Space-time finite element methods for elastodynamics: Formulations and error estimates.

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29.11.2016

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Keypoints of the paper

- consider elastodynamics: system of 2^{nd}-order hyperbolic equations in $\Omega \times [0, T]$
- no analysis of continuous formulation
- no investigation of right function spaces
- decomposition into time slaps Q_n
- conforming discretization for time slaps
- consistent formulation with stabilization terms
- error analysis
- numerical experiments

Overview

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Motivation

- common approach for time-dependent problems:
  1. Semi-discretization : FEM in Space
  2. Full-discretization : Runge-Kutta
- idea is to use also FEM in time
- semi-discrete equation is multiplied with testfunction + integration over $[0, T]$ $\rightsquigarrow$ *structured* space time meshes. (Cartesian product)
- this approach permits also *unstructured* meshes (useful in adaptivity)

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- this approach permits also *unstructured* meshes (useful in adaptivity)

Motivation - Adaptivity

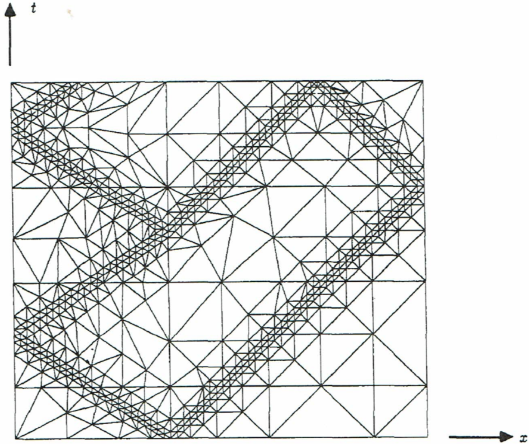

Figure : Space-time mesh for two material elastic rod problem

Problem formulation

Find u:

$$
\begin{aligned}
\rho \ddot{u} - \operatorname{div}(\sigma(\nabla u)) &= f \quad \text{on } Q := \Omega \times (0,T),\\
u &= g \quad \text{on } P^D := \Gamma^D \times (0,T),\\
\sigma(\nabla u)n &= h \quad \text{on } P^N := \Gamma^N \times (0,T),\\
u(x,0) &= u_0(x) \quad x \in \Omega,\\
\dot{u}(x,0) &= v_0(x) \quad x \in \Omega,
\end{aligned}
$$

where $\sigma(\nabla u) := \mathrm{C}\,\nabla u$ (Hooke's law) and $\Gamma := \partial\Omega = \overline{\Gamma^D \cup \Gamma^N}$.

- f ...*body forces*
- ρ ...*density*
- g ...*boundary displacement*
- h ...*boundary traction*
- C ...*stress tensor*

Overview

- Introduction
- Discrete Formulation
- Error Analysis
- Alternative Formulations
- Numerical Results

Preliminaries

- partition (0,T) into N time intervals $I_n = (t_n, t_{n+1})$
- time slaps:

$$Q_n := \Omega \times I_n$$
$$P_n := \Gamma \times I_n$$
$$P_n^N := \Gamma^N \times I_n \quad P_n^D := \Gamma^D \times I_n$$

- introduce $(n_e)_n$ elements $\{Q_n^e\}_e$ with boundaries $(P_n^e)_e$ in Q_n:

$$\tilde{Q}_n := \bigcup_{e=1}^{(n_e)_n} Q_n^e \quad \text{(element interior)}$$

$$P_n^{\text{int}} := \bigcup_{e=1}^{(n_e)_n} P_n^e - P_n \quad \text{(interior element boundary)}$$

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Jump operators

Normal vector n*:* normal to $P_n^e \cap \{t\}$ in the spatial plane $Q_n^e \cap \{t\}$.

- *spatial jump:*
  - $[\![w(x)]\!] := w(x^+) - w(x^-)$
  - $[\![\sigma(\nabla w)(x)]\!]n := (\sigma(\nabla w)n)(x^+) - (\sigma(\nabla w)n)(x^-)$
- *temporal jump:*
  - $[\![w(t)]\!] := w(t^+) - w(t^-)$
  - $[\![w(0)]\!] := w(0^+)$ and $[\![w(T)]\!] := -w(T^-)$.

Additional Notation

$$
\begin{aligned}
(u,w)_\Omega &:= \int_\Omega u \cdot w \, d\Omega, \\
(u,w)_{Q_n} &:= \int_{Q_n} u \cdot w \, dQ := \int_{I_n} \int_\Omega u \cdot w \, d\Omega dt, \\
(u,w)_{\tilde{Q}_n} &:= \sum_{e=1}^{(n_e)_n} \int_{Q_n^e} u \cdot w \, dQ, \\
(u,w)_{P_n^N} &:= \int_{P_n^N} u \cdot w \, ds, \quad (u,w)_{P_n^{\text{int}}} := \int_{P_n^{\text{int}}} u \cdot w \, ds, \\
a(u,w)_X &:= \int_X \sigma(\nabla u) \cdot \nabla w \, dX,
\end{aligned}
$$

where $X \in \{\Omega, Q_n, \tilde{Q}_n, P_n^N, P_n^{\text{int}}\}$.

Variational equation - Motivation

Consider a sufficiently smooth u

$$\rho \ddot{u} - \operatorname{div}(\sigma(\nabla u)) = f \quad \text{on } Q := \Omega \times (0,T).$$

We introduce $U = \{u_1, u_2\}$, $u_1 := u$ and $u_2 := \dot{u}_1 = \dot{u}$

$$\begin{aligned} \mathcal{L}_2 U &:= \rho \dot{u}_2 - \operatorname{div}(\sigma(\nabla u_1)) &&= f, \\ \mathcal{L}_1 U &:= \dot{u}_1 - u_2 &&= 0, \end{aligned}$$

$u_1 \ldots$ *displacement*
$u_2 \ldots$ *velocity*
On each Q_n we test with a smooth $\{w_1, w_2\}$:

$$\begin{aligned} (\rho \dot{u}_2, w_2)_{Q_n} + (\operatorname{div}(\sigma(\nabla u_1)), w_2)_{Q_n} &= (f, w_2)_{Q_n} && \forall w_2, \\ a(\mathcal{L}_1 U, w_1)_{Q_n} &= 0 && \forall w_1. \end{aligned}$$

