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Comput. Methods Appl. Mech. Engrg. 190 (2001) 3669–3699

Computer methods in applied mechanics and engineering

www.elsevier.com/locate/cma

Integration of elastic multibody systems by invariant conserving/dissipating algorithms. I. Formulation

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Received 27 July 1999

Abstract

This work presents a novel methodology for the dynamic analysis of general non-linear flexible multibody systems. In Part I we develop the 6-D compact representation of motion for those body models which motion may be described by a displacement field plus an independent rotation field. This approach explores the fundamental properties of rigid body motion, and in particular the coupled nature of linear and angular quantities in both kinematics and dynamics, inspiring a novel parameterization technique based on the exponential map. Using the proposed approach, we derive the governing equations for the case of multibody systems composed by rigid bodies and geometrically non-linear beams connected by holonomic constraints. These equations provide the starting point for the derivation of a class of numerical algorithms characterized by non-conventional conservation properties. In Part II of this work we develop the algorithms and illustrate their properties with the aid of some numerical applications.

1. Introduction

In the analysis of flexible multibody system dynamics, one must approach the solution of *stiff*, non-linear differential-algebraic equations (DAEs). The *stiffness* arises from the assembling of rigid and deformable bodies via *kinematic constraints*, which may be seen as limiting cases of elastic joints of increasing spring constant, and by the fact that flexible members introduce high-frequency components in the response. Both sources of stiffness are well known and various methodologies have been presented to deal with the associated numerical difficulties. Non-linearities are related to the large displacements and rotations involved in the motion (*geometrical non-linearity*). A further source of non-linear effects may be the inclusion of non-linear constitutive equations for the deformable components of the system (*material non-linearity*). In the present work, we shall limit our developments to the linear elastic case. This essentially means that we consider *small linear and angular strains*, while we allow the linear and angular displacement to be arbitrarily large.

In our view, the ability of accurately and efficiently describing a rigid motion, regardless of that fact that the body is rigid or deformable, should represent a crucial feature of any methodology devoted to multibody dynamics numerical analysis. In fact, rigid motion is endowed by properties that can be suitably exploited in order to make the numerical algorithms on the one hand more efficient and robust, and on the other hand capable of a higher degree of consistency. In other words, by a convenient formulation of the equations of rigid motion, and consequently of the equations of motion for a rigid body, one may

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algorithmically exploit their rich structure in order to obtain numerical schemes that *naturally* possess non-conventional preservation properties. We remark that, by this procedure, one can extend these algorithms to the equations governing the dynamics of a large class of deformable bodies of primary interest in multibody systems analysis, which are those described by an independent orientation field in addition to the displacement field. These body models encompass the Cosserat continua, such as geometrically non-linear beams and shells, and polar 3-D continua.

The methodology presented here is based on a set of notions which are more or less familiar to any analyst involved in the classical and structural mechanics fields. However, while the bulk of the literature on the subject places its attention on methods for solving the equations of motion, our approach is somehow inverted. We derive the equations in a form which, by its very nature, inspires the way to numerically solve them through a discretization process that guarantees the preservation of a number of *qualitative features* of the solution. Furthermore, some of these features may be usefully connected to the robustness of the algorithms, and eventually they lead to rigorous non-linear unconditional stability properties.

To summarize the outcome of this methodology, we point out that, starting from the *base pole* equations of motion we obtain a family of numerical methods that, among other properties, guarantee:

- the rigorous *integration on the configuration manifold*, without using a posteriori projection methods, additional constraints or other add-on techniques;
- the discrete preservation of the *rule of reduction of torques*, which implies the correct coupling between linear and angular quantities at the discretized level; and
- the exact discrete *conservation of linear and angular momenta* in the case of vanishing loads.

From this class of schemes, we develop two implicit low-order algorithms, namely the *energy-preserving* (EP) and the *energy-decaying* (ED) schemes, which display

- *non-linear unconditional stability*, based on the exact discrete conservation/dissipation of the total mechanical energy in the case of vanishing loads,

and, for the ED scheme,

- *selective dissipation* of the high-frequency modes with asymptotic annihilation.

These results are carried out in the two parts of this work. In the present one we develop the formulation for general flexible multibody systems composed of rigid and deformable members (the case of beams is addressed) linked by holonomic constraints. In the second part we develop the formulation of modified Runge–Kutta (RK) methods for multibody dynamics, and we investigate the EP and ED methods in some detail. The main features of the methodology are then assessed through numerical applications.

The theoretical formulation is described in detail in [9], while these ideas were initially developed by the authors in [11,12] for geometrically non-linear beams. Also related to the subject are [2,3,7,8,10], and the works of Simo and collaborators [13–15] and Bauchau and collaborators [4–6].

1.1. Overview of Part I

As previously said, Part I of this work is devoted to the derivation of the governing equations for general flexible multibody systems. Both rigid and flexible bodies fall in a general class of body models characterized by linear and rotational tensorial fields describing kinematic (displacements, velocities) and co-kinematic (forces, momenta) quantities. The algebraic structure of rigid motion allows to cast these equations in a *6-D compact representation*, in view of both an improved synthesis in the formulæ and an increased ability in capturing the many remarkable underlying properties. In fact, the group of rigid displacements is a *Lie group*, defined as the semi-direct product of the space of linear displacements and the group of rotations. Both components are Lie groups, the first representing a linear, and therefore almost trivial, example, while the second implying features such as non-linearity, non-commutativity, etc. While many are familiar with the special structure of rotations, and use this to derive computational techniques, it appears that not many authors have done the same with respect to the structure of rigid motions. As a matter of fact, many of the well-known operations and concepts connected to rotations may be usefully extended to rigid motions.

We first introduce the concept of ‘‘frame configuration tensor’’, a quantity describing both the position and orientation of a ‘‘frame’’, which is defined by the set formed by a material point in space with its associated triad of unit vectors. The ‘‘frame generalized velocity’’ is then defined as the quantity describing both linear and angular velocities of the frame itself. This leads to a special notion of linear velocity, the "base pole velocity" which is directly related to the derivative of the configuration tensor. The operation of "generalized cross product" extends the action of the ordinary cross product in 3-D space to the 6-D kinematic space, and sheds light on its particular structure as a *Lie algebra*. This consideration leads to the formulation of a novel parameterization based on the *exponential map of motion*. This technique allows for a linearly-angularly coupled representation of the frame configuration through a set of six parameters, namely the scalar components of a "generalized screw vector". All this formally behaves exactly as in the case of the exponential map of rotation and the rotation vector, due to the strict analogy existing between rotations and complete rigid motions as elements of Lie groups.

We remark that, even if the derivation could have followed a quite abstract differential-geometric approach, all the notions involved are here introduced and justified by standard linear algebra procedures, and do not require any knowledge of the peculiar methods of Lie group analysis.

"Base pole balance equations", or dynamic balance equations referred to the base pole, are derived for rigid bodies and geometrically non-linear beams, as a simple but meaningful example of a deformable Cosserat continuum. Suitable measures of strain are presented, and the theory briefly touches classical issues such as differential balance equations and energy conservation. Similar results can be derived for the geometrically non-linear shell. These equations are ready to be combined with the constraint equations to yield the DAE setting that governs the dynamics of general flexible multibody systems with arbitrary topology. The algorithms described in Part II shall take the base pole balance equations as starting points for the discretization process.

Constraints are accounted for by algebraic relations between suitably chosen material frames in the constrained bodies. The description of relative motion in the 6-D compact representation highlights the meaning of the displacement tensor and of the relative generalized velocity as the fundamental quantities that are subject to the limitations imposed by the joints. The balance equations are modified with the inclusion of constraint reaction forces via Lagrange multipliers. Many holonomic constraints of common use, such as the spherical, prismatic, revolute, and other joints are treated in detail. The expressions for their configuration and velocity-level equations represent the basis for the developments of Part II.

1.2. Summary of Part II

Part II is devoted to the development of a family of algorithms that discretize the governing equations of multibody systems as derived in Part I, and to the assessment of their properties and performances through a gallery of numerical applications.

The algorithmic framework is concerned with the solution of ODEs and DAEs in time. A most general finite element method (FEM) or finite volume method (FVM) may be employed to accomplish the spatial-discretization process. In fact, the results we are interested in, such as accuracy, stability and invariant preservation, do not depend on the particular spatial discretization adopted. The algorithms heavily rely on the adoption of an incremental parameterization technique, the exponential parameterization introduced in Part I.

The outcome is the definition of a class of numerical schemes that is as large as that of the RK methods, being a modification of these algorithms that by design guarantees the "geometrical integration" of the equations. This means that the numerical solution stays on the non-linear frame configuration manifold. Moreover, in the unconstrained case, linear and angular momenta are exactly preserved whenever the loads vanish.

A slight modification of the second-order Lobatto IIIA scheme (i.e., the trapezoidal rule) yields the EP method. This second-order non-linearly unconditionally stable scheme is detailed in the prototypal rigid body case, in the geometrically non-linear beam case, and eventually in the general case of flexible multibody systems. The final result is obtained by a special discretization of the constraints that guarantees the vanishing of the work of the constraint reaction forces. The same path is followed for the ED method. In this case, while second-order accuracy and non-linear unconditional stability are rigorously retained, one matches the computational drawback of an additional internal stage with the advantage of a *built-in low-pass algorithmic filter* that enables to cope with highly stiff applications.

The performances of these methods, and particularly of the ED scheme, are illustrated by some numerical examples. These simulations confirm the properties predicted in the analysis.

2. Theoretical framework

In this section we derive the basic relationships governing the kinematics of objects whose motions can be characterized by both linear and rotational tensorial fields. Moreover, we establish the fundamentals of the 6-D representation of motion. A central concept for these developments is the "base pole reduction" of 3-D and 6-D vectors, which leads to particularly simple expressions for the kinematic evolution equations, as well as for the dynamic balance equations. A more detailed discussion on the subject is given in [9].

2.1. Basic kinematic relations

Frame configuration. Consider a *frame*, defined as the set composed by a point, called *pole*, in the 3-D Euclidean manifold $\mathscr{E}^3$, and an orthonormal triad of vectors in $\mathbb{E}^3$, the translation space of $\mathscr{E}^3$, with origin in the pole. For our purposes it will suffice to consider a pair of such frames: the *base frame* $(\mathbf{o}, \{\mathbf{i}_k\}_{k=1,2,3})$ and the *moving frame* $(\mathbf{x}, \{\mathbf{e}_k\}_{k=1,2,3})$, the reasons for these names becoming apparent in the following. The position and orientation of the moving frame with respect to the base frame can be measured through the *frame position vector* $\mathbf{u} \in \mathbb{E}^3$ and the *frame orientation tensor* $\boldsymbol{\alpha} \in \mathrm{SO}(\mathbb{E}^3)$, defined as

$$\mathbf{u} := \mathbf{x} - \mathbf{o}, \tag{1}$$

$$\boldsymbol{\alpha} := \mathbf{e}_k \otimes \mathbf{i}_k \tag{2}$$

(summation over repeated indices, regardless to their position, is implied throughout this work). The above definition implies that the orientation tensor transforms the base triad into the moving triad as follows:

$$\mathbf{e}_k = \boldsymbol{\alpha}\,\mathbf{i}_k, \qquad k = 1, 2, 3. \tag{3}$$

The *rotation group* $\mathrm{SO}(\mathbb{E}^3) \subset \mathrm{Lin}(\mathbb{E}^3)$, the group of all "Special Orthogonal" transformations on $\mathbb{E}^3$, is a Lie group of dimension 3. Its elements are endowed with the well-known properties of unimodularity, $\det(\boldsymbol{\alpha}) = +1$, and orthogonality, $\boldsymbol{\alpha}^{-1} = \boldsymbol{\alpha}^{\mathrm{T}}$.

The pair $(\mathbf{u}, \boldsymbol{\alpha}) \in \mathbb{E}^3 \times \mathrm{SO}(\mathbb{E}^3)$ completely represents the *configuration* of the moving frame with respect to the base frame, accounting for both linear and angular informations: the relative displacement between the poles $\mathbf{o}$ and $\mathbf{x}$ and the relative rotation between the triads $\{\mathbf{i}_k\}_{k=1,2,3}$ and $\{\mathbf{e}_k\}_{k=1,2,3}$. Therefore, the set $\mathbb{E}^3 \times \mathrm{SO}(\mathbb{E}^3)$ is the *configuration space* of the moving frame with respect to the base frame.

One may also look at $(\mathbf{u}, \boldsymbol{\alpha})$ as a representation of the *rigid transformation* $\boldsymbol{C}$ from the base frame to the moving frame (a change of *framing*). This is understood here as the affine transformation that brings any placement $\bullet \in \mathscr{E}^3$ into the new placement $\boldsymbol{C}(\bullet) := \mathbf{o} + \mathbf{u} + \boldsymbol{\alpha}(\bullet - \mathbf{o}) \in \mathscr{E}^3$, or

$$\boldsymbol{C}(\bullet) = \mathbf{x} + \boldsymbol{\alpha}(\bullet - \mathbf{o}). \tag{4}$$

This map transforms the space $\mathscr{E}^3$ as seen by an observer connected with the base frame into that seen by an observer connected with the moving frame. Eq. (4) clearly shows that such a transformation consists of a rotation by $\boldsymbol{\alpha}$ followed by a translation by $\mathbf{u}$. The order in these operations is crucial, since all possible rigid transformations form a non-commutative group. This set is termed the *group of rigid displacements* of $\mathbb{E}^3$, or sometimes the *Euclidean group*, and denoted by $\mathrm{SE}(\mathbb{E}^3)$. This is isomorphic to the configuration space $\mathbb{E}^3 \times \mathrm{SO}(\mathbb{E}^3)$. It can be shown that both the sets $\mathrm{SE}(\mathbb{E}^3)$ and $\mathbb{E}^3 \times \mathrm{SO}(\mathbb{E}^3)$ are Lie groups. This particular algebraic structure allows to extend many remarkable kinematic properties of rotations to rigid displacements, thus leading to a convenient, unified treatment for both linear and angular quantities.

Frame velocity. Consider the pair $(\mathbf{u}, \boldsymbol{\alpha})$ as a smooth function of a single parameter $t \in [0, T]$. In other words, consider a regular "motion" of the moving frame with respect to the base frame, *which remains at rest*. The point $\mathbf{x}(t)$ describes a smooth curve in $\mathscr{E}^3$, and we refer to its tangent vector as the *local linear frame velocity* $\mathbf{v}_{\mathbf{x}} \in \mathbb{E}^3$,

$$\mathbf{v}_{\mathbf{x}} := \dot{\mathbf{u}}. \tag{5}$$

The superposed dot is used to indicate derivatives with respect to t. Tensor $\boldsymbol{\alpha}(t)$ represents a one-parameter family of special orthogonal transformations on $\mathbb{E}^3$, and its derivative with respect to the parameter t is given by

$$\dot{\boldsymbol{\alpha}} = \boldsymbol{\omega} \times \boldsymbol{\alpha}. \tag{6}$$

Vector $\boldsymbol{\omega} \in \mathbb{E}^3$ is the *angular frame velocity*. Note that, given definition (2), Eq. (6) is equivalent to the Poisson's formulæ:

$$\dot{\mathbf{e}}_k = \boldsymbol{\omega} \times \mathbf{e}_k, \qquad k = 1, 2, 3. \tag{7}$$

In the present work, the symbol $\times$ is used to denote both the ordinary cross product operation between 3-D vectors, and the standard isomorphism between vectors in $\mathbb{E}^3$ and skew-symmetric tensors in $\mathrm{Lin}(\mathbb{E}^3)$. The skew-symmetric tensors form a linear subspace of $\mathrm{Lin}(\mathbb{E}^3)$ denoted by $\mathrm{so}(\mathbb{E}^3)$. This means that $(\bullet\times) \in \mathrm{so}(\mathbb{E}^3)$ is the unique skew-symmetric tensor associated to vector $\bullet \in \mathbb{E}^3$. The inverse isomorphism is denoted by $\mathrm{axial}_\times(\bullet)$, so that the definition of the frame angular velocity $\boldsymbol{\omega}$ reads

$$\boldsymbol{\omega} := \mathrm{axial}_\times(\dot{\boldsymbol{\alpha}}\,\boldsymbol{\alpha}^{-1}). \tag{8}$$

The linear subspace $\mathrm{so}(\mathbb{E}^3)$ coincides with the Lie algebra of the Lie group $\mathrm{SO}(\mathbb{E}^3)$, as Eq. (6) shows when we look at it from a differential geometry standpoint. Since $\mathbb{E}^3$ and $\mathrm{so}(\mathbb{E}^3)$ are isomorphic, also $\mathbb{E}^3$ has the structure of a Lie algebra, and its *commutator* [1] is the ordinary cross product operation, or $[\bullet, \bigstar] := \bullet \times \bigstar, \forall \bigstar, \bullet \in \mathbb{E}^3$.

Eq. (6) may be written in "convected form" as

$$\dot{\boldsymbol{\alpha}} = \boldsymbol{\alpha}\,\overline{\boldsymbol{\omega}}\times, \tag{9}$$

the convected frame angular velocity $\overline{\boldsymbol{\omega}} := \boldsymbol{\alpha}^{-1}\boldsymbol{\omega}$ being defined as the convected image [2] of the frame angular velocity $\boldsymbol{\omega}$, or

$$\boldsymbol{\omega} = \boldsymbol{\alpha}\,\overline{\boldsymbol{\omega}}. \tag{10}$$

Clearly, an alternative definition for the convected angular velocity is given by

$$\overline{\boldsymbol{\omega}} = \mathrm{axial}_\times(\boldsymbol{\alpha}^{-1}\dot{\boldsymbol{\alpha}}). \tag{11}$$

The linear quantity corresponding to $\overline{\boldsymbol{\omega}}$ is the convected linear velocity $\overline{\mathbf{v}}_\mathbf{x}$, defined as the convected image of the frame local linear velocity $\mathbf{v}_\mathbf{x} := \boldsymbol{\alpha}^{-1}\mathbf{v}_\mathbf{x}$, or

$$\mathbf{v}_\mathbf{x} = \boldsymbol{\alpha}\,\overline{\mathbf{v}}_\mathbf{x}. \tag{12}$$

To complete the ingredients needed for the 6-D representation of motion we still need to take into account a different notion of linear velocity from those addressed by Eqs. (5) and (12).

