# SPACE–TIME FINITE ELEMENT FORMULATION FOR THE DYNAMICAL EVOLUTIONARY PROCESS

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**Abstract**

In the paper a new formula for the space time finite element approach to dynamic solution of solid is developed. One degree of freedom system was analyzed to find an unconditionally stable scheme of integration. Relations are expressed in terms of velocity. The geometry of the domain analysed is updated in every time step. The procedure can be efficient for geometrically non linear problems of mechanics of continuum.

# 1 Introduction

Nowadays practical problems of manufacturing require more accurate determination of phenomena which govern the process to be investigated. The increase of speed of calculations also is required since they become more and more complex. Particular requirements are imposed to dynamic computations, where the solution is repeated for each time step. Several completely new methods of calculations were elaborated to meet all the requirements imposed to the solution.

Recently the space–time element method has been developed. It can be applied to a dynamic modeling of mechanical problems, both to rigid and deformable body. It can be considered as an extension and generalization of the commonly known finite element method. The main feature is that time variable is considered in the same way as spatial variables. Since the space–time discretization is applied to quite new engineering problems, a short review of the state-of-the-art can introduce us to a subject.

First attempts of the space–time modelling of physical problems were published in (Gurtin, 1964; Herrera and Bielak, 1974). The definition of the minimized functional allowed to derive the relation between time variable and spatial variables in space–time subdomains. These subdomains can be regarded as space–time finite elements. Oden (Oden, 1969) proposed a general approach to the finite element method. He extended the imagine of the structure on time variable. Unfortunately, this interesting idea of the nonstationary partition of structure on subspaces

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proposed was not continued. Fried, Argyris, Scharpf and Chan (Fried, 1989; Argyris and Scharpf, $1969_1$, $1969_2$; Argyris and Chan, 1972) have formulated problems with space and time treated equally. However, in the papers of Kuang and Atluri, for example (Kuang and Atluri, 1985), the final discretization was carried on separately for time and space. For a long time dynamical problems were solved with separation of time variable and spatial variables. The physical space of the structure was discretized by one method (for example the finite element method, finite difference method) while time derivatives were integrated with the use of another (Runge–Kutta, Newmark, Wilson method etc.) A great number of published papers concerned the direct integration of the differential equation of motion assuming the stationary discretization.

Independently of the researches mentioned above Kączkowski in his papers (Kączkowski, 1975, 1976, 1979) introduced for the first time some abstract physical terms to mechanics: the equation of time–work, mass as a vector quantity or a space–time rigidity. A synthesis of the space–time element method can be found in (Kączkowski and Langer, 1980) and stability considerations in (Langer, 1979; Bajer, $1987_2$). Space–time elements which lead to unconditionally stable solution schemes were described in (Kacprzyk, 1984; Kacprzyk and Lewiński, 1983). Unfortunately they could be applied for only space–time forms rectangular in time, obtained as a vector product of spatial domain and time interval. In next works researchers turned to non rectangular shapes of elements. Triangular elements of string were elaborated (Witkowski, 1983, 1985). Then non stationary partition of the structure and non rectangular space–time elements (Bajer, 1986, $1987_2$) enabled to solve quite new group of problems by the space–time element method: contact problems (Bajer and Bogacz, 1989), problems with adaptive mesh (Bajer, $1987_1$; Bajer, 1989; Bajer, 1990).

Together with works which developed the method in many papers the estimation of the accuracy and efficiency of the space–time element method in different technical problems was described (Brzeziński and Pietrzakowski, 1979; Kacprzyk, 1981; Bajer, Burkhardt and Taltello, 1987, 1989; Taltello and Burkhardt, 1988; Kączkowski, 1988). Non linear effects: geometric (Witkowski, 1983; Podhorecka, 1988) and material (Podhorecki, 1986; Podhorecka and Podhorecki, 1985; Bajer, Bogacz and Bonthoux, 1991), were also considered. In the group of reviews we can find the works (Bajer and Podhorecki, 1989; Bajer and Bonthoux, 1988, 1991) and the introduction to publication (Witkowski, 1983).

The main advantage of the space–time element method is that the approximation in time is continuous. This condition in general case requires more computational effort than in the case of non continuous approximation. Here we will try to reduce the general problem of time integration of the differential equation with continuity of displacements and velocities which describe the system, to the problem which requires the same amount of arithmetical operations as the non continuous solution. However, here we still preserve the continuity in time. In this paper a new formula for the space time finite element approach is developed. One degree of freedom system was analyzed to find the unconditionally stable scheme of time integration of the differential equation. The velocity is assumed as a quantity which describes

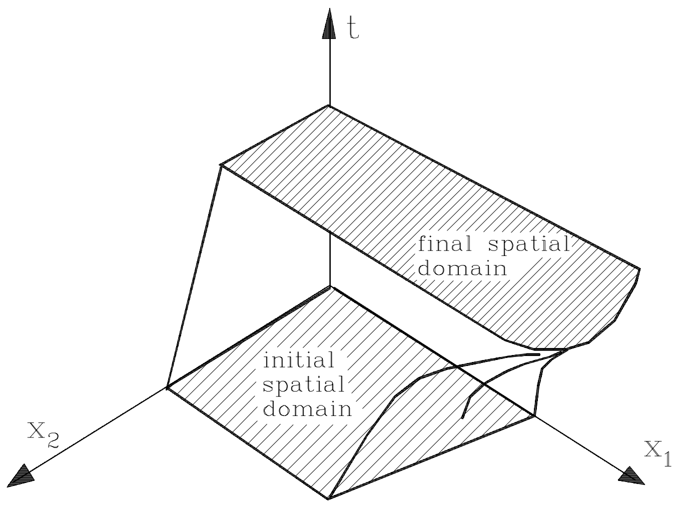


Figure 1: Space–time domain.

the process. Then the geometry of the domain analysed can easily be updated. The space–time element with the spatial geometry which changes in time is depicted in Fig. 1.

