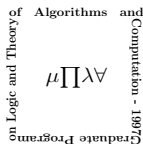


# o-Minimality and its Variations

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# o-Minimality

## Assumptions

- $\mathcal{M} = (M, <, \dots)$
- $<$  is dense, linear, without endpoints
- definability with parameters

## Definition

The structure  $\mathcal{M}$  is called *o-minimal* if every definable subset of  $M$  is a finite union of singletons and open intervals with endpoints in  $M_\infty := M \cup \{-\infty, +\infty\}$ .

A theory  $T$  is called *o-minimal* if every model  $\mathcal{M}$  of  $T$  is o-minimal.

# o-Minimality

The class of o-minimal structures is,

- closed under reducts (if  $<$  still remains in the language)
- closed under expansions by constants

## Some o-minimal structures

- $(\mathbb{Q}, <)$
- $(\mathbb{Q}, <, +)$
- $\mathcal{R} = (\mathbb{R}, <, +, -, \cdot, 0, 1)$

$(\mathbb{Q}, <, +, \cdot, 0, 1)$  is not o-minimal

The infinite discrete set of perfect squares is definable.

# Monotonicity and Finiteness

## Monotonicity Theorem

Let  $\mathcal{M}$  be an o-minimal structure and  $f : (a, b) \rightarrow M$  be a definable function with domain  $(a, b)$  (possibly  $a = -\infty$  or  $b = +\infty$ ). Then, there are points  $a = a_0 < a_1 < \dots < a_{k+1}$  s.t. for each  $j = 0, \dots, k$ ,  $f|_{(a_j, a_{j+1})}$  is either,

- constant, or
- a strictly monotonic and continuous bijection to an interval.

## Finiteness Lemma

Let  $A \subseteq M^2$  be definable and suppose that for each  $x \in M$  the fiber  $A_x := \{y \in M \mid (x, y) \in A\}$  is finite. Then there is  $N < \omega$  s.t.  $|A_x| \leq N$  for all  $x \in M$ .

## Corollary

Let  $f : (a, b) \rightarrow M$  be definable and continuous. Then  $f$  takes a maximum and minimum value on  $[a, b]$ .

## Exchange Principle

Let  $\mathcal{M}$  be o-minimal. Let  $b, c, a_1, \dots, a_n \in \mathcal{M}$ . If  $b$  is definable over  $c, a_1, \dots, a_n$ , and  $b$  is not definable over  $a_1, \dots, a_n$ , then  $c$  is definable over  $b, a_1, \dots, a_n$ .

# Motivation

## Question

What happens with the definable subsets of  $M^n$ ?

## Notation

Given definable  $X \subseteq M^n$

- $C(X) := \{f : X \rightarrow M \mid f \text{ is definable and continuous}\}$
- $C_\infty(X) := C(X) \cup \{-\infty, +\infty\}$
- for  $f, g \in C_\infty(X)$  s.t.  $(\forall \bar{x} \in X)(f(\bar{x}) < g(\bar{x}))$  then

$$(f, g)_X := \{(\bar{x}, y) \in X \times M \mid f(\bar{x}) < y < g(\bar{x})\}$$

Let  $(i_1, \dots, i_n)$  be a binary sequence. Then a  $(i_1, \dots, i_n)$ -*cell* is a definable subset of  $M^n$  defined as follows,

- a (0)-cell is a singleton
- a (1)-cell is an open interval

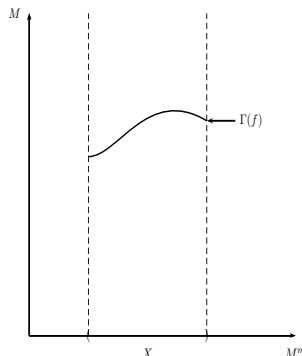
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Suppose that we have defined  $(i_1, \dots, i_{n-1})$ -cells, then

- a  $(i_1, \dots, i_{n-1}, 0)$ -cell is the graph of some  $f \in C(X)$ , where  $X$  is a  $(i_1, \dots, i_{n-1})$ -cell