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Variational equation - Motivation

Integration by parts:

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\begin{aligned}
(\rho \dot{u}_2, w_2)_{Q_n} + \underbrace{(\sigma(\nabla u_1), \nabla w_2)_{Q_n}}_{a(u_1, w_2)_{Q_n}} &= (f, w_2)_{Q_n} + (h, w_2)_{P_n^N} \qquad \forall w_2, \\
a(\mathcal{L}_1 U, w_1)_{Q_n} &= 0 \qquad \forall w_1.
\end{aligned}
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- Piecewise smooth solution U with respect to Q_n.
- Enforce continuity conditions
  - $u_1(t_n^+) = u_1(t_n^-)$, with $u_1(t_0^-) := u_0$
  - $u_2(t_n^+) = u_2(t_n^-)$, with $u_2(t_0^-) := v_0$
- Enforce them weakly in L^2 and $a(\cdot,\cdot)$ inner product.
  - $a(u_1(t_n^+), w_1(t_n^+))_\Omega = a(u_1(t_n^-), w_1(t_n^+))_\Omega \quad \forall w_1$
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Summarizing, we have for piecewise smooth U:

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a(\mathcal{L}_1 U, w_1)_{Q_n} &= 0 && \forall w_1,\\
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Only first derivatives appear $\rightsquigarrow$ we consider $H^1(Q_n)$ conforming discrete subspaces

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Only first derivatives appear $\rightsquigarrow$ *we consider* $H^1(Q_n)$ *conforming discrete subspaces*

Discrete Spaces

- $V^1_{h,g} := \{u_{1|Q_n} \in C^0(Q_n), u_1|_{Q_n^e} \in \mathcal{P}^k(Q_n^e), u_1 = g$ on $P_g\}$
- $V^2_{h,g} := \{u_{2|Q_n} \in C^0(Q_n), u_2|_{Q_n^e} \in \mathcal{P}^l(Q_n^e), u_2 = \dot{g}$ on $P_g\}$
- $V^1_{h,0}$ and $V^2_{h,0}$ $\rightsquigarrow$ homogeneous versions

$\rightsquigarrow$ Find $\{u_1^h, u_2^h\} \in V^1_{h,g} \times V^2_{h,g}$

$$
\begin{aligned}
(\rho \dot{u}_2^h, w_2^h)_{Q_n} + a(u_1^h, w_2^h)_{Q_n} &= (f, w_2^h)_{Q_n} + (h, w_2^h)_{P_n^N} && \forall w_2^h \in V^2_{h,0},\\
a(\mathcal{L}_1 U^h, w_1^h)_{\tilde{Q}_n} &= 0 && \forall w_1^h \in V^1_{h,0},\\
a(u_1^h(t_n^+), w_1^h(t_n^+))_\Omega &= a(u_1^h(t_n^-), w_1^h(t_n^+))_\Omega && \forall w_1^h \in V^1_{h,0},\\
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a(\mathcal{L}_1 U^h, w^h_1)_{\tilde{Q}_n} &= 0 && \forall w^h_1 \in V^1_{h,0},\\
a(u^h_1(t^+_n), w^h_1(t^+_n))_\Omega &= a(u^h_1(t^-_n), w^h_1(t^+_n))_\Omega && \forall w^h_1 \in V^1_{h,0},\\
(\rho u^h_2(t^+_n), w^h_2(t^+_n))_\Omega &= (\rho u^h_2(t^-_n), w^h_2(t^+_n))_\Omega && \forall w^h_2 \in V^2_{h,0}.
\end{aligned}
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Stabilization terms

- We add additional terms to improve stability of the system.
- Should be zero for a sufficiently smooth exact solution ("Consistency")

We have for sufficiently smooth u

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\begin{aligned}
(\mathcal{L}_2 U =) \quad \rho \dot{u}_2 - \operatorname{div}(\sigma(\nabla u_1)) &= f && \text{in } \tilde{Q}_n, \\
(\mathcal{L}_1 U =) \quad \dot{u}_1 - u_2 &= 0 && \text{in } \tilde{Q}_n, \\
[\![\sigma(\nabla u_1)(x)]\!] n &= 0 && \text{at } P_n^{\text{int}}, \\
\sigma(\nabla u_1) n &= h && \text{at } P_n^N
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We have for sufficiently smooth solution $\{u_1, u_2\}$

$$
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(\mathcal{L}_2 U, \rho^{-1}\tau_2 \mathcal{L}_2 W^h)_{\tilde{Q}_n} - (f, \rho^{-1}\tau_2 \mathcal{L}_2 W^h)_{\tilde{Q}_n} &= 0, \\
(\mathcal{L}_1 U, \tau_1 \mathcal{L} W^h)_{\tilde{Q}_n} &= 0, \\
([\![\sigma(\nabla u_1)(x)]\!] n, \rho^{-1} s [\![\sigma(\nabla w_1)(x)]\!] n)_{P_n^{\mathsf{int}}} &= 0, \\
(\sigma(\nabla u_1) n, \rho^{-1} s \sigma(\nabla w_1) n)_{P_n^N} - (h, \rho^{-1} s \sigma(\nabla w_1) n)_{P_n^N} &= 0,
\end{aligned}
$$

where τ_1, τ_2 and s are arbitrary $d \times d$ positive-definite matrices.

