Unit - I One dimensional Random Variables.

Random variables :

It is a function that assigns a real number $X(s)$ $s \in S$ where

$S \rightarrow$ sample space corresponding to random experiment 'E'.

Examples :

1. The number of telephone calls received in one hour.

X	0	1	2	3	4	5

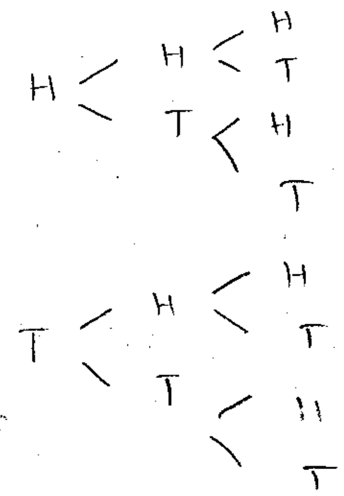
2. Tossing two coins at a time
The result are non numerical in nature.

$$S = \{HH, HT, TH, TT\}$$

Let X be the no of heads or tails

X	2	1	1	0

when tossing three coins at a time



{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT}

Discrete Random Variable (countable)

If x is a Random variable which can take finite or countably infinite number of values, x is called discrete random variable.

continuous Random Variable.

If x is a Random variable which can take values (Infinite number of values) in a interval, then x is called a continuous Random variable.

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HTT,
TT}

Probability function :

If x is a discrete Random variable which can take the values $x_1, x_2, x_3 \ldots$ such that

$$P[x = x_i] = P_i$$

then P_i is called the probability function (or) probability mass function (or) point probability function satisfies the following conditions

le)
which
finite
ed

(i) $P_i \geq 0 \quad \forall i \quad \{i = 0, 1, 2 \ldots\}$

(ii) $\sum P_i = 1$

It is denoted as [Discrete]

X	x_1	x_2	x_3	......
$P(x)$	P_1	P_2	P_3	......

which
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is

Related formulas :

1. To find constant $\sum P(x) = 1$
2. Mean: $E(X) = \sum x \, P(x)$
3. Second moment about origin
$$E(X^2) = \sum x^2 P(x)$$
4. r^{th} moment $E[X^r] = \sum x^r P(x)$

Variance $E[X^2] = E(X^2) - [E(X)]^2$

5. Moment generating function

$$M_x(t) = E[e^{tx}] = \Sigma e^{tx} p(x)$$

6. cumulative distribution function $F(x) = P[X \le x]$

1. For random variables x takes the values 1, 2, 3, 4 such that $2P(X=1) = 3P(X=2) = P(X=3) = 5P(X=4)$. Find the probability distribution and cumulative distribution.

Here x is discrete Random variable

X	1	2	3	4
$P(X)$	$k/2$	$k/3$	k	$k/5$

$2P(X=1) = 3P(X=2) = P(X=3) = 5(P(X=4))$
$= k$

$2P(X=1) = k$ $\quad$ $3P(X=2) = k$

$P(X=1) = k/2$ $\quad$ $P(X=2) = k/3$

$P(X=3) = k$ $\quad$ $5P(X=4) = k$

$P(X=4) = k/5$

W.K.T $\Sigma P(X) = 1$

$$\frac{k}{2} + \frac{k}{3} + k + \frac{k}{5} = 1$$

$$k\left(\frac{1}{2} + \frac{1}{3} + 1 + \frac{1}{5}\right) = 1$$

$x \le n]$

the

$=1)$

$)$

and

$x=4)$

$= k$

$$k\left(\frac{15+10+30+6}{30}\right) = 1$$

$$k\left(\frac{61}{30}\right) = 1$$

$$k = \frac{30}{61}.$$

Probability distribution

X	1	2	3	4
$P(X)$	$\frac{30}{61\times 2}$	$\frac{30}{61\times 3}$	$\frac{30}{61}$	$\frac{30}{61\times 5}$
	$=\frac{15}{61}$	$\frac{10}{61}$	$\frac{30}{61}$	$\frac{6}{61}$

Cumulative distribution function

X	1	2	3	4
$P(X)$	$\frac{15}{61}$	$\frac{10}{61}$	$\frac{30}{61}$	$\frac{6}{61}$

$F(X)$ $\quad P(X\le 1) = P(1) = \frac{15}{61}$

$P(X \le 2) = P(2) + P(1) = \frac{10}{61} + \frac{15}{61} = \frac{25}{61}$

$P(X \le 3) = P(3) + P(2) = \frac{30}{61} + \frac{25}{61} = \frac{55}{61}$

$P(X \le 4) = P(4) + P(3) = \frac{6}{61} + \frac{55}{61} = \frac{61}{61} = 1$

2. A Random variable has the following probability distribution function

X	-2	-1	0	1	2	3
P(x)	0.1	K	0.2	2K	0.3	3K

Find (i) K (ii) $P[X < 2]$ (iii) $P[-2 < x < 2]$

(iv) C.D.F of x (v) Mean of X.

Soln :

(i) $\sum P(x) = 1$

$0.1 + K + 0.2 + 2K + 0.3 + 3K = 1$

$6K + 0.6 = 1$

$6K = 1 - 0.6 = 0.4$

$6K = 0.4$

$K = \frac{0.4}{6} = \frac{0.2}{3} = 0.0666$

(ii) $P[X < 2] = P[X=1] + P[X=0] + P[X=-1] + P[X=-2]$

$= 2K + 0.2 + K + 0.1$

$= 0.3 + 3K$

$= 0.3 + 3\left(\frac{0.2}{3}\right) = 0.5$

wing

(iii) $P[-2 < x < 2] = P[X=-1] + P[X=0] + P[X=1]$

$= K + 0.2 + 2K$

$= 0.2 + 3K = 0.2 + 3\left(\frac{0.2}{3}\right)$

$= 0.4$

$= 2]$

(iv) C.D of X

x	-2	-1	0	1	2	3
$P(x)$	0.1	0.0666	0.2	0.1332	0.3	0.1998
$F(X)$	0.1					

$F(-1) = P(-1) + F(-2)$
$= 0.1 + 0.0666$
$= 0.1666$

$F(0) = P(0) + F(-1)$
$= 0.2 + 0.1666$
$= 0.3666$

$F(1) = P(1) + F(0)$
$= 0.1332 + 0.3666$
$= 0.4998$

$F(2) = P(2) + F(1)$
$= 0.3 + 0.4998$
$= 0.7998$

$F(3) = P(3) + F(2)$
$= 0.1998 + 0.7998$
$= 0.9996 = 1.$

(v) Mean $E(X) = \Sigma x\,(P(X))$

$= -2\,P(-2) + (-1)\,P(-1) + 0\,(P(0)) + 1\,P(1) + 2\,P(2) + 3\,P(3)$

$= -2(0.1) + (-1)(0.0666) + 0 + (1 \times 0.1332) + 2(0.3) + 3(0.1998)$

$= -0.2 - 0.0666 + 0.1332 + 0.6 + 0.5994$

$= 1.066$

3. The probability distribution of an infinite discrete Random variable

$P[X = j] = \frac{1}{2^j}$, $j = 1, 2, 3 \ldots \infty$. verify the total probability is 1 and also find the mean

(ii) $P[\text{even}]$, $P[X \geq 5]$, $P[X \div 3]$

Soln:

To verify $\sum_{j=1}^{\infty} P(x_j) = 1$

$= \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \frac{1}{2^5} + \ldots\ldots$

$= \frac{1}{2} + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 + \left(\frac{1}{2}\right)^4 + \ldots$

$= \frac{1}{2}\left[1 + \frac{1}{2} + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 + \ldots\right]$

W.K.T $1 + x + x^2 + x^3 = (1-x)^{-1}$

$= \frac{1}{2}\left[1 - \frac{1}{2}\right]^{-1}$

$= \frac{1}{2}\left[\frac{1}{2}\right]^{-1}$

$= \frac{1}{2} \times \frac{2}{1} = 1 \Rightarrow$ hence verified.

Mean: $E(X) = \sum x P(x)$

$= \left(1 \times \frac{1}{2}\right) + 2 \times \left(\frac{1}{2}\right)^2 + 3\left(\frac{1}{2}\right)^3 + \ldots$

$= \frac{1}{2}\left[1 + 2\left(\frac{1}{2}\right) + 3\left(\frac{1}{2}\right)^2 + \ldots\right)$

Its is in $1 + 2x + 3x^2 = (1-x)^{-2}$

$= \frac{1}{2}\left[1 - \frac{1}{2}\right]^{-2}$

finite

$$= \frac{1}{2}\left[\frac{1}{2}\right]^{-2}$$

$$= \frac{1}{2} \times \frac{2^2}{1^2}$$

$$= 2$$

so

$P[\text{even}] = P[X = \text{even nos}]$

$= P[x=2] + P[x=4] + P[x=6] + \cdots\cdots$

$= \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^4 + \left(\frac{1}{2}\right)^6 + \left(\frac{1}{2}\right)^8 + \cdots\cdots$

$= \left(\frac{1}{2}\right)^2 \left[1 + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^4 + \left(\frac{1}{2}\right)^6 + \cdots\right]$

$\rightarrow \left(\frac{1}{2}\right)^2 \left[1 + \left(\frac{1}{2}\right)^2 + \left(\left(\frac{1}{2}\right)^2\right)^2 + \left(\left(\frac{1}{2}\right)^2\right)^3 + \cdots\right]$

$= \frac{1}{4}\left[1 - \frac{1}{4}\right]^{-1}$

$= \frac{1}{4}\left[\frac{3}{4}\right]^{-1}$

$= \frac{1}{4}\left[\frac{4}{3}\right] = \frac{1}{3}$

$P[x \geq 5] = P[x = 5, 6, 7 \cdots \infty]$

$P[x=5] + P[x=6] + P[x=7] + \cdots\cdots$

$\frac{1}{2^5} + \frac{1}{2^6} + \frac{1}{2^7} + \frac{1}{2^8} + \cdots$

$\frac{1}{2^5}\left[1 + \frac{1}{2} + \left(\frac{1}{2}\right)^2 + \cdots\right]$

$\frac{1}{2^5}\left[1 - \frac{1}{2}\right]^{-1}$

$\frac{1}{2^5}\left[\frac{2-1}{2}\right]^{-1}$

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$\frac{1}{2^5}\left[\frac{1}{2}\right]^{-1}$

$\frac{1}{32}\left[\frac{2}{1}\right] = \frac{1}{16} = 0.0625$

$^3 + \cdots\cdots$

$\cdot\cdot)$

$\cdot x)^{-2}$

$P[x \div \text{by } 3]$

$= P[x = 3, 6, 9, 12 \cdots]$

$= P[x=3] + P[x=6] + P[x=9] + P[x=12] + \cdots$

$= \frac{1}{2^3} + \frac{1}{2^6} + \frac{1}{2^9} + \frac{1}{2^{12}} + \cdots$

$= \frac{1}{2^3}\left[1 + \frac{1}{2^3} + \frac{1}{2^6} + \frac{1}{2^9} + \cdots\right]$

$= \frac{1}{2^3}\left[1 + \left(\frac{1}{2}\right)^3 + \left[\left(\frac{1}{2}\right)^3\right]^2 + \left(\left(\frac{1}{2}\right)^3\right)^3 + \cdots\right]$

$= \frac{1}{2^3}\left[\left(1 - \frac{1}{2^3}\right)\right]^{-1}$

$= \frac{1}{8} \times \left(\frac{7}{8}\right)^{-1} = \frac{1}{8} \times \frac{8}{7} = \frac{1}{7}$.

4. A random variable x has the following Probability distribution function

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x	0	1	2	3	4	5	6	7
$P(x)$	0	K	$2K$	$2K$	$3K$	K^2	$2K^2$	$7K^2+K$

Find (i) the value of K (ii) $P(1.5 < x < 4.5 / x > 2)$

(iii) The smallest value of λ for which

$P[x \le \lambda] > 1/2$

Soln: To find constant $\Sigma P(x) = 1$

$0 + K + 2K + 2K + 3K + K^2 + 2K^2 + 7K^2 + K = 1$

$10K^2 + 9K = 1$

=12] + . . .

$10K^2 + 9K - 1 = 0$

$K = 0.1 , K = -1$

$K = -1$ is negligible

$K = 0.1$

(ii) $P[1.5 < x < 4.5 / x > 2]$

conditional probability

$$P(A/B) = \frac{P(A \cap B)}{P(B)}$$

$$\frac{P[(1.5 < x < 4.5) \cap (x > 2)]}{P[x > 2]} = \frac{P[x = 3, 4]}{P[x = 3, 4, 5, 6, 7]}$$

leaving

$$= \frac{P[x=3] + P[x=4]}{P[x=3] + P[x=4] + P[x=5] + P[x=6] + P[x=7]}$$

$$= \frac{0.2 + 0.3}{0.2 + 0.3 + 0.01 + 0.02 + 0.17}$$

7

$$= \frac{0.5}{0.7} = \frac{5}{7} = 0.7142$$

$7K^2 + K$

(iii) $P[x \le \lambda] > \frac{1}{2}$ smallest value

.5 / x > 2)

when $x = 0$

which

$P[x \le 0] = P[x = 0] = 0 \not> 1/2$

when $x = 1$

$P[x \le 1] = P[x = 1] + P[x = 0] = 0.1 + 0$

$= 0.1 \not> 1/2$

when $x = 2$

1

$P[x \le 2] = P(2) + P(1) + P(0) = 0.2 + 0.1 + 0$

$= 0.3 \not> 1/2$

when $x = 3$

$P[x \leq 3] = P(3) + P(2) + P(1) + P(0)$

$= 0.2 + 0.2 + 0.1 + 0 = 0.5 \not> 1/2$

when $x = 4$

$P[x \leq 4] = P(4) + P(3) + P(2) + P(1) + P(0)$

$= 0.3 + 0.2 + 0.2 + 0.1 + 0$

$= 0.8 > 1/2$

$P[x \leq 4] > 1/2$

$\lambda = 4$

H.W sums:

1. The probability distribution of x is given as

x	1	2	3	4
$P(x)$	0.4	0.3	0.2	0.1

Find $P[\frac{1}{2} < x < 7/2 \,/\, x > 1]$

2.

x	-2	-1	0	1
$P(x)$	0.4	k	0.2	0.3

Find k (ii) Mean

3.

x	0	1	2	3	4	5	6	7	8
$P(x)$	a	3a	5a	7a	9a	11a	13a	15a	17a

(i) a (ii) $P[x < 3]$, variance and C.D.F of x.

$\not> 1/2$

1\. When the probability mass function of a Random variable x is defined as $P[x=0] = 3c^2$, $P[x=1] = 4c-10c^2$ & $P[x=2] = 5c-1$, where $c>0$. Find
(i) c (ii) Mean (iii) variance of x.
(iv) The smallest value of x from which $F(x) > 1/2$ (v) The largest value of x for which $F(x) \not< 1/2$.

$)(0)$

soln :

x	0	1	2
$P(x)$	$3c^2$	$4c-10c^2$	$5c-1$

$$3c^2+4c-10c^2+5c-1 = 1$$

$$9c-7c^2-1-1=0$$

$$-7c^2+9c-2=0$$

$$7c^2-9c+2=0.$$

un

$c=1$ $\quad c=0.285$

$$c = 0.285$$

8

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x	0	1	2
$P(x)$	0.2436	0.327	0.425

Mean : $E(x) = \sum_{x=0}^{2} x\, P(x)$

$E(x) = (0)P(0) + (1)P(1) + (2)P(2)$

$= 0(0.2436) + (1)(0.327) + (2)(0.425)$

$= 0.327 + 0.85 = 1.177$

Variance : $E(x^2) - [E(x)]^2$

$E(x^2) = \sum x^2 P(x)$

$= (0)(0.2437) + (1)(0.327) + 2^2(0.425)$

$= 0.327 + 1.7 = 2.02$

Variance $= 2.02 - (1.177)^2$

$= 2.02 - 1.385$

$= 0.634$

standard deviation $(\sigma) = \sqrt{\text{variance}}$

$= \sqrt{0.634} = 0.796$

To find $F(x)$

x	0	1	2
P(x)	0.2436	0.327	0.425
F(x)	0.2436	0.5706	1

$P[X \leq x] > 1/2$

when $x = 0$,

$P[X \leq 0] = P(0) = 0.2436 \not> 1/2$.

when $x = 1$,

$P[X \leq 1] = 0.5702 > 1/2$.

$P[X \leq 1] > 1/2$.

is $P[X \leq x] > 1/2$.

$\therefore x = 1$.

The smallest value of x for which

$F(x) > 1/2 = 1$.

The largest value of $F(x) < 1/2$.

$F(0) = P[X \leq 0] = P(X=0) = 0.2436 < 1/2$

when $x = 1$

$F(1) = P[X \leq 1] = P(X=0) + P(X=1) = 0.5702 \not< \frac{1}{2}$

$\therefore x = 0$

6

Probability density function:

Notation $\rightarrow$ $f(x)$.

If x is a continuous random variable such that the probability of

$$P\left\{x - \frac{dx}{2} \leq x \leq x + \frac{dx}{2}\right\} = f(x)\,dx$$

then $f(x)$ is called probability density function (p.d.f)

Provided $f(x)$ satisfies the following conditions.

(i) $f(x) \geq 0$ (ii) $\int_{-\infty}^{\infty} f(x)\,dx = 1$.

Related formulas:

(i) To find constant $\int_{-\infty}^{\infty} f(x)\,dx = 1$

(ii) $P[X = a] = \int_{a}^{a} f(x)\,dx = 0$

(iii) Mean $E(X) = \int_{-\infty}^{\infty} x\, f(x)\,dx$

(iv) $E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)\,dx$

(v) r^{th} moment $E[X^r] = \int_{-\infty}^{\infty} x^r f(x)\,dx$

(vi) MGF $M_X(t) = E[e^{tx}] = \int_{-\infty}^{\infty} e^{tx} f(x)\,dx$

(vii) Cumulative distribution function CDF

$$F(x) = P[X \leq x] = \int_{-\infty}^{x} f(x)\,dx$$

(viii) Variance of $X = E(X^2) - [E(X)]^2$.

(ix) $P[a < x < b] = P[a \leq x < b] = P[a < x \leq b]$

$$= P[a \leq x \leq b] = \int_{a}^{b} f(x)\,dx$$

aing

1. Examine whether $f(x) = \frac{1}{\pi} \frac{1}{1+x^2}$, $-\infty < x < \infty$ can be a P.D.F of a continuous random variable.

Soln:

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{\infty} \frac{1}{\pi} \frac{1}{1+x^2}\,dx$$

$$= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{dx}{1+x^2}$$

$$= \frac{2}{\pi} \int_{0}^{\infty} \frac{dx}{1+x^2} \qquad \tan 0^\circ = 0, \quad \tan 90^\circ = \infty.$$

$$= \frac{2}{\pi} \left[\tan^{-1}\left(\frac{x}{\cancel{1}}\right) \right]_0^{\infty}$$

$$= \frac{2}{\pi} \left[\tan^{-1}(\infty) - \tan^{-1}(0) \right]$$

$$= \frac{2}{\pi} \times \frac{\pi}{2} = 1.$$

Given $f(x)$ is p.d.f

DF

2. A continuous random variable x can assume any values between $x=2$ & $x=5$ as a density function given by $f(x) = K(1+x)$ find (i) K (ii) $P[x<4]$

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b]

dx

$$f(x) = K(1+x)$$

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

$$\int_{2}^{5} K(1+x)\,dx = 1$$

$$K\left[x + \frac{x^2}{2}\right]_2^5 = 1$$

$$K\left[5 + \frac{5^2}{2} - \left(2 + \frac{2^2}{2}\right)\right] = 1$$

$$K\left[5 + \frac{25}{2} - 2 - 2\right] = 1$$

$$K\left[1 + \frac{25}{2}\right] = 1$$

$$K\left[\frac{27}{2}\right] = 1$$

$$K = \frac{2}{27}$$

$$P(X < 4) = \int_2^4 \frac{2}{27}(1+x)\,dx$$

$$= \frac{2}{27}\int_2^4 (1+x)\,dx$$

$$= \frac{2}{27}\left[x + \frac{x^2}{2}\right]_2^4$$

$$= \frac{2}{27}\left[4 + \frac{16}{2} - 2 - \frac{4}{2}\right]$$

$$= \frac{2}{27}\left[4 + 8 - 2 - 2\right]$$

$$= \frac{2}{27}[8] = \frac{16}{27} = 0.59$$

3. A continuous random variable X has the p.d.f of $f(x) = Kx^2e^{-x}$, $x \geq 0$. Find (i) K, Mean, variance, r^{th} moment about origin.

Given $f(x) = kx^2 e^{-x}, \; x > 0$.

$$\text{w.k.T} \int_{-\infty}^{\infty} k x^2 e^{-x} \, dx = 1$$

$$\int_{0}^{\infty} k x^2 e^{-x} \, dx = 1 \qquad e^{-\infty} = 0$$

$$k \int_{0}^{\infty} x^2 e^{-x} \, dx = 1$$

$$k \left[x^2 \left(\frac{e^{-x}}{-1} \right) - (2x) \left(\frac{e^{-x}}{(-1)^2} \right) + (2) \left(\frac{e^{-x}}{(-1)^3} \right) - 0 \right]$$

$$k \left(0 - \left(0 - 0 + 2 \left(\frac{e^0}{-1} \right) \right) \right] = 1.$$

$$k \left(\frac{2}{1} \right) = 1$$

$$k = 1/2$$

$$\text{Mean} \quad E(X) = \int_{-\infty}^{\infty} x f(x) \, dx$$

$$= \int_{0}^{\infty} x \cdot k x^2 e^{-x} \, dx$$

$$= \frac{1}{2} \int_{0}^{\infty} x^3 e^{-x} \, dx$$

$$= \frac{1}{2} \left[(x^3) \left(\frac{e^{-x}}{-1} \right) - (3x^2) \left(\frac{e^{-x}}{(-1)^2} \right) + (6x) \left(\frac{e^{-x}}{(-1)^3} \right) - (6) \left(\frac{e^{-x}}{(-1)^4} \right) \right]_0^{\infty}$$

$$= \frac{1}{2} \left[0 - \left(-\frac{6}{1} \right) \right] = \frac{1}{2} \left[\frac{6}{1} \right] = \frac{6}{2} = 3$$

Mean $= 3$.

$E(X) = 3$

x has

>0.

ent

Variance : $E(x^2) - [E(x)]^2$.

$$E(x^2) = \int_{-\infty}^{\infty} x^2 f(x)\, dx.$$

$$\int_0^{\infty} x^2 K x^2 e^{-x}\, dx$$

$$\frac{1}{2}\int_0^{\infty} x^4 e^{-x}\, dx$$

$$\frac{1}{2}\left[x^4\left(\frac{e^{-x}}{-1}\right) - (4x^3)\left(\frac{e^{-x}}{1}\right) + (12x^2)\left(\frac{e^{-x}}{-1}\right) + 24x\left(\frac{e^{-x}}{1}\right) + 24\left(\frac{e^{-x}}{-1}\right)\right]_0^{\infty}$$

$$= \frac{1}{2}\left[24\left(\frac{e^{-x}}{-1}\right)\right]_0^{\infty}$$

$$= \frac{1}{2}\left[24\left(0 - \frac{1}{-1}\right)\right]$$

$$= \frac{1}{2}\left[\frac{24}{1}\right] = \frac{24}{2} = 12.$$

$$= 12 - (3)^2 = 12 - 9 = 3.$$

r^{th} moment :

$$E(x^r) = \int_{-\infty}^{\infty} x^r f(x)\, dx$$

$$= \int_0^{\infty} x^r \frac{1}{2} x^2 e^{-x}\, dx \qquad \int_0^{\infty} e^{-x} x^{n-1}\, dx = \sqrt{n}\ \text{(}\Gamma(n)\text{)} = (n-1)!$$

$$= \frac{1}{2}\int_0^{\infty} x^{r+2} e^{-x}\, dx$$

$$= \frac{1}{2}\sqrt{r+3} = \frac{(r+2)!}{2}$$

$E(x^r) =$

$r = 1 \Rightarrow$ Mean

$r = 2 \Rightarrow E(x^2)$

1\. A continuous random variable as a p.d.f $f(x) = 3x^2, \; 0 \le x \le 1$.

