o-Minimality and its Variations

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o-Minimality

Assumptions

- $\mathcal{M} = (M, <, \dots)$
- $<$ is dense, linear, without endpoints
- definability with parameters

Definition

The structure $\mathcal{M}$ is called *o-minimal* if every definable subset of M is a finite union of singletons and open intervals with endpoints in $M_\infty := M \cup \{-\infty, +\infty\}$.
A theory T is called *o-minimal* if every model $\mathcal{M}$ of T is o-minimal.

o-Minimality

The class of o-minimal structures is,

- closed under reducts (if $<$ still remains in the language)
- closed under expansions by constants

Some o-minimal structures

- $(\mathbb{Q}, <)$
- $(\mathbb{Q}, <, +)$
- $\mathcal{R} = (\mathbb{R}, <, +, -, \cdot, 0, 1)$

$(\mathbb{Q}, <, +, \cdot, 0, 1)$ is not o-minimal

The infinite discrete set of perfect squares is definable.

Monotonicity and Finiteness

Monotonicity Theorem [Pillay & Steinhorn, 1986]

Let $\mathcal{M}$ be an o-minimal structure and $f : (a, b) \to M$ be a definable function with domain (a, b)(possibly $a = -\infty$ or $b = +\infty$). Then, there are points $a = a_0 < a_1 < \cdots < a_{k+1}$ s.t. for each $j = 0, \ldots, k$, $f|_{(a_j, a_{j+1})}$ is either,

- constant, or
- a strictly monotonic and continuous bijection to an interval.

Finiteness Lemma [Knight *et al.* , 1986]

Let $A \subseteq M^2$ be definable and suppose that for each $x \in M$ the fiber $A_x := \{y \in M | (x, y) \in A\}$ is finite. Then there is $N < \omega$ s.t. $|A_x| \leq N$ for all $x \in M$.

Applications

Corollary

Let $f : (a, b) \to M$ be definale and continuous. Then f takes a maximum and minimum value on $[a, b]$.

Exchange Principle [Pillay & Steinhorn, 1986]

Let $\mathcal{M}$ be o-minimal. Let $b, c, a_1, \ldots, a_n \in \mathcal{M}$. If b is definable over $c, a_1, \ldots, a_n$, and b is not definable over $a_1, \ldots, a_n$, then c is definable over $b, a_1, \ldots, a_n$.

Motivation

Question

What happen with the definable subsets of M^n?

Notation

Given definable $X \subseteq M^n$

- $C(X) := \{f : X \to M | f \text{ is definable and continuous}\}$
- $C_\infty(X) := C(X) \cup \{-\infty, +\infty\}$
- for $f, g \in C_\infty(X)$ s.t. $(\forall \bar{x} \in X)(f(\bar{x}) < g(\bar{x}))$ then

$$(f, g)_X := \{(\bar{x}, y) \in X \times M | f(\bar{x}) < y < g(\bar{x})\}$$

Cells

Let $(i_1, \ldots, i_n)$ be a binary sequence. Then a $(i_1, \ldots, i_n)$-*cell* is a definable subset of M^n defined as follows,

- a (0)-cell is a singleton
- a (1)-cell is an open interval

Suppose that we have defined $(i_1, \ldots, i_{n-1})$-cells, then

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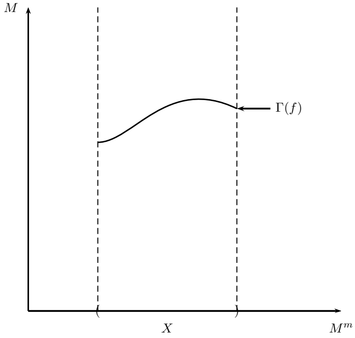
- a $(i_1, \ldots, i_{n-1}, 0)$-cell is the graph of some $f \in C(X)$, where X is a $(i_1, \ldots, i_{n-1})$-cell



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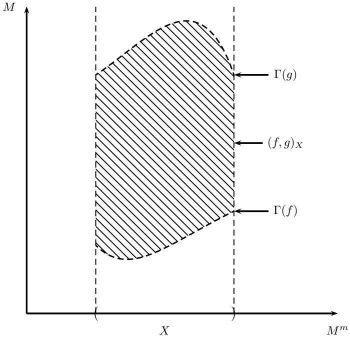
- a $(i_1, \ldots, i_{n-1}, 1)$-cell is the set $(f, g)_X$ where X is a $(i_1, \ldots, i_{n-1})$-cell and $f, g \in C_\infty(X)$



Decompositions

A *decomposition* of M^n is a partition of M^n into finitely many cells. It is defined recursively.

