Photon Trajectories around Schwarzschild Blackholes

Naman Soni
Department of Physics
IISER Bhopal

Fall 2025

Contents

1 Circular Orbit radius from Geodesic Equation

Step 1: The General Geodesic Equation

We begin with the general geodesic equation. This describes the path (geodesic) a free particle follows through curved spacetime:

$$\frac{d^2x^\alpha}{d\lambda^2} + \Gamma^\alpha_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} = 0$$

Here, $\Gamma^\alpha_{\mu\nu}$ are the Christoffel symbols, which encode the curvature of spacetime.

Schwarzschild Metric:

Using full SI units (where G and c are included):

$$ds^2 = -\left(1 - \frac{2GM}{rc^2}\right)c^2dt^2 + \left(1 - \frac{2GM}{rc^2}\right)^{-1}dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)$$

Using geometric units (where $G = c = 1$):

$$ds^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + \left(1 - \frac{2M}{r}\right)^{-1}dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)$$

Step 2: The Condition for a Circular Orbit

We are looking for a circular orbit at a constant radius r. This imposes two powerful conditions:

1. **Radial velocity is zero:** $\dot{r} = \frac{dr}{d\lambda} = 0$
2. **Radial acceleration is zero:** $\ddot{r} = \frac{d^2r}{d\lambda^2} = 0$

Select the Radial Component

We are interested in the conditions for a *circular* orbit, which means we care about the radial motion. We select the equation for the radial component by setting $\alpha = r$:

$$\frac{d^2r}{d\lambda^2} + \Gamma^r_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} = 0$$

The second term is a sum over all combinations of μ and ν (i.e., t, r, θ, ϕ). Written out, it looks like this:

$$\frac{d^2r}{d\lambda^2} + \Gamma^r_{tt}\left(\frac{dt}{d\lambda}\right)^2 + \Gamma^r_{rr}\left(\frac{dr}{d\lambda}\right)^2 + \Gamma^r_{\theta\theta}\left(\frac{d\theta}{d\lambda}\right)^2 + \Gamma^r_{\phi\phi}\left(\frac{d\phi}{d\lambda}\right)^2 + (\text{cross terms})\cdots = 0$$

(Note: For the Schwarzschild metric, all "cross terms" like $\Gamma^r_{t\phi}$ are zero, which simplifies this from the start.)

We focus on the radial ($\alpha = r$) component of the geodesic equation. Applying our two conditions (and assuming an orbit in the equatorial plane, so $\theta = \pi/2$ and $\dot{\theta} = 0$), the equation simplifies dramatically:

$$\Gamma^r_{tt}\left(\frac{dt}{d\lambda}\right)^2 + \Gamma^r_{\phi\phi}\left(\frac{d\phi}{d\lambda}\right)^2 = 0$$

This is the general condition for **any particle** to have a circular orbit.

Step 3: The Photon (Null Geodesic) Condition

This is our second, independent condition. A photon must travel on a null path, where the total spacetime interval ds^2 is always zero.

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = 0$$

For our circular orbit ($\dot{r} = 0, \dot{\theta} = 0$), this equation becomes:

$$g_{tt}\left(\frac{dt}{d\lambda}\right)^2 + g_{\phi\phi}\left(\frac{d\phi}{d\lambda}\right)^2 = 0$$

Step 4: Substitute the Metric components and Christoffel Symbols

Now we insert the known values for the Schwarzschild spacetime:

Metric Components:

- $g_{tt} = -\left(1 - \frac{2M}{r}\right)$
- $g_{\phi\phi} = r^2$
- $g_{rr} = \left(1 - \frac{2M}{r}\right)^{-1}$
- $g^{rr} = \left(1 - \frac{2M}{r}\right)$

Christoffel Symbols:

$$\boxed{\Gamma^{\rho}_{\mu\nu} = \frac{1}{2} g^{\rho\sigma} \left(\frac{\partial g_{\sigma\nu}}{\partial x^{\mu}} + \frac{\partial g_{\sigma\mu}}{\partial x^{\nu}} - \frac{\partial g_{\mu\nu}}{\partial x^{\sigma}} \right)}$$

Derivation of Γ^r_{tt}

$$\Gamma^r_{tt} = \frac{1}{2} g^{rr} (-\partial_r g_{tt})$$

$$= \frac{1}{2} \left(1 - \frac{2M}{r}\right) \left(- \left(- \frac{2M}{r^2} \right) \right)$$

$$= \frac{1}{2} \left(1 - \frac{2M}{r}\right) \left(\frac{2M}{r^2} \right)$$

$$\boxed{\Gamma^r_{tt} = \frac{M}{r^2} \left(1 - \frac{2M}{r}\right)}$$

Derivation of $\Gamma^r_{\phi\phi}$

This derivation assumes the coordinate swap where $g_{\phi\phi} = r^2$.

$$\Gamma^r_{\phi\phi} = \frac{1}{2} g^{rr} (-\partial_r g_{\phi\phi})$$

$$= \frac{1}{2} g^{rr} \left(-\partial_r (r^2) \right)$$

$$= \frac{1}{2} g^{rr} (-2r)$$

$$= \frac{1}{2} \left(1 - \frac{2M}{r}\right) (-2r)$$

$$\boxed{\Gamma^r_{\phi\phi} = -r \left(1 - \frac{2M}{r}\right)}$$

Step 5: Create the System of Two Equations

We now write out our two conditions from Step 2 and Step 3 using these components.