Base pole frame velocity. Applying the time convective derivative [3] to the frame position vector $\mathbf{u}$, we obtain the *base pole linear frame velocity* $\mathbf{v}_\mathbf{o} \in \mathbb{E}^3$,

$$\mathbf{v}_\mathbf{o} := \mathring{\mathbf{u}}. \tag{13}$$

[1] The commutator $[\,,]$ of a Lie algebra is a bilinear operation that satisfies the two *Jacobi identities*: $[\bullet, \bigstar] + [\bigstar, \bullet] = \mathbf{0}$ (antisymmetry) and $[[\bullet, \bigstar], \clubsuit] + [[\bigstar, \clubsuit], \bullet] + [[\clubsuit, \bullet], \bigstar] = \mathbf{0}$. Note that the commutator for $\mathrm{so}(\mathbb{E}^3)$ is the operation defined by $[\clubsuit, \spadesuit] := \clubsuit\spadesuit - \spadesuit\clubsuit$, $\forall \clubsuit, \spadesuit \in \mathrm{so}(\mathbb{E}^3)$. Equation $(\bullet\times)\,(\bigstar\times) - (\bigstar\times)\,(\bullet\times) = (\bullet \times \bigstar)\times$ expresses the relation between the commutators for $\mathrm{so}(\mathbb{E}^3)$ and $\mathbb{E}^3$.

[2] Convected images are defined as $\overline{\bullet} := \boldsymbol{\alpha}^{-1}\bullet$ for a vector $\bullet \in \mathbb{E}^3$, and $\overline{\bullet} := \boldsymbol{\alpha}^{-1}\bullet\,\boldsymbol{\alpha}$ for a tensor $\bullet \in \mathrm{Lin}(\mathbb{E}^3)$. We make use of the superposed bar to indicate convected quantities, which have scalar components with respect to the base triad that coincide with those of the "original" quantities with respect to the moving triad, i.e., $\mathbf{e}_k \cdot \bullet = \mathbf{i}_k \cdot \overline{\bullet}$ for a vector $\bullet \in \mathbb{E}^3$, and $\mathbf{e}_k \cdot (\bullet\,\mathbf{e}_h) = \mathbf{i}_k \cdot (\overline{\bullet}\,\mathbf{i}_h)$ for a tensor $\bullet \in \mathrm{Lin}(\mathbb{E}^3)$.

[3] Convective derivatives with respect to t are defined as $\mathring{\bullet} := \boldsymbol{\alpha}\,\dot{\overline{\bullet}}$ in the case of a vector $\bullet \in \mathbb{E}^3$, and $\mathring{\bullet} := \boldsymbol{\alpha}\,\dot{\overline{\bullet}}\,\boldsymbol{\alpha}^{-1}$ in the case of a tensor $\bullet \in \mathrm{Lin}(\mathbb{E}^3)$. We make use of the superposed circle to indicate convective derivatives with respect to time. This operation can be seen as a three-step procedure:

1. pull-back of the vector/tensor with tensor $\boldsymbol{\alpha}$, obtaining its convected image;
2. ordinary derivative of the convected image with respect to t;
3. push-forward of the result with tensor $\boldsymbol{\alpha}$.

The convective derivative is clearly the derivative taken by an observer at rest in the moving frame.

Note that the notation for $\mathbf{v_o}$ does not obviously relate to the velocity of the base pole $\mathbf{o}$ (which is null at all times), but is inspired by the relation that holds between $\mathring{\mathbf{u}}$ and the local linear velocity $\dot{\mathbf{u}}$. In fact, from the definition of convected derivative we get

$$\mathring{\mathbf{u}} := \boldsymbol{\alpha}\frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{\alpha}^{-1}\mathbf{u}) \tag{14}$$

$$= \boldsymbol{\alpha}(\boldsymbol{\alpha}^{-1}\dot{\mathbf{u}} - \boldsymbol{\alpha}^{-1}\dot{\boldsymbol{\alpha}}\boldsymbol{\alpha}^{-1}\mathbf{u}) \tag{15}$$

$$= \dot{\mathbf{u}} - \boldsymbol{\omega}\times\mathbf{u}. \tag{16}$$

The last equation, rewritten as

$$\mathbf{v_o} = \mathbf{v_x} + (\mathbf{x}-\mathbf{o})\times\boldsymbol{\omega}, \tag{17}$$

allows to interpret vector $\mathbf{v_o}$ as the linear velocity of $\mathbf{x}$ reduced to the base pole $\mathbf{o}$.

Note that the preceding equation represents a *spatial field* of linear velocity $\mathbf{v} : \mathscr{E}^3 \to \mathbb{E}^3$, which is formally defined by

$$\mathbf{v}_\star := \dot{\boldsymbol{C}}\big(\boldsymbol{C}^{-1}(\star)\big) \tag{18}$$

$\forall\star\in\mathscr{E}^3$. The base pole linear velocity $\mathbf{v_o}$ is thus the velocity of the image under $\boldsymbol{C}$ of the point $\boldsymbol{C}^{-1}(\mathbf{o})$. In other words, it is the velocity of the point that, as a result of the time-varying rigid transformation $\boldsymbol{C}(t)$, happens to pass through the position $\mathbf{o}$ at the given time instant. This also explains why we termed the vector $\mathbf{v_x}$ the "local" linear velocity, a way to remind that it coincides with the linear velocity of the moving pole $\mathbf{x}$ "reduced" to the point $\mathbf{x}$ itself. As a result, the base pole linear velocity $\mathbf{v_o}$ can be interpreted as an "Eulerian" description of motion, rather than a "Lagrangian" description, related to the local linear velocity $\mathbf{v_x}$.

Note that the base pole linear velocity $\mathbf{v_o}$ represents a "global" measure of linear speed of the moving frame. In fact, the velocity of any point rigidly connected to the moving frame, when reduced to the base pole, is the same. In formulæ this is expressed by

$$\mathbf{v_o} = \frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{x}-\mathbf{o}) + (\mathbf{x}-\mathbf{o})\times\boldsymbol{\omega}, \tag{19}$$

$$= \frac{\mathrm{d}}{\mathrm{d}t}(\star-\mathbf{o}) + (\star-\mathbf{o})\times\boldsymbol{\omega} + \frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{x}-\star) + (\mathbf{x}-\star)\times\boldsymbol{\omega}, \tag{20}$$

$$= \mathbf{v}_\star + (\star-\mathbf{o})\times\boldsymbol{\omega}, \tag{21}$$

since, given any two points $\bullet,\star\in\mathscr{E}^3$ rigidly connected to the moving frame, we have $\mathrm{d}(\bullet-\star)/\mathrm{d}t = \boldsymbol{\omega}\times(\bullet-\star)$. Thus, as we have a *unique* angular speed measure $\boldsymbol{\omega}$ for the frame as a single kinematic object, we can resort to the base pole velocity as a *unique* linear speed measure. This has some interesting implications that shall be clarified in the following.

2.2. 6-D representation

Generalized velocity – I. In order to convey the complete information of velocity for a frame, we define the following 6-D vectors

$$\mathbf{w_o} = \begin{bmatrix}\mathbf{v_o}\\ \boldsymbol{\omega}\end{bmatrix}, \qquad \mathbf{w_x} := \begin{bmatrix}\mathbf{v_x}\\ \boldsymbol{\omega}\end{bmatrix}, \qquad \overline{\mathbf{w}}_\mathbf{x} := \begin{bmatrix}\overline{\mathbf{v}}_\mathbf{x}\\ \overline{\boldsymbol{\omega}}\end{bmatrix}, \tag{22}$$

termed the *base pole generalized frame velocity* $\mathbf{w_o}\in\mathbb{K}^6$, the *local generalized frame velocity* $\mathbf{w_x}\in\mathbb{K}^6$, and the *convected generalized frame velocity* $\overline{\mathbf{w}}_\mathbf{x}\in\mathbb{K}^6$, respectively. The linear space $\mathbb{K}^6$, termed the *kinematic space*, is given by $\mathbb{K}^6 := \mathbb{E}^3\times\mathbb{E}^3$. Elements in $\mathbb{K}^6$, or "generalized velocities", are sometimes referred to as "twists" (as in [1]). Even if these quantities are understood and formally treated as "column vectors", we shall denote them simply as ordered pairs, such as $\mathbf{w_o} = (\mathbf{v_o},\boldsymbol{\omega})$, for the sake of a lighter notation. The ordered 3-D pairs represent the linear and angular vector components, respectively denoted by the subscripts L and A. This means that, for example, $\mathbf{w}_{\mathbf{o}_\mathsf{L}} = \mathbf{v_o}$ and $\mathbf{w}_{\mathbf{o}_\mathsf{A}} = \boldsymbol{\omega}$. We shall use the standard matrix algebra in the operations concerning 6-D vectors, since it leads to meaningful, although redundant, matricial expressions for 6-D tensors.

Let us look now at the relations holding between these generalized velocities. Vectors $\mathbf{w_o}$ and $\mathbf{w_x}$ are related by the equation

$$\mathbf{w_o} = \mathscr{T}(\mathbf{u})\,\mathbf{w_x}, \tag{23}$$

where the *translation operator* $\mathscr{T} : \mathbb{E}^3 \to \mathrm{Lin}(\mathbb{K}^6)$ is defined as

$$\mathscr{T}(\bullet) := \begin{bmatrix} \mathbf{I}_3 & \bullet\times \\ \mathbf{O}_3 & \mathbf{I}_3 \end{bmatrix} \tag{24}$$

$\forall \bullet \in \mathbb{E}^3$. The symbols $\mathbf{I}_3$ and $\mathbf{O}_3$ denote the identity and null tensors on $\mathbb{E}^3$, respectively. The translation operator performs the reduction of the generalized velocity to different poles in space.

Vectors $\mathbf{w_x}$ and $\overline{\mathbf{w}}_\mathbf{x}$ are related by the equation

$$\mathbf{w_x} = \mathscr{A}(\boldsymbol{\alpha})\,\overline{\mathbf{w}}_\mathbf{x}, \tag{25}$$

where the *convection operator* $\mathscr{A} : \mathrm{Lin}(\mathbb{E}^3) \to \mathrm{Lin}(\mathbb{K}^6)$ is defined as

$$\mathscr{A}(\bullet) := \begin{bmatrix} \bullet & \mathbf{O}_3 \\ \mathbf{O}_3 & \bullet \end{bmatrix} \tag{26}$$

$\forall \bullet \in \mathrm{Lin}(\mathbb{E}^3)$. The convection operator performs the rotation of the generalized velocity to different triads in space.

Configuration tensor. By combining Eqs. (23) and (25) we are led to equation

$$\mathbf{w_o} = \mathbf{C}\,\overline{\mathbf{w}}_\mathbf{x}, \tag{27}$$

where the global (i.e., linear and angular) displacement from the base frame to the moving frame is given by the *frame configuration tensor* $\mathbf{C} \in \mathrm{SR}(\mathbb{K}^6)$. This quantity is defined by

$$\mathbf{C} := \mathscr{T}(\mathbf{u})\,\mathscr{A}(\boldsymbol{\alpha}), \tag{28}$$

and thus its matricial form reads

$$\mathbf{C} := \begin{bmatrix} \boldsymbol{\alpha} & \mathbf{u}\times\boldsymbol{\alpha} \\ \mathbf{O}_3 & \boldsymbol{\alpha} \end{bmatrix}. \tag{29}$$

We term *rigid displacement tensors*, or simply *displacement tensors*, all the 6-D tensors sharing the structural decomposition into the operators $\mathscr{T}$ and $\mathscr{A}$, highlighted in Eq. (28), and *displacement group* the set $\mathrm{SR}(\mathbb{K}^6) \subset \mathrm{Lin}(\mathbb{K}^6)$ of all displacement tensors. This is the group of all "Special Rigid" transformations on $\mathbb{K}^6$, that is isomorphic to the configuration space $\mathbb{E}^3 \times \mathrm{SO}(\mathbb{E}^3)$. This means that the configuration tensor $\mathbf{C}$ is a one-to-one representation of the configuration pair $(\mathbf{u}, \boldsymbol{\alpha})$, and also of the rigid transformation $\boldsymbol{C}$ that brings the base frame into the moving frame. Although other representations may be used (see for example [9]), we favor the present one, employing the 6-D configuration tensor, since all of the relevant properties of the orientation tensor and its derivatives may be extended in a straightforward manner to this quantity, as it is shown in the following.

The set $\mathrm{SR}(\mathbb{K}^6)$ is a Lie group of dimension 6 and implies the property of unimodularity, $\det(\mathbf{C}) = +1$. Tensor $\mathbf{C}$ includes in a single quantity the complete information of position and orientation of the moving frame with respect to the base frame. As a linear operator on the kinematic space $\mathbb{K}^6$, we look at the action of $\mathbf{C}$ as a "generalized convection" process, extending the "convection" process performed by tensor $\boldsymbol{\alpha}$. In fact, in the 6-D case we understand it as a transformation that not only involves the change of "basis" from the moving triad to the base triad, but also the change of "origin" from the moving pole to the base pole. In this way, we look at Eq. (27), where the configuration tensor relates the base pole generalized velocity to the convected image of the local generalized velocity, as the 6-D extension of Eq. (10), where the orientation tensor relates the angular velocity with its convected image.

Generalized velocity – II. Consider now the derivative of the configuration tensor $\mathbf{C}$ with respect to parameter t. It is easy to check by direct calculation that this quantity may be expressed in the following form:

$$\dot{\mathbf{C}} = \mathbf{w_o} \times\!\!\cdot\, \mathbf{C}. \tag{30}$$

The above equation clearly shows a formal structure that recalls that of Eq. (6), where the base pole generalized velocity $\mathbf{w_o}$ acts in place of the angular velocity $\boldsymbol{\omega}$, the ordinary cross product $\times$ being replaced by the *North-East cross product* $\times\!\!\cdot$. This is defined by the following matricial structure,

$$\bullet \times\!\!\cdot = \begin{bmatrix} \bullet_{\mathsf{A}} \times & \bullet_{\mathsf{L}} \times \\ \mathbf{O}_3 & \bullet_{\mathsf{A}} \times \end{bmatrix}, \tag{31}$$

where $\bullet = (\bullet_{\mathsf{L}}, \bullet_{\mathsf{A}}) \in \mathbb{K}^6$. As for the ordinary cross product, the symbol $\times\!\!\cdot$ is used here to indicate both an internal, bilinear operation between 6-D vectors, and an isomorphism between vectors in $\mathbb{K}^6$ and tensors in $\mathrm{Lin}(\mathbb{K}^6)$ belonging to a particular subset. These tensors, sharing the structure highlighted in Eq. (31) and simply termed *North-East cross product tensors*, form a linear subspace of $\mathrm{Lin}(\mathbb{K}^6)$ denoted $\mathrm{sr}(\mathbb{K}^6)$. This means that $(\bullet \times\!\!\cdot) \in \mathrm{sr}(\mathbb{K}^6)$ is the unique North-East cross product tensor associated to vector $\bullet \in \mathbb{K}^6$. The inverse isomorphism is denoted by $\mathrm{axial}_{\times\!\cdot}(\bullet)$, so that an alternative definition of the base pole generalized velocity $\mathbf{w_o}$ is given by

$$\mathbf{w_o} := \mathrm{axial}_{\times\!\cdot}(\dot{\mathbf{C}}\mathbf{C}^{-1}). \tag{32}$$

This result underlines the intrinsic character of the base pole velocity, as related to the time derivative of the configuration tensor.

The linear subspace $\mathrm{sr}(\mathbb{K}^6)$ coincides with the Lie algebra of the Lie group $\mathrm{SR}(\mathbb{K}^6)$. Since $\mathbb{K}^6$ and $\mathrm{sr}(\mathbb{K}^6)$ are isomorphic, also $\mathbb{K}^6$ has the structure of a Lie algebra, and its commutator [4] is exactly the North-East generalized cross product operation, or $[\bullet, \bigstar] := \bullet \times\!\!\cdot \bigstar, \forall \bigstar, \bullet \in \mathbb{K}^6$.

The convected form of Eq. (30) is obtained as

$$\dot{\mathbf{C}} = \mathbf{C}\, \overline{\mathbf{w}}_{\mathbf{x}} \times\!\!\cdot. \tag{33}$$

In this case, comparing the preceding equation with Eq. (9), the convected generalized velocity $\overline{\mathbf{w}}_{\mathbf{x}}$ acts as the convected angular velocity $\overline{\boldsymbol{\omega}}$. Thus, the alternative definition of the convected generalized velocity is

$$\overline{\mathbf{w}}_{\mathbf{x}} := \mathrm{axial}_{\times\!\cdot}(\mathbf{C}^{-1}\dot{\mathbf{C}}). \tag{34}$$

As mentioned earlier, only the base pole generalized velocity $\mathbf{w_o}$ and the convected generalized velocity $\overline{\mathbf{w}}_{\mathbf{x}}$ appear in the fundamental equation of the frame kinematic evolution, given in its two possible forms (30) and (33). In fact, from the point of view of this formulation, the local generalized velocity $\mathbf{w_x}$ is not intrinsically as meaningful as its base pole and convected versions.

We remark that Eq. (30), when decomposed in its 3-D components, yields nothing but the evolution equation for the position $\mathbf{u}$,

$$\dot{\mathbf{u}} = \boldsymbol{\omega} \times \mathbf{u} + \mathbf{v_o}, \tag{35}$$

and the evolution equation (6) for the orientation $\boldsymbol{\alpha}$, while Eq. (33) yields the evolution equation for the convected position $\overline{\mathbf{u}} := \boldsymbol{\alpha}^{-1}\mathbf{u}$,

$$\dot{\overline{\mathbf{u}}} = -\overline{\boldsymbol{\omega}} \times \overline{\mathbf{u}} + \overline{\mathbf{v}}_{\mathbf{x}}, \tag{36}$$

and the evolution equation (9) for the orientation $\boldsymbol{\alpha}$.

[4] Note that the commutator for $\mathrm{sr}(\mathbb{K}^6)$ is the operation defined by $[\clubsuit, \spadesuit] := \clubsuit\spadesuit - \spadesuit\clubsuit, \forall \clubsuit, \spadesuit \in \mathrm{sr}(\mathbb{K}^6)$. Thus equation $(\bullet \times\!\!\cdot)\,(\bigstar \times\!\!\cdot) - (\bigstar \times\!\!\cdot)\,(\bullet \times\!\!\cdot) = (\bullet \times\!\!\cdot \bigstar) \times\!\!\cdot$ expresses the relation between the commutators for $\mathrm{sr}(\mathbb{K}^6)$ and for $\mathbb{K}^6$, generalizing to the 6-D space that encountered in the 3-D case.

