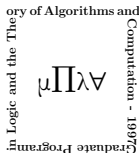


o-Minimality and its Variations

Vagios Vlachos



Graduate Program in Logic and Theory of Algorithms and Computation

2014

o-Minimality

Assumptions

- $\mathcal{M} = (M, <, \dots)$
- $<$ is dense, linear, without endpoints
- definability with parameters

Definition

The structure \mathcal{M} is called *o-minimal* if every definable subset of M is a finite union of singletons and open intervals with endpoints in $M_\infty := M \cup \{-\infty, +\infty\}$.

A theory T is called *o-minimal* if every model \mathcal{M} of T is o-minimal.

o-Minimality

The class of o-minimal structures is,

- closed under reducts (if $<$ still remains in the language)
- closed under expansions by constants

Some o-minimal structures

- $(\mathbb{Q}, <)$
- $(\mathbb{Q}, <, +)$
- $\mathcal{R} = (\mathbb{R}, <, +, -, \cdot, 0, 1)$

$(\mathbb{Q}, <, +, \cdot, 0, 1)$ is not o-minimal

The infinite discrete set of perfect squares is definable.

Monotonicity and Finiteness

Monotonicity Theorem [Pillay & Steinhorn, 1986]

Let \mathcal{M} be an o-minimal structure and $f : (a, b) \rightarrow M$ be a definable function with domain (a, b) (possibly $a = -\infty$ or $b = +\infty$). Then, there are points $a = a_0 < a_1 < \dots < a_{k+1}$ s.t. for each $j = 0, \dots, k$, $f|_{(a_j, a_{j+1})}$ is either,

- constant, or
- a strictly monotonic and continuous bijection to an interval.

Finiteness Lemma [Knight *et al.* , 1986]

Let $A \subseteq M^2$ be definable and suppose that for each $x \in M$ the fiber $A_x := \{y \in M \mid (x, y) \in A\}$ is finite. Then there is $N < \omega$ s.t. $|A_x| \leq N$ for all $x \in M$.

Applications

Corollary

Let $f : (a, b) \rightarrow M$ be definable and continuous. Then f takes a maximum and minimum value on $[a, b]$.

Exchange Principle [Pillay & Steinhorn, 1986]

Let \mathcal{M} be o-minimal. Let $b, c, a_1, \dots, a_n \in \mathcal{M}$. If b is definable over c, a_1, \dots, a_n , and b is not definable over a_1, \dots, a_n , then c is definable over b, a_1, \dots, a_n .

Motivation

Question

What happens with the definable subsets of M^n ?

Notation

Given a definable $X \subseteq M^n$

- $C(X) := \{f : X \rightarrow M \mid f \text{ is definable and continuous}\}$
- $C_\infty(X) := C(X) \cup \{-\infty, +\infty\}$
- for $f, g \in C_\infty(X)$ s.t. $(\forall \bar{x} \in X)(f(\bar{x}) < g(\bar{x}))$ then

$$(f, g)_X := \{(\bar{x}, y) \in X \times M \mid f(\bar{x}) < y < g(\bar{x})\}$$

Cells

Let (i_1, \dots, i_n) be a binary sequence. Then a (i_1, \dots, i_n) -*cell* is a definable subset of M^n defined as follows,

- a (0) -cell is a singleton
- a (1) -cell is an open interval

Suppose that we have defined (i_1, \dots, i_{n-1}) -cells, then

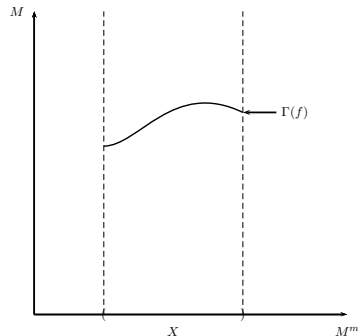
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- a $(i_1, \dots, i_{n-1}, 0)$ -cell is the graph of some $f \in C(X)$, where X is a (i_1, \dots, i_{n-1}) -cell



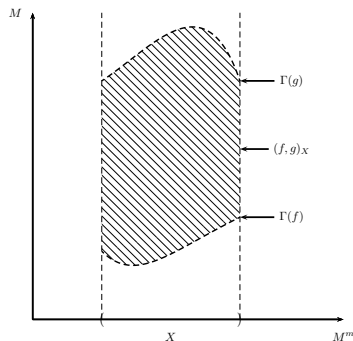
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- a $(i_1, \dots, i_{n-1}, 1)$ -cell is the set $(f, g)_X$ where X is a (i_1, \dots, i_{n-1}) -cell and $f, g \in C_\infty(X)$



Decompositions

A *decomposition* of M^n is a partition of M^n into finitely many cells. It is defined recursively.