- A decomposition of M is a collection,

$$\{(-\infty, a_1), (a_1, a_2), \ldots, (a_k, +\infty), \{a_1\}, \ldots, \{a_k\}\}$$

  where $a_1 < \ldots < a_k$ are points in M;
- a decomposition of M^{n+1} is a finite partition of M^{n+1} into cells C s.t. the set of projections $\pi(C)$ is a decomposition of M^n.

A decomposition $\mathcal{D}$ of M^n is said to *partition* a set $S \subseteq M^n$ if S is a union of cells in $\mathcal{D}$.

Cell Decomposition Theorem

Uniform Finiteness Property [Knight *et al.* , 1986]

Let $Y \subseteq M^{n+1}$ be definable and also for every $\bar{x} \in M^n$,

$$Y_{\bar{x}} := \{r \in M | (\bar{x}, r) \in Y\}$$

is finite. Then there exists $N < \omega$ s.t. $Y_{\bar{x}} \leq N$ for all $\bar{x} \in M^n$.

Cell Decomposition Theorem [Knight *et al.* , 1986]

1. Given any definable sets $A_1, \ldots, A_k \subseteq M^n$ there is a decomposition of M^n partitioning each of $A_1, \ldots, A_k$.
2. For each definable function $f : A \to M$, $A \subseteq M^n$, there is a decomposition $\mathcal{D}$ of M^n partitioning A s.t. the restriction $f|B : B \to M$ to each cell $B \in \mathcal{D}$ with $B \subseteq A$ is continuous.

Applications

Theorem [Knight *et al.* , 1986]

If $\mathcal{M}$ is an o-minimal structure, the $\mathrm{Th}(\mathcal{M})$ is an o-minimal theory.

We will show that if $\mathcal{N} \equiv \mathcal{M}$ then $\mathcal{N}$ is o-minimal.

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Let $\phi(x_1, \ldots, x_m, y)$ be a formula.
By o-minimality for all $\bar{a} \in M^m$ the set,

$$\phi(\bar{a}, y)^M := y \in M | \mathcal{M} \vDash \phi(\bar{a}, y)$$

is finite union of singletons and intervals. Consider now $\psi(\bar{x}, z)$ s.t.

$$\psi(\bar{x}, z)^M := (\bar{x}, z) | z \in \mathrm{bd}(y | \mathcal{M} \vDash \phi(\bar{x}, y))$$

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- ★ boundary is definable
- ★ boundary is finite for definable subsets of M
- ★ ψ is finite over M^m

Applications

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If $\mathcal{M}$ is an o-minimal structure, the $\mathrm{Th}(\mathcal{M})$ is an o-minimal theory.

We will show that if $\mathcal{N} \equiv \mathcal{M}$ then $\mathcal{N}$ is o-minimal.
By UFP, $\psi(\bar{x}, z)$ will be uniformly finite over M^m.
Thus for some $\kappa < \omega$ we have that,

$$\mathcal{M} \vDash (\forall \bar{x})(\{y | \phi(\bar{x}, y)\} \text{ has at most } \kappa \text{ boundary points})$$

Therefore for any $\mathcal{N} \equiv \mathcal{M}$ and $\bar{a}$, $\phi(\bar{a}, y)^N$ has finite boundary points and so it will be a finite union of points and intervals. $\square$

Prime models

Definition

Let $\mathcal{M} \models T$ and $A \subseteq \mathcal{M}$. $\mathcal{M}$ is said to be *prime over* A if for any $\mathcal{M}' \models T$ with $A \subseteq \mathcal{M}'$, there is an elementary mapping $f : \mathcal{M} \to \mathcal{M}'$ that is identity on A.

Theorem [Pillay & Steinhorn, 1986]

Let $A \subseteq \mathcal{M} \models T$, where T is an o-minimal theory. Then there is an a model $\mathcal{M}' \models T$, $A \subseteq \mathcal{M}'$ that is prime over A, and is unique up to isomorphism over A.

Ressayre's Theorem

Definition

Let $\mathcal{M}$ be an L-structure and let $A \subseteq M$. Let δ be an ordinal and $(a_\alpha : \alpha < \delta)$ be a sequence of elements from M. Let $A_\alpha = A \cup \{a_\beta : \beta < \alpha\}$. We call $(a_\alpha : \alpha < \delta)$ a *construction* over A if $\mathrm{tp}(a_\alpha / A_\alpha)$ is isolated for all $\alpha < \delta$.

Theorem [Ressayre]

Suppose that $A \subseteq \mathbb{M}$, $\mathcal{M} \prec \mathbb{M}$, $\mathcal{N} \prec \mathbb{M}$, and $\mathcal{M}$ and $\mathcal{N}$ are constructible over A. Then the identity map on A extends to an isomorphism between $\mathcal{M}$ and $\mathcal{N}$.