In conclusion, the introduction of the configuration tensor and of the generalized velocity leads to a unified treatment of both position and orientation, and their respective time-rates, for a general frame motion (which coincides with a general *rigid* motion). We remark that the formalism employed here shares almost identical formal appearance and properties with that used in the case of the reduced problem of pure rotational motion.

2.3. Kinematic and co-kinematic spaces

Generalized force. We turn now our attention to *co-vectors*, or vectors in the dual space of $\mathbb{E}^3$, denoted by $\mathbb{E}^{3*}$. These vectors serve to represent linear and rotational momenta, and forces and torques. We consider a system of co-vectors, for example a system of forces, and denote with $\mathbf{n} \in \mathbb{E}^{3*}$ the *resultant force*, and with $\mathbf{m_x} \in \mathbb{E}^{3*}$ the *resultant torque* with respect to point $\mathbf{x} \in \mathscr{E}^3$. We stack these quantities to form a "generalized" force vector reduced to pole $\mathbf{x}$, namely the *local generalized force* $\mathbf{f_x} := (\mathbf{n}, \mathbf{m_x}) \in \mathbb{K}^{6*}$. Generalized forces are sometimes termed *wrenches* in the literature (as in [1]). They form a linear space, termed *co-kinematic space* and denoted by $\mathbb{K}^{6*}$, which is the dual of the kinematic space $\mathbb{K}^6$.

The reduction of the resultant torque from pole $\mathbf{x}$ to pole $\mathbf{o}$ yields

$$\mathbf{m_o} = \mathbf{m_x} + (\mathbf{x} - \mathbf{o}) \times \mathbf{n}. \tag{37}$$

This leads to the transport relation for 6-D co-vectors as

$$\mathbf{f_o} = \mathscr{T}(\mathbf{u})^{-\mathrm{T}}\, \mathbf{f_x}, \tag{38}$$

where $\mathbf{f_o} := (\mathbf{n}, \mathbf{m_o}) \in \mathbb{K}^{6*}$ is the *base pole generalized force*. The relationship between 6-D co-vectors and their convected images reads

$$\mathbf{f_x} = \mathscr{A}(\boldsymbol{\alpha})\, \overline{\mathbf{f}}_\mathbf{x}, \tag{39}$$

where $\overline{\mathbf{f}}_\mathbf{x} := (\overline{\mathbf{n}}, \overline{\mathbf{m}}_\mathbf{x}) \in \mathbb{K}^{6*}$ is the *convected generalized force*, composed by the *convected resultant force* $\overline{\mathbf{n}} := \boldsymbol{\alpha}^{-1}\mathbf{n}$ and by the *convected resultant torque* $\overline{\mathbf{m}}_\mathbf{x} := \boldsymbol{\alpha}^{-1}\mathbf{m_x}$. Clearly, the above relations (38) and (39) are the co-kinematic versions of Eqs. (23) and (25). The corresponding equation to Eq. (27) is then

$$\mathbf{f_o} = \mathbf{C}^{-\mathrm{T}}\, \overline{\mathbf{f}}_\mathbf{x}. \tag{40}$$

This equation extends the action of the configuration tensor $\mathbf{C}$ over 6-D co-vectors. The "inverse-transpose" in Eqs. (38) and (40) highlights the duality in the rules of reduction to the different poles of linear vectors and rotational co-vectors as given by Eqs. (17) and (37).

Conjugation and identification. Generally speaking, co-vectors are associated to ordinary vectors through some bilinear conjugation function $\langle , \rangle_{\mathbb{E}^3} : \mathbb{E}^3 \times \mathbb{E}^{3*} \to \mathbb{R}$, namely, a *scalar product*, that is required to be indifferent with respect to rigid transformations. This means in particular that the conjugation of quantities evaluated with respect to different vector bases for $\mathbb{E}^3$ and $\mathbb{E}^{3*}$ must yield the same result. Then, considering the base and the moving triads as vector bases, one may write

$$\langle \bigstar, \bullet \rangle_{\mathbb{E}^3} = \langle \spadesuit \bigstar, \spadesuit \bullet \rangle_{\mathbb{E}^3} \tag{41}$$

$\forall \bullet \in \mathbb{E}^3$, $\forall \bigstar \in \mathbb{E}^{3*}$ and $\forall \spadesuit \in \mathrm{SO}(\mathbb{E}^3)$. As it is frequently done in mechanics, we may identify the spaces $\mathbb{E}^3$ and $\mathbb{E}^{3*}$ and write the conjugation in terms of the inner product on $\mathbb{E}^3$ as

$$\langle \bigstar, \bullet \rangle_{\mathbb{E}^3} := \bigstar \cdot \bullet. \tag{42}$$

In fact, the inner product complies with the requirement of invariance with respect to rigid displacements. This is easily proven given the orthogonality of tensor $\spadesuit$ in Eq. (41). This scalar product is clearly related to the fundamental notions of mechanical power and energy.

Although on $\mathbb{K}^6$ we do not need any metric structure, and thus any inner product, we may define the scalar product $\langle , \rangle_{\mathbb{K}^6} : \mathbb{K}^6 \times \mathbb{K}^{6*} \to \mathbb{R}$, between a kinematic vector and a co-kinematic vector simply as

$$\langle \bigstar, \bullet \rangle_{\mathbb{K}^6} := \langle \bigstar_\mathsf{L}, \bullet_\mathsf{L} \rangle_{\mathbb{E}^3} + \langle \bigstar_\mathsf{R}, \bullet_\mathsf{R} \rangle_{\mathbb{E}^3} \tag{43}$$

$\forall \bullet = (\bullet_{\mathsf{L}}, \bullet_{\mathsf{R}}) \in \mathbb{K}^6$ and $\forall \bigstar = (\bigstar_{\mathsf{L}}, \bigstar_{\mathsf{R}}) \in \mathbb{K}^{6*}$. It is easily checked that Eq. (43) defines a "natural" conjugation function, which reveals to be invariant not only with respect to rotations,

$$\langle \bigstar, \bullet \rangle_{\mathbb{K}^6} = \langle \mathscr{A}(\spadesuit)\bigstar, \mathscr{A}(\spadesuit)\bullet \rangle_{\mathbb{K}^6} \tag{44}$$

$\forall \spadesuit \in \mathrm{SO}(\mathbb{E}^3)$, but also to translations,

$$\langle \bigstar, \bullet \rangle_{\mathbb{K}^6} = \langle \mathscr{T}(\clubsuit)^{-\mathrm{T}}\bigstar, \mathscr{T}(\clubsuit)\bullet \rangle_{\mathbb{K}^6} \tag{45}$$

$\forall \clubsuit \in \mathscr{E}^3$, and hence it is indifferent with respect to any rigid transformation.

The identification of the spaces $\mathbb{K}^6$ and $\mathbb{K}^{6*}$ is not as trivial as in the case of $\mathbb{E}^3$ and $\mathbb{E}^{3*}$, since 6-D vectors and co-vectors transform differently when subjected to a rigid transformation, as shown by Eqs. (27) and (40). However, given the (usually implicitly assumed) identification of the spaces $\mathbb{E}^3$ and $\mathbb{E}^{3*}$ and of the scalar product $\langle\, , \rangle_{\mathbb{E}^3}$ with the internal product in $\mathbb{E}^3$, we denote the scalar product $\langle\, , \rangle_{\mathbb{K}^6}$ as a "dot" product, in view of a lighter notation. We shall write then

$$\bigstar \cdot \bullet := \bigstar_{\mathsf{L}} \cdot \bullet_{\mathsf{L}} + \bigstar_{\mathsf{R}} \cdot \bullet_{\mathsf{R}} \tag{46}$$

$\forall \bullet \in \mathbb{K}^6, \forall \bigstar \in \mathbb{K}^{6*}$, in place of Eq. (43).

2.4. Base pole balance equations

Bodies and motions. From a theoretical standpoint, a *body* $\mathscr{B}$ is here understood as a differential manifold, whose elements are the *material particles* or *body points* $P \in \mathscr{B}$. A *motion* of $\mathscr{B}$ may be defined as a map $\mathsf{X} : \mathscr{B} \times [0, T] \to \mathscr{V}$, where $\mathscr{V}$ is a suitable differential manifold, namely the configuration space of body $\mathscr{B}$.

For a classical *Cauchy continuum*, the most general configuration space is simply $\mathscr{E}^3$, the 3-D Euclidean manifold of positions. Thus, the position of a material particle P at time t is given by

$$\mathsf{X}(P) = \mathbf{x}_P. \tag{47}$$

In this body model the material field of position $\mathbf{u}_P := \mathbf{x}_P - \mathbf{o}$ with respect to an arbitrary origin $\mathbf{o}$ determines the configuration of the body, and there is no "orientation" field. This is not the case with other body models, such as *polar continua*. A polar model of special interest for our purposes is represented by a *Cosserat continuum*. We refer to this body model as one implying the manifold $\mathbb{E}^3 \times \mathrm{SO}(\mathbb{E}^3)$ as configuration space. In other words, this is a continuum where the image of a material particle under the motion X is a frame,

$$\mathsf{X}(P) = \left(\mathbf{x}_P, \{\mathbf{e}_{P_k}\}_{k=1,2,3}\right) \tag{48}$$

$\forall P \in \mathscr{B}$. Therefore the body configuration is determined by a double material field, composed by the position field $\mathbf{u}_P := \mathbf{x}_P - \mathbf{o}$ and the orientation field $\boldsymbol{\alpha}_P := \mathbf{e}_{P_k} \otimes \mathbf{i}_k$ with respect to an arbitrary base frame $(\mathbf{o}, \{\mathbf{i}_k\}_{k=1,2,3})$.

Bodies are endowed with inertial properties. In the Cauchy continuum framework, the primitive notion is that of *mass*, defined as a non-negative scalar measure m on the parts of a body $\mathscr{B}$,

$$m := \int_{\mathscr{B}} \mathrm{d}m_P \tag{49}$$

being $P \in \mathscr{B}$ the dummy integration variable. For Cosserat continua this notion may be extended to a more general non-negative symmetric tensorial measure $\mathbf{M} \in \mathrm{Lin}(\mathbb{K}^6, \mathbb{K}^{6*})$ on the parts of $\mathscr{B}$,

$$\mathbf{M} := \int_{\mathscr{B}} \mathrm{d}\mathbf{M}_P, \tag{50}$$

termed the *generalized inertia tensor*.

Dynamic balance equations. Body $\mathscr{B}$ is characterized during its motion by two 3-D vectors: the resultants of *linear momentum* $\mathbf{l} \in \mathbb{E}^3$ and of *angular momentum* $\mathbf{h}_\mathbf{y} \in \mathbb{E}^3$ with respect to a generic point $\mathbf{y} \in \mathscr{E}^3$; and by a fundamental scalar quantity, the *kinetic energy* T. For the sake of simplicity, the definitions of kinetic momenta and energy are recalled hereafter in the case of a Cauchy continuum body model,

$$\mathbf{l} := \int_{\mathscr{B}} \dot{\mathbf{u}}_P \, \mathrm{d}m_P, \tag{51}$$

$$\mathbf{h}_\mathbf{y} := \int_{\mathscr{B}} (\mathbf{x}_P - \mathbf{y}) \times \dot{\mathbf{u}}_P \, \mathrm{d}m_P, \tag{52}$$

$$T := \frac{1}{2} \int_{\mathscr{B}} \dot{\mathbf{u}}_P \cdot \dot{\mathbf{u}}_P \, \mathrm{d}m_P, \tag{53}$$

where particle $P \in \mathscr{B}$ is the dummy integration variable. These *constitutive equations* for kinetic momenta and energy clearly take a different form in the case of a different body model. We shall encounter an application of a one-dimensional Cosserat continuum when we deal with the geometrically non-linear beam model.

Following a classical approach to dynamics, the motion of a body $\mathscr{B}$ is viewed as the result of the application of a system of forces on it. As already seen, this system is characterized by two fundamental 3-D vectors: the resultant force $\mathbf{n} \in \mathbb{E}^3$ and the resultant torque $\mathbf{m}_\mathbf{y} \in \mathbb{E}^3$ with respect to a generic point $\mathbf{y} \in \mathscr{E}^3$. The *force* and *torque balance equations* are given, for any body model, as

$$\dot{\mathbf{l}} = \mathbf{n}, \tag{54}$$

$$\dot{\mathbf{h}}_\mathbf{y} + \dot{\mathbf{y}} \times \mathbf{l} = \mathbf{m}_\mathbf{y}, \tag{55}$$

with respect to a generic time-varying point $\mathbf{y}$, as customary. These two independent equations may be derived from a single principle of indifference with respect to rigid displacements for the mechanical power (*Noll's fundamental axiom* as discussed in, e.g., [16]). The coupling between the linear and angular balance equations is apparent considering the term $\dot{\mathbf{y}} \times \mathbf{l}$.

To simplify and uncouple these equations one can use the instantaneous *center of mass* of body $\mathscr{B}$ as the pole for the reduction of torques. The center of mass is denoted by $\mathbf{x}_m$ and defined, in a Cauchy's continuum framework, as

$$\mathbf{x}_m := \mathbf{o} + \frac{1}{m} \int_{\mathscr{B}} \mathbf{u}_P \, \mathrm{d}m_P. \tag{56}$$

In fact, since the time derivative of the preceding equation yields $\dot{\mathbf{x}}_m = \mathbf{l}/m$, the coupling term $\dot{\mathbf{x}}_m \times \mathbf{l}$ vanishes and one obtains the *center of mass torque balance equation*

$$\dot{\mathbf{h}}_{\mathbf{x}_m} = \mathbf{m}_{\mathbf{x}_m}. \tag{57}$$

It may be useful to remark that this uncoupling strategy is based on the choice of a special point which

(a) does not necessarily coincide with the position of the same material particle during the motion (except in the rigid body case);

(b) in the case of certain body models, such as beams, may not be the more convenient choice of pole for the description of the material properties, in comparison to other remarkable points (as, for example, the elastic center).

We shall see next that another, more general way to obtain the desired uncoupling is possible, and plays a central role in the development of the algorithms discussed in Part II of this work.

Base pole reduction. A different strategy simply consists in taking a fixed point as pole for the reduction of torques. This is frequently done in rigid body dynamics in the particular case of a motion that leaves the position of a material point fixed in space. In reality, such a procedure is convenient in the general case, for any body and any motion. Furthermore, this procedure is consistent with the base pole reduction as discussed in the development of kinematics.

With respect to the constant-in-time base pole $\mathbf{o} \in \mathscr{E}^3$, due to the vanishing of the coupling term $\dot{\mathbf{o}} \times \mathbf{l}$, we obtain the *base pole torque balance equation*

$$\dot{\mathbf{h}}_{\mathbf{o}} = \mathbf{m}_{\mathbf{o}}. \tag{58}$$

Note that in this case, the uncoupling strategy is based on a point that:

(a) is as general as possible, and can be chosen (and changed, during computations) at will to simplify the problem at hand, on account of its topology;

(b) yields the uncoupled form (58) independently of the body model (rigid/deformable, polar/non-polar, etc.).

We can now reformulate the balance equations in the co-kinematic space $\mathbb{K}^{6*}$. We term the 6-D co-vector $\mathbf{p}_{\mathbf{o}} := (\mathbf{l}, \mathbf{h}_{\mathbf{o}}) \in \mathbb{K}^{6*}$ the *generalized base pole kinetic moment* and write the *base pole generalized force balance equation* simply as

$$\dot{\mathbf{p}}_{\mathbf{o}} = \mathbf{f}_{\mathbf{o}}. \tag{59}$$

Note that, in the 6-D representation, the generalized force balance equation reduced to a generic pole $\mathbf{y} \in \mathscr{E}^3$ assumes the more cumbersome form

$$\dot{\mathbf{p}}_{\mathbf{y}} + \frac{\mathrm{d}}{\mathrm{d}t}\left(\mathscr{T}(\mathbf{y} - \mathbf{o})^{-\mathrm{T}}\right)\mathbf{p}_{\mathbf{y}} = \mathbf{f}_{\mathbf{y}}. \tag{60}$$

The comparison between the last equations should easily convey the feeling that the implementation of numerical procedures is likely to significantly gain from the simplicity of the first, while use of the second leads to the problem of finding a suitable discretization of the coupling term.

2.5. Parameterization of motion

General features of the exponential map. Let us briefly review the main features of the *exponential map* $\exp(\bullet)$. This is defined, for linear operators on a generic vector space, by the classical formula

$$\exp(\bullet) = \sum_{k=0}^{\infty} \frac{\bullet^k}{k!}. \tag{61}$$

It is a function that relates a Lie group to its Lie algebra. Thus, the invertible operator $\exp(\bullet)$ is an element of the group corresponding to the element $\bullet$ of the algebra. Clearly, $\exp(\mathbf{O}) = \mathbf{I}$, where $\mathbf{O}$ and $\mathbf{I}$ are the null and the identity tensors on the relevant vector space.

Remarkable properties of the exponential map are

$$\exp(\bullet)^{-1} = \exp(-\bullet), \tag{62}$$

$$\exp(\bullet) = \operatorname{dexp}(\bullet)\operatorname{dexp}(-\bullet)^{-1}, \tag{63}$$

$$\exp(\bullet) = \mathbf{I} + \operatorname{dexp}(\bullet)\,\bullet, \tag{64}$$

where $\operatorname{dexp}(\bullet)$ denotes the *associated differential map* defined as

$$\operatorname{dexp}(\bullet) = \sum_{k=0}^{\infty} \frac{\bullet^k}{(k+1)!}. \tag{65}$$

Note that by Eq. (64) we justify the name for $\operatorname{dexp}(\bullet)$, as it may be "symbolically" interpreted as $\operatorname{dexp}(\bullet) = (\exp(\bullet) - \mathbf{I})/\bullet$. The differential map plays an important role in the following developments, since by using it the derivative of an element in the group can be related to the derivative of the corresponding element in the algebra, as it will be shown in the following.

Since the well-known exponential map of rotation is a particular case of the exponential map of motion, which accounts for coupled translation and rotation, a brief review of the first is given below to help introduce the second.