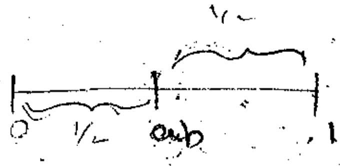
(i) $P[x \le a] = P[x > a]$

(ii) $P[x > b] = 0.05$ find a & b.

Soln:

(i) $P[x \le a] = P[x > a]$

$P[x \le a] = 1/2$



$$\int_{-\infty}^{a} f(x)\,dx = 1/2$$

$$\int_{0}^{a} 3x^2\,dx = 1/2$$

$$\left[\frac{3x^3}{3}\right]_0^a = 1/2$$

$$\frac{3a^3}{3} = 1/2$$

$$a^3 = 1/2$$

$$a = \left(\frac{1}{2}\right)^{1/3}$$

$\frac{e^{-x}}{-1}$

(ii) $P[x > b] = 0.05$

$$\int_{b}^{\infty} f(x)\,dx = 0.05$$

$$\int_{b}^{1} 3x^2\,dx = 0.05$$

$$\left[\frac{3x^3}{3}\right]_b^1 = 0.05$$

$n+1 = n b$

$1^3 - b^3 = 0.05$

$b^3 = 0.05 - 1$

$b^3 = 0.95$

$b = (0.95)^{1/3}$.

2. If $P(x) = \begin{cases} x e^{-\frac{x^2}{2}}, & x \geq 0 \\ 0, & \text{o.w} \end{cases}$

a) show that $p(x)$ is a p.d.f

b) find its distribution function

$[(F(x))]$

Soln:

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

$$= \int_0^{\infty} x e^{-\frac{x^2}{2}}\,dx$$

put $t = \frac{x^2}{2}$.

$dt = \frac{2x\,dx}{2}$

when $x = 0$, $t = 0$

when $x = \infty$, $t = \infty$

$$= \int_0^{\infty} e^{-t}\,dt$$

$$= \left(\frac{e^{-t}}{-1}\right)_0^{\infty}$$

$$= \left(\frac{e^{-\infty} - e^{-0}}{-1}\right) = 1.$$

So given $p(x)$ is p.d.f.

(ii) $F(x)$

$$F[x] = P[X \le x] = \int_{-\infty}^{x} f(x)\,dx$$

$$= \int_{0}^{x} x e^{-\frac{x^2}{2}}\,dx$$

put $t = \frac{x^2}{2}$.

$dt = \frac{2x\,dx}{2}$.

when $x = 0$, $t = 0$

when $x = x$, $t = \frac{x^2}{2}$.

$$= \int_{0}^{\frac{x^2}{2}} e^{-t}\,dt. \quad = \left(\frac{e^{-t}}{-1}\right)_{0}^{\frac{x^2}{2}}.$$

$$= \frac{e^{-\frac{x^2}{2}} - e^{-0}}{-1} = 1 - e^{-\frac{x^2}{2}}.$$

$$F(x) = 1 - e^{-\frac{x^2}{2}}.$$

3. If the density function of a continu-ous random variable is given by

$$f(x) = \begin{cases} ax & ; 0 \le x \le 1 \\ a & ; 1 \le x \le 2 \\ 3a - ax & ; 2 \le x \le 3 \\ 0 & ; \text{o.w} \end{cases}$$

Find the values of (i) a (ii) c.d.f of x soln : w.k.T $\int_{-\infty}^{\infty} f(x)\,dx = 1$

$\int_0^1 ax\,dx + \int_1^2 a\,dx + \int_2^3 (3a - ax)\,dx = 1$

$a\left(\frac{x^2}{2}\right)_0^1 + a(x)_1^2 + a\left(3x - \frac{x^2}{2}\right)_2^3 = 1$

$a\left(\frac{1}{2} - 0\right) + a(2-1) + a\left(9 - \frac{9}{2} - \left(6 - \frac{4}{2}\right)\right] = 1$

$a\left(\frac{1}{2} + 1 + 9 - \frac{9}{2} - 6 + \frac{4}{2}\right)\right] = 1$

$a\left(6 - \frac{8}{2}\right) = 1$

$a(6 - 4) = 1$

$2a = 1$

$a = 1/2$

C.D.F : $F(x) = P[X \leq x]$

interval $0 \leq x \leq 1$

$F(x) = P[X \leq x]$

$= \int_{-\infty}^{x} f(x)\,dx = \int_0^x ax\,dx$

$= a\left[\frac{x^2}{2}\right]_0^x = \frac{a}{2}\left[x^2 - 0\right] = \frac{a}{2}x^2$

interval $1 \leq x \leq 2$

$F(x) = P[X \leq x] = \int_{-\infty}^{x} f(x)\,dx$

$$= \int_0^x f(x)\,dx = \int_0^1 ax\,dx + \int_1^x a\,dx$$

$$= a\left(\frac{x^2}{2}\right)_0^1 + a\left(x\right)_1^x$$

$$= \frac{a}{2}(1-0) + a(x-1)$$

$$= \frac{a}{2} + ax - a = ax - \frac{a}{2}.$$

interval $2 \le x \le 3$.

$$F(x) = \int_{-\infty}^x f(x)\,dx = \int_0^1 ax\,dx + \int_1^2 a\,dx$$

$$+ \int_2^x (3a - ax)\,dx$$

$$= a\left(\frac{x^2}{2}\right)_0^1 + a\left(x\right)_1^2 + a\left(3x - \frac{x^2}{2}\right)_2^x$$

$$= \frac{a}{2} + a(2-1) + a\left(3x - \frac{x^2}{2} - (6-2)\right)$$

$$= \frac{a}{2} + a + a\left(3x - \frac{x^2}{2} - 4\right)$$

$$= \frac{a}{2} + a + 3ax - \frac{ax^2}{2} - 4a$$

$$= 3ax - \frac{ax^2}{2} - \frac{5a}{2}.$$

In the interval $x > 3$.

$$f(x) = \int_{\infty}^x f(x)\,dx = \int_0^1 ax\,dx + \int_1^2 a\,dx$$

$$+ \int_2^3 (3a - ax)\,dx + \int_3^x 0\,dx$$

 $= 1$ (C.D.F)

$$F(x) = \begin{cases} \frac{a}{2}x^2, & 0 \le x \le 1 \\ ax - \frac{a^2}{2}, & 1 \le x \le 2 \\ 3ax - \frac{ax^2}{2} - \frac{5a}{2}, & 2 \le x \le 3 \\ 1, & \text{o.w} \end{cases}$$

1. If x is a discrete $F(x) = P[X \le x]$ $= \sum_j^{x_j} P[X = x_j]$. If x is a continuous

$$F(x) = P[x \le x] = \int_{-\infty}^{x} f(x)\,dx$$

Related formulas :

(i) $F(-\infty) = 0$, $F(\infty) = 1$

(ii) $P[x = x_i] = f(x_i) - F(x_i - 1)$

(iii) $P[a \le x \le b] = F(b) - F(a)$

(iv) $\frac{d}{dx} F(x) = f(x)$. diff of $F(x)$ is $f(x)$

Inte of .

1. When the distribution function of random variable x is given by $F(x) = 1-(1+x)e^{-x}$, $x \ge 0$. Find the density function, mean, variance of x.

Soln :

p.d.f $\quad f(x) = \frac{d}{dx} F(x)$

$= \frac{d}{dx}\left[1 - (1+x)e^{-x}\right]$

$= 0 - \frac{d}{dx}(1+x)e^{-x}$

$= -\left[(1+x)e^{-x}(-1) + e^{-x}(0+1)\right]$

$= -\left[-e^{-x} - xe^{-x} + e^{-x}\right]$

$f(x) = xe^{-x}, \; x > 0$

$f(x) = xe^{-x}$

Mean : $E(X) = \int_{-\infty}^{\infty} x f(x)\,dx$

$= \int_0^{\infty} x \cdot xe^{-x}\,dx$

$= \int_0^{\infty} x^2 e^{-x}\,dx$

$= \left[x^2\left(\frac{e^{-x}}{-1}\right) - (2x)\left(\frac{e^{-x}}{(-1)^2}\right) + (2)\left(\frac{e^{-x}}{(-1)^3}\right)\right]_0^{\infty}$

$= 0 - \left(0 + 0 + \frac{2e^{0}}{-1}\right) = 2$

variance : $E(X^2) - [E(X)]^2$

$E(X^2) = \int x^2 f(x)\,dx$

$= \int_0^{\infty} x^2 x\, e^{-x}\,dx$

$= \int_0^\infty x^3 e^{-x} dx$

$= x^3\left(\frac{e^{-x}}{-1}\right) - 3x^2\left(\frac{e^{-x}}{(-1)^2}\right) + 6x\left(\frac{e^{-x}}{(-1)^3}\right) - 6\left(\frac{e^{-x}}{(-1)^4}\right)\Big]_0^\infty$

$= 0 - \left(0 - 6\left(\frac{e^{-0}}{1}\right)\right] = 6$

Variance of $x = E(x^2) - [E(x)]^2$

$= 6 - (2)^2 = 6 - 4 = 2$.

S.D of $x = \sqrt{\text{variance}} = \sqrt{2} = 1.414$

2. The C.D.F of a continuous random variable x is given by

$$F(x) = \begin{cases} 0 & , x < 0 \\ x^2 & , 0 < x < 1/2 \\ 1 - \frac{3}{25}(3-x)^2 & , \frac{1}{2} \le x < 3 \\ 1 & , x \ge 3 \end{cases}$$

Find the p.d.f of x, Evaluate $P(|x| \le 1)$ & $P\left(\frac{1}{3} \le x \le 4\right)$ using both p.d.f and c.d.f.

$f(x) = \frac{d}{dx} F(x)$

$$f(x) = \begin{cases} 0 & , x < 0 \\ 2x & , 0 < x < 1/2 \\ 0 - 3/25 \; 2(3-x)(-1) & , \frac{1}{2} < x < 3 \\ 0 & , x \ge 3 \end{cases}$$

$$f(x) = \begin{cases} 2x, & 0 < x < 1/2 \\ \frac{6}{25}(3-x), & \frac{1}{2} < x < 3 \\ 0, & x \geq 3. \end{cases}$$

(ii) $P[|x| \leq 1] = P[-1 \leq x < 1]$

$= \int_{-1}^{1} f(x)\,dx$

$= \int_{-1}^{0} 0\,dx + \int_{0}^{1/2} 2x\,dx + \int_{1/2}^{1} 6/25\,(3-x)\,dx$

$= 2\left(\frac{x^2}{2}\right)_0^{1/2} + \frac{6}{25}\left(3x - \frac{x^2}{2}\right)_{1/2}^{1}$

$= \left(\frac{1}{2}\right)^2 - 0^2 + \frac{6}{25}\left(3 - 1/2 - \left(3\left(\frac{1}{2}\right) - \frac{(1/2)^2}{2}\right)\right)$

$= \frac{1}{4} + \frac{6}{25}\left(3 - 1/2 - 3/2 + 1/8\right)$

$= \frac{1}{4} + \frac{6}{25}\left(1 + 1/8\right)$

$= \frac{1}{4} + \frac{6}{25} \times \frac{9}{8}$

$= \frac{1}{4} + \frac{27}{100} = \frac{52}{100} = 0.52.$

By using CDF

$P[a \leq x \leq b] = F(b) - F(a)$

$P(-1 \leq x \leq 1) = F(1) - F(-1)$

$= 1 - \frac{3}{25}(3-1)^2 - 0$

$= 1 - \frac{12}{25} = \frac{13}{25} = 0.52$

$$P\left(\frac{1}{3} \le x \le 4\right) = \int^{4} f(x)\,dx$$

$$= \int_{1/3}^{1/2} 2x\,dx + \int_{1/2}^{3} 6/25\,(3-x)\,dx + \int_{3}^{4} 0\,dx$$

$$= 2\left(\frac{x^2}{2}\right)_{1/3}^{1/2} + \frac{6}{25}\left(3x - \frac{x^2}{2}\right)_{1/2}^{3} + 0$$

$$= \left(\frac{1}{2}\right)^2 - \left(\frac{1}{3}\right)^2 + \frac{6}{25}\left(9 - \frac{9}{2} - \left(\frac{3}{2} - \frac{\left(\frac{1}{2}\right)^2}{2}\right)\right)$$

$$= \frac{1}{4} - \frac{1}{9} + \frac{6}{25}\left(9 - 4.5 - 1.5 + \frac{1}{8}\right]$$

$$= \frac{1}{4} - \frac{1}{9} + \frac{6}{25}\left(3 + \frac{1}{8}\right)$$

$$= \frac{1}{4} - \frac{1}{9} + \frac{6}{25}\left(\frac{25}{8}\right)$$

$$= \frac{1}{4} - \frac{1}{9} + \frac{3}{4} = 1 - \frac{1}{9} = \frac{8}{9}.$$

By using C.D.F $P\left(\frac{1}{3} \le x \le 4\right)$

$$= F(4) - F\left(\frac{1}{3}\right)$$

$$= 1 - \left(\frac{1}{3}\right)^2 = 1 - \frac{1}{9} = \frac{8}{9}$$

1. Suppose that the life time of certain radio tube (in hours) is a continuous random variable p.d.f

$$f(x) = \begin{cases} \dfrac{100}{x^2}, & x > 0 \\ 0 & \text{elsewhere} \end{cases}$$

a) what is the probability that a tube will last less than 200 hours. If it is known that the tube is still functioning after 150 hours of service.

b) what is the probability that will have to replace after 150 hours of service.

Soln:

Let x be the life time in hours.

b) $P[x > 150] = \int_{150}^{\infty} f(x)\,dx$

$= \int_{150}^{\infty} \frac{100}{x^2}\,dx$

$= 100 \int_{150}^{\infty} x^{-2}\,dx = 100 \left[\frac{x^{-2+1}}{-2+1}\right]_{150}^{\infty}$

$= 100 \left(\frac{x^{-1}}{-1}\right)_{150}^{\infty}$

$= -100 \left(\frac{1}{x}\right)_{150}^{\infty}$

$= -100 \left(\frac{1}{\infty} - \frac{1}{150}\right) = -100\left(0 - \frac{1}{150}\right)$

$= \frac{100}{150} = \frac{2}{3} = 0.666$

a) $P[X<200 / X>150]$

$$= \frac{P[(X<200) \cap (X>150)]}{P[X>150]}$$

$$= \frac{P(150<X<200)}{P[X>150]}$$

$$P(150<x<200) = \int_{150}^{200} \frac{100}{x^2}\,dx$$

$$= 100 \left(\frac{x^{-2+1}}{-2+1} \right)_{150}^{200}$$

$$= -100 \left(\frac{1}{x} \right)_{150}^{200} = -100 \left(\frac{1}{200} - \frac{1}{150} \right)$$

$$= -100 \left(\frac{3-4}{600} \right) = \frac{100}{600} = \frac{1}{6}$$

$$P[X>150] = \int_{150}^{\infty} \frac{100}{x^2}\,dx = \frac{2}{3}$$

$$P[X<200 / X>150] = \frac{1/6}{2/3}$$

$$= \frac{1}{6} \times \frac{3}{2} = \frac{3}{12} = \frac{1}{4}$$

$$= 0.25$$

(i)

(iii)

(iv)

(v)

Expectation & Moments.

If x is a discrete Random variable then excpectation

$\bar{X} = E(X) = \sum x\, p(x)$

If x is a continuous random variable

$$\bar{X} = E(X) = \int_{-\infty}^{\infty} x f(x)\, dx.$$

Properties:

(i) $E(a) = a$ (ii) $E(ax) = aE(x)$

(iii) $E(ax+b) = a\,E(X)+b$

(iv) $E(x+y) = E(x)+E(y)$

(v) $E(xy) = E(x)\cdot E(y)$ where x and y are independent.

where a and b are constants

Moments:

The n^{th} moment about origin of a random variable x is denoted as

$\mu_n' = E[x^n] = \sum x^n p(x) \rightarrow$ x is discrete

$\mu_n' = E(x^n) = \int x^n f(x)\, dx -$ x is continuous

$\frac{1}{4}$

n^{th} central moment (or) Moment about Mean.

$\mu_n = E[(x-\bar{x})^n] = \sum (x-\bar{x})^n p(x)$ - x is discrete

$\mu_n = E[(x-\bar{x})^n] = \int (x-\bar{x})^n f(x)\,dx$ - x is continuous

$\mu_1 = E[x-\bar{x}] = 0$

$\mu_2 = E[(x-\bar{x})^2] = \mu_2' - (\mu_1')^2 = \sigma^2$

$\mu_3 = E[(x-\bar{x})^3] = \underset{E(x^3)}{\mu_3'} - 3\underset{E(x^2)}{\mu_2'}\mu_1' + 2\underset{E(x)}{(\mu_1')^3}$

Relationship b/w μ_n' & μ_n.

$\mu_n = \mu_n' - nC_1 \mu_1' \mu_{n-1}' + nC_2 (\mu_1')^2 \mu_{n-2}' + \cdots$

$E(x-\bar{x}) = E(x) - E(\bar{x})$
$= \bar{x} - \bar{x} = 0.$

$E[(x-\bar{x})^2] = E(x^2 - 2x\bar{x} + \bar{x}^2)$
$= E(x^2) - 2\bar{x}E(x) + E[(\bar{x})^2]$
$= \mu_2' - 2\bar{x}\bar{x} + (\bar{x})^2$
$= \mu_2' - (\bar{x})^2$
$= \mu_2' - [E(x)]^2$
$= \mu_2' - (\mu_1')^2.$

Properties of Var(X):

1. $Var(X) \geq 0$.
2. $E(X^2) \geq [E(X)]^2$.
3. $Var(a) = a$, a is const.
4. $Var(X \pm a) = Var(X)$
5. $Var(aX \pm b) = a^2 Var(X)$
6. $Var(aX \pm bY) = a^2 Var(X) \pm b^2 Var(Y)$

1. For density fun $f(x) = \begin{cases} 1/4, & x = -1 \\ 1/4, & x = 0 \\ 1/2, & x = 2 \end{cases}$

calculate $E(X)$, $E(X^2)$, Var X, $E(|X|)$

$$E(X) = \sum x\,p(x)$$
$$= \left(-1 \times \frac{1}{4}\right) + 0\left(\frac{1}{4}\right) + 2\left(\frac{1}{2}\right)$$
$$= -\frac{1}{4} + 0 + 1 = \frac{3}{4}$$

$$E(X^2) = \sum x^2 p(x)$$
$$= (-1)^2\left(\frac{1}{4}\right) + 0^2\left(\frac{1}{4}\right) + 2^2\left(\frac{1}{2}\right)$$
$$= 1/4 + 0 + 4/2 = 9/4$$

$$Var(X) = E(X^2) - [E(X)]^2$$
$$= 9/4 - (3/4)^2 = \frac{9}{4} - \frac{9}{16}$$
$$= \frac{27}{16}$$

$E(|x|) = \sum |x| \, P(x)$

$= |-1| \, 1/4 + |0| \, 1/4 + |2| \, 1/2$

$= 1/4 + 0 + 1 = 5/4.$

2. Find mean & variance of the following density fun.

$$f(x) = \begin{cases} x \ ; & 0 \leq x < 1 \\ 2-x \ ; & 1 < x < 2 \\ 0 \ ; & \text{o.w} \end{cases}$$

$3 - \frac{10}{3}$

$\frac{9-10}{3}$

$\frac{1}{3}$

Mean $E(x) = \int_{-\infty}^{\infty} x f(x) \, dx$

$\text{var}(x) = E(x^2) - [E(x)]^2$

$\mu_2 = \mu_2' - (\mu_1')^2$

$E(x) = \int_0^1 x \cdot x \, dx + \int_1^2 x(2-x) \, dx$

$= \int_0^1 x^2 dx + \int_1^2 2x - x^2 dx$

$= \left(\frac{x^3}{3}\right)_0^1 + \left(\frac{2x^2}{2} - \frac{x^3}{3}\right)_1^2$

$= \left(\frac{1}{3}\right) + \left(4 - \frac{8}{3} - \left(1 - 1/3\right)\right]$

$= \frac{1}{3} + 4 - \frac{8}{3} - 1 + 1/3$

$= 3 - \frac{8+1+1}{3} = 3 - \frac{6}{3} = 3 - 2 = 1$

$E(x^2) = \int x^2 f(x)\, dx$

$= \int_0^1 x^2 \cdot x\, dx + \int_1^2 x^2(2-x)\, dx$

$= \int_0^1 x^3 dx + \int_1^2 2x^2 - x^3 dx$

$= \left(\frac{x^4}{4}\right)_0^1 + \left(\frac{2x^3}{3} - \frac{x^4}{4}\right)_1^2$

$= \left(\frac{1}{4} - 0\right) + 2\left(\frac{8}{3}\right) - \frac{16}{4} - \left(\frac{2}{3} - \frac{1}{4}\right)$

$= \frac{1}{4} + \frac{16}{3} - 4 - \frac{2}{3} + \frac{1}{4}$

$= 1/2 + \frac{10}{3} - 4$

$= \frac{3 + 28 - 24}{6} = 7/6$

$var(x) = \frac{7}{6} - (1)^2 = \frac{7-6}{6} = 1/6.$

H.W

1. The distribution fun of a conti s variable is given by

$f(x) = \begin{cases} 1/2\, x, & 0 < x < 2 \\ 0, & o.w \end{cases}$

find (i) $E(x)$ (ii) $var(x)$ (iii) $E(3x^2 - 2x)$

2. The x is a R V whose density fun

$f(x) = \begin{cases} Ae^{-x}, & x > 0 \\ 0, & o.w \end{cases}$ find (i) A (ii) mean (iii) $var(x)$ (iv) Third moment about mean.

$2 = 1$

3. A cont r.v has the p.d.f $f(x) = \begin{cases} \frac{1}{2}(x+1), & -1 < x < 1 \\ 0 & o.w \end{cases}$ find the mean and variance.

1. Two unbiased dice are thrown. Find the expected values of the sum of the numbers of the top faces of them.

Let x be the sum of the faces

x	2	3	4	5	6	7	8	9	10	11	12
P(x)	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

Mean $E(x) = \sum x\, p(x)$

$= 2\left(\frac{1}{36}\right) + 3\left(\frac{2}{36}\right) + 4\left(\frac{3}{36}\right) + 5\left(\frac{4}{36}\right) + 6\left(\frac{5}{36}\right) + 7\left(\frac{6}{36}\right) + 8\left(\frac{5}{36}\right) + 9\left(\frac{4}{36}\right) + 10\left(\frac{3}{36}\right) + 11\left(\frac{3}{36}\right) + 12\left(\frac{1}{36}\right)$

$= \frac{1}{36}(2+6+12+20+30+42+40+36+30+22+12)$

$= \frac{252}{36} = 7.$

Moment Generating function $M_x(t)$

$M_x(t) = E[e^{tx}]$

when x is discrete Random variable

$$M_x(t) = E[e^{tx}] = \sum e^{tx} p(x)$$

when x is continuous random variable

$$M_x(t) = E[e^{tx}] = \int_{-\infty}^{\infty} e^{tx} f(x) dx.$$

Properties :

1. $\frac{d}{dt} M_x(t) \Big|_{t=0} = E(x)$

2. $\frac{d^2}{dt^2} M_x(t)$ at $t=0 = E(x^2)$

$\frac{d^r}{dt^r} M_x(t)$ at $t=0 = E(x^r)$

3. $M_y(t) = e^{bt} M_x(at)$ if $y = ax + b$.

