O-MINIMALITY AND ITS VARIATIONS

VAGIOS S. VLACHOS

A survey on o-minimality and its variations

Graduate Program on Logic and Theory of Algorithms and Computation
$\mu\Pi\lambda\forall$
University of Athens
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Part I

O-MINIMALITY

1 O-MINIMAL STRUCTURES

1.1 INTRODUCTION

For this chapter we consider first-order structures $\mathcal{M} = (M, <, \dots)$ where M is the universe and $<$ is an interpretation of a dense linear order without endpoints on M. Also when we refer to definability we mean definability with parameters.

o-minimal structure

Definition 1.1. The structure $\mathcal{M}$ is called *o-minimal* if every definable subset of M is a finite union of singletons and open intervals with endpoints in $M_\infty := M \cup \{-\infty, +\infty\}$.

This definition tells us that the definable subsets of M are quantifier-free definable just with $=$ and $<$. From this we can observe that the class of o-minimal structures is closed under reducts (as long as $<$ remains in the language). Frequently when we have a structure on a rich language we show that it is o-minimal by quantifier elimination, and so all reducts will be o-minimal. Also o-minimality is closed under expansions by constants.

Examples. The following structures are o-minimal.

- $(\mathbb{Q}, <)$
- $(\mathbb{Q}, <, +)$
- $\mathcal{R} = (\mathbb{R}, <, +, -, \cdot, 0, 1)$. By Tarski's quantifier elimination, we need only to check that formulas with parameters define finite unions of intervals and singletons.
- $\mathcal{R} = (\mathbb{R}, <, +, -, \cdot, 0, 1, \exp)$. Here o-minimality follows from Wilkie's model completeness result [Wilkie 38].
- $(\mathbb{Q}, +, \cdot, 0, 1, <)$ is not o-minimal. The infinite discrete set of perfect squares is definable.

o-minimal theory

strongly o-minimal theory

Definition 1.2. A first-order theory T is called *o-minimal theory* (or *strongly o-minimal theory*) if every model $\mathcal{M}$ of T is o-minimal.

Later when we will have enough tools we will prove that (for the proof see theorem 2.10),

Theorem 1.3. [Knight et al. 12] *If $\mathcal{M}$ is o-minimal then* $\mathrm{Th}(\mathcal{M})$ *is strongly o-minimal.*

Therefore, by 1.3 we have that every structure which is elementary equivalent to an o-minimal structure is also o-minimal.

o-minimal structures have very nice properties. We will see an example right now.

Definably complete

Definition 1.4. An ordered structure $\mathcal{M}$ will be called *definably complete* if any parametrically definable subset of $\mathcal{M}$ that is bounded above (respectively, bounded below) in $\mathcal{M}$ has a supremum (respectively, infimum) in $\mathcal{M}$.

This is a definable analogue of the Dedekind completeness property that we already know holds in $(\mathbb{R}, <)$. So we have the following,

Proposition 1.5. *Any o-minimal structure is definably complete. Therefore, for every definable set $A \subseteq M$, $\inf(A)$ and $\sup(A)$ exist in M_∞.*

The converse of proposition 1.5 is not true. Indeed, the structure $(\mathbb{Q}, <, P)$, where P is a unary predicate interpreted as $P = \{1/n | n < \omega\}$, is definably complete but it is not o-minimal.

Let us give one more example of a nice property. A structure is said to have *discrete order type* if every element except the last (if it exists) has an immediate successor, and every element except the first (if it exists) has an immediate predecessor. We will see that these structures behave particularly well.

Proposition 1.6. *An o-minimal structure with a definable element, whose order type is discrete, has definable Skolem functions.*

Proof. Any interval in such structure that is not the entire structure has a least or greatest element. Fix a structure $\mathcal{M}$ as described, let $\varphi(\bar{x}, y)$ ($\bar{x}$ has arity n) be a formula and assume that $\mathcal{M} \vDash (\exists y)\, \varphi\left(\bar{b}, y\right)$. We have that the set $\left\{\bar{b}\right\} \times A \subset M^{n+1}$ where $\mathcal{M} \vDash \varphi\left(\bar{b}, a\right)$ for all $a \in A$ is definable and it will have a least or greatest element. Thus we can define a definable function $F_\varphi : M^n \to M$ which will return that least or greatest element and $\mathcal{M} \vDash \varphi\left(\bar{b}, F_\varphi\left(\bar{b}\right)\right)$. $\square$

We equip M with the *open interval topology*. This is the topology for which the open intervals form a base. We also equip each M^m, for $m > 0$, with the corresponding product topology, a base of which is formed by the open boxes in M^m. We remind here that an *open box* in M^m is a cartesian product of open intervals $(a_1, b_1) \times \cdots \times (a_m, b_m)$.

Now that we have a topology we can talk about the (topological) closure and the interior of a set $A \subseteq M^m$. We denote them by $\mathrm{cl}(A)$ and $\mathrm{int}(A)$ respectively. Then, for $A \subseteq M$ we call *boundary* of A the set,

$$\{x \in M | \text{each open interval containing } x \text{ intersects both } A \text{ and } M \backslash A\}$$

and denote it with $\mathrm{bd}(A)$.

Next we give two simple lemmas which will be useful later.

Lemma 1.7. *Let $A \subseteq M$ be definable. Then the boundary,* $\mathrm{bd}(A)$ *is finite. Moreover, if $a_1 < \cdots < a_n$ are the points of* $\mathrm{bd}(A)$ *then each interval (a_i, a_{i+1}), $i \in \{0, \ldots, n+1\}$ where $a_0 = -\infty$ and $a_{n+1} = +\infty$, is either part of A or disjoint from A.*

Proof. Because of the o-minimality of M we will have that A will be of the form

$$A = \{c_1, \ldots, c_k\} \cup (a_1, a_2) \cup \cdots \cup (a_{n-1}, a_n)$$

and without loss of generality we may assume that the intervals are disjoint. Then it is clear that $\mathrm{bd}(A) = \{c_1, \ldots, c_k, a_1, \ldots, a_n\}$ and therefore A will be finite. It is easy to verify the second part of the lemma. □

Lemma 1.8.

1. *If $A \subseteq M^m$ is definable, so are* $\mathrm{cl}(A)$ *and* $\mathrm{int}(A)$.
2. *If $A \subseteq B \subseteq M^m$ are definable sets, and A is open in B, then there is a definable open $U \subseteq M^m$ with $U \cap B = A$.*

Proof. To prove (1) we note that,

$$\begin{gathered}(x_1, \ldots, x_m) \in \mathrm{cl}(A) \iff \\ \forall y_1 \cdots \forall y_m \forall z_1 \cdots \forall z_m \left[(y_1 < x_1 < z_1 \wedge \cdots \wedge y_m < x_m < z_m) \rightarrow \right. \\ \left. \exists a_1 \cdots \exists a_m \left(y_1 < a_1 < z_1 \wedge \cdots \wedge y_m < a_m < z_m \wedge (a_1, \ldots, a_m) \in A\right)\right]\end{gathered}$$

and

$$\begin{gathered}(x_1, \ldots, x_m) \in \mathrm{int}(A) \iff \\ \exists a_1 \cdots \exists a_m \exists b_1 \cdots \exists b_m \left[(a_1 < x_1 < b_1 \wedge \cdots \wedge a_m < x_m < b_m) \wedge \right. \\ \left. (\forall y_1 \cdots \forall y_m \left(a_1 < y_1 < b_1 \wedge \cdots \wedge a_m < y_m < b_m\right) \rightarrow (y_1, \ldots, y_m) \in A)\right]\end{gathered}$$

For (2) we just need to take as U the union of all boxes in M^m whose intersection with B is contained in A. □

Now we introduce a notion of connectedness appropriate for definable sets.

Definition 1.9. A set $X \subseteq M^m$ is called *definably connected* if X is definable and is not the union of two disjoint nonempty definable subsets of X.

Definably connected

Lemma 1.10.

1. *The definably connected subsets of M are the following:*
   a) *the empty set*
   b) *the point sets*

c) the open intervals

d) the sets $[a,b)$ with $-\infty < a < b \leq +\infty$

e) the sets $(a,b]$ with $-\infty \leq a < b < +\infty$

f) the sets $[a,b]$ with $-\infty < a < b < +\infty$

2. *The image of a definably connected set $X \subseteq M^m$ under a definable continuous map $f : X \to M^n$ is definably connected.*

3. *If X and Y are definable subsets of M^m s.t. $X \subseteq Y \subseteq cl(X)$ and X is definably connected then Y is definably connected.*

4. *If X and Y are definably connected subsets of M^m and $X \cap Y \neq \emptyset$, then $X \cup Y$ is definably connected.*

Proof. 1. For the empty set and point sets the result holds trivially. Let us consider the case $I = (a,b)$. The other cases are treated similarly. Assume towards contradiction that I is not definably complete. Then there are U, V , $U \cap V = \emptyset$ s.t. $U \cap I$, $V \cap I$ are nonempty, open and cover I. By o-minimality, U, V are both finite unions of intervals and singletons. We can assume that $U = (a_0, b_0) \cup (a_1, b_1) \cup \cdots \cup (a_n, b_n)$ where $a_i < b_i \leq a_{i+1} < b_{i+i}$ for all $i = 0, \ldots, n-1$. Let $B = \{a_i, b_i | i \leq n\}$ be the set of the endpoints of these intervals. We claim that $I \cap B \neq \emptyset$. If not then for some i,

- $a_i \leq a < b \leq b_i$, or
- $b_i \leq a < b \leq a_{i+1}$, or
- $b \leq a_0$ or $b_n \leq a$.

In the first case $I \subseteq U$ except possibly for one or two endpoints. However, then V must contain such a point and since it is open, contains an interval around that point and so intersects U. Therefore $I \subseteq U$ but then $V \cap I$ is empty which is impossible. So the first case is out. In the other cases I is disjoint from U which is also impossible. Hence $B \cap I \neq \emptyset$. From this we derive that $I \cap U = (c_0, c_1) \cup \cdots \cup (c_{m-1}, c_m)$ where $c_j \in B \cup \{a, b\}$ for $j = 0, \ldots m$. It is not hard to see now that $(I \cap V) \cup (I \cap U) = I$ contradicts with V being open. Therefore I must be definably complete.

2. If f is continuous then the inverse image of an open set is open. Thus, if $f(X)$ were not definably connected, we could take inverse images U, V which gives us the failure of definable connectedness of X.

3. Suppose that Y is not definably connected. Then there are U, V open definable disjoint sets s.t. $Y = U \cup V$. Consider now $U' = U \cap X$ and $V' = V \cap X$ which are disjoint definable and open.

We claim that $V' \neq \emptyset$. Suppose not, then $V \subseteq \mathrm{cl}(X) \backslash X$, and let $a \in V$. Consider any open set B around a. Since $a \in \mathrm{cl}(X)$ there is $x \in X$ s.t. $x \in B$. For $B = V$ we have a contradiction. Therefore $V' \neq \emptyset$. The same way we show that $U' \neq \emptyset$. Thus U' and V' witness that X is not definably connected which is absurd.

4. Suppose that $X \cup Y$ is not definably connected. Then there exist U, V definable, open and disjoint s.t. $X \cup Y = U \cup V$. Then $U \cap X$ and $V \cap X$ are open and disjoint so one of them must be empty (because $(U \cap X) \cup (V \cap X) = X$ and then we have a contradiction with the hypothesis that X is definably connected). Without loss of generality, let $V \cap X = \emptyset$. We do the same for Y. Again we have two cases, $V \cap Y = \emptyset$ or $U \cap Y = \emptyset$. In the first case we have that U covers $X \cup Y$ which is impossible. But then in the second case we will have that $X \cap Y$ is in both U and V so they will not be disjoint which is absurd.

□

From this we can derive a definable analogue of Bolzano's Intermediate Value Theorem.

Theorem 1.11. *If the $f : [a, b] \to M$ is definable and continuous, then f assumes all values between $f(a)$ and $f(b)$.*

Proof. Assume that $M = (-\infty, +\infty)$. By the previous lemma 1.10 (1) and (2) we have that $[a, b]$ and $f([a, b])$ are definably connected. Therefore the graph $\Gamma(f) \subseteq [a, b] \times M$ is definably connected. Suppose now that there exists d s.t. $f(a) < d < f(b)$ and does not exist c s.t. $f(c) = d$. Now consider the sets,

$$\Gamma(f) \cap [a, b] \times (-\infty, d)$$
$$\Gamma(f) \cap [a, b] \times (d, +\infty)$$

They are nonempty, definable, disjoint, open subsets of $\Gamma(f)$ and they cover it. That is absurd. □

1.2 MONOTONICITY THEOREM

In this section we will prove the following very important theorem which describes the definable one-variable functions on o-minimal structures. Note that when we refer to definable we mean parametrically definable.

Theorem 1.12. [Monotonicity Theorem] [Pillay and Steinhorn 26] *Let $\mathcal{M}$ be an o-minimal structure and $f : (a, b) \to M$ be a definable function with domain (a, b) (possibly $a = -\infty$ or $b = +\infty$). Then there are points $a = a_0 < a_1 < \cdots < a_{k+1} = b$ s.t for each $j = 0, \ldots, k$, $f|_{(a_j, a_{j+1})}$ is either,*

- *constant, or*

- *a strictly monotonic and continuous bijection to an interval.*

We will prove first a technical lemma.

Lemma 1.13. *Let f be a parametrically definable unary function in an o-minimal structure $\mathcal{M}$ as described in the hypothesis of the Monotonicity Theorem 1.12. Then for any definable infinite interval $I \subseteq (a,b)$, there is an infinite $I^* \subseteq I$ on which f is either constant or strictly monotone.*

Proof. The argument will proceed through a case-by-case analysis based on the formulas below

$$\begin{aligned}
\varphi_0(x) \quad \equiv \quad & (\exists z)\left[(\exists y)(z \leq y < x \wedge y \in I \wedge x \in I)\right.\\
& \qquad \left.\wedge (\forall y)(z \leq y < x \rightarrow f(y) = f(x))\right]\\
& \vee (\exists z)\left[(\exists y)(x < y \leq z \wedge y \in I \wedge x \in I)\right.\\
& \qquad \left.\wedge (\forall y)(x < y \leq z \rightarrow f(y) = f(x))\right]\\
\varphi_1(x) \quad \equiv \quad & (\exists z)(\exists w)\left[(\exists u)(\exists v)(z \leq u < x < v \leq w \wedge u \in I \wedge v \in I)\right.\\
& \qquad \wedge (\forall u)(z \leq u < x \rightarrow f(u) < f(x))\\
& \qquad \left.\wedge (\forall v)(x < v \leq w \rightarrow f(x) < f(v))\right]\\
\varphi_2(x) \quad \equiv \quad & (\exists z)(\exists w)\left[(\exists u)(\exists v)(z \leq u < x < v \leq w \wedge u \in I \wedge v \in I)\right.\\
& \qquad \wedge (\forall u)(z \leq u < x \rightarrow f(u) > f(x))\\
& \qquad \left.\wedge (\forall v)(x < v \leq w \rightarrow f(x) > f(v))\right]\\
\varphi_3(x) \quad \equiv \quad & (\exists z)(\exists w)\left[(\exists u)(\exists v)(z \leq u < x < v \leq w \wedge u \in I \wedge v \in I)\right.\\
& \qquad \wedge (\forall u)(z \leq u \leq w \wedge u \neq x \rightarrow f(u) > f(x))\\
\varphi_4(x) \quad \equiv \quad & (\exists z)(\exists w)\left[(\exists u)(\exists v)(z \leq u < x < v \leq w \wedge u \in I \wedge v \in I)\right.\\
& \qquad \wedge (\forall u)(z \leq u \leq w \wedge u \neq x \rightarrow f(u) < f(x))
\end{aligned}$$

- φ_0 defines the points x where f is locally constant left or right of x,
- φ_1 and φ_2 define the points where f is locally strictly monotonic and
- φ_3 and φ_4 define the points that are local maxima or minima for f.

Using o-minimality we can see that $\varphi_0, \varphi_1, \varphi_2, \varphi_3, \varphi_4$ partition I into finitely many intervals in the structure $\mathcal{M}$ and thus one of the $\varphi_i(\mathcal{M}) := \{a \in M | \mathcal{M} \vDash \varphi_i(a)\}$, $i = 0, \ldots, 4$ must be infinite and hence must contain a definable interval $I_1 \subseteq I$.

Case I. $I_1 \subseteq \varphi_0(\mathcal{M})$. We can assume that for every $x \in I_1$, there is $z > x$ s.t. $(x,z) \neq \emptyset$ and $f \restriction [x,z)$ is constant. We assert that,

> (1) For some infinite, definable $I_2 \subseteq I_1$ and for every $x \in I_2$, there are $z < x < w$ s.t. $[z,x) \neq \emptyset$, $(x,w] \neq \emptyset$ and $f \restriction [z,w]$ is constant.

To establish this it suffices to show that the set

$$D = \{x \in I_1 | (1) \text{ does not hold for } x\}$$

is finite. Then $I_1 \backslash D$ would contain an interval as described in (1). D does not contain an infinite interval because $D \subseteq I_1 \subseteq \varphi_0(\mathcal{M})$and then for all $x \in I_1$ exists $z > x$ s.t. $(x, z] \neq \emptyset$ and $f \restriction [z, w]$ is constant. Thus by o-minimality, D must be finite. Therefore f is constant on I_2. For $d \in I_2$ let $C = \{x \in I_2 | f(d) = f(x)\}$. If $C = I_2$ we are done. If $C \subsetneq I_2$ then C is definable using d and by o-minimality we have that I_2 contains a boundary point e of C. However e does not satisfy (1) so f must be constant on it.

Case II. $I_1 \subseteq \varphi_1(\mathcal{M})$ or $I_1 \subseteq \varphi_2(\mathcal{M})$. The argument is identical for both cases so let us assume that $I_1 \subseteq \varphi_1(\mathcal{M})$. We claim that f is strictly increasing on I_1. Suppose that $a, b \in I_1$ s.t. $a < b$ and $f(b) < f(a)$. Consider the definable set

$$X = \{y \in I_1 | y > a \,\&\, f(a) > f(y)\}.$$

That should be non-empty (by our assumption). Since $I_1 \subseteq \varphi_1(\mathcal{M})$ we have that $X \neq (a, \infty) \cap I_1$. So let c be a boundary point of $X \cap I_1$ and without loss of generality assume that bounds it from the left. Now we check two possible cases. If $c \in I_1 \cap X$ then c can not satisfy $\varphi_1(x)$ because for some interval J on the left of c, if $x \in J$, $f(x) \geq f(a) > f(c)$. If $c \notin I_1 \cap X$ then c bounds an interval in $X \cap I_1$ from the left, and for some interval J to the right of c, if $x \in J$ then $f(x) < f(c)$. Therefore c could not satisfy $\varphi_1(x)$. In any event, X being non empty leads to impossibility, and so f must be strictly increasing as claimed.

Case III. $I_1 \subseteq \varphi_3(\mathcal{M})$ or $I_1 \subseteq \varphi_4(\mathcal{M})$. Again the argument is identical for both cases so we suppose that $I_1 \subseteq \varphi_3(\mathcal{M})$.

Claim. I_1 does not contain a subinterval finite or infinite on which f is constant.

If it contains such infinite an interval we are done. Suppose that it contains such finite interval. Then by o-minimality there can be only a finite number of finite subintervals of I_1, and so I_1 could be cut down (as we did when we picked I_1 from I).

Claim. I_1 is dense linear without endpoints.

We have density because if we had a pair x_0, x_1 s.t. $x_1 = \mathrm{succ}(x_0)$ then both of them could not satisfy $\varphi_3(x)$. Linearity comes from $\mathcal{M}$. If it has end points we can definably cut it to not contain them.

Next we assert that without loss of generality we can assume that,

$$(\forall x \in I_1)(\exists y)[y \in I_1 \wedge y > x \wedge f(x) > f(y)] \tag{2}$$

and

$$(\forall x \in I_1)(\exists y)[y \in I_1 \wedge y < x \wedge f(x) > f(y)] \tag{3}$$

For (2) consider the definable set,

$$X = \{x \in I_1 | \, (\forall y)\,(y > x \to f\,(x) \leq f\,(y))\}$$

If X is finite then I_1 can be refined appropriately. So, assume that X is infinite and hence contains an infinite definable interval J. However, since I_1 contains at least two elements on which f is constant, it is easy to see that f is strictly increasing on J, and then we are done. We work the same way for (3).

For any $c \in I_1$, $(c, \infty) \cap I_1$ can be partitioned into the following three sets

$$\begin{aligned} X^+ &= \{x \in I_1 | x > c \,\&\, f\,(x) > f\,(c)\} \\ X^= &= \{x \in I_1 | x > c \,\&\, f\,(x) = f\,(c)\} \\ X^- &= \{x \in I_1 | x > c \,\&\, f\,(x) < f\,(c)\} \end{aligned}$$

By o-minimality this partitioning consists of finitely many intervals in $\mathcal{M}$. We claim that the rightmost interval J belongs to X^-. First J cannot be contained in $X^=$ because by one of our previous claims we have that I_1 does not contain constant intervals for f. Let us assume that the claim does not hold and $J \subseteq X^+$ and let $d \in I_1$ be the left boundary of J. We start a new case analysis now.

Case 1. $d \notin J$ and $d \in X^- \cup X^=$. In this case d can not satisfy (2).

Case 2. $d \notin J$ and $d \in X^+$. Impossible because $d \notin J \subseteq X^+$.

Case 3. $d \in J$. To the left of d must be an interval in X^- (because I_1 is dense and contains no interval on which f is constant). If the interval belongs to X^+ or $X^=$, $\varphi_3\,(x)$ could not hold for d.

Since all the cases above exhaust all possibilities, we have shown that $J \subseteq X^-$, as asserted. Moreover $d \in J$. Let,

$$\begin{aligned} \varphi_5\,(u) \quad \equiv \quad & (\exists z)\,(\exists w)\,[(\exists x)\,(\exists y)\,(z < x < u < y < w) \\ & \qquad \wedge (\exists t)\,[(\forall s)\,(s \leq u \wedge s < w \to f\,(s) < t) \\ & \qquad\qquad \wedge (\forall s)\,(s \leq u \wedge z < s \to f\,(s) \geq t)]] \end{aligned}$$

If $c \in I_1$ and d is the lefthand boundary point of the rightmost interval in the partition of $(c, \infty) \cap I_1$, as above, into X^+, $X^=$, X^-, then clearly $\mathcal{M} \vDash \varphi_5\,(d)$.

We claim that $\varphi_5\,(\mathcal{M})$ must be infinite. If $\varphi_5\,(\mathcal{M})$ were finite, then we could cut down to an infinite interval $J \subseteq I_1$ on which $\neg\varphi_5\,(x)$ holds for each $x \in J$. But then for any point $c \in J$ and its corresponding "d" (as above, i.e. left hand boundary point of the rightmost interval in the partition of $(c, \infty) \cap I_1$ into X^+, $X^=$, X^-) it would follow that $\neg\varphi_5\,(d)$ holds, which is impossible. Consequently let $I_2 \subseteq I_1$ be an infinite definable interval on which $\varphi_5\,(x)$ is satisfied by all $x \in I_2$.

For any $c \in I_2$, the same argument as the one just given shows that there is some $d' \in I_2$, $d' < c$ and some interval $J' \subseteq (-\infty, c)$ with $d' \in J'$ and,

$$\begin{aligned} &(\forall x)\left(x \in J' \wedge x \leq d' \rightarrow f(x) < f(c)\right) \\ \wedge &(\forall x)\left(x \in J' \wedge x > d' \rightarrow f(x) \geq f(c)\right) \end{aligned} \tag{4}$$

However it must be true that $\mathcal{M} \vDash \varphi_5(d')$. Let t and $J = (z, w)$ be as guaranteed by $\varphi_5(d')$ holding. We now consider the interval $J^* = J \cap J'$ about d'. Wlog we assume that $f(x) \neq f(c)$ for all $x \in J^*$. Suppose first that $t \geq f(c)$. Then $x \in J^* \cap (-\infty, d')$, $f(x) \geq t$ since $\varphi_5(d')$ holds. But since (4) holds for $x \in J^* \cap (-\infty, d')$, $f(x) < f(c) < t$, which is impossible. Now suppose that $t < f(c)$. Then (4) implies that if $x \in J^* \cap (-\infty, d')$ then $f(x) \geq f(c)$. However since $\varphi_5(d')$ holds, $f(x) < t < f(c)$ for any $x \in J^* \cap (d', \infty)$. This again is impossible and so we have reached a contradiction under the assumption that $\varphi_3(\mathcal{M})$ is infinite. □

Now we are ready to proceed with the proof of the Monotonicity Theorem 1.12.

Proof. (of Monotonicity Theorem) Let $\theta(x)$ be the formula asserting,

> "On an interval of which x is the left endpoint, f is strictly monotone or constant, and there is no interval extending this interval on the left on which f is strictly monotone or constant."

$\theta(x)$ is expressible in first-order logic. We now claim that $\theta(\mathcal{M})$ must be finite. If it were infinite it would contain an infinite interval I. By Lemma 1.13 there would be an infinite interval $I^* \subseteq I$ where f would be strictly monotone or constant. However, any interior point of I^* could not satisfy $\theta(x)$, whence $\theta(\mathcal{M})$ must be finite.

Let $\theta(\mathcal{M}) = \{b_1, \ldots, b_{k-1}\}$ be enumerated in increasing order, and let $b_0 = -\infty$, $b_k = \infty$. We claim that f is strictly monotone or constant on each (b_{j-1}, b_j) for $j = 1, \ldots, k$. Because b_{j-1} satisfies $\theta(x)$ it is the left endpoint of an interval on which f is strictly monotone or constant. Let $I \subseteq (b_{j-1}, b_j)$ be the largest such interval. If $(b_{j-1}, b_j) \setminus I \neq \emptyset$ but is finite then clearly some member of $(b_{j-1}, b_j) \setminus I$ would satisfy $\theta(x)$ which is impossible. But if $(b_{j-1}, b_j) \setminus I$ were infinite then Lemma 1.13 would imply the existence of an infinite interval $J \subseteq (b_{j-1}, b_j) \setminus I$ on which f is strictly monotone or constant. Again it would imply that $\theta(\mathcal{M}) \cap (b_{j-1}, b_j) \neq \emptyset$ which can't be true. Therefore $I = (b_{j-1}, b_j)$.

Suppose now that $f \upharpoonright (b_{j-1}, b_j)$ is strictly monotone. The o-minimality of $\mathcal{M}$ implies that the image of f on (b_{j-1}, b_j) is the union of finitely many intervals in $\mathcal{M}$. Using the strict monotonicity of f on (b_{j-1}, b_j), we can subdivide (b_{j-1}, b_j) into finitely many intervals with appropriately chosen end points $b_{j-1} = b_0^j < b_1^j < \cdots < b_{k_j}^j = b_j$ s.t. the image of

f at $\left(b^j_{r-1}, b^j_r\right)$ is an interval and $f \restriction \left(b^j_{r-1}, b^j_r\right)$ is an order preserving or reversing bijection onto the image of f on $\left(b^j_{r-1}, b^j_r\right)$, for $r = 1, \ldots, k_j$.

Finally let $\{a_0, \ldots, a_n\}$ be an enumeration of all end points obtained as above, i.e.

$$\{a_0, \ldots, a_n\} = \{b_0, \ldots, b_k\} \cup \left(\bigcup_j \left\{ b^j_0, \ldots, b^j_{k_j} \right\} \right)$$

where $f \restriction (b_{j-1}, b_j)$ monotone. This concludes the proof. $\square$

The following theorem is a corollary of the Monotonicity Theorem, but we can also prove it without the use of it.