Exponential map of rotation. In the case of rotations we have $\exp : \mathrm{so}(\mathbb{E}^3) \to \mathrm{SO}(\mathbb{E}^3)$, and the exponential map is surjective. This means that given any rotation tensor $\mathbf{R} \in \mathrm{SO}(\mathbb{E}^3)$, we can find a *rotation vector* $\boldsymbol{\varphi} \in \mathbb{E}^3$ such that

$$\mathbf{R} = \exp(\boldsymbol{\varphi}\times) \tag{66}$$

with $\varphi := \|\boldsymbol{\varphi}\| \in [0, \pi]$. The rotation vector $\boldsymbol{\varphi}$ is an eigenvector corresponding to the unique real eigenvalue equal to $+1$ of the rotation tensor $\mathbf{R}$,

$$\mathbf{R}\boldsymbol{\varphi} = \boldsymbol{\varphi}. \tag{67}$$

The scalar components of the rotation vector $\boldsymbol{\varphi}$ with respect to some basis in $\mathbb{E}^3$ may then be assumed as exponential coordinates for the rotation tensor $\mathbf{R}$. The differential tensor associated to $\mathbf{R}$, denoted by $\mathbf{S} \in \mathrm{Lin}(\mathbb{E}^3)$,

$$\mathbf{S} := \mathrm{dexp}(\boldsymbol{\varphi}\times) \tag{68}$$

relates the derivatives of $\mathbf{R}$ with those of $\boldsymbol{\varphi}$. In fact, considering the equation $\dot{\mathbf{R}} = \boldsymbol{v} \times \mathbf{R}$, the rotational rate vector $\boldsymbol{v} := \mathrm{axial}_\times(\dot{\mathbf{R}}\mathbf{R}^{-1})$ may be expressed as

$$\boldsymbol{v} = \mathbf{S}\dot{\boldsymbol{\varphi}}. \tag{69}$$

A very important feature of the exponential map of rotation is that we can find useful finite form expressions for it and its associated differential map instead of the general definitions (61) and (65). This is obtained by using the recursive property of the ordinary cross product,

$$(\boldsymbol{\varphi}\times)^3 + \varphi^2(\boldsymbol{\varphi}\times) = \mathbf{O}_3, \tag{70}$$

$\forall\boldsymbol{\varphi} \in \mathbb{E}^3$, reducing thus the sums to only 3 terms. In fact, the well-known *Euler–Rodrigues formula* for the rotation tensor $\mathbf{R}$ and the similar finite form for the associated differential tensor $\mathbf{S}$ hold,

$$\mathbf{R} = \mathbf{I}_3 + \sum_{k=1}^{2} R_k\,(\boldsymbol{\varphi}\times)^k \tag{71}$$

$$\mathbf{S} = \mathbf{I}_3 + \sum_{k=1}^{2} S_k\,(\boldsymbol{\varphi}\times)^k. \tag{72}$$

The expressions of the scalar coefficients R_k, S_k, as functions of the sole rotation angle φ, are given by

$$\begin{aligned} R_1(\varphi) &:= \frac{1}{\varphi}\sin\varphi, \quad S_1(\varphi) := R_2(\varphi), \\ R_2(\varphi) &:= \frac{1}{\varphi^2}(1-\cos\varphi), \quad S_2(\varphi) := \frac{1}{\varphi^3}(\varphi - \sin\varphi). \end{aligned} \tag{73}$$

Exponential map of motion. In the case of general, complete (i.e., coupled translational and rotational) frame motions we have $\exp : \mathrm{sr}(\mathbb{K}^6) \to \mathrm{SR}(\mathbb{K}^6)$, and again the exponential map is surjective. This means that given any rigid displacement tensor $\mathbf{D} \in \mathrm{SR}(\mathbb{K}^6)$, we can find a *rigid displacement vector* $\boldsymbol{v}_\mathbf{o} \in \mathbb{K}^6$ such that

$$\mathbf{D} = \exp(\boldsymbol{v}_\mathbf{o}\times\!\!\!\times), \tag{74}$$

with the norm of its angular vector component subjected to the condition $\|\boldsymbol{v}_{\mathbf{o}_\mathrm{A}}\| \in [0, \pi]$. The rigid displacement vector $\boldsymbol{v}_\mathbf{o}$ is also termed *base pole generalized screw vector*, due to its relationship with the notion of *screw motion* (see [9] for details). Vector $\boldsymbol{v}_\mathbf{o}$ is an eigenvector corresponding to the double real eigenvalue equal to $+1$ of the displacement tensor $\mathbf{D}$:

$$\mathbf{D}\,\boldsymbol{v}_\mathbf{o} = \boldsymbol{v}_\mathbf{o}. \tag{75}$$

The scalar components of the generalized screw vector $\boldsymbol{v}_\mathbf{o}$ with respect to some basis in $\mathbb{K}^6$ may then be assumed as coordinates for the displacement tensor $\mathbf{D}$. The differential tensor associated to $\mathbf{D}$, denoted by $\mathbf{E} \in \mathrm{Lin}(\mathbb{K}^6)$,

$$\mathbf{E} := \mathrm{dexp}(\boldsymbol{v}_\mathbf{o}\times\!\!\!\!\nearrow), \tag{76}$$

relates the derivatives of $\mathbf{D}$ with those of $\boldsymbol{v}_\mathbf{o}$. In fact, considering the equation $\dot{\mathbf{D}} = \boldsymbol{\varpi}_\mathbf{o}\times\!\!\!\!\nearrow\mathbf{D}$, the screw rate vector $\boldsymbol{\varpi}_\mathbf{o} := \mathrm{axial}_{\times\!\!\!\nearrow}(\dot{\mathbf{D}}\,\mathbf{D}^{-1})$ may be expressed as

$$\boldsymbol{\varpi}_\mathbf{o} = \mathbf{E}\,\dot{\boldsymbol{v}}_\mathbf{o}, \tag{77}$$

in close analogy to the purely rotational case.

Let us denote with $\mathbf{t} \in \mathbb{E}^3$ and $\mathbf{R} \in \mathrm{SO}(\mathbb{E}^3)$ the vector and tensor that characterize the structural decomposition of the displacement tensor $\mathbf{D}$ as

$$\mathbf{D} = \mathscr{T}(\mathbf{t})\,\mathscr{A}(\mathbf{R}), \tag{78}$$

and define the base pole vector $\boldsymbol{\rho}_\mathbf{o}$ as

$$\boldsymbol{\rho}_\mathbf{o} := \mathbf{S}^{-1}\,\mathbf{t}, \tag{79}$$

with $\mathbf{S}$ given by Eq. (68). It follows that the 3-D angular vector component of the generalized screw vector $\boldsymbol{v}_\mathbf{o}$ coincides with the rotation vector $\boldsymbol{\varphi}$ corresponding to $\mathbf{R}$, Eq. (66), while the 3-D linear vector component is exactly $\boldsymbol{\rho}_\mathbf{o}$,

$$\boldsymbol{v}_\mathbf{o} = (\boldsymbol{\rho}_\mathbf{o}, \boldsymbol{\varphi}). \tag{80}$$

As for the case of rotations, the exponential map of motion has the considerable advantage of allowing for useful finite form expressions for the exponential map itself and its associated differential map. In fact, the North-east cross product satisfies a recursive property similar to (70), namely

$$(\boldsymbol{v}_\mathbf{o}\times\!\!\!\!\nearrow)^3 + \boldsymbol{\Phi}^2(\boldsymbol{v}_\mathbf{o}\times\!\!\!\!\nearrow) = \mathbf{O}_6 \tag{81}$$

$\forall \boldsymbol{v}_\mathbf{o} = (\boldsymbol{\rho}_\mathbf{o}, \boldsymbol{\varphi}) \in \mathbb{K}^6$. Tensor $\boldsymbol{\Phi} \in \mathrm{Lin}(\mathbb{K}^6)$ is a *generalized parallelism tensor* which is completely defined by the scalar pair (φ, τ),

$$\boldsymbol{\Phi} := \varphi \begin{bmatrix} \mathbf{I}_3 & \tau\,\mathbf{I}_3 \\ \mathbf{O}_3 & \mathbf{I}_3 \end{bmatrix}, \tag{82}$$

where $\tau := (\boldsymbol{\rho}_\mathbf{o} \cdot \boldsymbol{\varphi})/\varphi^2$ is the *pitch* of the generalized screw vector $\boldsymbol{v}_\mathbf{o}$. Application of the recursive property (81) to the general expressions (61) and (65) allows for the reductions of the sums to only 3 terms, as in the case of rotations. The following are the finite form expressions for the displacement tensor $\mathbf{D}$ and its associated differential operator $\mathbf{E}$,

$$\mathbf{D} = \mathbf{I}_6 + \sum_{k=1}^{2} \widehat{\mathbf{R}}_k\,(\boldsymbol{v}_\mathbf{o}\times\!\!\!\!\nearrow)^k, \tag{83}$$

$$\mathbf{E} = \mathbf{I}_6 + \sum_{k=1}^{2} \widehat{\mathbf{S}}_k\,(\boldsymbol{v}_\mathbf{o}\times\!\!\!\!\nearrow)^k. \tag{84}$$

The tensorial coefficients $\widehat{\mathbf{R}}_1$, $\widehat{\mathbf{R}}_2$ and $\widehat{\mathbf{S}}_1$, $\widehat{\mathbf{S}}_2$ are generalized parallelism tensors given by

$$\begin{aligned}\widehat{\mathbf{R}}_1(\varphi,\tau) &:= R_1(\varphi)\begin{bmatrix}\mathbf{I}_3 & \tau U_1(\varphi)\mathbf{I}_3\\ \mathbf{O}_3 & \mathbf{I}_3\end{bmatrix}, \quad \widehat{\mathbf{S}}_1(\varphi,\tau) := S_1(\varphi)\begin{bmatrix}\mathbf{I}_3 & \tau V_1(\varphi)\mathbf{I}_3\\ \mathbf{O}_3 & \mathbf{I}_3\end{bmatrix},\\ \widehat{\mathbf{R}}_2(\varphi,\tau) &:= R_2(\varphi)\begin{bmatrix}\mathbf{I}_3 & \tau U_2(\varphi)\mathbf{I}_3\\ \mathbf{O}_3 & \mathbf{I}_3\end{bmatrix}, \quad \widehat{\mathbf{S}}_2(\varphi,\tau) := S_2(\varphi)\begin{bmatrix}\mathbf{I}_3 & \tau V_2(\varphi)\mathbf{I}_3\\ \mathbf{O}_3 & \mathbf{I}_3\end{bmatrix}.\end{aligned} \tag{85}$$

Therefore, they are functions of the sole pitch τ and rotation angle φ through the scalar coefficients R_1, R_2, S_1, S_2 of the exponential map of rotation and the additional scalar coefficients U_1, U_2, V_1, V_2 defined as

$$\begin{aligned}U_1(\varphi) &:= \frac{\cos\varphi}{R_1(\varphi)} - 1, \quad V_1(\varphi) := U_2(\varphi),\\ U_2(\varphi) &:= \frac{R_1(\varphi)}{R_2(\varphi)} - 2, \quad V_2(\varphi) := \frac{S_1(\varphi)}{S_2(\varphi)} - 3.\end{aligned} \tag{86}$$

Note that $\widehat{\mathbf{S}}_1 = \widehat{\mathbf{R}}_2$, analogously to the 3-D case.

The interested reader is addressed to [9] for a thorough exposition on the subject, which can be meaningfully related to the solution and perturbation of constant coefficient tensorial ODEs. Detailed information on generalized parallelism in $\mathbb{K}^6$ and its relationship with screw displacement and the exponential map of motion is also provided there.

On parameterization of motion. The preceding considerations show how the exponential map of motion can be employed to define suitable global, i.e., coupled linear and angular, coordinates for frame motion accounting for both position and orientation. This parameterization shares similar formalism and properties as those of the exponential map of rotations, which turns out to be a particular case of the exponential map of motion when the general motion reduces to a pure fixed point rotation.

As the research on global parameterization procedures is still undergoing, we just mention that other techniques may be employed, such as *Cayley's parameterization*, based on Cayley's transform for linear operators. This represents the 6-D generalization of the Gibbs–Rodrigues parameterization of rotations. Details about this issue are given in [9].

Apart from the appealing synthesis in the formulæ and a certain didactical interest, in our view the main feature of a global parameterization technique as such lies in the fact that important geometric invariance properties are retained when continuous bodies are subjected to discretization. The result is numerical indifference to rigid displacements. In fact, this process does not affect the correct coupling between linear and angular quantities, as it is the case when separate parameterization techniques are adopted for the linear and angular components. This and related issues do have a considerable impact on the computational implementation and the numerical solution of flexible multibody systems dynamics.

3. Rigid body and beam dynamics

Among the body models of common application, the rigid body and the elastic beam play a central role in multibody systems simulation. For both of these models the relevant quantities such as linear and angular velocities and momenta, and forces and torques, may be conveniently described using the 6-D representation to yield compact constitutive and balance equations. The presentation shows the close formal analogy between the reduced case of purely rotational motion and the general one. The governing equations for rigid body and beam dynamics are cast in base pole form as ODEs in time, provided that, in the case of the beam, a semi-discretization process such as generic finite element methods (FEM) or finite volume methods (FVM) has been adopted.

3.1. Rigid body dynamics

Rigid body motion. A *rigid body* $\mathscr{B}$ may be described, on one hand, as a particular Cauchy continuum subjected to a material constraint that imposes constant relative (scalar) distances between its particles.

On the other hand, it can also be described as the simplest Cosserat continuum, composed by a single material particle B whose image, under the motion X, is the frame $(\mathbf{x}_B, \{\mathbf{e}_{B_k}\}_{k=1,2,3})$. Its configuration is described simply by the configuration pair $(\mathbf{u}_B, \boldsymbol{\alpha}_B)$, or by the configuration tensor $\mathbf{C}_B$. The two kinematic and dynamic descriptions are equivalent, provided the consistency of the inertial properties of the two models.

If we look at the rigid body $\mathscr{B}$ as a Cauchy continuum, the indeformability constraint that holds for any motion of $\mathscr{B}$ may be expressed as

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\mathbf{x}_Q - \mathbf{x}_P\| = 0, \tag{87}$$

for any pair of distinct particles $P, Q \in \mathscr{B}$. This entails the equiprojectivity of the velocity field,

$$(\dot{\mathbf{x}}_Q - \dot{\mathbf{x}}_P) \cdot (\mathbf{x}_Q - \mathbf{x}_P) = 0, \tag{88}$$

so that a unique time-dependent vector $\boldsymbol{\omega}(t)$ exists, the angular velocity of the whole rigid body $\mathscr{B}$, such that

$$\dot{\mathbf{u}}_Q = \dot{\mathbf{u}}_P + \boldsymbol{\omega} \times (\mathbf{u}_Q - \mathbf{u}_P), \tag{89}$$

$\forall P, Q \in \mathscr{B}$. Consider now any frame with a pole given by the position of a material particle $P \in \mathscr{B}$ and triad unit vectors satisfying the Poisson's equations (7) with such vector $\boldsymbol{\omega}$. We may affirm then that a unique time-dependent vector $\mathbf{v}_\mathbf{o}(t)$ exists, such that it represents the base pole linear velocity of any such frame. This vector represents thus a measure of the linear velocity for the whole rigid body $\mathscr{B}$.

This way we are able to apply the results developed thus far for frame kinematics to the rigid body motion, by simply choosing the position of an arbitrary material particle $B \in \mathscr{B}$ and following the evolution of the frame that satisfies equation

$$\dot{\mathbf{C}}_B = \mathbf{w}_\mathbf{o} \times \mathbf{C}_B \tag{90}$$

with a consistent initial condition $\mathbf{C}_B|_{t=0} \in \mathrm{SR}(\mathbb{K}^6)$.

Constitutive equations. The constitutive equations (51)–(53) in the case of a rigid body may be written in the form

$$\mathbf{l} = m\mathbf{v}_{\mathbf{x}_P} - \boldsymbol{\sigma}_{\mathbf{x}_P} \times \boldsymbol{\omega}, \tag{91}$$

$$\mathbf{h}_{\mathbf{x}_P} = \boldsymbol{\sigma}_{\mathbf{x}_P} \times \mathbf{v}_{\mathbf{x}_P} + \mathbf{J}_{\mathbf{x}_P}\boldsymbol{\omega}, \tag{92}$$

$$T = \frac{1}{2} m\mathbf{v}_{\mathbf{x}_P} \cdot \mathbf{v}_{\mathbf{x}_P} + \boldsymbol{\sigma}_{\mathbf{x}_P} \cdot \boldsymbol{\omega} + \frac{1}{2}\mathbf{J}_{\mathbf{x}_P}\boldsymbol{\omega} \cdot \boldsymbol{\omega}, \tag{93}$$

using the sole frame linear and angular velocities $\mathbf{v}_{\mathbf{x}_P}$ and $\boldsymbol{\omega}$ corresponding to any body point $P \in \mathscr{B}$. The scalar m is the mass of the rigid body, Eq. (49), while the vector $\boldsymbol{\sigma}_\mathbf{y} \in \mathbb{E}^3$, and the symmetric tensor $\mathbf{J}_\mathbf{y} \in \mathrm{Lin}(\mathbb{E}^3)$ with respect to a generic point $\mathbf{y} \in \mathscr{E}^3$ are the *static moment* and the *moment of inertia*, respectively defined as

$$\boldsymbol{\sigma}_\mathbf{y} := \int_{\mathscr{B}} (\mathbf{x}_P - \mathbf{y})\,\mathrm{d}m_P, \tag{94}$$

$$\mathbf{J}_\mathbf{y} := -\int_{\mathscr{B}} (\mathbf{x}_P - \mathbf{y}) \times (\mathbf{x}_P - \mathbf{y}) \times \,\mathrm{d}m_P. \tag{95}$$

In terms of the 6-D representation, Eqs. (91)–(93) are synthesized in the compact constitutive equations

$$\mathbf{p}_{\mathbf{x}_P} = \mathbf{M}_{\mathbf{x}_P}\mathbf{w}_{\mathbf{x}_P}, \tag{96}$$

$$T = \frac{1}{2}\mathbf{M}_{\mathbf{x}_P}\mathbf{w}_{\mathbf{x}_P} \cdot \mathbf{w}_{\mathbf{x}_P}. \tag{97}$$

The *generalized inertia tensor* reduced to the generic point $\mathbf{y} \in \mathscr{E}^3$, $\mathbf{M}_\mathbf{y} : \mathbb{K}^6 \to \mathbb{K}^{6*}$, is defined in matricial form as

$$\mathbf{M}_\mathbf{y} := \begin{bmatrix} m\,\mathbf{I}_3 & -\boldsymbol{\sigma}_\mathbf{y}\times \\ \boldsymbol{\sigma}_\mathbf{y}\times & \mathbf{J}_\mathbf{y} \end{bmatrix}. \tag{98}$$

Note that the indifference of the kinetic energy T with respect to rigid displacements can be shown by the following equivalent 6-D formulæ

$$T := \frac{1}{2}\,\mathbf{p}_\mathbf{o} \cdot \mathbf{w}_\mathbf{o}, \tag{99}$$

$$:= \frac{1}{2}\,\mathbf{p}_{\mathbf{x}_P} \cdot \mathbf{w}_{\mathbf{x}_P}, \tag{100}$$

$$:= \frac{1}{2}\,\overline{\mathbf{p}}_{\mathbf{x}_P} \cdot \overline{\mathbf{w}}_{\mathbf{x}_P}, \tag{101}$$

which can be easily demonstrated recalling Eqs. (44) and (45).

