0^H + \nu\left(A^L - A^H\right) \\
(\delta_F + b + (1-\alpha)\tau)B^H - \dot{B}^H &= 1 - \Omega_z^H - \alpha\tau p/(1+\kappa^{-1})B^H + \nu(B^L - B^H)
\end{aligned}
$$

and for the low type

$$
\begin{aligned}
(\delta_F + b + (1-\alpha)\tau)A^L - \dot{A}^L &= 1 + \Omega_x^L + bV_0^L + \alpha\tau^H p(A^H - A^L) \\
(\delta_F + b + (1-\alpha)\tau)B^L - \dot{B}^L &= 1 - \Omega_z^L - \alpha\tau^L p/(1+\kappa^{-1})B^L - \alpha\tau^H p(B^L - B^H/\lambda)
\end{aligned}
$$

The option values of innovation be simplified to

$$\Omega_x^i = \max_{x^i} \left\{ -\tilde{w} c^i(x^i) + x^i \bar{V}^i \right\}$$

$$\Omega_z^i = \max_{z^i} \left\{ -\tilde{w} d(z^i) + z^i B^i/(1 + \kappa^{-1}) \right\}$$

$$\Omega_0^i = \max_{z^i} \left\{ -\tilde{w} d(z^i) + z^i (A^i - B^i/(1 + \kappa) - V_0^i) \right\}$$

Now it can be verified that $B^H = B^L \equiv B$. As such we will also have $\Omega_z^H = \Omega_z^L \equiv \Omega_z$ and $z^H = z^H \equiv z$. As high type firms have a superior innovation technology, they will assume production in the case of sequential innovation. Between firms of a common type, it is ambiguous. The expected gain from innovation is given by

$$\bar{V}^H = \left[ (1-\alpha) + \alpha\mu_0 \right] \mathbb{E} V^H(\beta)$$
$$+ \alpha(1-p) \left[ \int_1^\infty (\mathbb{E} V^H(\beta\lambda) - V^H(\lambda)) d\mu_+^H(\lambda) + \int_1^\infty (\mathbb{E} V^H(\beta\lambda) - V^L(\lambda)) d\mu_+^L(\lambda) \right]$$
$$\bar{V}^L = \left[ (1-\alpha) + \alpha\mu_0 \right] \mathbb{E} V^L(\beta)$$
$$+ \alpha(1-p) \left[ \int_1^\infty (\mathbb{E} V^H(\beta\lambda) - V^L(\lambda)) d\mu_+^H(\lambda) + \int_1^\infty (\mathbb{E} V^L(\beta\lambda) - V^L(\lambda)) d\mu_+^L(\lambda) \right]$$

These can then be simplified to

$$\bar{V}^H = \left[ (1-\alpha) + \alpha\mu_0 \right] (A^H - B^H/(1+\kappa)) + \alpha(1-p) \left[ (\mu_+\Gamma_+ B^H/(1+\kappa^{-1}) + \mu_+^L(A^H - A^L) \right]$$
$$\bar{V}^L = \left[ (1-\alpha) + \alpha\mu_0 \right] (A^L - B^H/(1+\kappa)) + \alpha(1-p) \left[ (\mu_+\Gamma_+ B^H/(1+\kappa^{-1}) + \mu_+^H(A^H - A^L) \right]$$

Let there be a mass $L$ of researchers. The labor market clearing condition is

$$1 = \frac{\Gamma}{\tilde{w}} + (\mu^H + e\zeta^H)c(x^H) + (\mu^L + e\zeta^L)c(x^L) + \mu_0^H d(z_0^H) + \mu_0^L d(z_0^L) + \mu_+\Gamma_+ d(z)$$

**Steady State**   Imposing stationarity of values and state space elements, the firm value functions simplify to

$$B = \frac{1}{\delta_F + b + (1-\alpha)\tau + (\alpha\tau p + z\left(\frac{\delta+\tau+b}{\delta+\tau}\right)(1-1/\eta))/(1+\kappa^{-1})}$$

and

$$A^H = \frac{1 + \Omega_x^H + b V_0^H + \nu A^L}{\delta_F + b + (1-\alpha)\tau + \nu} \qquad A^L = \frac{1 + \Omega_x^L + b V_0^L + \alpha\tau^H p A^H}{\delta_F + b + (1-\alpha)\tau + \alpha\tau^H p}$$

The first part of Theorem 1 regarding the value of $\Gamma_+$ still holds in the environment, however the technological lead distribution is no longer tractable. As before the overall inverse technological lead is given by $\Gamma = \mu_0 + \mu_+ \Gamma_+$.