Discrete Variational Formulation

Find $U^h := \{u_1^h, u_2^h\} \in V_{g,h}^1 \times V_{g,h}^2$: for $n \in \{0, \ldots, N-1\}$

$$B_n(U^h, W^h) = L_n(W^h), \quad \forall W^h \in V_{0,h}^1 \times V_{0,h}^2,$$

$$\begin{aligned}
B_n(U^h, W^h) :=& (\rho \dot{u}_2^h, w_2^h)_{Q_n} + a(u_1^h, w_2^h)_{Q_n} + a(\mathcal{L}_1 U^h, w_1^h)_{\tilde{Q}_n} \\
&+ (\rho u_2^h(t_n^+), w_2^h(t_n^+))_\Omega + a(u_1^h(t_n^+), w_1^h(t_n^+))_\Omega \\
s_n(U^h, W^h) :=& \begin{cases} +(\mathcal{L}_2 U^h, \rho^{-1}\tau_2 \mathcal{L}_2 W^h)_{\tilde{Q}_n} + (\mathcal{L}_1 U^h, \tau_1 \mathcal{L} W^h)_{\tilde{Q}_n} \\ +([\![\sigma(\nabla u_1^h)(x)]\!] n, \rho^{-1} s [\![\sigma(\nabla u_1^h)(x)]\!] n)_{P_n^{\text{int}}} \\ +(\sigma(\nabla u_1^h) n, \rho^{-1} s \sigma(\nabla u_1^h) n)_{P_n^N} \end{cases} \\
L_n(W^h) :=& (f, w_2^h)_{Q_n} + (h, w_2^h)_{P_n^N} + (f, \rho^{-1}\tau_2 \mathcal{L}_2 W^h)_{\tilde{Q}_n} \\
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$$\begin{aligned}
L_n(W^h) &:= (f, w_2^h)_{Q_n} + (h, w_2^h)_{P_n^N} + (f, \rho^{-1}\tau_2 \mathcal{L}_2 W^h)_{\tilde{Q}_n} \\
&+ (h, \rho^{-1} s \sigma(\nabla u_1^h) n)_{P_n^N} + a(u_1^h(t_n^-), w_1^h(t_n^+))_\Omega \\
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$$B(U^h, W^h) = L(W^h), \quad \forall W^h \in V_{0,h}^1 \times V_{0,h}^2,$$

$$\begin{aligned} B(U^h, W^h) := \sum_{n=0}^{N-1} \Big[&(\rho \dot{u}_2^h, w_2^h)_{Q_n} + a(u_1^h, w_2^h)_{Q_n} + a(\mathcal{L}_1 U^h, w_1^h)_{\tilde{Q}_n} \\ &+ (\rho [\![u_2^h(t_n)]\!], w_2^h(t_n^+))_\Omega + a([\![u_1^h(t_n)]\!], w_1^h(t_n^+))_\Omega \\ &+ s_n(U^h, W^h) \Big] \end{aligned}$$

$$\begin{aligned} L(W^h) := \sum_{n=0}^{N-1} \Big[&(f, w_2^h)_{Q_n} + (h, w_2^h)_{P_n^N} + (f, \rho^{-1} \tau_2 \mathcal{L}_2 W^h)_{\tilde{Q}_n} \\ &+ (h, \rho^{-1} s \sigma(\nabla u_1^h) n)_{P_n^N} \Big] \end{aligned}$$

Properties

- For sufficiently smooth U: $B(U, W^h) = L(W^h)$.
- Let $E := U - U^h$: $B(E, W^h) = 0$. (*Galerkin Orthogonality*)
- We define the *total energy* at time t:

$$\mathcal{E}(W^h) := \frac{1}{2}(\rho w_2^h, w_2^h)_\Omega + \frac{1}{2}a(w_1^h, w_1^h)_\Omega$$

- Norm for convergence

$$|||W^h|||^2 := \sum_{n=0}^{N} \mathcal{E}([\![W^h(t_n)]\!]) + \sum_{n=0}^{N-1} s_n(W^h, W^h)$$

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- For sufficiently smooth U: $B(U, W^h) = L(W^h)$.
- Let $E := U - U^h$: $B(E, W^h) = 0$. (*Galerkin Orthogonality*)
- We define the *total energy* at time t:

$$\mathcal{E}(W^h) := \frac{1}{2}(\rho w_2^h, w_2^h)_\Omega + \frac{1}{2}a(w_1^h, w_1^h)_\Omega$$

- Norm for convergence

$$|\!|\!|W^h|\!|\!|^2 := \sum_{n=0}^{N} \mathcal{E}([\![W^h(t_n)]\!]) + \sum_{n=0}^{N-1} s_n(W^h, W^h)$$

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Theorem

Let the discrete Norm be defined as

$$|||W^h|||^2 := \sum_{n=0}^{N} \mathcal{E}([\![W^h(t_n)]\!]) + \sum_{n=0}^{N-1} s_n(W^h, W^h),$$

then it holds

$$|||W^h|||^2 = B(W^h, W^h) \quad \forall W^h \in V_{0,h}^1 \times V_{0,h}^2.$$

It immediately follows existence and uniqueness of a discrete solution U^h of

$$B(U^h, W^h) = L(W^h) \quad \forall W^h \in V_{0,h}^1 \times V_{0,h}^2.$$

Sketch of the proof

Recalling $|||W^h|||^2 := \sum_{n=0}^{N} \mathcal{E}([\![W^h(t_n)]\!]) + \sum_{n=0}^{N-1} s_n(W^h, W^h)$.

Since $B(W^h, W^h) = \sum_{n=0}^{N-1} [X_n(W^h) + s_n(W^h, W^h)]$, it is sufficient to show that

$$\sum_{n=0}^{N-1} X_n(W^h) = \sum_{n=0}^{N} \mathcal{E}([\![W^h(t_n)]\!]),$$

where

$$X_n(W^h) = (\rho \dot{w}_2^h, w_2^h)_{Q_n} + a(w_1^h, w_2^h)_{Q_n} + a(\mathcal{L}_1 W^h, w_1^h)_{\tilde{Q}_n} \\ + (\rho [\![w_2^h(t_n)]\!], w_2^h(t_n^+))_\Omega + a([\![w_1^h(t_n)]\!], w_1^h(t_n^+))_\Omega.$$

Due to $\mathcal{L}_1 W^h = \dot{w}_1^h - w_2^h$ and symmetry of $a(\cdot, \cdot)$ we obtain

$$a(w_1^h, w_2^h)_{Q_n} + a(\mathcal{L}_1 W^h, w_1^h)_{\tilde{Q}_n} = a(\dot{w}_1^h, w_1^h)_{Q_n}.$$

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Due to $\mathcal{L}_1 W^h = \dot{w}_1^h - w_2^h$ and symmetry of $a(\cdot,\cdot)$ we obtain