Energy balance. As an example of application of the present formulation, let us consider the energy balance for a rigid body. Conjugating the base pole balance equation (59) with the base pole generalized velocity $\mathbf{w}_\mathbf{o}$ on both sides yields

$$\dot{\mathbf{p}}_\mathbf{o} \cdot \mathbf{w}_\mathbf{o} = \mathbf{f}_\mathbf{o} \cdot \mathbf{w}_\mathbf{o}. \tag{102}$$

The right-hand side is the *power of the applied forces W*, which can be written in any of the following equivalent forms,

$$W := \mathbf{f}_\mathbf{o} \cdot \mathbf{w}_\mathbf{o}, \tag{103}$$

$$:= \mathbf{f}_{\mathbf{x}_P} \cdot \mathbf{w}_{\mathbf{x}_P}, \tag{104}$$

$$:= \overline{\mathbf{f}}_{\mathbf{x}_P} \cdot \overline{\mathbf{w}}_{\mathbf{x}_P}, \tag{105}$$

which show its indifference with respect to rigid displacements. With regard to the left-hand side of Eq. (102), note that the time derivative of the kinetic energy may be written using base pole quantities in the forms

$$\dot{T} := \mathbf{p}_\mathbf{o} \cdot \dot{\mathbf{w}}_\mathbf{o}, \tag{106}$$

$$:= \dot{\mathbf{p}}_\mathbf{o} \cdot \mathbf{w}_\mathbf{o}. \tag{107}$$

These results rely on the skew-symmetry property of the North-East cross product $\overset{\nearrow}{\times}$ as a bilinear operation on $\mathbb{K}^6$. In fact, the time derivative of the base pole generalized inertia tensor, given the time-independence of its convected image $\overline{\mathbf{M}}_{\mathbf{x}_P} := \mathbf{C}_P^{-\mathrm{T}} \mathbf{M}_\mathbf{o} \mathbf{C}_P^{-1}$, reads

$$\dot{\mathbf{M}}_\mathbf{o} = \mathbf{w}_\mathbf{o} \underset{\swarrow}{\times} \mathbf{M}_\mathbf{o} - \mathbf{M}_\mathbf{o}\, \mathbf{w}_\mathbf{o} \overset{\nearrow}{\times}, \tag{108}$$

and one gets

$$\dot{\mathbf{p}}_\mathbf{o} = \mathbf{w}_\mathbf{o} \underset{\swarrow}{\times} \mathbf{p}_\mathbf{o} + \mathbf{M}_\mathbf{o}\, \dot{\mathbf{w}}_\mathbf{o}, \tag{109}$$

so that in the end

$$\dot{\mathbf{p}}_\mathbf{o} \cdot \mathbf{w}_\mathbf{o} = \mathbf{p}_\mathbf{o} \cdot \dot{\mathbf{w}}_\mathbf{o}. \tag{110}$$

In Eqs. (108) and (109) we used the *South-West cross product* $\underset{\swarrow}{\times}$, an operator defined in terms of the North-East cross product $\overset{\nearrow}{\times}$ as

$$\bullet \underset{\swarrow}{\times} := -(\bullet \overset{\nearrow}{\times})^{\mathrm{T}}, \tag{111}$$

$\forall \bullet \in \mathbb{K}^6$. Note that taking a generic moving pole instead of the base pole, formulæ (106) and (107) above do not hold, since in such a case one cannot conveniently exploit the skew-symmetry property leading to $\mathbf{w}_\mathbf{o} \overset{\nearrow}{\times} \mathbf{w}_\mathbf{o} = \mathbf{0}_3$.

The energy balance, given Eqs. (103) and (107), finally takes the well-known expression

$$\dot{T} = W, \tag{112}$$

which represents the *theorem of kinetic energy conservation* for the rigid body.

Equations of motion. In the 6-D representation, the equations governing the general motion of the rigid body are then given by the evolution equation (30) for the configuration tensor, or its convected form (33), and the base pole balance equation (59). Making use of the constitutive equation (96) to eliminate the base pole generalized kinetic moment $\mathbf{p_o}$ and working with a constant-in-time convected local generalized inertia tensor $\overline{\mathbf{M}}_{\mathbf{x}_P}$, we get the ordinary differential equation (ODE) system

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{C}_P = \mathbf{C}_P \overline{\mathbf{w}}_{\mathbf{x}_P} \times\!\!\times, \tag{113}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathbf{C}_P^{-\mathrm{T}} \overline{\mathbf{M}}_{\mathbf{x}_P} \overline{\mathbf{w}}_{\mathbf{x}_P}\right) = \mathbf{f_o}. \tag{114}$$

These equations define an initial value problem (IVP) for variables $(\mathbf{C}_P, \overline{\mathbf{w}}_{\mathbf{x}_P})$ in the phase space $\mathrm{SR}(\mathbb{K}^6) \times \mathbb{K}^{6*}$.

Let us remark now that the equations of motion (113) and (114)

- are cast in base pole form, to enjoy the simplest possible differential operations on momenta;
- enforce the constitutive law in a way that does not require the update of the generalized inertia tensor;
- exhibit a striking resemblance to the pure rotational motion equations (i.e., motion around a fixed point), namely

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{\alpha}_\Omega = \boldsymbol{\alpha}_\Omega \overline{\boldsymbol{\varpi}}_{\mathbf{x}_\Omega} \times, \tag{115}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\boldsymbol{\alpha}_\Omega \overline{\mathbf{J}}_{\mathbf{x}_\Omega} \overline{\boldsymbol{\varpi}}_{\mathbf{x}_\Omega}\right) = \mathbf{m}_{\mathbf{x}_\Omega}, \tag{116}$$

  where $\Omega \in \mathscr{B}$ represents the body particle whose position $\mathbf{x}_\Omega$ is the fixed point of the motion and $\boldsymbol{\alpha}_\Omega$ is defined by an orthonormal triad with origin in $\mathbf{x}_\Omega$.

The last remark highlights the common structure of the pure rotational motion and the complete rigid motion problems. In fact, they can be both described as motions on Lie groups: the first, $\mathrm{SO}(\mathbb{E}^3)$, being a subgroup of the second, $\mathrm{SR}(\mathbb{K}^6) \sim \mathbb{E}^3 \times \mathrm{SO}(\mathbb{E}^3)$.

The strategy represented by Eqs. (113) and (114) seems to offer the best way to take advantage of the results achieved thus far, using the simplest representation for each ingredient in the framework. Note that one may choose to solve for the convected generalized kinetic moment $\overline{\mathbf{p}}_{\mathbf{x}_P}$ or for the convected generalized velocity $\overline{\mathbf{w}}_{\mathbf{x}_P}$ via the convected balance equation

$$\dot{\overline{\mathbf{p}}}_{\mathbf{x}_P} + \overline{\mathbf{w}}_{\mathbf{x}_P} \times\!\!\times \overline{\mathbf{p}}_{\mathbf{x}_P} = \overline{\mathbf{f}}_{\mathbf{x}_P}. \tag{117}$$

We do not favor this approach since, in view of the algorithmic schemes discussed in Part II of this work, it does not lead to kinetic moment and energy conservation in a straightforward manner. Note that equation (59) is nothing else than the 6-D version of the classical *Euler's equation* for the rigid body,

$$\dot{\overline{\mathbf{l}}} + \overline{\boldsymbol{\varpi}} \times \overline{\mathbf{l}} = \overline{\mathbf{n}}, \tag{118}$$

$$\dot{\overline{\mathbf{h}}}_{\mathbf{x}_P} + \overline{\boldsymbol{\varpi}} \times \overline{\mathbf{h}}_{\mathbf{x}_P} + \overline{\mathbf{v}}_{\mathbf{x}_P} \times \overline{\mathbf{l}} = \overline{\mathbf{m}}_{\mathbf{x}_P}, \tag{119}$$

which are often used as basis for the implementation of rigid body dynamic simulation algorithms.

3.2. Geometrically non-linear beam dynamics

Beam configuration. We define a *beam* as the solid $\mathscr{B}$ generated by the rigid motion in space of a reference "cross-section" $\mathscr{S}$, which is a region of a plane in $\mathscr{E}^3$. Such a motion is parameterized by the *material abscissa* $s \in [0, S]$. In other words, the material particles, whose images are the cross-sections along the beam, are in a one-to-one correspondence with the values of the material abscissa, so that in the following we will refer to s instead of P and to $[0, S]$ instead of $\mathscr{B}$.

By attaching a frame $(\mathbf{x}(s), \{\mathbf{e}_k(s)\}_{k=1,2,3})$ to the current cross-section, we are able to extend the framework already established to the description of beam kinematics. The configuration of the beam is characterized by the pair $(\mathbf{u}, \boldsymbol{\alpha})$, both smooth functions of s. We may look at this model in terms of a mono-dimensional Cosserat continuum, featuring a motion map $\mathsf{X} : [0, S] \times [0, T] \to \mathscr{E}^3 \times \mathrm{SO}(\mathbb{E}^3)$ associating a frame to a particle at a given material abscissa and time instant (s, t).

Point $\mathbf{x}(s)$ describes a smooth curve in $\mathscr{E}^3$, the *beam axis*, and its tangent vector $\boldsymbol{\tau}_\mathbf{x}$ is the derivative with respect to parameter s,

$$\boldsymbol{\tau}_\mathbf{x} := \mathbf{u}'. \tag{120}$$

The apex is used to indicate derivatives with respect to s. The orientation tensor $\boldsymbol{\alpha}(s)$ associated to the frame describes a one-parameter family of orthogonal transformations, and its derivative with respect to parameter s is given by

$$\boldsymbol{\alpha}' = \boldsymbol{\kappa} \times \boldsymbol{\alpha}, \tag{121}$$

or, in convected form, by

$$\boldsymbol{\alpha}' = \boldsymbol{\alpha}\, \overline{\boldsymbol{\kappa}} \times, \tag{122}$$

where the rate vectors $\boldsymbol{\kappa}$ and $\overline{\boldsymbol{\kappa}} := \boldsymbol{\alpha}^{-1}\boldsymbol{\kappa}$ are, respectively, the *curvature* and its convected image. To preserve the physical meaning of the quantities involved, we place a restriction on the orientation, requiring that $\boldsymbol{\tau}'_\mathbf{x} \cdot \mathbf{e}_3 > 0,\ \forall s \in [0, S]$.

In analogy to what has been done for the time derivatives, we consider the space convective derivative [5] of the position vector $\mathbf{u}$,

$$\boldsymbol{\tau}_\mathbf{o} := \mathbf{u}^\diamond, \tag{123}$$

and recognize it as the tangent vector reduced to the base pole,

$$\boldsymbol{\tau}_\mathbf{o} = \boldsymbol{\tau}_\mathbf{x} + (\mathbf{x} - \mathbf{o}) \times \boldsymbol{\kappa}. \tag{124}$$

It should be clear from these initial definitions that the description of the beam configuration in space is formally similar to the description of the motion of a rigid body (*Kirchhoff's analogy*). In fact, we pass to the 6-D representation by defining the 6-D vectors

$$\boldsymbol{\chi}_\mathbf{o} := (\boldsymbol{\tau}_\mathbf{o}, \boldsymbol{\kappa}), \qquad \overline{\boldsymbol{\chi}}_\mathbf{x} := (\overline{\boldsymbol{\tau}}_\mathbf{x}, \overline{\boldsymbol{\kappa}}), \tag{125}$$

termed the *base pole generalized curvature* $\boldsymbol{\chi}_\mathbf{o}$ and the *convected generalized curvature* $\overline{\boldsymbol{\chi}}_\mathbf{x}$, respectively. Again, the configuration tensor transforms $\overline{\boldsymbol{\chi}}_\mathbf{x}$ in $\boldsymbol{\chi}_\mathbf{o}$,

$$\boldsymbol{\chi}_\mathbf{o} = \mathbf{C}\, \overline{\boldsymbol{\chi}}_\mathbf{x}, \tag{126}$$

and the rate of change of the configuration tensor $\mathbf{C}$ with respect to s is

$$\mathbf{C}' = \boldsymbol{\chi}_\mathbf{o} \times \mathbf{C} \tag{127}$$

or, in convected form,

$$\mathbf{C}' = \mathbf{C}\, \overline{\boldsymbol{\chi}}_\mathbf{x} \times. \tag{128}$$

[5] Convective derivatives with respect to s are defined as $\bullet^\diamond := \boldsymbol{\alpha}\, \overline{\bullet}'$ in the case of a vector $\bullet \in \mathbb{E}^3$, and $\bullet^\diamond := \boldsymbol{\alpha}\, \overline{\bullet}'\, \boldsymbol{\alpha}^{-1}$ in the case of a tensor $\bullet \in \mathrm{Lin}(\mathbb{E}^3)$. We make use of the diamond apex to indicate convective derivatives with respect to space.

By taking mixed second derivatives of tensor $\mathbf{C}$ with respect to (s,t), we obtain that the base pole generalized velocity and the base pole generalized curvature are related by the compatibility condition

$$\dot{\boldsymbol{\chi}}_{\mathbf{o}} = \mathbf{w}'_{\mathbf{o}} + \mathbf{w}_{\mathbf{o}} \times \boldsymbol{\chi}_{\mathbf{o}}. \tag{129}$$

This equation plays an important role in the algorithmic developments, as it will be shown in Part II of this work. Its convected counterpart reads

$$\dot{\overline{\boldsymbol{\chi}}}_{\mathbf{x}} = \overline{\mathbf{w}}'_{\mathbf{x}} - \overline{\mathbf{w}}_{\mathbf{x}} \times \overline{\boldsymbol{\chi}}_{\mathbf{x}}. \tag{130}$$

Beam deformation. Consider now a constant-in-time *reference configuration* for the beam $\mathscr{B}$, denoted by the subscript N. A comparison between the current configuration $\mathbf{C}(s,t)$ and the reference configuration $\mathbf{C}_N(s)$, for which

$$\mathbf{C}'_N = \boldsymbol{\chi}_{N_{\mathbf{o}}} \times \mathbf{C}_N \tag{131}$$

holds, allows us to define convenient measures of linear and angular strains that are not influenced by any global rigid displacement of the beam. In fact, by looking at tensor $\mathbf{U}$, the *displacement tensor from the reference configuration*, defined as

$$\mathbf{U} := \mathbf{C}\,\mathbf{C}_N^{-1}, \tag{132}$$

we find that its derivative with respect to s is characterized by the 6-D vector $\boldsymbol{\varepsilon}_{\mathbf{o}}$,

$$\mathbf{U}' = \boldsymbol{\varepsilon}_{\mathbf{o}} \times \mathbf{U}. \tag{133}$$

We term $\boldsymbol{\varepsilon}_{\mathbf{o}}$ the *base pole generalized strain* and define it as

$$\boldsymbol{\varepsilon}_{\mathbf{o}} := \boldsymbol{\chi}_{\mathbf{o}} - \mathbf{U}\,\boldsymbol{\chi}_{N_{\mathbf{o}}}. \tag{134}$$

The base pole generalized strain is the desired deformation measure. In fact, when the displacement that brings $\mathbf{C}_N$ in $\mathbf{C}$ is rigid, i.e., independent from the material abscissa s, the deformation represented by the base pole generalized strain vanishes.

Note that Eq. (134) expresses the fact that the moving frame components of the generalized strain are the difference between the moving frame components of the local generalized curvature in the current configuration and the reference frame components of the local generalized curvature in the reference configuration. This is evident by looking at the convected counterpart of Eq. (134), given by

$$\overline{\boldsymbol{\varepsilon}}_{\mathbf{x}} := \overline{\boldsymbol{\chi}}_{\mathbf{x}} - \overline{\boldsymbol{\chi}}_{N_{\mathbf{x}_N}}, \tag{135}$$

with $\overline{\boldsymbol{\chi}}_{\mathbf{x}} := \mathbf{C}^{-1}\boldsymbol{\chi}_{\mathbf{o}}$, $\overline{\boldsymbol{\chi}}_{N_{\mathbf{x}_N}} := \mathbf{C}_N^{-1}\boldsymbol{\chi}_{N_{\mathbf{o}}}$, and $\overline{\boldsymbol{\varepsilon}}_{\mathbf{x}} := \mathbf{C}^{-1}\boldsymbol{\varepsilon}_{\mathbf{o}}$. From Eqs. (129) and (130) it can be easily proved that the generalized strain $\boldsymbol{\varepsilon}_{\mathbf{o}}$ satisfies the compatibility equation

$$\dot{\boldsymbol{\varepsilon}}_{\mathbf{o}} = \mathbf{w}'_{\mathbf{o}} + \mathbf{w}_{\mathbf{o}} \times \boldsymbol{\varepsilon}_{\mathbf{o}} \tag{136}$$

and its convected form

$$\dot{\overline{\boldsymbol{\varepsilon}}}_{\mathbf{x}} = \overline{\mathbf{w}}'_{\mathbf{x}} - \overline{\mathbf{w}}_{\mathbf{x}} \times \overline{\boldsymbol{\varepsilon}}_{\mathbf{x}}, \tag{137}$$

since the reference configuration quantities $\mathbf{C}_N$ and $\boldsymbol{\chi}_{N_{\mathbf{o}}}$ are time-independent.

The geometrical interpretation of the generalized strain vector is straightforward. In fact, $\boldsymbol{\varepsilon}_{\mathbf{x}_\mathrm{L}} \cdot \mathbf{e}_h = \overline{\boldsymbol{\varepsilon}}_{\mathbf{x}_\mathrm{L}} \cdot \mathbf{i}_h$, $(h = 1,2)$ and $\boldsymbol{\varepsilon}_{\mathbf{x}_\mathrm{L}} \cdot \mathbf{e}_3 = \overline{\boldsymbol{\varepsilon}}_{\mathbf{x}_\mathrm{L}} \cdot \mathbf{i}_3$ represent the shear and axial strains, respectively, while $\boldsymbol{\varepsilon}_{\mathbf{x}_\mathrm{A}} \cdot \mathbf{e}_h = \overline{\boldsymbol{\varepsilon}}_{\mathbf{x}_\mathrm{A}} \cdot \mathbf{i}_h$, $(h = 1,2)$ and $\boldsymbol{\varepsilon}_{\mathbf{x}_\mathrm{A}} \cdot \mathbf{e}_3 = \overline{\boldsymbol{\varepsilon}}_{\mathbf{x}_\mathrm{A}} \cdot \mathbf{i}_3$ represent the flexural and torsional strains, respectively.

