**PROBABILITY AND QUEUEING THEORY (MA 2262)**

**UNIT – I RANDOM VARIABLES**

# PART- A

**Problem 1** The probability density function of a continuous random variable $X$ is given by $f(x) = Ke^{-|x|}$. Find $K$ and C.D.F of X

**Solution:**

Since it is a probability density function,

$$\int_{-\infty}^{\infty} f(x)dx = 1$$

$$\int_{-\infty}^{\infty} Ke^{-|x|}dx = 1$$

$$2\int_{0}^{\infty} Ke^{-x}dx = 1$$

$$2K\left[\frac{e^{-x}}{-1}\right]_0^{\infty} = 1 \qquad 2K = 1 \qquad K = \frac{1}{2}.$$

Therefore,

$$f(x) = \frac{1}{2}e^{x}, \quad -\infty < x < 0$$

$$= \frac{1}{2}e^{-x}, \quad 0 < x < \infty$$

For $x \le 0$, the C.D.F is $F(x) = \int_{-\infty}^{x} \frac{1}{2}e^{x}dx = \frac{1}{2}e^{x}$

For $x > 0$, $F(x) = \int_{-\infty}^{0} \frac{1}{2}e^{x}dx + \int_{0}^{x} \frac{1}{2}e^{-x}dx = \frac{1}{2} + \frac{1}{2}(1 - e^{-x}) = \frac{1}{2}(2 - e^{-x})$

**Problem 2** $X$ and $Y$ are independent random variables with variance 2 and 3. Find the variance of $3X + 4Y$.

**Solution:**

$$V(3X + 4Y) = 9Var(X) + 16Var(Y) + 24Cov(XY)$$

$$= 9\times 2 + 16\times 3 + 0 \quad (\because X \,\&\, Y \text{ are independent } \operatorname{cov}(XY) = 0)$$

$$= 18 + 48 = 66.$$

**Problem 3** A Continuous random variable $X$ has a probability density function $F(x) = 3x^2;\ 0 \le x \le 1$. Find 'a' such that $P(x \le a) = P(x > a)$

**Solution:**

We know that the total probability =1

Given $P(X \le a) = P(X > a) = K(say)$

Then $K + K = 1$

$$K = \frac{1}{2}$$

i.e., $P(X \le a) = \frac{1}{2}$ & $P(X > a) = \frac{1}{2}$

Consider $P(X \le a) = \frac{1}{2}$

i.e., $\int_0^a f(x)\,dx = \frac{1}{2}$

$$\int_0^a 3x^2\,dx = \frac{1}{2}$$

$$3\left(\frac{x^3}{3}\right)_0^a = \frac{1}{2} \qquad a^3 = \frac{1}{2} \qquad a = \left(\frac{1}{2}\right)^{1/3}.$$

**Problem 4** A random variable $X$ has the p.d.f $f(x)$ given by $f(x) = \begin{cases} Cxe^{-x}; & if\ x > 0 \\ 0 & ; if\ x \le 0 \end{cases}$

Find the value of $C$ and cumulative density function of $X$.

**Solution:**

Since $\int_{-\infty}^{\infty} f(x)\,dx = 1$

$$\int_0^{\infty} Cxe^{-x}\,dx = 1$$

$$C\left[x\left(-e^{-x}\right) - \left(e^{-x}\right)\right]_0^{\infty} = 1 \qquad C = 1$$

$$\therefore f(x) = \begin{cases} xe^{-x}; x > 0 \\ 0 \quad ; x \le 0 \end{cases}$$

$$C.D.F\ F(x) = \int_0^x f(x)dt = \int_0^x te^{-t}\,dt = \left[-te^{-t} - e^{-t}\right]_0^x = -xe^{-x} - e^{-x} + 1$$

$= 1 - (1+x)e^{-x}$ for $x \ge 0$.

**Problem 5** If a random variable $X$ has the p.d.f $f(x) = \begin{cases} \frac{1}{2}(x+1); -1 < x < 1 \\ \quad 0 \quad ; otherwise \end{cases}$, find the mean and variance of $X$.

**Solution:**

$$\text{Mean} = \int_{-1}^{1} xf(x)\,dx = \frac{1}{2}\int_{-1}^{1} x(x+1)\,dx = \frac{1}{2}\int_{-1}^{1}\left(x^2 + x\right)dx$$

$$= \frac{1}{2}\left(\frac{x^3}{3} + \frac{x^2}{2}\right)_{-1}^{1} = \frac{1}{3}$$

$$\mu_2' = \int_{-1}^{1} x^2 f(x) dx = \frac{1}{2}\int_{-1}^{1}\left(x^3 + x^2\right) dx = \frac{1}{2}\left[\frac{x^4}{4} + \frac{x^3}{3}\right]_{-1}^{1}$$

$$= \frac{1}{2}\left[\frac{1}{4} + \frac{1}{3} - \frac{1}{4} + \frac{1}{3}\right]$$

$$= \frac{1}{2}.\frac{2}{3} = \frac{1}{3}$$

$$Variance = \mu_2' - \left(\mu_1'\right)^2$$

$$= \frac{1}{3} - \frac{1}{9} = \frac{3-1}{9} = \frac{2}{9}.$$

**Problem 6** A random variable $X$ has density function given by $f(x) = \begin{cases} 2e^{-2x}; x \geq 0 \\ 0 \quad ; x < 0 \end{cases}$.

Find the moment generating function.

**Solution:**

$$M_X(t) = E\left(e^{tx}\right) = \int_0^\infty e^{tx} f(x) dx = \int_0^\infty e^{tx} 2e^{-2x} dx$$

$$= 2\int_0^\infty e^{(t-2)x} dx$$

$$= 2\left[\frac{e^{(t-2)x}}{t-2}\right]_0^\infty = \frac{2}{2-t}, t < 2.$$

**Problem 7** If $X$ is a Poisson variate such that $P(X=2) = 9P(X=4) + 90P(X=6)$, find the variance.

**Solution:**

Given $P(X=2) = 9P(X=4) + 90P(X=6)$

$$\Rightarrow \frac{e^{-\lambda}\lambda^2}{2!} = 9\frac{e^{-\lambda}\lambda^4}{4!} + 90\frac{e^{-\lambda}\lambda^6}{6!}$$

$$\Rightarrow \lambda^4 + 3\lambda^2 - 4 = 0$$

$$\Rightarrow \left(\lambda^2 + 4\right)\left(\lambda^2 - 1\right) = 0$$

$$\Rightarrow \lambda^2 = 1 \ (or) \ \lambda^2 = -4$$

$\lambda = \text{Variance} = 1$ ($\because \lambda^2$ cannot be negative)

**Problem 8** Comment the following: "The mean of a binomial distribution is 3 and variance is 4

**Solution:**

In a binomial distribution, mean (np) > variance (npq).

Since variance $= 4$ & mean $= 3$, we have variance $<$ mean. Therefore, the given statement is wrong.