4\.

5\. If x & y are independent r.v and $z = x+y$ then $M_z(t) = M_x(t) \cdot M_y(t)$

problems :

1\. obtain the MGF for the distribution when $f(x) = \begin{cases} 2/3, & \text{at } x=1 \\ 1/3, & \text{at } x=2 \\ 0 & \text{o.w} \end{cases}$ also find mean & variance of x.

10	11	12
$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

$\left(\frac{5}{36}\right)+$

$\left(\frac{3}{36}\right)$

$M_x(t) = E[e^{tx}] = \sum e^{tx} p(x)$

$= e^{t(1)} p(1) + e^{t(2)} p(2) + e^{t(3)} p(3) + \ldots$

$= e^{t}\left(\frac{2}{3}\right) + e^{2t}\left(\frac{1}{3}\right)$

$M_x(t) = \frac{2}{3} e^{t} + \frac{1}{3} e^{2t}$

W.K.T

Mean $E(x) = \frac{d}{dt} M_x(t)$ at $t=0$

$\frac{d}{dt} M_x(t) = \frac{2}{3} e^{t} + \frac{2}{3} e^{2t}$

$\frac{d^2}{dt^2} M_x(t) = \frac{2}{3} e^{t} + \cancel{\frac{2}{3} e^{2t}} \cdot \frac{4}{3} e^{2t}$

Mean $E(x) = \frac{2}{3} e^{0} + \frac{2}{3} e^{0}$

$= \frac{4}{3}$

$Var(x) = E(x^2) - [E(x)]^2$

$E(x^2) = \frac{d^2}{dt^2} M_x(t)$ at $t=0$

$E(x^2) = \frac{2}{3} + \frac{4}{3} = \frac{6}{3} = 2$

$Var(x) = 2 - \left(\frac{4}{3}\right)^2$

$= 2 - \frac{16}{9} = \frac{18-16}{9} = \frac{2}{9}.$

H.W

1. The random variable x can assume the values 1 and -1 with probability $\frac{1}{2}$ each. Find MGF (ii) The first 4 moments about the origin.

x	1	-1
$P(x)$	$1/2$	$1/2$

2. Let x be the outcome when pair die is tossed. Find the MGF of X and hence find Mean & variance.

Problems:

1. Obtain MGF of the random variable X Have the p.d.f $f(x) = \begin{cases} x, & 0 \le x \le 1 \\ 2-x, & 1 \le x \le 2 \\ 0, & x > 2. \end{cases}$

$$M_x(t) = E[e^{tx}] = \int_{-\infty}^{\infty} e^{tx} f(x)\, dx.$$

$$= \int_0^1 x e^{tx} dx + \int_1^2 (2-x) e^{tx} dx$$

$$= \left(x\left(\frac{e^{tx}}{t}\right) - 1\left(\frac{e^{tx}}{t^2}\right)\right]_0^1 + \left[(2-x)\left(\frac{e^{tx}}{t}\right) - (-1)\left(\frac{e^{tx}}{t^2}\right)\right]_1^2$$

$$= \frac{e^t}{t} - \frac{e^t}{t^2} - \left(0 - \frac{e^0}{t^2}\right) + \left(0 + \frac{e^{2t}}{t^2} - \left(\frac{e^t}{t} + \frac{e^t}{t^2}\right)\right)$$

$$= \frac{e^t}{t} - \frac{e^t}{t^2} + \frac{1}{t^2} + \frac{e^{2t}}{t^2} - \frac{e^t}{t} - \frac{e^t}{t^2}$$

$$= \frac{e^{2t} - 2e^t + 1}{t^2}$$

$$M_x(t) = \frac{(e^t - 1)^2}{t^2}.$$

H.W

1. Find MGF

$$f(x) = \begin{cases} 1/3, & -1 \le x \le 2 \\ 0, & \text{elsewhere} \end{cases}$$

1. Obtain MGF of distribution fn $f(x) = \begin{cases} \frac{1}{2} e^{-x/2}, & x > 0 \\ 0, & x < 0 \end{cases}$ also find Mean and variance.

$$M_x(t) = E[e^{tx}] = \int_{-\infty}^{\infty} e^{tx} f(x)\, dx$$

$$= \int_0^{\infty} e^{tx} \frac{1}{2} e^{-x/2} dx$$

$$= \frac{1}{2} \int_0^{\infty} e^{-\frac{x}{2} + tx} dx$$

$$= \frac{1}{2} \int_0^{\infty} e^{-(\frac{1}{2} - t)x} dx$$

$$= \frac{1}{2} \left[\frac{e^{-(\frac{1}{2} - t)x}}{-(\frac{1}{2} - t)} \right]_0^{\infty}$$

$$= \frac{1}{2} \left[e^{-\infty} - \frac{e^0}{-(\frac{1}{2} - t)} \right]$$

$$= \frac{1}{2} \left[\frac{1}{1/2 - t} \right] = \frac{1}{2} \left[\frac{1}{\frac{1 - 2t}{2}} \right] = \frac{1}{1 - 2t}$$

$$M_x(t) = \frac{1}{1-2t}$$

Mean $E(x) = \frac{d}{dt} M_x(t)$ at $t = 0$

$\text{var}(x) = E(x^2) - [E(x)]^2$

$E(x^2) = \frac{d^2}{dt^2} M_x(t)$ at $t = 0$.

$$\frac{d}{dt}\left(\frac{1}{(1-2t)}'\right) = \frac{-1}{(1-2t)^2}(-2)$$

$$= \frac{2}{(1-2t)^2}$$

$$\frac{d^2}{dt^2} M_x(t) = 2\left(\frac{-2(-2)}{(1-2t)^3}\right) = \frac{8}{(1-2t)^3}$$

Mean $E(x) = \frac{2}{(1-0)^2} = \frac{2}{1} = 2$

$E(x^2) = \frac{8}{(1-0)^3} = 8.$

$Var(x) = 8 - (2)^2 = 4 \quad S.D = 2.$

H.W

1. Find the mean and var of $M_x(t) = \frac{2}{2-t}$

Problems:

1. If the probability density of x is given by $f(x) = \begin{cases} 2(1-x), & 0 < x < 1 \\ 0, & o.w \end{cases}$

(i) S.T $E[x^r] = \frac{2}{(r+1)(r+2)}$

(ii) using this result to evaluate $E[(2x+1)^2]$

Special distribution

1\. Binomial distribution : $B(n, p)$

Let A be the event associated with a random experiment E such that $P(A) = p$, $q = 1-p$, if we consider is independent repetation of 'E' and if the random variable x is called Binomial Random variable with parameter n and p, then it is represented as $B(n, p)$

$P[X = x] = nC_x p^x q^{n-x}$, $x = 0, 1, 2 \dots n$

1\. Derive the MGF of Binomial distribution and hence find its mean and variance.

$$M_x(t) = E[e^{tx}] = \sum_{x=0}^{n} e^{tx} P(x)$$

$$= \sum_{x=0}^{n} e^{tx} nC_x p^x q^{n-x}$$

$$= \sum_{x=0}^{n} nC_x q^{n-x} (e^t)^x p^x$$

$$= \sum_{x=0}^{n} nC_x q^{n-x} (pe^t)^x$$

$= nc_0 q^{n-0} (pe^t)^0 + nc_1 q^{n-1} (pe^t)^1 + nc_2 q^{n-2} (pe^t)^2$
$+ \cdots + nc_n q^{n-n} (pe^t)^n$

$= q^n + nc_1 q^{n-1} (pe^t) + nc_2 q^{n-2} (pe^t)^2 + \cdots$
$+ nc_n (pe^t)^n$

$M_x(t) = (q + pe^t)^n.$

Mean : $E(x) = \frac{d}{dt} M_x(t)$ at $t = 0$

$x^n + nc_1 x^{n-1} y + nc_2 x^{n-2} y^2 + \cdots nc_n y^n = (x+y)^n.$

Variance :

$E(x^2) = \frac{d^2}{dt^2} M_x(t)$ at $t = 0$

$M_x(t) = (q + pe^t)^n.$

$\frac{d}{dt} M_x(t) = n(q + pe^t)^{n-1} (0 + pe^t)$
$= np[e^t (q + pe^t)^{n-1}]$

$\frac{d^2}{dt^2} M_x(t) = np(e^t (n-1) (q + pe^t)^{n-2}$
$(0 + pe^t) + (q + pe^t)^{n-1} e^t]$

Mean $E(x) = np\, e^0 (q + pe^0)^{n-1}$
$= np (q+p)^{n-1}$ $\quad q + p = 1$
$= np.$

$E(x^2) = \frac{d^2}{dt^2} M_x(t)$ at $t = 0.$
$= np[(n-1)(q+p)^{n-2} (p) + (q+p)^{n-1}]$
$= np[(n-1)p + 1]$

$= np[np - p + 1]$

$E(x^2) = n^2p^2 - np^2 + np.$

$var(x) = n^2p^2 - np^2 + np - (np)^2$

$= np(1-p)$

$var(x) = npq$

1. The mean and variance of a binomial distributions are 4 & $\frac{4}{3}$ respectively. Find $P[x \geq 1]$

Given: mean $= 4$

variance $= 4/3$

$np = 4 \rightarrow (1)$

$npq = 4/3 \rightarrow (2).$

$p + q = 1$

$p = 1 - q$

$q = 1 - p$

sub (1) in (2)

$4q = 4/3$

$q = 1/3$

$p = 1 - q = 1 - 1/3 = 2/3$

sub p value in (1)

$n \times 2/3 = 4$

$n = \frac{12}{2} = 6$

$n = 6, \; p = 2/3, \; q = 1/3$

Binomial dist

$$P[X=x] = 6C_x\left(\frac{2}{3}\right)^x\left(\frac{1}{3}\right)^{6-x}, \; x = 0,1,2 \ldots 6$$

$$P[x \geq 1] = 1 - P[x < 1]$$
$$= 1 - P[X=0]$$
$$= 1 - 6C_0\left(\frac{2}{3}\right)^0\left(\frac{1}{3}\right)^{6-0}$$
$$= 1 - \left(1/3\right)^6$$
$$= 0.9986 \%$$

2. Out of 800 families with 4 children each how many families would be expected to have 2 boys & 2 girls 2) atleast (minimum) 1 boy 3) atmost (maximum) 2 girls 4) children of both sexes 5. all girls. Assume equal probability for boys and girls.

$n = 4$, $N = 800$

Let x be the no. of boys

prob of a boy $p = 1/2$.

$$q = 1 - p = 1 - 1/2 = 1/2$$

now Binomial distribution

$$P[X=x] = nC_x p^x q^{n-x}, \; x = 0,1,2 \ldots n.$$

$$P[X=x] = 4C_x\left(\frac{1}{2}\right)^x\left(\frac{1}{2}\right)^{4-x}, \; x = 0,1,2,3,4$$

(i) prob of 2 boys & 2 girls

$P[X=2] = 4C_2 \left(\frac{1}{2}\right)^2 \left(\frac{1}{2}\right)^{4-2}$

$= \frac{4\times 3}{1\times 2} \left(\frac{1}{2}\right)^4$

$= 6 \times \frac{1}{16}$

The no. of families having 2 boys & 2 girls

$= N \times P[X=2] = \overset{100}{\cancel{800}} \times \overset{3}{\cancel{6}} \times \frac{1}{\underset{2}{\cancel{16}}}$

$= 300$

(ii) Probability of atleast one boy

$= P[X \geq 1] = P[X=1] + P[X=2] + P[X=3] + P[X=4]$

$= 1 - P[X=0]$

$= 1 - 4C_0 \left(\frac{1}{2}\right)^0 \left(\frac{1}{2}\right)^{4-0} = 1 - \left(\frac{1}{2}\right)^4$

$= 1 - \frac{1}{16} = \frac{15}{16}$

The no. of families having atleast one boy $= \overset{\cancel{100}\ 50}{\cancel{800}} \times \frac{15}{\underset{2}{\cancel{16}}} = 750$

(iii) P[atmost 2 girls]

$= P$ [0 girls ~~or~~ 1 girl or 2 girls]

$= P$[0 girls] + P[1 girl] + P[2 girls]

$P[4 \text{ boys}] + P[3 \text{ boys}] + P[2 \text{ boys}]$

$= P[x=4] + P[x=3] + P[x=2]$

$= 4C_4 \left(\frac{1}{2}\right)^4 \left(\frac{1}{2}\right)^{4-4} + 4C_3 \left(\frac{1}{2}\right)^3 \left(\frac{1}{2}\right)^{4-3} + 4C_2 \left(\frac{1}{2}\right)^2 \left(\frac{1}{2}\right)^{4-2}$

$= 1 \cdot \left(\frac{1}{2}\right)^4 + 4\left(\frac{1}{2}\right)^4 + \frac{\cancel{4}^2 \times 3}{1 \times \cancel{2}} \left(\frac{1}{2}\right)^4.$

$= \left(\frac{1}{2}\right)^4 (1+4+6) = \frac{11}{16}.$

The no. of families having atmost 2 girls

$= 800 \times \frac{11}{16} = 550$

(iv) Probability of both sexes

$= P[1 \text{ boys} \ \& \ 3 \text{ girls}] + P[2 \text{ boys} \ \& \ 2 \text{ girls}] + P[3 \text{ boys} \ \& \ 1 \text{ girls}]$

$= P[x=1] + P[x=2] + P[x=3]$

$= 4C_1 \left(\frac{1}{2}\right)^1 \left(\frac{1}{2}\right)^{4-1} + 4C_2 \left(\frac{1}{2}\right)^2 \left(\frac{1}{2}\right)^{4-2} + 4C_3 \left(\frac{1}{2}\right)^3 \left(\frac{1}{2}\right)^{4-3}$

$= 4\left(\frac{1}{2}\right)^4 + 6\left(\frac{1}{2}\right)^4 + 4\left(\frac{1}{2}\right)^4$

$= (1/2)^4 (4+6+4) = \frac{1}{16} \times 14 = \frac{14}{16}.$

no. of families both sexes $= 800 \times \frac{14}{16}$

$= 700$

(v) Probability of all girls = $P[4 \text{ girls}]$

$= P[0 \text{ boy}] = P[x=0] = 4C_0 \left(\frac{1}{2}\right)^0 \left(\frac{1}{2}\right)^{4-0}$

$= \left(\frac{1}{2}\right)^4 = \frac{1}{16}$

The no. of. families having all girls $= 800 \times \frac{1}{16} = 50$.

2. A machine manufacturing screws is known to produce 5% defectives. In a random sample of 15 screws, what is the probability that there are (i) exactly three defectives (ii) exactly three screw. (iii) Not more than 3 defectives.

Let x be the no. of. defectives

P = prob of defectives $= 5\% = \frac{5}{100} = 0.05$

$q = 1 - p = 1 - 0.05 = 0.95$

$n = 15$

$P[x=x] = 15C_x (0.05)^x (0.95)^{15-x}, \quad x = 0, 1, 2 \dots 15$

(i) $P[x=3] = 15C_3 (0.05)^3 (0.95)^{15-3}$

$= 15C_3 (0.05)^3 (0.95)^{12}$

$= 0.0307$

(ii) P [not more than 3 defective]

$= P[x \leq 3] = P[x=3] + P[x=2] + P[x=1] + P[x=0]$

$= 0.0307 + 15C_2 (0.05)^2 (0.95)^{15-2} + 15C_1 (0.05)^1 (0.95)^{15-1} + 15C_0 (0.05)^0 (0.95)^{15-0}$

$= 0.0307 + 0.134 + 0.365 + 0.463$

$= 0.992.$

H.W

1) 6 dice are thrown 729 times. How many times do you except atleast 3 dices to show 5 or 6.

8m (X)

Let x be the no. of face values 5 or 6.

Let p be the prob of 5 or 6.

$P(5 \text{ or } 6) = P(5) + P(6) = \frac{1}{6} + \frac{1}{6} = \frac{2}{6} = 1/3.$

$p = 1/3 \quad q = 2/3.$

2) 10 ~~dice~~ coins are thrown simultaneously. Find the prob of getting 7 heads

8m (F)

(ii) Find the probability of atleast 7 heads.

Poisson Distribution.

If x is a discrete random variable that can assume the values $0, 1, 2 \ldots\ldots$ such that its probability mass function is given by

$$P[X = x] = \frac{e^{-\lambda}\lambda^{x}}{x!}, x = 0, 1, 2, 3 \ldots\ldots$$

1. Derive the MGF of poisson distribution and hence find its Mean and variance.

$$M_x(t) = E[e^{tx}] = \sum_{x=0}^{\infty} e^{tx} p(x)$$

$$= \sum_{x=0}^{\infty} e^{tx} e^{-\lambda}\lambda^{x}/x!$$

$$= e^{-\lambda} \sum_{x=0}^{\infty} \frac{(\lambda e^{t})^{x}}{x!}$$

$$= e^{-\lambda}\left[\frac{(\lambda e^{t})^{0}}{0!} + \frac{(\lambda e^{t})^{1}}{1!} + \frac{(\lambda e^{t})^{2}}{2!} + \cdots\right]$$

$$= e^{-\lambda}\left[1 + \frac{\lambda e^{t}}{1!} + \frac{(\lambda e^{t})^{2}}{2!} + \frac{(\lambda e^{t})^{3}}{3!} + \cdots\right]$$

$$= 1 + \frac{x^{1}}{1!} + \frac{x^{2}}{2!} + \frac{x^{3}}{3!} = e^{x}$$

$$M_x(t) = e^{-\lambda} e^{\lambda e^{t}} = e^{\lambda e^{t} - \lambda} = e^{\lambda(e^{t}-1)}$$

very

iab

8

$$M_x(t) = e^{\lambda(e^t - 1)}$$

Mean $E(x) = \frac{d}{dt} M_x(t)$ at $t=0$

Variance $E(x^2) = \frac{d^2}{dt^2} M_x(t)$ at $t=0$

$E(x^2) - [E(x)]^2$.

$M_x(t) = e^{-\lambda} e^{\lambda e^t}$ λe^t

$\frac{d}{dt} M_x(t) = e^{-\lambda}[e^{\lambda e^t} \cdot \lambda e^t]$

$= \lambda e^{-\lambda}[e^t . e^{\lambda e^t}]$

$\frac{d^2}{dt^2} M_x(t) = \lambda e^{-\lambda}(e^t e^{\lambda e^t} \lambda e^t + e^{\lambda e^t} e^t$

Mean $E(x) = \lambda e^{-\lambda}[e^0 e^{\lambda e^0}] = \lambda e^{-\lambda} \cdot e^{\lambda}$

$= \lambda e^{-\lambda + \lambda} = \lambda$

$E(x^2) = \lambda e^{-\lambda}[e^0 e^{\lambda e^0} \lambda e^0 + e^{\lambda e^0} e^0]$

$= \lambda e^{-\lambda}[e^{\lambda}\lambda + e^{\lambda}] = \lambda e^{-\lambda} e^{\lambda}[\lambda + 1]$

$E[x^2] = \lambda^2 + \lambda$.

Variance of $x = \lambda^2 + \lambda - (\lambda)^2 = \lambda$.

Poisson distribution is liming form of Binomial distribution :-

very (X)

Poisson distribution is limiting case of Binomial distribution under the following conditions.

1. n, the number of trails is indefinitely large (i.e) $(n \to \infty)$

(ii) p, the constant probability of success in each trial is very small. (i.e) $(p \to 0)$

(iii) $np = \lambda$ (is finite)

$p+q=1$ $q=1-$

$p = \frac{\lambda}{n}$ and $q = 1 - \frac{\lambda}{n}$.

and λ is a positive Real number.

Proof:

If x is a binomial parameter with n and p.

$$P[X=x] = nc_x \, p^x q^{n-x}, \quad x = 0,1,2 \ldots n$$

$$= \frac{n(n-1)(n-2)(n-3)\ldots(n-x+1)}{1\cdot 2\cdot 3\cdot 4 \ldots x} \left(\frac{\lambda}{n}\right)^x \left(1-\frac{\lambda}{n}\right)^{n-x}$$

$$= \frac{\lambda^x}{x!} \left[1\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\left(1-\frac{3}{n}\right)\ldots\left(1-\frac{x+1}{n}\right)\right] \left(1-\frac{\lambda}{n}\right)^{n-x}$$

$$= \frac{\lambda^x}{x!} \left[1\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\ldots\left(1-\frac{x+1}{n}\right)\right] \left(1-\frac{\lambda}{n}\right)^n \left(1-\frac{\lambda}{n}\right)^{-x}$$

taking lim $n \to \infty$ on both sides

$$\lim_{n\to\infty} nc_x \, p^x q^{n-x} = \lim_{n\to\infty} \frac{\lambda^x}{x!} \left(1\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\ldots\left(1-\frac{x+1}{n}\right)\left(1-\frac{\lambda}{n}\right)^n\left(1-\frac{\lambda}{n}\right)\right.$$

w.k.T $\lim_{n\to\infty}\left(1-\frac{\lambda}{n}\right)^n = e^{-\lambda}$

$$= \frac{\lambda^x}{x!} \lim_{n\to\infty}\left(1-\frac{\lambda}{n}\right)^n$$

$$P[X=x] = \frac{e^{-\lambda}\lambda^x}{x!} \quad x = 0, 1, 2, \ldots \infty$$

1. When is a poison variable such that $P[x=2] = 9P[x=4] + 90P[x=6]$. Find the variance of x.

Soln:

$$P[x=x] = \frac{e^{-\lambda}\lambda^x}{x!}$$

with

$$P[x=2] = 9P[x=4] + 90P[x=6]$$

$$\frac{e^{-\lambda}\lambda^2}{2!} = 9\frac{e^{-\lambda}\lambda^4}{4!} + 90\frac{e^{-\lambda}\lambda^6}{6!}$$

$$\frac{\cancel{e^{-\lambda}}\lambda^2}{2} = \cancel{e^{-\lambda}}\left[\frac{9\lambda^4}{4\cdot3\cdot2\cdot1} + \frac{90\lambda^6}{6\cdot5\cdot4\cdot3\cdot2\cdot1}\right]$$

$$\frac{\lambda^2}{2} = \lambda^4\left[\frac{3}{8} + \frac{\lambda^2}{8}\right]$$

$$\frac{1}{2} = \frac{\lambda^2}{8}\left[3 + \lambda^2\right]$$

$$4 = 3\lambda^2 + \lambda^4$$

$$\lambda^4 + 3\lambda^2 - 4 = 0$$

$$(\lambda^2)^2 + 3\lambda^2 - 4 = 0$$

$$(\lambda^2+4)(\lambda^2-1) = 0$$

-4

$4 \quad 1$

$\lambda^2 = a$

$a^2 + 3a - 4 = 0$

... x

$n-x$

$\left(1-\frac{\lambda}{n}\right)$

$\frac{x+1}{n}$

$-\frac{\lambda}{n})^{n-x}$

$-\frac{1}{1})$

$-\frac{\lambda}{n})^{-x}$

$-\frac{\lambda}{n})^{n}\left(1-\frac{\lambda}{n}\right)^{-x}$

$\lambda^2+4=0$ $\quad$ $\lambda^2=1$

$\lambda^2=-4$ $\quad$ $\lambda=\pm 1.$

$\lambda=\pm 2i$

Variance of poisson $\lambda=\pm 1.$

2. The number of monthly breakdown of a computer is a random variable having a poison distribution with mean $=1.8$ find the probability that this function for a month (i) without breakdown (ii) with only one breakdown (iii) atleast one breakdown.

Soln:

Mean $=1.8$

$\lambda=1.8$

poison distribution $P[X=x]=\dfrac{e^{-\lambda}\lambda^{x}}{x!}$

$x=0,1,2\ldots\infty.$

$$P[X=x]=\frac{e^{-1.8}(1.8)^{x}}{x!} \quad x=0,1,2\ldots\infty$$

Let X be the number of break down.

