Photon Trajectories around Schwarzschild and Kerr Black Holes

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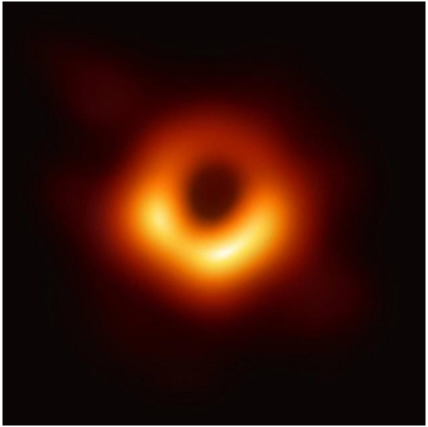
- Introduction
- Mathematical tools (Main Ingredients)
- Theory (Recipe)
- Conclusion

Motivation - Our Whatsapp group DP :)

First ever image of a Blackhole:

- Released in 2019 by EHT group.
- **Unbelievable !** So we **CHECK** the Physics ourselves.



M87* Supermassive Blackhole

Einstein Field Equations

- Einstein field equation:
$$G_{\mu\nu} = 8\pi T_{\mu\nu} \tag{1}$$
- For vacuum spacetime:
$$T_{\mu\nu} = 0 \tag{2}$$
- Solving these equations under symmetry assumptions gives 4 key black-hole solutions:
  - Schwarzschild (nonrotating, uncharged)
  - Kerr (rotating, uncharged)
  - Reissner–Nordström (charged, nonrotating)
  - Kerr-Newman Black Hole (charged, rotating)
- **Kerr is the most physically relevant solution for real black holes.**

Photon Trajectories around Schwarzschild Black Holes

We will first serve our appetizer !

What are we trying to answer/cook ?

We want to understand the **Motion of Photons** around **Schwarzschild Blackholes**:

1. Is there any **Possibility of a Circular Orbit** and what will be its **radius** ?
2. What is the **condition** for a photon to move in this orbit?
3. **Stability** of this orbit ?
4. How would **all possible trajectories** look like ?

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The metric is given as:

$$ds^2 = -\left(1 - \frac{2GM}{rc^2}\right) c^2 dt^2 + \left(1 - \frac{2GM}{rc^2}\right)^{-1} dr^2 + r^2 \left(d\theta^2 + \sin^2\theta d\phi^2\right)$$

Using geometric units (where $G = c = 1$):

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 \left(d\theta^2 + \sin^2\theta d\phi^2\right)$$

Geodesic equation and Null condition

The **geodesic equation** is:

$$\frac{d^2x^\alpha}{d\lambda^2} + \Gamma^\alpha_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} = 0$$

A photon must travel on a **null path**.

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = 0$$

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Circular orbit and its radius

- Select the **Radial Component**:

$$\frac{d^2 r}{d\lambda^2} + \Gamma^r_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda} = 0$$

- Expanding:

$$\frac{d^2 r}{d\lambda^2} + \Gamma^r_{tt} \left(\frac{dt}{d\lambda} \right)^2 + \Gamma^r_{rr} \left(\frac{dr}{d\lambda} \right)^2 + \Gamma^r_{\theta\theta} \left(\frac{d\theta}{d\lambda} \right)^2 + \Gamma^r_{\phi\phi} \left(\frac{d\phi}{d\lambda} \right)^2 + (\text{cross terms}) \cdots = 0$$

(Note: For the Schwarzschild metric, all "cross terms" like $\Gamma^r_{t\phi}$ are zero.)

Circular orbit and its radius

- Condition for a Circular Orbit:
  1. **Radial velocity is zero:** $\dot{r} = \frac{dr}{d\lambda} = 0$
  2. **Radial acceleration is zero:** $\ddot{r} = \frac{d^2 r}{d\lambda^2} = 0$
- Assuming an orbit in the equatorial plane, so $\theta = \pi/2$ and $\dot{\theta} = 0$ (Do we lose any generality?)
- This is the general condition for **any particle** to have a circular orbit:

$$\boxed{\Gamma^r_{tt}\left(\frac{dt}{d\lambda}\right)^2 + \Gamma^r_{\phi\phi}\left(\frac{d\phi}{d\lambda}\right)^2 = 0} \tag{A}$$

Circular orbit and its radius

- The Photon (Null Geodesic) Condition:

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = 0$$

- For circular orbit ($\dot{r} = 0, \dot{\theta} = 0$), this equation becomes:

$$\boxed{g_{tt} \left(\frac{dt}{d\lambda}\right)^2 + g_{\phi\phi} \left(\frac{d\phi}{d\lambda}\right)^2 = 0} \tag{B}$$