Equation A (from the Geodesic Equation):

$$\left[\frac{M}{r^2} \left(1 - \frac{2M}{r}\right) \right] \left(\frac{dt}{d\lambda} \right)^2 + \left[-r \left(1 - \frac{2M}{r}\right) \right] \left(\frac{d\phi}{d\lambda} \right)^2 = 0$$

We can simplify this by dividing the entire equation by $\left(1 - \frac{2M}{r}\right)$ and rearranging:

$$\frac{M}{r^2} \left(\frac{dt}{d\lambda} \right)^2 = r \left(\frac{d\phi}{d\lambda} \right)^2$$

Equation B (from the Photon Condition):

$$-\left(1-\frac{2M}{r}\right)\left(\frac{dt}{d\lambda}\right)^2+r^2\left(\frac{d\phi}{d\lambda}\right)^2=0$$

Rearranging this:

$$\left(1-\frac{2M}{r}\right)\left(\frac{dt}{d\lambda}\right)^2=r^2\left(\frac{d\phi}{d\lambda}\right)^2$$

Step 6: Solve the System

We now have two equations that must both be true, and our goal is to find the one value of r that makes this possible. The cleanest way is to divide Equation B by Equation A:

$$\frac{\left(1-\frac{2M}{r}\right)\left(\frac{dt}{d\lambda}\right)^2}{\frac{M}{r^2}\left(\frac{dt}{d\lambda}\right)^2}=\frac{r^2\left(\frac{d\phi}{d\lambda}\right)^2}{r\left(\frac{d\phi}{d\lambda}\right)^2}$$

The velocity terms ($\dot{t}^2$ and $\dot{\phi}^2$) cancel completely, leaving an equation purely about r:

$$\frac{1-\frac{2M}{r}}{\frac{M}{r^2}}=\frac{r^2}{r}$$

$$\frac{1-\frac{2M}{r}}{\frac{M}{r^2}}=r$$

Now, solve for r:

$$1-\frac{2M}{r}=r\left(\frac{M}{r^2}\right)$$

$$1-\frac{2M}{r}=\frac{M}{r}$$

$$1=\frac{M}{r}+\frac{2M}{r}$$

$$1=\frac{3M}{r}$$

$$\boxed{r=3M}$$

This is the radius of the **photon sphere** around a Schwarzschild black hole—the unique circular orbit where a photon can orbit the black hole.

2 Critical Impact Parameter from the Geodesic Equation

We aim to find the critical impact parameter (b_{crit}) for a photon in Schwarzschild spacetime. This is the specific value of $b = L/E$ that corresponds to the unstable circular orbit at the photon sphere radius ($r = 3M$). We will use the geodesic equation formalism.

Step 1: Conditions for a Circular Photon Orbit

From the analysis of the geodesic equation ($\frac{d^2x^\alpha}{d\lambda^2}+\Gamma^\alpha_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = 0$) and the null condition ($ds^2 = g_{\mu\nu}dx^\mu dx^\nu = 0$) for a circular orbit ($\dot{r} = 0, \ddot{r} = 0$) in the equatorial plane ($\dot{\theta} = 0$), we previously derived two necessary conditions:

1. **From the radial geodesic equation**: This gives the condition for any circular orbit. After substituting the relevant Christoffel symbols (Γ^r_{tt} and $\Gamma^r_{\phi\phi}$) and simplifying, we obtained:

$$\frac{M}{r^2}\dot{t}^2=r\dot{\phi}^2 \tag{1}$$

2. **From the null condition** ($ds^2 = 0$): This applies specifically to photons. After substituting the relevant metric components (g_{tt} and $g_{\phi\phi}$) and simplifying, we obtained:

$$\left(1 - \frac{2M}{r}\right)\dot{t}^2 = r^2\dot{\phi}^2 \tag{2}$$

Step 2: Incorporate Conserved Quantities

Lagrangian ($\mathcal{L}$) for a free particle in General Relativity is defined as:

$$\mathcal{L} = \frac{1}{2}g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu$$

When we expand this general definition for the Schwarzschild metric (in the equatorial plane, $\dot{\theta} = 0$), we get:

$$\mathcal{L} = \frac{1}{2}\left(g_{tt}\dot{t}^2 + g_{rr}\dot{r}^2 + g_{\phi\phi}\dot{\phi}^2\right)$$

$$\mathcal{L} = \frac{1}{2}\left[-\left(1 - \frac{2M}{r}\right)\dot{t}^2 + \left(1 - \frac{2M}{r}\right)^{-1}\dot{r}^2 + r^2\dot{\phi}^2\right]$$

The time-independence and axial symmetry of the Schwarzschild metric lead to conserved energy (E) and angular momentum (L) via the Euler-Lagrange equations (which are equivalent to the t and ϕ components of the geodesic equation):

$$\begin{aligned}
p_t &= \frac{\partial\mathcal{L}}{\partial\dot{t}} = -\left(1 - \frac{2M}{r}\right)\dot{t} = -E \\
&\implies \dot{t} = \frac{E}{1 - 2M/r} \\
p_\phi &= \frac{\partial\mathcal{L}}{\partial\dot{\phi}} = r^2\dot{\phi} = L \\
&\implies \dot{\phi} = \frac{L}{r^2}
\end{aligned}$$

Step 3: Use the Photon Sphere Radius

We know from solving the system (1) and (2) that the only radius where a circular photon orbit is possible is the photon sphere radius:

$$r = 3M$$

Step 4: Substitute into Orbit Equation

Let's substitute $r = 3M$ into one of the orbit equations, for example, Equation (2):

$$\begin{aligned}
\left(1 - \frac{2M}{3M}\right)\dot{t}^2 &= (3M)^2\dot{\phi}^2 \\
\left(1 - \frac{2}{3}\right)\dot{t}^2 &= 9M^2\dot{\phi}^2 \\
\frac{1}{3}\dot{t}^2 &= 9M^2\dot{\phi}^2 \qquad (*)
\end{aligned}$$