Theorem 1.14. [Exchange Principle for o-minimal Models] *Pillay and Steinhorn [26] Let $\mathcal{M}$ be o-minimal. Let $b, c, a_1, \ldots, a_n \in \mathcal{M}$. If b is algebraic over $c, a_1, \ldots, a_n$ and b is not algebraic over $a_1, \ldots, a_n$, then c is algebraic over $b, a_1, \ldots, a_n$.*

Proof. $\mathrm{Th}(\mathcal{M}, a_1, \ldots, a_n)$ is strongly o-minimal. Therefore we may suppose that $\{a_1, \ldots, a_n\} = \emptyset$. Thus suppose that b is algebraic over c and not algebraic over $\emptyset$. Since algebraic implies definable we may assume that there is some parameter free definable partial function f so that $f(c) = b$. Towards a contradiction suppose that c is not algebraic over b. Let $A = \{x \in M | f(x) = b\}$. By Monotonicity Theorem 1.12, f is piecewise strictly monotone or constant. The point c either is an endpoint of one of the intervals on which f is monotone or constant, or is in the interior of one such interval. If it is an endpoint then we are done because c is definable by the parameters used to define f. Suppose that c is in the interior of the kth interval, $I_k = (\alpha_{k-1}, \alpha_k)$ on which f is monotone or constant. If $f \restriction I_k$ is constant then b would be definable over A contrary to the hypothesis. If f is monotone on I_k then $f \restriction I_k$ has an inverse and the formula,

$$\text{"}x \in I_k \;\&\; x = f^{-1}(b)\text{"}$$

which is first order (the end points of I_k are definable and the inverse is also definable) defines c over $\{b\} \cup A$. $\square$

1.3 FINITENESS LEMMA

The following theorem is due to Knight et al. [12] and the proof we present is from Van den Dries [34]. It seems a little bit technical but we will use it later.

Theorem 1.15. [Finiteness Lemma] *Let $A \subseteq M^2$ be definable and suppose that for each $x \in M$ the fiber $A_x := \{y \in M | (x, y) \in A\}$ is finite. Then there is $N < \omega$ s.t. $|A_x| \leq N$ for all $x \in M$.*

Proof. First we will give some definitions. A point $(a, b) \in M^2$ will be called *normal* if there is a box $I \times J$ around it s.t.

- either $(I \times J) \cap A = \emptyset$ (hence $(a, b) \notin A$)
- or $(a, b) \in A$ and $(I \times J) \cap A = \Gamma(f)$ for some continuous function $f : I \to M$.

In the latter case f will be definable and unique. Indeed, definability of f of comes easily because the graph $\Gamma(f)$ is the intersection of A (which is definable) and a box $I \times J$. For uniqueness, suppose that there exists function g different form f, such that $(I' \times J') \cap A = \Gamma(g)$, where $(I' \times J')$ is a box around (a, b). Now f, g will intersect in (a, b), because that point belongs to their graphs. But then it would be impossible to separate f and g with open boxes around (a, b) and that means it could not be the case that $(I \times J) \cap A$ will be a graph of a function for any I, J around a and b respectively.

A point $(a, -\infty) \in M \times M_\infty$ is called normal if there is a box $I \times J$ disjoint from A s.t. $a \in I$ and $J = (-\infty, b)$ for some b. Finally, $(a, +\infty) \in M \times M_\infty$ is called normal if there is a box $I \times J$ disjoint from A with $a \in I$ and $J = (b, +\infty)$ for some b. By these definitions we have that,

Note. (1) The following sets are definable,

$$\begin{gathered}\left\{(a, b) \in M^2 \middle|\, (a, b) \text{ is normal}\right\}\\ \{a \in M|\, (a, -\infty) \text{ is normal}\}\\ \{a \in M|\, (a, +\infty) \text{ is normal}\}\end{gathered}$$

First let's see that $\{a \in M|\, (a, -\infty) \text{ is normal}\}$ is definable. We will describe the formula $\phi(x)$ that defines this set.

$$\text{“ } (\exists yzw)\,[(y < x < w) \wedge ((y, w) \times (-\infty, w)) \cap A = \emptyset] \text{ ”}$$

Because A is assumed definable this is a well defined formula. Similarly we can prove the definability of $\{a \in M|\, (a, +\infty) \text{ is normal}\}$. Next, we describe the formula $\psi(x, y)$ that defines the set $\left\{(a, b) \in M^2 \middle|\, (a, b) \text{ is normal}\right\}$.

$$\begin{gathered}\text{“ } (\exists x_1 x_2 y_1 y_2)\,[(x_1 < x < x_2) \wedge (y_1 < y < y_2)\\ \bigwedge [((x_1, x_2) \times (y_1, y_2) \cap A = \emptyset)\\ \bigvee (\forall y_3 y_4 \in \pi_2((x_1, x_2) \times (y_1, y_2) \cap A))\,(y_1 < y_3 < y_4 < y_2)\\ (\forall x_3 \in \pi_1((x_1, x_2) \times (y_3, y_4) \cap A))\\ (\exists x_4 x_5 \in \pi_1((x_1, x_2) \times (y_3, y_4) \cap A))\,(x_1 < x_4 < x_3 < x_5 < x_2)]] \text{ ”}\end{gathered}$$

where π_1 and π_2 are the projections on first and second coordinate. Because projections are definable functions and A is definable, the above describes a well defined formula.

Now we define countably many functions $f_1, f_2, \ldots, f_n, \ldots$ by

$$\begin{gathered}\operatorname{dom}(f_n) := \{x \in M|\, |A_x| \geq n\}\\ \text{and } f_n(x) := n^{\text{th}} \text{ element of } A_x.\end{gathered}$$

We observe here that every f_n is definable with possibly empty domain. Let $a \in M$ and $n \geq 0$ maximal with the property that $f_1, \ldots, f_n$ are defined and continuous on an interval containing a. We call the point a *good* if,

$$a \notin \mathrm{cl}(\mathrm{dom}(f_{n+1}))$$

and *bad* if,

$$a \in \mathrm{cl}(\mathrm{dom}(f_{n+1}))$$

Let $\mathcal{G}$ be the set of good points and $\mathcal{B}$ be the set of bad points. If $a \in \mathcal{G}$ and n has the property as above, then the domain of f_{n+1} is disjoint from an entire interval around a on which $f_1, \ldots, f_n$ are defined and continuous. This shows that for $a \in \mathcal{G}$ we have,

Note. (2) $|A_x|$ is constant on an interval around a, and

Note. (3) (a, b) is normal for all $b \in M_\infty$.

The key point of the proof is to show that $\mathcal{G}$ and $\mathcal{B}$ are definable sets. This is not clear from their definitions because the parameter n is depending on a. To achieve this we show first that,

Claim. If $a \in \mathcal{B}$ then there is a least $b \in M_\infty$ s.t. (a, b) is not normal.

To prove this claim let us define for $a \in \mathcal{B}$ and n as above,

$$\lambda(a,-) := \begin{cases} \lim\limits_{x \to a^-} f_{n+1}(x), & \text{if } f_{n+1} \text{ is defined on some interval } (t,a) \\ +\infty, & \text{otherwise} \end{cases}$$

$$\lambda(a,0) := \begin{cases} f_{n+1}(a), & \text{if } a \in \mathrm{dom}(f_{n+1}) \\ +\infty, & \text{otherwise} \end{cases}$$

$$\lambda(a,+) := \begin{cases} \lim\limits_{x \to a^+} f_{n+1}(x), & \text{if } f_{n+1} \text{ is defined on some interval } (a,t) \\ +\infty, & \text{otherwise} \end{cases}$$

Now let $\beta(a) := \min\{\lambda(a,-), \lambda(a,0), \lambda(a,+)\}$. Then by checking the various possibilities it is easy to see that $\beta(a)$ is the least $b \in M_\infty$ s.t. (a, b) is not normal. This proves our claim, and together with notes (1) and (3) we have that $\mathcal{G}$ and $\mathcal{B}$ are definable sets.

For the rest of the proof now, suppose that $\mathcal{B}$ is finite. Then $\mathcal{B} = \{a_1, \ldots, a_k\}$, with $-\infty = a_0 < a_1 < \cdots < a_k < a_{k+1} = +\infty$. We claim that $|A_x|$ is constant on each interval (a_i, a_{i+1}). To see this just take any point a in this interval and let $n = |A_a|$. Then by note (2) the set $\{x \in (a_i, a_{i+1}) \,|\, |A_x| = n\}$ is open (it will be either the whole interval or the empty set), and for the same reason the set, $\{x \in (a_i, a_{i+1}) \,|\, |A_x| \neq n\}$ is open too. Since both sets are definable the latter set must be empty.

Suppose now that $\mathcal{B}$ is not finite. We shall derive a contradiction. Define the sets,

$$\mathcal{B}_- := \{a \in \mathcal{B} | \exists y\,(y < \beta(a) \,\&\, (a,y) \in A)\}$$
$$\mathcal{B}_+ := \{a \in \mathcal{B} | \exists y\,(y > \beta(a) \,\&\, (a,y) \in A)\}$$

and the functions $\beta_- : \mathcal{B}_- \to M$ and $\beta_+ : \mathcal{B}_+ \to M$ by,

$$\beta_-(a) := \max\{y|y < \beta(a) \,\&\, (a,y) \in A\}$$
$$\beta_+(a) := \min\{y|y > \beta(a) \,\&\, (a,y) \in A\}$$

Since $\mathcal{B}$ is infinite by assumption, one of the sets $\mathcal{B}_- \cap \mathcal{B}_+$, $\mathcal{B}_- \backslash \mathcal{B}_+$, $\mathcal{B}_+ \backslash \mathcal{B}_-$, $\mathcal{B} \backslash (\mathcal{B}_- \cup \mathcal{B}_+)$ is infinite and each of these cases leads to contradiction. We will see the case that $\mathcal{B}_- \cap \mathcal{B}_+$ is infinite. Since β, β_- and β_+ are definable functions, there is by the monotonicity theorem an interval $I \subseteq \mathcal{B}_- \cap \mathcal{B}_+$ on which each of the functions β, β_- and β_+ is continuous. Note that $\beta_- < \beta < \beta_+$ on I. Now I splits into two subsets,

$$\{x \in I|\,(x, \beta(x)) \in A\} \text{ and}$$
$$\{x \in I|\,(x, \beta(x)) \notin A\}$$

and one of these subsets contains an interval. Replacing I by this subinterval we may assume that either $\Gamma(\beta|I) \subseteq A$ or $\Gamma(\beta|I) \cap A = \emptyset$. In either case it is clear that $\Gamma(\beta|I)$ consists of normal points, since β, β_- and β_+ are continuous on I. Now we have a contradiction since $(a, \beta(a))$ is never normal.

The other cases are similar, and that concludes the proof of 1.15. □

If we combine the Finiteness Lemma 1.15 and the Monotonicity Theorem 1.12 we easy get the following,

Corollary 1.16. *Let $A \subseteq M^2$ be definable such that the fiber $A_x := \{y \in M|\,(x,y) \in A\}$ is finite for each $x \in M$. Then there are points $a_1 < \cdots < a_k$ in M such that $A \cap ((a_i, a_{i+1}) \times M) = \Gamma(f_{i1}) \cup \cdots \cup \Gamma\left(f_{in(i)}\right)$ for certain definable continuous functions $f_{ij} : (a_i, a_{i+1}) \to M$ with $f_{i1}(x) < \cdots < f_{in(i)}(x)$ for each $x \in (a_i, a_{i+1})$.*

2 CELL DECOMPOSITION

o-minimality refers to definable sets in one variable. Here we will study the definable subsets of M^n. We will see that a definable subset of M^n splits into finitely many cells (definable sets of a simple form) and also each definable function on M^n is "cell-wise" continuous. For $n = 1$, for sets, this will be equivalent with the definition of o-minimal and for definable functions with the Monotonicity Theorem.

2.1 CELLS

Given a definable $X \subseteq M^n$, let

$$C(X) := \{f : X \to M | f \text{ is definable and continuous}\}$$

and let $C_\infty(X) := C(X) \cup \{-\infty, +\infty\}$ where $-\infty$ and $+\infty$ denotes the constant functions X which are taking the value $-\infty$ and $+\infty$ everywhere respectively. Suppose now that $f, g \in C_\infty(X)$ and also $(\forall \bar{x} \in X)(f(\bar{x}) < g(\bar{x}))$. We denote,

$$(f, g)_X := \{(\bar{x}, y) \in X \times M | f(\bar{x}) < y < g(\bar{x})\}$$

which we see is a definable set.

Definition 2.1. Let $(i_1, \ldots, i_m)$ be a sequence of 0's and 1's. Then an $(i_1, \ldots, i_m)$-*cell* is a definable subset of M^m defined as follows by recursion on m. *Cell*

1. A (0)-cell is a singleton of M, and a (1)-cell is a non-empty open interval possibly unbounded.
2. Suppose that $(i_1, \ldots, i_m)$-cells have been defined. Then an $(i_1, \ldots, i_m, 0)$-cell is the graph $\Gamma(f)$ of a function $f \in C(X)$ where X is an $(i_1, \ldots, i_m)$-cell. An $(i_1, \ldots, i_m, 1)$-cell is a set $(f, g)_X$, with X some $(i_1, \ldots, i_m)$-cell and $f, g \in C_\infty(X)$, with $f(\bar{x}) < g(\bar{x})$ for all $\bar{x} \in X$.

A *cell* in M^m is an $(i_1, \ldots, i_m)$-cell for some $i_1, \ldots, i_m \in \{0, 1\}$. The cells are all definable sets. Since the $(1, \ldots, 1)$-cells are exactly the cells which are open in their ambient space M^m we call them *open cells*.

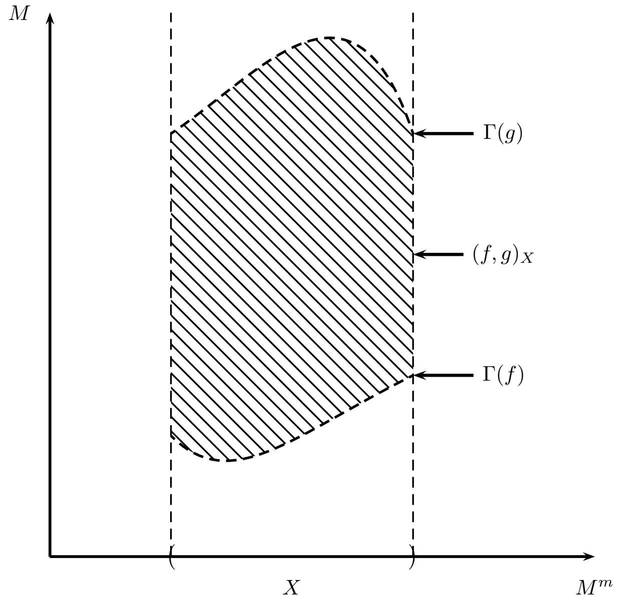

Figure 1: An $(i_1, \ldots, i_m, 1)$-cell

Proposition 2.2. *Each cell is locally closed (i.e. open in its closure).*

Proof. Let $C \subseteq M^{m+1}$ be a cell. Put $B := \pi(C) \subseteq M^m$ and assume inductively that the cell B is open in its closure $\mathrm{cl}(B)$, so that $\mathrm{cl}(B) \setminus B$ is a closed set. If $C = \Gamma(f)$ with $f : B \to M$ a definable continuous function, then $\mathrm{cl}(C) \setminus C$ is contained in $(\mathrm{cl}(B) \setminus B) \times M$, hence C is open in the closed set $C \cup ((\mathrm{cl}(B) \setminus B) \times M)$. If $C = (f, g)_B$ with $f, g : B \to M$ definable continuous functions on B, $f < g$, then one easily verifies that $\mathrm{cl}(C) \setminus C$ is contained in $\Gamma(f) \cup \Gamma(g) \cup ((\mathrm{cl}(B) \setminus B) \times M)$ and that C is open in the closed set $C \cup \Gamma(f) \cup \Gamma(g) \cup ((\mathrm{cl}(B) \setminus B) \times M)$ (See figure 2). The other cases are similar. □

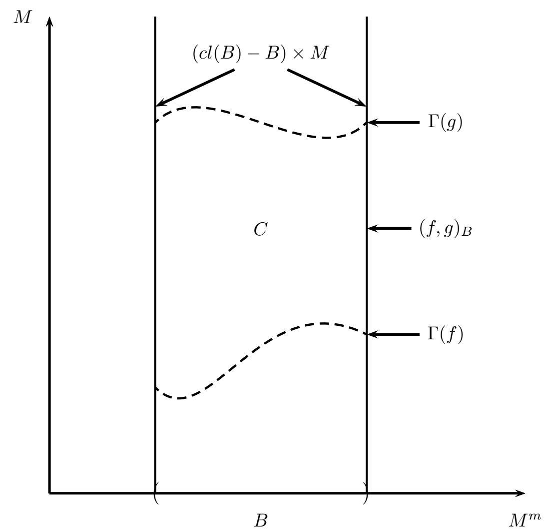

Figure 2

Remark. Each cell which is not a point (i.e. not a $(0, 0, \ldots, 0)$ cell) is homeomorphic, under a coordinate projection, to an open cell. We will make this explicit by defining this homeomorphism. Let $i = (i_1, \ldots, i_m)$ where, $i_j \in \{0, 1\}$ for $1 \le j \le m$. We define $p_i : M^m \to M^k$, $k \le m$ as follows:

- let $\lambda(1) < \cdots < \lambda(k)$ be the indices $\lambda \in \{1, \ldots, m\}$ for which $i_\lambda = 1$, where $k = i_1 + \cdots + i_m$;
- let $p_i(x_1, \ldots, x_m) := \left(x_{\lambda(1)}, \ldots, x_{\lambda(k)}\right)$.

It is easy to show by induction that p_i maps each cell C in M^m onto an open cell $p_i(C)$ in M^k. We denote $p_i(C)$ also by $p(C)$ and the homeomorphism $p_i|C : C \to p(C)$ by p_C. If C is an open cell, then $p_C = id_C$.

If A is a cell in M^{m+1} then $\pi(A)$ is a cell in M^m, where $\pi : M^{m+1} \to M^m$ is the projection on the first m coordinates. An application of this is the following proposition.

Proposition 2.3. *Each cell is definably connected.*

Proof. For intervals and points we have the result by lemma 1.10. If A is a cell in M^{m+1}, then we assume inductively that the cell $\pi(A)$ in M^m is definably connected and use the fact that each fiber $\pi^{-1}(x) \cap A$ is definably connected. $\square$

2.2 DECOMPOSITIONS

Definition 2.4. A *decomposition* of M^m is a special kind of partition of M^m into finitely many cells. The definition is by recursion on m. *Decomposition*

1. a decomposition of M is a collection,

$$\{(-\infty, a_1), (a_1, a_2), \ldots, (a_k, +\infty), \{a_1\}, \ldots, \{a_k\}\}$$

   where $a_1 < \cdots < a_k$ are points in M;

2. a decomposition of M^{m+1} is a finite partition of M^{m+1} into cells A s.t. the set of projections $\pi(A)$ is a decomposition of M^m. (Where π is the usual projection map.)

A decomposition $\mathcal{D}$ is said to *partition* a set $S \subseteq M^m$ if each cell in $\mathcal{D}$ is either part of S or disjoint from S, i.e. if S is a union of cells in $\mathcal{D}$. *Partition*

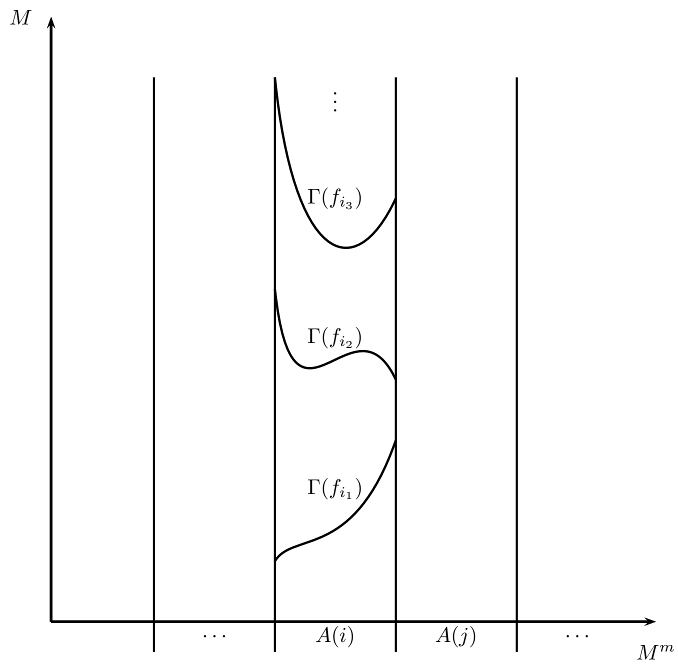

Figure 3

Remark. By the definition it is clear that given a decomposition of M^{m+1} we can get a decomposition of M^m using the projections. Now we will see how we can get a decomposition of M^{m+1} from a decomposition of M^m. Let $\mathcal{D} := \{A(1), \ldots, A(k)\}$ be a decomposition of M^m, $A(i) \neq A(j)$ if $i \neq j$, and let for each $i \in \{1, \ldots, k\}$ functions $f_{i_1} < \cdots < f_{i_{n(i)}}$ in $C(A_i)$ be given. Then

$$\begin{aligned} \mathcal{D}_i \quad := \quad & \left\{(-\infty, f_{i_1})_{A(i)}, (f_{i_1}, f_{i_2})_{A(i)}, \ldots, \left(f_{i_{n(i)}}, +\infty\right)_{A(i)}\right\} \\ & \cup \left\{\Gamma(f_{i_1}), \ldots, \Gamma\left(f_{i_{n(i)}}\right)\right\} \end{aligned}$$

is a partition of $A(i) \times M$ and we can easily check using the definition that $\mathcal{D}^* := \mathcal{D}_1 \cup \cdots \cup \mathcal{D}_k$ is a decomposition of M^{m+1}. Also every decomposition of M^{m+1} arises this way from a decomposition of M^m. We write $\mathcal{D} = \pi(\mathcal{D}^*)$. (See figure 3.)

2.3 CELL DECOMPOSITION THEOREM

Here we will prove the following fundamental theorem,

Theorem 2.5. [Cell Decomposition Theorem] [Knight et al. 12]

(I_m) *Given any definable sets $A_1, \ldots, A_k \subseteq M^m$ there is a decomposition of M^m partitioning each of $A_1, \ldots, A_k$.*

(II_m) *For each definable function $f : A \to M$, $A \subseteq M^m$, there is a decomposition $\mathcal{D}$ of M^m partitioning A s.t. the restriction $f|B : B \to M$ to each cell $B \in \mathcal{D}$ with $B \subseteq A$ is continuous.*

The proof is taken from Van den Dries [34] and it is by induction on m. For $m = 1$, (I_1) holds by o-minimality, and (II_1) follows from the Monotonicity Theorem 1.12.

Now assume that $(\mathrm{I}_1), \ldots, (\mathrm{I}_m)$ and $(\mathrm{II}_1), \ldots, (\mathrm{II}_m)$ hold. We shall derive first (I_{m+1}) and then (II_{m+1}).
First we will generalize the Finiteness Lemma 1.15. Call a set $Y \subseteq M^{m+1}$ *finite over* M^m if for each $x \in M^m$ the fiber $Y_x := \{r \in M | (x, r) \in Y\}$ is finite. Call Y *uniformly finite over* M^m if there is $N < \omega$ s.t. $|Y_x| \leq N$ for all $x \in M^m$. We shall use the finiteness lemma and the induction hypothesis to prove first,

Uniformly finite set

Lemma 2.6. [Uniform Finiteness Property] *Suppose the definable subset Y of M^{m+1} is finite over M^m. Then Y is uniformly finite over M^m.*

Definition 2.7. A structure which satisfies the uniform finiteness property will be called uniformly bounded.

Uniformly bounded structure

Proof. A box $B \subseteq M^m$ is called Y*-good* if for each point $(x, r) \in Y$ with $x \in B$ there is an interval I around r s.t. $Y \cap (B \times I) = \Gamma(f)$ for some continuous function $f : B \to M$. f is uniquely determined by B, I and Y, and is definable.

Claim 1. Suppose the box $B \subseteq M^m$ is Y-good. Then there are continuous definable functions $f_1 < \cdots < f_k$ in $C(B)$ s.t. $Y \cap (B \times M) = \Gamma(f_1) \cup \cdots \cup \Gamma(f_k)$.

To prove the claim let us fix $x \in B$ and write $Y_x = \{r_1, \ldots, r_k\}$ with $r_1 < \cdots < r_k$. Take intervals $I_1, \ldots, I_k$ around $r_1, \ldots, r_k$ respectively, and continuous functions $f_1, \cdots, f_k : B \to M$ s.t. $Y \cap (B \times I_j) = \Gamma(f_j)$, $j \in \{1, \ldots, k\}$.

We prove here that $f_1 < \cdots < f_k$. We will show that $f_1 < f_2$, the other inequalities following the same way. Suppose there is a point $p \in B$ with $f_1(p) = f_2(p)$. So $f_2(p) \in I_1$, and by continuity of f_2 there is a neighborhood $U \subseteq B$ of p s.t. $f_2(U) \subseteq I_1$. Since $\Gamma(f_2|U) \subseteq Y \cap (U \times I_1) = \Gamma(f_1|U)$ it follows that (by applying the same argument again for f_2, and get $\Gamma(f_1|U) \subseteq \Gamma(f_2|U)$) $f_1|U = f_2|U$. This argument shows that the set $\{p \in B | f_1(p) = f_2(p)\}$ is open (as reverse image of open intervals through continuous functions). Since $\{p \in B | f_1(p) < f_2(p)\}$ and $\{p \in B | f_1(p) > f_2(p)\}$ are also open and B is definably connected (Proposition 2.3), while $f_1(x) = r_1 < r_2 = f_2(x)$, it follows that $f_1 < f_2$.

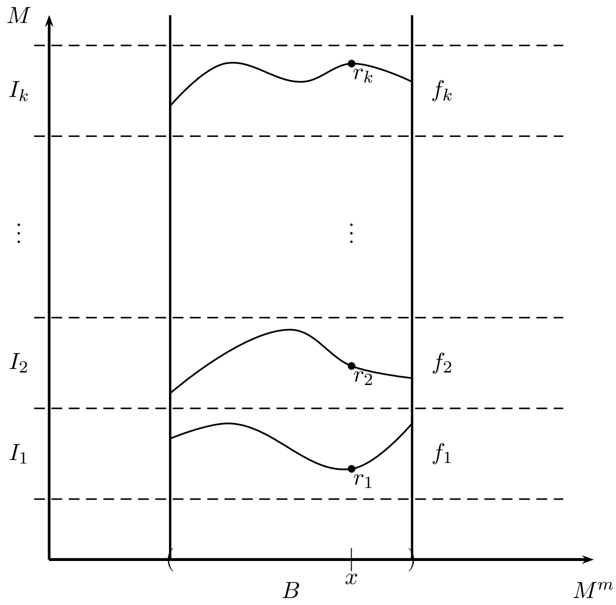

Figure 4

We prove now that $Y \cap (B \times M) = \Gamma(f_1) \cup \cdots \cup \Gamma(f_k)$. Take a point $(a, s) \in Y \cap (B \times M)$ and let $f : B \to M$ be continuous definable function s.t. $f(a) = s$ and $\Gamma(f) \subseteq Y$. Since $(x, f(x)) \in Y$ it follows that $f(x) = r_i = f_i(x)$ for some $i \in \{1, \ldots, k\}$. Because of $f_i < f_{i+1}$ for all $i \in \{1, \ldots, k\}$ we obtain from this $f = f_i$. This concludes the proof of claim 1.

A point $x \in M^m$ will be called Y*-good* if it belongs to a Y-good box. A set of Y-good points is definable (because Y-good sets are definable).

Claim 2. If $A \subseteq M^m$ is a definably connected set and all points of A are Y-good, then there are continuous functions $f_1 < \cdots < f_k$ in $C(A)$ s.t.

$$Y \cap (A \times M) = \Gamma(f_1) \cup \cdots \cup \Gamma(f_k)$$

To prove this claim, suppose first that $A \neq \emptyset$ and let $x \in A$, $k = |Y_x|$. Then x is Y-good and therefore there exists a box $B^x \subseteq M^m$ which is Y-good. We apply claim 1 for B^x and we have definable continuous functions $f_1^x < \cdots < f_k^x$ s.t. $Y \cap (B^x \times M) = \Gamma(f_1^x) \cup \cdots \cup \Gamma(f_k^x)$. It is clear that for every $x' \in B^x$, $|Y_{x'}| = k$. Now if we consider every B^{x_1} and B^{x_2} for $x_1, x_2 \in A$ we have from the definable connectedness of A that $(B^{x_1} \cap A) \cap (B^{x_2} \cap A) \neq \emptyset$. Therefore $|Y_a| = k$ for all $a \in A$. By "connecting" all these functions f_i^x for $x \in A$ we construct continuous functions f_i, $i = 1, \ldots, k$ as described by claim 2.

Claim 3. Each open cell in M^m contains a Y-good point.