# 2 One degree of freedom system

Let us consider free vibration of a material point, described by the equation

$$m\frac{\mathrm{d}v}{\mathrm{d}t} + kx = 0 \tag{1}$$

We assume the linear distribution of real velocity $v$ over the time interval $h$ ($0 \leq t \leq h$)

$$v = (1 - \frac{t}{h})v_0 + \frac{t}{h}v_1 \tag{2}$$

The displacement $x(t)$ is described by the integral

$$x(t) = \int_0^t v\,\mathrm{d}t = x_0 + \frac{h}{2}\left[1 - \left(1 - \frac{t}{h}\right)^2\right]v_0 + \frac{t^2}{2h}v_1 \tag{3}$$

We have the linear dependence on the velocity $v_0$ and $v_1$ determined at limits of the interval $[0, h]$. As a virtual velocity we assume the Dirac distribution which depends on the parameter $\alpha$ ($0 \leq \alpha \leq 1$) and only on the velocity $v_1$:

$$v^* = v_1\delta\left(\frac{t}{h} - \alpha\right) \tag{4}$$

Substitution of the above relations into (1) and integration over the time interval $[0, h]$ yields:

$$\int_0^h v^*\frac{1}{h}(v_1 - v_0)\,\mathrm{d}t + \int_0^h v^*\frac{k}{m}x(t)\mathrm{d}t = 0 \quad . \tag{5}$$

As a result we have:

$$v_1 = \frac{1 - \frac{kh^2}{2m}[1-(1-\alpha)^2]}{1+\frac{k\alpha^2h^2}{2m}} v_0 - \frac{k}{m}\frac{h}{\left(1+\frac{k\alpha^2h^2}{2m}\right)} x_0 \tag{6}$$

or symbolically

$$v_1 = T\,v_0 + Bx_0 \tag{7}$$

Displacement $x_1$ in a successive moment is determined from the velocity $v_0$ and $v_1$:

$$x_1 = x_0 + h[(1-\beta)v_0 + \beta v_1] \tag{8}$$

The accurate solution is obtained for $\beta = 1-\alpha$. With respect to this we can write

$$x_1 = x_0 + h[\alpha v_0 + (1-\alpha)v_1] \tag{9}$$

In the particular case of $\alpha = 1/2$ equation (9) is identical to the relation (3) assumed for $t = h$, that is $x_1 = x_0 + h(v_0+v_1)/2$.

Denoting $\kappa = h^2k/m$ one can write the transition to the successive moment in the following form:

$$\left\{ \begin{array}{c} v_1 \\ x_1 \end{array} \right\} = \left[ \begin{array}{cc} 1 - \frac{2\alpha\kappa}{2+\alpha^2\kappa} & -\frac{2\kappa}{h(\alpha^2\kappa+2)} \\ 3h - \frac{2h(\alpha\kappa+2)}{\alpha^2\kappa+2} & \frac{2\kappa(\alpha-1)}{\alpha^2\kappa+2}+1 \end{array} \right] \left\{ \begin{array}{c} v_0 \\ x_0 \end{array} \right\} \tag{10}$$

where the $2\times 2$ matrix is the transfer matrix $\mathbf{T}$. It allows to find the stability condition for $h \to \infty$. Eigenvalues of $\mathbf{T}$ in the case of $h \to \infty$ are as follows:

$$\lim_{h\to\infty} \lambda_{1/2} = \frac{\alpha^2-1}{\alpha^2} \pm \frac{i\sqrt{2\alpha^2-1}}{\alpha^2} \tag{11}$$

and their modules are:

$$\lim_{h\to\infty} |\lambda_{1/2}| = \left\{ \begin{array}{ll} 1, & \text{if } \sqrt{2}/2 \le \alpha \le 1 \\ \frac{1}{\alpha^2}\sqrt{\alpha^4-4\alpha^2+2}, & \text{if } 0 \le \alpha < \sqrt{2}/2 \end{array} \right. \tag{12}$$

Both the modules are equal to one when $\alpha \ge \sqrt{2}/2$. This important inequality allows us to assume for calculations the unconditionally stable procedure. Particularly, in problems of vibrations of systems composed of many degrees of freedom or if one wish to neglect the inertia effects in problems of the plastic flow of material, the unconditional stability is significant.