(i) $P[\text{without breakdown}]=P[X=0]=$

$$=\frac{e^{-1.8}(1.8)^{0}}{0!}$$

$=0.1653$

(ii) P

(iii) P

=

1. It a: be 2 wi Let P = He Bu (

(ii) P[with only one breakdown] = $P[X=1]$

$$= \frac{e^{-1.8}(1.8)^1}{1!} = 0.1653(1.8) = 0.2975$$

(iii) P[atleast one breakdown]

$= P[X \geq 1] = 1 - P[X < 1]$

$= 1 - P[X = 0]$

$= 1 - 0.1653 = 0.8347$

1. It is known that 5% of books bound at a certain bindary have defective bindings. Find the probability that 2 of 100 books bound bind its bindary will have defective bindings.

Let X be the no. of defective binding

P = probability of defective binding

$= 5\% = \frac{5}{100}$

$n = 100$

Let $\lambda = np = 100 \times \frac{5}{100} = 5$

By poisson distribution

$$P[X=x] = \frac{e^{-\lambda}\lambda^x}{x!}, \quad x = 0, 1, 2 \ldots$$

$$P[X=x] = \frac{e^{-5}5^x}{x!} \quad x = 0, 1, 2 \ldots$$

$$P\{2 \text{ defective}\} = P[X=2] = \frac{e^{-5}5^2}{2!} = 0.084$$

(or)

Binomial distribution

$n = 100$, $p = \frac{5}{100} = 0.05$ $q = \frac{95}{100} = 0.95$

$P[x = x] = 100C_x (0.05)^x (0.95)^{100-x}$

$P[x = 2] = 100C_2 (0.05)^2 (0.95)^{100-2}$

$= \frac{100 \times 99}{1 \times 2} (0.0025)(0.95)^{98}$

$= 4950 (0.0025)(0.95)^{98}$

$= 0.0811$.

4. H.W It is known that the probability of an item produced by a machine will be defective is $P(\text{def}) = 0.05$. ~~sent to~~ market produced items are in packets of 20. Find the no. of packets containing atleast, exactly and atmost 2 defective items. In a consignment of 1000 packets using (i) binomial (ii) poison.

$n = 20$, $p = 0.05$

Let x be the no. of defective items.

1. 6 coins are tossed 6400 times using poison distribution, determine the approximate probability of getting 6 heads.

$n = 6400$

Let p be the probability of 6 head $= \left(\frac{1}{2}\right)^6$

$P = \left(\frac{1}{2}\right)^6 = \frac{1}{64}$

$np = \lambda = 6400 \times \frac{1}{64} = 100$

$P[x = n] = \dfrac{e^{-\lambda}\lambda^{n}}{n!}$

$P[x = 6] = \dfrac{e^{-100}(100)^6}{6!} = 5.1667 \times 10^{-35}$

Geometric distribution :

If x is a discrete random variable such that that can assume the values

$1, 2, 3 \ldots \infty$

$P[x = n] = pq^{n-1}, \; n = 1, 2, 3 \ldots \infty$

such that its probability mass function

where $p + q = 1$ then x is said to follow as a Geometric distribution

Derive the MGF of Geometric distribution
and hence find its mean and variance.

soln :

M.G.F $M_x(t) = E[e^{tx}] = \sum_{x=1}^{\infty} e^{tx} p(x)$

$= \sum_{x=1}^{\infty} e^{tx} p q^{x-1}$

$= \sum_{x=1}^{\infty} e^{tx} p q^{x} q^{-1}$

$= \frac{p}{q} \sum_{x=1}^{\infty} q^{x} e^{tx}$

$= \frac{p}{q} \sum_{x=1}^{\infty} (qe^{t})^{x}$

$= \frac{p}{q} \left[(qe^t)^1 + (qe^t)^2 + (qe^t)^3 + \dots \right]$

$= \frac{pqe^t}{q} \left[1 + (qe^t) + (qe^t)^2 + \dots \right]$

$= pe^t \left[1 - qe^t \right]^{-1}$

$M_x(t) = \frac{pe^t}{1 - qe^t}$.

Mean $E(x) = \frac{d}{dt} M_x(t)$ at $t=0$.

$Var(x) = E(x^2) - [E(x)]^2$

$E(x^2) = \frac{d^2}{dt^2} M_x(t)$ at $t=0$.

$M_x(t) = \frac{pe^t}{(1-qe^t)}$

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$$\frac{d}{dt} M_x(t) = \frac{(1-qe^t)(pe^t) - pe^t(0-qe^t)}{(1-qe^t)^2}$$

$$= \frac{pe^t - pq(e^t)^2 + pq(e^t)^2}{(1-qe^t)^2} = \frac{pe^t}{(1-qe^t)^2}$$

$$\frac{d^2}{dt^2} M_x(t) = \frac{(1-qe^t)^2(pe^t) - pe^t 2(1-qe^t)(0-qe^t)}{(1-qe^t)^4}$$

now Mean $E(x) = \frac{pe^0}{(1-qe^0)^2} = \frac{p}{(1-q)^2}$

$$= p/p^2 = 1/p.$$

$$E(x^2) = \frac{(1-qe^0)^2(pe^0) + 2pqe^0e^0(1-qe^0)}{(1-qe^0)^4}$$

$$= \frac{(1-q)^2(p) + 2pq(1-q)}{(1-q)^4}.$$

only one valu p is given G.D

$$= \frac{(1-q)^2 p}{(1-q)^4} + \frac{2pq(1-q)}{(1-q)^4}$$

$$= \frac{p}{(1-q)^2} + \frac{2pq}{(1-q)^3}$$

$$= \frac{p(1-q)^1 + 2pq}{(1-q)^3}$$

$$= \frac{p - pq + 2pq}{(1-q)^3} = \frac{p+pq}{(1-q)^3} = \frac{p(1+q)}{p^3}$$

$$= \frac{(1+q)}{p^2}$$

Variance $[x] = E(x^2) - [E(x)]^2$

$$= \frac{1+q}{p^2} - \left(\frac{1}{p}\right)^2$$

$$= 1/p^2 + q/p^2 - 1/p^2$$

$$= q/p^2.$$

Memoryless property (Forget fullness)

$P[x > m+n / x > m] = P[x > n]$

If x is a discrete random variable, where m and n are any two positive integers

Proof :

Given $P[x = x] = pq^{x-1}, x = 1, 2, 3 \ldots$

$P[x > K] = P[x = K+1, K+2, K+3 \ldots]$

$$= \sum_{x=K+1}^{\infty} P(x)$$

$$= \sum_{x=K+1}^{\infty} pq^{x-1}$$

$$= pq^{K+1-1} + pq^{K+2-1} + pq^{K+3-1} + pq^{K+4-1} + \ldots$$

$$= pq^{K} + pq^{K+1} + pq^{K+2} + pq^{K+3} + \ldots$$

$$= pq^{K}[1 + q + q^2 + q^3 + \ldots]$$

$= pq^k [1-q]^{-1}$

$= \frac{pq^k}{p}$

$= q^k$.

$m=3,\ n=2$

$x > 3 = 4, 5, 6, 7,$

$x > 3+2 = 6, 7, 8$

$x > m+n$.

R.H.S:

$P[X > n] = q^n$

L.H.S

$P[X > m+n / x > m]$

$= \frac{P[(X > m+n) \cap (X > m)]}{P[X > m]}$

$P(A/B) = \frac{P(A \cap B)}{P(B)}$

$= \frac{P[X > m+n]}{P[X > m]} = \frac{q^{m+n}}{q^m}$

$= \frac{q^m \cdot q^n}{q^m} = q^n = R.H.S.$

11. The probability that applicant for a drivers licence with pass the road test on any given trial is 0.8. What is the probability that he will finally pass the test -

a) one the fourth trial

b) fewer than 4 trial.

Let x be the no of trials

p = prob of pass $= 0.8$

$q = 1 - p = 1 - 0.8 = 0.2$

W.K.T Geometric distribution

$P[X = x] = pq^{x-1}, \; x = 1, 2, 3 \ldots$

$P[X = x] = (0.8)(0.2)^{x-1}, \; x = 1, 2, 3 \ldots$

a) $P[X = 4] = (0.8)(0.2)^{4-1}$

$= (0.8)(0.2)^3$

$= (0.8)(0.008)$

$= 0.0064$

b) prob [few than 4 trials]

$P[X < 4] = P[X = 1] + P[X = 2] + P[X = 3]$

$= (0.8)(0.2)^{1-1} + (0.8)(0.2)^{2-1} + (0.8)(0.2)^{3-1}$

$= 0.8 + (0.8)(0.2) + (0.8)(0.04)$

$= 0.8 + 0.16 + 0.032$

$= 0.992$

2. If the probability that a target is destroyed on any one shoot is 0.5. What is the probability that it would be destroyed on the 6th attempt.

soln:

$p = 0.5$

$q = 1 - p = 1 - 0.5 = 0.5$

Geometric distribution is.

$P[X = x] = pq^{x-1}, \quad x = 0, 1, 2 \ldots$

Let X be the no. of trials

$P[X = x] = (0.5)(0.5)^{x-1}, \quad x = 1, 2, 3 \ldots$

$P[X = 6] = (0.5)(0.5)^{5}$

$= (0.5)^{6} = 0.0156.$

Negative Binomial distribution;

Note:

A Generalisation of Geometric distribution in which the Random variable is the number of Bernoullis trials required to obtain are success is the negative Binomial distribution.

If r trials (failures) have to occur are proceeding n^{th} success, $n+r$ trials required. As the first $n+r-1$ trials should result in r failures and $n-1$ success and $n+r$ th trial should result is a success, where $r = 0, 1, 2, 3 \ldots$

$r \rightarrow$ failures $\quad n \rightarrow$ success

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be

$$P[X=x] = {}^{x+r-1}C_{r-1} \cdot p^r \cdot q^x \,;\; x = 0,1,2\ldots$$

$$M_x(t) = E[e^{tx}] = \sum_{x=0}^{\infty} e^{tx} p(x)$$

$$= \sum_{x=0}^{\infty} e^{tx}\, {}^{x+r-1}C_{r-1}\, p^r q^x.$$

$$= p^r \sum_{x=0}^{\infty} {}^{x+r-1}C_{r-1} (qe^t)^x$$

$$= p^r \left[{}^{r-1}C_{r-1} (qe^t)^0 + {}^{r}C_{r-1} (qe^t)^1 + {}^{r+1}C_{r-1} (qe^t)^2 + \cdots \right]$$

$$= p^r \left[1 + {}^{r}C_{r-1} (qe^t) + {}^{r+1}C_{r-1} (qe^t)^2 + \cdots \right]$$

$$= p^r \left[1 - qe^t \right]^{-r}$$

$$M_x(t) = \frac{p^r}{(1-qe^t)^r}$$

$(1-qe^t)^{-r}$

$-r(1-qe^t)^{-r-1}$

Mean $E(x) = \frac{d}{dt} M_x(t) \,|_{t=0}$

$$\mathrm{Var}(x) = E(x^2) - [E(x)]^2$$

$$\frac{d^2}{dt^2} M_x(t) \, dt \; t=0 = E(x^2)$$

$$\frac{d}{dt} M_x(t) = p^r \frac{(-r(-qe^t))}{(1-qe^t)^{r+1}}$$

$\frac{1}{x^2} = -\frac{1}{x^2}$

$\frac{1}{x^r} = \frac{-r}{x^{r+1}}$

$$= rq\,p^r \left(\frac{e^t}{(1-qe^t)^{r+1}} \right)$$

$$\frac{d^2}{dt^2} M_x(t) = rqp^r \left[\frac{(1-qe^t)^{r+1} e^t - e^t \, (r+1)(1-qe^t)^{r+1-1}(-qe^t)}{\left((1-qe^t)^{r+1}\right)^2} \right]$$

$$\text{Mean } E(x) = rqp^r \frac{e^0}{(1-qe^0)^{r+1}} = \frac{p^r rq}{p^{r+1}}$$

$$E(x) = \frac{rq}{p}$$

$$\text{Variance } E(x^2) = rqp^r \left[\frac{(1-q)^{r+1} + (r+1)q(1-q)}{(1-q)^{2r+2}} \right]$$

$$= rqp^r \left[\frac{p^{r+1} + p^r(r+1)q}{p^{2r+2}} \right]$$

$$= \frac{rqp^r p^r [p + (r+1)q]}{p^{2r} \cdot p^2}$$

$$= \frac{rq[p + q + rq]}{p^2}$$

$$= \frac{rq[1 + rq]}{p^2}$$

$$\text{var}(x) = E(x^2) - [E(x)]^2$$

$$= \frac{rq + r^2q^2}{p^2} - \left(\frac{rq}{p}\right)^2$$

$$= \frac{rq}{p^2}.$$

1. The probability is 0.4 that a child exposed to a certain disease will catch it. This is disease what is the probability that a tenth child is exposed to the diseases will be the third to catch it.

soln:

Here x follows negative binomial distribution

$$P[X=x] = {}^{x+r-1}C_{r-1}\, p^r q^x,\ x=0,1,2\ldots$$

p = probability of disease to catch the child.

$= 0.4$

$q = 1-p = 1-0.4 = 0.6$

$p=0.4,\ q=0.6\quad x=10,\ r=3$

$$P[X=10] = {}^{10+3-1}C_{3-1}(0.4)^3(0.6)^{10}$$

$$= {}^{12}C_2 (0.4)^3(0.6)^{10}$$

$$= 0.0255$$

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2. A Pediatrician wishes recruit 5 couples each of whom is expecting the first child to participate in a new natural child birth regimen. If the probability that randomly selected couples agrees to participate is 0.2. What is the probability that atmost 15 couples must be asked before five are found to agree to participate.

Given:

$p = 0.2, \; q = 0.8 \quad r = 5$

$P(X = x) = {}^{x+r-1}C_{r-1} \, p^r q^x,$

Probability of atmost 10 failures $= P[X \leq 10]$

$= \sum_{x=0}^{10} \left({}^{x+5-1}C_{5-1} \right) (0.2)^5 (0.8)^x$

$= (0.2)^5 \left[{}^4C_4 (0.8)^0 + 5C_4 (0.8)^1 + 6C_4 (0.8)^2 + 7C_4 (0.8)^3 + 8C_4 (0.8)^4 + 9C_4 (0.8)^5 + 10C_4 (0.8)^6 + 11C_4 (0.8)^7 + 12C_4 (0.8)^8 + 13C_4 (0.8)^9 + 14C_4 (0.8)^{10} \right]$

$= 0.164$

Continuous distribution

1. Uniform distribution (or) Rectangular distribution.

If x follows a uniform distribution $a < x < b$.

$$f(x) = \begin{cases} \frac{1}{b-a}, & a < x < b \\ 0, & o.w \end{cases}$$

Derive the r^{th} moment about origin of uniform distribution and hence find its mean and variance.

$$\mu_r' = E[X^r] = \int_{-\infty}^{\infty} x^r f(x)\, dx$$

$$\mu_r' = \int_a^b x^r \cdot \frac{1}{b-a}\, dx$$

$$= \frac{1}{b-a} \int_a^b x^r \cdot dx$$

$$= \frac{1}{b-a} \left[\frac{x^{r+1}}{r+1} \right]_a^b$$

$$= \frac{1}{b-a} \left[\frac{b^{r+1} - a^{r+1}}{r+1} \right]$$

$$= \frac{b^{r+1} - a^{r+1}}{(r+1)(b-a)}$$

$$E(x) = \mu_1' = \frac{b^2 - a^2}{2(b-a)} =$$

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$$= \frac{(b-a)(b+a)}{2(b-a)}$$

$$E(x) = \frac{b+a}{2}.$$

when $r=2$

$$\mu_2' = E(x^2) = \frac{b^{2+1} - a^{2+1}}{(2+1)(b-a)} = \frac{b^3 - a^3}{3(b-a)}$$

$$E(x^2) = \frac{(b-a)(b^2+ab+a^2)}{3(b-a)}$$

$$\text{var}(x) = E(x^2) - [E(x)]^2$$

$$= \frac{b^2+ab+a^2}{3} - \left(\frac{b+a}{2}\right)^2$$

$$= \frac{b^2+ab+a^2}{3} - \frac{b^2+2ab+a^2}{4}$$

$$= \frac{4b^2+4ab+4a^2 - 3b^2 - 6ab - 3a^2}{12}$$

$$= \frac{b^2 - 2ab + a^2}{12} = \frac{(b-a)^2}{12}$$

1. x has uniform distribution $(-3, 3)$ find $P(|x-2|<2)$

Sln w.k.T uniform distribution.

$$f(x) = \frac{1}{3-(-3)}, -3 < x < 3.$$

$$f(x) = \begin{cases} \frac{1}{6}, & -3<x<3 \\ 0, & \text{otherwise} \end{cases}$$

$|x-2| < 2$

$-2 < |x-2| < 2$

adding 2

$-2+2 < x-2+2$

$0 < x < 4 < 2+2$

$$P(|x-2| < 2) = P(0<x<4)$$

$$= \int_0^3 1/6 \, dx + \int_3^4 0 \, dx$$

$$= \frac{1}{6}(x)_0^3 = \frac{3-0}{6} = \frac{3}{6} = \frac{1}{2}$$

$= 0.5$

2. If the MGF of the continuous Random variable $\frac{e^{5t}-e^{4t}}{t}$, $t \neq 0$. What is the distribution of x. What are its mean & variance?

(K) 2m

Here x is uniform distribution.

here $a=4, b=5$.

$$f(x) = \frac{1}{5-4}, 4<x<5$$

$$= 1, \; 4<x<5$$

Mean of uniform distribution $\Big\}$ $E(x) = \frac{b+a}{2} = \frac{5+4}{2}$

$= 9/2 = 4.5$

$$\text{variance} = \frac{(b-a)^2}{12} = \frac{(5-4)^2}{12} = 1/12$$

3. If x is uniformly distributed with mean $= 1$ and variance $4/3$. find $P(x<0)$

Given mean $= 1$

$$\frac{b+a}{2} = 1$$

$$b+a = 2 \rightarrow (1)$$

variance $= 4/3$

$$\frac{(b-a)^2}{12} = 4/3$$

$$(b-a)^2 = \frac{4 \times \cancel{12}^{\,4}}{\cancel{3}}$$

$$(b-a)^2 = 16$$

$$(b-a)^2 = 4^2$$

Finding sq. root

$$b-a = 4.$$

solving (1) & (2)

$$b+a = 2$$
$$b-a = 4$$
$$2b = 6$$
$$b = 3$$

sub $b=3$ in (1).

$$3+a = 2$$
$$a = 2-3$$
$$a = -1$$

$$f(x) = \frac{1}{3-(-1)} \quad -1<x<3$$

$= \frac{1}{4}, \quad -1 < x < 3$.

$P[X < 0] = \int_{-\infty}^{0} f(x)\,dx$

$= \int_{-1}^{0} 1/4 \, dx$

$= \frac{1}{4}(x)_{-1}^{0}$

$= \frac{1}{4}(0-(-1))$

$= \frac{1}{4}(1) = 1/4$.

Key words
Regularly,
uniformly

X. Subway trains on a certain line run every (uniform) half an hour b/w midnight and 6 in the morning. What is the probability that a man entering the station at a random time during this period will have to weight atleast 20 minutes.

Sm 1.

Soln.

0 10 20 30

Let x denote the time in minutes

(i.e) $f(x) = \frac{1}{30-0}, \quad 0 < x < 30$

$f(x) = \frac{1}{30}, \quad 0 < x < 30$

P[a waste has to wait atleast 20 minutes]

$P[0 < X < \cancel{30}] = \int_{0}^{10} f(x)\,dx$

$= \int_{0}^{10} 1/30 \, dx = 1/30 (x)_{0}^{10} = \frac{10-0}{30}$

$= \frac{10}{30} \Rightarrow 1/3 \Rightarrow 0.333$,

1) Buses arrive at specified stop at 15 min interval starting at 7 am that is they arrive at 7, 7.15, 7.30 7.45 and so on. If a passenger arrives at the stop at a random time that is uniformly distributed b/w 7 am and 7.30 am. Find the probability that he weights for a bus

(i) less than 5 mins for a bus
(ii) atleast 12 mins for a bus.

soln:

Let x denote the time in mins past 7.am

x is uniformly distributed

$$f(x) = \frac{1}{30}, \quad 0 < x < 30.$$

a) passenger has to wait less than 5 min is $P[7.10 \text{ to } 7.15 \text{ or } 7.25 \text{ to } 7.30]$

$$= P[10 < x < 15] + P[25 < x < 30]$$

$$= \int_{10}^{15} 1/30 \, dx + \int_{25}^{30} 1/30 \, dx$$

$$= \frac{1}{30}(x)_{10}^{15} + \frac{1}{30}(x)_{25}^{30}$$

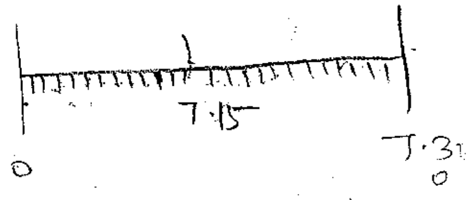
$= \frac{1}{30}\left[(15-10)+(30-25)\right]$

$= \frac{1}{30}(5+5) = \frac{10}{30} = 0.33$

b) P (atleast 12 min)

$= P[0 < x < 3] + P[15 < x < 18]$

$= \int_0^3 1/30\, dx + \int_{15}^{18} 1/30\, dx$



$= \frac{1}{30}(x)_0^3 + \frac{1}{30}(x)_{15}^{18}$

$= \frac{1}{30}\left(3-0+18-15\right]$

$= \frac{6}{30} = \frac{1}{5} = 0.2$

1. Show that for a Rectangular distribution

$f(x) = \begin{cases} 1/2a, & -a < x < a \\ 0, & \text{o.w.} \end{cases}$

the MGF about origin is $\frac{\sinh at}{at}$

$M_x(t) = E[e^{tx}] = \int_{-a}^{a} e^{tx} f(x)\, dx$

$= \int_{-a}^{a} e^{tx} \cdot \frac{1}{2a}\, dx$

$= \frac{1}{2a}\left(\frac{e^{tx}}{t}\right)_{-a}^{a}$ $\qquad (e^{\theta} - e^{-\theta} = 2\sinh \theta)$

$= \frac{1}{2at}\left(e^{at} - e^{-at}\right)$

$= \frac{2\sinh at}{2at} = \frac{\sinh at}{at}.$

Exponential distribution

A continuous random variable 'x' is said to follow an exponential distrib-ution, and (or) negative exponential distribution, with parameter ($x > 0$) if its p.d.f is given by

$$f(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0, \ \lambda > 0. \\ 0, & o.w \end{cases}$$

Derive the MGF of exponential distribution and hence find its mean and variance.

$$M_x(t) = E[e^{tx}] = \int_{-\infty}^{\infty} e^{tx} f(x)\, dx$$

$$= \int_0^{\infty} e^{tx} \lambda e^{-\lambda x} dx$$

$$= \lambda \int_0^{\infty} e^{tx} e^{-\lambda x} dx$$

$$= \lambda \int_0^{\infty} e^{-\lambda x + tx} dx$$

$$= \lambda \int_0^{\infty} e^{-(\lambda - t)x} dx$$

$$= \lambda \left[\frac{e^{-(\lambda - t)x}}{-(\lambda - t)} \right]_0^{\infty}$$

$$= \lambda \left[\frac{e^{-\infty} - e^{0}}{-(\lambda - t)} \right] \qquad e^{-\infty} = 0, \quad e^{0} = 1.$$

$$M_x(t) = \frac{\lambda}{\lambda - t}$$

Mean $E(x) = \frac{d}{dt} M_x(t) \big|_{t=0}$.

$$E(x^2) = \frac{d^2}{dt^2} M_x(t) \big|_{t=0}.$$

$$M_x(t) = \frac{\lambda}{\lambda - t}$$

$+(1-t)^{-2}$

$+[-2(1-t)^{-3}(-1)]$

$$\frac{d}{dt} M_x(t) = \frac{-\lambda}{(\lambda-t)^2}(-1) = \frac{\lambda}{(\lambda-t)^2}$$

$+\frac{(-2)}{(1-t)^3}$

$$\frac{d^2}{dt^2} M_x(t) = \frac{-2\lambda}{(\lambda-t)^3}(-1)$$

$$= \frac{2\lambda}{(\lambda-t)^3} = \frac{2\lambda}{(\lambda-0)^3} = \frac{2}{\lambda^2}$$

Mean $E(x) = \frac{\lambda}{(\lambda-0)^2} = \frac{\lambda}{\lambda^2} = \frac{1}{\lambda}$.

$$\text{var}(x) = \frac{2}{\lambda^2} - \left(\frac{1}{\lambda}\right)^2 = \frac{2}{\lambda^2} - \frac{1}{\lambda^2}$$

$$= 1/\lambda^2.$$

Memoryless property of exponential distribution.