**Problem 9** If $X$ and $Y$ are independent binomial variates $B\left(5,\frac{1}{2}\right)$ and $B\left(7,\frac{1}{2}\right)$ find $P\left[X+Y=3\right]$

**Solution:**

$X+Y$ is also a binomial variate with parameters $n_1+n_2=12$ & $p=\frac{1}{2}$

$$\therefore P\left[X+Y=3\right]=12C_3\left(\frac{1}{2}\right)^3\left(\frac{1}{2}\right)^9$$

$$=\frac{55}{2^{10}}$$

**Problem 10** If $X$ is uniformly distributed with Mean 1 and Variance $\frac{4}{3}$, find $P\left[X>0\right]$

**Solution:**

If $X$ is uniformly distributed over $(a,b)$, then

$$E(X)=\frac{b+a}{2} \ and \ V(X)=\frac{(b-a)^2}{12}$$

$$\therefore \frac{b+a}{2}=1 \Rightarrow a+b=2$$

$$\Rightarrow \frac{(b-a)^2}{12}=\frac{4}{3} \Rightarrow (b-a)^2=16$$

$\Rightarrow a+b=2$ & $b-a=4$ We get $b=3, a=-1$

$\therefore a=-1$ & $b=3$ and probability density function of $x$ is

$$f(x)=\begin{cases}\frac{1}{4}; -1<x<3\\ 0\ ; Otherwise\end{cases}$$

$$P\left[x>0\right]=\int_0^3 \frac{1}{4}dx=\frac{1}{4}\left[x\right]_0^3=\frac{3}{4}.$$

**Problem 11** If $X$ is $N(2,3)$ ind $P\left[Y\geq\frac{3}{2}\right]$ where $Y+1=X$.

**Solution:**

$$P\left[Y\geq\frac{3}{2}\right]=P\left[X-1\geq\frac{3}{2}\right]$$

$=P\left[X\geq 2.5\right]=P\left[Z\geq 0.17\right]$, where $Z=\frac{X-2}{3}$

$$=0.5-P\left[0\leq Z\leq 0.17\right]$$

$$=0.5-0.0675=0.4325$$

**Problem 12** If the probability is $\frac{1}{4}$ that a man will hit a target, what is the chance that he will hit the target for the first time in the $7^{th}$ trial?

**Solution:**

The required probability is

$$P[FFFFFFS] = P(F)P(F)P(F)P(F)P(F)P(F)P(S)$$

$$= q^6 p = \left(\frac{3}{4}\right)^6 \cdot \left(\frac{1}{4}\right) = 0.0445.$$

Here $p =$ probability of hitting target and $q = 1 - p$.

## PART - B

**Problem 13** A random variable $X$ has the following probability function:

Values of

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| $X$ | : | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
| $P(X)$ | : | 0 | $K$ | $2K$ | $2K$ | $3K$ | $K^2$ | $2K^2$ | $7K^2+K$ |

Find (i) $K$, (ii) Evaluate $P(X<6), P(X \geq 6)$ and $P(0<X<5)$

(iii). Determine the distribution function of $X$.

**Solution:**

(i)

$$\text{Since } \sum_{x=0}^{7} P(x) = 1,$$

$$K + 2K + 2K + 3K + K^2 + 2K^2 + 7K^2 + K = 1$$

$$10K^2 + 9K - 1 = 0$$

$$K = \frac{1}{10} \quad or \quad K = -1$$

As $P(x)$ cannot be negative $K = \frac{1}{10}$

(ii)

$$P(X<6) = P(X=0) + P(X=1) + \ldots + P(X=5)$$

$$= \frac{1}{10} + \frac{2}{10} + \frac{2}{10} + \frac{3}{10} + \frac{1}{100} + \ldots = \frac{81}{100}$$

Now $P(X \geq 6) = 1 - P(X<6)$

$$= 1 - \frac{81}{100} = \frac{19}{100}$$

Now $P(0<X<5) = P(X=1) + P(X=2) + P(X=3) + P(X=4)$

$$= K + 2K + 2K + 3K$$

$$= 8K = \frac{8}{10} = \frac{4}{5}.$$

(iii) The distribution of X is given by $F_X(x)$ defined by

$$F_X(x) = P(X \le x)$$

| $X$ | : | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|---|---|
| $F_X(x)$ | : | 0 | $\frac{1}{10}$ | $\frac{3}{10}$ | $\frac{5}{10}$ | $\frac{4}{5}$ | $\frac{81}{100}$ | $\frac{83}{100}$ | 1 |

**Problem 14** (i). If the probability distribution of $X$ is given as

| $X$ | : | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| $P(X)$ | : | 0.4 | 0.3 | 0.2 | 0.1 |

Find $P(1/2 < X < 7/2 / X > 1)$

(ii). If $P(x) = \begin{cases} \frac{x}{15}; x = 1,2,3,4,5 \\ 0 \quad ; elsewhere \end{cases}$

find (a) $P\{X = 1 or 2\}$ *and* (b) $P\{1/2 < X < 5/2 / x > 1\}$

**Solution:**

(i) $P\{1/2 < X < 7/2 / X > 1\} = \dfrac{P\{(1/2 < X < 7/2) \cap X > 1\}}{P(X > 1)}$

$$= \frac{P(X = 2 \, or \, 3)}{P(X = 2,3 \, or \, 4)}$$

$$= \frac{P(X=2) + P(X=3)}{P(X=2) + P(X=3) + P(X=4)}$$

$$= \frac{0.3 + 0.2}{0.3 + 0.2 + 0.1} = \frac{0.5}{0.6} = \frac{5}{6}.$$

(ii) (a) $P(X = 1 \, or \, 2) = P(X=1) + P(X=2)$

$$= \frac{1}{15} + \frac{2}{15} = \frac{3}{15} = \frac{1}{5}$$

(b) $P\left(\frac{1}{2} < X < \frac{5}{2} / x > 1\right) = \dfrac{P\left\{\left(\frac{1}{2} < X < \frac{5}{2}\right) \cap (X > 1)\right\}}{P(X > 1)}$

$$= \frac{P\{(X = 1 or 2) \cap (X > 1)\}}{P(X > 1)}$$

$$= \frac{P(X = 2)}{1 - P(X = 1)}$$

$$= \frac{2/15}{1 - (1/15)} = \frac{2/15}{14/15} = \frac{2}{14} = \frac{1}{7}.$$

**Problem 15** A random variable $X$ has the following probability distribution

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| $X$ | : | $-2$ | $-1$ | $0$ | $1$ | $2$ | $3$ |
| $P(X)$ | : | $0.1$ | $K$ | $0.2$ | $2K$ | $0.3$ | $3K$ |

a) Find $K$, b) Evaluate $P(X<2)$ and $P(-2<X<2)$

b) Find the cdf of $X$ and d) Evaluate the mean of $X$.

**Solution:**

a) Since $\sum P(X)=1$

$$0.1+K+0.2+2K+0.3+3K=1$$
$$6K+0.6=1$$
$$6K=0.4$$
$$K=\frac{0.4}{6}=\frac{1}{15}$$

b) $P(X<2)=P(X=-2,-1,0\ or\ 1)$

$$=P(X=-2)+P(X=-1)+P(X=0)+P(X=1)$$
$$=\frac{1}{10}+\frac{1}{15}+\frac{1}{5}+\frac{2}{15}$$
$$=\frac{3+2+6+4}{30}=\frac{15}{30}=\frac{1}{2}$$

$P(-2<X<2)=P(X=-1,0\ or\ 1)$

$$=P(X=-1)+P(X=0)+P(X=1)$$
$$=\frac{1}{15}+\frac{1}{5}+\frac{2}{15}$$
$$=\frac{1+3+2}{15}=\frac{6}{15}=\frac{2}{5}$$

c) The distribution function of $X$ is given by $F_X(x)$ defined by

$F_X(x)=P(X\le x)$

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| $X$ | : | $-2$ | $-1$ | $0$ | $1$ | $2$ | $3$ |
| $F_X(x)$ | : | $\frac{1}{10}$ | $\frac{1}{6}$ | $\frac{11}{30}$ | $\frac{1}{2}$ | $\frac{4}{5}$ | $1$ |

d) Mean of $X$ is defined by $E(X)=\sum xP(x)$

$$E(X)=\left(-2\times\frac{1}{10}\right)+\left(-1\times\frac{1}{15}\right)+\left(0\times\frac{1}{5}\right)+\left(1\times\frac{2}{15}\right)+\left(2\times\frac{3}{10}\right)+\left(3\times\frac{1}{5}\right)$$
$$=-\frac{1}{5}-\frac{1}{15}+\frac{2}{15}+\frac{3}{5}+\frac{3}{5}=\frac{16}{15}.$$

**Problem 16** $X$ is a continuous random variable with pdf given by

$$F(X)=\begin{cases} Kx & in \quad 0\le x\le 2 \\ 2K & in \quad 2\le x\le 4 \\ 6K-Kx & in \quad 4\le x\le 6 \\ 0 & elsewhere \end{cases}$$

Find the value of $K$ and also the cdf $F_X(x)$.