Tests performed for the one-degree-of-freedom system with the initial conditions $x_0 = 0$ and $v_0 = 1$ in the case of $\alpha = 0.5$ are presented in Fig. 2 and in the case of $\alpha = 1.0$ in Fig. 3. The error of the displacement amplitude for selected values of time step related to the period of vibrations $T$ is depicted in Fig. 4. It should be emphasized that the amplitude of the velocity is almost exact. In turn the error of the amplitude of displacements arises from the phase error, it means the elongation of the period of vibrations, which always appears when large time step $h$ is applied. In such a case the system starts to be more elastic and it responds increasing the displacement amplitude.

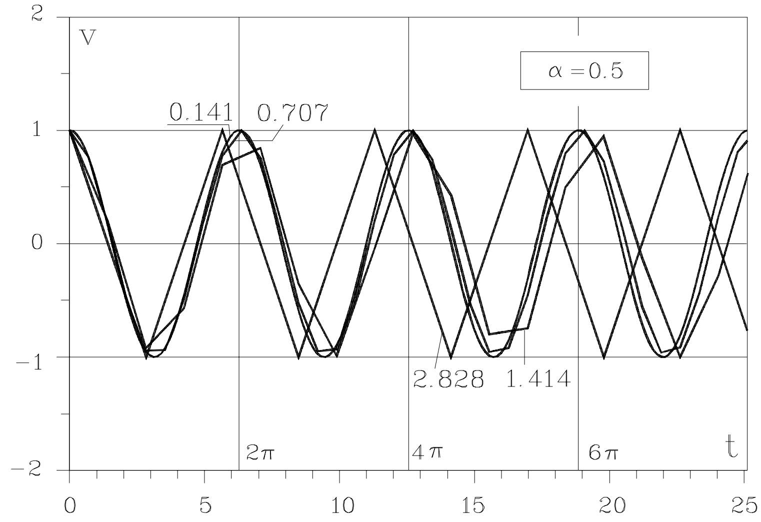


Figure 2: Velocity $v$ calculated with different time step for $\alpha = 0.5$.

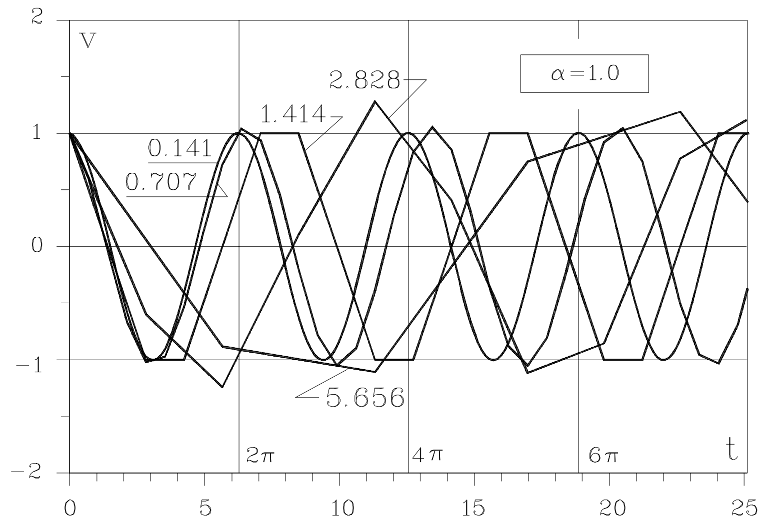


Figure 3: Velocity $v$ calculated with different time step for $\alpha = 1.0$.

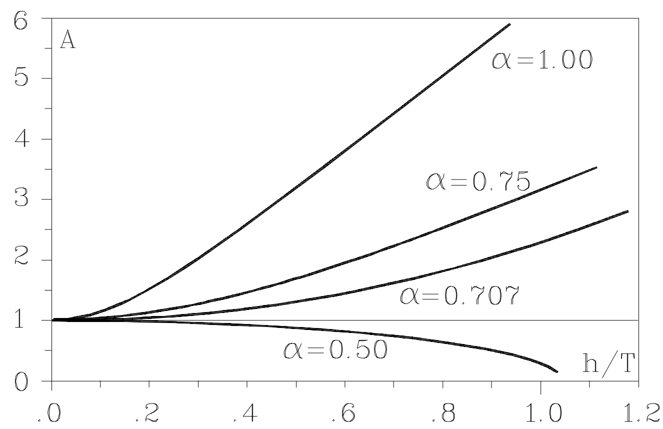


Figure 4: The displacement amplitude for selected parameters $\alpha$.

# 3 Finite element of a bar

Let us consider a bar vibrating axially, in an initial state described in a continuous domain $0 \le x \le l$. We will construct the mathematical model of the problem in a form of a discrete system, composed of one finite element.

We start from the differential equation of motion

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0 \tag{13}$$

$c^2 = E/\rho$ is the wave velocity in an elastic medium. Time derivative of the displacement $u$ is replaced by a velocity $v$, then the equation is multiplied by the function of distribution of the virtual velocity $v^*$. By integration we will balance the energy in the domain of the space–time element $\Omega = \{x, t : 0 \le x \le l(t), 0 \le t \le h\}$:

$$\int_\Omega v^* \frac{\partial v}{\partial t} \mathrm{d}\Omega - \int_\Omega v^* c^2 \frac{\partial^2 u}{\partial x^2} \mathrm{d}\Omega = 0 \tag{14}$$