If x is exponentially distributed

then $P[x > s+t \,/\, x > s] = P[x > t]$

proof:

$$P[x > k] = \int_k^\infty f(x)\,dx$$

$$= \int_k^\infty \lambda e^{-\lambda x}\,dx$$

$$= \lambda \left[\frac{e^{-\lambda x}}{-\lambda} \right]_{k}^{\infty}$$

$s = 3$

$t = 5$

$s + 5 = 8 = 9, 10, 11, 12$

$x > s = 4, 5, 6, 7, 8, 9, 10, 11, 12$

$$= \left[\frac{e^{-\lambda\infty} - e^{-k\lambda}}{-1} \right]$$

$$= e^{-k\lambda}$$

R.H.S $\quad P[x > t] = e^{-t\lambda} \rightarrow (1)$.

L.H.S $\quad P[x > s+t / x > s] = \dfrac{P[(x > s+t) \cap (x > s)]}{P[x > s]}$

$$= \frac{P[x > s+t]}{P[x > s]}$$

$$= \frac{e^{-(s+t)\lambda}}{e^{-s\lambda}} = \frac{e^{-s\lambda} \cdot e^{-t\lambda}}{e^{-s\lambda}} = e^{-\lambda t} = \text{R.H.S}$$

1. (7) The Mileage which car owners get with a certain kind of radial tyre is a Random Variable having an exponential distribution with a mean 40,000 km. Find the probability with one of these tyres is last

(i) atleast 20,000 km

(ii) atmost 30,000 km

Let x be the life time of radial tyre

Given Mean of exp $= 40,000$

$1/\lambda = 40,000$

$\frac{1}{40,000} = \lambda$

Exponential distribution.

$f(x) = \lambda e^{-\lambda x}, \; x > 0$

$= \frac{1}{40000} e^{-x/40000}, \; x > 0.$

(i) Probability atleast 20,000 km

$P[X > 20000] = \int_{20000}^{\infty} f(x)\,dx$

$= \int_{20000}^{\infty} \frac{1}{40000} e^{-\frac{x}{40000}}\,dx$

$= \frac{1}{40000} \left[\frac{e^{-x/40000}}{-1/40000} \right]_{20000}^{\infty}$

$= \frac{e^{-\infty} - e^{-\frac{20000}{40000}}}{-1}$

$= e^{-1/2} = 0.606$

(ii) P [atmost 30,000 km]

$P[X \leq 30,000] = \int_{-\infty}^{30000} f(x)\,dx$

$= \int_{0}^{30000} \frac{1}{40000} e^{-x/40000}\,dx$

$= \frac{1}{40000} \left[\frac{e^{-x/40000}}{-1/40000} \right]_{0}^{30000}$

$$= \left[\frac{e^{\frac{-30000}{40000}} - e^{-0/40000}}{-1} \right]$$

$$= 1 - e^{-3/4} = 0.578.$$

2. The time [in hours] required to repair a machine is exponentially distributed with parameter $\lambda = 1/2$.

a) what is the probability required time exceeds 2 hour.

b) what is conditional probability repair takes atleast 10 hours given that its duration exceeds 9 hours.

It is exponential distribution

$$f(x) = \lambda e^{-\lambda x}, \quad x > 0.$$

$$f(x) = 1/2\, e^{-x/2}, \quad x > 0.$$

Let x be the repair time of a machine

P[repair time exceeds 2 hours]

$$= P[X > 2] = \int_2^\infty f(x)\,dx$$

$$= \int_2^\infty 1/2\, e^{-x/2}\,dx$$

$$\frac{1}{2} \left[\frac{e^{-x/2}}{-1/2} \right]_2^\infty$$

$$= \left(\frac{e^{-\infty} - e^{-T/2}}{-1} \right) = e^{-1} = 0.3678$$

(ii) $P[x > 10 / x > 9]$

By memory less property

$P[x > s+t / x > s] = P[x > t]$

$$P[x > 1] = \int_1^\infty f(x)\,dx = \int_1^\infty \tfrac{1}{2} e^{-x/2}\,dx$$

$$= \tfrac{1}{2}\left[\frac{e^{-x/2}}{-1/2}\right]_1^\infty = \tfrac{1}{2}\left(\frac{e^{-\infty} - e^{-1/2}}{-1/2}\right)$$

$$= e^{-0.5} = 0.606$$

3. The daily consumption of milk in excess of 20,000 litres in a town is approximately exponential distr with parameter $1/3000$. The town has daily stock of 35,000 c. what is the probability that the stock of 2 days selected at random, the stock is in sufficient for both days.

Let y be the daily stock

Let x be the daily consumption of milk here $y = x + 20000$

Here prob 35,000 is insufficient.

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If $P(y > 35,000)$

$= P[x + 20000 > 35000]$

$= P[x > 35000 - 20000]$

$= P[x > 15000]$

Here $\lambda = 1/3000$

$f(x) = \lambda e^{-\lambda x}, \; x > 0$

$f(x) = 1/3000 \; e^{-x/3000}, \; x > 0$

$P[x > 35000] = P[x > 15000]$

$= \int_{15000}^{\infty} f(x)\,dx$

$= \int_{15000}^{\infty} 1/3000 \; e^{-x/3000}\,dx$

$\frac{1}{3000}\left[\frac{e^{-x/3000}}{-1/3000}\right]_{15000}^{\infty}$

$= \frac{e^{-\infty} - e^{-\frac{15000}{3000}}}{-1} = e^{-5} = 0.0067$

P (stock is sufficient for both the days) $= p^2$

$= (e^{-5})^2$

$= e^{-10}$.

4. A continuous Random Variable has the density function $ce^{-x/5}, x>0$ find (i) 'c'
(ii) $E(x)$ & $var(x)$.

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

$$\int_{0}^{\infty} c e^{-x/5}\,dx = 1$$

$$c\left[\frac{e^{-x/5}}{-1/5}\right]_0^{\infty} = 1$$

$$c\left[\frac{e^{-\infty} - e^{0}}{-1/5}\right] = 1$$

$$c\left[\frac{1}{1/5}\right] = 1$$

$$c = 1/5$$

$$c\left[\frac{5}{1}\right] = 1$$

$$5c = 1$$

$$c = 1/5.$$

$$f(x) = 1/5\, e^{-x/5},\ x>0$$

It is exponentially distributed with parameter

Mean of Exponential $= 1/\lambda = 1/\frac{1}{5} = 5$

Var (x) of Exponential $= 1/\lambda^2 = 1/\left(\frac{1}{5}\right)^2$

$$= 1/\frac{1}{25} = \frac{25}{1}$$

$$= 25.$$

has the
(i) ^c'

Gamma Distribution (or) Erlang distribution:

A continuous random variable X is said to follow a Gamma distribution with parameter λ. If its probability distribution function is given by

$$f(x) = \frac{e^{-x} x^{\lambda-1}}{\Gamma\lambda}, \quad \lambda > 0, \; 0 < x < \infty.$$

where $\Gamma\lambda = \int_0^\infty e^{-x} x^{\lambda-1} dx.$

1. Derive the M.G.F of Gamma distribution and hence find the Mean and variance of Gamma distribution.

$$M_x(t) = E(e^{tx}) = \int_{-\infty}^{\infty} e^{tx} f(x)\,dx$$

$$= \int_0^\infty e^{tx} \frac{e^{-x} x^{\lambda-1}}{\Gamma\lambda} dx$$

para

$$= \int_0^\infty \frac{e^{-x+tx}}{\Gamma\lambda} x^{\lambda-1} dx$$

$$= \int_0^\infty \frac{e^{-(1-t)x} x^{\lambda-1}}{\Gamma\lambda} dx$$

Put $(1-t)x = u$

when $x = 0, \; u = 0$

$x = \infty, \; u = \infty$

$(1-t)dx = du \Rightarrow dx = \frac{du}{1-t}$

$x = \frac{u}{1-t}$

$$= \frac{1}{\Gamma\lambda} \int_0^\infty e^{-u} \left(\frac{u}{1-t}\right)^{\lambda-1} \cdot \frac{du}{(1-t)}$$

$$= \frac{1}{\Gamma\lambda} \int_0^\infty e^{-u} \cdot \frac{u^{\lambda-1}}{(1-t)^{\lambda-1+1}} du$$

$$= \frac{1}{\Gamma\lambda} \frac{1}{(1-t)^\lambda} \int_0^\infty e^{-u} u^{\lambda-1} du$$

$$= \frac{1}{\Gamma\lambda} \cdot \frac{\Gamma\lambda}{(1-t)^\lambda}$$

$$M_x(t) = \frac{\Gamma\lambda}{\Gamma\lambda (1-t)^\lambda} = \frac{1}{(1-t)^\lambda}$$

Mean $E(x) = \frac{d}{dt} M_x(t) \Big|_{t=0}$

$Var(x) = E(x^2) - [E(x)]^2$

$E(x^2) = \frac{d^2}{dt^2} M_x(t) \Big|_{t=0}$

$$M_x(t) = \frac{1}{(1-t)^\lambda}$$

$(1-t)^{-\lambda}$

$-\lambda(1-t)^{-(\lambda+1)} \cdot (-1)$

$$\frac{d}{dt} M_x(t) = \frac{-\lambda(-1)}{(1-t)^{\lambda+1}} = \frac{\lambda}{(1-t)^{\lambda+1}}$$

$(1-t)^{-(\lambda+1)}$

$-(\lambda+1)(1-t)^{-(\lambda+2)}$

$$\frac{d^2}{dt^2} M_x(t) = \lambda \left(\frac{-(\lambda+1)(-1)}{(1-t)^{\lambda+2}}\right)$$

$$= \frac{\lambda(\lambda+1)}{(1-t)^{\lambda+2}}$$

$$E(x) = \frac{\lambda}{(1-0)^{\lambda+1}} = \frac{\lambda}{1^{\lambda+1}} = \lambda$$

$$E(x^2) = \frac{\lambda(\lambda+1)}{(1-0)^{\lambda+2}} = \lambda(\lambda+1)$$

$$var(x) = E(x^2) - [E(x)]^2$$
$$= \lambda(\lambda+1) - \lambda^2$$
Gre
$$= \lambda^2 + \lambda - \lambda^2 = \lambda.$$

$\frac{\Gamma\lambda}{(1-t)^{\lambda}}$

General Gamma Distribution (or) Erlang distribution (when 2 parameters are given).

A continuous random variable x is said to follow an Erlang distribution with parameter $\lambda > 0$ and $k > 0$. If its pdf is given by

$$f(x) = \frac{\lambda^k x^{k-1} e^{-\lambda x}}{\Gamma k}, \ x > 0.$$

$\lambda(1-t)^{-(\lambda+1)} (-1)$

1\. A life time of a particular machine is a continuous random variable with range $0 < x < \infty$ following parameter $\lambda = 1$ & $k = 2$. Find the probability of life time exceeds 2 hours.

$\lambda+1$

$(1-t)^{-(\lambda+1)}$

$(\lambda+1)(1-t)^{-(\lambda+2)} (-1)$

Given: $\lambda = 1$ & $k = 2$.

$$f(x) = \frac{\lambda^k x^{k-1} e^{-\lambda x}}{\Gamma k}, \ x \geq 0$$

$$f(x) = \frac{1^2 x^{2-1} e^{-x}}{\Gamma 2} = \frac{x e^{-x}}{\Gamma 2}$$

Let X be the life time of machine.

$P[X > 2 \text{ hours}] = P[X > 2]$

$$= \int_2^\infty f(x)\,dx \qquad \Gamma n = (n-1)!$$

$$= \int_2^\infty \frac{x e^{-x}}{\Gamma 2}\,dx \qquad \Gamma 2 = (2-1)! = 1! = 1$$

$$= \left[x\left(\frac{e^{-x}}{-1}\right) - 1\left(\frac{e^{-x}}{(-1)^2}\right)\right]_2^\infty$$

$$= 0 - \left(\frac{2e^{-2}}{-1} - \frac{e^{-2}}{1}\right)$$

$$= -\left[-2e^{-2} - e^{-2}\right]$$

$$= 3e^{-2} = 0.406.$$

2. The daily consumption of bread in the hostel is excess of 2000 loaves approximately Gamma Distribution with parameter $k = 2$, $\lambda = 1/1000$. The hostel has a daily stock of 3000 loaves. What is the probability that the stock is insufficient on a day.

Let Y be the daily stock

Let X be the daily consumption of bread.

Let $y = x + 2000$

$$P[Y > 3000] = P[X + 2000 > 3000]$$

$$= P[X > 1000] = \int_{1000}^{\infty} f(x)\,dx$$

$$f(x) = \frac{\lambda^k x^{k-1} e^{-\lambda x}}{\Gamma k}, \quad x \geq 0.$$

$$= \frac{\left(\frac{1}{1000}\right)^2 x^{2-1} e^{-x/1000}}{\Gamma 2}, \quad x \geq 0.$$

$$f(x) = \left(\frac{1}{1000}\right)^2 x\, e^{-x/1000}$$

$$P[X > 1000] = \int_{1000}^{\infty} \left(\frac{1}{1000}\right)^2 x\, e^{-\frac{x}{1000}}\,dx$$

$$= \left(\frac{1}{1000}\right)^2 \left[x \left(\frac{e^{-x/1000}}{-1/1000} \right) - 1\left(\frac{e^{-x/1000}}{(-1/1000)^2} \right) \right]_{1000}^{\infty}$$

$$= \left(\frac{1}{1000}\right)^2 \left[1000 \frac{e^{-\frac{1000}{1000}}}{-1/1000} - \frac{e^{-\frac{1000}{1000}}}{(1/1000)^2} \right]$$

$$= \frac{1}{(1000)^2} \left(0 - \left(-1000^2 e^{-1} - 1000^2 e^{-1}\right)\right)$$

$$= \frac{-1000^2}{1000^2} \left(e^{-1} + e^{-1}\right)$$

$$= 2e^{-1} = 0.7357$$

Weibull distribution:

The Random variable x is said to follow weibull distribution with two parameters $\beta > 0$, $\alpha > 0$.

$$f(x) = \alpha \beta x^{\beta - 1} e^{-\alpha x^{\beta}}, \quad x > 0.$$

Derive the r^{th} moment about origin of weibull distribution and hence find its Mean and variance.

$$\mu_r' = E[X^r] = \int_{-\infty}^{\infty} x^r f(x)\,dx$$

$$= \int_0^{\infty} x^r \alpha \beta x^{\beta-1} e^{-\alpha x^{\beta}}\,dx$$

put $\alpha x^{\beta} = t$

$$x^{\beta} = \frac{t}{\alpha}$$

$$x = \left(\frac{t}{\alpha}\right)^{1/\beta}$$

when $x = 0$, $t = 0$

$x = \infty$, $t = \infty$

$$\alpha \beta x^{\beta-1}\,dx = dt$$

$$= \int_0^{\infty} \left(\left(t/\alpha\right)^{1/\beta}\right)^r e^{-t}\,dt \qquad \Gamma n = \int_0^{\infty} e^{-x} x^{n-1}\,dx$$

$$= \int_0^{\infty} \frac{t^{r/\beta}}{\alpha^{r/\beta}} e^{-t}\,dt$$

)

two

$$E[x^r] = \frac{1}{\alpha^{r/\beta}} \Gamma\left(\frac{r}{\beta}+1\right)$$

Mean : E(x) put $r=1$ in $E[x^r]$

$$E[x] = \frac{1}{\alpha^{1/\beta}} \Gamma\left(\frac{1}{\beta}+1\right)$$

$$\text{var}(x) = E(x^2) - [E(x)]^2$$

$$\mu_2 = \mu_2' - (\mu_1')^2$$

cgis

find

put $r=2$ in $E(x^r)$

$$E(x^2) = \frac{1}{\alpha^{2/\beta}} \left(\Gamma\left(\frac{2}{\beta}+1\right)\right)$$

$$\text{var}(x) = \frac{1}{\alpha^{2/\beta}} \Gamma\left(\frac{2}{\beta}+1\right) - \left(\frac{1}{\alpha^{1/\beta}} \Gamma\left(\frac{1}{\beta}+1\right)\right)^2$$

$$= \frac{1}{\alpha^{2/\beta}} \left[\left(\Gamma\left(\frac{2}{\beta}+1\right) - \left(\Gamma\left(\frac{1}{\beta}+1\right)\right)^2\right]$$

1. Each of the 6 tubes of a radio set as a life length (in years) which follows a weibull distribution with parameters $\alpha = 25$, $\beta = 2$. If these tubes function independently of one another, what is the probability that no tube will have to replace during the first 2 months of operations.

$x^{n-1} dx$

Let X be the life time of ... 2.

$\alpha = 25, \quad \beta = 2$

$f(x) = \alpha \beta x^{\beta - 1} e^{-\alpha x^{\beta}}, \; x > 0$

$f(x) = (25)(2) x^{2-1} e^{-25x^2}, \; x > 0$

$f(x) = 50 x e^{-25x^2}, \; x > 0$

$P[X > 2 \text{ months}] = P[X > \frac{2}{12} \text{ years}]$

$P[X > 1/6] = \int_{1/6}^{\infty} f(x)\,dx$

$= \int_{1/6}^{\infty} 50 x e^{-25x^2} dx$

put $t = 25x^2$

when $x = 1/6$ $t = 25\left(\frac{1}{6}\right)^2 = \frac{25}{36}$

$x = \infty, \; t = \infty$

$dt = 25(2) x\,dx$

$= 50x\,dx$

$= \int_{25/6}^{\infty} e^{-t} dt = \left(\frac{e^{-t}}{-1}\right)_{25/36}^{\infty}$

$= \left(\frac{e^{-\infty} - e^{-25/36}}{-1}\right) = e^{-25/36}$

$= 0.4993.$

stulie

2. The life time of a component measured in hour follows weibull distribution with parameter $\alpha = 0.2$ & $\beta = 0.5$. Find the mean life time of component.

$\alpha = 0.2$ & $\beta = 0.5$

soln:

W.K.T Mean of weibull distribution

$$E[X] = \frac{1}{\alpha^{1/\beta}} \Gamma(1/\beta + 1)$$

$$= \frac{1}{(0.2)^{1/0.5}} \Gamma\left(1/0.5 + 1\right) = \frac{1}{(0.2)^{1/0.5}} \Gamma 3$$

$$= \frac{1}{0.04} 2! = 25 \times 2 = 50$$

Another method:

$$f(x) = \alpha\beta x^{\beta-1} e^{-\alpha x^{\beta}}, x > 0$$

$$= (0.2)(0.5) x^{0.5-1} e^{-0.2x^{0.5}}$$

$$f(x) = (0.1) x^{-0.5} e^{-0.2x^{0.5}}, x > 0$$

Mean: $E(x) = \int_0^{\infty} x f(x) dx$

$$= \int_0^{\infty} x \cdot (0.2)(0.5) x^{0.5-1} e^{-0.2x^{0.5}} dx$$

put $t = 0.2 x^{0.5}$

$dt = 0.2 \; 0.5 \; x^{0.5-1} dx$

$t/0.2 = x^{0.5}$

$$\frac{t^2}{(0.2)^2} = x$$

$$= \int_0^\infty \frac{t^2}{0.04} e^{-t} dt$$

$$= \frac{1}{0.04} \int_0^\infty t^2 e^{-t} dt$$

$$= \frac{1}{0.04} \Gamma 3$$

$$= \frac{2!}{0.04} = 50$$

Unit II : TWO DIMENSIONAL RANDOM VARIABLES.

Definition :

Let S be the sample space associated with a random experiment E. Let $X = x(S)$ and $y = Y(S)$ be two functions each assigning a real number to each outcomes of $s \in S$. Then (X, Y) is called a two dimensional random variable.

Example :

Tossing a coin & Rolling a dice at a time.

Let X be tossing a coin

Let Y be Rolling a die.

X/Y	1	2	3	4	5	6
H	1/12	1/12	1/12	1/12	1/12	1/12
T	1/12	1/12	1/12	1/12	1/12	1/12

Joint probability function : $(P(x, y))$

If (X, Y) is two dimensional discrete Random variable such that

$P[X = x_i, Y = y_j] = P_{ij}$ then P_{ij} is called Probability mass function of x, y Provided the following conditions are satisfied.

(i) $P_{ij} \geq 0 \; \forall \, i, j$ (ii) $\sum_j \sum_i P_{ij} = 1$

Related formulas :

1. Marginal probability function of x

$$P[X = x_i] = \sum_j P(x_i, y_j)$$

2. Marginal Probability function of y

$$P[Y = y_j] = \sum_i P(x_i, y_j)$$

3. conditional probability of x given y is

$$P[x_i / y_j] = \frac{P[x_i, y_j]}{P[y_j]}$$

4. conditional Probability of y given x is

$$P[y_j / x_i] = \frac{P[x_i, y_j]}{P[x_i]}$$

5. Independent

$$P[x_i, y_j] = P[x_i] \times P[y_j]$$

6. Mean $E[X] = \sum x_i P(x_i)$

7. Mean $E[Y] = \sum y_j P(y_j)$

8. $E(xy) = \sum_i \sum_j x_i y_j \, P(x_i, y_j)$

Problems :

1. For the Bivariant probability distribution of x, y given below. Find $P[x \le 1]$, $P[y \le 3]$, $P[x \le 1, y \le 3]$, $P[x \le 1 / y \le 3]$, $P[y \le 3 / x \le 1]$, $P[x + y \le 4]$.

x|y	1	2	3	4	5	6
0	0	0	1/32	2/32	2/32	3/32
1	1/16	1/16	1/8	1/8	1/8	1/8
2	1/32	1/32	1/64	1/64	0	2/64

and also check whether it is independent or not.

Soln :

(i) $P[x \le 1] = P[x = 0] + P[x = 1]$

Marginal of x :

when $x = 0$

$P[x = 0] = \sum_j P(x = 0, y_j)$

$= P(0,1) + P(0,2) + P(0,3) + P(0,4) + P(0,5) + P(0,6)$

$= 0 + 0 + 1/32 + 2/32 + 3/32 + 2/32 = 8/32$

Marginal of $x=1$:

$P(x=1) = \sum_j P(x=1, y_j)$

$= P(1,1) + P(1,2) + P(1,3) + P(1,4) + P(1,5) + P(1,6)$

$= 1/16 + 1/16 + 1/8 + 1/8 + 1/8 + 1/8$

$= \frac{10}{16}$

$P[x=2] = \sum_j P[x=2, y_j]$

$= P(2,1) + P(2,2) + \cdots + P(2,6)$

$= 1/32 + 1/32 + 1/64 + 1/64 + 0 + 2/64$

$= 8/64.$

Marginal probability of Y:

When $y=1$

$P[y=1] = \sum_i P(x_i, y=1)$

$= P(0,1) + P(1,1) + P(2,1)$

$= 0 + 1/16 + 1/32 = 3/32$

$P[y=2] = \sum_i P[x_i, y=2)$

$= P(0,2) + P(1,2) + P(2,2)$

$= 0 + 1/16 + 1/32 = 3/32$

$P[y=3] = \sum_i P(x_i, y=3)$

$= P(0,3) + P(1,3) + P(2,3)$

$= 1/32 + 1/8 + 1/64 = 11/64$

$P[Y=4] = \sum_i P(x_i, y=4)$

$= P(0,4) + P(1,4) + P(2,4)$

$= 2/32 + 1/8 + 1/64 = 13/64$

$P[y=5] = \sum_i P(x_i, y=5)$

$= P(0,5) + P(1,5) + P(2,5)$

$= 2/32 + 1/8 + 0 = 6/32$

$P[y=6] = \sum_i P(x_i, y=6)$

$= P(0,6) + P(1,6) + P(2,6)$

$= 3/32 + 1/8 + 2/64 = \frac{16}{64}$.