It is enough to show that each box B in M^m contains a Y-good point, because these are exactly the open cells. Let B be a box in M^m, then $B = B' \times (a, b)$ for a B' box in M^{m-1}. Let $p \in B'$ and consider the set,

$$Y(p) := \Big\{(r, s)\, M^2 | a < r < b \,\&\, (p, r, s) \in Y\Big\}$$

which is finite over M (because Y is finite over M^m). We now apply the corollary 1.16 on $Y(p)$, and we have that there are $a_1 < \cdots < a_k$ in M s.t.

$$Y(p) \cap ((a_i, a_{i+1}) \times M) = \Gamma(f_1) \cup \cdots \cup \Gamma(f_n)$$

for some definable continuous functions $f_1 < \cdots < f_n$. From this we conclude that the set $\{r \in M | r \text{ is not } Y(p)\text{-good}\}$ is finite because at most the elements $a_1, \ldots, a_k$ will belong to it. Therefore the definable set,

$$\mathrm{Bad}(Y) := \{(p, r) \in B | r \text{ is not } Y(p)\text{-good}\}$$

has no interior points. By the inductive assumption (I_m) there is a decomposition of M^m which partitions B and $\mathrm{Bad}(Y)$. Consider an open cell C of this partition s.t. $C \subseteq B$. Then $C \cap \mathrm{Bad}(Y) = \emptyset$ because $\mathrm{Bad}(Y)$ has no interior points. So if we go back and instead of B we use a box contained in C we will have that $\mathrm{Bad}(Y) = \emptyset$. Therefore we can apply claim 2 (for $Y(p)$ instead of Y) to find a number $k(p) < \omega$ s.t. $|Y_x| = k(p)$ for each point $x = (p, r) \in B$. Now we have to bound all these numbers $k(p)$, $p \in B'$.

To achieve this we pick an $r \in (a, b)$ and consider the set,

$$Y^r := \{(p, s) \,|\, (p, r, s) \in Y\} \subseteq M^m$$

Y^r will be finite over M^{m-1}, because Y is finite over M^m. By the inductive assumption Y^r will be uniformly finite over M^{m-1}, so there will be an $N < \omega$ such that for each $p \in B'$, $|\{s \in M | (p, s) \in Y^r\}| \leq N$. That is $\left|Y_{(p,r)}\right| \leq N$ for all $p \in B'$, hence $k(p) \leq N$ for all $p \in B'$. Thus $|Y_x| \leq N$ for all $x \in B$.

Now we are going to show our claim. For each $i \in \{0, \ldots, N\}$ let $B_i := \{x \in B | \, |Y_x| = i\}$, and define the functions $f_{i1} < \cdots < f_{ii}$ and $Y_x = \{f_{i1}(x), \ldots, f_{ii}(x)\}$. Apply now the induction hypothesis (II_m) on each f_{ij} and then use (I_m) to find a common refinement of the decompositions we obtained (from the application of (II_m)). Let's call this decomposition of M^m, $\mathcal{D}$. $\mathcal{D}$ partitions each of the sets B_i s.t. for each $A \in \mathcal{D}$, if $A \subseteq B_i$ then $f_{ij}|A$ is continuous for $j \in \{0, \ldots, i\}$. Since B is open and is partitioned by $\mathcal{D}$ there is an open cell $A \in \mathcal{D}$ s.t. $A \subseteq B$. But $B = \bigcup_i B_i$, so $A \subseteq B_i$ for some i. Therefore $f_{i1}, \ldots, f_{ij}$ are continuous on A. Hence each point of A is Y-good. Since $A \subseteq B$ this establishes claim 3.

We are now ready to finish the proof of the lemma. Take a decomposition $\mathcal{D}$ of M^m partitioning the set of Y-good points. Let $A \in \mathcal{D}$. If A is open then by claim 3 the cell A contains a Y-good point, so all points of A are Y-good. By claim 2 then there is a number $N_A < \omega$ s.t. $|Y_x| \leq N_A$ for all $x \in A$. By a previous remark in section 2.1 we have that every cell is homeomorphic under a coordinate projection to an open cell. Therefore by using that definable homeomorphism we can find such a number N_A for even the non-empty cells of $\mathcal{D}$. If we choose now $N := \max\{N_A | A \in \mathcal{D}\}$ we have that $|Y_x| \leq N$ for all $x \in M^m$. $\square$

Now we can proceed with the proof of (I_{m+1}). But first let's give a definition. For a definable set $A \subseteq M^{m+1}$ we put

$$\mathrm{bd}_m(A) := \left\{(x,r) \in M^{m+1} | r \in \mathrm{bd}(A_x)\right\}$$

and we note that $\mathrm{bd}_m(A)$ is a definable set, which is finite over M^m (by lemma 1.7) so we can apply the uniform finiteness property we proved above.

Proof of (I_{m+1}). Let $A_1, \ldots, A_k$ be definable subsets of M^{m+1}. Let

$$Y := \mathrm{bd}_m(A_1) \cup \cdots \cup \mathrm{bd}_m(A_k) .$$

Then $Y \subseteq M^{m+1}$ is definable and finite over M^m, and by the uniform finiteness property 2.6 it will be uniformly finite over M^m so there is $N < \omega$ s.t. $|Y_x| \leq N$ for all $x \in M^m$. For each $i \in \{0, \ldots, N\}$ let $B_i := \{x \in M^m | \, |Y_x| = i\}$ and define functions $f_{i1}, \ldots, f_{ii}$ on B_i by

$$Y_x = \{f_{i1}(x), \ldots, f_{ii}(x)\} \text{ and } f_{i1}(x) < \cdots < f_{ii}(x) .$$

Further put $f_{i0} := -\infty$ and $f_{ii+1} := +\infty$ (on B_i). Finally we define for each $\lambda \in \{1, \ldots, k\}$, $i \in \{0, \ldots, N\}$ and $1 \leq j \leq i$

$$C_{\lambda ij} := \{x \in B_i | f_{ij}(x) \in (A_\lambda)_x\}$$

and for each $\lambda \in \{1, \ldots, k\}$, $i \in \{0, \ldots, N\}$ and $0 \leq j \leq i$

$$D_{\lambda ij} := \{x \in B_i | \left(f_{ij}(x), f_{ij+1}(x)\right) \subseteq (A_\lambda)_x\} .$$

Using the inductive assumptions (I_m) and (II_m) we can get a decomposition $\mathcal{D}$ of M^m

- which partitions each of the B_i, $C_{\lambda ij}$ and $D_{\lambda ij}$
- and for each $E \in \mathcal{D}$ that is contained in a B_i, $f_{i1}|E, \ldots, f_{ii}|E$ are continuous functions.

For each cell $E \in \mathcal{D}$ we let $\mathcal{D}_E$ be the following partition of $E \times M$,

$$\{(f_{i0}|E, f_{i1}|E), \ldots, (f_{ii}|E, f_{ii+1}|E), \Gamma(f_{i1}|E), \ldots, \Gamma(f_{ii}|E)\}$$

where $i \in \{0, \ldots, N\}$ is s.t. $E \subseteq B_i$. Then $\mathcal{D}^* := \bigcup \{\mathcal{D}_E | E \in \mathcal{D}\}$ is a decomposition of M^{m+1} which partitions each set $A_1, \ldots, A_k$. $\square$

The proof of (II_{m+1}) will be based on the following lemma.

Lemma 2.8. *Let X be a topological space, $(R_1, <)$, $(R_2, <)$ dense linear orderings without endpoints and $f : X \times R_1 \to R_2$ a function s.t. for each $(x,r) \in X \times R_1$*

1. *$f(x, \cdot) : R_1 \to R_2$ is continuous and monotone on R_1,*
2. *$f(\cdot, r) : X \to R_2$ is continuous at x.*

Then f is continuous.

Proof. Let $(x, r) \in X \times R_1$ and $f(x, r) \in J$ for an interval J in R_2. We will show that the inverse image of J via f will be an open set in $X \times R_1$. So we need to find a neighborhood U of x and an interval I around r s.t. $f(U \times J) \subseteq J$. By (1) there are r_-, r_+ in R_1 s.t. $r_- < r < r_+$ and $f(x, r_-), f(x, r_+) \in J$. By (2) we can get a neighborhood U of x such that $f(U \times \{r_-\}), f(U \times \{r_+\}) \subseteq J$. We will show that for $I = (r_-, r_+)$, $f(U \times I) \subseteq J$.

Let $x' \in U$ and $r_- < r' < r_+$. Assume that $f(x', \cdot)$ is increasing (the decreasing case is similar). Then $f(x', r_-) \leq f(x', r') \leq f(x', r_+)$ and both $f(x', r_+)$ and $f(x', r_-)$ are in J. Hence $f(x', r') \in J$. □

Proof of (II_{m+1}). Let $f : A \to M$ be a definable function and $A \subseteq M^{m+1}$ definable. We have to show that f is cell-wise continuous. Because we have already proved (I_{m+1}) it suffices to show that

> (*) A can be partitioned into finitely many definable sets $A_1, \ldots, A_k$ such that $f|A_i : A_i \to M$ is continuous for $i = 1, \ldots, k$.

By (I_{m+1}) the set A can be partitioned into finitely many cells. So in order to prove (*) we may assume that A is already a cell. Now we consider two cases.

If the cell A is not open in M^{m+1}, then we use the definable homeomorphism $p_A : A \to p(A)$ as we defined it on the remark in section 2.1. Since $p(A) \subseteq M^n$ for some $n < m+1$ it follows from the inductive assumption (II_n) that the set $p(A)$ can be partitioned into definable set $B_1, \ldots, B_k$ s.t. $\left(f \circ p_A^{-1}\right)|B_j$ is continuous for each j. Hence A is partitioned into $p_A^{-1}(B_1), \ldots, p_A^{-1}(B_k)$, and the restriction of f to each of these sets is continuous. This establishes (*) for A non-open cell.

Suppose now that A is an open cell. We will call f *well-behaved at a point* $(p, r) \in A$ if $p \in C$ for some box $C \subseteq M^m$ and $a < r < b$ for some $a, b \in M$ s.t.

- $C \times (a, b)$ is contained in A,
- for all $x \in C$ the function $f(x, \cdot)$ is continuous and monotone on (a, b),
- the function $f(\cdot, r)$ is continuous at p.

Let A^* be the set of all points in A at which f is well-behaved. Note that A^* is definable.

Claim. A^* is dense in A.

To prove this it suffices to show that, given any box $B \subseteq M^m$ and $-\infty < a < c < +\infty$ such that $B \times (a, c)$ is contained in A, the box $B \times (a, c)$ intersects A^*. By the monotonicity theorem 1.12 for each

$x \in B$ there is a largest $\lambda(x) \in (a, c]$ such that the one-variable function $f(x, \cdot)$ is continuous and monotone on $(a, \lambda(x))$. Since $\lambda : B \to M$ is definable, there is by (II_m) a box $C \subseteq B$ on which λ is continuous. Taking C small enough we can assume that $b \leq \lambda(x)$ for all $x \in C$ and a fixed $b \in (a, c)$. Now choose an element $r \in (a, b)$. The function $f(\cdot, r) : C \to M$ is continuous on some smaller box, by (II_m). Replacing C by this smaller box, we see that f is well-behaved at each point (p, r) with $p \in C$. This establishes the claim.

Now by (I_{m+1}) we pick a decomposition $\mathcal{D}$ of M^{m+1} which partitions both A and A^*. Let $D \in \mathcal{D}$ be any open cell contained in A. To finish the proof we just need to show that f is continuous on D.

Because $D \subseteq A$ we have that $D \subseteq A^*$ since by claim 2.3 D intersects A^*, and $\mathcal{D}$ partitions A^*. In particular for each point $(p, r) \in D$ the function $f(\cdot, r)$ is continuous at p. Therefore D is the union of boxes $C \times (a, b)$ satisfying the conditions of the good behavior definition above for each $p \in C$, $a < r < b$. By lemma 2.8 the function f is continuous on each such box, hence f is continuous on D.

This concludes the proof of cell decomposition theorem. □

2.4 APPLICATIONS

Now we will use the cell decomposition theorem 2.5 in order to prove some very interesting theorems about o-minimal structures.

Theorem 2.9. *Let $\mathcal{M}$ be o-minimal. Then any definable $X \subseteq M^n$ is a disjoint union of finitely many definably connected definable sets.*

Proof. Let $X \subseteq M^n$ be definable set. By (I_n) of 2.5 we get a decomposition $\mathcal{D}$ of M^n which partitions X. Then $X = \bigcup \{D \in \mathcal{D} | D \subseteq X\}$. Because a decomposition contains finitely many cells and by proposition 2.3 each cell is definably connected we have proved the theorem. □

Next we will prove theorem 1.3 which stated in Chapter 1.

Theorem 2.10. [Knight et al. 12] *If $\mathcal{M}$ is o-minimal then* $\mathrm{Th}(\mathcal{M})$ *is strongly o-minimal.*

Proof. Let $\phi(x_1, \ldots, x_n, y)$ be a formula. By o-minimality for all $\bar{a} \in M^m$ the set $\phi(\bar{a}, y)^M := \{y \in M | \mathcal{M} \vDash \phi(\bar{a}, y)\}$ is a finite union of singletons and intervals. Consider now a formula $\psi(\bar{x}, z)$ such that,

$$\psi(\bar{x}, z)^M := \{(\bar{x}, z) \,|\, z \in \mathrm{bd}(\{y | \phi(\bar{x}, y)\})\}.$$

This is a well defined formula (since the boundary is definable as we saw before) and moreover finite over M^m (using lemma 1.7). Therefore by the uniform finiteness property 2.6, $\psi(\bar{x}, z)$ will be uniformly finite over M^m. Thus for some $k < \omega$ we have that,

$$\mathcal{M} \vDash (\forall \bar{x}) \left(\{y | \phi(\bar{x}, y)\} \text{ has at most } k \text{ boundary points} \right).$$

So for any $\mathcal{N} \equiv \mathcal{M}$ and $\bar{a} \in N^m$, $\phi(\bar{a}, y)^N$ is a finite union of points and intervals (because of the finite number of boundary points). $\square$

Definition 2.11. A *definably connected component* of a nonempty definable set $X \subseteq M^m$ is a maximal definably connected subset of X.

Definably connected component

Theorem 2.12. *Let $X \subseteq M^m$ be a nonempty definable set. Then X has only finitely many definably connected components. They are open and closed in X and form a partition of X.*

Proof. Let $T = \mathrm{Th}(\mathcal{M})$ where $\mathcal{M}$ is o-minimal. Thus, by theorem 2.10 we have that, theorem 2.9 holds for any $\mathcal{N} \vDash T$. Instead of working with definable sets let us work with formulas. Let $\phi(x_1, \ldots, x_n, \bar{y})$ be any formula. Applying (I_n) of the cell decomposition theorem 2.5 we have that, for any $\bar{a}$ in a model $\mathcal{N}$ of T, $\phi(\bar{x}, \bar{a})^N$ is a finite (disjoint) union of cells in N^n.

From the definition of cells, it follows that there is a formula $\psi(\bar{y})$ (without parameters) such that $\psi(\bar{a})$ expresses the fact that $\phi(\bar{x}, \bar{a})^N$ is a particular finite union of cells in N^n.

Let $\Psi(\bar{y})$ be the set of all possible such formulas $\psi(\bar{y})$, as $\bar{a}$ ranges over models of T. Thus $T \vDash (\forall \bar{y})(\bigvee \Psi(\bar{y}))$. By compactness, $T \vDash (\forall \bar{y})(\bigvee \Psi'(\bar{y}))$, where $\Psi'(\bar{y})$ is some finite subset of $\Psi(\bar{y})$. It clearly follows that for some $k < \omega$, for any $\bar{a}$ in M (in fact for any model of T), $\phi(\bar{x}, \bar{a})^M$ is a finite disjoint union of at most k cells in M^n. By proposition 2.3 we have that each cell is definably connected, so clearly $\phi(\bar{x}, \bar{a})^M$ has at most k definably connected components. $\square$

Theorem 2.13. *Suppose that $\mathcal{M}$ is an o-minimal expansion of $(\mathbb{R}, <)$. Then, we can replace definably connected with connected (with the usual topological sense) on Theorem 2.12.*

Proof. If the underlying order is $(\mathbb{R}, <)$ we can see that any cell in M^n is actually connected (with the topological sense on $\mathbb{R}$). Thus, $\phi(\bar{x}, \bar{a})^M$ has at most finitely many connected components, as $\bar{a}$ ranges over M^m. $\square$

Corollary 2.14. *Let $\mathcal{M}$ be an o-minimal structure over the ordered set $(\mathbb{R}, <)$. Then for every definable set $X \subseteq \mathbb{R}^m$, the following are equivalent.*

1. *X is definably connected.*
2. *X is connected with the usual topological sense.*

3 DIMENSION

3.1 DEFINABLE CLOSURE

Definition 3.1. For $A \subset M$,

- the *algebraic closure* $\mathrm{acl}(A)$ of A is the union of the finite A-definable sets, *Algebraic closure*
- and the *definable closure* $\mathrm{dcl}(A)$ of A is the union of the A-definable singletons. *Definable closure*

Alternatively, we can see algebraic closure as,

$$\mathrm{acl}(A) := \{c \in M | c \text{ has finitely many conjugates over } A\}$$

and definable closure as,

$$\mathrm{dcl}(A) := \{c \in M | c \text{ is fixed by any automorphism of } \mathcal{M} \text{ fixing } A\}$$

The reason is the following lemma,

Lemma 3.2. *A definable class $\mathcal{D}$ is definable over A iff $\mathcal{D}$ is invariant under all automorphisms over A (i.e. automorphisms which leave every element of A fixed).*

In general, $\mathrm{dcl}(A) \subseteq \mathrm{acl}(A)$ but in o-minimal structures they are equal. Indeed, in an o-minimal structure, in a finite definable set we can define the least element, the next least element, etc. Thus we have all the elements of it as definable singletons.

3.2 PREGEOMETRIES

Notation. If $A \subseteq M$ and $\bar{a} \in M^n$, with $\bar{a} = (a_1, \dots, a_n)$, we abuse the notation by writing $A\bar{a}$ for $A \cup \{a_1, \dots, a_n\}$.

Definition 3.3. A *pregeometry* on a set X is a function $\mathrm{cl} : \mathcal{P}(X) \to \mathcal{P}(X)$ (where $\mathcal{P}(X)$ is the power set of X) which satisfies the following conditions. *Pregeometry*

1. for all $A \subseteq X$, $A \subseteq \mathrm{cl}(A)$
2. for all $A \subseteq X$, $\mathrm{cl}(\mathrm{cl}(A)) = \mathrm{cl}(A)$
3. for all $A \subseteq X$, $\mathrm{cl}(A) = \bigcup \{\mathrm{cl}(F) \,| F \subseteq A,\, F \text{ finite}\}$
4. (Exchange Property) if $A \subseteq X$ and $b, c \in X$ with $b \in \mathrm{cl}(Ac) \setminus \mathrm{cl}(A)$, then $c \in \mathrm{cl}(Ab)$.

In any structure, algebraic closure satisfies (1)-(3). By theorem 1.14 (exchange principle for o-minimal models) we have that (4) holds when we are in o-minimal structures.

Geometric structure

Definition 3.4. A first order structure $\mathcal{M}$ is *geometric* if algebraic closure has the exchange property (so defines a pregeometry) in all models of $\mathrm{Th}(\mathcal{M})$, and $\mathcal{M}$ has the uniform finiteness property (i.e. lemma 2.6 holds).

Corollary 3.5. *Every o-minimal structure is geometric.*

Proof. By theorem 2.10 we have that every model of $\mathrm{Th}(\mathcal{M})$ is o-minimal and by theorem 1.14 algebraic closure has the exchange property in o-minimal structures. Finally by lemma 2.6 every o-minimal structure has the uniform finiteness property. □

3.3 DIMENSION

In a geometric structure $\mathcal{M}$, there is a general notion of independence. We can say that $I \subseteq M$ is independent if, for all $x \in I$, $x \notin \mathrm{acl}(I \setminus \{x\})$. If $A \subseteq M$ then we can talk of I being independent over A (regard the elements of A as being interpreted by new constants).

$\dim(\bar{a}/A)$

Dimension of a type

Definition 3.6. Let $\mathcal{M}$ be geometric, $A \subseteq M$ and $\bar{a} \in M^n$. Then $\dim(\bar{a}/A)$ is the least cardinality of a subtuple (initial segment) $\bar{a}'$ of $\bar{a}$ such that $\bar{a} \subseteq \mathrm{acl}(A\bar{a}')$. If $p(\bar{x}) \in S_n(A)$ (the set of complete n-types over A), then $\dim(p) = \dim(\bar{a}/A)$, for any $\bar{a}$ realizing p in an elementary extension of $\mathcal{M}$.

Lemma 3.7. *Pillay [24] Let $\mathcal{M}$ be a geometric structure.*

1. $\dim(\bar{a}/A)$ *is the cardinality of any maximal independent (over A) subtuple of $\bar{a}$.*

2. *If $A \subseteq B$, then* $\dim(\bar{a}/A) \geq \dim(\bar{a}/B)$

3. $\dim\left(\bar{a}\bar{b}/A\right) = \dim\left(\bar{a}/A\bar{b}\right) + \dim\left(\bar{b}/A\right)$

4. *If $p(\bar{x}) \in S_n(A)$ and $A \subseteq B$ then there is $p'(\bar{x}) \in S_n(B)$ such that $p \subseteq p'$ and* $\dim(p) = \dim(p')$.

Proof. (1)-(3) are easy to see by the definitions and the use of the exchange property. To show (4) we observe first that by (2) it is enough to find $p' \in S_n(B)$ with $\dim(p') \geq \dim(p)$. Suppose that $\dim(p) = k$. Let $\bar{a}$ realize p. Without loss of generality, $a_1, \ldots, a_k$ are algebraically independent over A. By saturating M enough we can find inductively $a'_1, \ldots, a'_k \in M$, algebraically independent over B with $\mathrm{tp}(a'_1, \ldots, a'_k/A) = \mathrm{tp}(a_1, \ldots, a_k/A)$. This is clearly sufficient. □

We can also use algebraic closure to get a notion of dimension for definable sets, mimicking the Zariski dimension for constructible sets in algebraically closed fields. For the rest of the section we shall assume $\mathcal{M}$ be to *sufficiently saturated*, i.e. $|A|^+$-saturated for any set of parameters A we might care to mention. By $|A|^+$ we denote the next cardinal after $|A|$. This way we are sure that certain tuples (for example tuples from A) will exist in our model. Without this assumption we can still define dimension for definable sets but we will have to quantify over elementary extensions of the model, as a realization in the definable set of the appropriate dimension may not exist in the model.

Definition 3.8. Let $\mathcal{M}$ be geometric and sufficiently saturated, and $X \subseteq M^n$ be A-definable. Then, *Dimension of a definable set*

$$\dim(X) := \max\{\dim(\bar{a}/A)\,|\bar{a} \in X\}.$$

Alternatively we can write,

$$\dim(X) := \max\{\dim(p)\,|p \in S_n(A) \text{ and } p \text{ is realized in } X\}.$$

We will say that $\bar{a}$ is *a generic point of X over A*, if $\dim(\bar{a}/A) = \dim(X)$. *Generic point*
Also $\mathrm{tp}(\bar{a}/A)$ will be *a generic type of X over A*, if $\dim(\mathrm{tp}(\bar{a}/A)) = \dim(X)$. We also set, $\dim(\emptyset) = -\infty$. *Generic type*

Let us note here that by (4) of lemma 3.7, $\dim(X)$ does not depend on A.

The next lemmas provide some properties of this dimension.

Lemma 3.9. *Let $\mathcal{M}$ be a sufficiently saturated geometric structure.*

1. $\dim(\{a\}) = 0$, *for* $a \in M$ *and* $\dim(M) = 1$.
2. $\dim$ *is invariant under permutation of coordinates.*
3. *If* $X \subseteq Y \subseteq M^n$ *definable, then* $\dim(X) \leq \dim(Y)$.
4. *If* $X, Y \subseteq M^n$ *definable, then*

$$\dim(X \cup Y) = \max\{\dim(X), \dim(Y)\}.$$

Proof. (1), (2) and (3) are easy to see by the definitions.

For (4) we have that,

$$\begin{aligned}
\dim(X \cup Y) &= \max\{\dim(\bar{a}/A)\,|\bar{a} \in X \cup Y\} \\
&= \max\{\{\dim(\bar{a}/A)\,|\bar{a} \in X\}, \{\dim(\bar{a}/A)\,|\bar{a} \in Y\}, \\
&\qquad \{\dim(\bar{a}/A)\,|\bar{a} \in X \cap Y\}\} \\
&\overset{(3)}{=} \max\{\{\dim(\bar{a}/A)\,|\bar{a} \in X\}, \{\dim(\bar{a}/A)\,|\bar{a} \in Y\}\} \\
&= \max\{\dim(X), \dim(Y)\}
\end{aligned}$$

□

In the following three propositions we also add the assumption that $\mathcal{M}$ is uniformly bounded.

Proposition 3.10. (Definability of dimension) *Let $\mathcal{M}$ be a sufficiently saturated, uniformly bounded, geometric structure and $X \subseteq M^{m+n}$ be a definable set. For each $\bar{a} \in M^m$ let $X_{\bar{a}} := \{\bar{y} \in M^n | (\bar{a}, \bar{y}) \in X\}$. For each $i = 0, \ldots, n$ let $X(i) := \{\bar{x} \in M^m | \dim(X_{\bar{x}}) = i\}$. Then each set $X(i)$ is definable, and for each i,*

$$\dim(\{(\bar{x}, \bar{y}) \in X | \bar{x} \in X(i)\}) = \dim(X(i)) + i.$$

Proof. We will prove it by induction on n. For $n = 1$ we have that $X \subseteq M^{m+1}$ definable and for each $\bar{a} \in M^m$, $X_{\bar{a}} := \{y \in M | (\bar{a}, y) \in X\} \subseteq M$ definable. Because $\mathcal{M}$ is uniformly bounded, $X_{\bar{a}}$ will be finite over M^m, hence $|X_{\bar{a}}| \leq N$, $N < \omega$ for all $\bar{a} \in M^m$. Then $\dim(X_{\bar{a}}) = 0$ for all $X_{\bar{a}}$ because of lemma 3.9 (2). Therefore $X(0) = \{\bar{x} \in M^m | \dim(X_{\bar{x}}) = 0\} = X$ and $X(1) = \emptyset$ are both definable, and

$$\dim(\{(\bar{x}, \bar{y}) \in X | \bar{x} \in X(0)\}) = \dim(X) = \dim(X(0)) + 0$$
$$\dim(\{(\bar{x}, \bar{y}) \in X | \bar{x} \in X(1)\}) = \dim(\emptyset) = \dim(X(1)) + 1$$

Suppose that it holds for n. Let $X \subseteq M^{m+n+1}$ definable. For all $\bar{a} \in M^m$ let $X_{\bar{a}} := \{(\bar{y}, z) \in M^n \times M | (\bar{a}, \bar{y}, z) \in X\} \subseteq M^{n+1}$ and also for all $(\bar{a}, \bar{y}) \in M^{m+n}$ let $X_{(\bar{a},\bar{y})} := \{z \in M | (\bar{a}, \bar{y}, z) \in X\} \subseteq M$.

Notice that, the possible values for $\dim\left(X_{(\bar{a},\bar{y})}\right)$, for all $(\bar{a}, \bar{y}) \in M^{m+n}$ are 0,1 or $-\infty$ (for the empty set). We define two disjoint sets,

$$B_0 := \left\{(\bar{a}, \bar{y}) \in M^{m+n} | \dim\left(X_{(\bar{a},\bar{y})}\right) = 0\right\}$$
$$B_1 := \left\{(\bar{a}, \bar{y}) \in M^{m+n} | \dim\left(X_{(\bar{a},\bar{y})}\right) = 1\right\}$$

Let $\pi_1 : M^{m+n+1} \to M^{m+n}$ be the projection on the first $m + n$ coordinates. So, $\pi_1(X) = B_0 \cup B_1$ and $\pi_1(X_{\bar{a}}) = (B_0)_{\bar{a}} \cup (B_1)_{\bar{a}}$. Applying the induction hypothesis and lemma 3.9 we have,

$$\dim(X_{\bar{a}}) = \max\{\dim((B_0)_{\bar{a}}), \dim((B_1)_{\bar{a}}) + 1\}$$

Consider now the set, $X(i) = \{\bar{a} \in M^m | \dim(X_{\bar{a}}) = i\}$. We define again two sets,

$$C_0 = \{\bar{a} \in M^m | \dim((B_0)_{\bar{a}}) = i \,\&\, \dim((B_1)_{\bar{a}}) + 1 < i\}$$
$$C_1 = \{\bar{a} \in M^m | \dim((B_0)_{\bar{a}}) < i \,\&\, \dim((B_1)_{\bar{a}}) + 1 = i\}$$

which are definable because of the induction hypothesis. Let $\pi_2 : M^{m+n} \to M^m$ be the projection on the first m coordinates.