Integration by parts of the second term, with the values determined at ends of the interval gives the relation in the following form:

$$\int_\Omega v^* \frac{\partial v}{\partial t} \mathrm{d}\Omega + \int_\Omega \frac{\partial v^*}{\partial x} c^2 \frac{\partial u}{\partial x} \mathrm{d}\Omega = 0 \tag{15}$$

Displacement $u(x,t)$ will be expressed by velocity

$$u(x,t) = u(x,0) + \int_0^t v(x,t)\, \mathrm{d}t \tag{16}$$

We assume the linear interpolation of the velocity in the interior of the element

$$v(x,t) = \mathbf{N}(x,t)\mathbf{v} \tag{17}$$

$\mathbf{N}$ is the matrix of interpolation functions and $\mathbf{v}$ is a velocity vector described at nodal points. Then equation (17) has a following form

$$u(x,t) = u_0 + \int_0^t \mathbf{N}(x,t)\mathrm{d}t\ \mathbf{v} \tag{18}$$

The differentiation of (18) results in the following equation:

$$\frac{\partial u}{\partial x} = \frac{h}{16a_1^2}\ln\left(1 - \frac{a_1 t}{h(x_1 - x_2)}\right)[(v_1 - v_2)(x_4 - x_3) + (v_3 - v_4)(x_2 - x_1)] + \\ \frac{t}{4a_1}(v_1 - v_2 - v_3 + v_4) + \frac{\mathrm{d}u_0}{\mathrm{d}x} \tag{19}$$

The derivative $\mathrm{d}u_0/\mathrm{d}x$ is the initial deformation $\varepsilon_0$ determined for $t = 0$. Equation (19) can be written in a short form:

$$\frac{\partial u}{\partial x} = [\mathbf{N}'_1, \cdots, \mathbf{N}'_4]\ \mathbf{v} + \boldsymbol{\varepsilon}_0 \tag{20}$$

In equation (19) and followings we use constants:

$$\begin{aligned} a_1 &= (x_1 - x_2 - x_3 + x_4)/4 \\ a_2 &= (-x_1 + x_2 - x_3 + x_4)/4 \\ a_3 &= (-x_1 - x_2 + x_3 + x_4)/4 \\ a_4 &= (x_1 + x_2 + x_3 + x_4)/4 \end{aligned} \tag{21}$$

The derivative $\partial v/\partial t$ can also be simply determined (analytically or numerically):

$$\frac{\partial v}{\partial t} = \frac{\partial \mathbf{N}}{\partial t}\mathbf{v} \tag{22}$$

Successive derivatives of the shape functions $\partial N_i/\partial t$ have more complex form and we will only mention here that they depend on the geometry of the element ($x_i$, $i = 1, ...4$; $h$) and variables $x$ and $t$.

We should assume the distribution of the virtual displacement $v^*$. In our considerations Dirac delta $\delta(t/h - \alpha)$ is put on the plane extended between the values $v_3$ and $v_4$, which are brought to points $x_L$ and $x_P$, respectively (Fig. 5). Since $x_L = x_2 + \alpha(x_3 - x_1)$ and $x_P = x_4 + \alpha(x_4 - x_2)$, where $\alpha = t/h$, $0 \leq \alpha \leq 1$, velocity $v^*$:

$$v^* = \left[v_4 + \frac{x - x_2 - \alpha(x_4 - x_2)}{x_2 - x_1 + \alpha(x_1 - x_2 - x_3 + x_4)}(v_4 - v_3)\right]\ \delta\Big(\frac{t}{h} - \alpha\Big) \tag{23}$$

and its spatial derivative:

$$\frac{\partial v^*}{\partial x} = \frac{v_4 - v_3}{x_2 - x_1 + \alpha(x_1 - x_2 - x_3 + x_4)}\ \delta\Big(\frac{t}{h} - \alpha\Big) \tag{24}$$

can be easily described.

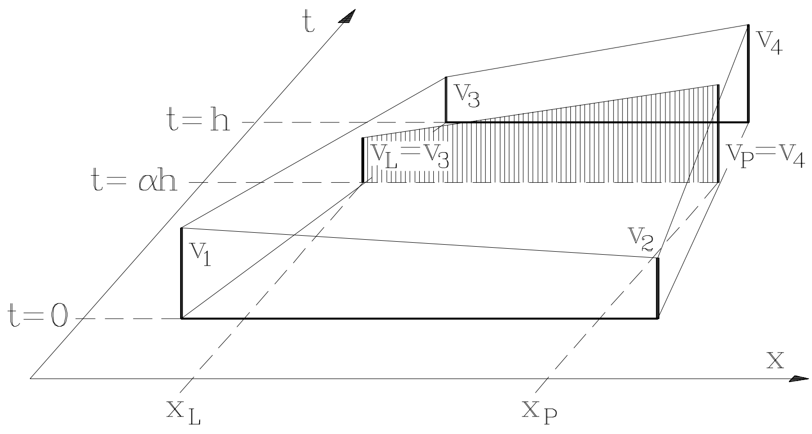


Figure 5: Scheme of the distribution of the virtual velocity.