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$\square$

Overview

- Introduction
- Discrete Formulation
- Error Analysis
- Alternative Formulations
- Numerical Results

Basics

Let $\tilde{U}^h$ be an interpolant of U , then we can estimate:

$$\begin{aligned} |||E||| := |||U^h - U||| &= |||U^h - \tilde{U}^h + \tilde{U}^h - U||| \\ &\leq |||\underbrace{U^h - \tilde{U}^h}_{=:E^h}||| + |||\underbrace{\tilde{U}^h - U}_{=:H}||| \\ &= |||E^h||| + |||H||| \end{aligned}$$

- $H = \{\eta_1, \eta_2\} \ldots$ *Interpolation error*
- $E^h = \{e_1^h, e_2^h\}$
- $E = \{e_1, e_2\}$

$$\begin{aligned} |||E^h|||^2 &= B(E^h, E^h) = B(E - H, E^h) \\ &= -B(H, E^h) \leq |B(H, E^h)| \leq \ldots \end{aligned}$$

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Let $\tilde{U}^h$ be an interpolant of U , then we can estimate:

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- $H = \{\eta_1, \eta_2\} \ldots$ *Interpolation error*
- $E^h = \{e_1^h, e_2^h\}$
- $E = \{e_1, e_2\}$

$$\begin{aligned} |\!|\!| E^h |\!|\!|^2 = B(E^h, E^h) &= B(E - H, E^h) \\ &= -B(H, E^h) \leq |B(H, E^h)| \leq \ldots \end{aligned}$$

Technical Lemmas

Lemma

$$\sum_{n=0}^{N-1} (\rho \dot{\eta}_2, e_2^h)_{Q_n} + \sum_{n=0}^{N-1} (\rho [\![\eta_2(t_n)]\!], e_2^h(t_n^+))_\Omega$$
$$= -\sum_{n=0}^{N-1} (\rho \eta_2, \dot{e}_2^h)_{Q_n} - \sum_{n=1}^{N} (\rho \eta_2(t_n), [\![e_2^h(t_n^+)]\!])_\Omega$$

$$\sum_{n=0}^{N-1} a(\dot{\eta}_1, e_1^h)_{Q_n} + \sum_{n=0}^{N-1} a([\![\eta_1(t_n)]\!], e_1^h(t_n^+))_\Omega$$
$$= -\sum_{n=0}^{N-1} a(\eta_1, \dot{e}_1^h)_{Q_n} - \sum_{n=1}^{N} a(\eta_1(t_n), [\![e_1^h(t_n^+)]\!])_\Omega$$

Proof by integration by parts + index shifts. $\square$

Technical Lemmas

Lemma

$$
\begin{aligned}
a(\rho\eta_2, \dot{e}_2^h)_{Q_n} + a(\eta_2, e_1^h)_{Q_n} \\
&= (\eta_2, \mathcal{L}_2 E^h)_{\tilde{Q}_n} + (\eta_2, [\![\sigma(\nabla\, e_1^h)(x)]\!] n)_{P_n^{int}} + (\eta_2, \sigma(\nabla\, e_1^h) n)_{P_n^N}
\end{aligned}
$$

Proof.

Integration by parts + symmetry:

$$
\begin{aligned}
a(\eta_2, e_1^h)_{Q_n} = a(e_1^h, \eta_2)_{Q_n} &= (\sigma(\nabla\, e_1^h), \nabla\, \eta_2)_{Q_n} \\
&= -(\eta_2, \mathrm{div}(\sigma(\nabla\, e_1^h)))_{\tilde{Q}_n} + (\eta_2, [\![\sigma(\nabla\, e_1^h)(x)]\!] n)_{P_n^{int}} \\
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\end{aligned}
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$\square$

Technical Lemmas

Lemma

$$a(\rho\eta_2, \dot{e}_2^h)_{Q_n} + a(\eta_2, e_1^h)_{Q_n}$$
$$= (\eta_2, \mathcal{L}_2 E^h)_{\tilde{Q}_n} + (\eta_2, [\![\sigma(\nabla\, e_1^h)(x)]\!] n)_{P_n^{int}} + (\eta_2, \sigma(\nabla\, e_1^h) n)_{P_n^N}$$

Proof.

Integration by parts + symmetry:

$$\begin{aligned} a(\eta_2, e_1^h)_{Q_n} &= a(e_1^h, \eta_2)_{Q_n} = (\sigma(\nabla\, e_1^h), \nabla\, \eta_2)_{Q_n} \\ &= -(\eta_2, \mathrm{div}(\sigma(\nabla\, e_1^h)))_{\tilde{Q}_n} + (\eta_2, [\![\sigma(\nabla\, e_1^h)(x)]\!] n)_{P_n^{\mathrm{int}}} \\ &\quad + (\eta_2, \sigma(\nabla\, e_1^h) n)_{P_n^N} \end{aligned}$$

$\square$

Technical Lemmas

Lemma

$$\sum_{n=1}^{N}\Big[-(\rho\eta_2(t_n^-),[\![e_2^h(t_n)]\!])_\Omega - a(\eta_1(t_n^-),[\![e_1^h(t_n)]\!])_\Omega\Big]$$

$$\leq \frac{1}{2}\sum_{n=1}^{N}\Big[\mathcal{E}([\![E^h(t_n)]\!]) + 4\mathcal{E}(H(t_n^-))\Big],$$

where $\mathcal{E}(W) = \frac{1}{2}(\rho w_2, w_2)_\Omega + \frac{1}{2}a(w_1, w_1)$.

Proof: Apply Young's inequality $|ab| \leq \frac{1}{2}(\frac{1}{\epsilon}a^2 + \epsilon b^2)$ with $\epsilon := 2$.

