UIT1501 Finite Automata Theory

Turing Machines

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Outline

1. What have we seen so far?
   - Context-free Grammar
   - Pushdown Automata
   - Relationship between CFG and PDA
   - Normal Forms
   - Pumping Lemma and Closure Properties

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2. Turing Machines
   - Informal Introduction
   - Formal Definitions
   - Examples
   - Computing integer-valued functions
   - Languages of Turing Machines

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Context-Free Grammar

- We have discussed about context-free grammars.
- We have seen some examples of constructing CFGs.
- We have also seen the formal definitions of derivations, starred derivations, leftmost derivations and rightmost derivations.
- We have seen the formal definition of the language of a CFG and formal definition of context-free languages.
- We have discussed about inferring strings and parse trees for context-free grammars
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- Any Questions?

Pushdown Automata

- We have seen the pushdown automata model — formal definitions, transition diagrams, instantaneous descriptions
- We have discussed two different models of PDA — one that accepts by entering a final state and the other by emptying its stack.
- We have shown that both these models are equivalent.
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PDA, CFG, and DPDA

- We have seen how a PDA can be systematically constructed to accept the words of language of a given CFG.
- We have also seen the converse — to systematically construct an equivalent CFG from a given PDA
- Thus the set of languages of PDA is exactly same as the set of languages of CFG — Context-free Languages
- We have seen a deterministic version of PDA (DPDA).
- DPDA do not have the same expressive power as PDA but they are more powerful than FA.
- Thus, we have $\mathcal{R} \subset DCFL \subset CFL$
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- We have seen how a CFG can be simplified by
  - eliminating all ϵ-productions
  - eliminating all unit productions
  - eliminating all useless symbols
- We have seen Chomsky Normal Form (CNF) and how to convert any grammar to its CNF
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- We have seen the pumping lemma for context-free languages
- Using the pumping lemma, we have shown the existence of non-context-free languages.
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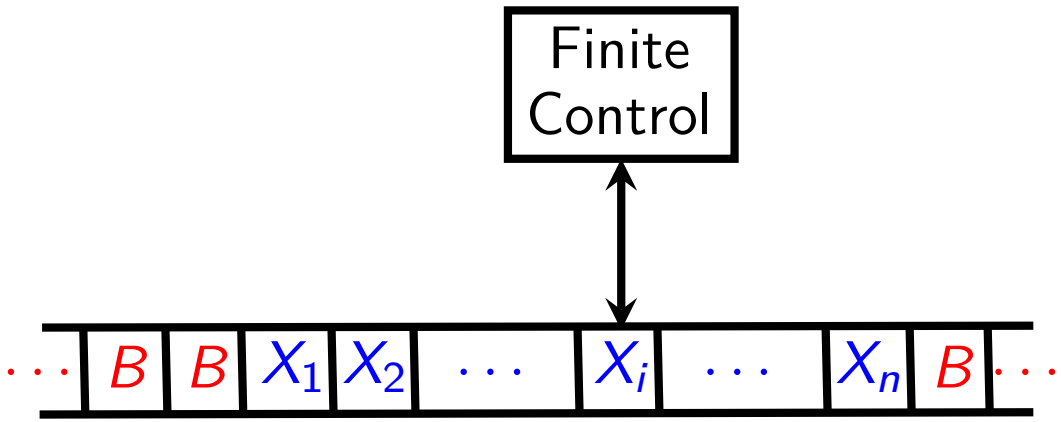
Informal Introduction