Now, since we have all the terms which are required, we can compute the respective integrals:

$$\mathbf{v}^T \int_\Omega (\mathbf{N}^*)^T \rho \, \frac{\partial \mathbf{N}}{\partial t} \, \mathrm{d}\Omega \cdot \mathbf{v} + \mathbf{v}^T \int_\Omega \left( \frac{\partial \mathbf{N}^*}{\partial x} \right)^T E \, \mathbf{N}' \, \mathrm{d}\Omega \cdot \mathbf{v} + \mathbf{v}^T \int_\Omega \left( \frac{\partial \mathbf{N}^*}{\partial x} \right)^T E \, \varepsilon_0 \, \mathrm{d}\Omega = 0 \tag{25}$$

We obtain the following form:

$$\left[ \int_\Omega (\mathbf{N}^*)^T \rho \, \frac{\partial \mathbf{N}}{\partial t} \, \mathrm{d}\Omega + \int_\Omega \left( \frac{\partial \mathbf{N}^*}{\partial x} \right)^T E \, \mathbf{N}' \, \mathrm{d}\Omega \right] \mathbf{v} + \int_\Omega \left( \frac{\partial \mathbf{N}^*}{\partial x} \right)^T E \, \varepsilon_0 \, \mathrm{d}\Omega = \mathbf{0} \tag{26}$$

or shortly:

$$(\mathbf{M} + \mathbf{K}) \, \mathbf{v} + \mathbf{e} = \mathbf{0} \tag{27}$$

The final form of the matrices $\mathbf{M}$, $\mathbf{K}$ and the vector $\mathbf{e}$ exhaust our formulation:

$$\mathbf{M} = \rho \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ m_{3,1} & m_{3,2} & m_{3,3} & m_{3,4} \\ m_{4,1} & m_{4,2} & m_{4,3} & m_{4,4} \end{bmatrix} \tag{28}$$

where:

$$m_{3,i} = - \int_{x_1 + \alpha(x_3 - x_1)}^{x_2 + \alpha(x_4 - x_2)} \frac{x - x_2 - \alpha(x_4 - x_2)}{x_2 - x_1 + \alpha(x_1 - x_2 - x_3 + x_4)} \cdot \frac{\partial N_i}{\partial t} \mathrm{d}x \tag{29}$$

$$m_{4,i} = \int_{x_1 + \alpha(x_3 - x_1)}^{x_2 + \alpha(x_4 - x_2)} \left[ \frac{x - x_2 - \alpha(x_4 - x_2)}{x_2 - x_1 + \alpha(x_1 - x_2 - x_3 + x_4)} + 1 \right] \cdot \frac{\partial N_i}{\partial t} \mathrm{d}x \tag{30}$$

The stiffness matrix $\mathbf{K}$ has a form:

$$\mathbf{K} = E \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ s_1 & -s_1 & s_2 & -s_2 \\ -s_1 & s_1 & -s_2 & s_2 \end{bmatrix} \tag{31}$$

where:

$$s_1 = \frac{1}{8a_1^2}h(a_1+a_2)\ln\frac{a_1(1-2\alpha)-a_2}{a_1-a_2} - \frac{h\alpha}{4a_1} \tag{32}$$

$$s_2 = \frac{1}{8a_1^2}h(a_1+a_2)\ln\frac{a_1(1-2\alpha)-a_2}{a_1-a_2} + \frac{h\alpha}{4a_1} \tag{33}$$

The vector $\mathbf{e}$ contains the initial strain (initial for the time interval actually considered):

$$\mathbf{e} = E\left\{\begin{array}{c} 0 \\ 0 \\ -\varepsilon_0 \\ \varepsilon_0 \end{array}\right\} \tag{34}$$

The testing calculations were carried on for the single element, fixed at one end. The initial length is equal to 1.0. The initial velocity $v_0$=1.0. When large time step $h$=1.0 is assumed (Fig. 6) we can observe large errors, although the solution is still stable. The velocity is damped and the length of the element decreases (Fig. 7). The system is overstiffned and the phenomenon dramatically grows up. It is not difficult to predict that such a large change of the element length (and the geometry of the space–time element) during one time step (in practice equal to the length of the element) has to flow unprofitably on the solution. However, the decrease of the time step considerably improves results. Two successive figures (Fig. 8 and 9) show results for lower initial velocity $v_0 = 0.1$. Time step $h = 0.1$ allows to obtain sufficiently good approach. In practice much more lower both time step and initial velocities are applied to engineering problems. All the tests were performed for $\alpha = 1.0$. If a smaller value of $\alpha$ is applied ($\sqrt{2}/2 < \alpha < 1$) results are still better.

Here we must say that numerical integration by Gauss quadrature usually used to compute integrals when the spatial domain changes and our space–time element is not a rectangular one, is accurate only for polynomes. That is why the order of numerical integration should be increased when the domain integrated differs considerably from the multiplex form (obtained by multiplication of spatial domain by time interval).

# 4 General case of elasticity

Here we will discuss a more general approach which allows to discretize the arbitrary problem of dynamics of a continuous system.