The relevant product mass distributions have flow equations

$$\dot{\mu}_0^H = b\mu_+^H - (\tau + z_0^H + \nu)\mu_0^H$$
$$\dot{\mu}_0^L = b\mu_+^L + \nu\mu_0^H - (\tau + z_0^L)\mu_0^L$$
$$\dot{\mu}_+^H = \tau^H \mu_0 + z_0^H \mu_0^H + \tau^H \mu_+ - (1-\alpha)\tau^L \mu_+^H - \nu\mu_+^H - b\mu_+^H$$
$$\dot{\mu}_+^L = \tau^L \mu_0 + z_0^L \mu_0^L + (1-\alpha)\tau^L \mu_+^H - \tau^H \mu_+^L + \nu\mu_+^H - b\mu_+^L$$

The overall mass distributions by type are

$$\dot{\mu}^H = \tau^H (1-\mu^H) - \tau^L \mu_0^H - (1-\alpha)\tau^L \mu_+^H - \nu\mu^H$$
$$\dot{\mu}^L = \tau^L \mu_0^H - \tau^H \mu^L + (1-\alpha)\tau^L \mu_+^H + \nu\mu^H$$

A bit of algebra reveals that the mass of high type firms can be expressed as

$$\mu^H = \frac{\tau^H}{\tau^H + \tau^L \left[ \frac{b+(1-\alpha)(\tau+z_0^H+\nu)}{b+\tau+z_0^H+\nu} \right] + \nu}$$

with $\mu^L = 1 - \mu^H$. The fractions of products that are expired conditional on type are then

$$\mu_0^H = \left[ \frac{b}{b + \tau + z_0^H + \nu} \right] \mu^H$$
$$\mu_0^L = \left[ \frac{b + \nu\mu_0^H}{b + \tau + z_0^L} \right] \mu^L$$

The average inverse technological lead for patent protected product lines resolves to

$$\Gamma_+ = \frac{(1-\alpha)\tau + (\tau + z)\mu_0}{(1-\alpha)\tau + b + \kappa(\tau + z + b)}$$

These can be used to determine the labor market clearing condition.

For simulations, we also need the conditional step distributions. For high type

$$\dot{\mu}_0^H = b\mu^H - (b + \tau + z_0^H + \nu)\mu_0^H$$
$$\dot{\mu}_1^H = (1-\alpha)\tau^H + \alpha\tau^H \mu_0 + z_0^H \mu_0^H - (b + \tau + z + \nu)\mu_1^H$$
$$\dot{\mu}_n^H = \alpha\tau^H \mu_{n-1} + (\alpha\tau^L + z)\mu_{n-1}^H - (b + \tau + z + \nu)\mu_n^H$$

and for low type

$$\dot{\mu}_0^L = \nu\mu_0^H + b\mu^L - (b + \tau + z_0^L)\mu_0^L$$
$$\dot{\mu}_1^L = \nu\mu_1^H + (1 - \alpha)\tau^L + \tau^L\mu_0 + z_0^L\mu_0^L - (b + \tau + z)\mu_1^L$$
$$\dot{\mu}_n^L = \nu\mu_n^H + (\alpha\tau^L + z)\mu_{n-1}^L - (b + \tau + z)\mu_n^L$$

These can be solved for iteratively with foreknowledge of $\mu^H$ and $\mu^L$.

# C   Algorithm

**Equilibrium**   The basic models without industry heterogeneity can be solved in a straight-forward fashion by setting up systems of equations consisting of first order conditions and the labor market clearing condition. These will depend on the aggregate innovation rates, so as to allow for the direct computation of product distributions, and the wage rate.