Step 5: Substitute Conserved Quantities (Evaluated at $r = 3M$)

Now, we substitute the expressions for $\dot{t}$ and $\dot{\phi}$ in terms of E and L into Equation (*). We need to evaluate $\dot{t}$ at $r = 3M$:

$$\begin{aligned}
\dot{t}(r = 3M) &= \frac{E}{1 - 2M/3M} = \frac{E}{1/3} = 3E \\
\dot{\phi}(r = 3M) &= \frac{L}{(3M)^2} = \frac{L}{9M^2}
\end{aligned}$$

Substituting these into (*):

$$\frac{1}{3}(3E)^2 = 9M^2\left(\frac{L}{9M^2}\right)^2$$

Step 6: Solve for the Impact Parameter $b = L/E$

Simplify the equation:

$$\frac{1}{3}(9E^2) = 9M^2\left(\frac{L^2}{81M^4}\right)$$
$$3E^2 = \frac{9M^2L^2}{81M^4}$$
$$3E^2 = \frac{L^2}{9M^2}$$

Now, rearrange to isolate L^2/E^2:

$$27M^2 = \frac{L^2}{E^2}$$

Since the impact parameter is defined as $b = L/E$, we have $b^2 = L^2/E^2$. Therefore:

$$b^2 = 27M^2$$

Taking the square root (impact parameter is positive):

$$\boxed{b = \sqrt{27}M}$$

This is the critical impact parameter b_{crit}, derived from the geodesic formalism and the properties of the Schwarzschild metric.

3 Equations for Photon Trajectory

1. **The Radial Velocity Equation:** This defines the radial velocity variable $\dot{r}$ used to make the system first-order.

$$\boxed{\frac{dr}{d\lambda} = \dot{r}}$$

   (In the code, this is `dr_dlambda = r_dot`)

2. **The Angular Velocity Equation:** This describes how the angle changes and comes from the conservation of angular momentum L.

$$\boxed{\dot{\phi} = \frac{d\phi}{d\lambda} = \frac{L}{r^2}}$$

   (In the code, this is dphi_dlambda = L / r**2)

3. **The Radial Acceleration Equation:** This describes how the radial velocity changes.

$$\boxed{\frac{d\dot{r}}{d\lambda} = \frac{L^2}{r^3} - \frac{3ML^2}{r^4}}$$

   (This is equivalent to $\frac{d^2r}{d\lambda^2} = \ldots$ In the code, this is `dr_dot_dlambda = (L**2 / r**3) - (3 * M * L**2 / r**4)`)

Deriving the Acceleration equation:

1. **The Radial Geodesic Equation:**
   We start with the radial component ($\alpha = r$) of the general geodesic equation. For an equatorial orbit ($\dot{\theta} = 0$), this is:

$$\frac{d^2r}{d\lambda^2} + \Gamma^r_{tt}(\dot{t})^2 + \Gamma^r_{rr}(\dot{r})^2 + \Gamma^r_{\phi\phi}(\dot{\phi})^2 = 0$$

   (Using $\dot{x} = dx/d\lambda$ and $\ddot{r} = d^2r/d\lambda^2$.)

2. **Substitute Christoffel Symbols:**
   For the Schwarzschild metric, the required Christoffel symbols are:

$$\Gamma^r_{tt} = \frac{M}{r^2}\left(1 - \frac{2M}{r}\right)$$

$$\Gamma^r_{\phi\phi} = -r\left(1 - \frac{2M}{r}\right)$$

   Derivation of Γ^r_{rr}:

$$\begin{aligned}
\Gamma^r_{rr} &= \frac{1}{2} \cdot g^{rr} \cdot \left(\frac{\partial g_{rr}}{\partial r}\right) \\
\frac{\partial g_{rr}}{\partial r} &= \frac{\partial}{\partial r}\left(1 - \frac{2M}{r}\right)^{-1} \\
\frac{\partial g_{rr}}{\partial r} &= -\frac{2M}{r^2}\left(1 - \frac{2M}{r}\right)^{-2}
\end{aligned}$$

$$\Gamma^r_{rr} = \frac{1}{2} \cdot \left[1 - \frac{2M}{r}\right] \cdot \left[-\frac{2M}{r^2}\left(1 - \frac{2M}{r}\right)^{-2}\right]$$

$$\boxed{\Gamma^r_{rr} = -\frac{M}{r^2}\left(1 - \frac{2M}{r}\right)^{-1}}$$

   Plugging these into the equation from Step 1 gives:

$$\ddot{r} + \left[\frac{M}{r^2}\left(1 - \frac{2M}{r}\right)\right](\dot{t})^2 - \left[\frac{M}{r^2\left(1 - \frac{2M}{r}\right)}\right](\dot{r})^2 - \left[r\left(1 - \frac{2M}{r}\right)\right](\dot{\phi})^2 = 0$$

3. **Substitute Conserved Quantities (E and L):** Lagrangian ($\mathcal{L}$) for a free particle in General Relativity is defined as:

$$\mathcal{L} = \frac{1}{2} g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu$$

   When we expand this general definition for the Schwarzschild metric (in the equatorial plane, $\dot{\theta} = 0$), we get:

$$\mathcal{L} = \frac{1}{2}\left(g_{tt}\dot{t}^2 + g_{rr}\dot{r}^2 + g_{\phi\phi}\dot{\phi}^2\right)$$

$$\mathcal{L} = \frac{1}{2}\left[-\left(1 - \frac{2M}{r}\right)\dot{t}^2 + \left(1 - \frac{2M}{r}\right)^{-1}\dot{r}^2 + r^2\dot{\phi}^2\right]$$