Let's assume that both C_0 and C_1 are nonempty. Notice that for all $\bar{a} \in C_0$, $\dim((B_0)_{\bar{a}}) = i$. Therefore using the induction hypothesis $\dim\left(\pi_2^{-1}(C_0) \cap B_0\right) = \dim(C_0) + i$. The same way we get that

$\dim\left(\pi_2^{-1}(C_0)\cap B_1\right) < \dim(C_0)+i$. By the base of our induction we finally have,

$$\dim\left(\pi_1^{-1}\left(\pi_2^{-1}(C_0)\cap B_0\right)\cap X\right)=\dim(C_0)+i$$
$$\dim\left(\pi_1^{-1}\left(\pi_2^{-1}(C_0)\cap B_1\right)\cap X\right)\leq\dim(C_0)+i$$

If we work the same way for C_1 we also get,

$$\dim\left(\pi_1^{-1}\left(\pi_2^{-1}(C_1)\cap B_0\right)\cap X\right)\leq\dim(C_0)+i$$
$$\dim\left(\pi_1^{-1}\left(\pi_2^{-1}(C_1)\cap B_1\right)\cap X\right)=\dim(C_0)+i$$

Notice now that,

$$\begin{aligned}\left\{(\bar{a},\bar{y},z)\in X|\dim\left(X_{(\bar{a},\bar{y})}\right)=i\right\} \quad = \quad & \left(\pi_1^{-1}\left(\pi_2^{-1}(C_0)\cap B_0\right)\cap X\right)\cap\\ & \left(\pi_1^{-1}\left(\pi_2^{-1}(C_0)\cap B_1\right)\cap X\right)\cap\\ & \left(\pi_1^{-1}\left(\pi_2^{-1}(C_1)\cap B_0\right)\cap X\right)\cap\\ & \left(\pi_1^{-1}\left(\pi_2^{-1}(C_1)\cap B_1\right)\cap X\right)\end{aligned}$$

Hence, $\dim\left(\left\{(\bar{a},\bar{y},z)\in X|\dim\left(X_{(\bar{a},\bar{y})}\right)=i\right\}\right) = \dim(C_0\cup C_1)+i = \dim(X(i))+i$. $\square$

Corollary 3.11. (Coordinate free version of 3.10) *Let $\mathcal{M}$ be a sufficiently saturated, uniformly bounded, geometric structure. Let also $S\subseteq M^m$ definable, $f: S\to M^n$ be a definable map and $i\in\{0,\ldots,m\}$. Then the set $B(i)=\{y\in M^n|\dim(f^{-1}(y))=i\}$ is definable and*

$$\dim\left(f^{-1}(B(i))\right)=\dim(B(i))+i.$$

Proof. Let $X=\{(f(x),x)\,|x\in S\}$ be the "reversed" graph of f. By interchanging the roles of m and n and applying lemma 3.10 on X we get the result of the corollary. $\square$

Proposition 3.12. *Let $\mathcal{M}$ be a sufficiently saturated, uniformly bounded, geometric structure. If $f: M^n\to M^m$ is a definable partial function, and $A\subseteq M^n$ is definable, then $\dim(f(A))\leq\dim(A)$, with equality if f is injective.*

Proof. We will prove it first for injective f. Applying proposition 3.10 on $X=\Gamma(f)$, for $i=0$ we get that $\dim(\Gamma(f))=\dim(A)$. By replacing f by its inverse we also get that, $\dim(\Gamma(f^{-1}))=\dim(f(A))$. Now because of lemma 3.9 (2), we have $\dim(A)=\dim(f(A))$.

Now for general f, notice that,

$$A=\bigcup_{i=0}^{m}f^{-1}(B(i))\,,\ f(A)=\bigcup_{i=0}^{m}B(i)$$

so by applying corollary 3.11 we have the desired inequality. $\square$

3.4 TOPOLOGICAL DIMENSION

In a structure $\mathcal{M}$ which carries a topology with a uniformly definable basis there is also a notion of topological dimension for definable sets.

Topological dimension

Definition 3.13. If $X \subseteq M^n$ is definable, then we define as *topological dimension* and denote by $\text{tdim}(X)$, the greatest $k \leq n$ s.t. for some projection $\pi : M^n \to M^k$, $\pi(X)$ has non-empty interior in M^k.

Proposition 3.14. *Let $\mathcal{M}$ be an o-minimal structure, $a_1, \ldots, a_n \in M$ and $A \subseteq M$. Then $\{a_1, \ldots, a_n\}$ is A-independent iff every A-definable set containing $(a_1, \ldots, a_n)$ has non-empty interior in M^n.*

Proof. ($\Leftarrow$) Suppose that every A-definable set containing $(a_1, \ldots, a_n)$ has non-empty interior in M^n and $\{a_1, \ldots, a_n\}$ is not A-independent. So for some i, $a_i \in \text{acl}(A \cup \{a_j | j \neq i\})$. Let $\phi(x_1, \ldots, x_n)$ be the $\mathcal{L}(A)$ formula (A-definable formula) s.t. a_i is the unique element satisfying $\phi(a_1, \ldots, a_{i-1}, x_i, a_{i+1}, \ldots, a_n)$. Let now $\psi(x_1, \ldots, x_n)$ be the formula which says that "x_i is the unique element satisfying $\phi(x_1, \ldots, x_n)$". Then the set $\psi^{\mathcal{M}}$ of all the points satisfying ψ contains only $(a_1, \ldots, a_n)$ and has no interior in M^n.

($\Rightarrow$)By induction on n. For $n = 1$ first, if $a_1 \notin \text{acl}(A)$ then a_1 is not in any A-definable finite set and by o-minimality every infinite set will have non-empty interior. Assume now that it holds for n, and suppose that $\{a_1, \ldots, a_{n+1}\}$ is A-independent. Let $X \subseteq M^{n+1}$ be an A-definable set containing $(a_1, \ldots, a_{n+1})$ we have to show that it has non-empty interior.

By the cell decomposition theorem 2.5 we can have a partition of X into cells. We can also assume that X is by itself a cell. We consider two cases.

Case 1. $X = \Gamma(f)$ for some definable f over some cell $Y \subseteq M^n$. Y and f are also A-definable, because X is. Then $a_{n+1} = f(a_1, \ldots, a_n)$, so $a_{n+1} \in \text{acl}(A \cup \{a_1, \ldots, a_n\})$ which is a contradiction.

Case 2. $X = (f, g)_Y$, where f, g are definable and continuous, $Y \subseteq M^n$ is A-definable (because X is) and $(a_1, \ldots, a_n) \in Y$. By induction hypothesis $(a_1, \ldots, a_n)$ is A-independent and Y has non-empty interior in M^n. By continuity of f, g we have that X has non-empty interior in M^{n+1}.

This concludes the proof. □

The previous proposition gives the following corollary.

Corollary 3.15. *Let $\mathcal{M}$ be o-minimal, and $X \subseteq M^n$ a definable set. Then* $\dim(X) = \text{tdim}(X)$.

The following proposition gives us a useful topological characterization of genericity. It says, very roughly, that if an A-definable property holds for an $\bar{a}$ generic over A then it holds throughout a neighbourhood of itself in the definable set.

Proposition 3.16. *Let $\mathcal{M}$ be o-minimal, $X \subseteq M^n$ A-definable and $\bar{b}$ be a generic point of X over A. Then $\dim(X) = k$ iff there is an open rectangular neighbourhood $Y \subseteq M^n$ of $\bar{b}$ and a projection $\pi : M^n \to M^k$ inducing a homeomorphism from $X \cap Y$ onto an open subset of M^k.*

Proof. ($\Rightarrow$) Assume that $\dim(X) = k$, and let $\bar{b} = (b_1, \ldots, b_n)$ be a generic point of X over A. Then $\dim(X) = \operatorname{tdim}(X) = \dim\left(\bar{b}/A\right)$. By the definition of topological dimension we have that there exists a projection $\pi : M^n \to M^k$ s.t. $\pi(X)$ has non-empty interior. Let us call that non-empty interior $B \subseteq \pi(X)$. By proposition 3.14 we have that $(b_1, \ldots, b_k) \in B$.

We know that X can be covered by open subsets $Y_1, \ldots, Y_s$. Projection is a continuous function thus $\pi^{-1}(X) \subseteq X$ is an open set and a subset of X, so it can be covered by some of the Y_j, $j \in J \subseteq \{1, \ldots, s\}$. Let $Y = \cup_j Y_j$. Now $\bar{b} \in Y$, Y is open and $\pi|(X \cap Y)$ is a function onto $B \subseteq M^k$.

($\Leftarrow$) By the assumptions and proposition 3.14 we have that $\dim(X) \geq k$. Assume now that $\dim(X) \geq k+1$. Then there is a projection $\pi_1 : M^n \to M^{k+1}$ s.t. $\pi_1(X)$ has non-empty interior in M^{k+1} and is A-definable. Let $\bar{b}' = (b_1, \ldots, b_k)$. By the assumption for the dimension and because $\bar{b}$ is generic of X over A we get that $\{b_1, \ldots, b_k\}$ is an independent set. Now consider the set

$$(\pi_1(X))_{\bar{b}'} := \left\{x \in M | \left(\bar{b}', x\right) \in \pi(X)\right\}.$$

Because $\mathcal{M}$ is uniformly bounded and because of its definition, $(\pi_1(X))_{\bar{b}'}$ will be a finite $A\bar{b}'$-definable set and $b_{k+1} \in (\pi_1(X))_{\bar{b}'}$. That means $b_{k+1} \in \operatorname{acl}\left(A\bar{b}'\right)$, hence $\{b_1, \ldots, b_k, b_{k+1}\}$ is not independent and $\dim(X) < k+1$. This concludes the proof. $\square$

Consider now an o-minimal expansion of the ordered set $\mathbb{R}$ (in a countable language). Then, as the next theorem shows, any definable set X has a generic point in the model over any countable set of parameters, even though $(\mathbb{R}, <)$ is not ω_1-saturated.

Nowhere dense set

First we have to give some definitions from topology. A *nowhere dense set*, is a set which closure has empty interior. Also a set A is called *meager* or *of first category*, if there exists a collection $\{Y_i | i \in \mathbb{N}\}$ of nowhere dense sets s.t. $A \subseteq \bigcup_i Y_i$. A topological space is called a *Baire space*, if the interior of every union of countably many closed nowhere dense sets is empty (we say then, that it has the Baire property).

Meager set

Baire space

The Baire Category Theorem gives us that $\mathbb{R}$ is a Baire space. A property of Baire spaces is that every open subspace of a Baire space is also a Baire space. It is not hard to see using these facts that every open subset of $\mathbb{R}$ is not meager.

Theorem 3.17. *Let M be $\mathbb{R}$. Let A be a countable subset of M, and let X be an A-definable subset of M^n. Then X contains a generic point over A, i.e. there is an $\bar{a} \in X$ such that* $\dim(X) = \dim(\bar{a}/A)$.

Proof. Because $\mathcal{M}$ is not ω_1-saturated we can not use the usual notion of dimension. We will use the fact, from corollary 3.15, that $\dim(X) = \operatorname{tdim}(X)$ for any definable $X \subseteq M^n$. Suppose that $\operatorname{tdim}(X) = k = \dim(X)$. Therefore there is a projection $\pi : X \to M^k$ onto, s.t. $\pi(X)$ has non-empty interior. Hence $\pi(X)$ contains an open subset, let's say B. We know from Baire category theorem that B is not meager. Thus we can not write B as countable union of sets whose closure has empty interior. Therefore B (and also $\pi(X)$) can't be contained in a union of A-definable sets of dimension less than k. So there is $\bar{b} \in \pi(X)$ such that $\dim\left(\bar{b}/A\right)$ and $\bar{b}$ extends to $\bar{a} \in X$ s.t. $\dim(\bar{a}/A) = k$. □

Finally we will give a consequence of the cell decomposition theorem.

Large set

Definition 3.18. Let $Y \subseteq X \subseteq M^n$. We say that Y is *large* in X if $\dim(X \backslash Y) < \dim(X)$.

A useful characterization of large sets, which follows easily by the definitions, is the following.

Lemma 3.19. *Let $Y \subseteq X$ definable sets. Then Y is large in X iff for every A over which X, Y are definable, every generic point $\bar{a}$ of X over A is in Y.*

Proposition 3.20. *Let $\mathcal{M}$ be o-minimal, B be a subset of M^k with* $\dim(B) = k$, *and $f : B \to M$ be a definable function. Assume that both B and f are both definable over a set A. Then there is an A-definable large subset S of A, open in M^k, s.t. $f|S$ is continuous.*

Proof. By the cell decomposition theorem there is a decomposition $\mathcal{D} = \{D_i | i \in I\}$ of M^k partitioning B s.t. for each $D \in \mathcal{D}$ with $D \subseteq B$, $f|D$ is continuous. Because $\dim(B) = k$, B must have non-empty interior in M^k and also because of the partitioning $B^\circ = \bigcup_j \{D_j | j \in J \subseteq I\}$ for some J. B° is open. Now consider the set $B \backslash B^\circ = \partial B$. It clearly has no interior in M^k, hence $\dim(\partial B) < k$. This concludes the proof. □

3.5 CELL DIMENSION

Now we will define a dimension for cells (definition 2.1). This definition will be recursive, as it is the definition of a cell.

Cell dimension

Definition 3.21. Let $\mathcal{M}$ be an o-minimal structure. We define the *cell dimension* of a cell, and denote it by cdim( ), as below

1. singletons have dimension 0 and intervals 1,

2. Suppose that D is a cell with dimension k. Then,

a) $\Gamma(f)$, for $f \in C(D)$, has dimension k, and

b) $(f,g)_D$, for $f, g \in C_\infty(D)$ with $f < g$, has dimension $k+1$.

It is not hard to see that by this definition we can also write for a cell $C = (i_1, \ldots, i_n)$ that,

$$\operatorname{cdim}(C) = i_1 + \cdots + i_n$$

By the cell decomposition theorem 2.5, any definable set X on $\mathcal{M}$ can be written as a union of cells, let's say $C_1, \ldots, C_n$. We will define a dimension for the definable sets using the cell dimension and we will denote it temporarily as $\dim'(X)$ as follows,

$$\dim'(X) = \max\left\{\operatorname{cdim}(C_1), \ldots, \operatorname{cdim}(C_n)\right\}$$

and we let $\dim'(\emptyset) = -\infty$. The following are two easy observations, which follow from the previous discussion.

Corollary 3.22. *Let $X \subseteq M^n$. Then $\dim'(X) = n$ iff X has non-empty interior in M^n.*

Corollary 3.23. *Let $X \subseteq M^n$. Then $\dim'(X) = k$ iff there is some projection $\pi : M^n \to M^k$ s.t. $\pi(X)$ has non-empty interior.*

Now we can derive the following,

Theorem 3.24. *Let $\mathcal{M}$ be o-minimal and $X \subseteq M^n$ definable. Then,*

$$\dim(X) = \operatorname{tdim}(X) = \dim'(X).$$

4 PRIME MODELS

4.1 BACKGROUND

Let us recall first the following definition.

Prime model over a set

Definition 4.1. If $A \subseteq \mathcal{M} \models T$, then a model $\mathcal{M} \models T$ such that $A \subseteq \mathcal{M}$ is said to be *prime over* A if for any $\mathcal{M}' \models T$ with $A \subseteq \mathcal{M}'$, there is an elementary mapping $f : \mathcal{M} \to \mathcal{M}'$ that is identity on A.

Equivalently, $\mathcal{M}$ is *prime over* A if whenever $\mathcal{N} \models T$ and $f : A \to \mathcal{N}$ is partial elementary, there is an elementary $f^* : \mathcal{M} \to \mathcal{N}$, extending f.

Our goal here is to show that for any subset of a model of a strongly o-minimal theory, there exists a model of the theory that is prime over the given subset and also that it is unique, up to isomorphism. This is the following theorem.

Theorem 4.2. [Pillay and Steinhorn 26] *Let $A \subseteq \mathcal{M} \models T$, where T is a strongly o-minimal theory. Then there is an $\mathcal{M} \models T$, $A \subseteq \mathcal{M}$, that is prime over A, and is unique up to isomorphism over A.*

This theorem can be considered as a generalization of the following,

Theorem 4.3. *The real closure R of an ordered field F is prime over F. Moreover, it is also unique up to isomorphism over F.*

because the theory of real closed fields is contained in the class of o-minimal theories.

We remark here that Shelah [30] proved the following,

κ-stable theory

Definition 4.4. A theory T is called κ-stable (for an infinite cardinal κ) if for every set A of cardinality κ the set of complete types over A has cardinality κ. We call T ω-stable if κ is $\aleph_0$.

Theorem 4.5. [Shelah 30] *Let $A \subseteq \mathcal{M} \models T$, where T is a ω-stable theory. Then there is an $\mathcal{M} \models T$, $A \subseteq \mathcal{M}$, that is prime over A, and is unique up to isomorphism over A.*

Note here that the class of ω-stable theories does not contain o-minimal theories. As we can see theorem 4.2 is an analogous of 4.5 for o-minimal theories, and this shows some analogy between o-minimality and stability.

An observation we can make is that theorem 4.2 seems to be quite tight. Just the existence of prime model over a substructure can fail even for a structure that is "very close" to be o-minimal, as we see in the next example.

Example 4.6. Consider the structure $\mathcal{M} = (Q, <, P)$, where P is a unary predicate such that both P and $Q \backslash P$ are dense in Q. Now $\mathcal{M}$ is not o-minimal but it can break into two substructures P and $Q \backslash P$ that are both o-minimal. We will see that $\mathcal{M}$ does not have a prime model over P. Let $f : P \to \mathcal{M}$ be a partial embedding. By Tarski-Vaught test A.10 we have that f can not be extended to an elementary embedding.

We will prove theorem 4.2 in two parts (existence and uniqueness) in the three following sections. Let us give some definitions before we close this section.

Definition 4.7. Let $T = \mathrm{Th}_A(\mathcal{M})$ be the set of all $\mathcal{L}_A$-sentences true in $\mathcal{M}$. Let p be the set of $\mathcal{L}_A$-formulas in free variables $v_1, \ldots, v_n$. We call p an *n-type* if $p \cup \mathrm{Th}_A(\mathcal{M})$ is satisfiable.

n-type

We say that p is a *complete n-type* if $\phi \in p$ or $\neg\phi \in p$ for all $\mathcal{L}_A$-formulas ϕ with free variables from $v_1, \ldots, v_n$. We let $S_n^{\mathcal{M}}(A)$ be the set of all complete n-types.

Complete n-type

A complete type $p \in S_n^{\mathcal{M}}(A)$ is called a *principal type*, if there is a formula $\phi(\bar{x}) \in p$, such that $T \vDash \forall \bar{x} (\phi(\bar{x}) \to \psi(\bar{x}))$ for every formula $\psi(\bar{x})$ in p.

Principal type

There is a natural topology on the space of complete n-types $S_n^{\mathcal{M}}(A)$. For ϕ an $\mathcal{L}_A$-formula with free variables from $v_1, \ldots, v_n$, let

$$[\phi] := \left\{ p \in S_n^{\mathcal{M}}(A) \,|\, \phi \in p \right\}.$$

If p is a complete type and $\phi \vee \psi \in p$ then $\phi \in p$ or $\psi \in p$. Thus $[\phi \vee \psi] = [\phi] \cup [\psi]$. Similarly, $[\phi \wedge \psi] = [\phi] \cap [\psi]$.

The *Stone topology* on $S_n^{\mathcal{M}}(A)$ is the topology generated by taking the sets $[\phi]$ as basic open sets.

Stone topology

Definition 4.8. Let $T = \mathrm{Th}_A(\mathcal{M})$ and $p(x)$ be a type (not necessarily complete). We say that $p(x)$ is *isolated* (by $\phi(x)$) if there is a formula $\phi(x)$ with the property that $\forall \psi(x) \in p(x), T \vDash \phi(x) \to \psi(x)$. Clearly if $p(x)$ is a complete type then $p(x)$ is a principal type.

Isolated type

Alternatively, using the previous formulation, we say that a type $p \in S_n^{\mathcal{M}}(A)$ is *isolated* if $\{p\}$ is an open subset of $S_n^{\mathcal{M}}(A)$.

Note here that isolated types are realized in every elementary substructure or extension and hence they cannot be omitted.

4.2 EXISTENCE OF PRIME MODELS

We are ready to prove the existence of prime models over a set $A \subseteq \mathcal{M}$ where $\mathcal{M}$ is o-minimal. The core of the proof is the following lemma which states that isolated types are dense in the Stone space.

Lemma 4.9. *Let $\mathcal{M}$ be an o-minimal structure, and $A \subseteq \mathcal{M}$. Then for any formula $\phi(\bar{x}, \bar{a})$ having parameters from A, there is a formula $\psi(\bar{x}, \bar{a}')$ also with parameters from A, so that,*

$$\mathcal{M} \vDash (\forall \bar{x}) \left[\psi(\bar{x}, \bar{a}') \to \phi(\bar{x}, \bar{a}) \right]$$

and for every formula $\theta\left(\bar{x}, \bar{b}\right)$ *with parameters from* A*, exactly one of*

$$\mathcal{M} \vDash (\forall \bar{x}) \left[\psi\left(\bar{x}, \bar{a}'\right) \to \theta\left(\bar{x}, \bar{b}\right)\right]$$

or

$$\mathcal{M} \vDash (\forall \bar{x}) \left[\psi\left(\bar{x}, \bar{a}'\right) \to \neg\theta\left(\bar{x}, \bar{b}\right)\right]$$

holds. In other words, the isolated types of $\mathrm{Th}(\mathcal{M}, a)_{a \in A}$ *are dense. (i.e., any formula with parameters from* A *is implied by a complete formula with parameters from* A*.)*

Proof. By induction on the number of free variables in $\bar{x}$. We start with the basis of our induction, $n = 1$. By o-minimality the set $\Phi = \{x \in M | \mathcal{M} \vDash \phi(x, \bar{a})\}$ is a finite union of open intervals and singletons. If Φ contains a singleton then the definition of this singleton, using just the parameters $\bar{a}$, yields a complete formula. Therefore wlog let us assume that Φ is a union of finite open intervals. Let $\phi_0(x, \bar{a})$ be the formula which defines exactly the leftmost such interval. If $\phi_0(x, \bar{a})$ is not yet complete then, by the negation of the definition of complete formulas, there is some formula $\psi(x, \bar{a}')$ with parameters from A such that,

$$\mathcal{M} \vDash (\exists x)\left(\phi_0(x, \bar{a}) \wedge \psi(x, \bar{a}')\right) \wedge (\exists x)\left(\phi_0(x, \bar{a}) \wedge \neg\psi(x, \bar{a}')\right).$$

Then a boundary point of $\psi(x, \bar{a}')$ must lie inside the open interval defined by $\phi_0(x, \bar{a})$. But then this boundary point will be definable by some formula $\psi^*(x, \bar{a}')$, and thus we can take this ψ^* as the complete formula we are looking for. □

Theorem 4.10. *Let* $A \subseteq \mathcal{M} \vDash T$*, where* T *is a complete o-minimal theory. Then there is* $\mathcal{M}_0 \prec \mathcal{M}$*, a prime model over* A*. Moreover we can choose* $\mathcal{M}_0$ *such that, every element of* $\mathcal{M}_0$ *realizes an isolated type over* A*.*

Proof. We will find an ordinal δ and build a sequence of sets $\langle A_\alpha | \alpha \leq \delta \rangle$ where $A_\alpha \subseteq M$ and

- $A_0 = A$,
- if α is a limit ordinal, then $A_\alpha = \bigcup_{\beta < \alpha} A_\beta$,
- if there is an element a_α, in $M \backslash A_\alpha$, realizing an isolated type over A_α, and set $A_{\mathrm{succ}(\alpha)} = A_\alpha \cup \{a_\alpha\}$,
- if no element of $M \backslash A_\alpha$ realizes an isolated type over A_α, we stop and let $\delta = \alpha$.

Let $\mathcal{M}_0$ be the substructure of $\mathcal{M}$ with universe A_δ.

Claim 1. $\mathcal{M}_0 \prec \mathcal{M}$.

Suppose that $\mathcal{M} \vDash \exists x \phi(x, \bar{a})$, where $\bar{a}$ a tuple of elements of A_δ. By lemma 4.9, the isolated types $S^{\mathcal{M}}(A_\delta)$ are dense. Thus there is a $b \in M$, such that $\mathcal{M} \vDash \phi(b, \bar{a})$, and $\mathrm{tp}^{\mathcal{M}}(b/A_\delta)$ is isolated. By the construction of A_δ we have that $b \in A_\delta$. Hence, $\mathcal{M}_0 \vDash \phi(x, \bar{a})$ and by the Tarski-Vaught test A.10, $\mathcal{M}_0 \prec \mathcal{M}$.

Claim 2. $\mathcal{M}_0$ is a prime model over A.

Suppose that $\mathcal{N} \vDash T$ and $f : A \to \mathcal{N}$ is a partial elementary function. We show by induction that there are $f = f_0 \subset f_1 \subset \cdots \subset f_\alpha \subset \cdots \subset f_\delta$, where $f_\alpha : A_\alpha \to \mathcal{N}$ is elementary.

If α is a limit ordinal, we let $f_\alpha = \bigcup_{\beta<\alpha} f_\beta$.

Given $f_\alpha : A_\alpha \to \mathcal{N}$ partial elementary, let $\phi(x, \bar{a})$ isolate $\mathrm{tp}^{\mathcal{M}_0}(a_\alpha/A_\alpha)$. Because f_α is partial elementary, by lemma A.15, $\phi(x, f_\alpha(\bar{a}))$ isolates $f_\alpha\left(\mathrm{tp}^{\mathcal{M}_0}(a_\alpha/A_\alpha)\right)$ in $S_1^{\mathcal{N}}(f_\alpha(A))$. Also because f_α is partial elementary, there is $b \in N$ s.t. $\mathcal{N} \vDash \phi(b, f_\alpha(\bar{a}))$. Thus, $f_{\alpha+1} = f_\alpha \cup \{(a_\alpha, b)\}$ is elementary.

In particular, $f_\delta : \mathcal{M}_0 \to \mathcal{N}$ is elementary. Thus, $\mathcal{M}_0$ is a prime model extension of A.

We will show now that every $\bar{a}$ in A_α realizes an isolated type over A, for all $\alpha < \delta$. We argue by induction on α. For α a limit ordinal, this is clear. For successor ordinals, it follows from the following lemma.

Lemma 4.11. *Suppose that $A \subseteq B \subseteq \mathcal{M} \vDash T$ and every $\bar{b} \in B^m$ realizes an isolated type in $S_m^{\mathcal{M}}(A)$. Suppose that $\bar{a} \in M^n$ realizes an isolated type in $S_n^{\mathcal{M}}(B)$. Then, $\bar{a}$ realizes an isolated type in $S_n^{\mathcal{M}}(A)$.*

Proof. Let $\phi(\bar{x}, \bar{y})$ be an $\mathcal{L}$-formula and $\bar{b} \in B^m$ such that $\phi\left(\bar{x}, \bar{b}\right)$ isolates $\mathrm{tp}^{\mathcal{M}}(\bar{a}/B)$. Let $\theta(\bar{y})$ be an $\mathcal{L}_A$-formula isolating $\mathrm{tp}^{\mathcal{M}}\left(\bar{b}/A\right)$. We first claim that $\phi(\bar{x}, \bar{y}) \wedge \theta(\bar{y})$ isolates $\mathrm{tp}^{\mathcal{M}}\left(\bar{a}, \bar{b}/A\right)$.