**Solution:**

$$\text{Since } \int_{\infty}^{\infty} F(x)\,dx=1$$

$$\int_0^2 Kx\,dx+\int_2^4 2K\,dx+\int_4^6 (6k-kx)\,dx=1$$

$$K\left[\left(\frac{x^2}{2}\right)_0^2+(2x)_2^4+\int_4^6\left(6x-\frac{x^2}{2}\right)_4^6\right]=1$$

$$K\left[\cancel{2}+\cancel{8}-4+36-18-24+8\right]=1$$

$$8K=1$$

$$K=\frac{1}{8}$$

We know that $F_X(x)=\int_{-\infty}^{x} f(x)\,dx$

If $x<0$, then $F_X(x)=\int_{-\infty}^{x} f(x)\,dx=0$

If $x\in(0,2)$, then $F_X(x)=\int_{-\infty}^{x} f(x)\,dx$

$$F_X(x)=\int_{-\infty}^{0} f(x)\,dx+\int_0^x f(x)\,dx$$

$$=\int_{-\infty}^{0} 0\,dx+\int_0^x Kx\,dx=\int_{-\infty}^{0} 0\,dx+\frac{1}{8}\int_0^x x\,dx$$

$$=\left(\frac{x^2}{16}\right)_0^x=\frac{x^2}{16},0\le x\le 2$$

If $x\in(2,4)$, then $F_X(x)=\int_{-\infty}^{x} f(x)\,dx$

$$F_X(x)=\int_{-\infty}^{0} f(x)\,dx+\int_0^2 f(x)\,dx+\int_2^x f(x)\,dx$$

$$=\int_{-\infty}^{0} 0\,dx+\int_0^2 Kx\,dx+\int_2^x 2K\,dx$$

$$= \int_0^2 \frac{x}{8}dx + \int_2^x \frac{1}{4}dx = \left(\frac{x^2}{16}\right)_0^2 + \left(\frac{x}{4}\right)_2^x$$

$$= \frac{1}{4} + \frac{x}{4} - \frac{1}{2}$$

$$= \frac{x}{4} - \frac{4}{16} = \frac{x-1}{4}, \ 2 \le x < 4$$

If $x \in (4,6)$, then $F_X(x) = \int_{-\infty}^{0} 0dx + \int_0^2 Kxdx + \int_2^4 2Kdx + \int_4^x k(6-x)dx$

$$= \int_0^2 \frac{x}{8}dx + \int_2^4 \frac{1}{4}dx + \int_4^x \frac{1}{8}(6-x)dx$$

$$= \left(\frac{x^2}{16}\right)_0^2 + \left(\frac{x}{4}\right)_2^4 + \left(\frac{6x}{8} - \frac{x^2}{16}\right)_4^x$$

$$= \frac{1}{4} + 1 - \frac{1}{2} + \frac{6x}{8} - \frac{x^2}{16} - 3 + 1$$

$$= \frac{4 + 16 - 8 + 12x - x^2 - 48 + 16}{16}$$

$$= \frac{-x^2 + 12x - 20}{16}, 4 \le x \le 6$$

If $x > 6$, then $F_X(x) = \int_{-\infty}^{0} 0dx + \int_0^2 Kxdx + \int_2^4 2Kdx + \int_4^6 k(6-x)dx + \int_6^x 0dx$

$= 1, \ x \ge 6$

$$\therefore F_X(x) = \begin{cases} 0 & ; x \le 0 \\ \frac{x^2}{16} & ; 0 \le x \le 2 \\ \frac{1}{4}(x-1) & ; 2 \le x \le 4 \\ \frac{-1}{16}\left(20 - 12x + x^2\right) & ; 4 \le x \le 6 \\ 1 & ; x \ge 6 \end{cases}$$

**Problem 17** A random variable $X$ has density function $f(x) = \begin{cases} \frac{K}{1+x^2}, & -\infty < x < \infty \\ 0 & , \ Otherwise \end{cases}$

Determine $K$ and the distribution function. Evaluate the probability $P(x \ge 0)$.

**Solution:**

Since $\int_{-\infty}^{\infty} f(x)dx = 1$

$$\int_{-\infty}^{\infty} \frac{K}{1+x^2} dx = 1$$

$$K\int_{\infty}^{\infty} \frac{dx}{1+x^2} = 1$$

$$K\left(\tan^{-1} x\right)_{-\infty}^{\infty} = 1$$

$$K\left(\frac{\pi}{2}-\left(-\frac{\pi}{2}\right)\right) = 1$$

$$K\pi = 1$$

$$K = \frac{1}{\pi}$$

$$F_X(x) = \int_{-\infty}^{x} f(x)dx = \int_{-\infty}^{x} \frac{K}{1+x^2} dx$$

$$= \frac{1}{\pi}\left(\tan^{-1} x\right)_{-\infty}^{x}$$

$$= \frac{1}{\pi}\left[\tan^{-1} x - \left(-\frac{\pi}{2}\right)\right]$$

$$= \frac{1}{\pi}\left[\frac{\pi}{2} + \tan^{-1} x\right], -\infty < x < \infty$$

$$P(X \geq 0) = \frac{1}{\pi}\int_{0}^{\infty} \frac{dx}{1+x^2} = \frac{1}{\pi}\left(\tan^{-1} x\right)_{0}^{\infty}$$

$$= \frac{1}{\pi}\left(\frac{\pi}{2} - \tan^{-1} 0\right) = \frac{1}{2}.$$

**Problem 18** If $X$ has the probability density function $f(x) = \begin{cases} Ke^{-3x}, x > 0 \\ 0 \quad , otherwise \end{cases}$ find $K$, $P[0.5 \leq X \leq 1]$ and the mean of $X$.