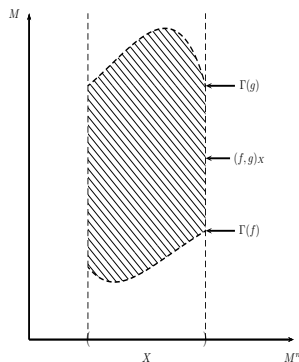


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- a  $(i_1, \dots, i_{n-1}, 1)$ -cell is the set  $(f, g)_X$  where  $X$  is a  $(i_1, \dots, i_{n-1})$ -cell and  $f, g \in C_\infty(X)$



# Decompositions

A *decomposition* of  $M^n$  is a partition of  $M^n$  into finitely many cells. It is defined recursively.

- A decomposition of  $M$  is a collection,

$$\{(-\infty, a_1), (a_1, a_2), \dots, (a_k, +\infty), \{a_1\}, \dots, \{a_k\}\}$$

where  $a_1 < \dots < a_k$  are points in  $M$ ;

- a decomposition of  $M^{n+1}$  is a finite partition of  $M^{n+1}$  into cells  $C$  s.t. the set of projections  $\pi(C)$  is a decomposition of  $M^n$ .

A decomposition  $\mathcal{D}$  of  $M^n$  is said to *partition* a set  $S \subseteq M^n$  if  $S$  is a union of cells in  $\mathcal{D}$ .

# Cell Decomposition Theorem

## Uniform Finiteness Property

Let  $Y \subseteq M^{n+1}$  be definable and also for every  $\bar{x} \in M^n$ ,

$$Y_{\bar{x}} := \{r \in M \mid (\bar{x}, r) \in Y\}$$

is finite. Then there exists  $N < \omega$  s.t.  $|Y_{\bar{x}}| \leq N$  for all  $\bar{x} \in M^n$ .

## Cell Decomposition Theorem

- ① Given any definable sets  $A_1, \dots, A_k \subseteq M^n$  there is a decomposition of  $M^n$  partitioning each of  $A_1, \dots, A_k$ .
- ② For each definable function  $f : A \rightarrow M$ ,  $A \subseteq M^n$ , there is a decomposition  $\mathcal{D}$  of  $M^n$  partitioning  $A$  s.t. the restriction  $f|_B : B \rightarrow M$  to each cell  $B \in \mathcal{D}$  with  $B \subseteq A$  is continuous.

## Theorem

If  $\mathcal{M}$  is an o-minimal structure, the  $\text{Th}(\mathcal{M})$  is an o-minimal theory.

Proof...

# Prime models

## Definition

Let  $\mathcal{M} \models T$  and  $A \subseteq \mathcal{M}$ .  $\mathcal{M}$  is said to be *prime over  $A$*  if for any  $\mathcal{M}' \models T$  with  $A \subseteq \mathcal{M}'$ , there is an elementary mapping  $f : \mathcal{M} \rightarrow \mathcal{M}'$  that is identity on  $A$ .

## Theorem

Let  $A \subseteq \mathcal{M} \models T$ , where  $T$  is an o-minimal theory. Then there is an a model  $\mathcal{M}' \models T$ ,  $A \subseteq \mathcal{M}'$  that is prime over  $A$ , and is unique up to isomorphism over  $A$ .

# Ressayre's Theorem

## Definition

Let  $\mathcal{M}$  be an  $L$ -structure and let  $A \subseteq M$ . Let  $\delta$  be an ordinal and  $(a_\alpha : \alpha < \delta)$  be a sequence of elements from  $M$ . Let  $A_\alpha = A \cup \{a_\beta : \beta < \alpha\}$ . We call  $(a_\alpha : \alpha < \delta)$  a *construction* over  $A$  if  $\text{tp}(a_\alpha/A_\alpha)$  is isolated for all  $\alpha < \delta$ .