$P[X \leq 1] = P[X=0] + P[X=1]$

$= 8/32 + 10/16 = \frac{28}{32}$.

$P[y \leq 3] = P[y=1] + P[y=2] + P[y=3]$

$= 3/32 + 3/32 + 11/64 = 23/64$.

$P[x \leq 1, y \leq 3] = P[x=0, y=1] + P[x=0, y=2]$

$+ P[x=0, y=3] + P[x=1, y=1] +$

$P[x=1, y=2] + P[x=1, y=3]$

$= 0 + 0 + 1/32 + 1/16 + 1/16 + 1/8 = 9/32$

$P[x \leq 1 \mid y \leq 3] = \frac{P[x \leq 1, y \leq 3]}{P[y \leq 3]} = \frac{9/32}{23/64}$

$= \frac{9}{32} \times \frac{64}{23} = \frac{18}{23}$

$$P[y \le 3 / x \le 1] = \frac{P[y \le 3, x \le 1]}{P[x \le 1]}$$

$$= \frac{9/32}{28/32} = 9/28$$

$$P[x+y \le 4] = P[x+y=4] + P[x+y=3] + P[x+y=2] + P[x+y=1]$$

$$= P[x=0, y=4] + P[x=1, y=3] + P[x=2, y=2] + P[x=0, y=3] + P[x=1, y=2] + P[x=2, y=1] + P[x=0, y=2] + P[x=1, y=1] + P[x=0, y=1]$$

$$= 2/32 + 1/8 + 1/32 + 1/32 + 1/16 + 1/32 + 0 + 1/16 + 0$$

$$= \frac{2+4+1+1+2+1+2}{32} = \frac{13}{32}.$$

Independent:

$$P[x=0] \times P[y=1] = \frac{8}{32} \times \frac{3}{32} = \frac{24}{(32)^2} \ne 0$$

ie $P[x=0] \times P[y=1] \ne P[x=0, y=1]$

so it is not an independent.

2. The Join probability mass function x, y is given by

very (X) 16mrd

$P(x,y) = k(2x+3y)$, $x = 0, 1, 2$ $y = 1, 2, 3$.

Find all the marginals and the conditional probability also find the probability distribution $x+y$ and also check wheth

$=3]$

$2, y=2]$

$x=2, y=1]$

$0, y=1]$

$+ 1/16 + 0$

$=0$

$]$

y is

$,3$.

utional

Whether

it is independent or not :

y\x	0	1	2
1	3k	5k	7k
2	6k	8k	10k
3	9k	11k	13k

$P(x,y) = k(2x+3y)$

$P(0,1) = k(0+3) = 3k$

$P(1,1) = k(2+3) = 5k$

$P(0,2) = k(0+6) = 6k$.

w.k.T $\sum p(x,y) = 1$

$15k + 24k + 33k = 1$

$72k = 1$

$k = 1/72$

Marginal of x.

when $x=0$

$P[x=0] = P(0,1) + P(0,2) + P(0,3)$

$= 3k + 6k + 9k = 18k$

when $x=1$

$P[x=1] = P(1,1) + P(1,2) + P(1,3)$

$= 5k + 8k + 11k$

$= 24k$

when $x=2$

$P[x=2] = P(2,1) + P(2,2) + P(2,3)$

$= 7k + 10k + 13k = 30k$

Marginal Probability of y.

when $y=1$

$$P[y=1] = P(0,1) + P(1,1) + P(2,1)$$
$$= 3k + 5k + 7k = 15k$$

$$P[y=2] = P(0,2) + P(1,2) + P(2,2)$$
$$= 6k + 8k + 10k = 24k$$

$$P[y=3] = P(0,3) + P(1,3) + P(2,3)$$
$$= 9k + 11k + 13k = 33k$$

Conditional probability of x given y:

$$P[x_i/y=1] = \frac{P[x_i, y=1]}{P[y=1]}$$

when $x=0$, $\frac{P(0,1)}{P[y=1]} = \frac{3k}{15k} = 1/5$

when $x=1$, $\frac{P(1,1)}{P[y=1]} = \frac{5k}{15k} = \frac{1}{3}$

when $x=2$, $\frac{P(2,1)}{P[y=1]} = \frac{7k}{15k} = \frac{7}{15}$

when $x=0$, $\frac{P[0,2]}{P[y=2]} = \frac{6k}{24k} = \frac{1}{4}$

when $x=1$, $\frac{P[1,2]}{P[y=2]} = \frac{8k}{24k} = \frac{1}{3}$

when $x=2$, $\frac{P[2,2]}{P[y=2]} = \frac{10k}{24k} = \frac{10}{24}$

when $x=0$, $\frac{P[0,3]}{P[y=3]} = \frac{9k}{33k} = \frac{9}{33}$

when $x=1$, $\frac{P[1,3]}{P[y=3]} = \frac{11k}{33k} = \frac{11}{33}$

when $x=2$, $\frac{P[2,3]}{P[y=3]} = \frac{13k}{33k} = \frac{13}{33}$

$\frac{11}{13}$
$\frac{24}{9}$
$\frac{9}{33}$

conditional Probability of Y given X.

$P\left[x_i / y=2\right] = \frac{P[x_i, y=2]}{P[y=2]}$

when $x=0$, $P(0,$

$P[x=0/y_j] = P[x=0, y_j]$

$P[y_j/x=0] = \frac{P[y_j, x=0]}{P[x=0]}$

when $y=1 \Rightarrow \frac{P(0,1)}{P[x=0]} = \frac{3k}{18k} = \frac{1}{6}$

when $y=2 \Rightarrow \frac{P(0,2)}{P[x=0]} = \frac{6k}{18k} = \frac{1}{3}$

when $y=3 \Rightarrow \frac{P(0,3)}{P[x=0]} = \frac{9k}{18k} = \frac{1}{2}$

when $y=1 \Rightarrow \frac{P(1,1)}{P[x=1]} = \frac{5k}{28k} = \frac{5}{28}$

when $y=2 \Rightarrow \frac{P(1,2)}{P[x=1]} = \frac{8k}{28k} = \frac{8}{28}$

H.W

when $y=3 \Rightarrow \dfrac{P(1,3)}{P[X=1]} = \dfrac{11k}{25k} = \dfrac{11}{25}$

when $y=1 \Rightarrow \dfrac{P(2,1)}{P[X=2]} = \dfrac{7k}{30k} = \dfrac{7}{30}$

when $y=2 \Rightarrow \dfrac{P(2,2)}{P[X=2]} = \dfrac{10k}{30k} = \dfrac{10}{30} = \dfrac{1}{3}$

when $y=3 \Rightarrow \dfrac{P(2,3)}{P[X=2]} = \dfrac{13k}{30k} = \dfrac{13}{30}$

Independent:

$P[X=0] * P[Y=1]$

$= \dfrac{18}{72} \times \dfrac{15}{72} = \dfrac{270}{5184} \neq \dfrac{3}{72}$.

So it is not an independent.

Probability distribution of $x+y$

$x+y$	1	2	3	4	5
$P(x+y)$	$P(0,1)$ $3k$	$P(0,2) + P(1,1)$ $6k+5k$ $= 11k$	$P(0,3) + P(1,2) + P(2,1)$ $= 9k + 8k + 7k$ $= 24k$	$P(1,3) + P(2,2)$ $= 11k + 10k$ $= 21k$	$P(3,2)$ $= 13k$

H.W The two dimensional R.V x, y has the joint density function $f(x, y) = \frac{x+2y}{27}$ $x = 0, 1, 2$, $y = 0, 1, 2$. Find the conditional distribution of y when $x = 2$. Also find conditional distribution of x when $y = 1$.

$= \frac{1}{3}$

problems :

1. Three balls are drawn at random without replacement from a box containing 2 white, 3 red, 4 black balls. If x denotes number of white balls drawn and y denotes the number of red balls drawn. Find the probability distribution of x, y.

Y | X	0	1	2
0	4/84	12/84	4/84
1	18/84	24/84	3/84
2	12/84	6/84	0
3	1/84	0	0

$P(3, 2) = 13K$

$$P[x=0, y=0] = \frac{4C_3}{9C_3} = \frac{4}{84}$$

$$P[x=0, y=1] = \frac{3C_1 \times 4C_2}{9C_3} = \frac{18}{84}$$

$$P[x=0, y=2] = \frac{3c_2 \times 4c_1}{9c_3} = \frac{12}{84}$$

$$P[x=1, y=0] = \frac{2c_1 \times 4c_2}{9c_3} = \frac{12}{84}$$

$$P[x=1, y=1] = \frac{2c_1 \times 3c_1 \times 4c_1}{9c_3} = \frac{24}{84}$$

$$P[x=1, y=2] = \frac{2c_1 \times 3c_2}{9c_3} = 6/84$$

$$P[x=1, y=3] = 0$$

$$P(x=2, y=0) = \frac{2c_2 \times 4c_1}{9c_3} = 4/84$$

$$P(x=2, y=1) = \frac{2c_2 \times 3c_1}{9c_3} = 3/84$$

$$P(x=2, y=2) = 0$$

$$P(x=2, y=3) = 0$$

$$P(x=0, y=3) = \frac{3c_3}{9c_3} = 1/84$$

Independent:

$$P[x=0] \times P[y=1]$$

$$= \frac{35}{84} \times \frac{45}{84} = \frac{1575}{(84)^2} = \frac{1575}{7056}$$

$P[X=0] = P(0,1) + P(0,1) + P(0,2) + P(0,3)$
$+ P(0,0)$
$= 4/84 + 18/84 + 12/84 + 1/84$
$= \frac{35}{84}$

$\frac{1}{4}$

$P[Y=1] = P(0,1) + P(1,1) + P(2,1)$
$= \frac{18}{84} + \frac{24}{84} + \frac{3}{84} = \frac{45}{84}$

Jain Probability density function $f_{xy}(x,y)$:-
If (x,y) is a two ~~con~~ dimensional continuous random variable such that

$$P\left\{x - \frac{dx}{2} \le x \le x + \frac{dx}{2},\ y - \frac{dy}{2} \le y \le y + \frac{dy}{2}\right\}$$

$= f_{xy}(x,y)\, dx\, dy$. Then $f(x,y)$ is called the jain p.d.f of (x,y). Provided $f(x,y)$ satisfies the following conditions.

(i) $f(x,y) \ge 0,\ \forall x,y$

(ii) $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\, dx\, dy = 1.$

Related Formulas :

1. To find constant $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = 1$.

2. Marginal density function of x.

$$f_x(x) = \int_{-\infty}^{\infty} f(x,y)\,dy.$$

3. Marginal density function of y.

$$f_y(y) = \int_{-\infty}^{\infty} f(x,y)\,dx.$$

4. conditional density function of y given x

$$f(y/x) = \frac{f(x,y)}{f(x)}$$

5. conditional density function of x given y

$$f(x/y) = \frac{f(x,y)}{f(y)}$$

6. Independent :

$$f_x(x)\,f_y(y) = f_{xy}(x,y)$$

7. $P(a \le x \le b,\ c \le y \le d) = \int_a^b \int_c^d f(x,y)\,dx\,dy$

(ii)

1. The joint probability density function of a two dimensional Random variable is given by $f(x,y) = xy^2 + \frac{x^2}{8}$, $0 \le x \le 2$, $0 \le y \le 1$. Compute (i) $P[x>1]$, $P[y<1/2]$, $P[x>1/y<1/2]$, $P[y<1/2 / x>1]$, $P[x<y]$ & $P[x+y \le 1]$

(i) $P[X>1] = \int_1^{\infty} f(x)\,dx$

Marginal density fun of x

$f(x) = \int_{-\infty}^{\infty} f(x,y)\,dy$

$f(x) = \int_0^1 \left(xy^2 + \frac{x^2}{8}\right) dy$

$= \left(\frac{xy^3}{3} + \frac{x^2 y}{8}\right)_0^1 = \frac{x}{3} + \frac{x^2}{8} - 0$

$f(x) = \frac{x}{3} + \frac{x^2}{8}$.

$P[X>1] = \int_1^2 \left(\frac{x}{3} + \frac{x^2}{8}\right) dx$

$= \left(\frac{x^2}{3 \times 2} + \frac{x^3}{8 \times 3}\right)_1^2$

$= \frac{2^2}{6} + \frac{2^3}{24} - \left(\frac{1}{6} + \frac{1}{24}\right)$

$= \frac{4}{6} + \frac{8}{24} - \frac{1}{6} - \frac{1}{24}$

$= \frac{3}{6} + \frac{7}{24} = \frac{12+7}{24} = \frac{19}{24} = 0.791$.

(ii) $P[Y<1/2] = \int_{-\infty}^{1/2} f(y)\,dy$.

Marginal density fun of y:

$f(y) = \int_{-\infty}^{\infty} f(x,y)\,dx$.

$f(y) = \int_0^2 \left(xy^2 + \frac{x^2}{8}\right) dx$

$= \left(\frac{x^2 y^2}{2} + \frac{x^3}{24}\right)_0^2 = \left(\frac{4y^2}{2} + \frac{8}{24} - 0\right)$

$f(y) = 2y^2 + 1/3$

inner limit curve
outer limit points

(iii)

$P[y < 1/2] = \int_0^{1/2} (2y^2 + 1/3)\,dy$

$= \left(\frac{2y^3}{3} + \frac{1}{3}y\right)_0^{1/2}$

$= \frac{2}{3}\left(\frac{1}{2}\right)^3 + \frac{1}{3}\left(\frac{1}{2}\right) - 0$

$= \frac{2}{24} + \frac{1}{6} = \frac{1}{12} + \frac{1}{6} = \frac{1+2}{12} = \frac{3}{12} = \frac{1}{4}$

(iii) $P[x \geq 1 / y < 1/2]$

$$\frac{P[x \geq 1, y < 1/2]}{P[y < 1/2]}$$

(iv)

$P[x \geq 1, y < 1/2] = \int_1^2 \int_0^{1/2} f(x,y)\,dy\,dx$

$= \int_1^2 \int_0^{1/2} \left(xy^2 + \frac{x^2}{8}\right) dy\,dx$

$= \int_1^2 \left(\frac{xy^3}{3} + \frac{x^2}{8}y\right)_0^{1/2} dx$

$= \int_1^2 \left(\frac{x\left(\frac{1}{2}\right)^3}{3} + \frac{x^2}{8}\left(\frac{1}{2}\right) - 0\right) dx$

$= \int_1^2 \left(\frac{x}{24} + \frac{x^2}{16}\right) dx = \left(\frac{x^2}{48} + \frac{x^2}{48}\right)_1^2$

$= \frac{2^2}{48} + \frac{2^3}{48} - \left(\frac{1}{48} + \frac{1}{48}\right)$

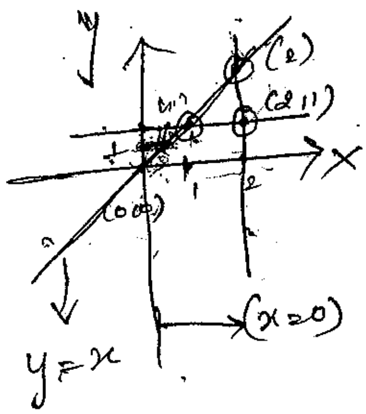
$= \frac{4+8-2}{48} = \frac{10}{48} = \frac{5}{24} = 0.208$

limit
true
limit

$$P[x>1 \mid y<1/2] = \frac{5/24}{1/4} = \frac{5}{24}\times\frac{4}{1} = \frac{5}{6}$$
$$= 0.833.$$

(iii) $P[x<y] = \int_0^1 \int_0^y f(x,y)\,dx\,dy$

$$= \int_0^1 \int_0^y \left(xy^2 + \frac{x^2}{8}\right) dx\,dy$$



$$= \int_0^1 \left(\frac{x^2y^2}{2} + \frac{x^3}{24}\right)_0^y dy.$$

$\frac{1}{4}$

$$= \int_0^1 \left(\frac{y^4}{2} + \frac{y^3}{24}\right) dy.$$

$$= \left[\frac{y^5}{10} + \frac{y^4}{4\times 24}\right]_0^1$$

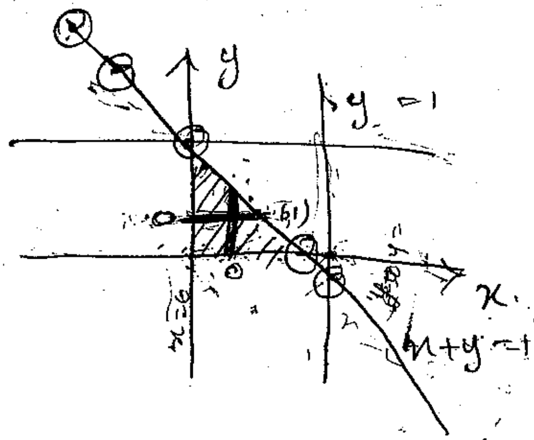
$$= \frac{1}{10} + \frac{1}{96} - 0 = \frac{96+10}{960} = \frac{106}{960}.$$

(iv) $P[x+y \le 1]$

$x+y = 1$

$y = 1-x$

x	-2	-1	0	1	2
$y=1-x$	3	2	1	0	-1



$$P[x+y\le 1] = \int_0^1 \int_0^{1-x} f(x,y)\,dy\,dx$$

$(a-b)^3 = a^3 - b^3 - 3a^2b + 3ab^2$

$$= \int_0^1 \int_0^{1-x} \left(xy^2 + \frac{x^2}{8}\right) dy\,dx$$

$$= \int_0^1 \left(\frac{xy^3}{3} + \frac{x^2}{8}(y)\right]_0^{1-x} dx$$

$$= \int_0^1 \left(\frac{x}{3}(1-x)^3 + \frac{x^2}{8}(1-x) - 0 \right) dx$$

$$= \int_0^1 \frac{x}{3}(1-3x+3x^2-x^3) + \frac{1}{8}(x^2-x^3) dx.$$

$$= \int_0^1 \frac{1}{3}(x-3x^2+3x^3-x^4) + \frac{1}{8}(x^2-x^3) dx$$

$$= \frac{1}{3}\left[\frac{x^2}{2} - \frac{3x^3}{3} + \frac{3x^4}{4} - \frac{x^5}{5}\right]_0^1 + \frac{1}{8}\left[\frac{x^3}{3} - \frac{x^4}{4}\right]_0^1$$

$$= \frac{1}{3}\left[\frac{1}{2} - 1 + \frac{3}{4} - \frac{1}{5}\right] + \frac{1}{8}\left[\frac{1}{3} - \frac{1}{4}\right]$$

$$= \frac{1}{3}\left[\frac{10-20+15-4}{20}\right] + \frac{1}{8}\left[\frac{4-3}{12}\right]$$

$$= \frac{1}{3}\left[\frac{1}{20}\right] + \frac{1}{8}\left[\frac{1}{12}\right] = \frac{1}{60} + \frac{1}{96}$$

2. The joint p.d.f of a continuous Random variable (x,y) is $f(x,y) = kxye^{-(x^2+y^2)}$ $x>0, y>0$. Find the value of k, check whether i.e x and y are independent.

(ii)

Soln

w.k.T $\iint f(x,y)\,dx\,dy = 1$

$$\int_0^\infty \int_0^\infty kxy e^{-(x^2+y^2)} dx\, dy = 1$$

$$k \int_0^\infty \int_0^\infty xy e^{-x^2} e^{-y^2} dx\, dy = 1$$

$$k \int_0^\infty x e^{-x^2} dx \int_0^\infty y e^{-y^2} dy = 1$$

In .

dx

$-\frac{x^4}{4}\Big]_0^1$

ndom $\cdot(x^2+y^2)$

eek

put $t = x^2$ put $u = y^2$

when $x=0, t=0$ when $y=0, u=0$

$x=\infty, t=\infty$ when $y=\infty, u=\infty$

$dt = 2x\,dx$ $du = 2y\,dy$

$\frac{dt}{2} = x\,dx$ $\frac{du}{2} = y\,dy$.

$$k\int_0^\infty e^{-t}\frac{dt}{2}\int_0^\infty e^{-u}\frac{du}{2} = 1$$

$$\frac{k}{4}\left[\left(\frac{e^{-t}}{-1}\right)_0^\infty\left(\frac{e^{-u}}{-1}\right)_0^\infty\right] = 1$$

$$\frac{k}{4}\left[\left(\frac{e^{-\infty}-e^0}{-1}\right)\left(\frac{e^{-\infty}-e^0}{-1}\right)\right] = 1$$

$$\frac{k}{4} = 1 \Rightarrow k = 4$$

(iii) Independent $f(x,y) = f(x)\cdot f(y)$

Marginal density function of x $\Big\}$ $f(x) = \int_{-\infty}^{\infty} f(x,y)\,dy$

$$= \int_0^\infty 4xye^{-x^2}e^{-y^2}\,dy$$

$$= 4xe^{-x^2}\int_0^\infty ye^{-y^2}\,dy.$$

put $u = y^2$

$du = 2y\,dy$

$\frac{du}{2} = y\,dy$

$\frac{du}{2} = y\,dy$

$$f(x) = 4xe^{-x^2}\int_0^\infty e^{-u}\frac{du}{2}$$

$$= 2xe^{-x^2}\left[\left(\frac{e^{-u}}{-1}\right)_0^\infty\right] = 2xe^{-x^2}\left(\frac{e^{-\infty}-e^0}{-1}\right)$$

$$= 2xe^{-x^2}$$

iii) $f(y) = 2ye^{-y^2}, \; y > 0$

$$f(x)f(y) = 2xe^{-x^2} \cdot 2ye^{-y^2}$$
$$= 4xy e^{-x^2} e^{-y^2}$$
$$= 4xy \, e^{-x^2 - y^2}$$
$$= 4xy \, e^{-(x^2+y^2)}$$

$f(x)f(y) = f(x,y)$

So, it is Independent.

H.W

1. The Joint p.d.f of a cont R.V (X,Y) is $f(x,y) = ke^{-(x+y)}, \; 0 \le x,y \le \infty$. find (i) k (ii) Marginal distributions (iii) conditional densities (iv) Are x & y are independent.

1. The joint p.d.f of two dimensional R.V

$$f(x,y) = \begin{cases} \frac{8}{9}xy, & 1 < x < y < 2 \\ 0, & \text{o.w} \end{cases}$$

find all Marginal and conditional density functions.

Soln.