$\square$

Interpolation estimates

- We assume: $\tau_1 = O(h^\alpha)$, $\tau_2 = O(h^\beta)$, $s = O(h^\gamma)$
- If $U \in H^{\max(k,l)+1}(Q)$: interpolation error $H = \{\eta_1, \eta_2\}$ fulfils
  - $\sum_{n=0}^{N-1} (\eta_2, \rho\tau_2^{-1}\eta_2)_{Q_n} \leq O(h^{2l+2-\beta})$
  - $\sum_{n=0}^{N-1} a(\eta_1, \tau_1^{-1}\eta_1)_{Q_n} \leq O(h^{2k+\alpha})$
  - $\sum_{n=0}^{N-1} (\mathcal{L}_2 H, \rho^{-1}\tau_2\mathcal{L}_2 H)_{\tilde{Q}_n} \leq O(h^{\min(2k-2+\beta, 2l+\beta)})$
  - $\sum_{n=0}^{N-1} a(\mathcal{L}_1 H, \rho^{-1}\tau_1\mathcal{L}_1 H)_{\tilde{Q}_n} \leq O(h^{\min(2k-2+\alpha, 2l+\alpha)})$
  - $\sum_{n=0}^{N-1} (\eta_2, \rho s^{-1}\eta_2)_{P_n^N \cup P_n^{\text{int}}} \leq O(h^{2l+1-\gamma})$
  - $$\sum_{n=0}^{N-1} \Big[(\llbracket\sigma(\nabla\eta_1)(x)\rrbracket n, \rho^{-1}s\llbracket\sigma(\nabla\eta_1)(x)\rrbracket n)_{P_n^{\text{int}}} + (\sigma(\nabla\eta_1)n, \rho^{-1}s\sigma(\nabla\eta_1)n)_{P_n^N} \Big] \leq O(h^{2k-1+\gamma})$$
  - $\sum_{n=0}^{N} \Big[\mathcal{E}(H(t_n^-)) + \mathcal{E}(H(t_n^+)) \Big] \leq O(h^{\min(2k-1, 2l+1)})$, where $\mathcal{E}(H(t_0^-)) = \mathcal{E}(H(t_N^+)) = 0$.

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- We assume: $\tau_1 = O(h^\alpha)$, $\tau_2 = O(h^\beta)$, $s = O(h^\gamma)$
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  - $\sum_{n=0}^{N-1} \Big[([\![\sigma(\nabla\eta_1)(x)]\!]n, \rho^{-1} s [\![\sigma(\nabla\eta_1)(x)]\!]n)_{P_n^{\text{int}}}$
    $+ (\sigma(\nabla\eta_1)n, \rho^{-1} s\sigma(\nabla\eta_1)n)_{P_n^N} \Big] \le O(h^{2k-1+\gamma})$
  - $\sum_{n=0}^{N} \Big[\mathcal{E}(H(t_n^-)) + \mathcal{E}(H(t_n^+)) \Big] \le O(h^{\min(2k-1, 2l+1)})$,
    where $\mathcal{E}(H(t_0^-)) = \mathcal{E}(H(t_N^+)) = 0$.

Main Theorem

Theorem

Let $U \in H^{\max(k,l)+1}(Q)$*,* τ_1, τ_2 *and* s *be chosen, such that*

$$|\tau_1| = |\tau_2| = O(h), \quad |s| = O(1) \qquad (\alpha = \beta = 1, \gamma = 0),$$

then we have

$$|||E|||^2 \leq O(h^{\min(2k-1, 2l+1)}).$$

Practical choices for τ_1, τ_2 and s are:

1. $\tau_1 = \tau_2 = \frac{\Delta x}{2c} I, \qquad s = \frac{1}{2c} I$
2. $\tau_1 = \tau_2 = \frac{\Delta t}{2} I, \qquad s = \frac{\Delta t}{2 \Delta x} I$

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$$
\begin{aligned}
&\leq \sum_{n=0}^{N-1} \Big[\frac{1}{4}(\mathcal{L}_2 E^h, \rho^{-1}\tau_2\mathcal{L}_2 E^h)_{\tilde{Q}_n} + (\eta_2, \rho\tau_2^{-1}\eta_2)_{\tilde{Q}_n} \\
&\quad + \frac{1}{4}a(\mathcal{L}_1 E^h, \tau_1\mathcal{L}_1 E^h)_{\tilde{Q}_n} + a(\eta_1, \tau_1^{-1}\eta_1)_{\tilde{Q}_n} \\
&\quad + \frac{1}{4}a(\mathcal{L}_2 E^h, \rho^{-1}\tau_2\mathcal{L}_2 E^h)_{\tilde{Q}_n} + a(\mathcal{L}_2 H, \rho^{-1}\tau_2\mathcal{L}_2 H)_{\tilde{Q}_n} \\
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&\quad + \frac{1}{4}([\![\sigma(\nabla e_1^h)(x)]\!]n, \rho^{-1}s[\![\sigma(\nabla e_1^h)(x)]\!]n)_{P_n^{\text{int}}} + (\eta_2, \rho s^{-1}\eta_2)_{P_n^{\text{int}}} \\
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&+ \frac{1}{4}([\![\sigma(\nabla e_1^h)(x)]\!]n, \rho^{-1}s[\![\sigma(\nabla e_1^h)(x)]\!]n)_{P_n^{\text{int}}} + ([\![\sigma(\nabla \eta_1^h)(x)]\!]n, \rho^{-1}s[\![\sigma(\nabla \eta_1^h)(x)]\!]n)_{P_n^{\text{int}}} \\
&\quad + \frac{1}{4}(\sigma(\nabla e_1^h)n, \rho^{-1}s\sigma(\nabla e_1^h)(x)n)_{P_n^N} + (\sigma(\nabla \eta_1^h)n, \rho^{-1}s\sigma(\nabla \eta_1^h)(x)n)_{P_n^N}\Big] \\
&\quad + \frac{1}{2}\sum_{n=1}^{N}\mathcal{E}([\![E^h(t_n)]\!]) + 2\sum_{n=1}^{N}\mathcal{E}(H^h(t_n^-))
\end{aligned}
$$