Suppose that $\mathcal{M} \vDash \psi\left(\bar{a}, \bar{b}\right)$. Because $\phi\left(\bar{x}, \bar{b}\right)$ isolates $\mathrm{tp}^{\mathcal{M}}(\bar{a}/B)$,

$$\mathrm{Th}_A(\mathcal{M}) \vDash \phi\left(\bar{x}, \bar{b}\right) \to \psi\left(\bar{x}, \bar{b}\right).$$

Thus, because $\theta(\bar{y})$ isolates $\mathrm{tp}^{\mathcal{M}}\left(\bar{b}/A\right)$

$$\mathrm{Th}_A(\mathcal{M}) \vDash \theta(\bar{y}) \to (\phi(\bar{x}, \bar{y}) \to \psi(\bar{x}, \bar{y}))$$

and

$$\mathrm{Th}_A(\mathcal{M}) \vDash (\theta(\bar{y}) \wedge \phi(\bar{x}, \bar{y})) \to \psi(\bar{x}, \bar{y}).$$

Hence, $\phi(\bar{x}, \bar{y}) \wedge \theta(\bar{y})$ isolates $\mathrm{tp}^{\mathcal{M}}\left(\bar{a}, \bar{b}/A\right)$ as claimed.

Because $\mathrm{tp}^{\mathcal{M}}\left(\bar{a}, \bar{b}/A\right)$ is isolated, so is $\mathrm{tp}^{\mathcal{M}}(\bar{a}/A)$ by lemma A.16. $\square$

This concludes the proof of the theorem. $\square$

4.3 CONSTRUCTIBLE MODELS

We will now define constructible models over a set A which are somehow a generalization of atomic models.

Definition 4.12. Let $A \subseteq B \subseteq \mathcal{M}$. Then B is said to be *atomic over* A provided that for every $n < \omega$, every n-tuple of elements from B realizes a principal type over A, in $\mathcal{M}$.

Atomic model

Definition 4.13. Let $\mathcal{M}$ be an $\mathcal{L}$-structure and let $A \subseteq M$. Let δ be an ordinal and $\langle a_\alpha | \alpha < \delta \rangle$ be a sequence of elements from M. Let $A_\alpha := A \cup \{a_\beta | \beta < \alpha\}$. We call $\langle a_\alpha | \alpha < \delta \rangle$ a *construction over* A if $\mathrm{tp}(a_\alpha / A_\alpha)$ is isolated for all $\alpha < \delta$.

Construction

We say that $B \subseteq M$ is *constructible over* A if there is a construction $\langle a_\alpha | \alpha < \delta \rangle$ over A such that $B = A \cup \{a_\alpha | \alpha < \delta\}$. We say that a model $\mathcal{M}$ is *constructible over* A, if M is constructible over A.

Constructible set

Constructible model

Observe that if B is constructible over A then it is also atomic over A. Also if $\langle a_\alpha | \alpha < \delta \rangle$ is a construction over A, then it is also construction over A_α, for all $\alpha < \delta$.

Our goal is to show that constructible models are also prime over a set A, o-minimal theories have a constructible model and that constructible models are unique for any theory.

Observing the proof of theorem 4.10 and the definition of constructible models we have the following corollary.

Corollary 4.14.

1. *If T o-minimal theory, $\mathcal{M} \vDash T$, and $A \subseteq M$, there is $\mathcal{N} \prec \mathcal{M}$ constructible over A.*

2. *If $\mathcal{M}$ is constructible over A, then $\mathcal{M}$ is prime over A and every type realized in $\mathcal{M}$ is isolated over A.*

We will now prove Ressayre's theorem 4.17 which states that constructible extensions are unique for any theory.

Definition 4.15. Suppose that $\langle a_\alpha | \alpha < \delta \rangle$ is a construction of $\mathcal{M}$ over A. For each $\alpha < \delta$, let $\theta_\alpha(x)$ be a formula with parameters from A_α isolating $\mathrm{tp}(a_\alpha / A_\alpha)$. We say that $C \subseteq M$ is *sufficient* if whenever $a_\alpha \in C$, then all parameters from θ_α are in C.

Sufficient set

Lemma 4.16. *Suppose that $\langle a_\alpha | \alpha < \delta \rangle$ is a construction of $\mathcal{M}$ over A.*

1. *Each A_α is sufficient.*

2. *If C_i is sufficient for all $i \in I$, so is $\bigcup_{i \in I} C_i$.*

3. *If $X \subseteq M$ is finite, then there is a finite sufficient $C \supseteq X$.*

4. If $C \subseteq M$ is sufficient, then M is constructible over $A \cup C$.

Proof. (1) and (2) are derived from the definitions.

We prove (3) by induction on α that if $X \subseteq A_\alpha$ is finite, then there is a finite sufficient $C \supseteq X$. That is clear for $\alpha = 0$ and α limit ordinal. Suppose that $\alpha = \mathrm{succ}(\beta)$ and $X = X_0 \cup \{a_\beta\}$ where $X_0 \subseteq A_\alpha$. Let $B \subseteq A_\alpha$ be the parameters of θ_β. By induction hypothesis there is a finite sufficient $C \supseteq X_0 \cup B$. Then, $C \cup \{a_\beta\}$ is a finite sufficient set containing X.

To prove (4) we must show that, $\mathrm{tp}(a_\alpha / A_\alpha \cup C)$ is isolated for all $\alpha < \delta$. If $a_\alpha \in C$ this is trivial. So let's assume that $a_\alpha \notin C$. We claim that θ_α isolates $\mathrm{tp}(a_\alpha / A_\alpha \cup C)$. Suppose not, then there is an $\mathcal{L}_A$-formula $\psi(x, \bar{y})$ and $\bar{b} \in C$ such that $\psi\left(a_\alpha, \bar{b}\right)$, but

$$\mathrm{Th}_{A_\alpha \cup C}(\mathcal{M}) \nvDash \theta_\alpha(x) \rightarrow \psi\left(x, \bar{b}\right).$$

Thus, θ_a does not isolate $\mathrm{tp}\left(a_\alpha / A_\alpha, \bar{b}\right)$. By (2) and (3) we may, without loss of generality, assume that A_α,$\bar{b}$ is sufficient. We may assume that $\bar{b} = (a_{\alpha_1}, \ldots, a_{\alpha_n})$, where $\alpha < \alpha_1 < \cdots < \alpha_n$. Note that $A_\alpha \cup \{a_{\alpha_1}, \ldots, a_{\alpha_m}\}$ is sufficient for $m = 1, \ldots, n$.

Claim . For each $i = 1, \ldots, m$, there is an $\mathcal{L}_{A_\alpha}$-formula isolating $\mathrm{tp}\left(a_{\alpha_1}, \ldots, a_{\alpha_m} / A_{\mathrm{succ}(\alpha)}\right)$.

We prove this claim by induction on m. Suppose that the claim holds from $l < m$. The formula $\theta_{a_{l+1}}$ isolates $\mathrm{tp}(a_\alpha / A_{\alpha_{l+1}})$. All the parameters occurring in $\theta_{\alpha_{l+1}}$ are in $A_\alpha \cup \{a_{\alpha_1}, \ldots, a_{\alpha_l}\}$. Thus, $\theta_{\alpha_{l+1}}$ isolates $\mathrm{tp}\left(a_{\alpha_{l+1}} / A_{\mathrm{succ}(\alpha)} \cup \{a_{\alpha_1}, \ldots, a_{\alpha_l}\}\right)$.

Let $\theta_{\alpha_{l+1}}$ be $\psi(a_{\alpha_1}, \ldots, a_{\alpha_l}, x_{l+1})$, where $\psi(x_1, \ldots, x_{l+1})$ is an $\mathcal{L}_{A_\alpha}$-formula. By induction hypothesis, there is an $\mathcal{L}_{A_\alpha}$-formula $\phi(x_1, \ldots, x_l)$ isolating $\mathrm{tp}\left(a_{\alpha_1}, \ldots, a_{\alpha_l} / A_{\mathrm{succ}(\alpha)}\right)$. As in the proof of lemma 4.11, $\mathrm{tp}\left(a_{\alpha_1}, \ldots, a_{\alpha_{l+1}} / A_{\mathrm{succ}(\alpha)}\right)$ is isolated by $\phi(x_1, \ldots, x_l) \wedge \psi(x_1, \ldots, x_{l+1})$. This proves the claim.

Let $\psi(\bar{y})$ be an $\mathcal{L}_{A_\alpha}$-formula isolating $\mathrm{tp}\left(\bar{b} / A_{\mathrm{succ}(\alpha)}\right)$. Then, $\theta_\alpha(x) \wedge \psi(\bar{y})$ isolates $\mathrm{tp}\left(a_\alpha, \bar{b} / A_\alpha\right)$. Because a_α does not occur as a parameter in ψ, $\theta_\alpha(x)$ isolates $\mathrm{tp}\left(a_\alpha / A_\alpha, \bar{b}\right)$, as desired. □

Theorem 4.17. [Ressayre, unpublished] (Uniqueness of Constructible Models) *Suppose that $A \subseteq \mathbb{M}$, $\mathcal{M} \prec \mathbb{M}$, $\mathcal{N} \prec \mathbb{M}$, and $\mathcal{M}$ and $\mathcal{N}$ are constructible over A. The identity map on A extends to an isomorphism between $\mathcal{M}$ and $\mathcal{N}$.*

Proof. Let $\langle a_\alpha | \alpha < \delta \rangle$ and $\langle b_\alpha | \alpha < \gamma \rangle$ be the constructions of $\mathcal{M}$ and $\mathcal{N}$ over A respectively (and $|M| = \kappa$). We define the following collection of partial elementary functions,

$$\begin{aligned} I \;=\; & \{f : X \rightarrow N | f \text{ is partial elementary, } A \subset X,\ X \text{ is sufficient} \\ & \text{in } \mathcal{M},\ f|A \text{ is the identity, and } \mathrm{img}(X) \text{ is sufficient in } \mathcal{M}\}. \end{aligned}$$

The identity map is in I and by lemma 4.16 (2), the union of a chain of elements of I is also an element of I. Therefore, we can apply Zorn's Lemma to get a maximal element of I. Let's call this maximal element $f_{\sup} : X \to N$. We claim that $f_{\sup}$ is an isomorphism between $\mathcal{M}$ and $\mathcal{N}$ as the one we are seeking. It suffices to show that $\mathrm{dom}(f_{\sup}) = M$ and $\mathrm{img}(f_{\sup}) = N$.

Let us show first that $\mathrm{dom}(f_{\sup}) = M$. Suppose that it is not true and let $a \in M \backslash X$. Let C_0 be a finite sufficient subset of M, with $a \in C_0$. By lemma 4.16 (4) we have that M will be constructible over $A \cup C_0$. Then by corollary 4.14 (2) we have that $\mathrm{tp}(C_0/X)$ is isolated. Thus, we extend $f_{\sup}$ to a partial elementary $f_0 : X \cup C_0 \to N$. Let $D_0 \supseteq f_0(C_0)$ be a finite sufficient subset of N. By a similar argument the type $\mathrm{tp}(D_0/f(X))$ is isolated, so by lemma A.17 $\mathrm{tp}(D_0/f(X \cup C_0))$ is isolated too. Hence we can find a finite $C_0' \supseteq C_0$ and a surjective partial elementary function $f_0' : X \cup C_0' \to f(X) \cup D_0$. Let C_1 be a finite sufficient containing C_0'. By the same argument $\mathrm{tp}(C_1/X \cup C_0')$ is isolated and we may extend f_0' to a partial elementary $f_1 : C_1 \to N$.

In this manner we construct $C_0 \subseteq C_1 \subseteq C_2 \subseteq \cdots$, a sequence of finite sufficient subsets of M, and $f_{\sup} \subseteq f_0 \subseteq f_1 \subseteq f_2 \subseteq \cdots$, where $f_i : X \cup C_i \to N$ is partial elementary and $f_i(C_i)$ is contained in a sufficient subset of $f_{i+1}(C_{i+1})$. If $g := \bigcup_i f_i$, then $g \in I$, which contradicts the maximality of $f_{\sup}$.

A symmetric argument shows that $\mathrm{img}(f_{\sup}) = N$. Thus $f_{\sup}$ is the desired isomorphism. $\square$

4.4 UNIQUENESS OF PRIME MODELS

Now that we know by corollary 4.14 (2) that constructible models are prime, and by Ressayre's theorem 4.17 that constructible models are unique up to isomorphism we just need to establish that prime models are also constructible for o-minimal theories, and we will have the desired result about the uniqueness of prime models.

We will start to prove a series of lemmas in order to assist us with this proof.

Lemma 4.18. *Let T be an o-minimal theory, $A \subseteq \mathcal{M} \models T$ and $a, b \in cl(A)$.*

1. *Suppose that the formula $\phi(x) = (a < x) \wedge (x < b)$ isolates a complete non-algebraic type over A. Then $I = (a, b)^{\mathcal{M}} := \{c \in \mathcal{M} | a < c < b\}$ is either a discrete linear order without endpoints or a dense linear order without endpoints.*

2. *Suppose that I is as above, and I' is any other open interval in $\mathcal{M}$ all of whose elements satisfy that same non-algebraic principal type over A. If for some $b \in I$, $cl_A(\{b\}) \cap I' \neq \emptyset$, then there exists a monotone bijection $g : I \to I'$ that is definable with parameters*

from A. Moreover, any $f : I \to I$ that is definable over A must be a monotone increasing bijection.

3. *Suppose that I and $f : I \to I'$ are as in (2), and $X \subseteq I$ is atomic and algebraically independent over A. Then $f(X)$ is an atomic and algebraically independent subset of I'. Furthermore if A is maximal so is $f(X)$.*

4. *Let I and I' be as above, and $\{b_1, \ldots, b_{n+1}\}$ be an atomic and algebraically independent subset of I. Suppose also that $cl_A(\{b_1, \ldots, b_{n+1}\}) \cap I' = \emptyset$ but $cl_A(\{b_1, \ldots, b_{n+1}\}) \cap I \neq \emptyset$. Then there is a $J \subseteq I$ such that $b_{n+1} \in J$ and every element of J satisfies the same principal type of $A \cup \{b_1 \ldots, b_n\}$, and a monotone bijection $g : J \to I'$ that is definable from parameters in $A \cup \{b_1, \ldots, b_n\}$.*

Proof. Let's start with (1). Because $a, b \in \mathrm{cl}(A)$, a, b will be definable over A, so ϕ can be viewed as a formula over A. Suppose that I is not discrete. Then without loss of generality there will be a $c \in I$ with no immediate successor. Then c cannot have any immediate predecessor either. Indeed, if d is an immediate predecessor of c, then d must satisfy the same type over A with c which means that d cannot have an immediate successor which is a contradiction. Since every element in I satisfies the same type over A as c, we conclude that no element in I has an immediate successor or predecessor, and consequently I must be dense.

For (2). Suppose that $\psi(x, b)$ is an algebraic formula with parameters from A such that for some $c \in I'$, $\mathcal{M} \models \psi(c, b)$. Since $\mathrm{cl}_A(\{b\}) = \mathrm{dcl}_A(\{b\})$ (because of o-minimality), we may also assume that

$$\mathcal{M} \models (\exists! x)\, \psi(x, b) \wedge \text{“}x \in I'\text{”}$$

Since the formula $(\exists! x)\, \psi(x, y) \wedge \text{“}x \in I'\text{”}$ has parameters from A, it follows that for any $b' \in I$,

$$\mathcal{M} \models (\exists! x)\, \psi(x, b') \wedge \text{“}x \in I'\text{”}.$$

Let $g : I \to I'$ be the function defined with parameters from A by $f(b) = c$ iff $\mathcal{M} \models \psi(c, b)$. We will show that g must be a monotone bijection. By monotonicity theorem 1.12, g must be piecewise monotone or constant. However g cannot change its behavior because a boundary point of this change would be an interior point of I that is definable over A. Thus, g must be monotone or constant. Suppose that g is constant, then the image of g would be a point in I' definable over A, hence g has to be monotone. Lastly, since the image of g is definable over A, it is apparent that the image of g must be all I'. Therefore g is a monotone bijection.

Now to finish with (2) we have to show that any $f : I \to I$ definable over A, must be a monotonically increasing bijection. If f were decreasing then both $D^+ := \{b \in I | f(b) \geq b\}$ and $D^- := \{b \in I | f(b) < b\}$

would be nonempty subsets of I that are definable over A and so would yield a boundary point in I that also would be definable over A.

For (3) we will show first that $f(X)$ is atomic. Let $\bar{a}$ be any n-tuple of elements of $f(X)$. We will show that is realizes a principal type over A. Let $\bar{b}$ be the n-tuple of the pre images of the elements of $\bar{a}$. Every element of $\bar{b}$ is in X and because X is atomic there is a principal type $p(\bar{y})$ which is realized by $\bar{b}$. Let $\phi(\bar{y})$ be the formula that axiomatize $p(\bar{y})$. Now if f is increasing because of o-minimality $f\left(\bar{b}\right) = \bar{a}$ will realize $\phi(\bar{y})$ and thus $p(\bar{y})$ too. If f is decreasing $\bar{a}$ will realize $\neg\phi(\bar{y})$ and therefore will realize the principal type which is complement of $p(\bar{y})$. In any case we have that $f(X)$ is atomic.

Now we will show that $f(X)$ is algebraically independent. Let $y \in f(X)$ and let $z \in X$ s.t. $f(z) = y$. Suppose that $y \in \mathrm{acl}(f(X) \setminus \{y\}) = \mathrm{acl}(f(X) \setminus f(z))$, so there is a formula $\phi(x, \bar{a})$ s.t. $\mathcal{M} \vDash \phi(f(z), \bar{a})$ but then from the previous part of the proof there exists $\psi(x, \bar{a}')$ s.t. $\mathcal{M} \vDash \psi(z, \bar{a}')$ which means $z \in \mathrm{acl}(X \setminus \{z\})$ and that contradicts with our hypothesis that X is algebraically independent.

For (4). Since $\mathrm{cl}_A(\{b_1, \ldots, b_n\}) \cap I' = \emptyset$, we conclude that every element of I' satisfies the same non-algebraic principal type over $A \cup \{b_1, \ldots, b_n\}$. Moreover, since $\{b_1, \ldots, b_{n+1}\}$ is atomic and independent over A, it follows that there is some open $J \subseteq I$ so that $b_{n+1} \in J$ and every element of J satisfies the same non-algebraic principal type over $A \cup \{b_1 \ldots, b_n\}$. Applying (2) to J and I' then yields the desired conclusion. $\square$

Lemma 4.19. *Suppose that $A \subseteq \mathcal{N} \vDash T$, where T is o-minimal theory, $a, b \in \mathrm{cl}(A)$ and the formula*

$$\phi(x) = (a < x) \wedge (x < b)$$

isolates a non-algebraic complete type over A. Also assume that $X \subseteq I = (a, b)^{\mathcal{N}}$ and that $cl_A(X) \cap I$ is dense, or discrete if I is discrete. Then, if $\mathcal{M}$ is atomic over $cl_A(X)$, it follows that $(a, b)^{\mathcal{M}} \subseteq cl_A(X)$.

Proof. Let $c \in (a, b)^{\mathcal{M}}$. Then c satisfies an atomic type over $\mathrm{cl}_A(X)$. If this type is algebraic then clearly $c \in \mathrm{cl}_A(X)$. The assumption that the type that c satisfies over $\mathrm{cl}_A(X)$ is non-algebraic and thus of the form "$e_1 < x < e_2$" for some $e_1, e_2 \in \mathrm{cl}_A(X)$ leads to a contradiction. Indeed, our hypothesis, in the case that $(e_1, e_2)^{\mathcal{N}}$ is either dense or discrete (and infinite) implies that $(e_1, e_2)^{\mathcal{N}} \cap \mathrm{cl}_A(X) \neq \emptyset$. But then it can not be the case that "$e_1 < x < e_2$" isolates a complete type over $\mathrm{cl}_A(X)$, contrary to what we supposed. That completes the proof. $\square$

Lemma 4.20. *Let $A \subseteq \mathcal{N} \vDash T$, where T is an o-minimal theory, and $a, b \in cl(A)$ such that the formula "$a < x < b$" isolates a complete type over A. Then there exists $\mathcal{M} \vDash T$ s.t. $A \subseteq \mathcal{M}$ and*

$$\dim_A(\{c \in \mathcal{M} | a < c < b\}) \leq \aleph_0$$

.

Proof. By corollary 4.14 (1) there exists a constructible model over any subset of a model of T. Since by a previous observation in section 4.3, constructible models over a subset A are also atomic over A, applying lemma 4.19 to a model $\mathcal{M}$ that is constructible over $\mathrm{cl}(A)$ we shall be done if we find some $X \subseteq (a,b)^{\mathcal{M}}$ so that $\mathrm{cl}_A(X)$ is dense or discrete if $(a,b)^{\mathcal{N}}$ is and $\dim_A(X) \leq \aleph_0$.

If $(a,b)^{\mathcal{N}}$ is discrete, then simply let X be any singleton, and we are done. We thus may assume that $(a,b)^{\mathcal{N}}$ is dense (linear order without endpoints). We construct X as the union of an increasing chain of finite sets X_n, $n < \omega$. For notational convenience let $X_{-1} := \emptyset$.

At stage $n = 0$, choose some $x_0 \in (a,b)^{\mathcal{N}}$ and let $X_0 := \{x_0\}$. Also, let $\mathscr{I}_0$ enumerate in some fixed order all non-algebraic principal types over $\mathrm{cl}_A(X_0)$ with an endpoint in X_0 (i.e. formulas of the form "$c < x < x_0$" or "$x_0 < x < c$", where $c \in \mathrm{cl}_A(X_0)$).

Now suppose that we have the sets X_i, $i < n$, and $\mathscr{I}_i$, $i < n$. The construction of X_n breaks up in two cases.

Case 1. There is no member of $\bigcup_{i<n} \mathscr{I}_i$, that is still isolated over $\mathrm{cl}_A(\bigcup_{i<n} X_i)$. In this case, let $X_n := X_{n-1}$ and $\mathscr{I} = \langle\,\rangle$ (empty sequence).

Case 2. There is some member of $\bigcup_{i<n} \mathscr{I}_i$, that is still isolated over $\mathrm{cl}_A(\bigcup_{i<n} X_i)$. Let J be the least such member of $\mathscr{I}_0 \frown \mathscr{I}_1 \frown \cdots \frown \mathscr{I}_{n-1}$ (concatenation), and choose $x_n \in J$. Then let $X_n := X_{n-1} \cup \{x_n\}$, and let $\mathscr{I}_n$ be an enumeration of all non-algebraic principal types over $\mathrm{cl}_A(X_n)$ that have an endpoint in X_n.

Setting now $X := \bigcup_{n<\omega} X_n$ we assert that that $\mathrm{cl}_A(X) \cap (a,b)^{\mathcal{N}}$ is dense. Indeed, suppose that $c, d \in \mathrm{cl}_A(X) \cap (a,b)^{\mathcal{N}}$, $c < d$ and let n be such that $c, d \in \mathrm{cl}_A(X_n)$. Without loss of generality assume that $c \in \mathrm{cl}_A(X_n) \setminus \mathrm{cl}_A(X_{n-1})$, and so $X_n = X_{n-1} \cup \{x_n\}$. Let the formula "$e_1 < x < e_2$" isolate the complete type over $A \cup X_{n-1}$ that x_n realizes. There exists a function f with domain $(e_1, e_2)^{\mathcal{N}}$ that is definable with parameters in $A \cup X_{n-1}$ such that $f(x_n) = c$.

Since "$e_1 < x < e_2$" isolates a complete type over $A \cup X_{n-1}$, f must be monotone or constant on $(e_1, e_2)^{\mathcal{N}}$. But since $c \in \mathrm{cl}_A(X_n) \setminus \mathrm{cl}_A(X_{n-1})$, f must be monotone on $(e_1, e_2)^{\mathcal{N}}$. Also we may infer that the image $R = f\left((e_1, e_2)^{\mathcal{N}}\right)$ must be an interval in $\mathcal{N}$ again because every element of $(e_1, e_2)^{\mathcal{N}}$ satisfies the same type over $A \cup X_{n-1}$. Thus f is an order preserving or reversing bijection from $(e_1, e_2)^{\mathcal{N}}$ onto $R = (e_1', e_2')^{\mathcal{N}}$ and c is an interior point of $(e_1', e_2')^{\mathcal{N}}$. Without loss of generality we suppose that f is increasing.

If $e_2' < d$, then since we would have $c < e_2' < d$ and e_2' already definable over $A \cup X_{n-1}$, we would be done (X will be as desired).

We suppose that $d \leq e_2'$. Therefore

$$\mathrm{cl}_A(X) \cap (c,d)^{\mathcal{N}} \neq \emptyset \iff \begin{cases} \mathrm{cl}_A(X) \cap \left(x_n, f^{-1}(d)\right)^{\mathcal{N}} \neq \emptyset, & \text{if } d < e_2', \\ \mathrm{cl}_A(X) \cap (x_n, e_2)^{\mathcal{N}} \neq \emptyset, & \text{if } d = e_2'. \end{cases}$$

If there is no non-algebraic principal type over $\mathrm{cl}_A(X_n)$ with left endpoint x_n, then $\mathrm{cl}_A(X) \cap (c,d)^{\mathcal{N}} \neq \emptyset$. Suppose now that there exists such a type given by the interval I. It follows that,

$$I \subseteq \begin{cases} \left(x_n, f^{-1}(d)\right)^{\mathcal{N}}, & \text{if } d < e_2' \\ (x_n, e_2)^{\mathcal{N}}, & \text{if } d = e_2' \end{cases}$$

and by some stage in the construction of X, either I is no longer isolated, or a point chosen from I is put into X. In either event, $\mathrm{cl}_A(X) \cap I \neq \emptyset$ and the proof is complete. □

The following lemma is due to Harrington and its proof may be found in [Sacks 28].

Lemma 4.21. [Harrington] *Let T be a complete theory. For any subset A of a model of T, suppose also that every formula with parameters from A is implied by a complete formula with parameters from A. Then, if $B \subseteq C \subseteq D$ are all submodels of a model of T, D is atomic over B, and for any complete type over B, either all or none of its realizations in D are in C, it follows that D is atomic over C.*

Now the following proposition will give us the missing information to finally obtain 4.2, using with it of course corollary 4.14 and Ressayre's theorem 4.17.

Proposition 4.22. *Let $\mathcal{M}$ be a model of a o-minimal theory T, that is prime over a subset A of a model of T. Then $\mathcal{M}$ is constructible over A.*

Proof. Let $\mathcal{M} \models T$ be prime over A. Let $\langle I_\alpha | \alpha < \lambda \rangle$ enumerate all open intervals in $\mathcal{M}$ having the property that all elements in the intervals satisfy the same non-algebraic type over A. Observe that $\mathcal{M} = \mathrm{cl}(A) \cup \bigcup_{\alpha<\lambda} I_\alpha$. Now, any such I_α consists of the set of elements in $\mathcal{M}$ satisfying the formula "$a_\alpha < x < b_\alpha$", for some $a_\alpha, b_\alpha \in \mathrm{cl}(A)$. By lemma 4.20, there exists a model $\mathcal{N}_\alpha$ containing A in which $\dim_A(\{c \in N_\alpha | \mathcal{N}_\alpha \models a_\alpha < c < b_\alpha\})$ is countable. Since $\mathcal{M}$ is prime over A, we can elementarily embed $\mathcal{M}$ into $\mathcal{N}_\alpha$ over A, from which it follows that $\dim_A(I_\alpha)$ is countable. For each $\alpha < \lambda$, we now fix an enumeration $C_\alpha = \langle c_n^\alpha | n < \omega \rangle$ of a maximal algebraically independent and (since $\mathcal{M}$ is prime and hence atomic) atomic subset I_α. We now show that $\mathcal{M}$ is constructible.

We shall enumerate $\mathcal{M} \backslash A$ as the concatenation of a recursively defined sequence of sequences of elements of $\mathcal{M}$, $\langle D_\alpha | \alpha < \lambda \rangle$, which will be seen to be a construction. Moreover, we will show that, for each $\alpha < \lambda$

(4.21.1) $I_\alpha \subseteq \bigcup_{\beta \leq \alpha} D_\beta$

(4.21.2) $\bigcup_{\beta \leq \alpha} D_\beta \cup A$ is algebraically, and hence, definably closed, and

(4.21.3) for each $\gamma < \lambda$, if $I_\gamma \nsubseteq \bigcup_{\beta \leq \alpha} D_\beta$, then $I_\gamma \cap \bigcup_{\beta \leq \alpha} D_\beta = \emptyset$.

