o-Minimality and its Variations

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o-Minimality

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Weakly o-minimal structures

Let $\mathcal{M} = (M, <, \dots)$ be a linearly ordered structure.

Definition

A set $C \subseteq M$ is called *convex* , if for any $a, b \in C$ with $a < b$, and $c \in M$ such that $a < c < b$, then $c \in C$.

Definition

A structure $\mathcal{M}$ will be called *weakly o-minimal* , if the definable subsets of $\mathcal{M}$ are finite unions of convex sets in $(M, <)$.
We say that a complete theory T is *weakly o-minimal* if every model of T is weakly o-minimal.

Theorem

Expanding an o-minimal structure with unary predicates for convex subsets yields a structure with weakly o-minimal theory.

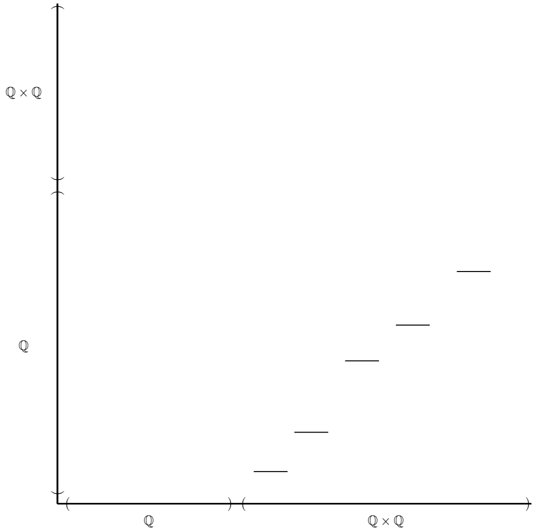
Monotonicity

Let $\mathcal{M} = (M, <, P, Q, f)$ such that.

- M is the disjoint union of the interpretations of the unary relations P and Q
- P is the interpretation of $\mathbb{Q}$ with the usual order
- Q is the interpretation of $\mathbb{Q} \times \mathbb{Q}$, lexicographically ordered
- P proeceds Q in $<$ on M
- $f : Q \to P$, $f((n, m)) = n$ for all $n, m \in \mathbb{Q}$

M is weakly o-minimal and also $\mathrm{Th}(\mathcal{M})$ is weakly o-minimal.

Monotonicity



Weakly o-minimal structures

- ★ We have a local monotonicity theorem.
- ★ Weakly o-minimal structures do not neccesserily have weakly o-minimal theory.
- ★ Weakly o-minimal structures do not neccesserily have prime models.

Theorem

Every weakly o-minimal ordered group is divisible and abelian.

Theorem

Every weakly o-minimal ordered field is real closed.

A definition for minimality

Let $L \subset L^+$ be languages, and $\mathcal{K}$ be an elementary class of L-structures.

Definition

An L^+-structure $\mathcal{M}$ is $\mathcal{K}$-*minimal* if the the reduct $\mathcal{M}|_L$ is in $\mathcal{K}$ and every L^+-definable subset of M is definable by a quantifier-free L-formula. A complete L^+-theory is $\mathcal{K}$-*minimal* if all its models are $\mathcal{K}$-minimal.

- o-minimality is a special case of the above definition but not weak o-minimality.
- $\mathcal{K}$-minimality is closed under reducts to languages containg L, and under expansion by constants.

C-minimality

Let $C(x; y, z)$ be a ternary realation, $L = \{C\}$, and $\mathcal{K}_C$ be the class of L-structures satisfying the following axioms.

- $(\forall xyz)[C(x; y, z) \to C(x; z, y)]$
- $(\forall xyz)[C(x; y, z) \to C(y; x, z)]$
- $(\forall xyzw)[C(x; y, z) \to (C(w; y, z) \vee C(x; w, z))]$
- $(\forall xy)[x \neq y \to (\exists z \neq y)C(x; y, z)]$
- $(\exists xy)(x \neq y)$

Definition

A structure $\mathcal{M} = (M, C, \ldots)$ is *C-minimal* if its theory is $\mathcal{K}_C$-minimal.

P-minimality

Definition

Let $L = (+, -, \cdot, 0, 1, (P_n)_{n>1})$, where P_n are unary predicates. Regard $\mathbb{Q}_p$ as an L-structure, letting P_n picking the n^{th} powers in $mathboldQ_p$. Let $\mathcal{K}_P$ be the class of L-structures elementarily equivalent to $\mathbb{Q}_p$. Then if $L^+ \supseteq L$, an L^+-structure is *P-minimal* if all models of its theory are $\mathcal{K}_P$-minimal

Summary

	o-minimal	weakly	C-minimal	P-minimal
Monotonicity	✓	local	✓	✓
CDT	✓	✓	✓	iff it has Skolem functions
Prime Model	✓	✗	✗	✗
Groups	DAG	DAG		
Fields	RCF	RCF	ACVF	
Exchange	✓	✗	✗	✓
IP	✗	✗	✗	✗