Constitutive equations. Following a frequently used approach in the analysis of hyper-elastic continua (those characterized by a linear relationship between stresses and strains), we postulate the existence of two frame-indifferent scalar quantities, the *kinetic energy*, T, and the *deformation energy*, U, defined as

$$T := \int_0^S T_l\,\mathrm{d}s, \qquad U := \int_0^S U_l\,\mathrm{d}s. \tag{138}$$

The linear densities $T_l(s)$ and $U_l(s)$ depend on local quantities in such a way that their Hessians with respect to $\overline{\mathbf{w}}_\mathbf{x}$ and $\overline{\boldsymbol{\varepsilon}}_\mathbf{x}$ define constant-in-time, symmetric, positive definite tensors. These tensors are the linear density of the *convected generalized inertia tensor* $\overline{\mathbf{M}}_{l_\mathbf{x}}$ and the *convected generalized stiffness tensor* $\overline{\mathbf{K}}_\mathbf{x}$:

$$\overline{\mathbf{M}}_{l_\mathbf{x}} := \frac{\partial^2}{\partial \overline{\mathbf{w}}_\mathbf{x}^2} T_l, \qquad \overline{\mathbf{K}}_\mathbf{x} := \frac{\partial^2}{\partial \overline{\boldsymbol{\varepsilon}}_\mathbf{x}^2} U_l. \tag{139}$$

The energy densities are then expressed by the following expressions:

$$T_l = \frac{1}{2}\,\overline{\mathbf{p}}_{l_\mathbf{x}} \cdot \overline{\mathbf{w}}_\mathbf{x}, \qquad U_l = \frac{1}{2}\,\overline{\mathbf{c}}_\mathbf{x} \cdot \overline{\boldsymbol{\varepsilon}}_\mathbf{x}, \tag{140}$$

where the linear density of *convected generalized kinetic moment* $\overline{\mathbf{p}}_l$ and the *convected generalized internal force* $\overline{\mathbf{c}}_\mathbf{x}$ are defined as

$$\overline{\mathbf{p}}_{l_\mathbf{x}} := \frac{\partial}{\partial \overline{\mathbf{w}}_\mathbf{x}} T_l, \qquad \overline{\mathbf{c}}_\mathbf{x} := \frac{\partial}{\partial \overline{\boldsymbol{\varepsilon}}_\mathbf{x}} U_l, \tag{141}$$

respectively. The equivalent forms

$$T_l = \frac{1}{2}\,\mathbf{p}_{l_\mathbf{o}} \cdot \mathbf{w}_\mathbf{o}, \qquad U_l = \frac{1}{2}\,\mathbf{c}_\mathbf{o} \cdot \boldsymbol{\varepsilon}_\mathbf{o} \tag{142}$$

hold, similarly to the rigid body case. Thus,

$$\mathbf{p}_{l_\mathbf{o}} = \mathbf{M}_{l_\mathbf{o}}\mathbf{w}_\mathbf{o}, \qquad \mathbf{c}_\mathbf{o} = \mathbf{K}_\mathbf{o}\boldsymbol{\varepsilon}_\mathbf{o}, \tag{143}$$

represent the base pole constitutive laws for the linear density of generalized kinetic moment and for the generalized internal force.

Differential form of the balance equations. In the case of a beam, the base pole balance equation (59) may be used, under suitable regularity assumptions for the quantities involved, to derive the differential balance equations reduced to the base pole.

In fact, the system of forces acting on the beam are given by the sum of two contributions: (1) the integral of the "generalized body force" $\mathbf{b}_\mathbf{o}$ acting on each section along the beam, and (2) the sum of the "generalized contact forces" $\mathbf{c}_\mathbf{o}$ acting at the beam boundary, i.e., the initial and final sections $s = 0, S$. Hence, we define the base pole generalized force as the sum

$$\mathbf{f}_\mathbf{o} := \int_0^S \mathbf{b}_\mathbf{o}\,\mathrm{d}s + \mathbf{c}_\mathbf{o}\big|_0^S. \tag{144}$$

The base pole balance equation (59) is then equivalent to

$$\int_0^S \left(\dot{\mathbf{p}}_{l_\mathbf{o}} - \mathbf{c}'_\mathbf{o} - \mathbf{b}_\mathbf{o}\right)\mathrm{d}s = \mathbf{0}. \tag{145}$$

Setting the integrand to vanish identically entails the *differential form* of the base pole balance equation as

$$\dot{\mathbf{p}}_{l_\mathbf{o}} - \mathbf{c}'_\mathbf{o} = \mathbf{b}_\mathbf{o}. \tag{146}$$

Note, similarly to the rigid body case, the remarkable simplicity of the preceding equation when compared to the differential form of the convected local balance equation,

$$\left(\dot{\overline{\mathbf{p}}}_{l_\mathbf{x}} + \overline{\mathbf{w}}_\mathbf{x} \times \overline{\mathbf{p}}_{l_\mathbf{x}}\right) - \left(\overline{\mathbf{c}}'_\mathbf{x} + \overline{\boldsymbol{\chi}}_\mathbf{x} \times \overline{\mathbf{c}}_\mathbf{x}\right) = \overline{\mathbf{b}}_\mathbf{x}. \tag{147}$$

This equation represents the 6-D extension of the 3-D equations usually adopted in the literature.

Energy balance. Consider the energy balance for the beam. We shall follow similar procedures to those already carried out for the rigid body, Conjugating equation (146) with the base pole generalized velocity $\mathbf{w}_\mathbf{o}$ on both sides and integrating along the beam yields

$$\int_0^S \left(\dot{\mathbf{p}}_{l_o} - \mathbf{c}_o' - \mathbf{b}_o\right) \cdot \mathbf{w}_o \, ds = 0 \tag{148}$$

or, after integration by parts,

$$\int_0^S \dot{\mathbf{p}}_{l_o} \cdot \mathbf{w}_o \, ds + \int_0^S \mathbf{c}_o \cdot \mathbf{w}_o' \, ds = \int_0^S \mathbf{b}_o \cdot \mathbf{w}_o \, ds + (\mathbf{c}_o \cdot \mathbf{w}_o)|_0^S. \tag{149}$$

The right-hand side is the sum of the *power of the external forces* $W := W^B + W^C$, defined as the sum of the *power of body forces*,

$$W^B := \int_0^S \mathbf{b}_o \cdot \mathbf{w}_o \, ds, \tag{150}$$

and the *power of contact forces*,

$$W^C := (\mathbf{c}_o \cdot \mathbf{w}_o)|_0^S. \tag{151}$$

The left-hand side is the time derivative of the total mechanical energy $E := T + U$. In fact, the following remarkable equations hold:

$$\dot{T}_l = \dot{\mathbf{p}}_{l_o} \cdot \mathbf{w}_o, \qquad \dot{U}_l = \mathbf{c}_o \cdot \mathbf{w}_o'. \tag{152}$$

Note that the first may be obtained in the same way as for the rigid body case, while the second relies on the compatibility equation (136).

Given the above equations, the energy balance equation (149) yields

$$\dot{E} = W, \tag{153}$$

or the *theorem of total mechanical energy conservation* for the beam.

Equations of motion. In the 6-D representation, the equations governing the general motion of the beam are given by the equation of evolution in time (30) for the configuration tensor, the equation for its spatial variation (127), or their respective convected forms (33) and (128), and the differential base pole balance equation (146). We make use of the constitutive equations (143) to eliminate the base pole generalized kinetic moment $\mathbf{p}_{l_o}$ and the base pole generalized internal force $\mathbf{c}_o$, in order to work with the constant-in-time convected generalized inertia tensor $\overline{\mathbf{M}}_{l_x}$ and the convected generalized stiffness tensor $\overline{\mathbf{K}}_{l_x}$. We get the following system of partial differential equations (PDE),

$$\frac{\partial}{\partial t}\mathbf{C} = \mathbf{C}\,\overline{\mathbf{w}}_x \times\!\!\!\cdot, \tag{154}$$

$$\frac{\partial}{\partial s}\mathbf{C} = \mathbf{C}\,\overline{\boldsymbol{\chi}}_x \times\!\!\!\cdot, \tag{155}$$

$$\frac{\partial}{\partial t}\left(\mathbf{C}^{-T}\overline{\mathbf{M}}_{l_x}\overline{\mathbf{w}}_x\right) - \frac{\partial}{\partial s}\left(\mathbf{C}^{-T}\overline{\mathbf{K}}_x\left(\overline{\boldsymbol{\chi}}_x - \overline{\boldsymbol{\chi}}_{N_{x_N}}\right)\right) = \mathbf{b}_o. \tag{156}$$

These equations may be complemented by suitable initial and boundary conditions so that, when further coupled with the compatibility equation (130) to eliminate $\overline{\boldsymbol{\chi}}_x$, they define an initial/boundary value problem (IBVP) for the variables $(\mathbf{C}_P, \overline{\mathbf{w}}_x)$ in the phase space $\mathrm{SR}(\mathbb{K}^6) \times \mathbb{K}^{6*}$.

Eqs. (154)–(156) may be subjected to a *semi-discretization* process, or discretization in space, to finally yield a system of s in the time variable only. This may be done through several approaches, such as enforcing finite element (FEM) or finite volume (FVM) processes. In any case, a weak form is derived from the partial differential base pole equation of dynamic balance (146) by conjugating with suitable test functions $\boldsymbol{\pi}$ and integrating along the beam,

$$\int_0^S \boldsymbol{\pi} \cdot \left(\dot{\mathbf{p}}_{l_o} - \mathbf{c}_o' - \mathbf{b}_o\right) ds = 0. \tag{157}$$

Then, integrating by parts one gets the desired ODE

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^S \boldsymbol{\pi}\cdot\mathbf{p}_{\mathrm{l_o}}\,\mathrm{d}s+\int_0^S \boldsymbol{\pi}'\cdot\mathbf{c}_{\mathbf{o}}\,\mathrm{d}s=\int_0^S \boldsymbol{\pi}\cdot\mathbf{b}_{\mathbf{o}}\,\mathrm{d}s+(\boldsymbol{\pi}\cdot\mathbf{c}_{\mathbf{o}})|_0^S. \tag{158}$$

In Part II of this work we shall start from this equation, together with Eq. (154), to derive unconditionally stable schemes for non-linear beam dynamics.

4. Constrained systems dynamics

In this section we extend the presented formulation to the problem of constrained mechanical systems, composed by rigid and/or flexible bodies assembled via mechanical joints. The description of these joints is developed in terms of quantities describing the relative kinematics between frames. We derive the complete set of equations of motion for a general multibody system via the Lagrange multiplier technique, enforcing the constraints by direct assembling of their descriptive algebraic equations together with the equations of motion.

Various mechanical joints of common application are formulated starting from simple prototypal cases. The equations for the configuration-level and the velocity-level constraints are given with respect to convenient measures of linear and angular displacements and velocities. We shall make use of these formulæ to derive suitable approximations in Part II of this work.

4.1. Relative kinematics

Relative frame configuration. Consider two frames $(\mathbf{x}_A,\{\mathbf{e}_{A_k}\}_{k=1,2,3})$ and $(\mathbf{x}_B,\{\mathbf{e}_{B_k}\}_{k=1,2,3})$, whose configuration with respect to the base frame $(\mathbf{o},\{\mathbf{i}_k\}_{k=1,2,3})$ is represented by the configuration tensors $\mathbf{C}_A$ and $\mathbf{C}_B$, corresponding to the configuration pairs $(\mathbf{u}_A,\boldsymbol{\alpha}_A)$ and $(\mathbf{u}_B,\boldsymbol{\alpha}_B)$, all smooth functions of time.

In the 6-D framework we are led to consider the relative configuration of the frame B with respect to frame A by means of the *displacement tensor* $\mathbf{D}\in \mathrm{SR}(\mathbb{K}^6)$ defined as

$$\mathbf{D}:=\mathbf{C}_B\,\mathbf{C}_A^{-1}. \tag{159}$$

The structural decomposition of this quantity is given by

$$\mathbf{D}=\mathscr{T}(\mathbf{t})\,\mathscr{A}(\mathbf{R}), \tag{160}$$

where $\mathbf{t}\in\mathbb{E}^3$ denotes the *translation vector* defined as

$$\mathbf{t}:=\mathbf{u}_B-\mathbf{R}\,\mathbf{u}_A, \tag{161}$$

while $\mathbf{R}\in\mathrm{SO}(\mathbb{E}^3)$ is the *rotation tensor* defined as

$$\mathbf{R}:=\boldsymbol{\alpha}_B\,\boldsymbol{\alpha}_A^{-1}. \tag{162}$$

We stress the global character intrinsic to the displacement tensor $\mathbf{D}$. In fact, it completely defines the relative configuration of the two frames, accounting for both linear and angular relative displacements (in particular, it defines the relative configuration of two rigid bodies).

The pair $(\mathbf{t},\mathbf{R})$ is a representation of the rigid transformation $\boldsymbol{D}(\bullet):=\mathbf{o}+\mathbf{t}+\mathbf{R}\,(\bullet-\mathbf{o})\in\mathscr{E}^3$ that brings frame A into frame B, or

$$\boldsymbol{D}(\bullet)=\mathbf{x}_A+\mathbf{R}\,(\bullet-\mathbf{o}). \tag{163}$$

This map transforms the space $\mathscr{E}^3$ as seen by an observer connected with frame A into that seen by an observer connected with frame B. Clearly, $\boldsymbol{D}=\boldsymbol{C}_B\circ\boldsymbol{C}_A^{-1}$, where the symbol $\circ$ denotes the composition of maps, so that the relationship between $\mathbf{D}$ and $\boldsymbol{D}$ corresponds exactly to that between $\mathbf{C}$ and $\boldsymbol{C}$.

Note that the *displacement vector* $\Delta\mathbf{u}\in\mathbb{E}^3$ defined by

$$\Delta\mathbf{u}:=\mathbf{u}_B-\mathbf{u}_A, \tag{164}$$

does not play an explicit role in the 6-D representation based on tensor $\mathbf{D}$, Eq. (160). In fact, the relative translation $\mathbf{t}$ represents a global measure of the linear displacement, while $\Delta\mathbf{u}$ does not. To prove this, let us take any point $\mathbf{y}_A$ rigidly connected to frame A, and its image $\mathbf{y}_B$ under the rigid displacement that takes frame A into frame B. The rigidity of the connection implies that $(\mathbf{y}_B - \mathbf{x}_B) = \mathbf{R}(\mathbf{y}_A - \mathbf{x}_A)$, so that

$$(\mathbf{y}_B - \mathbf{o}) - \mathbf{R}(\mathbf{y}_A - \mathbf{o}) = (\mathbf{y}_B - \mathbf{x}_B) - \mathbf{R}(\mathbf{y}_A - \mathbf{x}_A) + (\mathbf{x}_B - \mathbf{o}) - \mathbf{R}(\mathbf{x}_A - \mathbf{o}), \tag{165}$$

$$= (\mathbf{x}_B - \mathbf{o}) - \mathbf{R}(\mathbf{x}_A - \mathbf{o}), \tag{166}$$

$$= \mathbf{t}. \tag{167}$$

The displacement tensor $\mathbf{D}$, or equivalently the pair $(\mathbf{t}, \mathbf{R})$, allow for a meaningful description of the relative configuration. However, another description may be set considering the quantity $\overline{\mathbf{D}}$ defined as

$$\overline{\mathbf{D}} := \mathbf{C}_A^{-1}\mathbf{C}_B. \tag{168}$$

Note that this tensor may be obtained by evaluating the generalized convected image of the displacement tensor $\mathbf{D}$ with respect to frame A or frame B, equivalently:

$$\overline{\mathbf{D}} = \mathbf{C}_A^{-1}\mathbf{D}\,\mathbf{C}_A, \tag{169}$$

$$= \mathbf{C}_B^{-1}\mathbf{D}\,\mathbf{C}_B. \tag{170}$$

The preceding equations allow us to term tensor $\overline{\mathbf{D}}$ the *convected displacement tensor* without specification of the frame (A or B) with respect to which the generalized convection is performed. This also justifies our notation employing the overbar, with no need of further indications.

The structural decomposition of the convected displacement tensor is given by

$$\overline{\mathbf{D}} = \mathscr{T}(\boldsymbol{\alpha}_A^{-1}\Delta\mathbf{u})\,\mathscr{A}(\overline{\mathbf{R}}), \tag{171}$$

featuring frame A convected image of the displacement vector $\Delta\mathbf{u}$ and the *convected rotation tensor* $\overline{\mathbf{R}}$. This is the 3-D analog to $\overline{\mathbf{D}}$, defined as

$$\overline{\mathbf{R}} := \boldsymbol{\alpha}_A^{-1}\boldsymbol{\alpha}_B, \tag{172}$$

and equivalently obtained as the convected image of the rotation tensor $\mathbf{R}$ with respect to frames A or B,

$$\overline{\mathbf{R}} = \boldsymbol{\alpha}_A^{-1}\mathbf{R}\,\boldsymbol{\alpha}_A, \tag{173}$$

$$= \boldsymbol{\alpha}_B^{-1}\mathbf{R}\,\boldsymbol{\alpha}_B. \tag{174}$$

Note that the scalar components $\{\overline{R}_{hk}\}_{h,k=1,2,3}$ of the convected rotation tensor $\overline{\mathbf{R}}$ with respect to the base triad $\{\mathbf{i}_k\}_{k=1,2,3}$ form the matrix of the *direction cosines* of frame B with respect to frame A,

$$\overline{\mathbf{R}} = (\mathbf{e}_{A_h} \cdot \mathbf{e}_{B_k})\,\mathbf{i}_h \otimes \mathbf{i}_k. \tag{175}$$

The interest in the use of the convected version $\overline{\mathbf{D}}$ of the displacement tensor is easily explained by the fact that it is a constant-in-time quantity whenever frames A and B are rigidly connected, that is, whenever they move as parts of a single rigid system. Indeed, in such a case both the A-frame components of the displacement vector $\mathbf{e}_{A_k} \cdot \Delta\mathbf{u} = \mathbf{i}_k \cdot (\boldsymbol{\alpha}_A^{-1}\mathbf{u})$ and the direction cosines $\overline{R}_{hk} = \mathbf{i}_h \cdot \overline{\mathbf{R}}\,\mathbf{i}_k$ are all constant-in-time quantities. This aspects inspire the use of the convected displacement tensor $\overline{\mathbf{D}}$ as a convenient measure to describe the relative configuration, especially in the case of constrained systems. Another interesting point arises when we look at the time derivative of $\overline{\mathbf{D}}$.