We now build $\langle D_\alpha | \alpha < \lambda \rangle$. First we enumerate D_0. Let $\mathrm{cl}(A) \setminus A = \mathrm{dcl}(A) \setminus A$ be enumerated as $F_0 \langle f_\gamma | \gamma < \beta_0 \rangle$. It is obvious that F_0 constitutes a construction over A. Notice that $F_0 \cap I_\alpha = \emptyset$ for all $\alpha < \lambda$ and thus that each element of any given I_α satisfies the same principal type over $A \cup F_0$. Since $\mathcal{M}$ is atomic over A, clearly $\mathcal{M}$ is atomic over $A \cup F_0$. Therefore since the order type of $C_0 = \langle c^0_n | n < \omega \rangle$ is ω, it is a matter of routine to verify that C_0 is a construction over $A \cup F_0$. Next, let $E_0 = \langle e^0_\nu | \nu < \delta_0 \rangle$ enumerate $\mathrm{cl}_A(F_0 \cup C_0) \setminus A \cup F_0 \cup C_0$. Again, obviously E_0 is a construction over $A \cup F_0 \cup C_0$. Finally, let $D_0 = F_0 \frown C_0 \frown E_0$. Observe first that by maximality of C_0, $I_0 \subseteq D_0$. Furthermore, $D_0 \cup A$ is algebraically closed. Lastly, by lemma 4.18 ((3) and (4)) for any $\gamma < \lambda$, if $I_\gamma \nsubseteq D_0$, then $I_\gamma \cap D_0 = \emptyset$. Hence, (4.21.1), (4.21.2) and (4.21.3) hold for $\alpha = 0$.

Next suppose that (4.21.1)-(4.21.3) hold for all $\beta < \alpha$. We build D_α so these will continue to be satisfied. Observe that $\bigcup_{\beta < \alpha} D_\beta \cup A$ is algebraically closed also for each $\gamma < \lambda$, if $I_\gamma \nsubseteq \bigcup_{\beta < \alpha} D_\beta$, then $I_\gamma \cap \bigcup_{\beta < \alpha} D_\beta = \emptyset$. For any $\gamma < \lambda$, it follows that either $I_\gamma \subseteq \bigcup_{\beta < \alpha} D_\beta$ or $I_\gamma \cap \bigcup_{\beta < \alpha} D_\beta = \emptyset$, and in the latter event that each member of I_γ satisfies the same non-algebraic principal type over $A \cup \bigcup_{\beta < \alpha} D_\beta$(recall that endpoints of I_γ are in $\mathrm{cl}(A)$). This fact, together with the lemma 4.9, suffices to show that $A \subseteq A \cup \bigcup_{\beta < \alpha} D_\beta \subseteq \mathcal{M}$ satisfy the hypothesis of lemma 4.21. Hence $\mathcal{M}$ is atomic over $A \cup \bigcup_{\beta < \alpha} D_\beta$. If $I_\alpha \subseteq \bigcup_{\beta < \alpha} D_\beta$, then let $D_\alpha = \langle \, \rangle$, in which case (4.21.1)-(4.21.3) trivially continue to hold. Thus assume that $I_\alpha \cap \bigcup_{\beta < \alpha} D_\beta = \emptyset$. Since $C_\alpha = \langle c^\alpha_n | n < \omega \rangle$ has order type ω, it is of course a construction over $A \cup \bigcup_{\beta < \alpha} D_\beta$. Let now $E_\alpha = \langle e^\alpha_\nu | \nu < \delta_\alpha \rangle$ enumerate $\mathrm{cl}_A \left(\bigcup_{\beta < \alpha} D_\beta \cup C_\alpha \right)$. Again, E_α constitutes a construction over $A \cup \bigcup_{\beta < \alpha} D_\beta$. Finally let $D_\alpha := C_\alpha \frown E_\alpha$. Since C_α is a maximal independent set subset of I_α over A, we have that $I_\alpha \subseteq \bigcup_{\beta < \alpha} D_\beta$ so that (4.21.1) is satisfied. (4.21.2) is satisfied too. Lastly, (4.21.3) holds because of lemma 4.18 (3) and (4). This completes the definition of $\langle D_\alpha | \alpha < \lambda \rangle$, which is indeed a construction of $\mathcal{M}$ over A and that concludes the proof. □

5 DEFINABLE TYPES

5.1 DEFINITIONS

Let us start this chapter with some definitions that we will use. Let $\mathcal{L}$ be a first order language and $\mathcal{M}$ a structure on $\mathcal{L}$. We denote by $\mathcal{L}_M$ the expansion of the language $\mathcal{L}$ with constants for each element of M.

Definition 5.1. A type $p(\bar{x}) \in S_n$ is *definable* if for any $\mathcal{L}$-formula $\theta(\bar{x}, \bar{w})$, there is an $\mathcal{L}_M$-formula $d\theta(\bar{w})$ such that for all $\bar{a} \in M$, $\theta(\bar{x}, \bar{a}) \in p(\bar{x})$ iff $\mathcal{M} \vDash d\theta(\bar{a})$. The formula $d\theta$ is called *definition* of p.

Definable type

Next we define a special kind of structure extension.

Definition 5.2. If $\mathcal{M} \prec \mathcal{N}$, we say that $\mathcal{N}$ is a conservative extension of $\mathcal{M}$ if for any n and any $\mathcal{L}_N$-definable $S \subseteq N^n$, $S \cap M^n$ is $\mathcal{L}_M$-definable in $\mathcal{M}$.

Conservative extension

The following characterization of conservative extensions is easy to prove.

Proposition 5.3. *Let $\mathcal{M} \prec \mathcal{N}$. Then $\mathcal{N}$ is a conservative extension of $\mathcal{M}$ iff every $\mathcal{M}$-type realized in $\mathcal{N}$ is definable.*

Definable types play a central role in stability theory and have proven useful in the study of models of arithmetic.

Our goal here is to provide a characterization of definable types over o-minimal structures, as it was done by Marker and Steinhorn [18].

Van den Dries in [32] studied definable types over real closed fields and proved the following,

Theorem 5.4. [van den Dries 32]

1. *Every type over $(\mathbb{R}, +, \cdot, 0, 1)$ is definable.*
2. *Let F and K be real closed fields and $F \subset K$. Then the following are equivalent,*
   a) *every element of K that is bounded in absolute value by an element of F is infinitely close (in the sense of F) to an element of F.*
   b) *K is a conservative extension of F.*

In this section we will generalize that result by proving that every type over an o-minimal expansion of $\mathbb{R}$ is definable.

Let us continue with some more definitions. From now on again, $\mathcal{L}$ contains a binary symbol $<$, T is a complete o-minimal theory extending the theory of dense linear order and $\mathcal{M}$, $\mathcal{N}$ are models of T.

Definition 5.5. Let $\mathcal{M} \vDash T$. Let $p(x) \in S_1(M)$. We say that $p(x)$ is a *cut* over $\mathcal{M}$

Cut

iff for any $a \in M$, if "$a < x$" $\in p(x)$, then there is a $b \in M$ s.t. "$b < x$" $\in p(x)$ and $a < b$, and similarly if "$x < a$" $\in p(x)$ then there is a $c < a$, $c \in M$ s.t. "$x < c$" $\in p(x)$,

iff there are nonempty disjoint subsets C_0 and C_1 of M, s.t. $C_0 \cup C_1 = M$, C_0 has no greatest element, C_1 has no least element, for all $c \in C_0$, "$c > x$" $\in p(x)$, and for all $c \in C_1$, "$x < c$" $\in p(x)$.

Noncut

If $p(x)$ is not isolated and is not a cut, we call $p(x)$ a *noncut*.

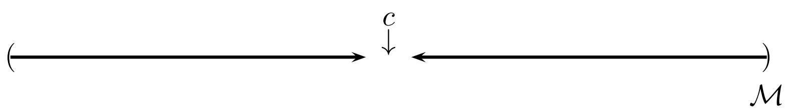

Figure 5: A cut.

Example 5.6. Let $\tilde{\mathbb{Q}}$ be the field of real algebraic numbers. We know that $\pi \in \mathbb{R} \backslash \tilde{\mathbb{Q}}$ and we have that $\operatorname{tp}\left(\pi, \tilde{\mathbb{Q}}\right)$ is a cut, and that is also true for every $r \in \mathbb{R} \backslash \tilde{\mathbb{Q}}$. On the other hand, if t is an infinite hyperreal or t is infinitesimally close to an element of $\tilde{\mathbb{Q}}$, then $\operatorname{tp}\left(t, \tilde{\mathbb{Q}}\right)$ is a noncut.

Let us also assume that all the types are non-principal and hence they are not realized in the original model $\mathcal{M}$. For this fact, we have that there are only the four following kinds of noncuts,

1. For some $q \in \mathcal{M}$, $\{m < x < q | m \in \mathcal{M}, m < q\}$
2. For some $q \in \mathcal{M}$, $\{q < x < m | m \in \mathcal{M}, q < m\}$
3. $\{x < m | m \in \mathcal{M}\}$
4. $\{m < x | m \in \mathcal{M}\}$

From now on we will refer to the noncuts as of form 1-4.

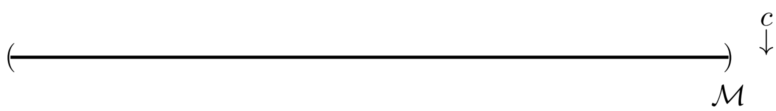

Figure 6: A noncut of form 4.

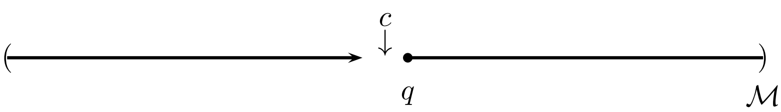

Figure 7: A noncut of form 1.

We can give a characterization of o-minimal structures using the notion of cut.

Proposition 5.7. *Let $\mathcal{M}$ be a linearly ordered structure. If $\mathcal{M}$ is o-minimal then, for any cut C in $\mathcal{M}$, there is a unique complete one-type with parameters from $\mathcal{M}$ which extends C.*

Proof. Let $\phi(x, \bar{m})$ be a formula with parameters from $\mathcal{M}$. Then $\phi(x, \bar{m})$ partitions $\mathcal{M}$ into finitely many intervals and point-sets such that for any I of them either $\phi(x, \bar{m})$ holds, or $\neg\phi(x, \bar{m})$ holds for every $x \in I$. Since only one interval can be consistent with a cut, then the value of $\phi(x, \bar{m})$ in any type extending C is determined. □

Moreover, the inverse of the last proposition is also true, but we will omit its proof. This gives us the following characterization of o-minimal structures.

Theorem 5.8. [Pillay and Steinhorn 26] *Let $\mathcal{M}$ be a linearly ordered structure. Then $\mathcal{M}$ is o-minimal if and only if, for any cut C in $\mathcal{M}$, there is a unique complete one-type with parameters from $\mathcal{M}$ which extends C.*

5.2 OMITTING TYPES

We will prove here two results about the realization of cuts and noncuts. Specifically we will see that realizing a noncut does not force us to realize a cut and vice versa. The results in this section are due to Marker [16].

Let us add some notation for convenience reasons. Let $\mathcal{M}$ be a model of an o-minimal theory. If $p(x) \in S_n(M)$ and $\bar{a}$ realizes p, then by theorem 4.2 we know that there exists a unique (up to isomorphism) prime model over $M \cup \{\bar{a}\}$. We let $P(\mathcal{M} \cup \{\bar{a}\})$ denote that model.

Proposition 5.9. *Let $\mathcal{M} \vDash T$. Let $\sigma(x) \in S_1(M)$ be a cut. Let $\tau(x) \in S_1(M)$ be a noncut. Let also b be a realization of $\tau(x)$. Then $\sigma(x)$ is omitted in $P(\mathcal{M} \cup \{b\})$.*

Proof. Let $\mathcal{M}'$ be $P(\mathcal{M} \cup \{b\})$, and towards a contradiction suppose that a $c \in \mathcal{M}'$ realizes $\sigma(x)$. $\mathcal{M}'$ as prime model is also atomic, thus as c is an element of $\mathcal{M}'$ then the type $\mathrm{tp}(c)$ will be isolated by some formula and let us assume that this formula is the $\beta_0 < x < \beta_1$, where β_0, β_1 are in $\mathrm{cl}(\mathcal{M} \cup \{b\})$. Then these β_0, β_1 are contained in the cut σ, i.e. both of them realize the type $\sigma(x)$. Therefore we may assume that $c \in \mathrm{cl}(\mathcal{M} \cup \{b\})$. That said, let f be a definable function s.t. $f(b) = c$.

Suppose now that the noncut τ is of the form 1, i.e.

$$\tau(x) = \{m < x < q | m \in \mathcal{M},\, m < q\}$$

for some $q \in \mathcal{M}$. The other cases are treated similarly.

Using monotonicity theorem 1.12 we can find an $a \in \mathcal{M}$ such that f in monotone and continuous on (a, q). As $\mathcal{M} \prec \mathcal{M}'$, this remains true in $\mathcal{M}'$. Without loss of generality, let us assume that f is increasing on (a, q). Consider

$$Y = \{y \in \mathcal{M} | \text{ there is an } x \in \mathcal{M} \text{ s.t. } a < x < q \text{ and } f(x) = y\}$$

the image of f on the set (a, q). To reach the desired contradiction we examine two possible cases.

Case 1. Y is unbounded in $\mathcal{M}$. Then there is some $d \in \mathcal{M}$ such that $a < d < q$ and $f(d) > c$. Since $\mathcal{M}$ and $\mathcal{M}'$ both satisfy $\forall x > d\,(x < q \to f(x) > f(d))$, then $\mathcal{M}' \vDash f(b) > f(d) > c$ which is a contradiction.

Case 2. Y is bounded in $\mathcal{M}$. Then by the monotonicity and continuity of f, Y is an interval in $\mathcal{M}$, let's say (β, δ), where β is possibly $-\infty$. Clearly the cut σ is contained in the interval (β, δ). Since σ is a cut, there is an $e \in \mathcal{M}$ such that $e < \delta$ and $x < e \in \sigma(x)$. Therefore there exists an $d \in \mathcal{M} \cap (a, q)$ such that $f(d) = e$. By monotonicity of f, $\mathcal{M}, \mathcal{M}' \vDash \forall x > \delta\,(x < q \to f(x) > e)$. Thus $f(b) = c$ cannot realize $\sigma(x)$ which is a contradiction.

□

Proposition 5.10. *If $\sigma(x), \tau(x) \in S_1(M)$, $\sigma(x)$ is a cut, $\tau(x)$ is a noncut and b is a realization of $\sigma(x)$, then $\tau(x)$ is omitted in $P(\mathcal{M} \cup \{b\})$.*

Proof. Let $\mathcal{M}'$ be $P(\mathcal{M} \cup \{b\})$, and towards a contradiction suppose that a $c \in \mathcal{M}'$ realizes $\tau(x)$. Using the same argument as before, let f be a definable function such that $f(b) = c$. We consider cases regarding the form of the noncut. We will check again only the case that $\tau(x)$ is of the form 1.

Let $g : \mathcal{M} \to \mathcal{M}$, be the following function

$$g(x) = \begin{cases} f(x), & \text{if } f(x) < q \\ \text{undefined}, & \text{otherwise} \end{cases}$$

By monotonicity there are $a, c \in \mathcal{M}$ such that $b \in (a, c)$ and f is monotonic and continuous on (a, c). Without loss of generality we also assume that f is also increasing on (a, c). Let $d \in \mathcal{M}$ such that $b < d < c$. By monotonicity we have, $\mathcal{M}, \mathcal{M}' \vDash \forall x\,(a < x < d \to f(x) < f(d))$. But then $c = f(b) < f(d) < q$. So c does not realize τ, which is a contradiction. □

5.3 DEDEKIND COMPLETE STRUCTURES

We can now define Dedekind completeness (definably complete, see definition 1.4) using the cut definition.

Definition 5.11. If $\mathcal{M} \subset \mathcal{N}$, we say that $\mathcal{M}$ is *Dedekind complete in* $\mathcal{N}$ if no cut in $S_1(M)$ is realized in $\mathcal{N}$. *Dedekind complete*

We say that $\mathcal{M}$ is *Dedekind complete* if it is Dedekind complete in every elementary extension.

Now we state the main result of this section, and we will omit its proof.

Theorem 5.12. [Marker and Steinhorn 18] *Let $p(\bar{x}) \in S_n(M)$. Then $p(\bar{x})$ is definable iff for any $\bar{a}$ realizing $p(\bar{x})$ (in some elementary extension of $\mathcal{M}$), it is the case that $\mathcal{M}$ is Dedekind complete in $\mathcal{M}(\bar{a})$.*

Easily from this we get the following corollary.

Corollary 5.13. *If $\mathcal{M}$ is Dedekind complete, then every $p(\bar{x}) \in S_n(M)$ is definable.*

Using theorem 5.12 we can obtain a result which demonstrates a similarity in the behavior of definable types in an o-minimal theory and types in a stable theory. First recall the following definition.

Definition 5.14. Let $M \subset A$, $p \in S_n(M)$, $q \in S_n(A)$ and $p \subset q$. The type q is a *coheir* of p if and only if for all $\phi(\bar{x}, \bar{a}) \in q$, there is an $\bar{m} \in M^n$ so that $\phi(\bar{m}, \bar{a})$ is true in the monster model. *Coheir*

In a stable theory every type over M is definable and has a unique coheir over $A \supset M$. The following is due to Pillay and a consequence of 5.12.

Theorem 5.15. *Let $\mathcal{M}$ be a densely ordered o-minimal structure and let $p(\bar{x}) \in S_n(M)$. Then p is definable if and only if p has a unique coheir over any $\mathcal{N} \succ \mathcal{M}$.*

5.4 STABLE THEORIES AND INDEPENDENCE PROPERTY

We will talk now about the independence property. Let T be a complete theory. Let $\phi(x_1, \dots, x_n, y_1, \dots, y_n)$ be a formula. An *n-ladder* for ϕ is a sequence $\left(\bar{a}_0, \dots, \bar{a}_{n-1}, \bar{b}_0, \dots, \bar{b}_{n-1}\right)$ of tuples in some model $\mathcal{M}$ of T, such that

$$\forall i, j < n,\ \mathcal{M} \models \phi\left(\bar{a}_i, \bar{b}_j\right) \iff i \leq j.$$

We say that ϕ is a *stable formula* (for T, or for $\mathcal{M}$) if there is some $n < \omega$, such that no n-ladder for ϕ exists. Otherwise we call it *unstable formula*. A theory T is called *unstable* if there is an unstable formula for T. T is called stable if it is not unstable. *Unstable theory*

We also say that a structure $\mathcal{M}$ is stable or unstable if its theory $\mathrm{Th}(\mathcal{M})$ is stable or unstable.

Independence property

Definition 5.16. Let T be a complete L-theory. A formula $\phi(\bar{x}, \bar{y})$ is said to have the *independence property* if in every model $\mathcal{M}$ of T, there is for each $n < \omega$ a family of tuples $\bar{b}_0, \ldots, \bar{b}_{n-1}$ such that for every subset X of n there is a tuple $\bar{a}$ in $\mathcal{M}$ for which $\mathcal{M} \vDash \phi\left(\bar{a}, \bar{b}_i\right) \iff i \in X$. T has the independence property if some formula has the independence property for T.

It is easy to see that every complete theory with the independence property is unstable. The independence property is a rough measure of instability because if a theory has the independence property it shares fewer properties with stable theories than a theory that lacks it.

We will show here that o-minimal theories does not have the independence property. We state here the following lemma which is due to B. Poizat.

Lemma 5.17. *A theory T has the independence property if and only if for any $\kappa \geq \omega$, there is an $\mathcal{M} \vDash T$ of power κ, $\mathcal{N} \succ \mathcal{M}$ and type $p(x)$ over $\mathcal{M}$ such that,*

$$|\{q | q \text{ is a type over } \mathcal{N} \text{ that is coheir of } p\}| \geq 2^{2^{\kappa}}.$$

Let us prove the following theorem.

Theorem 5.18. *Let T be an o-minimal theory, $\mathcal{M} \vDash T$ and $p(x)$ over $\mathcal{M}$. Then for any $\mathcal{N} \succ \mathcal{M}$, p has at most two coheirs over $\mathcal{N}$.*

Proof. Let $p(x)$ be a complete type over $\mathcal{M}$ and let $\mathcal{N} \succ \mathcal{M}$. Let $C(x)$ be the cut over $\mathcal{M}$ determined by $p(x)$. There will be only one such cut. Indeed, by theorem 5.8 we have that there will be at least one such cut, and because of the completeness of $p(x)$ we have that it must be unique. Let $A \subseteq \mathcal{N}$ be the set of elements that satisfy $C(x)$. If a cut $C'(x)$ is to be included in a coheir of $p(x)$ then, by the definition of coheir, $C'(x)$ must contain either $C^+ = \{a < x | a \in A\}$ or $C^- = \{a > x | a \in A\}$ and furthermore, that $C'(x)$ is completely determined by $C(x) \cup C^+(x)$ or $C(x) \cup C^-(x)$. Hence, there are at most two cuts that are included in any coheir of $p(x)$ over $\mathcal{N}$. Therefore by theorem 5.8, $p(x)$ has at most two coheirs over $\mathcal{N}$. $\square$

Now using the lemma 5.17 we get the following corollary which is the result we were looking for in this section.

Corollary 5.19. *No o-minimal theory T has the independence property.*

6 DEFINABLE GROUPS AND FIELDS

6.1 ORDERED GROUPS AND FIELDS

The goal of this chapter is to present a Trichotomy theorem and some results on fields and groups in o-minimal structures. The treatment follows somehow §3 of [Macpherson 13].

Let us show some basic results on ordered groups and fields as they are presented in Pillay and Steinhorn [26].

Theorem 6.1. *Let $\mathcal{G} = (G, +, 0, <)$ be an o-minimal ordered group. Then $\mathcal{G}$ is a divisible ordered abelian group.*

Proof. First we claim that $\mathcal{G}$ has no proper non-trivial definable subgroups, i.e. the only definable (parametrically) subgroups of $\mathcal{G}$ are $\{0\}$ and $\mathcal{G}$.

Lemma 6.2. *Let $\mathcal{G} = (G, +, 0, <)$ be an o-minimal ordered group. Then, the only parametrically definable subgroups of $\mathcal{G}$ are, $\{0\}$ and $\mathcal{G}$.*

Proof (of Lemma 6.2). Suppose that $\mathcal{H}$ is a proper non-trivial, parametrically definable subgroup, and let it be given by

$$\{h \in G | \mathcal{G} \vDash \phi(h, g_1, \dots, g_n)\}.$$

$\mathcal{H}$ was assumed to be non-trivial, thus $\mathcal{H}$ is infinite since the infinite set $\{nh | n \in \mathbb{N}\}$ is a subset of H, for any $h \in H$, $h \neq 0$. By o-minimality, $\mathcal{H}$ contains a non-trivial interval J. It follows that $\mathcal{H}$ contains a largest non-trivial interval I about 0, which without loss of generality we may assume that is symmetric around 0. Since $\mathcal{H}$ is proper ($\mathcal{H} \subsetneq \mathcal{G}$), the o-minimality of $\mathcal{G}$ implies that I must be of the form $(-h, h)$ or $[-h, h]$ for some $h \in G$. We will show that neither case can occur and that way we will reach to contradiction.

First, assume that $I = [-h, h]$. If there is no $h' \in G \backslash H$ such that $h < h' < 2h$, then $[-2h, 2h] \subseteq H$ which contradicts with the maximal character of I. But, if $h < h' < 2h$ then $0 < h' - h < h$, whence $h' - h \in H$. It follows that $(h' - h) + h = h' \in H$ which is again a contradiction as before.

Now assume that $I = (-h, h)$. Since $I \neq \emptyset$, there is some $h' \in I$ such that $h' > 0$. It follows that $0 < h - h' < h$, so $h - h' \in H$. But then $(h - h') + h' = h \in H$ which is absurd. Thus the lemma has been proved. □

We will establish now that $\mathcal{G}$ is abelian. Let $g \in G$ and consider the parametrically definable subgroup $C(g) = \{h \in G | h + g = g + h\}$. Now $C(g)$ is abelian and by the previous lemma, $C(g) = G$. Thus $\mathcal{G}$ is abelian.

Now consider the (parametrically definable) subgroup $nG = \{ng | g \in G\}$. nG is divisible and again because it is non empty, $nG = G$. Therefore $\mathcal{G}$ will be divisible and this completes the proof. $\square$

Theorem 6.3. *Let $R = (R, +, \cdot, 1, <)$ be an o-minimal ordered ring. Then R is a real closed field.*

Proof. For the proof we will need two lemmas. The first is just an observation.

Lemma 6.4. *A parametrically definable convex substructure of an o-minimal structure is also o-minimal.*

The second lemma is a well known result from algebra. Recall that, an ordered ring R is said to have the intermediate value property if for any polynomial $p(x) \in R[x]$ and any $a, b \in R$ with $a < b$ and $p(a) \cdot p(b) < 0$, there is some $c \in R$ such that $a < c < b$ and $p(c) = 0$.

Lemma 6.5. *An ordered field is real closed iff it has the intermediate value property.*

We will show first that R is a field. For this it is enough to show that the set of non-zero positive elements of R, R^+, forms an abelian group under multiplication. Let $r \in R$, $r \neq 0$. Notice that $rR := \{r \cdot s | s \in R\}$ forms a non-trivial ordered subgroup of R under addition. Then by lemma 6.2, $rR = R$. In particular there is some $s \in R$ such that $r \cdot s = 1$. If $r > 0$ then s is positive, thus R^+ forms an ordered group under multiplication. Since R^+ is definable and convex in R, by lemma 6.4 it is also o-minimal. Now again by lemma 6.2 R^+ is abelian as desired.

We will now show that R satisfies the intermediate value property and by lemma 6.5 follow that it will be real closed. Let $a, b \in R$, $a < b$ such that $p(a) \cdot p(b) < 0$, where $p(x) \in R[x]$. Without loss of generality let us assume that $p(a) > 0$ and $p(b) < 0$. If there were no $c \in R$, $a < c < b$, satisfying $p(c) = 0$, then $(a, b)^R = P^+ \cup P^-$, where P^+ and P^- are the parametrically definable sets,

$$P^+ := \{d \in R | a < d < b \,\&\, p(d) > 0\}$$
$$P^- := \{d \in R | a < d < b \,\&\, p(d) < 0\}.$$

Since it has been shown that R is a field, $(a, b)^R$ must be a densely ordered set. Also, using the ordered field axioms, one can verify that polynomials in $R[x]$ are continuous parametrically definable functions in R under the topology given by the ordering. Now, if $P^+ = (a, b)$, then we can see that the continuity of $p(x)$ is violated at $x = b$. For a similar reason it cannot be the case that $P^- = (a, b)$. Thus, by the o-minimality of R, there must be some $c \in (a, b)$ which is a boundary point between P^+ and P^-. We show that $c \notin P^+ \cup P^-$. Indeed, suppose that $c \in P^+$. Any open interval I containing c, must intersect both P^+ and P^-. Therefore,

$$p^{-1}\left(\left\{d \in R | \frac{1}{2}p\left(c\right) < d < \frac{3}{2}p\left(c\right)\right\}\right)$$

does not contain any open interval, contradicting the continuity of p. Similarly, we have that $c \notin P^-$. But then $(a, b) \neq P^+ \cup P^-$ which contradicts to our hypothesis. This completes the proof of the theorem. $\square$

6.2 A TRICHOTOMY AND A DICHOTOMY THEOREM

Here, we will provide an answer to the following,

Question. *Is there a sense in which an o-minimal structure is either "trivial" (like* $(\mathbb{Q}, <)$*), or group-like, or field-like?*

Any answer to this question has to be "local". Consider the following example,

Example. We could form an o-minimal structure with three parts, L (the leftmost part), M (the middle part) and R (the rightmost part), with L carrying the structure of a pure dense linear order, M that of divisible abelian ordered group and R that of a real closed field. We also need to add a point between L and M, and a point between M and R, to ensure o-minimality.

An answer to this question has been given by Peterzil and Starchenko [22].

Definition 6.6. Let $\mathcal{M}$ be an o-minimal structure, and $a \in M$.