The time-independence and axial symmetry of the Schwarzschild metric lead to conserved energy (E) and angular momentum (L) via the Euler-Lagrange equations (which are equivalent to the t and ϕ components of the geodesic equation):

$$
\begin{aligned}
p_t &= \frac{\partial \mathcal{L}}{\partial \dot{t}} = -\left(1 - \frac{2M}{r}\right)\dot{t} = -E \\
&\implies \dot{t} = \frac{E}{1 - 2M/r} \\
p_\phi &= \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = r^2\dot{\phi} = L \\
&\implies \dot{\phi} = \frac{L}{r^2}
\end{aligned}
$$

Substitute these into the equation from Step 2:

$$\ddot{r} + \left[\frac{M(1-2M/r)}{r^2}\right]\left(\frac{E}{1-2M/r}\right)^2 - \left[\frac{M}{r^2(1-2M/r)}\right](\dot{r})^2 - [r(1-2M/r)]\left(\frac{L}{r^2}\right)^2 = 0$$

Now, simplify the terms:

$$\ddot{r} + \frac{ME^2}{r^2(1-2M/r)} - \frac{M(\dot{r})^2}{r^2(1-2M/r)} - \frac{L^2(1-2M/r)}{r^3} = 0$$

$$\ddot{r} + \frac{M}{r^2(1-2M/r)}\left(E^2 - \dot{r}^2\right) - \frac{L^2(1-2M/r)}{r^3} = 0$$

4. **Use the Photon (Null) Condition to Simplify:**

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = 0$$

We write out the full ds^2 for the Schwarzschild metric (in the equatorial plane, $d\theta = 0$):

$$ds^2 = g_{tt}\,dt^2 + g_{rr}\,dr^2 + g_{\phi\phi}\,d\phi^2 = 0$$

Since, the coordinates are functions of λ:

$$x^\mu = x^\mu(\lambda)$$

$$dx^\mu = \dot{x}^\mu d\lambda$$

Using Einstein summation convention:

$$g_{\mu\nu}(dx^\mu)(dx^\nu) = 0$$

$$g_{\mu\nu}(\dot{x}^\mu d\lambda)(\dot{x}^\nu d\lambda) = 0$$

Here, $\mu = \nu$, therefore:

$$g_{\mu\mu}(\dot{x}^\mu d\lambda)(\dot{x}^\mu d\lambda) = 0$$

$$g_{\mu\mu}(\dot{x}^\mu)^2(d\lambda)^2 = 0$$

Dividing by $d\lambda^2$ (where λ is the affine parameter) gives:

$$g_{tt}(\dot{t})^2 + g_{rr}(\dot{r})^2 + g_{\phi\phi}(\dot{\phi})^2 = 0$$

Now, substitute the metric components:

$$g_{tt} = -\left(1 - \frac{2M}{r}\right)$$

$$g_{rr} = \left(1 - \frac{2M}{r}\right)^{-1}$$

$$g_{\phi\phi} = r^2$$

Plugging these in, our starting equation is:

$$-\left(1 - \frac{2M}{r}\right)\dot{t}^2 + \left(1 - \frac{2M}{r}\right)^{-1}\dot{r}^2 + r^2\dot{\phi}^2 = 0$$

Now, substitute the expressions for $\dot{t}$ and $\dot{\phi}$:

$$-\left(1 - \frac{2M}{r}\right)\left[\frac{E}{1 - 2M/r}\right]^2 + \left(1 - \frac{2M}{r}\right)^{-1}\dot{r}^2 + r^2\left[\frac{L}{r^2}\right]^2 = 0$$

To isolate $\dot{r}^2$, multiply the entire equation by $\left(1 - \frac{2M}{r}\right)$:

$$\left(1 - \frac{2M}{r}\right)\left[-\frac{E^2}{1 - \frac{2M}{r}} + \frac{\dot{r}^2}{1 - \frac{2M}{r}} + \frac{L^2}{r^2}\right] = 0$$

$$-E^2 + \dot{r}^2 + \frac{L^2}{r^2}\left(1 - \frac{2M}{r}\right) = 0$$

This gives us a crucial substitution for the term $E^2 - \dot{r}^2$:

$$E^2 - \dot{r}^2 = \frac{L^2}{r^2}\left(1 - \frac{2M}{r}\right)$$

5. **Substitute and Solve:**

   Now, substitute the expression for $(E^2 - \dot{r}^2)$ into the equation obtained in step 3:

$$\ddot{r} + \frac{M}{r^2(1 - 2M/r)}\left[\frac{L^2}{r^2}\left(1 - \frac{2M}{r}\right)\right] - \frac{L^2(1 - 2M/r)}{r^3} = 0$$

   The $\left(1 - \frac{2M}{r}\right)$ terms in the first part cancel out beautifully:

$$\ddot{r} + \frac{ML^2}{r^4} - \frac{L^2(1 - 2M/r)}{r^3} = 0$$

$$\ddot{r} = -\frac{ML^2}{r^4} + \frac{L^2(1 - 2M/r)}{r^3}$$

$$\ddot{r} = -\frac{ML^2}{r^4} + \left(\frac{L^2}{r^3} - \frac{2ML^2}{r^4}\right)$$

   Combine the r^4 terms:

$$\ddot{r} = \frac{L^2}{r^3} - \left(\frac{ML^2}{r^4} + \frac{2ML^2}{r^4}\right)$$

$$\ddot{r} = \frac{L^2}{r^3} - \frac{3ML^2}{r^4}$$

   Since $\ddot{r} = \frac{d\dot{r}}{d\lambda}$, we have the final equation:

$$\boxed{\frac{d\dot{r}}{d\lambda} = \frac{L^2}{r^3} - \frac{3ML^2}{r^4}}$$

4 Stability Analysis of the Photon Orbit using the Geodesic Equation

We can determine the stability of the circular photon orbit by performing a linear stability analysis directly on the radial geodesic equation of motion. This method involves introducing a small perturbation to the orbit and observing whether the perturbation grows or decays.