Marginal density function of x $f(x) = \int_{-\infty}^{\infty} f(x,y) \, dy$

$$f(x) = \int_x^2 \frac{8}{9} xy \, dy = \frac{8}{9} x \left(\frac{y^2}{2} \right)_x^2$$
$$= \frac{4}{9} x (2^2 - x^2)$$
$$= \frac{4}{9} x (4 - x^2) = \frac{4}{9} (4x - x^3)$$

Marginal density of y :

$$f(y) = \int_{-\infty}^{\infty} f(x,y)\,dx$$

$$= \int_{1}^{y} \frac{8}{9} xy\,dx = \frac{8}{9} y\left(\frac{x^2}{2}\right)_1^y$$

$$= \frac{4}{9} y\,(y^2 - 1^2) = \frac{4}{9}(y^3 - y)$$

conditional density of y given x

$$f(y/x) = \frac{f(x,y)}{f(x)} = \frac{\cancel{8}^{2}/\cancel{9}\,\cancel{x}\,y}{\frac{\cancel{4}}{\cancel{9}}\cancel{x}(4-x^2)}$$

$$= \frac{2y}{4-x^2}$$

conditional density of x given y

$$f(x/y) = \frac{f(x,y)}{f(y)} = \frac{\frac{\cancel{8}^{2}}{\cancel{9}} x\cancel{y}}{\frac{\cancel{4}}{\cancel{9}}\cancel{y}(y^2-1)} = \frac{2x}{y^2-1}.$$

H.w

1. (X) $f(x,y) = \begin{cases} 2 ; & 0 < x < y < 1 \\ 0 , & \text{o.w} \end{cases}$ find the Marginal and conditional densities.

1. The joint p.d.f of (x,y) is given by

$$f(x,y) = \begin{cases} 8xy & 0 \le x \le 1 ,\ 0 \le y \le x \\ 0 & \text{o.w} \end{cases}$$

find (i) $f_x(x)$ (ii) $f_y(y)$ (iii) $f(y/x)$ (iv) Independent.

Marginal density of x $f_x(x) = \int_{-\infty}^{\infty} f(x,y)\,dy$

$$f(x) = \int_{0}^{x} 8xy\,dy$$

$= \int_0^x 8xy\, dy$

$8\int_0^x xy\, dy$

$8x\left[\frac{y^2}{2}\right]_0^x$

$\cancel{8}^4 x\left[\frac{x^2}{\cancel{2}}\right] = 4x^3.$

Marginal density of y $f_y(y) = \int_{-\infty}^{\infty} f(x,y)\, dx$

$f_y(y) = \int_0^1 8xy\, dx$

$8y\int_0^1 x\, dx$

$8y\left[\frac{x^2}{2}\right]_0^1$

$\cancel{8}^4 y\left[\frac{1}{\cancel{2}}\right] = 4y.$

conditional density of y given x

$f(y/x) = \frac{f(x,y)}{f(x)}$

$= \frac{8\cancel{x}y}{4x^{\cancel{3}}} = \frac{\cancel{8}y}{\cancel{4}x^2} = \frac{2y}{x^2}.$

Independent $f(x) \times f(y) = f(x,y)$

$4x^3 \times 4y = 8xy$

$16x^3y \neq 8xy$

so it is not an Indenendent.

y) dx

1. $f(x,y) = \begin{cases} x e^{-x(y+1)}, & 0 \le x, y \le \infty \\ 0 & 0.\omega \end{cases}$ verify whether it is independent or not where joint p.d.f is ~~f(x,y)~~.

Marginal density function of x

$$f_x(x) = \int_{-\infty}^{\infty} f(x,y)\,dy$$

$$f(x) = \int_0^{\infty} x e^{-x(y+1)}\,dy \qquad \cancel{x} e^{-x}\left[\frac{e^{-xy}}{-\cancel{x}}\right] \qquad e^{-x}[0+1] = e^{-x}$$

$$= \cancel{x}\left[\frac{e^{-x(y+1)}}{-\cancel{x}}\right]_0^{\infty}$$

$$= 0 - \frac{e^{-x(0+1)}}{-1} = e^{-x}$$

M.d.f of y

$$f(y) = \int_{-\infty}^{\infty} f(x,y)\,dx = \int_0^{\infty} x e^{-x(y+1)}\,dx$$

$$= x\left(\frac{e^{-x(y+1)}}{-(y+1)}\right) - 1\left(\frac{e^{-x(y+1)}}{(-(y+1))^2}\right]_0^{\infty}$$

$$= 0 - \left(0 - \frac{e^0}{(y+1)^2}\right) = \frac{1}{(y+1)^2}$$

Independent :

$$f(x)\cdot f(y) = e^{-x}\cdot\frac{1}{(y+1)^2} = \frac{e^{-x}}{(y+1)^2} \ne f(x,y)$$

∴ so it is not an Independent

)

ent.

1. The joint p.d.f of (x, y) is given by

$$f(x,y) = \begin{cases} \dfrac{6-x-y}{8}, & 0<x<2,\ 2<y<4 \\ 0 & \text{o.w} \end{cases}$$

find $f(y/x=2)$.

soln :

$f(y/x=2)$ = conditional density of y given x.

W.K.T $f(y/x) = \dfrac{f(x,y)}{f(x)}$

M.d.f of x $\quad f(x) = \int_{-\infty}^{\infty} f(x,y)\,dy$.

$$= \int_2^4 \left(\frac{6-x-y}{8}\right) dy$$

$$= \frac{1}{8}\left[\left(6y - xy - \frac{y^2}{2}\right)\right]_2^4$$

$$= \frac{1}{8}\left[\left(24 - 4x - \frac{16}{2}\right) - \left(12 - 2x - \frac{4}{2}\right)\right]$$

$$f(x) = 1/8\left[24 - 4x - 8 - 12 + 2x + 2)\right]$$

$$= 1/8\left[6 - 2x\right]$$

$$f(y/x) = \frac{1/8\,(6-x-y)}{1/8\,(6-2x)}$$

Put $x=2$

$$f(y/x=2) = \frac{6-2-y}{6-2(2)} = \frac{4-y}{2}$$

by $< y$

2. Check whether x and y are Independent

H.W $f(x,y) = \begin{cases} \frac{1}{4}(1+xy), & |x| \le 1, |y| \le 1 \\ 0, & o.w. \end{cases}$

1. Find M.d.f and conditional density function given that joint density fun

given x.

$$f(x,y) = \begin{cases} \frac{x(1+3y^2)}{4}, & 0<x<2,\ 0<y<1 \\ 0 & o.w \end{cases}$$

Find also $P(\frac{1}{4}<x<\frac{1}{2} / y=\frac{1}{3})$.

Marginal density fun of x: $f(x) = \int_{-\infty}^{\infty} f(x,y)\,dy$

$$f(x) = \int_0^1 \frac{x(1+3y^2)}{4}\,dy$$

$$= \frac{x}{4}\int_0^1 (1+3y^2)\,dy$$

$$= \frac{x}{4}\left[y + \frac{\cancel{3}y^3}{\cancel{3}}\right]_0^1$$

$$= \frac{x}{4}\left[(1+1) - 0\right]$$

$$= \frac{x}{4}(2-0) = \frac{2x}{4} = \frac{x}{2}.$$

$\frac{P(\frac{1}{4}<x<\frac{1}{2},\ y=\frac{1}{3})}{P(y=\frac{1}{3})}$

$\int_{1/4}^{1/2} \cancel{\frac{1}{4}(1+}\ dx$

$P($

$$f(y) = \int_0^2 \frac{x(1+3y^2)}{4}\,dx$$

$$\frac{1}{4}\left[\int_0^2 (x+3y^2x)\,dx\right]$$

$\frac{3y^2x^2}{2}$

$$\frac{1}{4}\left[\frac{x^2}{2} + \frac{3y^2x^2}{2}\right]_0^2$$

$$\frac{1}{4}\left[\frac{4}{2} + 6y^2\right] = \frac{1}{4}\left[2+6y^2\right] = \frac{2}{4}\left[1+3y^2\right]$$

$$= \frac{1+3y^2}{2}.$$

(6 - 4)

conditional density of y given x

$$f(y/x) = \frac{f(x,y)}{f(x)}$$

$$\frac{\cancel{x}(1+3y^2)}{\cancel{4}\,2} \times \frac{2}{\cancel{x}}$$

$$= \frac{\frac{x(1+3y^2)}{4}}{\frac{x}{2}} = \frac{1+3y^2}{2}$$

$$\frac{x(1+3y^2)}{4} \times \frac{2}{}$$

conditional density of x given y

$$f(x/y) = \frac{f(x,y)}{f(y)} = \frac{\frac{x(1+\cancel{3y^2})}{\cancel{4}\,2}}{\frac{1+3y^2}{\cancel{2}}} = \frac{x}{2}$$

$P[1/4 < x < 1/2 \mid y = 1/3]$

sub $y = 1/3$ in $f(x/y)$

$$f\left(\frac{x}{y=1/3}\right) = \frac{x}{2}$$

$$P[1/4 < x < 1/2 \mid y = 1/3] = \int_{1/4}^{1/2} x/2 \, dx$$

$$= 1/2 \left[\frac{x^2}{2}\right]_{1/4}^{1/2}$$

$$= 1/4 \left[\frac{\left(\frac{1}{2}\right)^2}{2} - \frac{\left(\frac{1}{4}\right)^2}{2}\right]$$

$\frac{3}{64}$

$$= \frac{1}{4}\left[\frac{1}{4} - \frac{1}{16}\right]$$

$$= \frac{1}{4}\left[\frac{4-1}{16}\right] = \frac{4-1}{16} = \frac{1}{4}\left[\frac{3}{16}\right] = \frac{3}{64}$$

Joint cumulative distribution function :

X, Y is two dimensional random variable is (discrete / continuous) $F(x, y) = P[X \leq x, Y \leq y]$ is called c.d.f of x, y.

In discrete case

$$F(x,y) = \sum_{y_j \leq y} \sum_{x_i \leq x} P_{ij}$$

In continuous case.

$$F(x,y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x,y)\, dy\, dx$$

$$= \int_{-\infty}^{y} \int_{-\infty}^{x} f(x,y)\, dy\, dx.$$

Properties of c.d.f :-

(i) $F(-\infty, y) = F(x, -\infty) = 0$

$F(\infty, \infty) = 1$

(ii) $P[a < x < b, Y \leq y] = F(b,y) - f(a,y)$

(iii) $P[X \leq x, c \leq y < d] = F(x,d) - F(x,c)$

(iv) $P(a < x < b, c < y < d) = F(b,d) - F(a,d) - F(b,c) + F(a,c)$

(v) At point of continuity of (x,y)

$$f(x,y) = \frac{\partial^2}{\partial x \partial y} F(x,y)$$

1. If the join distribution function of (x,y) is given by $F(x,y) = \begin{cases} (1-e^{-x})(1-e^{-y}), & x>0, y>0 \\ 0, & \text{otherwise} \end{cases}$

Find (i) Marginal density function of x & y

(ii) Are x & y are independent (iii) $P(1<x<3, 1<y<2)$

Soln:

Marginal density of x $f(x) = \int_{-\infty}^{\infty} f(x,y)\,dy$

W.K.T $f(x,y) = \dfrac{\partial^2}{\partial x \partial y} F(x,y)$

$= \dfrac{\partial^2}{\partial x \partial y} (1-e^{-x})(1-e^{-y})$

$= \dfrac{\partial}{\partial x} (1-e^{-x})\left(0-e^{-y}(-1)\right)$

$= e^{-y}(0-e^{-x})(-1)$

$= e^{-y}.e^{-x} = e^{-(x+y)}, \quad x>0, y>0.$

(i) Marginal density of x $f(x) = \int_0^{\infty} e^{-x} e^{-y}\,dy$

$= e^{-x}\left(\dfrac{e^{-y}}{-1}\right)_0^{\infty} = e^{-x}\left(\dfrac{e^{-\infty}-e^{0}}{-1}\right) = e^{-x}.$

Marginal density of y $f(y) = \int_0^{\infty} e^{-x}.e^{-y}\,dx$

$= e^{-y}\left(\dfrac{e^{-x}}{-1}\right)_0^{\infty} = e^{-y}\left(\dfrac{e^{-\infty}-e^{0}}{-1}\right) = e^{-y}$

(ii) Are x and y are independent

$f(x).\,f(y) = e^{-x}.\,e^{-y} = e^{-(x+y)}$

$= f(x,y)$

$\therefore$ so it is independent.

) is

$0, y>0$

$1<y<2)$

(iii) By using joint p.d.f

$P(1<x<3,\ 1<y<2) = \int_1^3\int_1^2 f(x,y)\,dy\,dx$

$= \int_1^3\int_1^2 e^{-x}\cdot e^{-y}\,dy\,dx$

$= \int_1^3 e^{-x}\left(\frac{e^{-y}}{-1}\right)_1^2 dx = \int_1^3 e^{-x}\left(\frac{e^{-2}-e^{-1}}{-1}\right) dx.$

$= \left(\frac{e^{-2}-e^{-1}}{-1}\right)\left(\frac{e^{-x}}{-1}\right)_1^3$

$= \left(\frac{e^{-2}-e^{-1}}{-1}\right)\left(\frac{e^{-3}-e^{-1}}{-1}\right)$

$= (e^{-2}-e^{-1})(e^{-3}-e^{-1})$

$= 0.0739$.

By using joint c.d.f

$F(x,y)$

$P(a<x<b,\ c<y<d) = F(b,d) - F(a,d) - F(b,c) + F(a,c)$

$-y\,dy$

$P(1<x<3,\ 1<y<2) = F(3,2) - F(1,2) - F(3,1) + F(1,1)$

$)\,dx$

$= (1-e^{-3})(1-e^{-2}) - (1-e^{-1})(1-e^{-2}) - (1-e^{-3})(1-e^{-1}) + (1-e^{-1})(1-e^{-1})$

$= (1-e^{-2})\left[1-e^{-3}-(1-e^{-1})\right] + (1-e^{-1})\left[1-e^{-1}-(1-e^{-3})\right]$

$= (1-e^{-2})(1-e^{-3}-1+e^{-1}) + (1-e^{-1})(1-e^{-1}-1+e^{-3})$

$= (1-e^{-2})(e^{-1}-e^{-3}) + (1-e^{-1})(e^{-3}-e^{-1})$

$= (1-e^{-2})(e^{-1}-e^{-3}) - (e^{-1}-e^{-3})(1-e^{-1})$

$= (e^{-1} - e^{-3}) (1 - e^{-2} - (1 - e^{-1}))$

$= (e^{-1} - e^{-3}) (1 - e^{-2} \cancel{+} 1 + e^{-1}) = (e^{-1} - e^{-3}) (e^{-1} - e^{-2})$

$= 0.0739.$

Covariance - Correlation :

(i) The correlation between two variables x and y is defined as

$$E(xy) = \sum_j \sum_i x_i y_j \, P(x_i, y_j) \text{ - } x \,\&\, y \text{ are discrete.}$$

$$E(xy) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} xy \, f(x,y) \, dx \, dy \text{ - } x \,\&\, y \text{ are continuous}$$

(ii) Two r.v are uncorrelated to each other with the correlation between x & y are equal to product of mean.

$$E(xy) = E(x) \cdot E(y)$$

(iii) Two r.v are orthogonal to each other the correlation between x and y is equal to zero.

$$E(xy) = 0$$

Covariance :

The covariance between two random variables x and y is defined as

Simplification

$$\text{cov}(x,y) = E\{(x-\bar{x})(y-\bar{y})\}$$

$$\text{cov}(x,y) = \sum_j \sum_i (x_i - \bar{x})(y_j - \bar{y})\, P(x_i, y_j)$$

$$\text{cov}(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x-\bar{x})(y-\bar{y})\, f(x,y)\, dx\, dy$$

$$\text{cov}(x,y) = E(xy) - E(x)\cdot E(y).$$

Proof :

$$\text{cov}(x,y) = E[(x-\bar{x})(y-\bar{y})]$$

$$= E[xy - x\bar{y} - \bar{x}y + \bar{x}\bar{y}]$$

$$= E[xy] - \bar{y}\,E(x) - \bar{x}\,E[y] + \bar{x}\,\bar{y}$$

$$= E(xy) - \bar{y}\bar{x} - \bar{x}\bar{y} + \bar{x}\bar{y}$$

$$\text{cov}(x,y) = E(xy) - \bar{y}\bar{x}$$

$$\text{cov}(x,y) = E(xy) - E(x)\,E(y)$$

Co-relation coefficient :

It is the measurement of Relationship.

$-1 \leq r \leq 1$.

$r = -1 \Rightarrow$ perfectly negatively co-related.

$r = 0 \Rightarrow$ no correlation

$r = 1 \Rightarrow$ Perfectly positively co-related.

It is denoted by r

$$r(x,y) = \frac{\text{cov}(x,y)}{\sigma_x \cdot \sigma_y} = \frac{E(xy) - E(x) \cdot E(y)}{\sigma_x \cdot \sigma_y}$$

$$\sigma_x = \sqrt{E(x^2) - [E(x)]^2}$$

$$\sigma_y = \sqrt{E(y^2) - [E(y)]^2}.$$

$E(x), E(y)$ = Mean of x & y

properties :-

1. $|r| \le 1 \quad (-1 \le r \le 1)$

   $r = -1 \Rightarrow$ perfectly negatively corelated

   $r = 0 \Rightarrow$ no correlation

   $r = 1 \Rightarrow$ perfectly positively corelated.

2. corelation coefficient is independent of change of origin and scale.

   if $u = \frac{x-a}{h}$, $v = \frac{y-b}{k}$ then $r_{xy} = r_{uv}$

Note:

1. $\text{cov}(ax, by) = ab\, \text{cov}(x,y)$

2. $\text{cov}(a+x, b+y) = \text{cov}(x,y)$

3. When σ_x & σ_y are known

$$r = \frac{\sigma_x^2 + \sigma_y^2 - \sigma_{(x-y)}^2}{2\sigma_x \sigma_y}$$

E(Y)

cov (ax, by) :

$$\operatorname{cov}(x, y) = E(xy) - E(x)\cdot E(y)$$

$$\operatorname{cov}(ax, by) = E(ax\,by) - E(ax)\cdot E(by)$$

$$= E(abxy) - aE(x)\cdot bE(y)$$

$$= ab\,E(xy) - ab\,E(x)\cdot E(y)$$

$$= ab\left[E(xy) - E(x)\cdot E(y)\right]$$

$$= ab\,\operatorname{cov}(x, y)$$

hence proved.

where x & y are discrete :

d

$$r(x, y) = \rho_{xy} = \frac{E(xy) - E(x)\cdot E(y)}{\sigma_x \cdot \sigma_y}$$

$$E(x) = \frac{\Sigma x}{n} \qquad E(y) = \frac{\Sigma y}{n}$$

ted .

of

$$E(xy) = \frac{\Sigma xy}{n} \qquad \sigma_x = \sqrt{E(x^2) - [E(x)]^2}$$

$$E(x^2) = \frac{\Sigma x^2}{n} \qquad \sigma_y = \sqrt{E(y^2) - [E(y)]^2}$$

: r_{uv}

$$E(y^2) = \frac{\Sigma y^2}{n}$$

1. calculate the correlation coefficient for the following (in inches) of father x and y and their sons y.

x	65	66	67	~~67~~ 68	69	70	72	
y	67	68	65	68	72	72	69	71

x	y	x^2	y^2	xy
65	67	4225	4489	4355
66	68	4356	4624	4488
67	65	4489	4225	4355
67	68	4489	4624	4556
68	72	4624	5184	4896
69	72	4761	5184	4968
70	69	4900	4761	4830
72	71	5184	5041	5112

$$E(x) = \frac{\Sigma x}{n} = \frac{544}{8} = 68$$

$$\frac{\Sigma y}{n} = E(y) = \frac{552}{8} = 69$$

$$E(x^2) = \frac{\Sigma x^2}{n} = \frac{37028}{8} = 4628.5$$

$$E(y^2) = \frac{\Sigma y^2}{n} = \frac{38132}{8} = 4766.5$$

$$E(xy) = \frac{\Sigma xy}{n} = \frac{37560}{8} = 4695$$

$$\sigma_x = \sqrt{E(x^2) - E(x)^2}$$

$$\sigma_x = \sqrt{4628.5 - (68)^2}$$

$$= \sqrt{4628.5 - 4624} = \sqrt{4.5} = 2.121$$

$$\sigma_y = \sqrt{4766.5 - (69)^2}$$

$$= \sqrt{4766.5 - 4761} = \sqrt{5.5} = 2.345$$

$= r(x,y) = \rho_{xy} = \dfrac{E(XY) - E(X)\cdot E(Y)}{\sigma_x \cdot \sigma_y}$

$= \dfrac{4695 - 68 \times 69}{(2.121)(2.345)} = \dfrac{4695 - 4692}{4.9608} = \cancel{\dfrac{273}{4.9608}}$

$\cancel{= 55.031}$. $= \dfrac{3}{4.973} = 0.603$.

2. Find the corelation coeff b/w x & y from

x	10	14	18	22	26	30
y	18	12	24	06	30	36

x	y	xy	x^2	y^2
10	18	180	100	324
14	12	168	196	144
18	24	432	324	576
22	06	132	484	36
26	30	780	676	900
30	36	1080	900	1296

$E(x) = \dfrac{\Sigma x}{n} = \dfrac{120}{6} = 20$

$E(y) = \dfrac{\Sigma y}{n} = \dfrac{126}{6} = 21$

$E(xy) = \dfrac{\Sigma xy}{n} = \dfrac{2772}{6} = 462$

$E(x^2) = \dfrac{\Sigma x^2}{n} = \dfrac{2680}{6} = 446.66$

2) $\cancel{\Sigma}E(x)^2$

2.121

345

$$E(y^2) = \frac{\Sigma y^2}{n} = \frac{3276}{6} = 546$$

8m

$$\sigma_x = \sqrt{E(x^2) - [E(x)]^2}$$
$$= \sqrt{446.66 - (20)^2}$$
$$= \sqrt{446.66 - 400} = \sqrt{46.66} = 6.830$$

$$\sigma_y = \sqrt{E(y^2) - [E(y)]^2}$$
$$= \sqrt{546 - (21)^2} = \sqrt{546 - 441} = \sqrt{105}$$
$$= 10.246$$

$$r(x,y) = \frac{462 - \cancel{446.66}\,(20)(21)}{(6.830)(10.246)}$$

$$= \frac{462 - 420}{69.980} = \frac{42}{69.980} = 0.600.$$

H.W

1. Find the correlation relation b/w IQ level of Tamilnadu & Bihar states:

x_1 Tamilnadu	6.7	6.9	7.9	2.3	8.6	1.9
Bihar	6.3	6.5	2.4	7.0	2.3	~~7~~.6

1. Two variables x & y are related as $y = 4x + 9$. Find the correlation coeff b/w x & y.

Soln: Correlation coefficient b/w x & y

$$r(x,y) = \frac{E(xy) - E(x) \cdot E(y)}{\sigma_x \cdot \sigma_y}$$

= 6.830

$$E(y) = E(4x+9) = 4E(x) + 9$$

$$\sigma_y = \sqrt{E(y^2) - [E(y)]^2}$$

$$E(y^2) = E[(4x+9)^2] = E[16x^2 + 81 + 72x]$$

$$= 16E(x^2) + 81 + 72E(x)$$

$\sqrt{105}$

10.246

$$\sigma_y = \sqrt{16E(x^2) + 81 + 72E(x) - (4E(x)+9)^2}$$

$$= \sqrt{16E(x^2) + 81 + 72E(x) - [16(E(x))^2 + 81 + 72E(x)]}$$

$$= \sqrt{16E(x^2) + 81 + 72E(x) - 16(E(x))^2 - 81 - 72E(x)}$$

level

$$\sigma_y = \sqrt{16[E(x^2) - (E(x))^2]} = 4\sqrt{\sigma_x^2} = 4\sigma_x$$

$$E(xy) = E(x(4x+9))$$

1.9

$$= E(4x^2 + 9x) = 4E(x^2) + 9E(x)$$

7.6

$$r = \frac{4E(x^2) + 9E(x) - E(x)(4(E(x)+9))}{\sigma_x \cdot 4\sigma_x}$$

$$= \frac{4E(x^2) + 9E(x) - 4(E(x))^2 - 9E(x)}{4\sigma_x^2}$$

$$= \frac{4E(x^2) - 4(E(x))^2}{4\sigma_x^2} = \frac{4\left(E(x^2) - (E(x))^2\right)}{4\sigma_x^2}$$

$\sqrt{\sigma x}$

$$= \frac{4\sigma_x^2}{4\sigma_x^2} = 1.$$

2. Two variables x & y have the joint density function $f(x,y) = \begin{cases} 2-x-y, & 0<x<1, \ 0<y<1 \\ 0, & \text{otherwise.} \end{cases}$

16m

S.t $r_{xy} = -1/11$.