- A decomposition of M is a collection,

$$\{(-\infty, a_1), (a_1, a_2), \dots, (a_k, +\infty), \{a_1\}, \dots, \{a_k\}\}$$

where $a_1 < \dots < a_k$ are points in M ;

- a decomposition of M^{n+1} is a finite partition of M^{n+1} into cells C s.t. the set of projections $\pi(C)$ is a decomposition of M^n .

A decomposition \mathcal{D} of M^n is said to *partition* a set $S \subseteq M^n$ if S is a union of cells in \mathcal{D} .

Cell Decomposition Theorem

Uniform Finiteness Property [Knight *et al.* , 1986]

Let $Y \subseteq M^{n+1}$ be definable and also for every $\bar{x} \in M^n$,

$$Y_{\bar{x}} := \{r \in M \mid (\bar{x}, r) \in Y\}$$

is finite. Then there exists $N < \omega$ s.t. $|Y_{\bar{x}}| \leq N$ for all $\bar{x} \in M^n$.

Cell Decomposition Theorem [Knight *et al.* , 1986]

- ① Given any definable sets $A_1, \dots, A_k \subseteq M^n$ there is a decomposition of M^n partitioning each of A_1, \dots, A_k .
- ② For each definable function $f : A \rightarrow M$, $A \subseteq M^n$, there is a decomposition \mathcal{D} of M^n partitioning A s.t. the restriction $f|_B : B \rightarrow M$ to each cell $B \in \mathcal{D}$ with $B \subseteq A$ is continuous.

Applications

Theorem [Knight *et al.* , 1986]

If \mathcal{M} is an o-minimal structure, then $\text{Th}(\mathcal{M})$ is an o-minimal theory.

We will show that if $\mathcal{N} \equiv \mathcal{M}$ then \mathcal{N} is o-minimal.

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Let $\phi(x_1, \dots, x_m, y)$ be a formula.

By o-minimality for all $\bar{a} \in M^m$ the set,

$$\phi(\bar{a}, y)^M := \{y \in M \mid \mathcal{M} \models \phi(\bar{a}, y)\}$$

is finite union of singletons and intervals. Consider now $\psi(\bar{x}, z)$ s.t.

$$\psi(\bar{x}, z)^M := \{(\bar{x}, z) \mid z \in \text{bd}(y \mid \mathcal{M} \models \phi(\bar{x}, y))\}$$

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- ★ boundary is definable
- ★ boundary is finite for definable subsets of M
- ★ ψ is finite over M^m

Applications

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If \mathcal{M} is an o-minimal structure, then $\text{Th}(\mathcal{M})$ is an o-minimal theory.

We will show that if $\mathcal{N} \equiv \mathcal{M}$ then \mathcal{N} is o-minimal.

By UFP, $\psi(\bar{x}, z)$ will be uniformly finite over M^m .

Thus for some $\kappa < \omega$ we have that,

$$\mathcal{M} \models (\forall \bar{x})(\{y | \phi(\bar{x}, y)\} \text{ has at most } \kappa \text{ boundary points})$$

Therefore for any $\mathcal{N} \equiv \mathcal{M}$ and \bar{a} , $\phi(\bar{a}, y)^N$ has finite boundary points and so it will be a finite union of points and intervals. □

Prime models

Definition

Let $\mathcal{M} \models T$ and $A \subseteq \mathcal{M}$. \mathcal{M} is said to be *prime over A* if for any $\mathcal{M}' \models T$ with $A \subseteq \mathcal{M}'$, there is an elementary mapping $f : \mathcal{M} \rightarrow \mathcal{M}'$ that is identity on A .

Theorem [Pillay & Steinhorn, 1986]

Let $A \subseteq \mathcal{M} \models T$, where T is an o-minimal theory. Then there is a model $\mathcal{M}' \models T$, $A \subseteq \mathcal{M}'$ that is prime over A , and is unique up to isomorphism over A .

Ressayre's Theorem

Definition

Let \mathcal{M} be an L -structure and let $A \subseteq M$. Let δ be an ordinal and $(a_\alpha : \alpha < \delta)$ be a sequence of elements from M . Let $A_\alpha = A \cup \{a_\beta : \beta < \alpha\}$. We call $(a_\alpha : \alpha < \delta)$ a *construction* over A if $\text{tp}(a_\alpha/A_\alpha)$ is isolated for all $\alpha < \delta$.

Theorem [Ressayre]

Suppose that $A \subseteq \mathbb{M}$, $\mathcal{M} \prec \mathbb{M}$, $\mathcal{N} \prec \mathbb{M}$, and \mathcal{M} and \mathcal{N} are constructible over A . Then the identity map on A extends to an isomorphism between \mathcal{M} and \mathcal{N} .