Step 1: The Radial Geodesic Equation

From the full geodesic formalism, we derived the equation for the radial acceleration ($\ddot{r} = \mathrm{d}^2r/\mathrm{d}\lambda^2$) of a photon:

$$\ddot{r} = \frac{L^2}{r^3} - \frac{3ML^2}{r^4} \tag{3}$$

Let's define the right-hand side as the **effective force** function, $f(r)$:

$$\ddot{r} = f(r) \quad \text{where} \quad f(r) = L^2r^{-3} - 3ML^2r^{-4}$$

Step 2: Find the Equilibrium Orbit

An equilibrium orbit (a circular orbit) exists when the radial acceleration is zero. We find this equilibrium radius, r_0, by setting $\ddot{r} = f(r_0) = 0$:

$$\begin{aligned} f(r_0) = \frac{L^2}{r_0^3} - \frac{3ML^2}{r_0^4} &= 0 \\ \frac{L^2}{r_0^3} &= \frac{3ML^2}{r_0^4} \end{aligned}$$

Assuming $L \neq 0$, we can cancel L^2 and multiply by r_0^4:

$$r_0 = 3M$$

This confirms our circular orbit is at $r_0 = 3M$.

Step 3: Perturb the Orbit

To test stability, we nudge the photon from its equilibrium orbit by a small, time-dependent amount, $\delta r(\lambda)$. The new, perturbed radius is:

$$r(\lambda) = r_0 + \delta r(\lambda) = 3M + \delta r(\lambda)$$

The acceleration of this perturbed radius is:

$$\ddot{r} = \frac{\mathrm{d}^2}{\mathrm{d}\lambda^2}(r_0 + \delta r) = \frac{\mathrm{d}^2(\delta r)}{\mathrm{d}\lambda^2} = \ddot{\delta r}$$

This acceleration must still obey the original geodesic equation:

$$\ddot{\delta r} = f(r_0 + \delta r)$$

Step 4: Linearize the Equation of Motion

We now use a Taylor expansion for $f(r)$ around the equilibrium point r_0. For a very small perturbation δr, we keep only the first-order terms:

$$f(r_0 + \delta r) \approx f(r_0) + \left[\frac{\mathrm{d}f}{\mathrm{d}r}\right]_{r=r_0} \cdot \delta r$$

Substitute this into our perturbed equation:

$$\ddot{\delta r} \approx f(r_0) + \left[\frac{\mathrm{d}f}{\mathrm{d}r}\right]_{r=r_0} \cdot \delta r$$

From Step 2, we know $f(r_0) = 0$ (it's the equilibrium point). This leaves the linearized equation for the perturbation:

$$\ddot{\delta r} \approx [f'(r_0)] \cdot \delta r$$

The stability now depends on the sign of the coefficient $f'(r_0) = f'(3M)$.

- If $f'(3M) < 0$, we have $\ddot{\delta r} = -k \cdot \delta r$. This is the equation for a simple harmonic oscillator (a stable spring), and the orbit is **stable**.
- If $f'(3M) > 0$, we have $\ddot{\delta r} = +k \cdot \delta r$. This is the equation for exponential growth, and the orbit is **unstable**.

Step 5: Calculate the Derivative $f'(r)$

We differentiate $f(r)$:

$$\begin{aligned} f'(r) = \frac{\mathrm{d}}{\mathrm{d}r} f(r) &= \frac{\mathrm{d}}{\mathrm{d}r}\left(L^2 r^{-3} - 3ML^2 r^{-4}\right) \\ &= L^2(-3)r^{-4} - 3ML^2(-4)r^{-5} \\ &= -\frac{3L^2}{r^4} + \frac{12ML^2}{r^5} \end{aligned}$$

Step 6: Evaluate $f'(r_0)$ at $r_0 = 3M$

Now, we plug the equilibrium radius $r_0 = 3M$ into the derivative:

$$\begin{aligned} f'(3M) &= -\frac{3L^2}{(3M)^4} + \frac{12ML^2}{(3M)^5} \\ &= -\frac{3L^2}{81M^4} + \frac{12ML^2}{243M^5} \\ &= -\frac{L^2}{27M^4} + \frac{12L^2}{243M^4} \\ &= -\frac{9L^2}{243M^4} + \frac{12L^2}{243M^4} \end{aligned}$$

$$\boxed{f'(3M) = +\frac{L^2}{81M^4}}$$

Step 7: Conclusion

The coefficient $f'(3M) = +\frac{L^2}{81M^4}$ is **positive**. Therefore, the circular photon orbit at $r = 3M$ is **unstable**.