Cuts and Noncuts

Let $\mathcal{M} \models T$. Let $p(x) \in S_1(M)$. We say that $p(x)$ is a *cut* over $\mathcal{M}$ iff there are non empty disjoint subsets C_0 and C_1 of M, s.t. $C_0 \cup C_1 = M$, C_0 has no greatest element, C_1 has no least element, for all $c \in C_0$, "$c < x$" $\in p(x)$, and for all $c \in C_1$, "$x < c$" $\in p(x)$.
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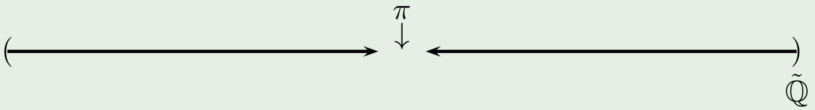
Example

Let $\tilde{\mathbb{Q}}$ be the field of real algebraic numbers. We know that $\pi \in \mathbb{R} \setminus \tilde{\mathbb{Q}}$, so $\mathrm{tp}(\pi, \tilde{\mathbb{Q}})$ is a cut.
If t is an infinite hyperreal or t is infinitesimally close to an element of $\tilde{\mathbb{Q}}$, then $\mathrm{tp}(t, \tilde{\mathbb{Q}})$ is a noncut.



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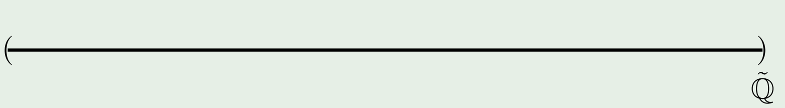

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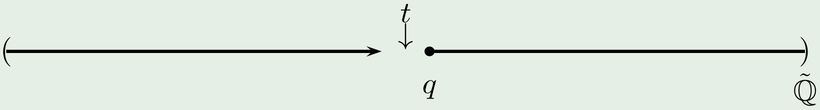

Cuts and Noncuts

Let $\mathcal{M} \models T$, where T is an o-minimal theory.

Theorem [Marker, 1986]

Let $\sigma(x) \in S_1(M)$ be a cut. Let $\tau(x) \in S_1(M)$ be a noncut, and b be a realization of $\tau(x)$. Then $\sigma(x)$ is omitted in $P(\mathcal{M} \cup \{b\})$.

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Ordered Groups and Fields

Theorem [Pillay & Steinhorn, 1986]

Let $\mathcal{G} = (G, +, 0, <)$ be an o-minimal ordered group. Then $\mathcal{G}$ is a divisible oredered abelian group.

Reminder

An abelian group G is divisible if, for every positive integer n and every $g \in G$, there exists $y \in G$ such that $ny = g$.

Theorem [Pillay & Steinhorn, 1986]

Let $\mathcal{R} = (R, +, \cdot, 1, <)$ be an o-minimal ordered ring. Then $\mathcal{R}$ is a real closed field.

Reminder

A field F is real closed if it is elementarily equivalent to $\mathbb{R}$.

Trichotomy Theorem

Question

Is there a sense in which an o-minimal structure is either "trivial", or group-like, or field-like?

Any answer to this has to be "local".

Trichotomy Theorem

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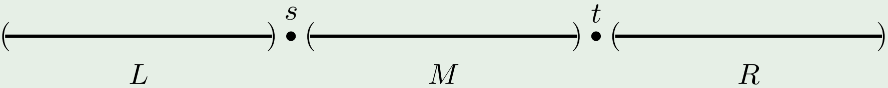
Is there a sense in which an o-minimal structure is either "trivial", or group-like, or field-like?

Any answer to this has to be "local".

Example

- L is carrying the structure of a pure dense linear order
- M is carrying the structure of a divisible abelian ordered group
- R is carrying the structure of a real closed field



Trichotomy Theorem

Trichotomy Theorem [Peterzil & Starchenko, 1998]

Let $\mathcal{M}$ be a sufficiently saturated o-minimal structure. Given an $a \in M$ one and only one of the following holds,

- a is trivial,
- the structure that $\mathcal{M}$ induces on some convex neighborhood of a is an ordered vector space over an ordered division ring.
- the structure that $\mathcal{M}$ induces over some open interval around a is an o-minimal expansion of a real closed field.

Weakly o-minimal structures

Let $\mathcal{M} = (M, <, \dots)$ be a linearly ordered structure.

Definition

A set $C \subseteq M$ is called *convex* , if for any $a, b \in C$ with $a < b$, and $c \in M$ such that $a < c < b$, then $c \in C$.

Definition

A structure $\mathcal{M}$ will be called *weakly o-minimal* , if the definable subsets of $\mathcal{M}$ are finite unions of convex sets in $(M, <)$.
We say that a complete theory T is *weakly o-minimal* if every model of T is weakly o-minimal.