Moving to a setting with industry heterogeneity, the algorithm can be split into two levels. First, a wage rate is proposed, then each industry equilibrium is solved individually as in the basic model. Finally, these solution vectors are aggregated to evaluate the overall labor market clearing condition and the aggregate growth rate. Solving this system for the wage rate and growth rate constitutes solving the equilibrium in its entirety. I use the Powell's hybrid method[10] described in Powell (1970) to solve equations at both the industry level and overall equilibrium level. For more information on solving nonlinear systems, see Zangwill and Garcia (1981).

**Simulations**   Simulating firms in an efficient manner involves a small amount of further derivation. In particular, when a firm undertakes a successful innovation, with probability $\alpha$ it must purchase rights to existing technology from the incumbent, assuming said incumbent's product has patent protection. In this case, if the innovating firm assumes production (i.e., with probability $q$ is the untyped case), the step size of the product they receive can be drawn from the steady state distribution. In addition there will be one further innovation on top of that.

Interestingly, since the realization of the step size value for a given patent does not affect future patenting dynamics, these two factors will be independent in steady state as well. Assuming step increments are Pareto distributed, the distribution of the technological lead for

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[10]The exact code used is from the MINPACK library through the Python wrapper in SciPy.

a product line with $n$ patents will be

$$\log(\lambda) \sim \text{Erlang}(n, 1/\kappa)$$

as the log of a Pareto random variable is exponentially distributed and the sum of i.i.d. exponentials random variables is Erlang distributed. Note that the Erlang distribution is simply the Gamma distribution with an integer curvature parameter.

**Social Planner's Problem**    The social planner's problem in the model with industry heterogeneity can be simplified in a manner similar to that of cost minimization techniques when dealing with multi-product firm problems. In the general setting, a social planner must choose vector $\vec{x}$ of consisting of aggregate innovation rates for each industry, i.e., $\vec{x} = (\tau_1^H, \tau_1^L, z_1, \ldots, \tau_M^H, \tau_M^L, z_M)$. Use the notation $x_{i1} = \tau_i^H$, $x_{i2} = \tau_i^L$, and $x_{i3} = z_i$. Define the following maximization problem for each industry $i$

$$\Delta_i(\lambda_L, \lambda_g) = \max_{\vec{x}_i} \left\{ -\Delta_i(\vec{x}_i) - \lambda_L L_R(\vec{x}_i) + \lambda_g g(\vec{x}_i) \right\}$$

where $\lambda_L$ and $\lambda_g$ represent the aggregate shadow values of labor and growth. Let the maximands of the above be denoted $\vec{x}_i(\lambda_L, \lambda_g)$. Finally, let $\vec{x}(\lambda_L, \lambda_g) = [\vec{x}_i(\lambda_L, \lambda_g)]_{i=1}^M$ and $\Delta(\lambda_L, \lambda_g) = \prod_{i=1}^M \Delta_i(\lambda_L, \lambda_g)$. Define the aggregate maximization

$$S = \max_{\lambda_L, \lambda_g} \left\{ \left[ \delta + (\sigma - 1) g(\vec{x}(\lambda_L, \lambda_g)) \right] \left( \frac{1 - L_R(\vec{x}(\lambda_L, \lambda_g))}{\Delta(\lambda_L, \lambda_g)} \right)^{\sigma - 1} \right\}$$

It can be shown that any maximizer of the above problem is socially optimal. The above formulation has the advantage of having computational complexity that scales linearly with the number of industries, rather than quadratically.