5 Effective Potential formulation

Taking the Lagrangian:

$$\mathcal{L} = \frac{1}{2} g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu = \frac{1}{2} g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda}$$

Since,

$$g_{tt} = -\left(1 - \frac{2M}{r}\right), \quad g_{rr} = \left(1 - \frac{2M}{r}\right)^{-1}, \quad g_{\phi\phi} = r^2$$

So,

$$\mathcal{L} = \frac{1}{2}\left(g_{tt}\dot{t}^2 + g_{rr}\dot{r}^2 + g_{\phi\phi}\dot{\phi}^2\right)$$

$$\mathcal{L} = \frac{1}{2}\left[-\left(1 - \frac{2M}{r}\right)\dot{t}^2 + \left(1 - \frac{2M}{r}\right)^{-1}\dot{r}^2 + r^2\dot{\phi}^2\right]$$

Deriving the Effective Potential Equation from the Lagrangian

1. **Applying the photon condition:** ($\mathcal{L} = 0$):

$$0 = \mathcal{L} = \frac{1}{2}\left[-\left(1 - \frac{2M}{r}\right)\dot{t}^2 + \left(1 - \frac{2M}{r}\right)^{-1}\dot{r}^2 + r^2\dot{\phi}^2\right]$$

2. **Time-Symmetry (t):** The Lagrangian doesn't depend on t, so the generalized momentum p_t is constant. We call this constant $-E$ (Energy).

$$p_t = \frac{\partial \mathcal{L}}{\partial \dot{t}} = -\left(1 - \frac{2M}{r}\right)\dot{t} = -E$$

$$\implies \dot{t} = \frac{E}{1 - 2M/r}$$

3. **Rotational-Symmetry (ϕ):** The Lagrangian doesn't depend on ϕ, so p_ϕ is constant. We call this constant L (Angular Momentum).

$$p_\phi = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = r^2\dot{\phi} = L$$

$$\implies \dot{\phi} = \frac{L}{r^2}$$

4. Now, substitute these into the equation above:

$$0 = -\left(1 - \frac{2M}{r}\right)\left(\frac{E}{1 - 2M/r}\right)^2 + \left(1 - \frac{2M}{r}\right)^{-1}\dot{r}^2 + r^2\left(\frac{L}{r^2}\right)^2$$

5. This simplifies to:

$$0 = -\frac{E^2}{1 - 2M/r} + \dot{r}^2 + \frac{L^2}{r^2}$$

6. Multiply the entire equation by $(1 - \frac{2M}{r})$ to clear denominators:

$$0 = -E^2 + \dot{r}^2(1 - \frac{2M}{r}) + L^2\left(\frac{1}{r^2}\right)(1 - \frac{2M}{r})$$

Rearranging to isolate the radial motion term:

$$\left(\frac{dr}{d\lambda}\right)^2 = E^2 - \frac{L^2}{r^2}\left(1 - \frac{2M}{r}\right)$$

This has the familiar form of a 1D motion problem:

$$\left(\frac{dr}{d\lambda}\right)^2 = E^2 - V_{\text{eff}}^2(r)$$

where the **Effective Potential Squared**, $V_{\text{eff}}^2(r)$, is defined as:

$$V_{\text{eff}}^2(r) = \frac{L^2}{r^2}\left(1 - \frac{2M}{r}\right)$$

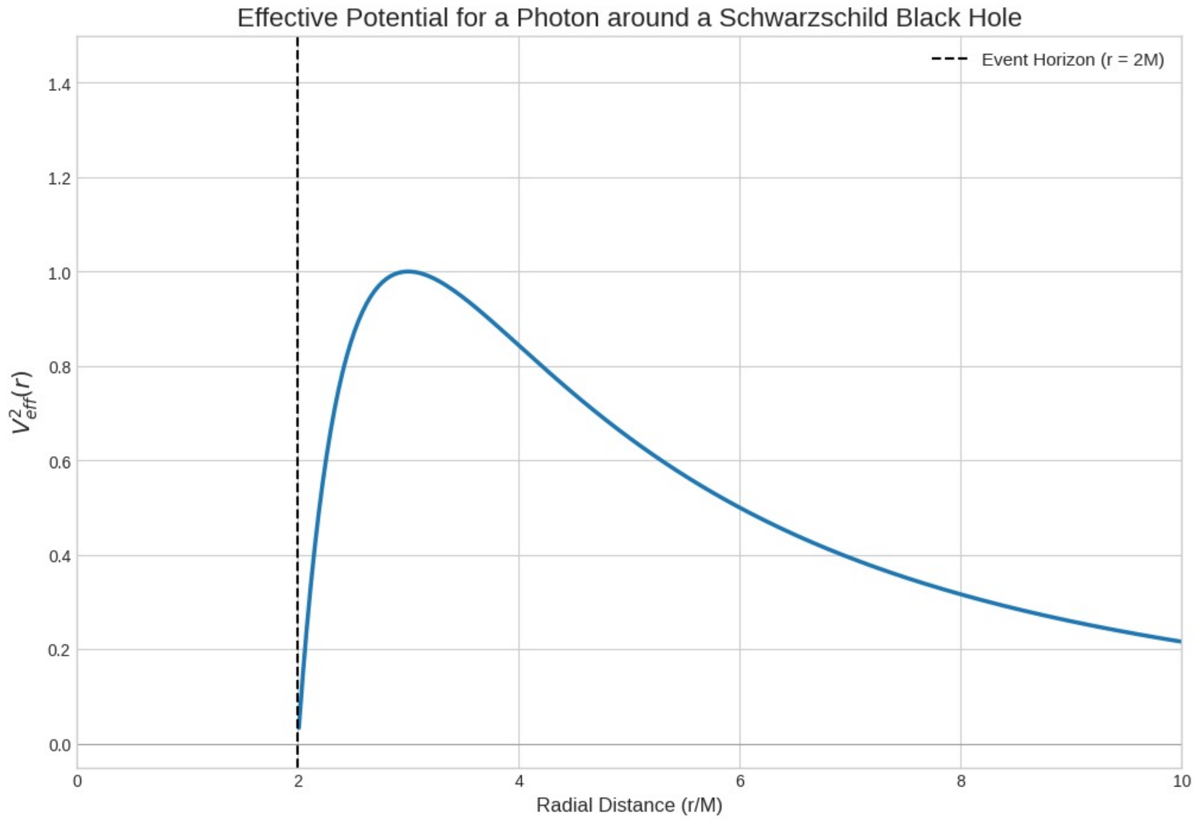

Figure 1: Effective potential Plot

Finding the Photon Sphere Radius (r_{ps})