$$
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\leq \sum_{n=0}^{N-1} \Big[& \frac{1}{4}(\mathcal{L}_2 E^h, \rho^{-1}\tau_2 \mathcal{L}_2 E^h)_{\tilde{Q}_n} + (\eta_2, \rho\tau_2^{-1}\eta_2)_{\tilde{Q}_n} \\
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& + \frac{1}{4} a(\mathcal{L}_2 E^h, \rho^{-1}\tau_2 \mathcal{L}_2 E^h)_{\tilde{Q}_n} + a(\mathcal{L}_2 H, \rho^{-1}\tau_2 \mathcal{L}_2 H)_{\tilde{Q}_n} \\
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& + \frac{1}{4}(\llbracket \sigma(\nabla e_1^h)(x) \rrbracket n, \rho^{-1} s \llbracket \sigma(\nabla e_1^h)(x) \rrbracket n)_{P_n^{\text{int}}} + (\eta_2, \rho s^{-1}\eta_2)_{P_n^{\text{int}}} \\
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& + \frac{1}{4}(\sigma(\nabla e_1^h)n, \rho^{-1} s \sigma(\nabla e_1^h)(x) n)_{P_n^N} + (\sigma(\nabla \eta_1^h)n, \rho^{-1} s \sigma(\nabla \eta_1^h)(x) n)_{P_n^N} \Big] \\
& + \frac{1}{2} \sum_{n=1}^{N} \mathcal{E}(\llbracket E^h(t_n) \rrbracket) + 2 \sum_{n=1}^{N} \mathcal{E}(H^h(t_n^-))
\end{aligned}
$$

$$
\begin{aligned}
\leq & \frac{1}{2}\|\!|E^h|\!\|^2 + \sum_{n=0}^{N-1} \Big[\\
& + (\eta_2, \rho\tau_2^{-1}\eta_2)_{\tilde{Q}_n} && \leq O(h^{2l+2-\beta}) \\
& + a(\eta_1, \tau_1^{-1}\eta_1)_{\tilde{Q}_n} && \leq O(h^{2k-\alpha}) \\
& + a(\mathcal{L}_2 H, \rho^{-1}\tau_2\mathcal{L}_2 H)_{\tilde{Q}_n} && \leq O(h^{\min(2k-2+\beta, 2l+\beta)}) \\
& + a(\mathcal{L}_1 H, \tau_1\mathcal{L}_1 H)_{\tilde{Q}_n} && \leq O(h^{\min(2k-2+\alpha, 2l+\alpha)}) \\
& + (\eta_2, \rho s^{-1}\eta_2)_{P_n^{\text{int}}} && \leq O(h^{2l+1-\gamma}) \\
& + (\eta_2, \rho s^{-1}\eta_2)_{P_n^N} && \leq O(h^{2l+1-\gamma}) \\
& + ([\![\sigma(\nabla\eta_1^h)(x)]\!]n, \rho^{-1}s[\![\sigma(\nabla\eta_1^h)(x)]\!]n)_{P_n^{\text{int}}} && \leq O(h^{2k-1+\gamma}) \\
& + (\sigma(\nabla\eta_1^h)n, \rho^{-1}s\sigma(\nabla\eta_1^h)(x)n)_{P_n^N}\Big] && \leq O(h^{2k-1+\gamma}) \\
& + 2\sum_{n=1}^{N} \mathcal{E}(H^h(t_n^-)) && \leq O(h^{\min(2k-1, 2l+1)})
\end{aligned}
$$

optimal choice: $\alpha = \beta = 1, \gamma = 0$

- $\|E^h\|^2 \leq \frac{1}{2}\|E^h\|^2 + O(h^{\min(2k-1,2l+1)})$
- $\rightsquigarrow \|E^h\|^2 \leq O(h^{\min(2k-1,2l+1)})$ for $\alpha = \beta = 1, \gamma = 0$

$$
\begin{aligned}
\|H^h\|^2 = \sum_{n=0}^{N-1} \Big[& \\
&+ a(\mathcal{L}_2 H, \rho^{-1}\tau_2 \mathcal{L}_2 H)_{\tilde{Q}_n} && \leq O(h^{\min(2k-2+\beta,2l+\beta)}) \\
&+ a(\mathcal{L}_1 H, \tau_1 \mathcal{L}_1 H)_{\tilde{Q}_n} && \leq O(h^{\min(2k-2+\alpha,2l+\alpha)}) \\
&+ ([\![\sigma(\nabla \eta_1^h)(x)]\!]n, \rho^{-1} s [\![\sigma(\nabla \eta_1^h)(x)]\!]n)_{P_n^{\mathrm{int}}} && \leq O(h^{2k-1+\gamma}) \\
&+ (\sigma(\nabla \eta_1^h)n, \rho^{-1} s \sigma(\nabla \eta_1^h)(x)n)_{P_n^N} \Big] && \leq O(h^{2k-1+\gamma}) \\
&+ 2\sum_{n=0}^{N} \mathcal{E}([\![H^h(t_n)]\!]) && \leq O(h^{\min(2k-1,2l+1)})
\end{aligned}
$$

- $\rightsquigarrow \|H^h\|^2 \leq O(h^{\min(2k-1,2l+1)})$ for $\alpha = \beta = 1, \gamma = 0$ $\square$

- $|||E^h|||^2 \leq \frac{1}{2}|||E^h|||^2 + O(h^{\min(2k-1,2l+1)})$
- $\rightsquigarrow |||E^h|||^2 \leq O(h^{\min(2k-1,2l+1)})$ for $\alpha = \beta = 1, \gamma = 0$

$$
\begin{aligned}
|||H^h|||^2 = \sum_{n=0}^{N-1} \Big[& \\
& + a(\mathcal{L}_2 H, \rho^{-1}\tau_2 \mathcal{L}_2 H)_{\tilde{Q}_n} && \leq O(h^{\min(2k-2+\beta,2l+\beta)}) \\
& + a(\mathcal{L}_1 H, \tau_1 \mathcal{L}_1 H)_{\tilde{Q}_n} && \leq O(h^{\min(2k-2+\alpha,2l+\alpha)}) \\
& + ([\![\sigma(\nabla\,\eta_1^h)(x)]\!] n, \rho^{-1} s [\![\sigma(\nabla\,\eta_1^h)(x)]\!] n)_{P_n^{\text{int}}} && \leq O(h^{2k-1+\gamma}) \\
& + (\sigma(\nabla\,\eta_1^h) n, \rho^{-1} s \sigma(\nabla\,\eta_1^h)(x) n)_{P_n^N} \Big] && \leq O(h^{2k-1+\gamma}) \\
& + 2\sum_{n=0}^{N} \mathcal{E}([\![H^h(t_n)]\!]) && \leq O(h^{\min(2k-1,2l+1)})
\end{aligned}
$$