## Theorem [Ressayre]

Suppose that  $A \subseteq \mathbb{M}$ ,  $\mathcal{M} \prec \mathbb{M}$ ,  $\mathcal{N} \prec \mathbb{M}$ , and  $\mathcal{M}$  and  $\mathcal{N}$  are constructible over  $A$ . Then the identity map on  $A$  extends to an isomorphism between  $\mathcal{M}$  and  $\mathcal{N}$ .

# Cuts and Noncuts

Let  $\mathcal{M} \models T$ . Let  $p(x) \in S_1(M)$ . We say that  $p(x)$  is a *cut* over  $\mathcal{M}$  iff there are non empty disjoint subsets  $C_0$  and  $C_1$  of  $M$ , s.t.  
 $C_0 \cup C_1 = M$ ,  $C_0$  has no greatest element,  $C_1$  has no least element, for all  $c \in C_0$ , " $c < x$ "  $\in p(x)$ , and for all  $c \in C_1$ , " $x < c$ "  $\in p(x)$ .  
 If  $p(x)$  is not isolated and is not a cut, we call  $p(x)$  a *noncut*.

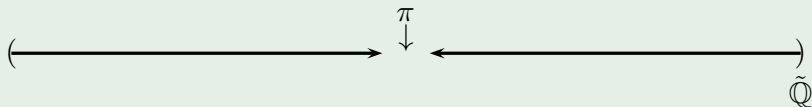
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## Example

Let  $\tilde{\mathbb{Q}}$  be the field of real algebraic numbers. We know that  $\pi \in \mathbb{R} \setminus \tilde{\mathbb{Q}}$ , so  $\text{tp}(\pi, \tilde{\mathbb{Q}})$  is a cut.

If  $t$  is an infinite hyperreal or  $t$  is infinitesimally close to an element of  $\tilde{\mathbb{Q}}$ , then  $\text{tp}(t, \tilde{\mathbb{Q}})$  is a noncut.





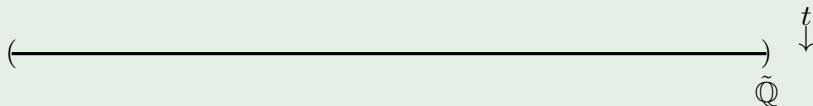
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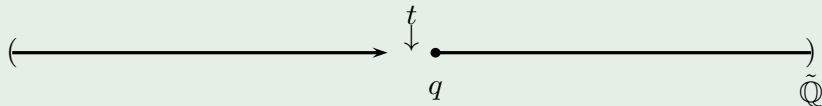
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# Cuts and Noncuts

Let  $\mathcal{M} \models T$ , where  $T$  is an o-minimal theory.

## Theorem

Let  $\sigma(x) \in S_1(M)$  be a cut. Let  $\tau(x) \in S_1(M)$  be a noncut, and  $b$  be a realization of  $\tau(x)$ . Then  $\sigma(x)$  is omitted in  $P(\mathcal{M} \cup \{b\})$ .

## Theorem

Let  $\tau(x) \in S_1(M)$  be a noncut. Let  $\sigma(x) \in S_1(M)$  be a cut, and  $b$  be a realization of  $\sigma(x)$ . Then  $\tau(x)$  is omitted in  $P(\mathcal{M} \cup \{b\})$ .

# Ordered Groups and Fields

## Theorem

Let  $\mathcal{G} = (G, +, 0, <)$  be an o-minimal ordered group. Then  $\mathcal{G}$  is a divisible ordered abelian group.

## Remind

An abelian group  $G$  is divisible if, for every positive integer  $n$  and every  $g \in G$ , there exists  $y \in G$  such that  $ny = g$ .

## Theorem

Let  $\mathcal{R} = (R, +, \cdot, 1, <)$  be an o-minimal ordered ring. Then  $\mathcal{R}$  is a real closed field.

## Remind

A field  $F$  is real closed if it is elementarily equivalent to  $\mathbb{R}$ .

# Trichotomy Theorem

## Question

Is there a sense in which an o-minimal structure is either “trivial”, or group-like, or field-like?

Any answer to this has to be “local”.