Cuts and Noncuts

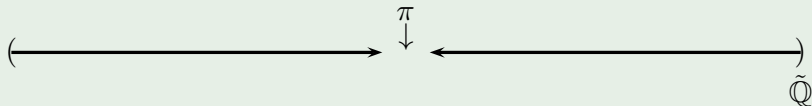
Let $\mathcal{M} \models T$. Let $p(x) \in S_1(M)$. We say that $p(x)$ is a *cut* over \mathcal{M} iff there are non empty disjoint subsets C_0 and C_1 of M , s.t.
 $C_0 \cup C_1 = M$, C_0 has no greatest element, C_1 has no least element, for all $c \in C_0$, “ $c < x$ ” $\in p(x)$, and for all $c \in C_1$, “ $x < c$ ” $\in p(x)$.
If $p(x)$ is not isolated and is not a cut, we call $p(x)$ a *noncut*.

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Example

Let $\tilde{\mathbb{Q}}$ be the field of real algebraic numbers. We know that $\pi \in \mathbb{R} \setminus \tilde{\mathbb{Q}}$, so $\text{tp}(\pi, \tilde{\mathbb{Q}})$ is a cut.
 If t is an infinite hyperreal or t is infinitesimally close to an element of $\tilde{\mathbb{Q}}$, then $\text{tp}(t, \tilde{\mathbb{Q}})$ is a noncut.



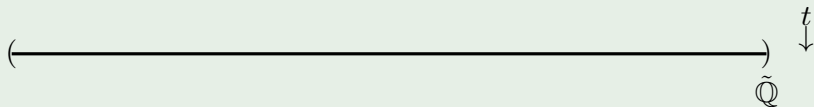
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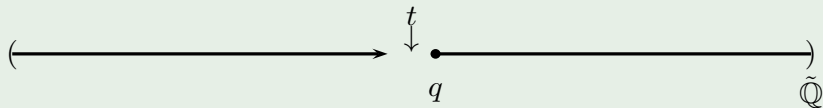


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Cuts and Noncuts

Let $\mathcal{M} \models T$, where T is an o-minimal theory.

Theorem [Marker, 1986]

Let $\sigma(x) \in S_1(M)$ be a cut. Let $\tau(x) \in S_1(M)$ be a noncut, and b be a realization of $\tau(x)$. Then $\sigma(x)$ is omitted in $P(\mathcal{M} \cup \{b\})$.

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Ordered Groups and Fields

Theorem [Pillay & Steinhorn, 1986]

Let $\mathcal{G} = (G, +, 0, <)$ be an o-minimal ordered group. Then \mathcal{G} is a divisible ordered abelian group.

Reminder

An abelian group G is divisible if, for every positive integer n and every $g \in G$, there exists $y \in G$ such that $ny = g$.

Theorem [Pillay & Steinhorn, 1986]

Let $\mathcal{R} = (R, +, \cdot, 1, <)$ be an o-minimal ordered ring. Then \mathcal{R} is a real closed field.

Reminder

A field F is real closed if it is elementarily equivalent to \mathbb{R} .

Trichotomy Theorem

Question

Is there a sense in which an o-minimal structure is either “trivial”, or group-like, or field-like?

Any answer to this has to be “local”.

Trichotomy Theorem

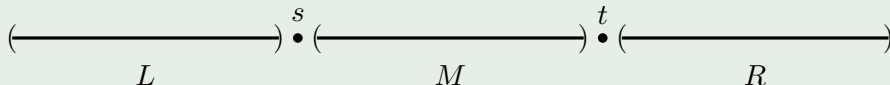
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Any answer to this has to be “local”.

Example

- L is carrying the structure of a pure dense linear order
- M is carrying the structure of a divisible abelian ordered group
- R is carrying the structure of a real closed field



Trichotomy Theorem

Trichotomy Theorem [Peterzil & Starchenko, 1998]

Let \mathcal{M} be a sufficiently saturated o-minimal structure. Given an $a \in M$ one and only one of the following holds,

- a is trivial,
- the structure that \mathcal{M} induces on some convex neighborhood of a is an ordered vector space over an ordered division ring.
- the structure that \mathcal{M} induces over some open interval around a is an o-minimal expansion of a real closed field.

Weakly o-minimal structures

Let $\mathcal{M} = (M, <, \dots)$ be a linearly ordered structure.

Definition

A set $C \subseteq M$ is called *convex*, if for any $a, b \in C$ with $a < b$, and $c \in M$ such that $a < c < b$, then $c \in C$.

Definition

A structure \mathcal{M} will be called *weakly o-minimal*, if the definable subsets of \mathcal{M} are finite unions of convex sets in $(M, <)$.

We say that a complete theory T is *weakly o-minimal* if every model of T is weakly o-minimal.