- $\rightsquigarrow |||H^h|||^2 \leq O(h^{\min(2k-1,2l+1)})$ for $\alpha = \beta = 1, \gamma = 0$ $\square$

- $\|\!|E^h|\!\|^2 \leq \frac{1}{2}\|\!|E^h|\!\|^2 + O(h^{\min(2k-1,2l+1)})$
- $\rightsquigarrow \|\!|E^h|\!\|^2 \leq O(h^{\min(2k-1,2l+1)})$ for $\alpha = \beta = 1, \gamma = 0$

$$
\begin{aligned}
\|\!|H^h|\!\|^2 = \sum_{n=0}^{N-1} \Big[& \\
&+ a(\mathcal{L}_2 H, \rho^{-1}\tau_2 \mathcal{L}_2 H)_{\tilde{Q}_n} && \leq O(h^{\min(2k-2+\beta,2l+\beta)}) \\
&+ a(\mathcal{L}_1 H, \tau_1 \mathcal{L}_1 H)_{\tilde{Q}_n} && \leq O(h^{\min(2k-2+\alpha,2l+\alpha)}) \\
&+ ([\![\sigma(\nabla\, \eta_1^h)(x)]\!]n, \rho^{-1} s [\![\sigma(\nabla\, \eta_1^h)(x)]\!]n)_{P_n^{\text{int}}} && \leq O(h^{2k-1+\gamma}) \\
&+ (\sigma(\nabla\, \eta_1^h)n, \rho^{-1} s\sigma(\nabla\, \eta_1^h)(x)n)_{P_n^N} \Big] && \leq O(h^{2k-1+\gamma}) \\
&+ 2\sum_{n=0}^{N} \mathcal{E}([\![H^h(t_n)]\!]) && \leq O(h^{\min(2k-1,2l+1)})
\end{aligned}
$$

- $\rightsquigarrow \|\!|H^h|\!\|^2 \leq O(h^{\min(2k-1,2l+1)})$ for $\alpha = \beta = 1, \gamma = 0$ $\square$

Overview

- Introduction
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- Error Analysis
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Simplified formulations - Single Field

- Assume that $\dot{u}_1^h - u_2^h = 0$ and $\dot{w}_1^h - w_2^h = 0$.
- Define $\tau := \tau_2$ and $\mathcal{L}u^h = \rho\ddot{u}^h - \operatorname{div}(\sigma(\nabla u^h))$

$$\begin{aligned}
b_n(u^h, w^h) :=\ & (\rho\ddot{u}^h, \dot{w}^h)_{Q_n} + a(u^h, \dot{w}^h)_{Q_n} + (\mathcal{L}u^h, \rho^{-1}\tau\mathcal{L}w^h)_{\tilde{Q}_n} \\
& + ([\![\sigma(\nabla u^h)(x)]\!]n, \rho^{-1}s[\![\sigma(\nabla u^h)(x)]\!]n)_{P_n^{\mathsf{int}}} \\
& + (\sigma(\nabla u^h)n, \rho^{-1}s\sigma(\nabla u^h)n)_{P_n^N} \\
& + (\rho\dot{u}^h(t_n^+), \dot{w}^h(t_n^+))_\Omega + a(u^h(t_n^+), w^h(t_n^+))_\Omega \\
l_n(w^h) :=\ & (f, \dot{w}^h)_{Q_n} + (h, \dot{w}^h)_{P_n^N} + (f, \rho^{-1}\tau\mathcal{L}w^h)_{\tilde{Q}_n} \\
& + (h, \rho^{-1}s\sigma(\nabla u^h)n)_{P_n^N} + a(u^h(t_n^-), w^h(t_n^+))_\Omega \\
& + (\rho\dot{u}^h(t_n^-), \dot{w}^h(t_n^+))_\Omega
\end{aligned}$$

Convergence theorem applies with $l = k - 1$.

Simplified formulations - time discontinuous Galerkin

$\tau_1 = \tau_2 = s = 0 \rightsquigarrow$

$$
\begin{aligned}
B_n(U^h, W^h) &:= (\rho \dot{u}_2^h, w_2^h)_{Q_n} + a(u_1^h, w_2^h)_{Q_n} + a(\mathcal{L}_1 U^h, w_1^h)_{\tilde{Q}_n} \\
&\quad + (\rho u_2^h(t_n^+), w_2^h(t_n^+))_\Omega + a(u_1^h(t_n^+), w_1^h(t_n^+))_\Omega \\
L_n(W^h) &:= (f, w_2^h)_{Q_n} + (h, w_2^h)_{P_n^N} \\
&\quad + a(u_1^h(t_n^-), w_1^h(t_n^+))_\Omega + (\rho u_2^h(t_n^-), w_2^h(t_n^+))_\Omega
\end{aligned}
$$

$$
\begin{aligned}
b_n(u^h, w^h) &:= (\rho \ddot{u}^h, \dot{w}^h)_{Q_n} + a(u^h, \dot{w}^h)_{Q_n} \\
&\quad + (\rho \dot{u}^h(t_n^+), \dot{w}^h(t_n^+))_\Omega + a(u^h(t_n^+), w^h(t_n^+))_\Omega \\
l_n(w^h) &:= (f, \dot{w}^h)_{Q_n} + (h, \dot{w}^h)_{P_n^N} + a(u^h(t_n^-), w^h(t_n^+))_\Omega \\
&\quad + (\rho \dot{u}^h(t_n^-), \dot{w}^h(t_n^+))_\Omega
\end{aligned}
$$

Not coverd by convergence theorem. (observed divergence for $l > k$)