If we denote the strain $\boldsymbol{\varepsilon}$ as

$$\boldsymbol{\varepsilon} = \mathcal{D}\mathbf{u} \ , \tag{35}$$

where $\mathcal{D}$ is a differential operator, and the stress $\boldsymbol{\sigma}$ as

$$\boldsymbol{\sigma} = \mathbf{E}\boldsymbol{\varepsilon} \ , \tag{36}$$

and if we assume the distribution of the virtual velocity $\mathbf{v}^*$, the equation of the virtual work expressed in terms of velocity will assume the following form:

$$\int_\Omega (\mathbf{v}^*)^T \rho \frac{\partial \mathbf{v}}{\partial t} \mathrm{d}\Omega + \int_\Omega (\dot{\boldsymbol{\varepsilon}})^T \boldsymbol{\sigma} \mathrm{d}\Omega + \int_\Omega (\mathbf{v}^*)^T \eta_z \mathbf{v} \, \mathrm{d}\Omega = \mathbf{0} \tag{37}$$

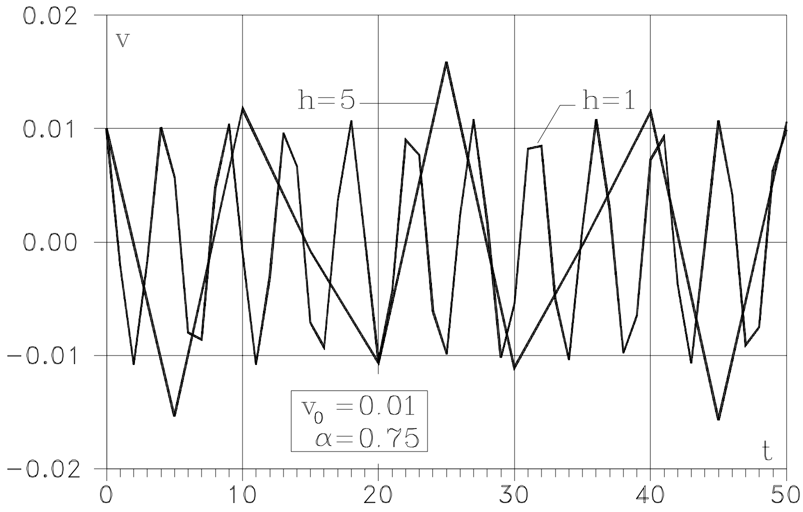


Figure 6: Velocity in time in the case of different steps of integration $h$ ($v_0 = 1.0$, $\alpha = 1.0$).

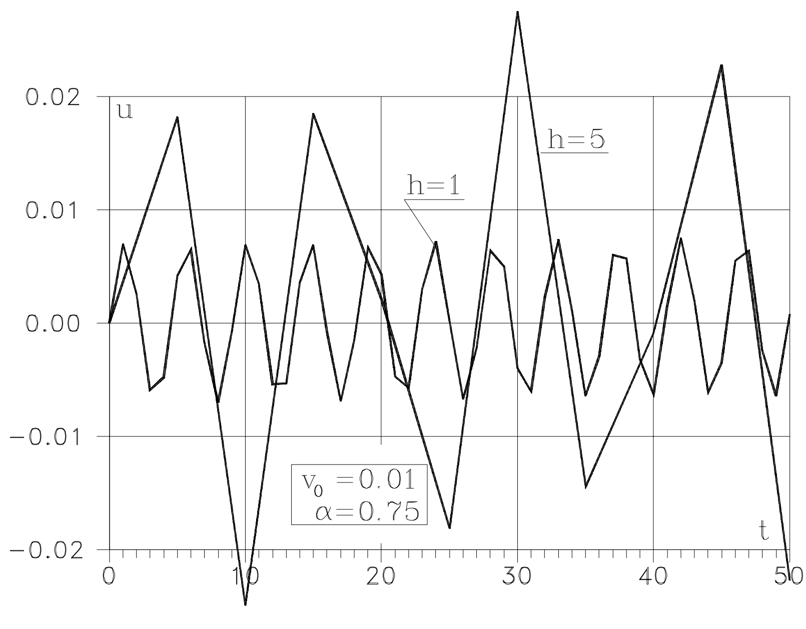


Figure 7: Length of the element in time in the case of different steps of integration $h$ ($v_0 = 1.0$, $\alpha = 1.0$).

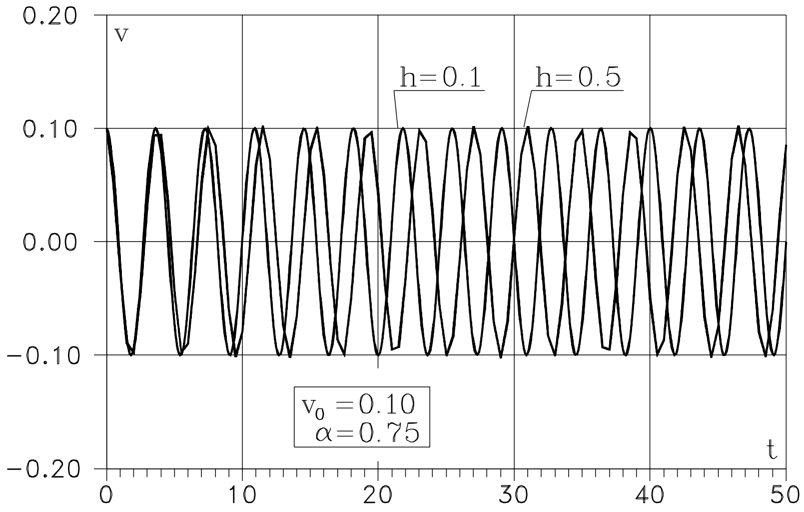


Figure 8: Velocity in time in the case of different steps of integration $h$ ($v_0 = 0.1$, $\alpha = 1.0$).