**Solution:**

Since $\int_{-\infty}^{\infty} f(x)dx = 1$

$$\int_{0}^{\infty} Ke^{-3x} dx = 1$$

$$K\left[\frac{e^{-3x}}{-3}\right]_{0}^{\infty} = 1 \qquad \frac{K}{3} = 1 \qquad K = 3$$

$$P(0.5 \leq X \leq 1) = \int_{0.5}^{1} f(x)dx = 3\int_{0.5}^{1} e^{-3x} dx = \cancel{3}\frac{e^{-3} - e^{-1.5}}{-\cancel{3}} = \left[e^{-1.5} - e^{-3}\right]$$

Mean of $X = E(x) = \int_0^\infty x f(x)\,dx = 3\int_0^\infty x e^{-3x}\,dx$

$$= 3\left[x\left(\frac{-e^{-3x}}{3}\right) - 1\left(\frac{e^{-3x}}{9}\right)\right]_0^\infty = \frac{3\times 1}{9} = \frac{1}{3}$$

**Problem 19** A random variable $X$ has the P.d.f $f(x) = \begin{cases} 2x, 0 < x < 1 \\ 0 \quad , Otherwise \end{cases}$

Find (i) $P\left(X < \frac{1}{2}\right)$ (ii) $P\left(\frac{1}{4} < x < \frac{1}{2}\right)$ (iii) $P\left(X > \frac{3}{4} / X > \frac{1}{2}\right)$

**Solution:**

(i) $P\left(x < \frac{1}{2}\right) = \int_0^{1/2} f(x)\,dx = \int_0^{1/2} 2x\,dx = 2\left(\frac{x^2}{2}\right)_0^{1/2} = \frac{2\times 1}{8} = \frac{1}{4}$

(ii) $P\left(\frac{1}{4} < x < \frac{1}{2}\right) = \int_{1/4}^{1/2} f(x)\,dx = \int_{1/4}^{1/2} 2x\,dx = 2\left(\frac{x^2}{2}\right)_{1/4}^{1/2}$

$$= 2\left(\frac{1}{8} - \frac{1}{32}\right) = \left(\frac{1}{4} - \frac{1}{16}\right) = \frac{3}{16}.$$

(iii) $P\left(X > \frac{3}{4} / X > \frac{1}{2}\right) = \dfrac{P\left(X > \frac{3}{4} \cap X > \frac{1}{2}\right)}{P\left(X > \frac{1}{2}\right)} = \dfrac{P\left(X > \frac{3}{4}\right)}{P\left(X > \frac{1}{2}\right)}$

$$P\left(X > \frac{3}{4}\right) = \int_{3/4}^{1} f(x)\,dx = \int_{3/4}^{1} 2x\,dx = 2\left(\frac{x^2}{2}\right)_{3/4}^{1} = 1 - \frac{9}{16} = \frac{7}{16}$$

$$P\left(X > \frac{1}{2}\right) = \int_{1/2}^{1} f(x)\,dx = \int_{1/2}^{1} 2x\,dx = 2\left(\frac{x^2}{2}\right)_{1/2}^{1} = 1 - \frac{1}{4} = \frac{3}{4}$$

$$P\left(X > \frac{3}{4} / X > \frac{1}{2}\right) = \frac{\frac{7}{16}}{\frac{3}{4}} = \frac{7}{16} \times \frac{4}{3} = \frac{7}{12}.$$

**Problem 20** A Man drawn 3 balls from an urn containing 5 white and 7 black balls. He gets Rs.10 for each white ball and Rs.5 for each black ball. Find his expectation.

**Solution:**

Let $X$ denotes the amount that he expects to receive

| $W$ | $B$ |
|---|---|
| 5 | 7 |

| $3B$ | $1W \,\&\, 2B$ | $2W \,\&\, 1B$ | $3W$ |
|---|---|---|---|
| $X = Rs15$ | $Rs20$ | $Rs25$ | $Rs30$ |

$$P(X=15)=P(3\ Black\ balls)=\frac{7C_3}{12C_3}=\frac{\frac{7\times6\times5}{1\times2\times3}}{\frac{12\times11\times10}{1\times2\times3}}=\frac{7}{44}$$

$$P(X=20)=P(2B1W)=\frac{7C_2.5C_1}{12C_3}=\frac{21}{44}$$

$$P(x=25)=P(2W1B)=\frac{7c2.5c2}{12c3}=\frac{14}{44}$$

$$P(X=30)=P(3W)=\frac{5C_3}{12C_3}=\frac{2}{44}$$

$$E(X)=\sum xP(x)$$

$$=15\times\frac{7}{44}+20\times\frac{21}{44}+25\times\frac{14}{44}+30\times\frac{2}{44}$$

$$=\frac{935}{44}=Rs.21.25$$

**Problem 21** From an urn containing 3 red and 2 black balls, a man is to draw 2 balls at random without replacement, being promised Rs.20/- for each red ball he draws and Rs.10/- for each black ball. Find his expectation.

**Solution:**

Let $X$ denotes the amount he receives

$$\boxed{\begin{matrix} R & B \\ 3 & 2 \end{matrix}}$$

$$\begin{array}{llll} & 2B & 1R\ 1B & 2R \\ X= & Rs20 & Rs30 & Rs40 \end{array}$$

$$P(X=20)=P(2\ Black\ balls)=\frac{2C_2}{5C_2}=\frac{\frac{2\times1}{1\times2}}{\frac{5\times4}{1\times2}}=\frac{1}{10}$$

$$P(X=30)=P(1\,\text{Re}\,d\ \ \&\ 1\ Black\ ball)=\frac{3C_1\times2C_1}{5C_2}=\frac{6}{10}$$

$$P(X=40)=P(2\,\text{Re}\,d\ balls)=\frac{3C_2}{5C_2}=\frac{3}{10}$$

$$E(x)=\sum xP(x)$$

$$=20\times\frac{1}{10}+30\times\frac{6}{10}+40\times\frac{3}{10}$$

$$=\frac{20+180+120}{10}=\frac{320}{10}$$

$$E(x)=Rs.32/-$$

**Problem 22** The elementary probability law of a continuous random variable is $f(x) = y_0 e^{-b(x-a)}, \quad a \le x < \infty, b > 0$ where a, b and $y_0$ are constants. Find $y_0$, the r$^{th}$ moment about the point $x = a$ and also find the mean and variance.

**Solution:**
Since the total probability is unity,

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

$$y_0 \int_0^{\infty} e^{-b(x-a)}\,dx = 1$$

$$y_0 \left[\frac{e^{-b(x-a)}}{-b}\right]_0^{\infty} = 1$$

$$y_0 \left(\frac{1}{b}\right) = 1$$

$$y_0 = b.$$

$\mu_r'$ ( r$^{th}$ moment about the point $x = a$ )= $\int_{-\infty}^{\infty} (x-a)^r f(x)\,dx$

$$= b\int_a^{\infty} (x-a)^r e^{-b(x-a)}\,dx$$

Put $x - a = t$, $dx = dt$, when $x = a, t = 0$ ; $x = \infty, t = \infty$

$$= b\int_0^{\infty} t^r e^{-bt}\,dt$$

$$= b\frac{\Gamma(r+1)}{b^{(r+1)}} = \frac{r!}{b^r}$$

In particular $r = 1$

$$\mu_1' = \frac{1}{b}$$

$$\mu_2' = \frac{2}{b^2}$$

Mean $= a + \mu_1' = a + \frac{1}{b}$

Variance $= \mu_2' - \left(\mu_1'\right)^2$

$$= \frac{2}{b^2} - \frac{1}{b^2} = \frac{1}{b^2}.$$

**Problem 23** The first four moments of a distribution about $x = 4$ are 1,4,10 and 45 respectively. Show that the mean is 5, variance is 3, $\mu_3 = 0$ and $\mu_4 = 26$ .