Circular orbit and its radius

Substituting Metric components and **Christoffel Symbols** in equations A and B:

- Equation A:

$$\boxed{\frac{M}{r^2}\left(\frac{dt}{d\lambda}\right)^2 = r\left(\frac{d\phi}{d\lambda}\right)^2}$$

- Equation B:

$$\boxed{\left(1-\frac{2M}{r}\right)\left(\frac{dt}{d\lambda}\right)^2 = r^2\left(\frac{d\phi}{d\lambda}\right)^2}$$

Circular orbit and its radius

- **Solving** these 2 equations:

$$\frac{\left(1-\frac{2M}{r}\right)\left(\frac{dt}{d\lambda}\right)^2}{\frac{M}{r^2}\left(\frac{dt}{d\lambda}\right)^2}=\frac{r^2\left(\frac{d\phi}{d\lambda}\right)^2}{r\left(\frac{d\phi}{d\lambda}\right)^2}$$

- $\dot{t}^2$ **and** $\dot{\phi}^2$ **cancel**, leaving an equation purely about r:

$$\frac{1-\frac{2M}{r}}{\frac{M}{r^2}}=\frac{r^2}{r}$$

Circular orbit and its radius

Simplifying further:

$$\boxed{r = 3M}$$

This is the radius of the **Photon Sphere/Ring** around a **Schwarzschild black hole**, the unique circular orbit where a photon can orbit the black hole.

Critical Impact Parameter

Starting with Conserved Quantities:

- Lagrangian ($\mathcal{L}$) for a free particle is defined as:

$$\mathcal{L} = \frac{1}{2} g_{\mu\nu} \dot{x}^{\mu} \dot{x}^{\nu}$$

- Using this for the Schwarzschild metric (in the equatorial plane, $\dot{\theta} = 0$), we get:

$$\mathcal{L} = \frac{1}{2} \left[-\left(1 - \frac{2M}{r}\right) \dot{t}^2 + \left(1 - \frac{2M}{r}\right)^{-1} \dot{r}^2 + r^2 \dot{\phi}^2 \right]$$

Critical Impact Parameter

3 Theory (Recipe)

Conserved Quantities:

- $$p_t = \frac{\partial \mathcal{L}}{\partial \dot{t}} = -\left(1 - \frac{2M}{r}\right)\dot{t} = -E$$
  $$\implies \dot{t} = \frac{E}{1 - 2M/r}$$

- $$p_\phi = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = r^2\dot{\phi} = L$$
  $$\implies \dot{\phi} = \frac{L}{r^2}$$

Critical Impact Parameter

3 Theory (Recipe)

- Substituting the Photon Sphere Radius $r = 3M$ in Equation B:

$$\left(1 - \frac{2M}{3M}\right)\dot{t}^2 = (3M)^2\dot{\phi}^2 \tag{*}$$

- Substitute Conserved Quantities (Evaluated at $r = 3M$):

$$\dot{t}(r = 3M) = \frac{E}{1 - 2M/3M} = \frac{E}{1/3} = 3E$$
$$\dot{\phi}(r = 3M) = \frac{L}{(3M)^2} = \frac{L}{9M^2}$$

Critical Impact Parameter

- Substituting these into (*):

$$\frac{1}{3}(3E)^2 = 9M^2\left(\frac{L}{9M^2}\right)^2$$

- Rearranging:

$$27M^2 = \frac{L^2}{E^2}$$

Critical Impact Parameter

- Now we define the ratio $\frac{L}{E}$ as b i.e. **Impact parameter**
- Let's call this b value for circular orbit to be b_{crit} i.e. **Critical Impact Parameter**:

$$\boxed{b_{crit} = \sqrt{27}M}$$

Equations for Photon Trajectory

1. **Radial Velocity Equation:**

$$\boxed{\frac{dr}{d\lambda} = \dot{r}}$$

2. **Angular Velocity Equation:**

$$\boxed{\dot{\phi} = \frac{d\phi}{d\lambda} = \frac{L}{r^2}}$$

3. **Radial Acceleration Equation:**

$$\boxed{\frac{d\dot{r}}{d\lambda} = \ddot{r} = \text{Oops ?}}$$

Equations for Photon Trajectory

3 Theory (Recipe)

Radial Acceleration Equation:

- Removing $\dot{r} = 0$ and $\ddot{r} = 0$ constraints on geodesic equation:

$$\frac{d^2 r}{d\lambda^2} + \Gamma^r_{tt}(\dot{t})^2 + \Gamma^r_{rr}(\dot{r})^2 + \Gamma^r_{\phi\phi}(\dot{\phi})^2 = 0$$

- Plugging in Christoffel Symbols, $\dot{t}$ and $\dot{\phi}$:

$$\ddot{r} + \left[\frac{M(1-2M/r)}{r^2}\right]\left(\frac{E}{1-2M/r}\right)^2 - \left[\frac{M}{r^2(1-2M/r)}\right](\dot{r})^2 - [r(1-2M/r)]\left(\frac{L}{r^2}\right)^2 = 0$$

Equations for Photon Trajectory

- On further simplification:

$$\ddot{r} + \frac{M}{r^2(1-2M/r)}\left(E^2 - \dot{r}^2\right) - \frac{L^2(1-2M/r)}{r^3} = 0 \tag{C}$$

- Using the Null condition:

$$g_{tt}(\dot{t})^2 + g_{rr}(\dot{r})^2 + g_{\phi\phi}(\dot{\phi})^2 = 0$$

- Plugging in metric components, $\dot{t}$ and $\dot{\phi}$:

$$-\left(1-\frac{2M}{r}\right)\left[\frac{E}{1-2M/r}\right]^2 + \left(1-\frac{2M}{r}\right)^{-1}\dot{r}^2 + r^2\left[\frac{L}{r^2}\right]^2 = 0$$

Equations for Photon Trajectory

- Rearranging:

$$E^2 - \dot{r}^2 = \frac{L^2}{r^2}\left(1 - \frac{2M}{r}\right)$$

- Substituting in C:

$$\ddot{r} + \frac{M}{r^2(1-2M/r)}\left[\frac{L^2}{r^2}\left(1 - \frac{2M}{r}\right)\right] - \frac{L^2(1-2M/r)}{r^3} = 0$$

- Simplifying this gives the **Radial Acceleration Equation:**

$$\boxed{\frac{d\dot{r}}{d\lambda} = \ddot{r} = \frac{L^2}{r^3} - \frac{3ML^2}{r^4}}$$

Equations for Photon Trajectory

1. **Radial Velocity Equation:**

$$\boxed{\frac{dr}{d\lambda} = \dot{r}}$$

2. **Angular Velocity Equation:**

$$\boxed{\dot{\phi} = \frac{d\phi}{d\lambda} = \frac{L}{r^2}}$$

3. **Radial Acceleration Equation:**

$$\boxed{\frac{d\dot{r}}{d\lambda} = \ddot{r} = \frac{L^2}{r^3} - \frac{3ML^2}{r^4}}$$

- **Analytical method** can only take us so far!

Stability Analysis of Circular orbit

- **Perturb the Orbit:**

$$r(\lambda) = r_0 + \delta r(\lambda) = 3M + \delta r(\lambda)$$

- Acceleration of this perturbed radius:

$$\ddot{r} = \frac{d^2}{d\lambda^2}(r_0 + \delta r) = \frac{d^2(\delta r)}{d\lambda^2} = \ddot{\delta r}$$

- Radial acceleration function:

$$\ddot{r} = f(r) \quad \text{where} \quad f(r) = L^2 r^{-3} - 3ML^2 r^{-4}$$

Stability Analysis of Circular orbit

- Taylor Expanding and keeping 1st order term:

$$\ddot{\delta r} = f(r_0 + \delta r) \approx f(r_0) + \left[\frac{df}{dr}\right]_{r=r_0} \cdot \delta r$$

- We know $f(r_0) = 0$:

$$\ddot{\delta r} \approx [f'(r_0)] \cdot \delta r$$

- Stability now depends on the sign of the coefficient $f'(r_0) = f'(3M)$.

- Calculating the Derivative $f'(r)$:

$$f'(r) = \frac{d}{dr}f(r) = -\frac{3L^2}{r^4} + \frac{12ML^2}{r^5}$$

- Evaluating $f'(r_0)$ at $r_0 = 3M$:

$$\boxed{f'(3M) = \frac{L^2}{81M^4}} \qquad \text{(Positive)}$$

This implies that **orbit is unstable**.

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Here's What We Cooked Up !

1. We found out the **circular orbit radius** $r = 3M$.
2. The photons orbiting this radius satisfy $\frac{L}{E} = \sqrt{27}M$
3. Obtained **equations of motion** of photons.
4. Watched the **simulation** of photons playing garba around the blackhole.
5. Found that **circular orbit is unstable** and any perturbation will result in photons either getting captured into the blackhole or escaping.