A circular orbit occurs when the radial velocity is zero, $\frac{dr}{d\lambda} = 0$, and the particle is at a stationary point (a maximum or minimum) of the effective potential, $\frac{d}{dr}V_{\text{eff}}^2(r) = 0$.
For a photon sphere (an unstable circular orbit), we need:

1. Radial motion is zero: $E^2 - V_{\text{eff}}^2(r) = 0$ (meaning $E^2 = V_{\text{eff}}^2(r)$)
2. Stationary point: $\frac{d}{dr}V_{\text{eff}}^2(r) = 0$

We focus on the stationary point condition:

$$0 = \frac{d}{dr}V_{\text{eff}}^2(r) = \frac{d}{dr}\left[\frac{L^2}{r^2}\left(1 - \frac{2M}{r}\right)\right]$$

Since L is constant, we can remove it from the derivative:

$$0 = L^2\frac{d}{dr}\left[\frac{1}{r^2} - \frac{2M}{r^3}\right]$$

Taking the derivative with respect to r:

$$0 = \frac{d}{dr}(r^{-2} - 2Mr^{-3})$$

$$0 = (-2)r^{-3} - 2M(-3)r^{-4}$$

$$0 = -\frac{2}{r^3} + \frac{6M}{r^4}$$

Rearranging the terms:

$$\frac{2}{r^3} = \frac{6M}{r^4}$$

Multiplying by r^4:

$$2r = 6M$$

$$\implies r = 3M$$

Conclusion

The radius of the circular orbit (the **Photon Sphere**) is found to be:

$$\mathbf{r_{ps} = 3M}$$

Restoring the constants G and c:

$$\mathbf{r_{ps}} = \frac{\mathbf{3GM}}{\mathbf{c^2}}$$

This is a maximum of $V_{\text{eff}}^2(r)$, indicating that the orbit is **unstable**.

6 Finding the Critical Impact Parameter

Step 1: The Radial Energy Equation

We start with the radial energy equation for a photon, which we derived from the Lagrangian:

$$\left(\frac{dr}{d\lambda}\right)^2 = E^2 - V_{\text{eff}}^2(r)$$

Where the effective potential (squared) is:

$$V_{\text{eff}}^2(r) = \frac{L^2}{r^2}\left(1 - \frac{2M}{r}\right)$$

Step 2: Introduce the Impact Parameter b

The impact parameter b is defined as the ratio of angular momentum L to energy E:

$$b = \frac{L}{E} \implies L = bE$$

Now, substitute $L = bE$ into the effective potential equation:

$$\begin{aligned} V_{\text{eff}}^2(r) &= \frac{(bE)^2}{r^2}\left(1 - \frac{2M}{r}\right) \\ &= E^2\left[\frac{b^2}{r^2}\left(1 - \frac{2M}{r}\right)\right] \end{aligned}$$

Our main energy equation becomes:

$$\left(\frac{dr}{d\lambda}\right)^2 = E^2 - E^2\left[\frac{b^2}{r^2}\left(1 - \frac{2M}{r}\right)\right]$$

We can divide the entire equation by E^2 to get a master equation that depends only on the impact parameter b:

$$\frac{1}{E^2}\left(\frac{dr}{d\lambda}\right)^2 = 1 - \frac{b^2}{r^2}\left(1 - \frac{2M}{r}\right)$$

This is often written by defining a new potential,

$$V_b^2(r) = \frac{b^2}{r^2}\left(1 - \frac{2M}{r}\right)$$

Step 3: The Critical Orbit Condition

The critical (circular) orbit at the photon sphere is an unstable equilibrium. This happens at the **peak** of the potential energy curve.

- Condition 1: **At the peak, the radial velocity is zero.** This means the kinetic energy term is zero, and the total energy equals the potential energy:

$$1 = V_b^2(r) \implies 1 = \frac{b^2}{r^2}\left(1 - \frac{2M}{r}\right)$$

- Condition 2: **The orbit is at the peak itself.** We find the location of the peak by taking the derivative of the potential and setting it to zero. We already solved this and found the peak is at:

$$r = 3M$$

Step 4: Solve for the Critical Impact Parameter ($b_{\mathbf{crit}}$)

Now we just plug the location of the peak ($r = 3M$) into the energy condition (Condition 1) to find the specific value of b that makes this orbit possible.

Set $r = 3M$ in the equation $1 = \frac{b^2}{r^2}\left(1 - \frac{2M}{r}\right)$:

$$\begin{aligned}
1 &= \frac{b_{\text{crit}}^2}{(3M)^2}\left(1 - \frac{2M}{3M}\right) \\
1 &= \frac{b_{\text{crit}}^2}{9M^2}\left(1 - \frac{2}{3}\right) \\
1 &= \frac{b_{\text{crit}}^2}{9M^2}\left(\frac{1}{3}\right) \\
1 &= \frac{b_{\text{crit}}^2}{27M^2}
\end{aligned}$$

Now, solve for b_{crit}:

$$\boxed{b_{\text{crit}} = \sqrt{27}M}$$

This is the critical impact parameter. Any photon with $b > b_{\text{crit}}$ will be scattered (lensed), and any photon with $b < b_{\text{crit}}$ will be captured.