**Solution:**

Given $\mu_1' = 1, \mu_2' = 4, \mu_3' = 10, \mu_4' = 45$

$\mu_r' = r^{th}$ moment about to value $x = 4$

Here $A = 4$

Hence mean $= A + \mu_1' = 4 + 1 = 5$

$$\text{Variance} = \mu_2 = \mu_2' - \left(\mu_1'\right)^2$$

$$= 4 - 1 = 3.$$

$$\mu_3 = \mu_3' - 3\mu_2'\mu_1' + 2\left(\mu_1'\right)^3$$

$$= 10 - 3(4)(1) + 2(1)^3 = 0$$

$$\mu_4 = \mu_4' - 4\mu_3'\mu_1' + 6\mu_2'\left(\mu_1'\right)^2 - 3\left(\mu_1'\right)^4$$

$$= 45 - 4(10)(1) + 6(4)(1)^2 - 3(1)^4$$

$$\mu_4 = 26.$$

**Problem 24** A continuous random variable X has the p.d.f $f(x) = kx^2e^{-x}$, $x \geq 0$. Find the r$^{th}$ moment of X about the origin. Hence find mean and variance of X.

**Solution:**

$$\text{Since } \int_0^\infty Kx^2e^{-x}dx = 1$$

$$K\left[x^2\left(\frac{e^{-x}}{-1}\right) - 2x\left(\frac{e^{-x}}{1}\right) + 2\left(\frac{e^{-x}}{-1}\right)\right]_0^\infty = 1$$

$$2K = 1 \qquad K = \frac{1}{2}.$$

$$\mu_r' = \int_0^\infty x^r f(x)dx$$

$$= \frac{1}{2}\int_0^\infty x^{r+2}e^{-x}dx$$

$$= \frac{1}{2}\int_0^\infty e^{-x}x^{(r+3)-1}dx = \frac{(r+2)!}{2}$$

Putting $r = 1$, $\mu_1' = \frac{3!}{2} = 3$

$r = 2$, $\mu_2' = \frac{4!}{2} = 12$

$\therefore$ Mean $= \mu_1' = 3$

$$\text{Variance} = \mu_2' - \left(\mu_1'\right)^2$$

$$\text{i.e., } \mu_2 = 12 - (3)^2 = 12 - 9 = 3$$

**Problem 25** Find the moment generating function of the random variable X, with probability density function $f(x) = \begin{cases} x & for \quad 0 \le x < 1 \\ 2 - x & for \quad 1 \le x < 2 \\ 0 & otherwise \end{cases}$. Also find $\mu_1', \mu_2'$.

**Solution:**

$$M_X(t) = \int_{-\infty}^{\infty} e^{tx} f(x) dx$$

$$= \int_0^1 e^{tx} x dx + \int_1^2 e^{tx}(2 - x) dx$$

$$= \left(\frac{xe^{tx}}{t} - \frac{e^{tx}}{t^2}\right)_0^1 + \left[(2 - x)\frac{e^{tx}}{t} - (-1)\frac{e^{tx}}{t^2}\right]_1^2$$

$$= \frac{e^t}{t} - \frac{e^t}{t^2} + \frac{1}{t^2} + \frac{e^{2t}}{t^2} - \frac{e^t}{t} - \frac{e^t}{t^2}$$

$$= \left(\frac{e^t - 1}{t}\right)^2$$

$$= \left[1 + \frac{t}{1!} + \frac{t^2}{2!} + \frac{t^3}{3!} + \ldots - 1\right]^2$$

$$= \left[1 + \frac{t}{2!} + \frac{t^2}{3!} + \frac{t^3}{4!} + \ldots\right]^2$$

$$\mu_1' = coeff.\ of\ \frac{t}{1!} = 1$$

$$\mu_2' = coeff.\ of\ \frac{t^2}{2!} = \frac{7}{6}.$$

**Problem 26** Find the moment generating function and r^th moments for the distribution whose p.d.f is $f(x) = Ke^{-x}$, $0 \le x \le \infty$. Find also standard deviation.

**Solution:**

Total probability=1

$$\therefore \int_0^\infty ke^{-x} dx = 1$$

$$k\left[\frac{e^{-x}}{-1}\right]_0^\infty = 1 \qquad k = 1$$

$$M_X(t) = E\left[e^{tx}\right] = \int_0^\infty e^{tx} e^{-x} dx = \int_0^\infty e^{(t-1)x} dx$$

$$= \left[\frac{e^{(t-1)x}}{t-1}\right]_0^\infty = \frac{1}{1-t}, \; t < 1$$

$$= (1-t)^{-1} = 1 + t + t^2 + \ldots + t^r + \ldots \infty$$

$$\mu_r' = coeff.\ of\ \frac{t^2}{r!} = r!$$

When $r = 1$, $\mu_1' = 1! = 1$

$r = 2$, $\mu_2' = 2! = 2$

Variance $= \mu_2' - \mu_1' = 2 - 1 = 1$

$\therefore$ Standard deviation = 1.

**Problem 27** Find the moment generating function for the distribution whose p.d.f is $f(x) = \lambda e^{-\lambda x}$, $x > 0$ and hence find its mean and variance.

**Solution:**

$$M_X(t) = E\left(e^{tx}\right) = \int_0^\infty e^{tx} f(x)\, dx = \int_0^\infty \lambda e^{-\lambda x} e^{tx} dx$$

$$= \lambda \int_0^\infty e^{-x(\lambda - t)} dx = \lambda \left[\frac{e^{-x(\lambda - t)}}{-(\lambda - t)}\right]_0^\infty = \frac{\lambda}{\lambda - t}$$

$$\text{Mean} = \mu_1' = \left[\frac{d}{dt} M_X(t)\right]_{t=0} = \left[\frac{\lambda}{(\lambda - t)^2}\right]_{t=0} = \frac{1}{\lambda}$$

$$\mu_2' = \left[\frac{d^2}{dt^2} M_X(t)\right]_{t=0} = \left[\frac{\lambda(2)}{(\lambda - t)^3}\right]_{t=0} = \frac{2}{\lambda^2}$$

$$\text{Variance} = \mu_2' - \left(\mu_1'\right)^2 = \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2}$$

**Problem 28** Let the random variable $X$ have the p.d.f $f(x) = \begin{cases} \frac{1}{2} e^{-\frac{x}{2}} & , x > 0 \\ 0 & , otherwise. \end{cases}$

Find the moment generating function, mean & variance of $X$.

**Solution:**

$$M_X(t) = E\left(e^{tx}\right) = \int_{-\infty}^{\infty} e^{tx} f(x)\, dx = \int_0^\infty e^{tx} \frac{1}{2} e^{-x/2} dx$$

$$= \frac{1}{2}\int_0^{\infty} e^{\left(t-\frac{1}{2}\right)x} dx = \frac{1}{2}\left[\frac{e^{\left(t-\frac{1}{2}\right)x}}{\left(t-\frac{1}{2}\right)}\right]_0^{\infty} = \frac{1}{1-2t}, \; if\; t < \frac{1}{2}.$$

$$E(X) = \left[\frac{d}{dt}M_X(t)\right]_{t=0} = \left[\frac{2}{(1-2t)^2}\right]_{t=0} = 2$$

$$E(X^2) = \left[\frac{d^2}{dt^2}M_X(t)\right]_{t=0} = \left[\frac{8}{(1-2t)^3}\right]_{t=0} = 8$$

$$Var(X) = E(X^2) - [E(X)]^2 = 8 - 4 = 4.$$

**Problem 29** a) Define Binomial distribution Obtain its m.g.f., mean and variance.
b) Six dice are thrown 729 times. How many times do you expect at least 3 dice show 5 or 6 ?