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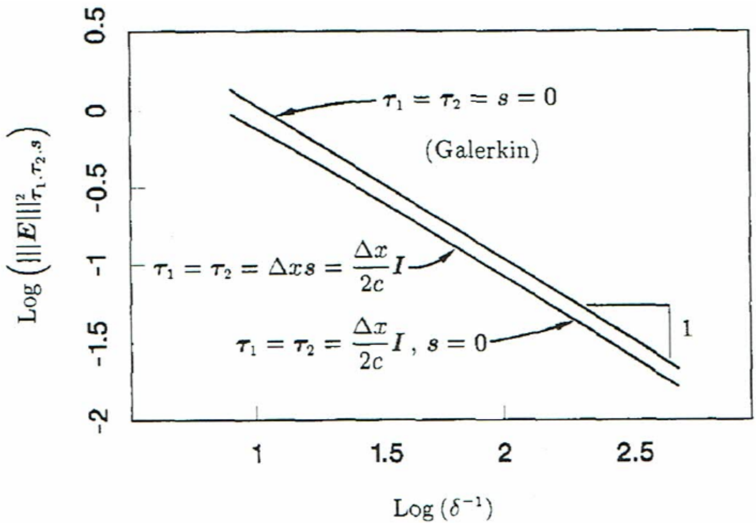
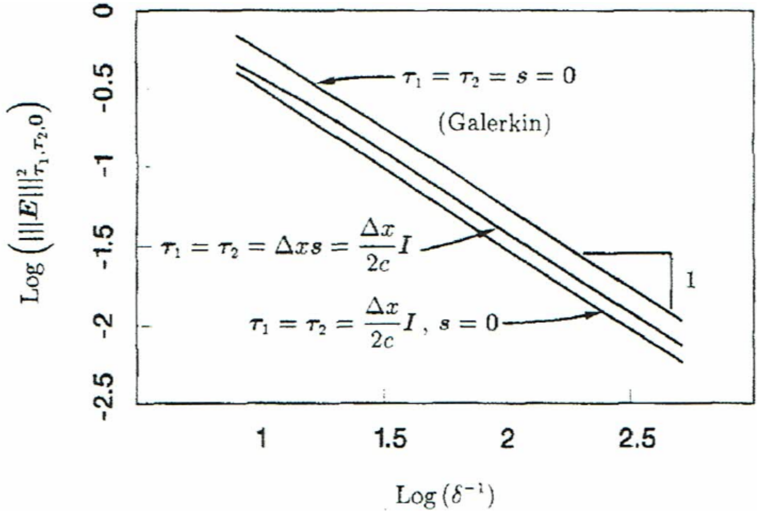
Setup for numerical experiments

- 1d elastic rod
- two ends are fixed, $f = 0$, $u_0 = 0$, v_0 ∼first harmonic.
- $\frac{c\Delta t}{\Delta x} = 1.2$
- consider $Qk - Ql$ standard elements with $k, l \in \{1, 2\}$
- consider also formulations with $s = 0$ and $s = \tau_1 = \tau_2 = 0$
- consider the three norms $\|\!|\cdot|\!\|_{\tau_1,\tau_2,s}$, $\|\!|\cdot|\!\|_{\tau_1,\tau_2,0}$ and $\|\!|\cdot|\!\|_{0,0,0}$
- Test cases:
  - $Q1 - Q1$ with $\|\!|\cdot|\!\|_{\tau_1,\tau_2,s}$, $\|\!|\cdot|\!\|_{\tau_1,\tau_2,0}$ and $\|\!|\cdot|\!\|_{0,0,0}$
  - $Qk - Ql$ with $k, l \in \{1, 2\}$ with $s = 0$

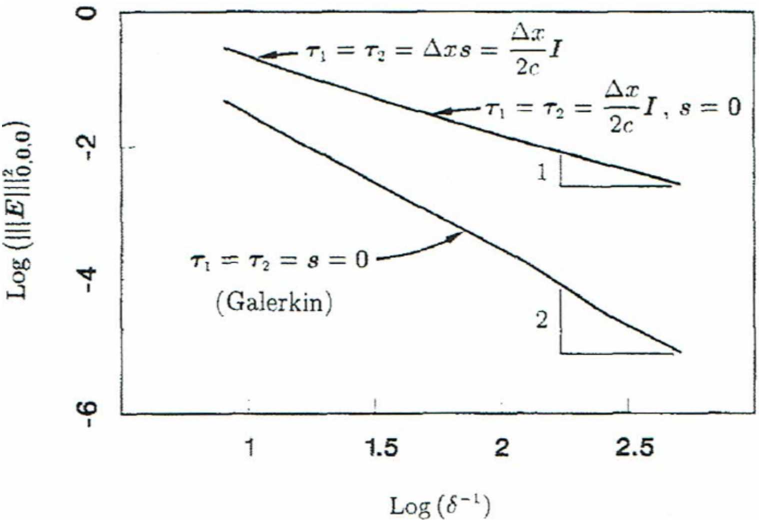
$Q1 - Q1$: Error in the $\|\!|\cdot|\!\|_{\tau_1,\tau_2,s}$ norm



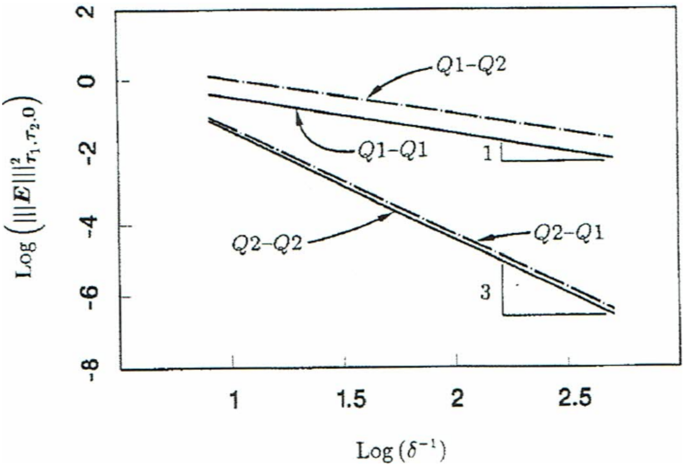
$Q1 - Q1$: Error in the $\|\cdot\|_{\tau_1,\tau_2,0}$ norm



$Q1 - Q1$: Error in the $|||\cdot|||_{0,0,0}$ norm



$Qk - Ql$: Error in the $\|\!|\cdot|\!\|_{\tau_1,\tau_2,0}$ norm



Same results for $\tau_1 = \tau_2 = 0$, but divergence for $l > k$