7 Equations for Photon Trajectory

1. **The Radial Velocity Definition:** This defines the radial velocity variable $\dot{r}$ used to make the system first-order.

$$\boxed{\frac{dr}{d\lambda} = \dot{r}}$$

(In the code, this is `dr_dlambda = r_dot`)

2. **The Angular Velocity Equation:** This describes how the angle changes and comes from the conservation of angular momentum L.

$$\boxed{\dot{\phi} = \frac{d\phi}{d\lambda} = \frac{L}{r^2}}$$

(In the code, this is dphi_dlambda = L / r**2)

3. **The Radial Acceleration Equation:** This describes how the radial velocity changes.

$$\boxed{\frac{d\dot{r}}{d\lambda} = \frac{L^2}{r^3} - \frac{3ML^2}{r^4}}$$

(This is equivalent to $\frac{d^2r}{d\lambda^2} = \ldots$ In the code, this is `dr_dot_dlambda = (L**2 / r**3) - (3 * M * L**2 / r**4)`)

Derivation: This derivation starts from the radial energy equation for a photon in Schwarzschild spacetime and finds the equation for its radial acceleration.

Step 1: Start with the Radial Energy Equation

From the Lagrangian formalism (or the $ds^2 = 0$ condition), we derived the equation relating the radial velocity $\left(\dot{r} = \frac{dr}{d\lambda}\right)$ to the photon's energy (E) and the effective potential squared ($V_{\text{eff}}^2(r)$):

$$\left(\frac{dr}{d\lambda}\right)^2 = E^2 - V_{\text{eff}}^2(r)$$

We can rewrite this as:

$$\dot{r}^2 + V_{\text{eff}}^2(r) = E^2$$

Step 2: Differentiate with Respect to λ

Now, we take the derivative of the entire equation with respect to the affine parameter λ. Remember that E^2 is a constant.

$$\frac{d}{d\lambda}\left(\dot{r}^2 + V_{\text{eff}}^2(r)\right) = \frac{d}{d\lambda}(E^2)$$

$$\frac{d}{d\lambda}(\dot{r}^2) + \frac{d}{d\lambda}(V_{\text{eff}}^2(r)) = 0$$

Step 3: Apply the Chain Rule

We need to evaluate these derivatives using the chain rule.

- For the first term:
$$\frac{d}{d\lambda}(\dot{r}^2) = 2\dot{r}\frac{d\dot{r}}{d\lambda} = 2\dot{r}\ddot{r} \quad (\text{where } \ddot{r} = \frac{d^2r}{d\lambda^2})$$
- For the second term:
$$\frac{d}{d\lambda}(V_{\text{eff}}^2(r)) = \frac{dV_{\text{eff}}^2}{dr}\frac{dr}{d\lambda} = \frac{dV_{\text{eff}}^2}{dr}\dot{r}$$

Substitute these back into the equation:

$$2\dot{r}\ddot{r} + \frac{dV_{\text{eff}}^2}{dr}\dot{r} = 0$$

Step 4: Simplify

$$\dot{r}[2\ddot{r} + \frac{dV_{\text{eff}}^2}{dr}] = 0$$

$$\ddot{r} + \frac{1}{2}\frac{dV_{\text{eff}}^2}{dr} = 0$$

Finally,

$$\boxed{\ddot{r} = -\frac{1}{2}\frac{dV_{\text{eff}}^2}{dr}}$$

Replacing $\ddot{r}$ with $\frac{d^2r}{d\lambda^2}$ gives the final form:

$$\boxed{\frac{d^2r}{d\lambda^2} = -\frac{1}{2}\frac{d}{dr}V_{\text{eff}}^2(r)}$$

Calculate the Derivative $\frac{d}{dr}V_{\text{eff}}^2(r)$

We had already found:

$$V_{\text{eff}}^2(r) = \frac{L^2}{r^2}\left(1 - \frac{2M}{r}\right)$$

We can expand this for easier differentiation:

$$V_{\text{eff}}^2(r) = \frac{L^2}{r^2} - \frac{2ML^2}{r^3} = L^2r^{-2} - 2ML^2r^{-3}$$

Now, we differentiate the expanded expression with respect to r :

$$\frac{d}{dr}V_{\text{eff}}^2(r) = \frac{d}{dr}(L^2r^{-2}) - \frac{d}{dr}(2ML^2r^{-3})$$

$$\frac{d}{dr}V_{\text{eff}}^2(r) = L^2(-2)r^{-3} - 2ML^2(-3)r^{-4}$$

$$\frac{d}{dr}V_{\text{eff}}^2(r) = -\frac{2L^2}{r^3} + \frac{6ML^2}{r^4}$$

Substitute the Derivative into the Acceleration Formula

Finally, substitute this derivative back into the formula from Step 1:

$$\frac{d^2r}{d\lambda^2} = -\frac{1}{2}\left(-\frac{2L^2}{r^3} + \frac{6ML^2}{r^4}\right)$$

$$\frac{d^2r}{d\lambda^2} = \left(-\frac{1}{2}\right)\left(-\frac{2L^2}{r^3}\right) + \left(-\frac{1}{2}\right)\left(\frac{6ML^2}{r^4}\right)$$

$$\boxed{\frac{d^2r}{d\lambda^2} = \frac{L^2}{r^3} - \frac{3ML^2}{r^4}}$$

Since $\frac{d^2r}{d\lambda^2} = \frac{d\dot{r}}{d\lambda}$, this is exactly the equation in your image:

$$\boxed{\frac{d\dot{r}}{d\lambda} = \frac{L^2}{r^3} - \frac{3ML^2}{r^4}}$$

8 References

1. A First Course in General Relativity - Bernard Schutz
2. Spacetime and Geometry An Introduction to General Relativity - Sean M. Carroll
3. Classical Mechanics - Herbert Goldstein, Charles P. Poole, John Safko
4. David Tong - Lectures on General Relativity
   https://www.damtp.cam.ac.uk/user/tong/gr.html