**Solution:**

a) A random variable $X$ is said to follow binomial distribution if it assumes only non-negative values and its probability mass function is given by $P(X = x) = nC_x p^x q^{n-x},\; x = 0,1,2,...,n \; and \; q = 1-p.$

M.G.F of Binomial Distribution about origin is

$$M_X(t) = E\left[e^{tx}\right] = \sum_{x=0}^{n} e^{tx} P(X = x)$$

$$= \sum_{x=0}^{n} nC_x\, x\, P^x q^{n-x} e^{tx}$$

$$= \sum_{x=0}^{n} nC_x \left(pe^t\right)^x q^{n-x}$$

$$M_X(t) = \left(q + pe^t\right)^n$$

Mean of Binomial distribution

$$\text{Mean} = E(X) = M_X'(0)$$

$$= \left[n\left(q+pe^t\right)^{n-1} pe^t\right]_{t=0} = np \text{ Since } q + p = 1$$

$$E(X^2) = M_X''(0)$$

$$= \left[n(n-1)\left(q+pe^t\right)^{n-2}\left(pe^t\right)^2 + npe^t\left(q+pe^t\right)^{n-1}\right]_{t=0}$$

$$E(X^2) = n(n-1)p^2 + np$$

$$= n^2p^2 + np(1-p) = n^2p^2 + npq$$

$$\text{Variance} = E(X^2) - [E[X]]^2 = npq$$

$$Mean = np \; ; Variance = npq$$

b) Let $X$ : the number of times the dice shown 5 or 6

$P[5\ or\ 6] = \frac{1}{6} + \frac{1}{6} = \frac{1}{3}$

$\therefore P = \frac{1}{3}$ and $q = \frac{2}{3}$

Here $n = 6$

To evaluate the frequency of $X \geq 3$

By Binomial theorem,

$$P[X = r] = 6C_r \left(\frac{1}{3}\right)^r \left(\frac{2}{3}\right)^{6-r} \text{ where } r = 0,1,2...6.$$

$$P[X \geq 3] = P(3) + P(4) + P(5) + P(6)$$

$$= 6C_3 \left(\frac{1}{3}\right)^3 \left(\frac{2}{3}\right)^3 + 6C_4 \left(\frac{1}{3}\right)^4 \left(\frac{2}{3}\right)^2 + 6C_5 \left(\frac{1}{3}\right)^5 \left(\frac{2}{3}\right) + 6C_6 \left(\frac{1}{3}\right)^6$$

$$= 0.3196$$

$\therefore$ Expected number of times at least 3 dies to show 5 or 6 $= N \times P[X \geq 3]$

$$= 729 \times 0.3196 = 233.$$

**Problem 30** a) Find the m.g.f. of the geometric distribution and hence find its mean and variance.

b) Six coins are tossed 6400 times. Using the Poisson distribution, what is the approximate probability of getting six heads $x$ times?

**Solution:**

a) M.G.F about origin $= M_X(t) = E\left[e^{tx}\right]$

$$= \sum_{x=0}^{\infty} e^{tx} \frac{\lambda^x e^{-\lambda}}{x!}$$

$$= e^{-\lambda} \sum_{x=0}^{\infty} \frac{\left(\lambda e^t\right)^x}{x!} \quad = e^{-\lambda} e^{\lambda e^t}$$

$$M_X(t) = e^{\lambda\left(e^t - 1\right)}$$

Mean and variance using M.G.F

$$\text{Mean} = E(X) = M_X'(0)$$

$$= \left[e^{\lambda\left(e^t-1\right)} \lambda e^t\right]_{t=0} = \lambda$$

$$E\left(X^2\right) = M_X''(0) = \left[\left(\lambda e^t\right)^2 e^{\lambda\left(e^t-1\right)} + e^{\lambda\left(e^t-1\right)} \lambda e^t\right]_{t=0}$$

$$= \lambda^2 + \lambda$$

$\therefore$ Variance $= E\left(x^2\right) - \left[E(X)\right]^2 = \lambda$

b) Probability of getting one head with one coin $= \frac{1}{2}$.

$\therefore$ The probability of getting six heads with six coins $\left(\frac{1}{2}\right)^6 = \frac{1}{64}$

$\therefore$ Average number of 6 heads with six coins in 6400 throws $= np = 6400 \times \frac{1}{64} = 100$

$\therefore$ Mean of the Poisson distribution $= \lambda = 100$

By Poisson distribution, the approximate probability of getting six heads $x$ times is given by

$$P[X = x] = \frac{\lambda^x e^{-\lambda}}{x!} = \frac{(100)^x e^{-100}}{x!},\ x = 0,1,2,...$$

**Problem 31** a) A die is cast until 6 appears. What is the probability that it must cast more than five times?

b) Suppose that a trainee soldier shoots a target in an independent fashion. If the probability that the target is shot on any one shot is 0.8.
   (i) What is the probability that the target would be hit on $6^{th}$ attempt?
   (ii) What is the probability that it takes him less than 5 shots?

**Solution:** Probability of getting six $= \frac{1}{6}$

$$\therefore p = \frac{1}{6}\ \ \&\ \ q = 1 - \frac{1}{6}$$

Let $x =$ Number of throws for getting the number 6. By geometric distribution $P[X = x] = q^{x-1}p, x = 1,2,3....$

Since 6 can be got either in first, second……throws.

To find $P[X > 5] = 1 - P[X \le 5]$

$$= 1 - \sum_{x=1}^{5}\left(\frac{5}{6}\right)^{x-1}.\frac{1}{6}$$

$$= 1 - \left[\left(\frac{1}{6}\right) + \left(\frac{5}{6}\right)\left(\frac{1}{6}\right) + \left(\frac{5}{6}\right)^2\left(\frac{1}{6}\right) + \left(\frac{5}{6}\right)^3\left(\frac{1}{6}\right) + \left(\frac{5}{6}\right)^4\left(\frac{1}{6}\right)\right]$$

$$= 1 - \frac{\frac{1}{6}\left[1 - \left(\frac{5}{6}\right)^5\right]}{1 - \frac{5}{6}} = \left(\frac{5}{6}\right)^5 = 0.4019$$

b) Here $p = 0.8,\ q = 1 - p = 0.2$

$$P[X = r] = q^{r-1}p, r = 0,1,2...$$

(i) The probability that the target would be hit on the $6^{th}$ attempt $= P[X = 6]$

$$= (0.2)^5(0.8) = 0.00026$$

(ii) The probability that it takes him less than 5 shots

$$= P[X < 5]$$

$$= \sum_{r=1}^{4} q^{r-1} p = 0.8 \sum_{r=1}^{4} (0.2)^{r-1}$$
$$= 0.8[1 + 0.2 + 0.04 + 0.008] = 0.9984$$

**Problem 32** a) If $X_1, X_2$ are two independent random variables each flowing negative binomial distribution with parameters $(r_1, p)$ and $(r_2, p)$, show that the sum also follows negative binomial distribution

b) If a boy is throwing stones at a target, what is the probability that his $10^{\text{th}}$ throw is his $5^{\text{th}}$ hit, if the probability of hitting the target at any trial is 0.5?

**Solution:**

a) Let $X_1$ be a negative binomial variate with $(r_1, p)$ and $X_2$ be another negative binomial variate wit $(r_2, p)$ and let them be independent.