Theorem

Expanding an o-minimal structure with unary predicates for convex subsets yields a structure with weakly o-minimal theory.

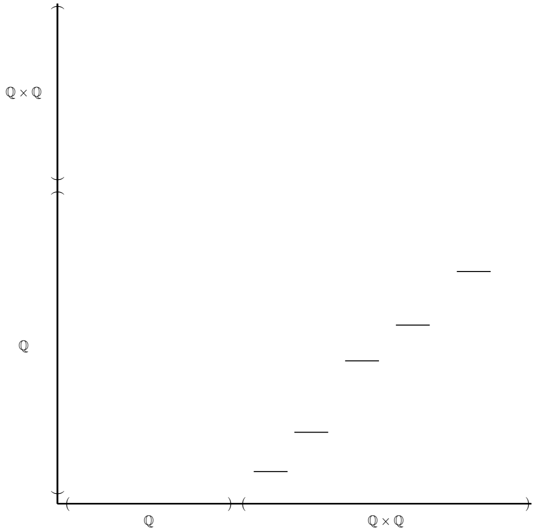
Monotonicity

Let $\mathcal{M} = (M, <, P, Q, f)$ such that.

- M is the disjoint union of the interpretations of the unary relations P and Q
- P is the interpretation of $\mathbb{Q}$ with the usual order
- Q is the interpretation of $\mathbb{Q} \times \mathbb{Q}$, lexicographically ordered
- P proeceds Q in $<$ on M
- $f : Q \to P$, $f((n, m)) = n$ for all $n, m \in \mathbb{Q}$

M is weakly o-minimal and also $\text{Th}(\mathcal{M})$ is weakly o-minimal.

Monotonicity



Weakly o-minimal structures

- ★ We have a local monotonicity theorem. [Arefiev, 1997]
- ★ Weakly o-minimal structures do not neccesserily have weakly o-minimal theory.
- ★ Weakly o-minimal structures do not neccesserily have prime models.

Theorem [Macpherson *et al.* , 2000]

Every weakly o-minimal ordered group is divisible and abelian.

Theorem [Macpherson *et al.* , 2000]

Every weakly o-minimal ordered field is real closed.

A definition for minimality

Let $L \subset L^+$ be languages, and $\mathcal{K}$ be an elementary class of L-structures.

Definition [Macpherson & Steinhorn, 1996]

An L^+-structure $\mathcal{M}$ is $\mathcal{K}$-*minimal* if the the reduct $\mathcal{M}|_L$ is in $\mathcal{K}$ and every L^+-definable subset of M is definable by a quantifier-free L-formula.
A complete L^+-theory is $\mathcal{K}$-*minimal* if all its models are $\mathcal{K}$-minimal.

- ★ o-minimality is a special case of the above definition but not weak o-minimality.
- ★ $\mathcal{K}$-minimality is closed under reducts to languages containg L, and under expansion by constants.

C-minimality

Let $C(x;y,z)$ be a ternary realation, $L = \{C\}$, and $\mathcal{K}_C$ be the class of L-structures satisfying the following axioms.

- $(\forall xyz)[C(x;y,z) \to C(x;z,y)]$
- $(\forall xyz)[C(x;y,z) \to C(y;x,z)]$
- $(\forall xyzw)[C(x;y,z) \to (C(w;y,z) \vee C(x;w,z))]$
- $(\forall xy)[x \neq y \to (\exists z \neq y)C(x;y,z)]$
- $(\exists xy)(x \neq y)$

Definition

A structure $\mathcal{M} = (M, C, \ldots)$ is *C-minimal* if its theory is $\mathcal{K}_C$-minimal.

P-minimality

Definition

Let $L = (+, -, \cdot, 0, 1, (P_n)_{n>1})$, where P_n are unary predicates. Regard $\mathbb{Q}_p$ as an L-structure, letting P_n picking the n^{th} powers in $\mathbf{Q_p}$. Let $\mathcal{K}_P$ be the class of L-structures elementarily equivalent to $\mathbb{Q}_p$. Then if $L^+ \supseteq L$, an L^+-structure is *P-minimal* if all models of its theory are $\mathcal{K}_P$-minimal

Summary

	o-minimal	weakly	C-minimal	P-minimal
Monotonicity	✓	local	✓	✓
CDT	✓	✓	✓	iff it has Skolem functions[1]
Prime Model	✓	✗	✗	✗
Groups	DAG	DAG		
Fields	RCF	RCF	ACVF	
Exchange	✓	✗	✗	✓
IP	✗[2]	✗[3]	✗[4]	✗[5]

[1] Mourgues [2009]
[2] Pillay & Steinhorn [1986]
[3] Macpherson *et al.* [2000]
[4] Macpherson & Steinhorn [1996]
[5] Haskell & Macpherson [1997]

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