Relative frame velocity. It should be clear that, in order to obtain a measure of relative velocity between frames endowed with a certain degree of invariance, we must deal with base pole velocities. In fact, such a measure would not suffer from the specific choice of the pole and triad, and therefore would be the same for any two frames rigidly connected to the original ones. This consideration inspires the definition of the *relative generalized velocity* of frame B with respect to frame A, as the vector $\Delta\mathbf{w_o} = (\Delta\mathbf{v_o}, \Delta\boldsymbol{\omega}) \in \mathbb{K}^6$ given by

$$\Delta\mathbf{w_o} := \mathbf{w}_{B_o} - \mathbf{w}_{A_o}. \tag{176}$$

The above definition identifies a unique relative generalized velocity for any two frames rigidly connected to frames A and B. Such a remarkable property relies on the notion of base pole reduction and is clearly lost when one assumes as measure of relative velocity the plain difference of the local generalized velocities of frames A and B.

The intrinsic meaning of the relative generalized velocity $\Delta\mathbf{w_o}$ becomes even clearer when we consider the time derivative of the convected displacement tensor. This can be easily computed as

$$\dot{\overline{\mathbf{D}}} = (\mathbf{C}_A^{-1}\Delta\mathbf{w_o})\times\overline{\mathbf{D}}. \tag{177}$$

As it is apparent, the kinematic vector characterizing the rate of change of $\overline{\mathbf{D}}$ is the A-frame generalized convected picture of the relative generalized velocity $\Delta\mathbf{w_o}$. Note that in 3-D terms we have the corresponding expression $\dot{\overline{\mathbf{R}}} = (\boldsymbol{\alpha}_A^{-1}\Delta\boldsymbol{\omega}) \times \overline{\mathbf{R}}$ for the derivative of the convected rotation tensor $\overline{\mathbf{R}}$. It is apparent from Eq. (177) that the relative generalized velocity $\Delta\mathbf{w_o}$ vanishes whenever frames A and B move as they were rigidly connected, and therefore it represents a very practical quantity from the point of view of applications.

On the other hand, the time derivative of the displacement tensor $\mathbf{D}$ defines a different measure of relative velocity, that may be termed the *co-rotational relative generalized velocity* $\boldsymbol{\varpi}_\mathbf{o} \in \mathbb{K}^6$,

$$\dot{\mathbf{D}} = \boldsymbol{\varpi}_\mathbf{o}\times\mathbf{D}. \tag{178}$$

Its expression is given by $\boldsymbol{\varpi}_\mathbf{o} := \mathbf{w}_{B_o} - \mathbf{D}\,\mathbf{w}_{A_o}$. The 3-D corresponding expression is given by $\dot{\mathbf{R}} = \boldsymbol{v} \times \mathbf{R}$, with $\boldsymbol{v} := \boldsymbol{\omega}_B - \mathbf{R}\,\boldsymbol{\omega}_A$, and $\boldsymbol{\varpi}_\mathbf{o} = (\boldsymbol{\eta}_\mathbf{o}, \boldsymbol{v})$, where $\boldsymbol{\eta}_\mathbf{o} := \dot{\mathbf{t}} + \mathbf{t} \times \boldsymbol{v}$. We shall not refer to $\boldsymbol{\varpi}_\mathbf{o}$ and its related quantities in the following, favoring instead the use of the relative generalized velocity $\Delta\mathbf{w_o}$.

4.2. Mechanical constraints between two bodies

Basic characterization of constraints. We consider now two generic bodies $\mathscr{B}_A$ and $\mathscr{B}_B$, and assume two *material frames* $(\mathbf{x}_A, \{\mathbf{e}_{A_k}\}_{k=1,2,3})$ and $(\mathbf{x}_B, \{\mathbf{e}_{B_k}\}_{k=1,2,3})$ to describe their relative motion. Generally speaking, we must consider two different classes of kinematic constraints between frames A and B:

1. the *holonomic* or *configuration-level constraints*, which impose limitations on their relative configuration, and
2. the *non-holonomic* or *velocity-level constraints*, which impose limitations on their relative generalized velocities.

In the framework proposed here, the equation that defines a holonomic time-dependent constraint may be written as

$$\boldsymbol{\phi}(\overline{\mathbf{D}}, t) = \mathbf{0}_K, \tag{179}$$

while the equation corresponding to a non-holonomic time-dependent constraint, given Eq. (177), reads

$$\boldsymbol{\psi}(\overline{\mathbf{D}}, \mathbf{C}_A^{-1}\Delta\mathbf{w_o}, t) = \mathbf{0}_K, \tag{180}$$

where $\boldsymbol{\phi} : \mathrm{SR}(\mathbb{K}^6) \times [0, T] \to \mathbb{R}^K$ and $\boldsymbol{\psi} : \mathbb{K}^6 \times \mathrm{SR}(\mathbb{K}^6) \times [0, T] \to \mathbb{R}^K$ are the constraint functions, to be specified by the particular constraint at hand. [6] The scalar K is the *number of constrained degrees of freedom*, which depends on the specific constraint type, ranging from $K = 0$ (the unconstrained case, i.e., the frames are free to move with respect to each other) to $K = 6$ (the totally constrained case, i.e., the frames are clamped to each other).

[6] Both of the above equations refer to *bilateral* constraints. The case of *unilateral* constraints, when the equal signs in Eqs. (179), (180) are replaced by inequality signs, is not addressed in this work.

In the following, we shall concentrate our attention on holonomic constraints. For these objects, both a configuration-level and a velocity-level formulation is available. In fact, the derivation of a configuration-level constraint with respect to time naturally yields a velocity-level constraint. [7] Given the configuration-level expression (179) of a holonomic constraint, its time derivative can be put in the following form

$$\mathbf{A}_{\mathbf{o}}^{\mathrm{T}} \Delta \mathbf{w}_{\mathbf{o}} + \mathbf{a} = \mathbf{0}_{K}, \tag{181}$$

using the relative generalized velocity $\Delta \mathbf{w}_{\mathbf{o}}$. Tensor $\mathbf{A}_{\mathbf{o}} : \mathbb{R}^{K} \to \mathbb{K}^{6}$ is termed the *base pole constraint tensor* and is related to the partial derivative of the constraint function $\boldsymbol{\phi}$ with respect to the convected displacement tensor $\overline{\mathbf{D}}$. Vector $\mathbf{a} \in \mathbb{R}^{K}$ is termed the *constraint velocity*, since it is simply the partial derivative of the constraint function $\boldsymbol{\phi}$ with respect to time t.

The constraint tensor $\mathbf{A}_{\mathbf{o}}$ defines the base pole reaction forces that arise in the case of *ideal* constraints. These are defined as constraints whose reactions perform null work for any admissible virtual displacement. For an ideal, bilateral, holonomic constraint, the base pole generalized reaction forces $\mathbf{f}_{\mathbf{o}}^{\mathrm{R}}$ are then expressed through a *Lagrange multiplier vector* $\boldsymbol{\lambda} \in \mathbb{R}^{K}$ as

$$\mathbf{f}_{\mathbf{o}}^{\mathrm{R}} := \mathbf{A}_{\mathbf{o}} \boldsymbol{\lambda}. \tag{182}$$

Then, denoting with $\mathbf{f}_{\mathbf{o}}^{\mathrm{A}}$ the "applied" contribution to the base pole generalized forces $\mathbf{f}_{\mathbf{o}}$, such that $\mathbf{f}_{\mathbf{o}} = \mathbf{f}_{\mathbf{o}}^{\mathrm{A}} + \mathbf{f}_{\mathbf{o}}^{\mathrm{R}}$, we may write the balance equation for a constrained rigid body as

$$\dot{\mathbf{p}}_{\mathbf{o}} = \mathbf{f}_{\mathbf{o}}^{\mathrm{A}} + \mathbf{A}_{\mathbf{o}} \boldsymbol{\lambda}, \tag{183}$$

and append the constraint equation (179) to determine the Lagrange multiplier vector $\boldsymbol{\lambda}$. In the case of a constrained beam, we have the boundary reaction forces $\mathbf{c}_{\mathbf{o}}^{\mathrm{R}} = \mathbf{A}_{\mathbf{o}} \boldsymbol{\lambda}$, such that the spatially-integrated balance equation reads

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{0}^{S} \boldsymbol{\pi} \cdot \mathbf{p}_{\mathrm{l}_{\mathbf{o}}} \, \mathrm{d}s + \int_{0}^{S} \boldsymbol{\pi}' \cdot \mathbf{c}_{\mathbf{o}} \, \mathrm{d}s = \int_{0}^{S} \boldsymbol{\pi} \cdot \mathbf{b}_{\mathbf{o}} \, \mathrm{d}s + (\boldsymbol{\pi} \cdot \mathbf{c}_{\mathbf{o}}^{\mathrm{A}})|_{0}^{S} + (\boldsymbol{\pi} \cdot \mathbf{A}_{\mathbf{o}} \boldsymbol{\lambda})|_{0}^{S}, \tag{184}$$

and the Lagrange multiplier vectors $\boldsymbol{\lambda}|_{s=0}$ and $\boldsymbol{\lambda}|_{s=S}$ are determined by appending the constraint equations (179) for each constrained boundary vertex.

Prototypal holonomic constraints. As we have seen, a way to enforce kinematic constraints on the relative configuration between frames may be formulated in terms of the convected displacement tensor $\overline{\mathbf{D}}$, or, equivalently, of the A-convected displacement vector $\boldsymbol{\alpha}_{A}^{-1} \Delta \mathbf{u}$ and the convected rotation tensor $\overline{\mathbf{R}}$.

Before examining some examples of ideal, time-independent, holonomic constraints of common application in multibody systems simulation, we develop the basic relations holding for the prototypal cases of the two joints $\mathscr{J}_{\mathscr{L}}$ and $\mathscr{J}_{\mathscr{A}}$ (the subscripts $_{\mathscr{L}}$ and $_{\mathscr{A}}$ stand for "prototypal linear joint" and "prototypal angular joint"), respectively defined by the following scalar constraint equations:

$$\begin{aligned} \boldsymbol{\phi}_{\mathscr{L}} &= \mathbf{j}_{\mathscr{L}_A} \cdot \Delta \mathbf{u} && (185) \\ &= \bar{\mathbf{j}}_{\mathscr{L}_A} \cdot \boldsymbol{\alpha}_{A}^{-1} \Delta \mathbf{u}, && (186) \end{aligned}$$

$$\begin{aligned} \boldsymbol{\phi}_{\mathscr{A}} &= \mathbf{j}_{\mathscr{A}_A} \cdot \mathbf{j}_{\mathscr{A}_B} && (187) \\ &= \bar{\mathbf{j}}_{\mathscr{A}_A} \cdot \overline{\mathbf{R}} \bar{\mathbf{j}}_{\mathscr{A}_B}. && (188) \end{aligned}$$

These primitive joints constrain only one degree of freedom, $K_{\mathscr{L}} = K_{\mathscr{A}} = 1$. The unit vectors $\mathbf{j}_{\mathscr{L}_A}, \mathbf{j}_{\mathscr{A}_A}, \mathbf{j}_{\mathscr{A}_B}$ represent material lines fixed in frames A and B, depending on the subscript, defined by the constraint at hand. Thus, their convected images $\bar{\mathbf{j}}_{\mathscr{L}} := \boldsymbol{\alpha}_{A}^{-1} \mathbf{j}_{\mathscr{L}}$, $\bar{\mathbf{j}}_{\mathscr{A}_1} := \boldsymbol{\alpha}_{A}^{-1} \mathbf{j}_{\mathscr{A}_1}$, $\bar{\mathbf{j}}_{\mathscr{A}_2} := \boldsymbol{\alpha}_{B}^{-1} \mathbf{j}_{\mathscr{A}_2}$ are constant in time. They represent directions of prescribed or infeasible translation and/or rotation.

[7] Those velocity-level constraints which cannot be integrated to obtain configuration-level constraints are termed *proper non-holonomic constraints* and are not treated here.

The velocity-level form of $\mathscr{J}_{\mathscr{L}}$ and $\mathscr{J}_{\mathscr{A}}$ may be obtained starting by the following relations:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{\alpha}_A^{-1}\Delta\mathbf{u}) &= \boldsymbol{\alpha}_A^{-1}(\dot{\mathbf{u}}_B - \dot{\mathbf{u}}_A) + \left(\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{\alpha}_A^{-1}\right)\Delta\mathbf{u},\\ &= \boldsymbol{\alpha}_A^{-1}(\mathbf{v}_{B_\mathbf{o}} - \mathbf{u}_B\times\boldsymbol{\omega}_B) - \boldsymbol{\alpha}_A^{-1}(\mathbf{v}_{A_\mathbf{o}} + \mathbf{u}_A\times\boldsymbol{\omega}_A) - \boldsymbol{\alpha}_A^{-1}\boldsymbol{\omega}_A\times\Delta\mathbf{u},\\ &= \boldsymbol{\alpha}_A^{-1}(\Delta\mathbf{v}_\mathbf{o} - \mathbf{u}_B\times\Delta\boldsymbol{\omega}),\end{aligned}$$

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\overline{\mathbf{R}}\,\mathbf{i}_k &= (\boldsymbol{\alpha}_A^{-1}\Delta\boldsymbol{\omega})\times\overline{\mathbf{R}}\,\mathbf{i}_k,\\ &= \boldsymbol{\alpha}_A^{-1}\Delta\boldsymbol{\omega}\times\boldsymbol{\alpha}_B\,\mathbf{i}_k,\\ &= \boldsymbol{\alpha}_A^{-1}\Delta\boldsymbol{\omega}\times\mathbf{e}_{B_k}.\end{aligned}$$

With the help of these results, possible forms for the constraint tensors $\mathbf{A}_{\mathscr{L}}$ and $\mathbf{A}_{\mathscr{A}}$, such that $\dot{\boldsymbol{\phi}}_{\mathscr{L}} = \mathbf{A}_{\mathscr{L}}^{\mathrm{T}}\Delta\mathbf{w}_\mathbf{o}$ and $\dot{\boldsymbol{\phi}}_{\mathscr{A}} = \mathbf{A}_{\mathscr{A}}^{\mathrm{T}}\Delta\mathbf{w}_\mathbf{o}$ are given by

$$\mathbf{A}_{\mathscr{L}} := \begin{bmatrix}\mathbf{j}_{\mathscr{L}_A}\\ \mathbf{u}_B\times\mathbf{j}_{\mathscr{L}_A}\end{bmatrix}, \tag{189}$$

$$\mathbf{A}_{\mathscr{A}} := \begin{bmatrix}\mathbf{0}_3\\ \mathbf{j}_{\mathscr{A}_A}\times\mathbf{j}_{\mathscr{A}_B}\end{bmatrix}. \tag{190}$$

The prototypal joints just examined may be combined to yield more complex holonomic constraints. In fact, a large class of them may be defined by requiring that K_{lin} mutually independent scalar components of $\boldsymbol{\alpha}_A^{-1}\Delta\mathbf{u}$ and K_{ang} mutually independent scalar components of $\overline{\mathbf{R}}$ are assigned as given functions of time. The integers K_{lin} and K_{ang} both range from 0 to 3 and the total number of constrained degrees of freedom is clearly $K = K_{\mathrm{lin}} + K_{\mathrm{ang}}$. Among the constraints that fall into this class one finds the six *lower pairs* (prismatic, revolute, cylindrical, helicoidal, spherical, and planar joints) and other typical constraints such as the universal and the distance joint.

Spherical joint. Given the preceding Eqs. (185) and (189), the formulæ for the *spherical joint* $\mathscr{J}_{\mathscr{S}}$ are straightforward. In fact, the spherical joint is a kinematic pair requiring that no displacement occurs, in any direction,

$$\Delta\mathbf{u} = \mathbf{0}_3 \iff \boldsymbol{\alpha}_A^{-1}\Delta\mathbf{u} = \mathbf{0}_3, \tag{191}$$

while any rotation is allowed. The configuration-level constraint reads then

$$\boldsymbol{\phi}_{\mathscr{S}} = \begin{bmatrix}\mathbf{i}_1\cdot\boldsymbol{\alpha}_A^{-1}\Delta\mathbf{u}\\ \mathbf{i}_2\cdot\boldsymbol{\alpha}_A^{-1}\Delta\mathbf{u}\\ \mathbf{i}_2\cdot\boldsymbol{\alpha}_A^{-1}\Delta\mathbf{u}\end{bmatrix} = \begin{bmatrix}\mathbf{e}_{A_1}\cdot\Delta\mathbf{u}\\ \mathbf{e}_{A_2}\cdot\Delta\mathbf{u}\\ \mathbf{e}_{A_3}\cdot\Delta\mathbf{u}\end{bmatrix}, \tag{192}$$

and the velocity-level constraint makes use of the constraint tensor

$$\mathbf{A}_{\mathscr{S}} = \begin{bmatrix}\mathbf{I}_3\\ \mathbf{u}_B\times\end{bmatrix}. \tag{193}$$

Clearly $K_{\mathscr{S}} = 3$, the only unconstrained degrees of freedom being the 3 parameters of rotation (such as the 3 scalar components of the rotation vector). The generalized screw vector $\boldsymbol{\nu}_\mathbf{o}$ corresponding to $\mathbf{D} = \exp(\boldsymbol{\nu}_\mathbf{o}\times)$ is given by $\boldsymbol{\nu}_\mathbf{o} = (\mathbf{0}_3, \boldsymbol{\varphi})$.