# D    Proofs

**Proof of proposition 2.**    It is now necessary to specify a form for the R&D cost function. As is common in the existing literature, I use a constant elasticity function with exponent $\eta$

$$c(x) = c x^\eta$$

Using the first order condition for innovation intensity, we can find an expression relating the option value of innovation and the expected return from successful innovation

$$\Omega = (1 - 1/\eta) x \bar{V} = \left( \frac{1 - 1/\eta}{1 + e} \right) \tau \bar{V}$$

Then we can construct an equation characterizing the relationship between $\bar{V}$ and $\tau$

$$\bar{V} = \frac{q_0 \left[ \left( \frac{1}{\delta + b + (1-\alpha)\tau} \right) - \left( \frac{1}{1+\kappa} \right) B \right] + \alpha(1-p)(\Gamma - \mu_0) \left( \frac{\kappa}{1+\kappa} \right) B}{1 - q_0 \left( \frac{1 - 1/\eta}{1+e} \right) \left( \frac{\tau}{\delta + \tau} \right) \left( \frac{\delta + b + \tau}{\delta + b + (1-\alpha)\tau} \right)}$$

where $q_0 = (1-\alpha) + \alpha \mu_0$ is the probability of not having to purchase patent rights from the existing incumbent upon a successful innovation. This expression can be shown to be well-defined and positive for any $\tau \geqslant 0$. Call this function $\bar{V}(\tau)$. The first order condition for innovation be rearranged to

$$L_R = c(1+e)^{1-\eta}\tau^\eta = \frac{\tau \bar{V}}{\eta \Gamma + \tau \bar{V}}$$

Notice the value on the left is the share of resources devoted to research and the value on the right is strictly less than one, so we are guaranteed to find such a $\tau$ satisfying the above equation that results in a mixture of production and research labor. Showing uniqueness, which can be assured with the concavity of $\bar{V}(\cdot)$, is a more difficult matter. However, this can be easily verified for particular sets of parameters.

**Proof of proposition 3.**    Using the expression for $\log(\Delta)$ in (8), we can derive

$$\frac{\partial \log(\Delta)}{\partial \tau} = \left( \frac{1}{b(1+m) + \tau + z_0} \right) - \left( \frac{\tau + z_0}{b(1+m) + \tau + z_0} \right) \left( \frac{1}{1+m} \right) \frac{\partial m}{\partial \tau}$$
$$- \frac{1}{b + \tau + z_0} + \frac{mb}{(b + \tau + z_0)^2} + \left( \frac{\tau + z_0}{b + \tau + z_0} \right) \frac{\partial m}{\partial \tau}$$

Equation (7) implies that $m$ increases with both $\tau$ and $\alpha$. Further, we can derive

$$\frac{\partial m}{\partial \tau} = \kappa \left[ \frac{\alpha b - (1-\alpha)z}{(b + (1-\alpha)\tau)^2} \right]$$

which implies

$$\frac{\partial}{\partial \alpha} \left[ \frac{\partial m}{\partial \tau} \right] > 0$$

The above expression simplifies to

$$\frac{\partial \log(\Delta)}{\partial \tau} = \frac{m}{(b + \tau + z_0)^2} \left( \frac{bm}{bm + b + \tau + z_0} \right.$$
$$\left. + \frac{\partial m}{\partial \tau} \left( \frac{\tau + z_0}{b + \tau + z_0} - \left( \frac{1}{1+m} \right) \left( \frac{\tau + z_0}{b(1+m) + \tau + z_0} \right) \right) \right)$$

which can be seen to be increasing in $\alpha$. As $\tau$ and $z_0$ enter into the above expression in the exact same manner, the same logic applies for $z_0$ as well. However, $z_0$ does not affect $m$, the resulting expression is simply

$$\frac{\partial \log(\Delta)}{\partial z_0} = \frac{m}{(b + \tau + z_0)^2} \left( \frac{bm}{bm + b + \tau + z_0} \right)$$

Conversely, $z$ affects $m$ positively but does not change the composition between expired and unexpired product lines, meaning we find

$$\frac{\partial \log(\Delta)}{\partial z} = \frac{\partial m}{\partial z} \left( \frac{\tau + z_0}{b + \tau + z_0} - \left( \frac{1}{1 + m} \right) \left( \frac{\tau + z_0}{b(1 + m) + \tau + z_0} \right) \right)$$

Varying $z$ and $z_0$ simultaneously would yield an expression equivalent to that for $\tau$, where both terms are present.