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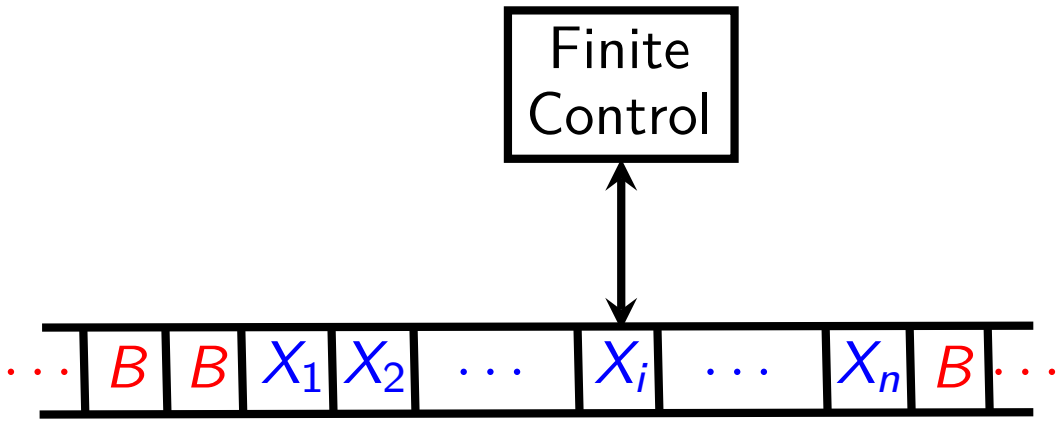

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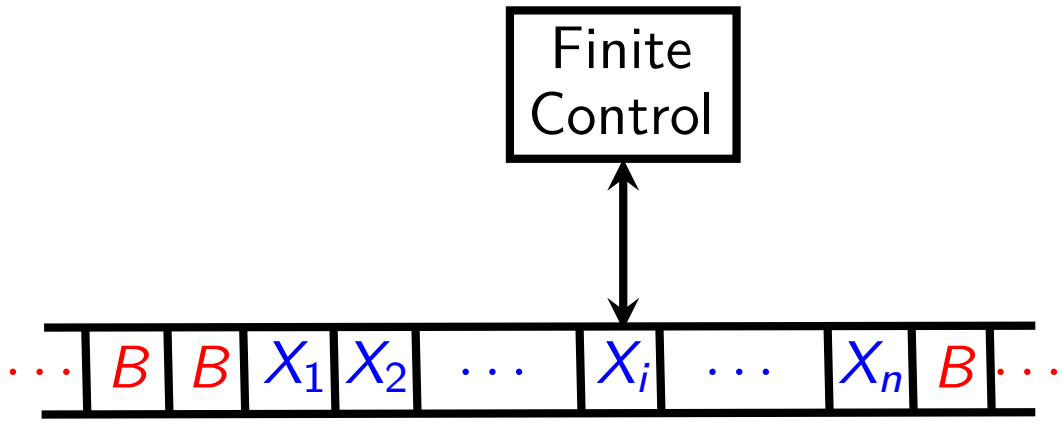

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- All the cells in the tape contain a special blank symbol B.

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- When the machine starts, some input string $X_1X_2 \cdots X_n$ is available on the tape and the finite control points to the cell containing the first symbol X_1. We say that the machine is scanning that cell. The machine starts in its designated start state.

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  - Move the finite control to the left or right.
- A "step" of operation of the machine as described above is prescribed by its "program" given by a transition function.

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- $\delta : (Q\ X\ \Gamma) \rightarrow (Q\ X\ \Gamma\ X\ D)$ is the **transition function**, where D is the set of directions $\{L, R\}$.

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- Suppose $\delta(q, X_i) = (p, Y, L)$. Then the move of the machine is captured as

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- As we have seen before, we can capture multiple moves of the machine $\vdash^*$ (zero or more moves) as reflective and transitive closure of $\vdash$.