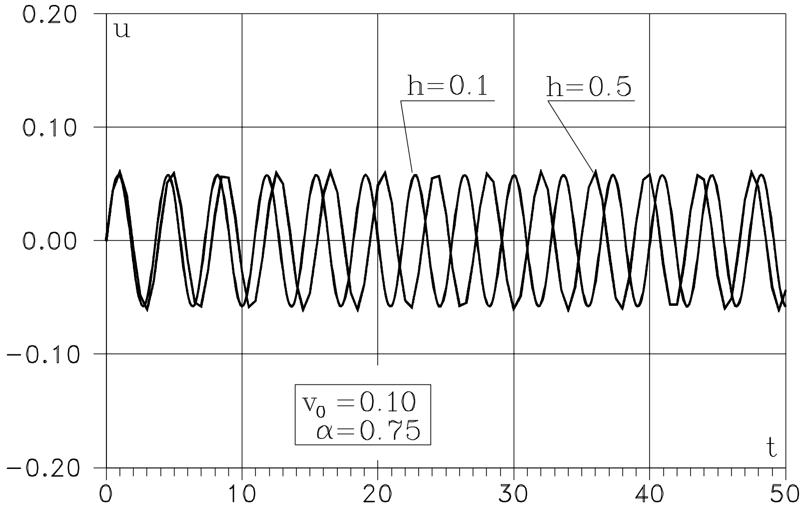


Figure 9: Length of the element in time in the case of different steps of integration $h$ ($v_0 = 0.1$, $\alpha = 1.0$).

Displacement $\mathbf{u}(t)$ is described by an integral

$$\mathbf{u}(t) = \mathbf{u}_0 + \int_0^t \mathbf{v}\;\mathrm{d}t\;. \tag{38}$$

With respect to (35), (36) and (38) we have:

$$\int_\Omega (\mathbf{v}^*)^T \rho \frac{\partial \mathbf{v}}{\partial t} \mathrm{d}\Omega + \int_\Omega (\mathcal{D}\mathbf{v}^*)^T \mathbf{E}\, \underbrace{\mathcal{D}\mathbf{u}_0}_{\boldsymbol{\varepsilon}_0} \mathrm{d}\Omega + \int_\Omega \left[ (\mathcal{D}\mathbf{v}^*)^T\, \mathbf{E}\, \mathcal{D} \int_0^t \mathbf{v}\;\mathrm{d}t \right] \mathrm{d}\Omega + \int_\Omega (\mathbf{v}^*)^T \eta_z \mathbf{v}\, \mathrm{d}\Omega = \mathbf{0} \tag{39}$$

The next step is to introduce the interpolation formulas:

$$\mathbf{v} = \mathbf{N}\mathbf{q} \quad \text{and} \quad \mathbf{v}^* = \mathbf{N}^*\mathbf{q}\;. \tag{40}$$

Finally we have:

$$\left\{ \int_\Omega \left[ (\mathcal{D}\mathbf{N}^*)^T\, \mathbf{E}\, \mathcal{D} \int_0^t \mathbf{N}\;\mathrm{d}t \right] \mathrm{d}\Omega + \int_\Omega (\mathbf{N}^*)^T \rho \frac{\partial \mathbf{N}}{\partial t}\, \mathrm{d}\Omega + \int_\Omega (\mathbf{N}^*)^T \eta_z\, \mathbf{N}\, \mathrm{d}\Omega \right\} \dot{\mathbf{q}} + \int_\Omega (\mathcal{D}\mathbf{N}^*)^T\, \mathbf{E}\, \boldsymbol{\varepsilon}_0\, \mathrm{d}\Omega = \mathbf{0} \tag{41}$$

If we assume as before the distribution of the virtual parameters which depend only on the nodal parameters determined for $t = h$, we will obtain in equation (41) the upper half of the matrices $\mathbf{M}$, $\mathbf{K}$ and the vector $\mathbf{e}$ equal to zero. Here we also can steer the properties of the procedure by the parameter $\alpha$.

A conclusion which concerns the numerical cost of the procedure should be emphasized. We must step back to the integration of the product of two functions of which one is Dirac function. Such an integration in a volume $\Omega$ in terms of variables $x, y, z, t$ reduces the computation to the integration for the surface $t = \alpha h$ over spatial variables $x, y, z$ only. It decreases the cost of computations comparing with the classical, linear interpolation of virtual parameters in time.