- The point a is *non-trivial* if there is an open interval $I \subset M$ such that $a \in I$ and a definable continuous function $f : I \times I \to M$ which is strictly monotonic in each variable.
- A *convex* $\bigwedge$*-definable group* in M is a group $(G, *)$, where $G \subseteq M$ is convex, and the group operation $*$ (regarded as ternary relation) is the intersection of a definable set with G^3.
- If $(G, <, +, 0)$ is a convex $\bigwedge$-definable group and $p \in G$ with $p > 0$, then a *group interval* is a structure $([-p, p], <, +, 0)$, where $+$ is the induced partial function $[-p, p] \times [-p, p] \to [-p, p]$.
- If I is an interval of M, then $\mathcal{M}|I$ is the structure with domain I, whose 0-definable sets are those of the form $I^k \cap U$ for definable $U \subseteq M^k$ (if I is closed then such sets are I-definable, by lemma 2.5 of [Peterzil and Starchenko 22]).

If every point of M is trivial, then by Mekler et al. [19], every definable set is a boolean combination of binary relations.

It is easy to see that if $(G, *)$ is a convex $\bigwedge$-definable ordered group in M, and $a \in G$, then $*$ witnesses that a is non-trivial. The following converse is much deeper.

Theorem 6.7. [Peterzil and Starchenko 22] *Let $\mathcal{M}$ be ω^+-saturated, and $a \in M$ be non-trivial. Then there is a convex $\bigwedge$-definable infinite group $G \subseteq M$, such that $a \in G$ and G is a divisible ordered abelian group.*

It follows from the theorem above that given a non-trivial $a \in M$, there is a closed interval I containing a on which a group-interval is definable.

Even without the saturation assumption, there is a closed interval I with $a \in M$ non-trivial and a definable group interval structure induced in I. Saturation allows us to find the whole domain of a group, on say an infinitesimal neighborhood about a, but this infinitesimal neighborhood may not be definable.

Theorem 6.8. [Peterzil and Starchenko 22] *Suppose that $(I, <, +, 0)$ is a 0-definable group interval in a sufficiently saturated o-minimal structure $\mathcal{M}$. Then precisely one of the following statements holds.*

1. *There is an ordered vector space $\mathcal{V} = (V, +, c, d(x))_{c \in C, d \in D}$ (with C a set of constants) over an ordered division ring D, an interval $[-p, p]$ in V and an order-preserving isomorphism of group intervals $\sigma : I \to [-p, p]$, such that $\sigma(S)$ is 0-definable in $\mathcal{V}$ for every 0-definable $S \subseteq I^n$.*

2. *There is a real closed field $\mathcal{R}$ definable in $\mathcal{M}$ with its domain a subinterval of I and its order compatible with $<$.*

Combining the theorems above we get the following,

Theorem 6.9. [Trichotomy Theorem] [Peterzil and Starchenko 22] *Let $\mathcal{M}$ be a sufficiently saturated o-minimal structure. Given an $a \in M$ one and only one of the following holds,*

- *a is trivial,*
- *the structure that $\mathcal{M}$ induces on some convex neighborhood of a is an ordered vector space over an ordered division ring,*
- *the structure that $\mathcal{M}$ induces over some open interval around a is an o-minimal expansion of a real closed field.*

We close this section by stating a dichotomy theorem which looks like Theorem 6.8 but it has global character rather than local. If $\mathcal{G} = (G, <, +, \ldots)$ is an expansion of an ordered group, we say that $\mathcal{G}$ is *linearly bounded* if for any definable function $f : G \to G$ there is a definable endomorphism λ of G such that, $|f(x)| \leq \lambda(x)$ for sufficiently large x.

Linearly bounded

Theorem 6.10. [Miller and Starchenko 20] *Let $\mathcal{G} = (G, <, +, \dots)$ be an o-minimal expansion of an ordered group. Then if $\mathcal{G}$ is not linearly bounded, there is a definable binary operation $\cdot$ such that $(G, <, +, \cdot)$ is an ordered field.*

Part II

VARIANTS OF O-MINIMALITY

7 WEAK O-MINIMALITY

7.1 DEFINITIONS

Let $\mathcal{M} = (M, <, \ldots)$ be a linearly ordered structure.

Definition 7.1. A set $C \subseteq M$ is called *convex*, if for any $a, b \in C$ with $a < b$ and $c \in M$ such that $a < c < b$, then $c \in C$.

Convex set

Using the notion of convex sets, Dickmann proposed the following variation of o-minimality.

Definition 7.2. A structure $\mathcal{M}$ will be called *weakly o-minimal* (*w.o.m.*), if the definable subsets of M are finite unions of convex sets in $(M, <)$.

Weakly o-minimal structure

We say that a complete theory T is *weakly o-minimal* if every model of T is weakly o-minimal.

Weakly o-minimal theory

Our goal in this chapter is to present various results about weakly o-minimal structures and compare them with the respectively results about o-minimal structures.

Definition 7.3. With $\overline{M}$ we will denote the *Dedekind completion* of $(M, <)$, that is the smallest complete lattice that contains the given partial order.

Dedekind completion

An *interval* in $\mathcal{M}$ will be a convex set having both a supremum and infimum in $M \cup \{-\infty, +\infty\}$. For subsets of M^n, all topological notions refer to the product topology induced by the order topology on $(M, <)$.

We need here to introduce the concept of a *sort in* $\overline{M}$. First we will see how this notion is forced upon us. Suppose that we want to prove a cell decomposition theorem as 2.5 but for w.o.m. structures. Let us try to see how we could prove the analogue of (I_n) of 2.5, that is, assuming that we have a definable set $S \subseteq M^{n+1}$ on a w.o.m. structure we want to express it as a finite union of "cells" (under some reasonable definition of "cell"). Let $\pi : M^{n+1} \to M^n$ be the projection on the first n coordinates, and $I = \pi(S)$. I will be a finite union of convex subsets of M^n. By weak o-minimality, for each $\bar{a} \in M^n$ the fiber $S_{\bar{a}} := \{y | (\bar{a}, y) \in S\}$ is the union of finitely many convex sets, possibly singletons. If each $S_{\bar{a}}$ is an interval (i.e. it has supremum and infimum in $M \cup \{-\infty, +\infty\}$) then there are definable functions $f, g : M^n \to M \cup \{\pm\infty\}$ where $f(\bar{x}) := \inf S_{\bar{x}}$ and $g(\bar{x}) := \sup S_{\bar{x}}$ and we have that $S = (f, g)_I$. In the o-minimal case we can also assume that f, g are also continuous and thus S will be a cell. In the weakly o-minimal case we will have difficulties to apply this if the sets $S_{\bar{a}}$ are convex but do not have endpoints in M (then infimum and supremum could possible be out of $M \cup \{\pm\infty\}$). To overcome this we consider f, g as definable functions $M \to \overline{M}$ and for this reason we introduce sorts as follows.

Sorts in $\overline{M}$

There is a natural notion of definable sort in $\overline{M}$. Let $Y \subseteq M^{n+1}$ be definable set, let $\pi : M^{n+1} \to M^n$ be the projection on the first n coordinates and let $Z = \pi(Y)$. For each $\bar{a} \in Z$ let $Y_{\bar{a}} := \{y | (\bar{a}, y) \in Y\}$. Suppose that for every $\bar{a} \in Z$ the set $Y_{\bar{a}}$ is bounded above but does not have a supremum in M. We let $\sim$ be the definable equivalence relation on M^n given by,

$$\bar{a} \sim \bar{b} \text{ for all } \bar{a}, \bar{b} \in M^n \backslash Z,$$
$$\bar{a} \sim \bar{b} \iff \sup Y_{\bar{a}} = \sup Y_{\bar{b}} \text{ if } \bar{a}, \bar{b} \in Z$$

In other words $\bar{a} \sim \bar{b}$ if $Y_{\bar{a}}$ and $Y_{\bar{b}}$ have a common final segment. Let $\bar{Z} := Z/\sim$. There is a definable total order on $M \cup \bar{Z}$ defined as follows. Let $\bar{a} \in Z$ and $c \in M$. Then

$$[\bar{a}] < c \text{ iff } w < c \text{ for all } w \in Y_{\bar{a}}.$$

If $\bar{a} \nsim \bar{b}$ then there is some $x \in M$ such that $[\bar{a}] < x < [\bar{b}]$ or $[\bar{b}] < x < [\bar{a}]$, and so $<$ induces a total order on $M \cup \bar{Z}$. We call such a set $\bar{Z}$ sort in $\overline{M}$ and view $\bar{Z}$ as naturally embedded in $\overline{M}$.

Similarly we can obtain a sort in $\overline{M}$ by considering the infimum of sets instead of supremum. Notice here that a sort depends on the set Y and not just on the equivalence relation defined in its projection. In this setting a definable function $f : M^n \to \overline{M}$ is understood to be a definable function $M^n \to \bar{Z}$, where $\bar{Z}$ is a sort in $\overline{M}$.

Lastly we give the first result on weak o-minimal structures of this section.

Theorem 7.4. *If T is a weakly o-minimal theory, then T does not have the independence property.*

7.2 MONOTONICITY

The theory of o-minimal structures begins with the monotonicity theorem 1.12. Let us see some examples now which demonstrate the behavior of definable functions in weakly o-minimal structures.

Example 7.5. Let $\mathcal{M} = (M, <, P, Q, f)$, where M is the disjoint union of the interpretations of the unary relations P and Q, with P preceding Q in the ordering $<$ on M. Let the interpretation of P be $\mathbb{Q}$ with the usual order and the interpretation of Q, $\mathbb{Q} \times \mathbb{Q}$ lexicographically ordered. The symbol f is interpreted by a unary function with domain Q which takes values in P. It is defined by $f((n, m)) = n$, for all $n, m \in \mathbb{Q}$. It can be shown that $\mathcal{M}$ is a weakly o-minimal structure and in fact all models of $\mathrm{Th}(\mathcal{M})$ are weakly o-minimal. The function f is everywhere continuous and locally constant, but we cannot partition its domain into finitely many convex sets on which f is constant or strictly monotonic.

Example 7.6. Let $\mathcal{M} = (M, <, f)$, where $M = \mathbb{Z} \times \mathbb{Q}$, $<$ is the lexicographic order on M and for all $(n, q) \in M$, $f((n, q)) = (-n, q)$. It can be shown that the structure $\mathcal{M}$ and also the theory $\mathrm{Th}(\mathcal{M})$ is weakly o-minimal. The definable function f is everywhere locally strictly monotonic but not piecewise monotonic (in a sense of what a monotonicity theorem we are looking for expect it to be.).

These examples show that we can expect at most a local monotonicity theorem for weakly o-minimal structures.

Definition 7.7. Let f be a function from a densely ordered set I to a densely ordered set K. We say that f is *tidy* if one of the following holds,

Tidy function

1. for all $x \in I$ there is an interval $J \subseteq I$ such that $x \in \mathrm{int}(J)$ and $f|J$ is strictly increasing, in which case f is said to be *locally increasing* on I;
2. for all $x \in I$ there is an interval $J \subseteq I$ such that $x \in \mathrm{int}(J)$ and $f|J$ is strictly decreasing, in which case f is said to be *locally decreasing* on I;
3. for all $x \in I$ there is an interval $J \subseteq I$ such that $x \in \mathrm{int}(J)$ and $f|J$ is constant, in which case f is said to be *locally constant* on I;

Definition 7.8. A weakly o-minimal structure $\mathcal{M}$ is said to have *monotonicity* if whenever $A \subseteq \overline{M}$ is a definable sort and $f : D \subseteq M \to A$ is a definable function, there is some $m \in \mathbb{N}$ and a partition of $D = \mathrm{dom}(f)$ into definable sets $X, I_1, \ldots, I_m$ such that X is finite, each I_i is convex and on each set I_i the function f is tidy.

Macpherson et al. [15] proved the following result under some extra hypotheses, and Arefiev [2] proved it in general.

Theorem 7.9. [Arefiev 2] *If $\mathcal{M}$ is a weakly o-minimal structure, then it has monotonicity.*

7.3 DIMENSION

Recall the definition of topological dimension 3.13. We say that *topological dimension is well-behaved in a class $\mathcal{K}$ of structures*, if for all $\mathcal{M} \in \mathcal{K}$, $m, n > 0$, and definable $X_1, \ldots, X_m \in M^n$,

$$\mathrm{tdim}(X_1 \cup \cdots \cup X_m) = \max\{\mathrm{tdim}(X_1), \ldots, \mathrm{tdim}(X_m)\}.$$

As we saw in chapter 3 topological dimension is well-behaved in the class of o-minimal structures. Using the monotonicity theorem for weakly o-minimal structures 7.9 we can take the following result.

Theorem 7.10. [Macpherson et al. 15] *Topological dimension is well-behaved in the class of weakly o-minimal structures.*

7.4 CELL DECOMPOSITION

In this section we will show a weak version of cell decomposition for those structures whose theories are weakly o-minimal. The theorem is due to Macpherson et al. [15]. First we will need a definition of cell for weakly o-minimal structures.

Cell in a w.o.m. structure

Definition 7.11. Let $\mathcal{M}$ be weakly o-minimal. A *cell* is a subset of M^n, defined as follows.

1. A 1-cell is a definable convex subset of M.
2. A set $X \subseteq M^{n+1}$ is an $(n+1)$-cell if there is an n-cell $Y \subseteq M^n$ such that,
   a) $X = \{(\bar{y}, f(\bar{y}) \,|\, \bar{y} \in Y)\}$, where $f : M^n \to M$ is a definable function, or
   b) $X = \{(\bar{y}, z) \,|\, \bar{y} \in Y,\ f_1(\bar{y}) < z < f_2(\bar{y})\}$, where A_1 and A_2 are sorts in $\overline{M}$ and $f_i : Y \to A_i$, for $i = 1, 2$ are definable functions such that $f_1(\bar{y}) < f_2(\bar{y})$ for all $\bar{y} \in Y$. We allow $\{-\infty\}$ to be the sort A_1 and $\{+\infty\}$ to be the sort A_2. In this case, we often denote the cell X as $(f_1, f_2)_Y$.

Decomposition in w.o.m. structures

Theorem 7.12. [Macpherson et al. 15] *Let $\mathcal{M}$ be a model of a weakly o-minimal theory, let $n > 0$, and let $X_1, \dots, X_r$ be definable subsets of M^n. Then there is a partition $\mathcal{P}$ of M^n into finitely many n-cells such that each of $X_1, \dots, X_r$ is a union of cells in $\mathcal{P}$. We call that partition a decomposition of M^n which partitions $X_1, \dots, X_r$.*

Proof. By induction on n. The result is immediate for $n = 1$, by the weak o-minimality of $\mathcal{M}$. As hypothesis, assume that the result is true for $n - 1$. Let $\pi : M^n \to M^{n-1}$ be the projection onto the first $n - 1$ coordinates, and $X_1, \dots, X_r$ be definable subsets of M^n. By the induction hypothesis there is a decomposition of M^{n-1} which partitions each $\pi(X_i)$. We may assume that $\pi(X_1) = \pi(X_2) = \cdots = \pi(X_r) = Y$. For each $\bar{a} \in Y$ we define the fiber, $X_{\bar{a},i} := \{x \in M | \, (\bar{a}, x) \in X_i\}$. Each of the $X_{\bar{a},i}$ is a definable subset of M and thus it would be a finite union of definable convex sets. The number of such convex sets is bounded as $\bar{a}$ ranges over Y, since otherwise (using a compactness argument) there would be an elementary extension of $\mathcal{M}$ for which infinitely many convex sets were needed, contradicting that fact $\mathrm{Th}(\mathcal{M})$ is weakly o-minimal. We can also assume that each convex set is a singleton or an open set. By decomposing Y further if necessary, we may suppose that there is some $K \in \mathbb{N}$ such that we have a uniformly definable family of partitions $\mathcal{P}_{\bar{a}}$, for $\bar{a} \in Y$, of M into K convex sets satisfying several conditions,

- for each $i \in \{1, \ldots, r\}$, if $\bar{a}, \bar{b} \in Y$ then $X_{\bar{a},i}$ and $X_{\bar{b},i}$ are unions of the same number of n_i definable convex sets in $\mathcal{P}_{\bar{a}}$ and $\mathcal{P}_{\bar{b}}$ respectively.
- for all $\bar{a} \in Y$ the jth convex set in $\mathcal{P}_{\bar{a}}$, in increasing order, is of the same type, i.e.
  - both or neither are singletons,
  - both or neither have infima in M, or in $\overline{M} \backslash M$, or $-\infty$,
  - similarly for suprema.
- as $\bar{a}$ ranges over Y we have that the jth convex set in $\mathcal{P}_{\bar{a}}$, in increasing order, is contained in exactly the same sets $X_{\bar{a},i}$ for $i \in \{1, \ldots, r\}$.

Now, for fixed $i \in \{1, \ldots, r\}$ and $j \in \{1, \ldots, n_i\}$, the set of the lower bounds of the jth convex set, in increasing order, in $X_{\bar{a},i}$, as $\bar{a}$ ranges over Y, is a sort A_{ij} in $\overline{M}$. Hence there is a definable function $f_{ij} : Y \to A_{ij}$ that maps each $\bar{a} \in Y$ to the lower bound of the jth convex set in $X_{\bar{a},i}$. Upper bounds are handled similarly. The theorem now follows easily using these sorts to define the cells which decompose M^n. □

7.5 WEAKLY O-MINIMAL GROUPS AND FIELDS

We will see here two results regarding the weakly o-minimal ordered groups and fields. These results are the w.o.m. analogues of the results we proved in section 6.1. These results are due to Macpherson et al. 15.

First we will prove a lemma.

Lemma 7.13. *Let $(G, <, \cdot, 1)$ be a weakly o-minimal ordered group, and let H be a definable subgroup of G. Then H is convex in G.*

Proof. If H is the trivial subgroup $\{1\}$ then the lemma clearly holds. So, suppose that H is a definable subgroup of G and $H \neq \{1\}$. By weak o-minimality of G, H is the union of finitely many convex subsets of G. Let X be the greatest of these components with respect to the ordering induced by $<$, and let $x \in X$ with $x > 1$. Let $y \in G$ such that $1 < y < x$. To prove the lemma if suffices to show that $y \in H$. We have that $x < yx < x^2$, and as X is the greatest component of H, $x^2 \in X$. Since X is convex, $yx \in X$. Therefore $yx \in H$, and $y \in H$. □

Theorem 7.14. [Macpherson et al. 15] *Every weakly o-minimal ordered group is divisible and abelian.*

Proof. Let $(G, <, \cdot, 1)$ be a weakly o-minimal ordered group. To show that it is abelian, consider $x, y \in G$ and, without loss of generality, assume that, $1 < y < x$. Consider the definable subgroup $C(x) := \{g \in G | gx = xg\}$. By the previous lemma it will be convex and it will contain 1 and x, hence $y \in C(x)$. Because y, x were chosen arbitrary, G will be abelian.

To show that it is divisible, let $n > 0$ and consider the definable subgroup of G, $G^n := \{g^n | g \in G\}$. Then G^n contains cofinal elements and by the previous lemma it is also convex in G. Hence, $G^n = G$ and this concludes the proof □

Theorem 7.15. [Macpherson et al. 15] *Every weakly o-minimal ordered field is real closed.*

The proof of this theorem is much more tedious than the previous and the analogue for the o-minimal context (Theorem 6.3), and it will be omitted.

8 OTHER VARIATIONS OF O-MINIMALITY

8.1 A GENERALIZATION OF O-MINIMALITY

We now consider a different generalization of o-minimality, from Macpherson and Steinhorn [14]. Suppose that $L \subset L^+$ are languages, and $\mathcal{K}$ is an elementary class of L-structures.

Definition 8.1. An L^+-structure $\mathcal{M}$ is $\mathcal{K}$-*minimal* if the reduct $\mathcal{M}|_L$ is in $\mathcal{K}$ and every L^+-definable subset of M is definable by a quantifier-free L-formula.

$\mathcal{K}$-minimal structure

A complete L^+-theory is $\mathcal{K}$-*minimal* if all its models are $\mathcal{K}$-minimal.

$\mathcal{K}$-minimal theory

O-minimality is included in this setting as a special case, indeed, let L have a single binary relation $<$, and $\mathcal{K}$ be the elementary class of all dense linear orders. Also the notion of strongly minimal theory is a special case of the above definition, let L be the language which contains only $=$, and $\mathcal{K}$ be the class of all infinite L-structures.

On the other hand, weak o-minimality, is not covered by this definition, because $\mathcal{K}$-minimality is closed under reducts of languages containing L and under expansions by constants.

In this chapter we will talk about C and P minimality which are included in the setting of the definition above.

8.2 C-MINIMALITY

Consider now a ternary relation $C(x; y, z)$ (the semicolon indicates that the first variable is distinguished by the other two). Let $L = \{C\}$, and $\mathcal{K}_C$ be the class of L-structures satisfying the following axioms.

- (C1) $(\forall xyz)\,[C(x; y, z) \to C(x; z, y)]$
- (C2) $(\forall xyz)\,[C(x; y, z) \to C(y; x, z)]$
- (C3) $(\forall xyzw)\,[C(x; y, z) \to (C(w; y, z) \vee C(x; w, z))]$
- (C4) $(\forall xy)\,[x \neq y \to (\exists z \neq y)\, C(x; y, z)]$
- (C5) $(\exists xy)\,(x \neq y)$

The axioms above were isolated by Adeleke and Neumann [1].

As an example, let $(T, \leq)$ be a semi-linearly ordered set (that is a partial ordered set where each two elements have a common lower bound, but the set of all lower bounds of an element is totally ordered). Let M be the set of all maximal chains of $(T, \leq)$. We now interpret the symbol $C(x; y, z)$ on M to hold if $x \neq y = z$ or if x, y, z are distinct chains and x branches below where y and z branch i.e. $y \cap x \subset y \cap z$, where we regard the chains as subsets of T. It can be shown that (M, C) satisfies the axioms (C1)-(C5). Moreover, Adeleke and Neumann [1] showed that the converse is also true. If $(M, C) \in \mathcal{K}_C$, then there is a semi-linear order $(T, \leq)$ interpretable in (M, C), such that M consists of a set of maximal chains of $(T, \leq)$.

By the previous example we can think of the $\mathcal{K}_C$-structures as sets of chains in a semi-linear order. Because of this we often refer to *nodes of* (M, C), meaning the internal nodes of the underlying semi-linear order.

C-minimal structure

Definition 8.2. Considering the class $\mathcal{K}_C$. We say that a structure $\mathcal{M} = (M, C, \dots)$ is *C-minimal* if its theory is $\mathcal{K}_C$-minimal.

This definition was introduced in [Macpherson and Steinhorn 14]. Note that we closed the condition, for a structure to be C-minimal, under elementary equivalence by the definition, unlike the o-minimal case.

8.2.1 *Model Theory of C-minimal Structures*

If $(M, C) \in \mathcal{K}_C$, we can define a Hausdorff topology on M, and we get a notion of topological dimension.

Theorem 8.3. [Haskell and Macpherson 9] *Topological dimension is well-behaved in any C-minimal structure. That means, if $X_1, X_2, \dots, X_n$ are definable subsets of M^m, then*

$$\operatorname{tdim}(X_1 \cup X_2 \cup \cdots \cup X_n) = \max \{\operatorname{tdim}(X_i) \,|\, i \in \{1, 2, \dots, n\}\}.$$

There is also a notion of "cell" for C-minimal structures, and a cell decomposition theorem.

Theorem 8.4. [Haskell and Macpherson 9] *Let $\mathcal{M}$ be a C-minimal structure, and $X_1, X_2, \dots, X_n$ be definable subsets of M^m. Then there is a partition of M^m in finitely many cells such that each X_i is the union of some of the cells.*

Using the same notion of "cell" one can prove the following.

Theorem 8.5. *Let $\mathcal{M}$ be a C-minimal structure, X be definable subsets of M^n, and $f : M^n \to M$ a definable partial function. Then X can be expressed as the disjoint union of finitely many cells on each of which f is continuous.*

It is also known, by Macpherson and Steinhorn [14], that in a C-minimal structure algebraic closure does not necessarily has the exchange property. However, Haskell and Macpherson at 9, showed that the exchange property can only fail if only there is a certain kind of definable "bad" function, between M and the set of internal nodes.

Therefore, if $\mathcal{M}$ has the exchange property, then $\mathcal{M}$ can be considered a geometric structure (in the sense of 3.4) so we can use algebraic closure to get a notion of dimension for definable sets. In this case, the notion of geometric and topological dimension coincide (Haskell and Macpherson 9).

The last result on the model theory of C-minimal structures is from Macpherson and Steinhorn [14].

Theorem 8.6. *If $\mathcal{M}$ is C-minimal, then $\mathcal{M}$ does not have the independence property.*

8.2.2 *C-minimal Groups and Fields*

Definition 8.7. A *C-group* is a structure $\mathcal{G} = (G, \cdot, 1, C)$ so that $(G, \cdot, 1)$ is a group, and C is a C-relation on G preserved by left and right multiplication, that is *C-group*

$$\mathcal{G} \vDash C(x; y, z) \longleftrightarrow C(uxv; uyv, uzv)$$

for all $x, y, z, u, v \in G$.

A *C-minimal group* is a C-group which is also a C-minimal structure. *C-minimal group*

We will see now an example of C-minimal group. Let $(G, +, 0)$ be the direct sum of copies of $\mathbb{Z}/2\mathbb{Z}$, indexed by $\mathbb{Q}$. We view elements of G as sequences of zeros and ones indexed by rational numbers with finitely many non-zero terms. We denote by a_r the rth element of $a \in G$. For $a, b, c \in G$, define

$$C(a; b, c) \iff (\exists r \in \mathbb{Q})(\forall q \leq r)[b_q = c_q \wedge a_r \neq b_r].$$

This C relation satisfies the axioms (C1)-(C5), thus $\mathcal{G} = (G, +, 0, C)$ is a C-group. It is possible to show that $\mathcal{G}$ is also a C-minimal group, by verifying that $\mathrm{Th}(\mathcal{G})$ admits elimination of quantifiers.

For an other example let p be a prime number, $\mathbb{Q}_p$ be the field of p-adic numbers, and $v : \mathbb{Q}_p \to \mathbb{Z} \cup \{\infty\}$ be the usual valuation. We define a C-relation on $\mathbb{Q}_p$ by

$$C(a; b, c) \iff v(c - b) > v(c - a)$$

for $a, b, c \in \mathbb{Q}_p$. Using quantifier elimination, given by Weispfenning [37], we can derive that this is a C-minimal group. Similarly, $(\mathbb{Z}_p, +, 0, C)$ is a C-minimal group, where $\mathbb{Z}_p$ are the p-adic integers.

Definition 8.8. A *C-field* is a structure $\mathcal{F} = (F, +, -, \cdot, 0, 1, C)$ so that $(F, +, -, \cdot, 0, 1,)$ is a field, C is a C-relation on F, and both by multiplication by non-zero elements and addition preserve the C-relation by left and right. *C-field*

A *C-minimal field* is a C-field which is also a C-minimal structure. *C-minimal field*

A small connection between C-minimal fields and o-minimality is the following proposition.

Proposition 8.9. *In a C-minimal field, the value group is o-minimal.*

Now we will give a characterization of C-minimal fields.

Theorem 8.10. [Macpherson and Steinhorn 14] *If $\mathcal{F}$ is an algebraically closed non-trivially valued field, then $(\mathcal{F}, C)$ is C-minimal.*

The converse also holds.

Theorem 8.11. [Haskell and Macpherson 9] *Every C-minimal field F is an algebraically closed valued field.*

8.3 P-MINIMALITY

In this section we will give a version of o-minimality which will support $\mathbb{Q}_p$ as field. C-minimality will not be able to achieve this. Indeed, the value group of $\mathbb{Q}_p$ which is $\mathbb{Z}$ has the definable subset $2\mathbb{Z}$ and thus, by proposition 8.9, $\mathbb{Q}_p$ can not be C-minimal field.

Let L be the language $(+, -, \cdot, 0, 1, (P_n)_{n>1})$, where P_n are unary predicates.

P-minimal structure

Definition 8.12. Regard $\mathbb{Q}_p$ as an L-structure and let P_n pick out the nth powers in $\mathbb{Q}_p$. Let $\mathcal{K}_P$ be the class of L-structures elementarily equivalent to $\mathbb{Q}_p$. Then if $L^+ \supseteq L$, an L^+-structure is *P-minimal* if all models of its theory are $\mathcal{K}_P$-minimal.

Mourgues [21], gives a "cell" definition and prove the following cell decomposition result for P-minimal structures.