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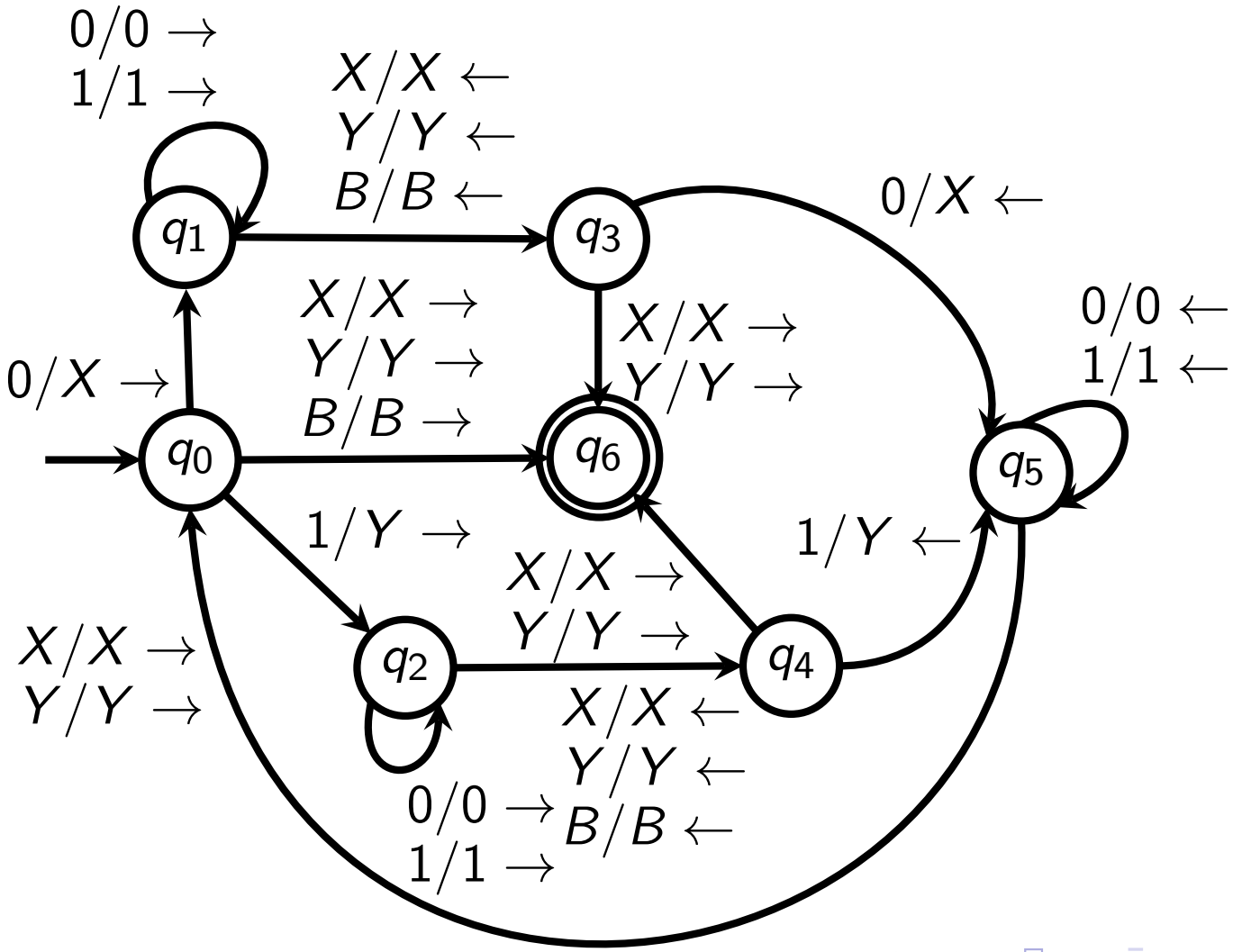
- Note that ϵ needs to be accepted. So, the machine halts in a final state in the beginning itself if the first symbol read is a blank B.

Transition Table

- Based on this idea, we can write the following "program".