Then $M_{X_1}(t) = \left(q - pe^t\right)^{-r_1}$

$M_{X_2}(t) = \left(q - pe^t\right)^{-r_2}$

Then $M_{X_1+X_2}(t) = M_{X_1}(t) M_{X_2}(t)$

$= \left(q - pe^t\right)^{-(r_1+r_2)}$, which is the m.g.f of a negative binomial variable with $r_1 + r_2$ as parameter. This proves the result.

b) Since the $10^{\text{th}}$ throw should result in the $5^{\text{th}}$ success

(i.e.) the $5^{\text{th}}$ hit, the first 9 throws should have resulted in 4 successes and 5 failures. Hence we have $x = 5, r = 5, p = q = \frac{1}{2}$

$\therefore$ Required probability $= P[X = 5]$

By N.B.D, $P[X = x] = \binom{x + r - 1}{x} p^r q^x$

$$= \binom{5+5-1}{5} p^5 q^5 = 9C_4 \left(\frac{1}{2}\right)^{10} = 0.123.$$

**Problem 33** a) State and prove the memoryless property of exponential distribution.

b) A component has an exponential time to failure distribution with mean of 10,000 hours.

(i) The component has already been in operation for its mean life. What is the probability that it will fail by 15,000 hours?

(ii) At 15,000 hours the component is still in operation. What is the probability that it will operate for another 5000 hours.

**Solution:**

a) Statement: If $X$ is exponentially distributed with parameters $\lambda$, then for any two positive integers s and t, $P[x > s + t / x > s] = P[x > t]$

Proof:

The p.d.f of X is $f(x)=\begin{cases}\lambda e^{-\lambda x}, x\geq 0\\ 0 \quad ,Otherwise\end{cases}$

$$\therefore P[X>t]=\int_t^{\infty}\lambda e^{-\lambda x}dx=\left[-e^{-\lambda x}\right]_t^{\infty}=e^{-\lambda t}$$

$$\therefore P[X>s+t/x>s]=\frac{P[x>s+t\cap x>s]}{P[x>s]}$$

$$=\frac{P[X>s+t]}{P[X>s]}=\frac{e^{-\lambda(s+t)}}{e^{-\lambda s}}=e^{-\lambda t}$$

$$=P[x>t]$$

b) Let $X$ denote the time to failure of the component then $X$ has exponential distribution with $Mean=1000$ hours.

$$\therefore \frac{1}{\lambda}=10,000\Rightarrow \lambda=\frac{1}{10,000}$$

The p.d.f. of $X$ is $f(x)=\begin{cases}\frac{1}{10,000}e^{-\frac{x}{10,000}} &, x\geq 0\\ 0 &, otherwise\end{cases}$

(i) Probability that the component will fail by 15,000 hours given it has already been in operation for its mean life $=P[x<15,000/x>10,000]$

$$=\frac{P[10,000<X<15,000]}{P[X>10,000]}$$

$$=\frac{\int_{10,000}^{15,000}f(x)dx}{\int_{10,000}^{\infty}f(x)dx}=\frac{e^{-1}-e^{-1.5}}{e^{-1}}$$

$$=\frac{0.3679-0.2231}{0.3679}=0.3936.$$

(ii) Probability that the component will operate for another 5000 hours given that it is in operation 15,000 hours $=P[X>20,000/X>15,000]$

$$=P[x>5000] \quad \text{[By memoryless property]}$$

$$=\int_{5000}^{\infty}f(x)dx=e^{-0.5}=0.6065$$

**Problem 34** The Daily consumption of milk in a city in excess of 20,000 gallons is approximately distributed as a Gamma variate with parameters $\alpha=2$ and

$\lambda = \frac{1}{10,000}$. The city has a daily stock of 30,000 gallons. What is the probability that the stock is insufficient on a particular day?

**Solution:**
Let $X$ be the r.v denoting the daily consumption of milk (is gallons) in a city
Then $Y = X - 20,000$ has Gamma distribution with p.d.f.

$$f(y) = \frac{1}{(10,000)^2 \Gamma(2)} y^{2-1} e^{-\frac{y}{10,000}}, y \geq 0$$

$$f(y) = \frac{ye^{-\frac{y}{10,000}}}{(10,000)^2}, y \geq 0.$$

$\therefore$ The daily stock of the city is 30,000 gallons; the required probability that the stock is insufficient on a particular day is given by

$$P[X > 30,000] = P[Y > 10,000]$$

$$= \int_{10,000}^{\infty} g(y)dy = \int_{10,000}^{\infty} \frac{ye^{-\frac{y}{10,000}}}{(10,000)^2} dy$$

Put $Z = \frac{y}{10,000}$, then $dz = \frac{dy}{10,000}$

$$\therefore P[X > 30,000] = \int_1^{\infty} ze^{-z} dz$$

$$= \left[-ze^{-z} - e^{-z}\right]_1^{\infty} = \frac{2}{e}$$

**Problem 35** a) suppose that the lifetime of a certain kind of an emergency backup battery (in hours) is a r.v. $X$, having the Weibull distribution with parameter $\alpha = 0.1$ and $\beta = 0.5$.

(i) Find the mean life time of these batteries

(ii)The probability that such a battery will last more than 800 hours.

b) Each of the 6 tubes of a radio set has the life length (in years) which may be considered as a r.v that follows a Weibull distribution with parameter $\alpha = 25$ and $\beta = 2$. If these tubes function independently of one another, what is the probability that no tube will have to be replaced during the first 2 months of service?

**Solution:**
a) For the p.d.f of $X$ in the form $f(x) = \alpha\beta x^{\beta-1}e^{-\alpha x^{\beta}}, x > 0$

$$\text{Mean} = \alpha^{-\frac{1}{\beta}} \Gamma\left(\frac{1}{\beta} + 1\right)$$

i) Mean life time $= \mu = (0.1)^{-\frac{1}{0.5}} \Gamma\left(1 + \frac{1}{0.5}\right)$

$$= (0.1)^{-2} \Gamma(3) = 100 \times 2$$

$= 200\ hours$

ii) Probability that a battery will last more than 300 hours $= P[X > 300]$

$$\int_{300}^{\infty} f(x)dx = \int_{300}^{\infty} \alpha\beta x^{\beta-1}e^{-\alpha x^{\beta}}\,dx$$

$$= \int_{300}^{\infty} (0.1)(0.5)x^{-0.5}e^{-(0.1)x^{0.5}}\,dx$$

Put $y = (0.1)x^{0.5}$

Then $dy = (0.1)(0.5)x^{-0.5}dx$

$$\therefore P[X > 300] = \int_{(0.1)(300)^{0.5}}^{\infty} e^{-y}\,dy$$

$$= e^{-(0.1)(300)^{0.5}} = 0.177$$

b) Let $X$ be life length of the tube

Then the p.d.f of $X$ is given by $f(x) = \alpha\beta x^{\beta-1}e^{-\alpha x^{\beta}}$, $x > 0$

i.e., $f(x) = 50xe^{-25x^2}$, $x > 0$

P [a tube is not replaced during the first 2 months]

$$= P\left[X > \frac{1}{6}\right], \quad \text{since two months} = \frac{1}{6}\,yrs$$

$$= \int_{\frac{1}{6}}^{\infty} 50xe^{-25x^2}dx$$

$$= \left(-e^{25x^2}\right)_{\frac{1}{6}}^{\infty} = e^{-\frac{25}{36}}$$

P[all 6 tube are not replaced during first 2 months]

$$= \left(e^{-25/36}\right)^6 = e^{-25/6} = 0.0155\,.$$

**Problem 36** Support that the service life (in hours) of a semi conductor is a random variable having the Weibull distribution with $\alpha = 0.025$ and $\beta = 0.5$, what is the probability that such a semi conductor will be in working condition after 4000 hours?