In the case of equation (41) the domain of integration is reduced from the space–time volume $\Omega$ to the spatial surface $A(\alpha h)$. The first integral contains the term integrated over the interval $[0, t]$. With respect to the above remark we must integrate in $[0, \alpha h]$. When the linear functions $\mathbf{N}$ are assumed, we can determine the value of the integrated term for the point $t = \alpha h/2$ and multiply the result by the length of interval $\alpha h$. Then the stiffness matrix, inertia matrix and the initial stress vector, which describe the space–time element, have the following forms:

$$\mathbf{K} = \iint_{A_{\alpha h}} (\mathcal{D}\mathbf{N}_{\alpha h}(x, y))^T\, \mathbf{E}\, \mathcal{D}\mathbf{N}(x, y, \alpha h/2)\, \mathrm{d}x\, \mathrm{d}y \;\cdot \alpha h \tag{42}$$

$$\mathbf{M} = \iint_{A_{\alpha h}} \mathbf{N}^T_{\alpha h}(x, y) \rho \frac{\partial \mathbf{N}(x, y, \alpha h)}{\partial t}\, \mathrm{d}x\, \mathrm{d}y \tag{43}$$

$$\mathbf{Z} = \iint_{A_{\alpha h}} \mathbf{N}^T_{\alpha h}(x, y)\, \eta_z\, \mathbf{N}(x, y, \alpha h)\, \mathrm{d}x\, \mathrm{d}y \tag{44}$$

$$\mathbf{e} = \iint_{A_{\alpha h}} (\mathcal{D}\mathbf{N}_{\alpha h}(x, y))^T\, \mathbf{E}\, \boldsymbol{\varepsilon}_0\, \mathrm{d}x\, \mathrm{d}y \tag{45}$$

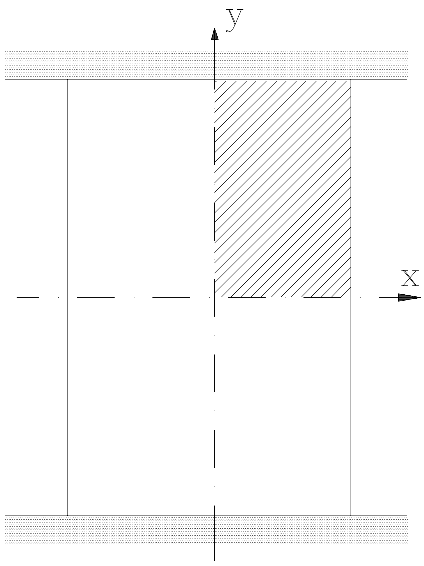


Figure 10: Rectangular sample compressed with a constant velocity.

for $t = \alpha h$. $\mathbf{N}_{\alpha h}$ is a matrix of the interpolation functions determined on the surface $A_{\alpha h}$ and $\mathbf{N}(x, y, \cdot)$ is a matrix of the interpolation functions described for the volume $\Omega$ and determined in a given moment $(\cdot)$. The change of limits of integration simplifies the formulas which become more convenient for numerical calculations.

Another attempt of the introduction of virtual functions different from the first order polynome was undertaken by Bohatier[1]. He assumed the virtual functions constant in time. They depend on the values at the end of the time interval $t = h$ and lead to the convergent scheme. However, the stability is limited to $h < 2\sqrt{3}\sqrt{m/k}$.

# 5 Numerical example

The initial rectangular domain cut from the greater one by the axis of symmetry is compressed with the constant velocity (Fig. 10). Viscoplastic behaviour of material was assumed as in the work (Bohatier, 1992). Dimensions of the sample: $h$=7.7 cm, $b$=5 cm. Other constants: $m$=0.1, $K$=0.01, $\rho$=0.0. All the nodes at the upper surface, except one at the right corner, are fixed. The nodes placed on the axis of symmetry can slide. Generalized deformation is depicted in Fig. 11. A good convergence with results obtained in (Bohatier, 1985) can be noticed.

[1] Unpublished communication

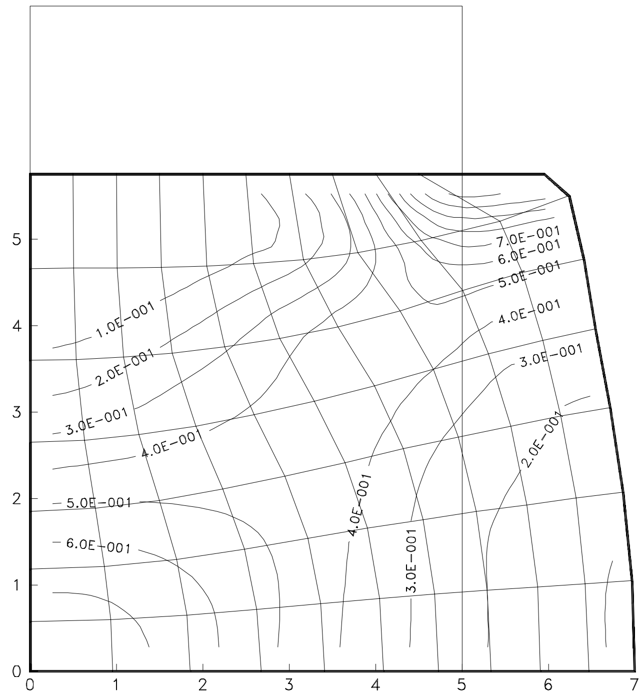

Figure 11: Generalized strain in the compressed rectangle.

# 6 Conclusions

In the paper we have found a new procedure for the time integration of the differential equation of motion of the dynamical systems with continuous distribution of displacements and velocities between two successive time intervals. Up to now the requirement of continuity could be fulfilled by the space–time element method which required the integration over time as well as over spatial domain. The obtained scheme of time integration proves that accurate results can be obtained also with a low numerical cost, preserving the continuity of investigated object in time.

More detailed analysis of the continuous approach would exhibit additional profits for geometrically non linear analysis. However, it is a separate problem.

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