	0	1	X	Y	B
$\rightarrow q_0$	(q_1, X, R)	(q_2, Y, R)	(q_6, X, R)	(q_6, Y, R)	(q_6, B, R)
q_1	$(q_1, 0, R)$	$(q_1, 1, R)$	(q_3, X, L)	(q_3, Y, L)	(q_3, B, L)
q_2	$(q_2, 0, R)$	$(q_2, 1, R)$	(q_4, X, L)	(q_4, Y, L)	(q_4, B, L)
q_3	(q_5, X, L)	—	(q_6, X, R)	(q_6, Y, R)	—
q_4	—	(q_5, Y, L)	(q_6, X, R)	(q_6, Y, R)	—
q_5	$(q_5, 0, L)$	$(q_5, 1, L)$	(q_0, X, R)	(q_0, Y, R)	—
$*q_6$	—	—	—	—	—

Transition Diagrams



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$\vdash Y q_5 X X Y \vdash Y X q_0 X Y$
$\vdash Y X X q_6 Y$

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$\vdash Y q_5 X X Y \vdash Y X q_0 X Y$
$\vdash Y X X q_6 Y$

- Since the machine has entered an accepting state, the input string 1001 is accepted. Note that in our example, the machine halts at the accepting state since no moves are defined for the accepting state q_6.

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- Since $\delta(q_3, 1)$ is not defined, the machine halts at q_3. Since q_3 is not an accepting state, the input string 1011 is rejected.

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q_2	(q_2, a, L)	—	(q_0, X, R)	(q_2, Y, L)	—
q_3	—	—	—	(q_3, Y, R)	(q_4, B, R)
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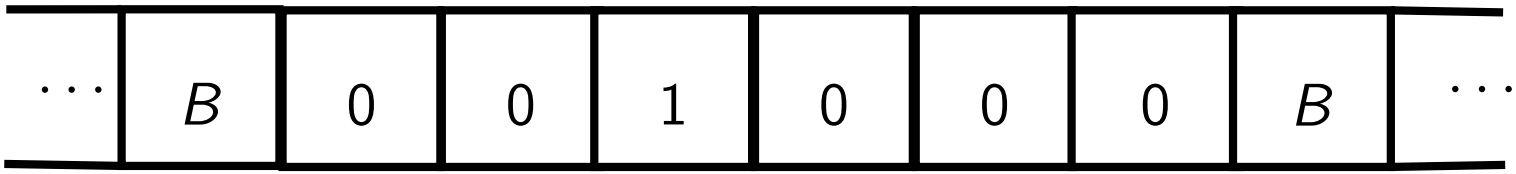
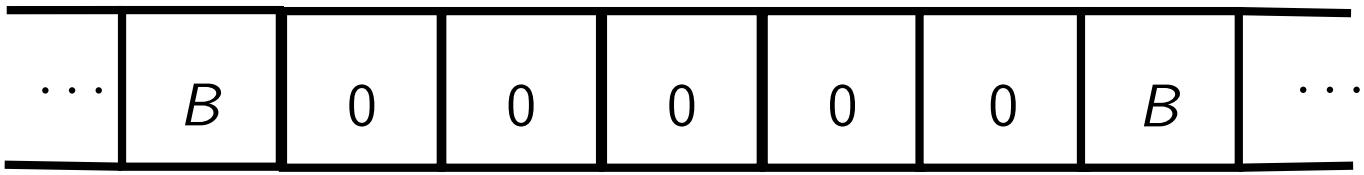
- A machine can be designed to read the arguments from the tape (separated by a separator symbol 1), and leave the answer on the tape when it halts.
- For example, we can let a TM start with 001000 on the tape and halt with 00000 to realize the addition function.





Example of integer computation

- Let us construct a TM to perform $\dot{-}$, the proper subtraction over integers, defined by $m \dot{-} n = max(m - n, 0)$. That is, $m \dot{-} n = m - n$ if $m > n$ and $m \dot{-} n = 0$ if $m \leq n$.