"Clamped triad" joint. One could define a *"clamped triad" joint* $\mathscr{J}_{\mathscr{T}}$ as the dual of the preceding case. This is such that any linear displacement between the frames are allowed, while no rotation at all is permitted,

$$\mathbf{R} = \mathbf{I}_3 \iff \overline{\mathbf{R}} = \mathbf{I}_3. \tag{194}$$

Starting from Eqs. (187) and (190), writing the configuration-level constraint as

$$\boldsymbol{\phi}_{\mathscr{T}} = \begin{bmatrix} \overline{R}_{12} \\ \overline{R}_{23} \\ \overline{R}_{31} \end{bmatrix} = \begin{bmatrix} \mathbf{e}_{A_1} \cdot \mathbf{e}_{B_2} \\ \mathbf{e}_{A_2} \cdot \mathbf{e}_{B_3} \\ \mathbf{e}_{A_3} \cdot \mathbf{e}_{B_1} \end{bmatrix}, \tag{195}$$

and the velocity-level constraint through the constraint tensor

$$\mathbf{A}_{\mathscr{T}} = \begin{bmatrix} \mathbf{O}_3 \\ \mathbf{I}_3 \end{bmatrix}. \tag{196}$$

Clearly $K_{\mathscr{T}} = 3$, the only unconstrained degrees of freedom being the 3 parameters of translation $\mathbf{t} = \Delta\mathbf{u}$ (such as the 3 scalar components of the displacement vector). The generalized screw vector $\boldsymbol{v}_{\mathbf{o}}$ corresponding to $\mathbf{D} = \exp(\boldsymbol{v}_{\mathbf{o}}\times)$ is given by $\boldsymbol{v}_{\mathbf{o}} = (\mathbf{t}, \mathbf{0}_3)$.

Prismatic joint. A *prismatic joint* $\mathscr{J}_{\mathscr{P}}$ is the kinematic pair which prescribes that frame B can only translate with respect to frame A along the direction defined by the unit vector $\mathbf{j}_{\mathscr{P}}$, fixed in both frames A and B. The two frames are initially coincident and placed at any point along the joint axis $\mathbf{j}_{\mathscr{P}}$.

The constraint states that the linear displacement between the frames is parallel to $\mathbf{j}_{\mathscr{P}}$,

$$\mathbf{j}_{\mathscr{P}} \times \Delta\mathbf{u} = \mathbf{0}_3 \Longleftrightarrow \bar{\mathbf{j}}_{\mathscr{P}} \times \boldsymbol{\alpha}_A^{-1}\Delta\mathbf{u} = \mathbf{0}_3, \tag{197}$$

and that no rotation is allowed,

$$\mathbf{R} = \mathbf{I}_3 \Longleftrightarrow \overline{\mathbf{R}} = \mathbf{I}_3. \tag{198}$$

Since no relative rotation occurs, the displacement vector and the translation vector coincide, and we have $\Delta\mathbf{u} = \mathbf{t} = s\,\mathbf{j}_{\mathscr{P}}$, where s, the scalar relative translation from frame A to frame B, is the only unconstrained degree of freedom ($K_{\mathscr{P}} = 5$).

If we assume that the $\mathbf{e}_3$ axis of both frames is initially (and hence at any time) aligned with the $\mathbf{j}_{\mathscr{P}}$ axis of the joint, the structural form of the displacement tensor is given by $\overline{\mathbf{D}} = \mathscr{T}(s\mathbf{i}_3)$ and the generalized convected screw vector is $\bar{\boldsymbol{v}}_{\mathbf{o}} = (s\mathbf{i}_3, \mathbf{0}_3)$. We may write the configuration-level constraint as

$$\boldsymbol{\phi}_{\mathscr{P}} = \begin{bmatrix} \mathbf{i}_1 \cdot \boldsymbol{\alpha}_A^{-1}\Delta\mathbf{u} \\ \mathbf{i}_2 \cdot \boldsymbol{\alpha}_A^{-1}\Delta\mathbf{u} \\ \overline{R}_{12} \\ \overline{R}_{23} \\ \overline{R}_{31} \end{bmatrix} = \begin{bmatrix} \mathbf{e}_{A_1} \cdot \Delta\mathbf{u} \\ \mathbf{e}_{A_2} \cdot \Delta\mathbf{u} \\ \mathbf{e}_{A_1} \cdot \mathbf{e}_{B_2} \\ \mathbf{e}_{A_2} \cdot \mathbf{e}_{B_3} \\ \mathbf{e}_{A_3} \cdot \mathbf{e}_{B_1} \end{bmatrix}, \tag{199}$$

and the velocity-level constraint is based on the constraint tensor

$$\mathbf{A}_{\mathscr{P}} = \left[\begin{array}{cc|c} \mathbf{e}_{A_1} & \mathbf{e}_{A_2} & \mathbf{A}_{\mathscr{T}} \\ \mathbf{0}_3 & \mathbf{0}_3 & \end{array}\right]. \tag{200}$$

Revolute joint. A *revolute joint* $\mathscr{J}_{\mathscr{R}}$ is the kinematic pair which prescribes that frame B can only rotate with respect to frame A about the direction defined by the unit vector $\mathbf{j}_{\mathscr{R}}$, fixed in both frames A and B. The two frames are initially coincident and placed at any point along the joint axis $\mathbf{j}_{\mathscr{R}}$.

The constraint states that no linear displacement between the frames is allowed,

$$\Delta\mathbf{u} = \mathbf{0}_3 \Longleftrightarrow \boldsymbol{\alpha}_A^{-1}\Delta\mathbf{u} = \mathbf{0}_3, \tag{201}$$

and that the rotation must occur about $\mathbf{j}_{\mathscr{R}}$,

$$(\mathbf{R} - \mathbf{I}_3)\,\mathbf{j}_{\mathscr{R}} = \mathbf{0}_3 \Longleftrightarrow (\overline{\mathbf{R}} - \mathbf{I}_3)\,\bar{\mathbf{j}}_{\mathscr{R}} = \mathbf{0}_3. \tag{202}$$

Since no linear displacement occurs, the translation vector is also null, $\mathbf{t} = \Delta\mathbf{u} = \mathbf{0}_3$, while we have $\mathbf{R} = \exp(\theta\mathbf{j}_{\mathscr{R}}\times)$, where θ, the scalar relative rotation from frame A to frame B, is the only unconstrained degree of freedom ($K_{\mathscr{R}} = 5$).

If we assume that the $\mathbf{e}_3$ axis of both frames is initially (and hence at any time) aligned with the $\mathbf{j}_{\mathscr{R}}$ axis of the joint, the structural form of the convected displacement tensor is given by $\overline{\mathbf{D}} = \mathscr{A}(\exp(\theta \mathbf{i}_3 \times))$ and the convected generalized screw vector is $\overline{\boldsymbol{v}}_{\mathbf{o}} = (\mathbf{0}_3, \theta \mathbf{i}_3)$. We may write the configuration-level constraint as

$$\boldsymbol{\phi}_{\mathscr{R}} = \begin{bmatrix} \mathbf{i}_1 \cdot \boldsymbol{\alpha}_A^{-1} \Delta\mathbf{u} \\ \mathbf{i}_2 \cdot \boldsymbol{\alpha}_A^{-1} \Delta\mathbf{u} \\ \mathbf{i}_3 \cdot \boldsymbol{\alpha}_A^{-1} \Delta\mathbf{u} \\ \overline{R}_{13} \\ \overline{R}_{23} \end{bmatrix} = \begin{bmatrix} \mathbf{e}_{A_1} \cdot \Delta\mathbf{u} \\ \mathbf{e}_{A_2} \cdot \Delta\mathbf{u} \\ \mathbf{e}_{A_3} \cdot \Delta\mathbf{u} \\ \mathbf{e}_{A_1} \cdot \mathbf{e}_{B_3} \\ \mathbf{e}_{A_2} \cdot \mathbf{e}_{B_3} \end{bmatrix}, \tag{203}$$

and the velocity-level constraint is based on the constraint tensor

$$\mathbf{A}_{\mathscr{R}} = \left[\mathbf{A}_{\mathscr{S}} \left| \begin{matrix} \mathbf{0}_3 & \mathbf{0}_3 \\ \mathbf{e}_{A_1} & \mathbf{e}_{A_2} \end{matrix} \right. \right]. \tag{204}$$

Planar joint. A *planar joint* $\mathscr{J}_{\mathscr{F}}$ is the kinematic pair which prescribes that frame B can only translate with respect to frame A on the plane normal to the unit vector $\mathbf{j}_{\mathscr{F}}$, fixed in both frames A and B, and rotate about that same vector. Assume that the two frames are initially coincident.

The constraint states that the linear displacement between the frames is parallel to the plane defined by $\mathbf{j}_{\mathscr{F}}$,

$$\mathbf{j}_{\mathscr{F}} \cdot \Delta\mathbf{u} = 0 \iff \bar{\mathbf{j}}_{\mathscr{F}} \cdot \boldsymbol{\alpha}_A^{-1} \Delta\mathbf{u} = 0. \tag{205}$$

and that the rotation must occur about $\mathbf{j}_{\mathscr{F}}$,

$$(\mathbf{R} - \mathbf{I}_3)\, \mathbf{j}_{\mathscr{F}} = \mathbf{0}_3 \iff (\overline{\mathbf{R}} - \mathbf{I}_3)\, \bar{\mathbf{j}}_{\mathscr{F}} = \mathbf{0}_3. \tag{206}$$

Since the linear displacement is tangential to the joint plane and the rotation is about the normal to the same plane, we have $\Delta\mathbf{u} = s_1 \mathbf{e}_{A_1} + s_2 \mathbf{e}_{A_2}$ and $\mathbf{R} = \exp(\theta \mathbf{j}_{\mathscr{F}} \times)$. The three scalars s_1, s_2, θ are the two relative translations and the relative rotation, respectively, from frame A to frame B. These are the only unconstrained degrees of freedom ($K_{\mathscr{F}} = 3$).

If we assume that the $\mathbf{e}_3$ axis of both frames is initially (and hence at any time) aligned with the $\mathbf{j}_{\mathscr{F}}$ axis of the joint, the structural form of the convected displacement tensor is given by $\overline{\mathbf{D}} = \mathscr{T}(s_1 \mathbf{i}_1 + s_2 \mathbf{i}_2) \mathscr{A}(\exp(\theta \mathbf{i}_3 \times))$ and the convected generalized screw vector is $\overline{\boldsymbol{v}}_{\mathbf{o}} = (r_1 \mathbf{i}_1 + r_2 \mathbf{i}_2, \theta \mathbf{i}_3)$, where the linear displacement parameters r_1, r_2 can be easily obtained as functions of s_1, s_2, θ and of the coordinates of $\mathbf{x}_A$. We may write the configuration-level constraint as

$$\boldsymbol{\phi}_{\mathscr{F}} = \begin{bmatrix} \mathbf{i}_3 \cdot \boldsymbol{\alpha}_A^{-1} \Delta\mathbf{u} \\ \overline{R}_{23} \\ \overline{R}_{31} \end{bmatrix} = \begin{bmatrix} \mathbf{e}_{A_3} \cdot \Delta\mathbf{u} \\ \mathbf{e}_{A_2} \cdot \mathbf{e}_{B_3} \\ \mathbf{e}_{A_3} \cdot \mathbf{e}_{B_1} \end{bmatrix}, \tag{207}$$

and the velocity-level constraint is based on the constraint tensor

$$\mathbf{A}_{\mathscr{F}} = \begin{bmatrix} \mathbf{e}_{A_3} & \mathbf{0}_3 & \mathbf{0}_3 \\ \mathbf{u}_B \times \mathbf{e}_{A_3} & \mathbf{e}_{A_1} & \mathbf{e}_{A_2} \end{bmatrix}. \tag{208}$$

Cylindrical joint. A *cylindrical joint* $\mathscr{J}_{\mathscr{C}}$, allowing only translation and rotation between the two frames along the joint axis $\mathbf{j}_{\mathscr{C}}$ is easily recovered by the superposition of a prismatic joint and a revolute joint aligned on the same axis $\mathbf{j}_{\mathscr{C}}$. With similar hypothesis of initially coincident frames and assuming $\mathbf{e}_3 = \mathbf{j}_{\mathscr{C}}$ for both frames, the configuration-level constraint reads

$$\boldsymbol{\phi}_{\mathscr{C}} = \begin{bmatrix} \mathbf{i}_1 \cdot \boldsymbol{\alpha}_A^{-1} \Delta\mathbf{u} \\ \mathbf{i}_2 \cdot \boldsymbol{\alpha}_A^{-1} \Delta\mathbf{u} \\ \overline{R}_{13} \\ \overline{R}_{23} \end{bmatrix} = \begin{bmatrix} \mathbf{e}_{A_1} \cdot \Delta\mathbf{u} \\ \mathbf{e}_{A_2} \cdot \Delta\mathbf{u} \\ \mathbf{e}_{A_1} \cdot \mathbf{e}_{B_3} \\ \mathbf{e}_{A_2} \cdot \mathbf{e}_{B_3} \end{bmatrix}, \tag{209}$$

while the velocity-level constraint tensor is

$$\mathbf{A}_{\mathscr{C}} = \begin{bmatrix} \mathbf{e}_{A_1} & \mathbf{e}_{A_2} & \mathbf{0}_3 & \mathbf{0}_3 \\ \mathbf{u}_B \times \mathbf{e}_{A_1} & \mathbf{u}_B \times \mathbf{e}_{A_2} & \mathbf{e}_{A_1} & \mathbf{e}_{A_2} \end{bmatrix}. \tag{210}$$

Universal joint. Another example of a useful kinematic constraint is the *universal joint* $\mathscr{J}_{\mathscr{U}}$, which allows the two frames to rotate with respect to each other about two intersecting perpendicular axes defined by the unit vectors $\mathbf{j}_{\mathscr{U}_A}$ and $\mathbf{j}_{\mathscr{U}_B}$, fixed in frames A and B, respectively. The joint can be thought of as the combination of two revolute joints. If the frames are initially coincident and the triads are assumed aligned with the joint axes in such a way that $\mathbf{e}_{A_1} = \mathbf{j}_{\mathscr{U}_A}$ and $\mathbf{e}_{B_2} = \mathbf{j}_{\mathscr{U}_B}$ the configuration-level constraint reads

$$\boldsymbol{\phi}_{\mathscr{U}} = \begin{bmatrix} \mathbf{i}_1 \cdot \boldsymbol{\alpha}_A^{-1} \Delta\mathbf{u} \\ \mathbf{i}_2 \cdot \boldsymbol{\alpha}_A^{-1} \Delta\mathbf{u} \\ \mathbf{i}_3 \cdot \boldsymbol{\alpha}_A^{-1} \Delta\mathbf{u} \\ \overline{R}_{12} \end{bmatrix} = \begin{bmatrix} \mathbf{e}_{A_1} \cdot \Delta\mathbf{u} \\ \mathbf{e}_{A_2} \cdot \Delta\mathbf{u} \\ \mathbf{e}_{A_3} \cdot \Delta\mathbf{u} \\ \mathbf{e}_{A_1} \cdot \mathbf{e}_{B_2} \end{bmatrix}, \tag{211}$$

while the velocity-level constraint tensor is

$$\mathbf{A}_{\mathscr{U}} = \left[\mathbf{A}_{\mathscr{S}} \middle| \begin{matrix} \mathbf{0}_3 \\ \mathbf{e}_{A_1} \times \mathbf{e}_{B_2} \end{matrix} \right]. \tag{212}$$

Generalization of the constraints and exponential parameterization. In the discussion above we considered kinematic constraints between two material frames, acting as moving frames that describe the motion of bodies $\mathscr{B}_A$ and $\mathscr{B}_B$. These may be rigid bodies, or boundary vertices of a flexible body, such as the ends of a beam. However, in the applications one needs a more general formulation in which the constrained frames, labeled, say, A^* and B^*, do not coincide with those, labeled A and B, chosen to describe the body, for example in view of its shape, its inertial or elastic properties. Frames A^* and B^* are rigidly connected to frames A and B, so that the following relationships hold:

$$\mathbf{C}_{A^*} = \mathbf{C}_A \overline{\mathbf{P}}_A, \tag{213}$$

$$\mathbf{C}_{B^*} = \mathbf{C}_B \overline{\mathbf{P}}_B, \tag{214}$$

where the two displacement tensors $\overline{\mathbf{P}}_A, \overline{\mathbf{P}}_B \in \mathrm{SR}(\mathbb{K}^6)$ are constant in time. In this case, all the formulæ given above for the configuration-level constraints still hold, if modified by substituting the position vectors $\mathbf{u}_A$, $\mathbf{u}_B$ and the triad unit vectors $\{\mathbf{e}_{A_k}\}_{k=1,2,3}$, $\{\mathbf{e}_{B_k}\}_{k=1,2,3}$ of frames A and B with those of frames A^* and B^*, denoted by $\mathbf{u}_{A^*}$, $\mathbf{u}_{B^*}$ and $\{\mathbf{e}_{A_k^*}\}_{k=1,2,3}$, $\{\mathbf{e}_{B_k^*}\}_{k=1,2,3}$. The same applies to the velocity-level constraints. In fact, due to the rigidity of the connection between frames A and A^*, and B and B^*, the generalized velocities $\mathbf{w}_{A_\mathbf{o}^*}$ and $\mathbf{w}_{B_\mathbf{o}^*}$ coincide with $\mathbf{w}_{A_\mathbf{o}}$ and $\mathbf{w}_{B_\mathbf{o}}$, respectively.

5. Conclusions

A novel formulation devoted to the analysis of general flexible multibody systems has been presented in Part I of this work. This serves as a basis for the development of a class of non-conventional algorithms for the dynamic simulation of such systems. The design process of these schemes and some representative numerical applications are the subject of Part II of this work.

The main features of the presented formulation lie in the way we approach the problem of representing and parameterizing the motion of material frames, i.e., rigid motion. Instead of considering separately the linear and rotational components, we draw a general procedure to treat these quantities in a coupled fashion, inspired by the underlying properties of the rigid displacements, which, remarkably, form a Lie group. This does not only imply the rewriting of the equations of motion in a 6-D compact representation, using such quantities as the frame configuration tensor and the frame generalized velocity, but also deeply impacts the parameterization strategy and the formulation of the dynamic balance equations.

The exponential map of motion has been introduced as the basis for a coupled parameterization which leads to the preservation of the correct interaction between linear and angular quantities even in the discretized case, thus maintaining the rule of reduction of torques. This leads to a rigid displacement invariance of the algorithms based on this procedure. The cases of the rigid body and the geometrically nonlinear elastic beam have been treated in detail, and their base pole balance equations have been derived and discussed, together with issues such as energy conservation.

Since the distinguishing feature of multibody analysis is represented by the presence of kinematic constraints, special attention has been given to relative motion, and particularly to the identification of convenient measures of relative rigid displacement and generalized velocity. Within this framework, general kinematic constraints have been considered in their configuration and velocity-level forms. Standard Lagrange multipliers have been introduced to account for constraint reaction forces in the base pole balance equations. Finally, a detailed presentation of the relevant formulæ for a number of holonomic constraints of primary interest has been given to complete the theoretical framework.

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