Theorem

Expanding an o-minimal structure with unary predicates for convex subsets yields a structure with weakly o-minimal theory.

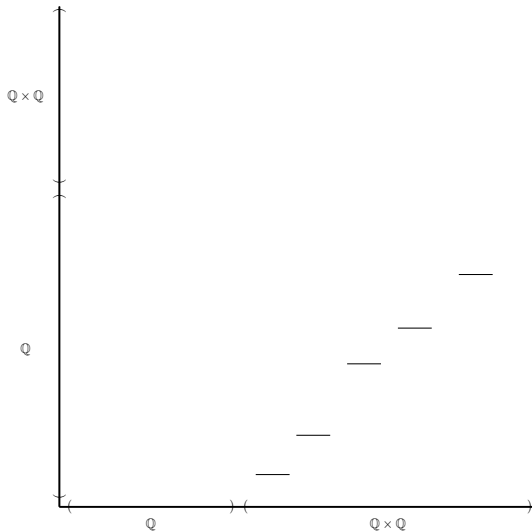
Monotonicity

Let $\mathcal{M} = (M, <, P, Q, f)$ such that.

- M is the disjoint union of the interpretations of the unary relations P and Q
- P is the interpretation of \mathbb{Q} with the usual order
- Q is the interpretation of $\mathbb{Q} \times \mathbb{Q}$, lexicographically ordered
- P precedes Q in $<$ on M
- $f : Q \rightarrow P$, $f((n, m)) = n$ for all $n, m \in \mathbb{Q}$

M is weakly o-minimal and also $\text{Th}(\mathcal{M})$ is weakly o-minimal.

Monotonicity



Weakly o-minimal structures

- ★ We have a local monotonicity theorem. [Arefiev, 1997]
- ★ Weakly o-minimal structures do not necessarily have weakly o-minimal theory.
- ★ Weakly o-minimal structures do not necessarily have prime models.

Theorem [Macpherson *et al.* , 2000]

Every weakly o-minimal ordered group is divisible and abelian.

Theorem [Macpherson *et al.* , 2000]

Every weakly o-minimal ordered field is real closed.

A definition for minimality

Let $L \subset L^+$ be languages, and \mathcal{K} be an elementary class of L -structures.

Definition [Macpherson & Steinhorn, 1996]

An L^+ -structure \mathcal{M} is \mathcal{K} -*minimal* if the reduct $\mathcal{M}|_L$ is in \mathcal{K} and every L^+ -definable subset of M is definable by a quantifier-free L -formula. A complete L^+ -theory is \mathcal{K} -*minimal* if all its models are \mathcal{K} -minimal.

- ★ o-minimality is a special case of the above definition but not weak o-minimality.
- ★ \mathcal{K} -minimality is closed under reducts to languages containing L , and under expansion by constants.

C-minimality

Let $C(x; y, z)$ be a ternary relation, $L = \{C\}$, and \mathcal{K}_C be the class of L -structures satisfying the following axioms.

- $(\forall xyz)[C(x; y, z) \rightarrow C(x; z, y)]$
- $(\forall xyz)[C(x; y, z) \rightarrow C(y; x, z)]$
- $(\forall xyzw)[C(x; y, z) \rightarrow (C(w; y, z) \vee C(x; w, z))]$
- $(\forall xy)(x \neq y \rightarrow (\exists z \neq y)C(x; y, z))$
- $(\exists xy)(x \neq y)$

Definition

A structure $\mathcal{M} = (M, C, \dots)$ is *C-minimal* if its theory is \mathcal{K}_C -minimal.

P-minimality

Definition

Let $L = (+, -, \cdot, 0, 1, (P_n)_{n>1})$, where P_n are unary predicates. Regard \mathbb{Q}_p as an L -structure, letting P_n picking the n^{th} powers in \mathbb{Q}_p . Let \mathcal{K}_P be the class of L -structures elementarily equivalent to \mathbb{Q}_p . Then if $L^+ \supseteq L$, an L^+ -structure is *P-minimal* if all models of its theory are \mathcal{K}_P -minimal

Summary

	o-minimal	weakly	C -minimal	P -minimal
Monotonicity	✓	local	✓	✓
CDT	✓	✓	✓	iff it has Skolem functions ¹
Prime Model	✓	✗	✗	✗
Groups	DAG	DAG		
Fields	RCF	RCF	ACVF	
Exchange	✓	✗	✗	✓
IP	✗ ²	✗ ³	✗ ⁴	✗ ⁵

¹Mourgues [2009]

²Pillay & Steinhorn [1986]

³Macpherson *et al.* [2000]

⁴Macpherson & Steinhorn [1996]

⁵Haskell & Macpherson [1997]

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