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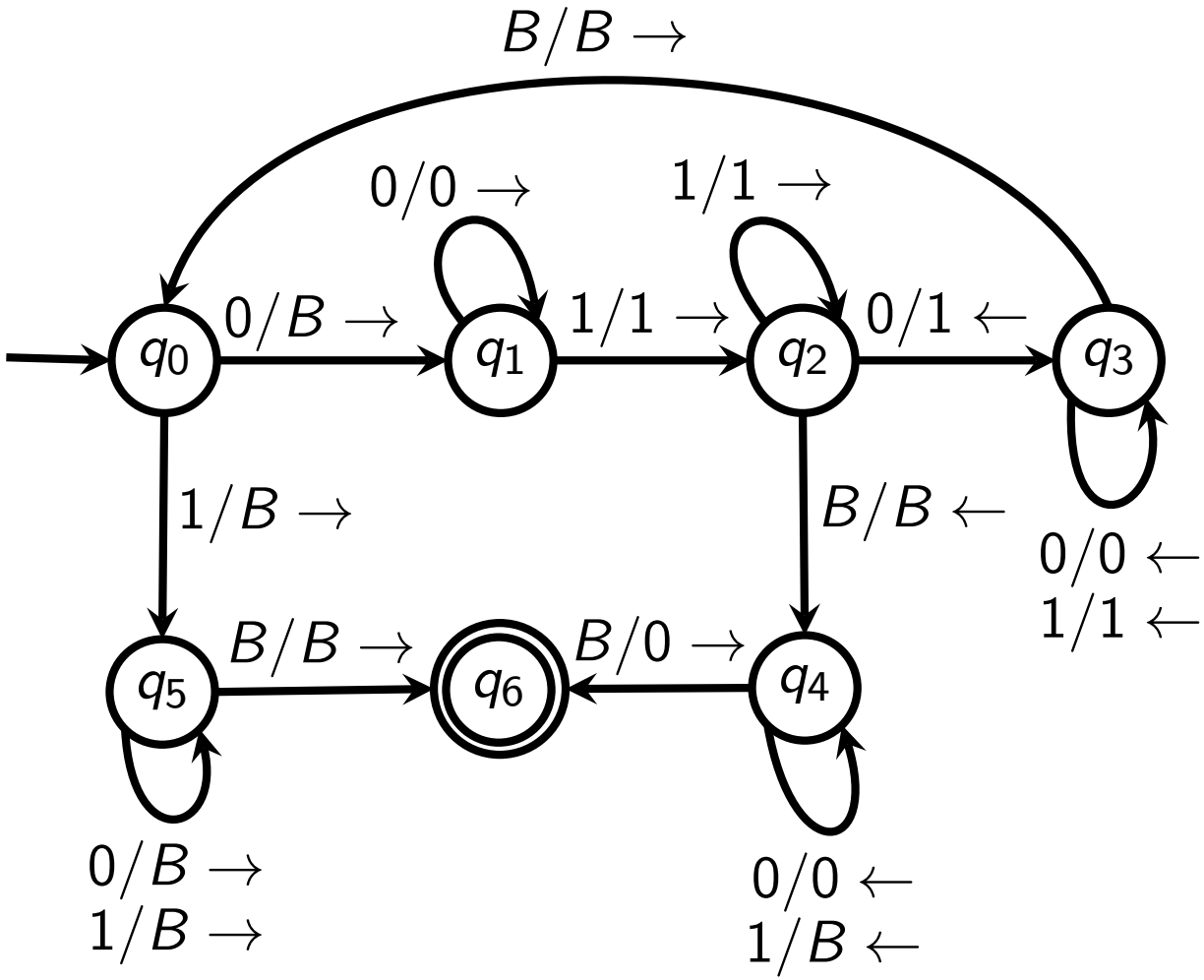
  - Searching for a 0 in the second argument, a blank B is seen. This arises when $m > n$. Now the machine comes back, changes the first 1 to a 0 and erases the remaining 1's by changing them to B's.
  - At the beginning of the iteration, the machine can not find any more 0 in the first argument. This arises when $m \leq n$. The machine blanks out all the 1's and 0's to halt with the blank tape.

Transition Table

- Following is a program for proper subtraction.

