o-Minimality and its Variations

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o-Minimality

Assumptions

- $\mathcal{M} = (M, <, \ldots)$
- $<$ is dense, linear, without endpoints
- definability with parameters

Definition

The structure $\mathcal{M}$ is called *o-minimal* if every definable subset of M is a finite union of singletons and open intervals with endpoints in $M_\infty := M \cup \{-\infty, +\infty\}$.
A theory T is called *o-minimal* if every model $\mathcal{M}$ of T is o-minimal.

o-Minimality

The class of o-minimal structures is,

- closed under reducts (if $<$ still remains in the language)
- closed under expansions by constants

Some o-minimal structures

- $(\mathbb{Q}, <)$
- $(\mathbb{Q}, <, +)$
- $\mathcal{R} = (\mathbb{R}, <, +, -, \cdot, 0, 1)$

$(\mathbb{Q}, <, +, \cdot, 0, 1)$ is not o-minimal

The infinite discrete set of perfect squares is definable.

Monotonicity and Finiteness

Monotonicity Theorem

Let $\mathcal{M}$ be an o-minimal structure and $f : (a, b) \to M$ be a definable function with domain (a, b)(possibly $a = -\infty$ or $b = +\infty$). Then, there are points $a = a_0 < a_1 < \cdots < a_{k+1}$ s.t. for each $j = 0, \ldots, k$, $f|_{(a_j, a_{j+1})}$ is either,

- constant, or
- a strictly monotonic and continuous bijection to an interval.

Finiteness Lemma

Let $A \subseteq M^2$ be definable and suppose that for each $x \in M$ the fiber $A_x := \{y \in M | (x, y) \in A\}$ is finite. Then there is $N < \omega$ s.t. $|A_x| \leq N$ for all $x \in M$.

Applications

Corollary

Let $f : (a, b) \to M$ be definale and continuous. Then f takes a maximum and minimum value on $[a, b]$.

Exchange Principle

Let $\mathcal{M}$ be o-minimal. Let $b, c, a_1, \ldots, a_n \in \mathcal{M}$. If b is definable over $c, a_1, \ldots, a_n$, and b is not definable over $a_1, \ldots, a_n$, then c is definable over $b, a_1, \ldots, a_n$.