Theorem 8.13. [Mourgues 21] *A P-minimal structure has cell decomposition if and only if it has definable Skolem functions.*

There is not yet a good analogue of the Monotonicity Theorem 1.12 for these structures.

In any P-minimal structure we can define a topology which gives us a well-behaved topological dimension. Furthermore, unlike the weakly o-minimal and the C-minimal case, algebraic closure has the exchange property in any P-minimal structure and thus such structures are always geometric and this provides us a notion of geometric dimension. This geometric notion of dimension coincides with the topological dimension.

Consider now the language L_{an}^D, which is extending the language L above. L_{an}^D has an m-place function symbol for all $f \in \mathbb{Q}_p\{x_1, \ldots, x_m\}$ and $m \geq 0$, which satisfies $f(x) \in I$ for all $(x_1, \ldots, x_m) \in I^m$. It also has a binary function symbol D which is interpreted in $\mathbb{Q}_p$ and $\mathbb{Z}_p$ as "restricted division",

$$D(x, y) = \begin{cases} \frac{x}{y}, & \text{if } 0 < |x| \leq |y| \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

The following is the main theorem of Van Den Dries et al. [36].

Theorem 8.14. *The L_{an}^{D}-structure $\mathbb{Q}_p$ is P-minimal.*

We close this section with a result from Haskell and Macpherson [10].

Theorem 8.15. *Let $\mathcal{F}$ be a P-minimal expansion of a model of* $\mathrm{Th}(\mathbb{Q}_p)$*. Then* $\mathrm{Th}(\mathcal{F})$ *does not have the independence property.*

Part III

APPENDIX

A

LANGUAGES AND FORMULAS

A.1 BASICS

Definition A.1. A *language* L is a triplet $(\mathcal{R}, \mathcal{F}, \mathcal{C})$ where, *Language*

1. $\mathcal{R}$ is a set of relation symbols R and each of them is equipped with a number $\text{arity}(R) \in \mathbb{N}$,
2. $\mathcal{F}$ is a set of function symbols f and each of them is equipped with a number $\text{arity}(f) \in \mathbb{N}$ and
3. $\mathcal{C}$ is a set of constant symbols.

Sometimes constant symbols are treated as 0-arity function symbols.

Definition A.2. An *L-structure* is a tuple *Structure*

$$\mathcal{M} \equiv \left(M, \left(R^{\mathcal{M}}\right)_{R \in \mathcal{R}}, \left(f^{\mathcal{M}}\right)_{f \in \mathcal{F}}, \left(c^{\mathcal{M}}\right)_{c \in \mathcal{C}}\right)$$

where,

1. M is a nonempty set called the *universe*, *domain* or *underlying set* of $\mathcal{M}$,
2. for each $R \in \mathcal{R}$, $R^{\mathcal{M}} \subseteq M^n$, where $n = \text{arity}(R)$
3. for each $f \in \mathcal{F}$, $f^{\mathcal{M}} : M^m \to M$, where $m = \text{arity}(f)$
4. for each $c \in \mathcal{C}$, $c^{\mathcal{M}} \in M$.

Usually, we will choose languages that closely correspond to the structure that we wish to study. The following are called *logical symbols*,

parentheses	), (
variables	$v_0, v_1, \ldots, v_n, \ldots$
connectives	$\wedge, \neg$
quantifier	$\forall$
identity relation	$\equiv$

Definition A.3. A *term* is a string of logical symbols and symbols from L which is defined as follows, *Term*

1. A variable is a term.
2. A constant symbol is a term.

3. If f is an m-arity function symbol and $t_1, \dots, t_m$ are terms, then $f(t_1, \dots, t_m)$ is a term.

4. A string of symbols is a term only if it can be shown to be a term by a finite number of applications of 1-3.

Atomic formula

Definition A.4. The *atomic formulas* of L are strings of the form given below,

1. $t_1 \equiv t_2$ is an atomic formula, where t_1, t_2 are terms of $\mathcal{L}$.

2. If R is a relation symbol with arity n and $t_1, \dots, t_n$ are terms, then $R(t_1, \dots, t_n)$ is an atomic formula.

Formula

Definition A.5. The *formulas* of L are defined as follows,

1. An atomic formula is a formula.

2. If φ and ψ are formulas, then $(\varphi \wedge \psi)$ and $(\neg\varphi)$ are formulas.

3. If v is a variable and φ is a formula, then $(\forall v)\,\varphi$ is a formula.

4. A string of symbols is a formula only if it can be shown to be a formula by a finite number of applications of 1-3.

Free variable

We say that a variable v *occurs freely* in a formula ϕ if it is not inside a quantifier $((\forall v)\,\phi$ or $(\neg\forall v)\,(\neg\phi) \equiv (\exists v)\,\phi)$. Otherwise we say that v is *bound*. We write $\phi(v_1, \dots, v_m)$ to denote that the free variables in ϕ are the $v_1, \dots, v_m$, or among them.

We call a formula, *sentence*, if it has no free variables.

Satisfiability

Definition A.6. Let ϕ be an formula with free variables from $\bar{v} = (v_1, \dots, v_m)$, and let $\bar{a} = (a_1, \dots, a_m) \in M^m$. We define $\mathcal{M} \vDash \phi(\bar{a})$ inductively as follows (t_i will denote some term).

1. If ϕ is $t_1 = t_2$, then $\mathcal{M} \vDash \phi(\bar{a})$ if $t_1^{\mathcal{M}}(\bar{a}) = t_2^{\mathcal{M}}(\bar{a})$.

2. If ϕ is $R(t_1, \dots, t_m)$, then $\mathcal{M} \vDash \phi(\bar{a})$ if $\left(t_1^{\mathcal{M}}(\bar{a}), \dots, t_m^{\mathcal{M}}(\bar{a})\right) \in R^{\mathcal{M}}$.

3. If ϕ is $\neg\psi$, then $\mathcal{M} \vDash \phi(\bar{a})$ if $\mathcal{M} \nvDash \psi(\bar{a})$.

4. If ϕ is $(\psi \wedge \theta)$, then $\mathcal{M} \vDash \phi(\bar{a})$ if $\mathcal{M} \vDash \psi(\bar{a})$ and $\mathcal{M} \vDash \theta(\bar{a})$.

5. If ϕ is $(\forall u)\,\psi(\bar{v}, u)$, then $\mathcal{M} \vDash \phi(\bar{a})$ if $\mathcal{M} \vDash \psi(\bar{a}, b)$ for all $b \in M$.

If $\mathcal{M} \vDash \phi(\bar{a})$ we say that $\mathcal{M}$ *satisfies* $\phi(\bar{a})$ or $\phi(\bar{a})$ is *true* in $\mathcal{M}$.

Definable set

Definition A.7. Let $\mathcal{M}$ be an L-structure. Then a subset $X \subseteq M^n$ is called *definable* if there exists a formula $\phi(v_1, \dots v_n, w_1, \dots, w_m)$ and $\bar{b} \in M^m$ such that $X = \left\{\bar{a} \in M^n | \mathcal{M} \vDash \phi\left(\bar{a}, \bar{b}\right)\right\}$. We say that $\phi\left(\bar{v}, \bar{b}\right)$ *defines* X.

We also say that X is A-*definable* or *definable over* A or *definable with parameters from* A, if there is a formula $\psi(\bar{v}, w_1, \dots w_l)$ and $\bar{b} \in A^l$ such that $\psi\left(\bar{v}, \bar{b}\right)$ defines X.

Definition A.8. We say that two L-structures $\mathcal{M}$ and $\mathcal{N}$ are *elementarily equivalent* if,

Elementary equivalence

$$\mathcal{M} \vDash \phi \text{ if and only if } \mathcal{N} \vDash \phi$$

for all L-sentences ϕ. We write then, $\mathcal{M} \equiv \mathcal{N}$.

Definition A.9. If $\mathcal{M}$ and $\mathcal{N}$ are L-structures, then an L-embedding $j : \mathcal{M} \to \mathcal{N}$ (a map that preserves relations, constants, and functions) is called *elementary embedding* if,

Elementary embedding

$$\mathcal{M} \vDash \phi\left(a_1, \ldots, a_n\right) \iff \mathcal{N} \vDash \phi\left(j\left(a_1\right), \ldots, j\left(a_n\right)\right)$$

for all L-formulas $\phi\left(v_1, \ldots, v_n\right)$ and all $a_1, \ldots, a_n \in M$.

If $\mathcal{M}$ is a substructure of $\mathcal{N}$, we say that it is an *elementary substructure* and write $\mathcal{M} \prec \mathcal{N}$ if the inclusion map is elementary. We also say that $\mathcal{N}$ is an *elementary extension* of $\mathcal{M}$.

Elementary extension

Theorem A.10. [Tarski-Vaught Test] *Suppose that $\mathcal{M}$ is a substructure of $\mathcal{N}$. Then, $\mathcal{M}$ is an elementary substructure if and only if, for any formula $\phi\left(v, \bar{w}\right)$ and $\bar{a} \in M$, if there is $b \in N$ such that $\mathcal{N} \vDash \phi\left(b, \bar{a}\right)$, then there is $c \in M$ such that $\mathcal{N} \vDash \phi\left(c, \bar{a}\right)$.*

Definition A.11. If $\mathcal{M}$, $\mathcal{N}$ are L-structures and $B \subseteq M$, we say that $f : B \to N$ is a *partial elementary map* if and only if

Partial elementary map

$$\mathcal{M} \vDash \phi\left(\bar{b}\right) \iff \mathcal{N} \vDash \phi\left(f\left(\bar{b}\right)\right)$$

for all L-formulas ϕ and all finite sequences $\bar{b}$ from B.

Theorem A.12. *If $\mathcal{M}$, $\mathcal{N}$ are L-structures, $B \subseteq M$ and $f : B \to N$ is a partial elementary map, then there is $\mathcal{N}'$ an elementary extension of $\mathcal{N}$ ($\mathcal{N} \prec \mathcal{N}'$) and an elementary embedding $g : \mathcal{M} \to \mathcal{N}'$ which extending f.*

A.2 TYPES

Let $\mathcal{M}$ be an L-structure and $A \subseteq M$. Let L_A be the language obtained by adding to L constant symbols for all $a \in A$.

Definition A.13. Let p be a set of L_A-formulas in free variables $v_1, \ldots, v_n$. We cal p an *n-type* if $p \cup \mathrm{Th}_A(\mathcal{M})$ is satisfiable. We say that p is a *complete n-type* if $\phi \in p$ or $\neg\phi \in p$ for all L_A-formulas ϕ with free variables form $v_1, \ldots, v_n$. We let $S_n^{\mathcal{M}}(A)$ denote the set of all complete n-types.

Type

Complete type

Let $\mathrm{tp}^{\mathcal{M}}(a_1, \ldots, a_n/A) = \{\phi(v_1, \ldots, v_n) \in L_A | \mathcal{M} \models \phi(a_1, \ldots, a_n)\}$. Then, $\mathrm{tp}^{\mathcal{M}}(a_1, \ldots, a_n/A)$ is a complete n-type. We write $\mathrm{tp}^{\mathcal{M}}(\bar{a})$ for $\mathrm{tp}^{\mathcal{M}}(\bar{a}/\emptyset)$.

There is a natural topology on the space of complete n-types $S_n^{\mathcal{M}}(A)$. For ϕ an L_A-formula with free variables among the $v_1, \ldots, v_n$, let

$$[\phi] := \left\{p \in S_n^{\mathcal{M}}(A) \,|\, \phi \in p\right\}.$$

If p is a complete type and $\phi \vee \psi \in p$, then $\phi \in p$ or $\psi \in p$. Thus $[\phi \vee \psi] = [\phi] \cup [\psi]$. Similarly, $[\phi \wedge \psi] = [\phi] \cap [\psi]$.

Stone topology

The *Stone topology* on $S_n^{\mathcal{M}}(A)$ is the topology generated by taking the sets $[\phi]$ as basic open sets. For complete types p, exactly one of ϕ and $\neg\phi$ is in p. Thus, $[\phi] = S_n^{\mathcal{M}}(A) \setminus [\neg\phi]$ is also closed.

Proposition A.14.

1. *If $A \subseteq B \subset M$ and $p \in S_n^{\mathcal{M}}(B)$, let $p|A$ be the set of L_A-formulas in p. Then $p|A \in S_n^{\mathcal{M}}(A)$ and $\sigma : p \mapsto p|A$ is a continuous map from $S_n^{\mathcal{M}}(B)$ onto $S_n^{\mathcal{M}}(A)$.*

2. *If $f : \mathcal{M} \to \mathcal{N}$ is an elementary embedding and $p \in S_n^{\mathcal{M}}(A)$, let*

$$f(p) := \{\phi(\bar{v}, f(\bar{a}) \,|\, \phi(\bar{v}, \bar{a}) \in p)\}.$$

Then, $f(p) \in S_n^{\mathcal{N}}(f(A))$ and the map $\tau : p \mapsto f(p)$ is continuous, 1-1 and onto.

Proof. 1. Because $p|A \cup \mathrm{Th}_A(\mathcal{M}) \subseteq p \cup \mathrm{Th}_B(\mathcal{M})$, $p|A \cup \mathrm{Th}_A(\mathcal{M})$ is satisfiable. Therefore $p|A$ is a type. Now because $p|A$ is the set of all L_A-formulas in p, and p is a complete type, $p|A$ is also complete. Thus, $p|A \in S_n^{\mathcal{M}}(A)$.

For the continuity of σ, let ϕ be an L_A-formula. Then $[\phi]$ is a basic open set of $S_n^{\mathcal{M}}(A)$, and

$$[\phi] = \left\{q \in S_n^{\mathcal{M}}(A) \,|\, \phi \in q\right\} = \left\{q \in S_n^{\mathcal{M}}(B) \,|\, \phi \in q\right\}.$$

Hence the inverse image of an open set of $S_n^{\mathcal{M}}(A)$ through σ will be an open set of $S_n^{\mathcal{M}}(B)$, and σ will be continuous.

To show that σ is onto, let $q \in S_n^{\mathcal{M}}(A)$. Then, there is an elementary extension $\mathcal{N}$ of $\mathcal{M}$ and $\bar{a} \in N^n$ realizing q. Then $p = \mathrm{tp}^{\mathcal{N}}(\bar{a}/B) \in S_n^{\mathcal{M}}(B)$ and $p|A = q$. Thus σ is surjective.

2. Let $\Delta = \{\phi_1(\bar{v}, f(\bar{a})), \ldots, \phi_n(\bar{v}, f(\bar{a}))\}$ be a finite subset of $f(p)$. Because $p \cup \mathrm{Th}_A(\mathcal{M})$ is satisfiable,

$$\mathcal{M} \models (\exists \bar{v}) \left(\bigwedge_{i=1}^{n} \phi_i(\bar{v}, \bar{a})\right)$$

and because f is elementary embedding,

$$\mathcal{N} \models (\exists \bar{v}) \left(\bigwedge_{i=1}^{n} \phi_i(\bar{v}, f(\bar{a}))\right).$$

Therefore, $\Delta \cup \mathrm{Th}_{f(A)}(\mathcal{N})$ is satisfiable and thus, $f(p) \cup \mathrm{Th}_{f(A)}(\mathcal{N})$ is satisfiable, and $f(p)$ is a type. It is also complete because p is. Hence $f(p) \in S_n^{\mathcal{N}}(f(A))$.

For continuity, note that, for

$$[\phi(\bar{v}, f(\bar{a}))] = \left\{f(p) \in S_n^{\mathcal{N}}(f(A)) \,|\, \phi(\bar{v}, f(\bar{a})) \in f(p)\right\}$$

which is a basic open set of $S_n^{\mathcal{N}}(f(A))$, we have that

$$\tau^{-1}([\phi(\bar{v}, f(\bar{a}))]) = \left\{p \in S_n^{\mathcal{M}}(A) \,|\, \phi(\bar{v}, \bar{a}) \in p\right\} = [\phi(\bar{v}, \bar{a})]$$

which is basic open in $S_n^{\mathcal{M}}(A)$. Therefore f is continuous.

To see that τ is 1-1, let $p_1, p_2 \in S_n^{\mathcal{M}}(A)$. Now we have that,

$$f(p_1) = \{\phi(\bar{v}, f(\bar{a}) \,|\, \phi(\bar{v}, \bar{a}) \in p_1)\}$$
$$f(p_2) = \{\phi(\bar{v}, f(\bar{a}) \,|\, \phi(\bar{v}, \bar{a}) \in p_2)\}$$

which means that, if $f(p_1) = f(p_2)$ then we must have that $p_1 = p_2$.

Finally we will show that τ is surjective. Let $p \in S_n^{\mathcal{N}}(f(A))$ and $p = \{\phi_1(\bar{v}, f(\bar{a})), \ldots, \phi_n(\bar{v}, f(\bar{a})), \ldots\}$. Now because $p \cup \mathrm{Th}_{f(A)}(\mathcal{N})$ is consistent, we have that $\{\phi_1(\bar{v}, f(\bar{a})), \ldots, \phi_n(\bar{v}, f(\bar{a}))\} \cup \mathrm{Th}_{f(A)}(\mathcal{N})$ will be also consistent. So,

$$\mathcal{N} \models (\exists \bar{v}) \left(\bigwedge_{i=1}^{n} \phi_i(\bar{v}, f(\bar{a}))\right)$$

and because f is elementary,

$$\mathcal{M} \models (\exists \bar{v}) \left(\bigwedge_{i=1}^{n} \phi_i(\bar{v}, \bar{a})\right).$$

Therefore, by compactness, $\{\phi_1(\bar{v}, \bar{a}), \ldots, \phi_n(\bar{v}, \bar{a}), \ldots\} \cup \mathrm{Th}_A(\mathcal{M})$ is consistent and $q = \{\phi_1(\bar{v}, \bar{a}), \ldots, \phi_n(\bar{v}, \bar{a}), \ldots\}$ is a type, and moreover complete. Finally $f(q) = p$.

□

Lemma A.15. *If $\mathcal{M}$, $\mathcal{N}$ are structures, $A \subseteq M$ and $f : A \to \mathcal{N}$ is partial elementary, then $S_n^{\mathcal{M}}(A)$ is homeomorphic to $S_n^{\mathcal{N}}(f(A))$.*

Proof. Recall first that a homeomorphism is a map between two topological spaces that is 1-1, onto, continuous and open. From proposition A.14 (2) we have that the map $\tau : p \mapsto f(p)$ is an 1-1, onto, continuous map between $S_n^{\mathcal{M}}(A)$ and $S_n^{\mathcal{N}}(f(A))$. Also by checking the proof we find that the hypothesis that f is an elementary embedding can be replaced by f being a partial elementary map from A to $\mathcal{N}$. To finish our proof we just need to show that τ is also an open map.

Let $[\phi(\bar{v}, \bar{a})]$ be a basic open set in $S_n^{\mathcal{M}}(A)$. Then,

$$\tau([\phi(\bar{v}, \bar{a})]) = \left\{f(p) \in S_n^{\mathcal{N}}(f(A)) \,|\, \phi(\bar{v}, f(\bar{a})) \in f(p)\right\} = [\phi(\bar{v}, f(\bar{a}))].$$

So $\tau([\phi(\bar{v}, \bar{a})])$ is a basic open set in $S_n^{\mathcal{N}}(f(A))$ and τ is open. □

Lemma A.16. *Suppose that* $\left(\bar{a},\bar{b}\right) \in M^{m+n}$ *realizes an isolated type in* $S_{m+n}(T)$*, then* $\bar{a}$ *realizes an isolated type in* $S_m(T)$*. Indeed, if* $A \subseteq M$ *and* $\left(\bar{a},\bar{b}\right) \in M^{m+n}$ *realizes an isolated type in* $S^{\mathcal{M}}_{m+n}(A)$*, then* $tp^{\mathcal{M}}(\bar{a}/A)$ *is isolated.*

Proof. Let $\phi(\bar{v},\bar{w})$ isolate $\mathrm{tp}^{\mathcal{M}}\left(\bar{a},\bar{b}/A\right)$. We claim that $\exists\bar{w}\phi(\bar{v},\bar{w})$ isolates $\mathrm{tp}^{\mathcal{M}}(\bar{a}/A)$. Let $\psi(\bar{v})$ be any L_A-formula, such that, $\mathcal{M} \vDash \psi(\bar{a})$. We must show that,

$$\mathrm{Th}_A(\mathcal{M}) \vDash (\exists\bar{w}\phi(\bar{v},\bar{w})) \to \psi(\bar{v})\,.$$

Suppose not. Then, there is $\bar{c} \in M^m$ such that

$$\mathcal{M} \vDash \exists\bar{w}\left(\phi(\bar{c},\bar{w}) \wedge \neg\psi(\bar{c})\right).$$

Let $\bar{d} \in M^n$ such that $\mathcal{M} \vDash \phi\left(\bar{c},\bar{d}\right) \wedge \neg\psi(\bar{c})$. Because $\phi(\bar{v},\bar{w})$ isolates $\mathrm{tp}^{\mathcal{M}}\left(\bar{a},\bar{b}/A\right)$,

$$\mathrm{Th}_A(\mathcal{M}) \vDash \phi(\bar{v},\bar{w}) \to \psi(\bar{v})\,.$$

This is a contradiction because

$$\psi(\bar{v}) \in \mathrm{tp}^{\mathcal{M}}(\bar{a}/A) \subset \mathrm{tp}^{\mathcal{M}}\left(\bar{a},\bar{b}/A\right).$$

□

Lemma A.17. *Suppose that* $A \subset M$*,* $\bar{a},\bar{b} \in M$ *such that* $tp^{\mathcal{M}}\left(\bar{a},\bar{b}/A\right)$ *is isolated. Show that* $tp^{\mathcal{M}}\left(\bar{a}/A,\bar{b}\right)$ *is isolated.*

BIBLIOGRAPHY

[1] S Samson Adepoju Adeleke and Peter M Neumann. *Relations related to betweenness: their structure and automorphisms*, volume 623. AMS Bookstore, 1998. (Cited on pages 71 and 72.)

[2] Roman Arefiev. On monotonicity for weakly o-minimal structures. *preprint*, 1997. (Cited on page 67.)

[3] Yerzhan Baisalov and Bruno Poizat. Paires de structures o-minimales. *The Journal of Symbolic Logic*, 63(2):570–578, 1998.

[4] Bektur Sembiuly Baizhanov. Expansion of a model of a weakly o-minimal theory by a family of unary predicates. *Journal of Symbolic Logic*, pages 1382–1414, 2001.

[5] Chen Chung Chang and H Jerome Keisler. *Model theory*. Access Online via Elsevier, 1990.

[6] Gregory Cherlin and Max A Dickmann. Real closed rings ii. model theory. *Annals of pure and applied logic*, 25(3):213–231, 1983.

[7] Jan Denef. P-adic semi-algebraic sets and cell decomposition. *J. Reine Angew. Math*, 369:154–166, 1986.

[8] Max A Dickmann. Elimination of quantifiers for ordered valuation rings. *The Journal of Symbolic Logic*, 52(1):116–128, 1987.

[9] Deirdre Haskell and Dugald Macpherson. Cell decompositions of c-minimal structures. *Annals of Pure and Applied Logic*, 66(2): 113–162, 1994. (Cited on pages 72, 73, and 74.)

[10] Deirdre Haskell and Dugald Macpherson. A version of o-minimality for the p-adics. *The Journal of Symbolic Logic*, 62 (4):1075–1092, 1997. (Cited on page 75.)

[11] Deirdre Haskell, Anand Pillay, and Charles Steinhorn. *Model theory, algebra, and geometry*, volume 39. Cambridge University Press, 2000.

[12] Julia F Knight, Anand Pillay, and Charles Steinhorn. Definable sets in ordered structures. ii. *Transactions of the American Mathematical Society*, 295(2):593–605, 1986. (Cited on pages 3, 12, 20, and 26.)

[13] Dugald Macpherson. Notes on o-minimality and variations. *Model theory, algebra, and geometry*, 39:97–130, 1999. (Cited on page 57.)

[14] Dugald Macpherson and Charles Steinhorn. On variants of o-minimality. *Annals of Pure and Applied Logic*, 79(2):165–209, 1996. (Cited on pages 71, 72, 73, and 74.)

[15] Dugald Macpherson, David Marker, and Charles Steinhorn. Weakly o-minimal structures and real closed fields. *Transactions of the American Mathematical Society*, 352(12):5435–5483, 2000. (Cited on pages 67, 68, 69, and 70.)

[16] David Marker. Omitting types in o-minimal theories. *The Journal of Symbolic Logic*, 51(1):63–74, 1986. (Cited on page 53.)

[17] David Marker. *Model theory: an introduction*. Springer, 2002.

[18] David Marker and Charles I Steinhorn. Definable types in o-minimal theories. *The Journal of Symbolic Logic*, 59(1):185–198, 1994. (Cited on pages 51 and 55.)

[19] Alan Mekler, Matatyahu Rubin, and Charles Steinhorn. Dedekind completeness and the algebraic complexity of o-minimal structures. *Canad. J. Math*, 44(4):843–855, 1992. (Cited on page 60.)

[20] Chris Miller and Sergei Starchenko. A growth dichotomy for o-minimal expansions of ordered groups. *Transactions of the American Mathematical Society*, 350(9):3505–3521, 1998. (Cited on page 61.)

[21] Marie-Hélène Mourgues. Cell decomposition for p-minimal fields. *Mathematical Logic Quarterly*, 55(5):487–492, 2009. (Cited on page 74.)

[22] Ya'acov Peterzil and Sergei Starchenko. A trichotomy theorem for o-minimal structures. *Proceedings of the London Mathematical Society*, 77(3):481–523, 1998. (Cited on pages 59 and 60.)

[23] Anand Pillay. Some remarks on definable equivalence relations in o-minimal structures. *The Journal of Symbolic Logic*, 51(3): 709–714, 1986.

[24] Anand Pillay. On groups and fields definable in o-minimal structures. *Journal of Pure and Applied Algebra*, 53(3):239 – 255, 1988. ISSN 0022-4049. doi: http://dx.doi.org/10.1016/0022-4049(88)90125-9. URL http://www.sciencedirect.com/science/article/pii/0022404988901259. (Cited on page 30.)

[25] Anand Pillay. *An introduction to stability theory*. Dover Publications, 2011.

[26] Anand Pillay and Charles Steinhorn. Definable sets in ordered structures. i. *Transactions of the American Mathematical Society*, 295(2):565–592, 1986. (Cited on pages 7, 12, 39, 53, and 57.)

[27] Anand Pillay and Charles Steinhorn. Definable sets in ordered structures. iii. *Transactions of the American Mathematical Society*, pages 469–476, 1988.

[28] Gerald E Sacks. *Saturated model theory.* World Scientific, 2010. (Cited on page 49.)

[29] Philip Scowcroft and Lou van den Dries. On the structure of semi-algebraic sets over p-adic fields. *The Journal of Symbolic Logic*, 53 (4):1138–1164, 1988.

[30] Saharon Shelah. Uniqueness and characterization of prime models over sets for totally transcendental first-order theories. *Journal of Symbolic Logic*, pages 107–113, 1972. (Cited on page 39.)

[31] Lou Van den Dries. A generalization of the tarski-seidenberg theorem, and some nondefinability results. *Bulletin of the American Mathematical Society*, 15(2):189–193, 1986.

[32] Lou van den Dries. Tarski's problem and pfaffian functions. *Studies in Logic and the Foundations of Mathematics*, 120:59–90, 1986. (Cited on page 51.)

[33] Lou Van Den Dries. T-convexity and tame extensions ii. *The Journal of Symbolic Logic*, 62(1):14–34, 1997.

[34] Lou Van den Dries. *Tame topology and o-minimal structures*, volume 248. Cambridge University Press, 1998. (Cited on pages 12 and 21.)

[35] Lou Van Den Dries and Adam H Lewenberg. T-convexity and tame extensions. *Journal of Symbolic Logic*, pages 74–102, 1995.

[36] Lou Van Den Dries, Deirdre Haskell, and Dugald Macpherson. One-dimensional p-adic subanalytic sets. *Journal of the London Mathematical Society*, 59(1):1–20, 1999. (Cited on page 75.)

[37] Volker Weispfenning. Quantifier elimination and decision procedures for valued fields. In *Models and sets*, pages 419–472. Springer, 1984. (Cited on page 73.)

[38] Alex J Wilkie. A theorem of the complement and some new o-minimal structures. *Selecta Mathematica, New Series*, 5(4):397–421, 1999. (Cited on page 3.)