	0	1	B
$\rightarrow q_0$	(q_1, B, R)	(q_5, B, R)	—
q_1	$(q_1, 0, R)$	$(q_2, 1, R)$	—
q_2	$(q_3, 1, L)$	$(q_2, 1, R)$	(q_4, B, L)
q_3	$(q_3, 0, L)$	$(q_3, 1, L)$	(q_0, B, R)
q_4	$(q_4, 0, L)$	(q_4, B, L)	$(q_6, 0, R)$
q_5	(q_5, B, R)	(q_5, B, R)	(q_6, B, R)
$*q_6$	—	—	—

Transition Diagram



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- Let $M = (Q, \Sigma, \Gamma, \delta, q_0, B, F)$ be a Turing machine. Then $Lang(M)$, the language of the machine M is defined as follows:

$$Lang(M) = \left\{ w \mid w \in \Sigma^* \text{ and } q_0 w \vdash^* \alpha p \beta \text{ for some } p \in F \text{ and } \alpha, \beta \in \Gamma \right.$$

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- Note that there is no condition that the machine has seen all the input symbols. The machine may accept w just by seeing a prefix of it.
- The class of languages described by Turing machines is termed as **recursively enumerable languages**.

$$RE = \left\{ L \mid \text{ there exists a TM } M \text{ such that } L = Lang(M) \right\}$$

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- However, it is not possible to ensure that M always halts when it is not accepting the input string.
- Suppose we insist that a Turing machine must halt on all inputs. This is important because such machines stand for "algorithms". Languages of such always-halting TM's are referred to as **recursive languages**.

$$RC = \{L \mid \text{ there exists TM } M \text{ that halts on all inputs and } L = Lang(M$$

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- It remains to be seen if $RC = RE$???
- And of course the big question: Is $RE = 2^{\Sigma^*}$???

Exercises

8.2.1 Show the ID's of the palindrome checking TM if the input tape contains:

- 00
- 100101
- 01110

8.2.2 Design TMs for the following languages:

- The set of strings with an equal number of 0's and 1's
- $\left\{a^n b^n c^n \mid n \geq 1\right\}$
- The set of all strings of balanced parentheses.

Exercises

8.2.5 Consider the TM

$$M = \left(\{q_0, q_1, q_2, q_3, q_f\}, \{0, 1\}, \{0, 1, B\}, \delta, q_0, B, \{q_f\}\right)$$

Informally but clearly describe the language $L(M)$ if δ consists of the following sets of rules:

- $\delta(q_0, 0) = (q_1, 1, R); \delta(q_1, 1) = (q_0, 0, R); \delta(q_1, B) = (q_f, B, R)$.
- $\delta(q_0, 0) = (q_0, B, R); \delta(q_0, 1) = (q_1, B, R); \delta(q_1, 1) = (q_1, B, R); \delta(q_1, B) = (q_f, B, R)$.
- $\delta(q_0, 0) = (q_1, 1, R); \delta(q_1, 1) = (q_2, 0, L); \delta(q_2, 1) = (q_0, 1, R); \delta(q_1, B) = (q_f, B, R)$.

Summary

- We have seen an informal introduction to Turing machines and their formal definitions.
- We have seen some examples of constructing Turing machines.
- Turing machines can be seen as computing integer functions.
- We have seen an example of integer computing with Turing machines.
- We have seen the definition of language of a Turing machine — recursively enumerable languages
- Recursive languges are those for which Turing machines that halt on all inputs can be constructed.
- It remains to be seen if $RC \subset RE$?? and $RE \subset 2^{\Sigma^*}$???

What next?

- Review regular expressions and finite automata
- Review context-free grammars and pushdown automata
- Review pumping lemmas and their applications
- Review closure and decision properties of regular and context-free languages
- Review the Turing Machines
- Work out the exercises given during the lectures!
- Review the basics of complexity analysis of algorithms
- In the next class, we will continue our discussions on Turing Machines