**Solution:**

Let $X$ be the service life of the conductor.

Then $X$ has Weibull distribution with parameter $\alpha$ and $\beta$ and whose density is given by

$$f(x) = \alpha\beta x^{\beta-1}e^{-\alpha x^{\beta}}$$

Here $\alpha$ =0.025 and $\beta$ =0.5

P [Conductor will be in working after 4000 hours] $= P[x \geq 4000] = \int_{4000}^{\infty} f(x)dx$

We know that $P[X \geq a] = e^{-\alpha a^{\beta}}$

$\therefore$ Required probability $= e^{-(0.025)(4000)^{0.5}}$

$= e^{-1.58} = 0.2057$

**Problem 37** a) The amount of time that a camera will run without having to be reset is a random variable having exponential distribution with $\theta = 50$ days. Find the probability that such a camera will (i) have to be reset in less than 20 days (ii) not has to be reset in at least 60 days.

b) Subway trains on a certain line run every half an hour between midnight and six in the morning. What is the probability that a man entering the station at a random time during this period will have to wait at least 20 minutes.

**Solution:**

a) Let $X$ be the time that the camera will run without having to be reset.

The $X$ is a random variable with exponentially distributed with

$\theta = 50$ days. The p.d.f of X is given by

$$f(x) = \begin{cases} \frac{1}{\theta} e^{-\frac{x}{\theta}}, x \geq 0 \\ 0 \qquad x < 0 \end{cases}$$

(i) P[Camera will have to be reset in less than 20 days]

$= P[\textit{Camera will run for less than } 20 \textit{ days}]$

$= P[X < 20]$

$$= \int_0^{20} \frac{1}{50} e^{-\frac{x}{50}} dx = 1 - e^{-\frac{2}{5}} = 0.3297$$

(ii) P[Camera will not have to be reset in at least 60 days]

$= P[\textit{camera will run for atleast } 60 \textit{days}]$

$$= P[X > 60] = \int_{60}^{\infty} \frac{1}{50} e^{-\frac{x}{50}} dx = e^{-\frac{6}{5}} = 0.3012$$

b) Let $X$ denote the waiting time in minutes for the next train. Under the assumption that a man arrives at the station at random time, X is uniformly distributed on (0, 30) with

p.d.f. $f(x) = \begin{cases} \frac{1}{30}, 0 < x < 30 \\ 0 \quad , Otherwise \end{cases}$

The probability that he has to wait at least 20 minutes

$= P[X \geq 20]$

$$= \int_{20}^{30} f(x)\, dx = \frac{1}{30}(30 - 20) = \frac{1}{3}$$

**Problem 38** If $X$ is a random variable uniformly distribution in (-1,1), find the p.d.f. of $y = \cos \pi x$

**Solution:**

The p.d.f of $X$ is $f_X(x) = \begin{cases} \frac{1}{2}, & -1 < x < 1 \\ o, & otherwise \end{cases}$

$$y = cos\pi x$$

$$\frac{dy}{dx} = -\pi sin\pi x = -\pi\sqrt{1-y^2}$$

$$\therefore \left|\frac{dx}{dy}\right| = \frac{1}{\pi\sqrt{1-y^2}}$$

Range of $Y$: $-1 < x < 1 \Rightarrow -1 < \frac{1}{\pi}cos^{-1}y < 1$

$$\Rightarrow -\pi < cos^{-1}y < \pi$$

$$\Rightarrow -1 < y < 1$$

$\therefore$ p.d.f of $Y$ is $f_Y(y) = \begin{cases} \frac{1}{2\pi\sqrt{1-y^2}}, & -1 < y < 1 \\ 0, & Otherwise \end{cases}$

**Problem 39** If $X$ is uniformly distributed in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, find the p.d.f of $Y = \tan X$.

**<u>Solution:</u>**

a) The p.d.f of $X$ is $f_X(x) = \begin{cases} \frac{1}{\pi}, & -\frac{\pi}{2} < x < \frac{\pi}{2} \\ 0, & Otherwise \end{cases}$

$$y = \tan x \Rightarrow x = \tan^{-1}(y)$$

$$\therefore \frac{dx}{dy} = \frac{1}{1+y^2}$$

$\therefore$ p.d.f. of $f_Y(y) = f_X(x)\left|\frac{dx}{dy}\right|$

$$= \frac{1}{\pi}\frac{1}{1+y^2}, -\infty < y < \infty$$

(This distribution of y is called Cauchy distribution)

**Problem 40** a) Given the r.v. $X$ with density function $f_X(x) = \begin{cases} 2x, & 0 < x < 1 \\ 0, & otherwise \end{cases}$.

Find the p.d.f of (i) $Y = 8X^3$ (ii) $Y = 3X + 1$.

b) If the r.v. $X$ follows an exponential distribution with parameter 2, prove that $Y = X^3$ follows Weibull distribution with parameters 2 and 1/3.

**<u>Solution:</u>**

a) (i) $y = 8x^3 \Rightarrow \dfrac{dy}{dx} = 8.3x^2 = 24x^2 = 24\left(\dfrac{y}{8}\right)^{\frac{2}{3}} = 6y^{\frac{2}{3}}$

$$= \left|\frac{dy}{dx}\right| = \frac{1}{6}y^{-\frac{2}{3}}$$

The p.d.f of $Y$ $f_Y(y) = f_X(x)\left|\dfrac{dx}{dy}\right| = 2x - \dfrac{1}{6}y^{-\frac{2}{3}}$

$$= y^{\frac{1}{3}}.\frac{1}{6}y^{-\frac{2}{3}}$$

$$\Rightarrow f_Y(y) = \frac{1}{6}y^{-\frac{1}{3}},\ 0 < y < 8$$

(ii) $y = 3x + 1 \Rightarrow \dfrac{dx}{dy} = \dfrac{1}{3}$ and $x = \dfrac{y-1}{3}$

p.d.f of $Y$ $f_Y(y) = f_X(x)\left|\dfrac{dx}{dy}\right|$

$$= \frac{2}{3}(y-1)\left(\frac{1}{3}\right) = \frac{2}{9}(y-1),\ 1 < y < 4$$

b) The p.d.f of $X$ is $f_X(x) = \begin{cases} 2e^{-2x}, & x > 0 \\ 0 & , otherwise \end{cases}$

Then $\dfrac{dy}{dx} = 3x^2 \Rightarrow \dfrac{dx}{dy} = \dfrac{1}{3x^2} = \dfrac{1}{3}y^{-\frac{2}{3}}$

The p.d.f of $Y$ is $f_Y(y) = f_X(x)\left|\dfrac{dx}{dy}\right|$

$$= \frac{1}{3}y^{-\frac{2}{3}}.2e^{-2x}$$

$$= \frac{1}{3}y^{-\frac{2}{3}}2e^{-2y^{\frac{1}{3}}}$$

$$\therefore f_Y(y) = 2\left(\frac{1}{3}\right)y^{\frac{1}{3}-1}e^{-2y^{\frac{1}{3}}}$$

Taking $\alpha = 2$ and $\beta = \dfrac{1}{3}$ the p.d.f of $y$ is in the form

$f_Y(y) = \alpha\beta y^{\beta-1}e^{-\alpha y^{\beta}}$ . Since $x > 0$ , $y > 0$

$\therefore$ The r.v. $Y = X^3$ follows Weibull distribution with parameters 2 and $\dfrac{1}{3}$.