Soln:

$$r(x,y) = \frac{E(xy) - E(x)\cdot E(y)}{\sigma_x \cdot \sigma_y}$$

1.5833
5.840
$\sqrt{4.257}$

Mean of x:

$$E(x) = \int_{-\infty}^{\infty} x f(x)\,dx$$

M.d.f of x: $f(x) = \int_{-\infty}^{\infty} f(x,y)\,dy$

$$f(x) = \int_0^1 f(x,y)\,dy$$

$$f(x) = \int_0^1 (2-x-y)\,dy$$

$$= \left(2y - xy - \frac{y^2}{2}\right)_0^1 = 2 - x - 1/2 = \frac{3-2x}{2}$$

$2 - \frac{2x-1}{2}$

$4 - \frac{2x-1}{2}$

$3 - \frac{2x}{2}$

$$E(x) = \int_0^1 x\left(\frac{3-2x}{2}\right)dx$$

$$= \frac{1}{2}\int_0^1 3x - \frac{2x^2}{2}\,dx$$

$$= \frac{1}{2}\left[\frac{3x^2}{2} - \frac{2x^3}{3}\right]_0^1$$

$$= \frac{1}{2}\left[\frac{3}{2} - \frac{2}{3} - 0\right]$$

$$= 1/2\left[\frac{9-4}{6}\right] = 5/12$$

$-(E(x))^2$

$\bar{x}^2$

density

x.

1.5833
5.840
$\sqrt{-4.257}$

$\frac{2x-1}{2}$

$-\frac{2x-1}{2}$

$3-\frac{2x}{2}$

$$\sigma_x = \sqrt{E(x^2)-(E(x))^2}$$

$$E(x^2) = \int_0^1 x^2 f(x)\,dx$$

$$= \int_0^1 x^2\left(\frac{3-2x}{2}\right)dx$$

$$= \frac{1}{2}\int_0^1 (3x^2-2x^3)\,dx$$

$$= \frac{1}{2}\left(\frac{3x^3}{3}-\frac{2x^4}{4}\right)_0^1$$

$$= \frac{1}{2}\left(1-\tfrac{1}{2}\right) = \tfrac{1}{2}\left(\tfrac{1}{2}\right) = \tfrac{1}{4}$$

$$\sigma_x = \sqrt{\frac{1}{4}-\left(\frac{5}{12}\right)^2} = \sqrt{0.0764} = 0.276$$

Mean of Y $E(Y) = \int_{-\infty}^{\infty} y\, f(y)\,dy.$

Marginal d.f of y $f(y) = \int_{-\infty}^{\infty} f(x,y)\,dx$

$$= \int_0^1 (2-x-y)\,dx = \left(2x-\frac{x^2}{2}-yx\right)_0^1$$

$$= 2-\tfrac{1}{2}-y$$

$$f(y) = \frac{3-2y}{2}$$

$$E(Y) = \int_0^1 y\, f(y)\,dy$$

$$= \int_0^1 y\left(\frac{3-2y}{2}\right)dy$$

$$= \frac{1}{2}\int_0^1 3y-2y^2\,dy$$

$$= \frac{1}{2}\left(\frac{3y^2}{2}-\frac{2y^3}{3}\right)_0^1$$

$$= \frac{1}{2}\left(\frac{3}{2}-\frac{2}{3}\right)$$

$= \frac{1}{2}\left(\frac{9-4}{6}\right) = 5/12$

$\frac{1}{2} - \frac{2}{6}$

$E(y^2) = \int_{-\infty}^{\infty} y^2 f(y)\, dy$

$= \int_0^1 y^2 \left(\frac{3-2y}{2}\right) dy$

$= 1/2 \int_0^1 3y^2 - 2y^3\, dy$

$= 1/2 \left(\frac{3y^3}{3} - \frac{2y^4}{4}\right)_0^1$

$= 1/2\left(1 - 1/2\right) = \frac{1}{4}$

$\sigma_y = \sqrt{\frac{1}{4} - \left(\frac{5}{12}\right)^2} = 0.276$

$E(xy) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\, f(x,y)\, dx\, dy$

$= \int_0^1 \int_0^1 xy\,(2-x-y)\, dx\, dy$

$= \int_0^1 \int_0^1 (2xy - x^2y - xy^2)\, dx\, dy$

$= \int_0^1 \left(2y\frac{x^2}{2} - \frac{yx^3}{3} - \frac{x^2y^2}{2}\right)_0^1 dy$

$= \int_0^1 \left(y - \frac{y}{3} - \frac{y^2}{2}\right) dy$

$= \left(\frac{y^2}{2} - \frac{y^3}{6} - \frac{y^3}{6}\right)_0^1 = 1/2 - 1/6 - 1/6 = 1/2 - \frac{2}{6}$

$= \frac{3-2}{6} = 1/6$

$r = \frac{1/6 - \frac{5}{12}\cdot\frac{5}{12}}{0.276 \times 0.276}$

$$= \frac{0.1666 - 0.1736}{0.0761} = -0.0919 = -\frac{1}{11}.$$

$\frac{2}{6}$

1. If X & Y be the random variables whose Join p.d.f is $\begin{cases} x+y, & 0<x<1, 0<y<1 \\ 0, & o.w. \end{cases}$ computing correlation coeff b/w x & y and covariance of (x,y).

16m

$$r(x,y) = \frac{E(XY) - E(X)\cdot E(Y)}{\sigma_x \cdot \sigma_y}$$

mean of X: $E(X) = \int_{-\infty}^{\infty} x\, f(x)\, dx$

$$f(x) = \int_0^1 f(x,y)\, dy$$

$$= \int_0^1 (x+y)\, dy$$

$$= \left[xy + \frac{y^2}{2}\right]_0^1 = \left[x + \frac{1}{2}\right]$$

$$E(X) = \int_0^1 x\left(x + \frac{1}{2}\right) dx$$

$$= \int_0^1 \left(x^2 + \frac{x}{2}\right) dx$$

$$= \left(\frac{x^3}{3} + \frac{x^2}{4}\right)_0^1 = \frac{1}{3} + \frac{1}{4} = \frac{4+3}{12} = \frac{7}{12}$$

$-\frac{2}{6}$,

$$\sigma_x = \sqrt{E(X^2) - (E(X))^2}$$

$$E(X^2) = \int_0^1 x^2\left(x + \frac{1}{2}\right) dx$$

$$= \int_0^1 \left(x^3 + \frac{x^2}{2}\right) dx = \left[\frac{x^4}{4} + \frac{x^3}{6}\right]_0^1$$

$$= \frac{1}{4} + \frac{1}{6} = \frac{3+2}{12} = 5/12$$

0.4166
0.3402

$$\sigma_x = \sqrt{\frac{5}{12} - \left(\frac{7}{12}\right)^2}$$

$$= \sqrt{0.0763} = 0.276.$$

$$E(y) = \int_{-\infty}^{\infty} y\, f(y)\, dy$$

$$f(y) = \int_0^1 (x+y)\, \cancel{dy}\, dx$$

$$= \left[\frac{x^2}{2} + yx\right]_0^1 = \left[\frac{1}{2} + y\right]$$

2 | 6, 4
3 | 3, 2
2 | 1, 2
1, 1

$$E(y) = \int_0^1 y\left(\frac{1}{2} + y\right) dy$$

$$= \int_0^1 \left(\frac{y}{2} + y^2\right) dy$$

$$= \left(\frac{y^2}{4} + \frac{y^3}{3}\right)_0^1 = \frac{1}{4_{\times 3}} + \frac{1}{3} = \frac{3+4}{12} = \frac{7}{12}.$$

H.W
1.

$$E(y^2) = \int_0^1 y^2\left(\frac{1}{2} + y\right) dy$$

$$= \int_0^1 \left(\frac{y^2}{2} + y^3\right) dy$$

$$= \left[\frac{y^3}{6} + \frac{y^4}{4}\right]_0^1 = \frac{1}{6} + \frac{1}{4} = \frac{2+3}{12} = \frac{5}{12}.$$

$$\sigma_y = \sqrt{\frac{5}{12} - \left(\frac{7}{12}\right)^2} = 0.276$$

$$r(x,y) = \frac{\left(\frac{1}{3}\right) - \left(\frac{7}{12}\right)\left(\frac{7}{12}\right)}{(0.276)(0.276)} = -0.0919$$

$$\begin{array}{r} 0.333 \\ -\ (0.340) \\ \hline 0.0761 \end{array}$$

66
02

$$E(xy) = \int_0^1\int_0^1 (xy)(x+y)\,dx\,dy$$

$$= \int_0^1\int_0^1 (x^2y + xy^2)\,dx\,dy$$

$$= \int_0^1 \left(\frac{x^3}{3}y + \frac{x^2}{2}y^2\right)_0^1 dy$$

$$= \int_0^1 \left(\frac{y}{3} + \frac{y^2}{2}\right) dy$$

$$= \left(\frac{y^2}{6} + \frac{y^3}{6}\right)_0^1$$

$$= \frac{1}{6} + \frac{1}{6} = \frac{2}{6} = \frac{1}{3}.$$

$\mathrm{COV}(x, y) = -0.007$

$\frac{7}{12}$.

H.W

1. $f(x,y) = \begin{cases} \frac{3}{2}(x^2+y^2), & 0 \le x \le 1,\ 0 \le y \le 1, \\ 0, & \text{o.w} \end{cases}$

Find the correlation coeff b/w x & y.

$- = \frac{5}{12}$.

1. Two independent variables x & y have the p.d.f $f(x) = \begin{cases} 4ax, & 0 < x < 1 \\ 0, & \text{o.w} \end{cases}$

$f(y) = \begin{cases} 4by, & 0 < y < 1 \\ 0, & \text{o.w} \end{cases}$ Find the correlation coeff b/w x & y and also covar(x, y)

333
.340)
0761

$\text{cov}(x, y) = E(XY) - E(X) \cdot E(Y)$

$E(X) = \int x f(x)\, dx$

$= \int_0^1 x \, 4ax \, dx = \int_0^1 4ax^2 dx$

$= 4a\left(\frac{x^3}{3}\right)_0^1 = \frac{4a}{3}.$

$E(Y) = \int_0^1 y \cdot 4by \, dy = \int_0^1 4by^2 dy$

$= 4b\left(\frac{y^3}{3}\right)_0^1 = \frac{4b}{3}.$

$E(xy) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} xy \, f(x, y)\, dx\, dy$

Are x & y are Independent

$f(x, y) = f(x) f(y)$

$f(x, y) = 4ax \cdot 4by, \quad 0 < x < 1, \; 0 < y < 1$

$= 16abxy$

$E(xy) = \int_0^1 \int_0^1 xy \cdot 16abxy \, dx\, dy$

$= 16ab \int_0^1 \int_0^1 x^2 y^2 \, dx\, dy$

$= 16ab \int_0^1 \left(\frac{x^3}{3}\right)_0^1 y^2 dy = 16ab \int_0^1 \frac{y^2}{3} dy$

$= \frac{16ab}{3}\left[\frac{y^3}{3}\right]_0^1 = \frac{16ab}{9}.$

$$cov(x,y) = \frac{16ab}{9} - \frac{4a}{3} \cdot \frac{4b}{3}$$

$$= \frac{16ab}{9} - \frac{16ab}{9} = 0.$$

correlation coeff

$$r(x,y) = \frac{cov(x,y)}{\sigma_x \cdot \sigma_y} = \frac{0}{\sigma_x \cdot \sigma_y} = 0.$$

2. Let x, y, z be uncorrelated R.V with 0 means and S.D 5, 12, 9 respectively. If ~~xxx~~ $U = x+y$ & $V = y+z$. Find the correlation coefficient between U & V. (r_{uv})

soln:

C.C b/w U & V

$$r(u,v) = \frac{E(UV) - E(U) \cdot E(V)}{\sigma_U \cdot \sigma_V}$$

$0<y<1$

where x & y are uncorrelated with zero means

$$E(XY) = E(X) \cdot E(Y) = 0 \cdot 0 = 0$$

$$E(YZ) = E(Y) \cdot E(Z) = 0$$

$$E(ZX) = E(Z) \cdot E(X) = 0.$$

$$E(U) = E(X+Y) = E(X) + E(Y) = 0$$

$$E(V) = E(y+z) = E(y) + E(z) = 0.$$

$$E(UV) = E((x+y)(y+z))$$

$$= E(xy + xz + y^2 + yz)$$

$= E(xy) + E(xz) + E(y^2) + E(yz)$

$E(UV) = E(y^2)$

$\sigma_y = 12$

$\sqrt{E(y^2) - (E(y))^2} = 12.$

$E(y^2) = 12^2$

$E(y^2) = 144.$

$\sigma_U = \sqrt{E(u^2) - (E(u))^2}.$

$E(u^2) = E((x+y)^2) = E(x^2 + y^2 + 2xy)$

$= E(x^2) + 2E(xy) + E(y^2)$

$\sigma_V = \sqrt{E(v^2) - (E(v))^2}$

$= \sqrt{E(y+z)^2}$

$E((y+z)^2) = E(y^2 + 2yz + z^2)$

$= E(y^2) + 2E(yz) + E(z^2)$

$\sigma_x = 5$ $\qquad$ $\sigma_z = 9$

$\sqrt{E(x^2) - (E(x))^2} = 5$ $\qquad$ $\sqrt{E(z^2) - (E(z))^2} = 9$

$E(x^2) = 25$ $\qquad$ $E(z^2) = 81.$

$E(UV) = E(y^2) = 144$

$\sigma_U = \sqrt{E(x^2) + E(y^2)} = \sqrt{25+144} = \sqrt{169} = 13$

$\sigma_V = \sqrt{E(y^2) + E(z^2)} = \sqrt{144+81} = \sqrt{225} = 15$

$r(u,v) = \dfrac{144 - 0\times 0}{13 \times 15} = 0.738.$

2. The following general table gives the join probability distribution of two r.v x & y. find $E(x)$, $E(y)$ & $E(x,y)$ and also find the corelation coeff.

$y \backslash x$	-1	1	
0	$1/8$	$3/8$	$4/8 = 1/2$
1	$2/8$	$2/8$	$4/8 = 1/2$
	$3/8$	$5/8$	$1/2 \Rightarrow 1$

Soln :

Corelation coefficient

$$r(x,y) = \frac{E(xy) - E(x)\cdot E(y)}{\sigma_x \cdot \sigma_y}$$

$E(x) = \sum x\, p(x)$

$P(X)$ = Marginal prob function of x

$P(x=-1) = \sum P(-1, y_j) = P(-1,0) + P(-1,1)$
$= 1/8 + 2/8 = 3/8$

$P(x=1) = \sum P(1, y_j) = P(1,0) + P(1,1)$
$= 3/8 + 2/8 = 5/8$

$P(y=0) = \sum P(x, y=0) = P(-1,0) + P(1,0)$
$= 1/8 + 3/8 = 4/8$

$P(y=1) = \sum P(x, y=1) = P(-1,1) + P(1,1)$
$= 2/8 + 2/8 = 4/8$

Mean $E(x) = \sum x\, P(x) = (-1)\, P(x=-1) + 1\, P(x=1)$
$= (-1)\left(\frac{3}{8}\right) + 1 \times \frac{5}{8} = \frac{2}{8}$

9

$\bar{?} = 13$

$\bar{?} = 15$

$$\sigma_x = \sqrt{E(x^2) - (E(x))^2}$$

$$E(x^2) = \sum x^2 p(x)$$

$$= (-1)^2 \times \frac{3}{8} + 1^2 \times \frac{5}{8} = \frac{3}{8} + \frac{5}{8} = 1$$

$$\sigma_x = \sqrt{1 - (1/4)^2} = 0.968$$

$$E(y) = \sum y\, P(y) = 0\, P(y=0) + 1\, P(y=1)$$

$$= 0 + 1 \times 4/8 = 1/2$$

$$E(y^2) = \sum y^2 p(y) = 0^2 P(0) + 1^2 P(y=1) = \frac{4}{8} = 1/2$$

$$\sigma_y = \sqrt{E(y^2) - (E(y))^2} = \sqrt{1/2 - (1/2)^2} = 0.5$$

$$E(xy) = \sum xy\, P(x,y) = -1 \times 0\, P(-1,0) +$$

$$(-1 \times 1)\, P(-1,1) + 1 \times 0\, P(1,0) + 1 \times 1\, P(1,1)$$

$$= -1 \times \frac{2}{8} + \frac{2}{8} = 0.$$

$$r(x,y) = \frac{0 - 1/4 \times 1/2}{0.5 \times 0.968} = -0.258$$

Regression lines:

Regression is a mathematical measure of average relationship between two or more variables in terms of original limits of the data.

The line of regression of y on x is given by

$\frac{5}{8} = 1$

$$y - \bar{y} = r \frac{\sigma_y}{\sigma_x} (x - \bar{x})$$

(or) $y - \bar{y} = b_{yx} (x - \bar{x})$

The line of regression of x on y.

$$x - \bar{x} = r \frac{\sigma_x}{\sigma_y} (y - \bar{y})$$

(or)

$$x - \bar{x} = b_{xy} (y - \bar{y})$$

$\frac{4}{8} = \frac{1}{2}$

here $\bar{x}, \bar{y}$ are mean of x & y

σ_x, σ_y are S.D of x & y.

0.5

r - correlation coefficient

+

$(1,1)$

b_{xy} & b_{yx} are regression coefficient

Note:

Both the lines of regression passes through $(\bar{x}, \bar{y})$

Angle b/w the two lines of regression

$$\tan\theta = \frac{1 - r^2}{r} \left(\frac{\sigma_x \sigma_y}{\sigma_x^2 + \sigma_y^2} \right)$$

line of more te of

1. From the following data. the two regression equation

Marks in Economics	25	28	35	32	31	36	29	38	34	32
Marks in statistics	43	46	49	41	36	32	31	30	33	39

(ii) Most likely marks in statistics (ii) when marks in economics are 30 (iii) c. coeff b/w marks statistics and Economics. (iv) the two regression equations (v)

x	y	xy	x^2	y^2
25	43	1075	625	1849
28	46	1288	784	2116
35	49	1715	1225	2401
32	41	1312	1024	1681
31	36	1116	961	1296
36	32	1152	1296	1024
29	31	899	841	961
38	30	1140	1444	900
34	33	1122	1156	1089
32	39	1248	1024	1521
320	380	12067	10380	14838

Regression lines of y on x.

$$y - \bar{y} = r \frac{\sigma_y}{\sigma_x} (x - \bar{x})$$

$$r = \frac{E(xy) - E(x)E(y)}{\sigma_x \sigma_y}$$

when coeff b/w

the

$$E(x) = \frac{\Sigma x}{n} = \frac{320}{10} = 32$$

$$E(y) = \frac{\Sigma y}{n} = \frac{380}{10} = 38$$

$$E(xy) = \frac{\Sigma xy}{n} = \frac{12067}{10} = 1206.7$$

$$E(x^2) = \frac{\Sigma x^2}{n} = \frac{10380}{10} = 1038$$

$$E(y^2) = \frac{\Sigma y^2}{n} = \frac{14838}{10} = 1483.8$$

$$\sigma_x = \sqrt{E(x^2) - (E(x))^2}$$

$$= \sqrt{1038 - (32)^2} = 3.7415$$

$$\sigma_y = \sqrt{E(y^2) - (E(y))^2}$$

$$= \sqrt{1483.8 - (38)^2} = 6.308$$

$$r = \frac{1206.7 - 32 \times 38}{3.7415 \times 6.308}$$

$$r = -0.394$$

Regression lines of y on x.

$$y - 38 = -0.394\left(\frac{6.308}{3.7415}\right)(x-32)$$

$$y - 38 = -0.6643\,(x-32)$$

$$y - 38 = -0.6643x + 21.25$$

$$y = -0.6643x + 21.25 + 38$$

$$y = -0.6643x + 59.25$$

Regression lines of x on y

$$x - \bar{x} = r \frac{\sigma_x}{\sigma_y} (y - \bar{y})$$

$$x - 32 = -0.394 \left(\frac{3.7415}{6.308}\right) (y - 38)$$

$$x - 32 = -0.2336\, y + 8.82$$

$$x = -0.2336\, y + 40.88.$$

(iii) The most likely marks in Statistics (y) are 30.
Where Marks in Economics (x)

Regression line of y put $x = 30$

$$y = -0.6643x + 59.25.$$

$$y = -0.6643 \times 30 + 59.25$$

$$y = 39.32$$

(7.) Q. The heights of father and sons are
H.W given in cm

(X) Height of Father	150	152	155	157	160	161	166
(Y) Height of son	154	156	158	159	160	162	164

Find the two lines of Regression and calculate the excepted average height of son. when the height of the father is 154 cm.

3. The two lines of Regression are $8x - 10y + 66 = 0$ and $40x - 18y - 214 = 0$. The variance of x is 9. (i) Find Mean values of x & y. (ii) correlation coeff b/w x & y. $(r(x, y))$.

Soln:

W.K.T. The lines of Regression passes through $\bar{x}$ & $\bar{y}$.

Regression line of y on x

$$y - \bar{y} = b_{yx}(x - \bar{x})$$

$$y - \bar{y} = r\frac{\sigma_y}{\sigma_x}(x - \bar{x})$$

Regression line of x on y

$$x - \bar{x} = b_{xy}(y - \bar{y})$$

$$y - \bar{y} = r\frac{\sigma_x}{\sigma_y}(y - \bar{y})$$

$$b_{xy} \times b_{yx} = r\frac{\sigma_x}{\sigma_y} \times r\frac{\sigma_y}{\sigma_x}$$

$$b_{xy} \times b_{yx} = r^2$$

$$r = \pm\sqrt{b_{xy} \times b_{yx}}$$

$$8\bar{x} - 10\bar{y} = -66 \rightarrow (1)$$

$$40\bar{x} - 18\bar{y} = 214 \rightarrow (2)$$

$$\bar{x} = 13$$

$$\bar{y} = 17$$

Assume eqn (1) Regression line of y on x.

$$-10y = -8x - 66$$

$$y = \frac{-8}{-10}x - \frac{66}{-10}$$

$$y = \frac{8x}{10} + \frac{66}{10} \Rightarrow y = mx + c$$

$$m = b_{yx}$$

$$\therefore b_{yx} = \frac{8}{10}.$$

Let eqn (2) be Regression line of x on y

$$40x - 18y - 214 = 0$$

$$40x = 18y + 214$$

$$x = \frac{18}{40}y + \frac{214}{40}.$$

$$b_{xy} = \frac{18}{40}$$

W.K.T $r = +\sqrt{b_{yx} * b_{xy}}$

$$= \sqrt{\frac{18}{40} \times \frac{8}{10}}$$

$$r = 0.6$$

4. The following data are available $\bar{x} = 970$, $\bar{y} = 18$, $\sigma_x = 38$, $\sigma_y = 2$. correlation coefficient $r = 0.6$. Find the line of regression and obtain the value of x when $y = 20$.

The line of regression y on x

$$y - \bar{y} = r \frac{\sigma_y}{\sigma_x} (x - \bar{x})$$

$$y - 18 = 0.6\left(\frac{2}{38}\right)(x - 970)$$

$$y - 18 = 0.0315x - 30.63$$

$$y = 0.0315x - 12.63$$

The line of regression of x on y

$$x - \bar{x} = r \frac{\sigma_x}{\sigma_y} (y - \bar{y})$$

$$x - 970 = 0.6\left(\frac{38}{2}\right)(y - 18)$$

$$x - 970 = 11.4y - 205.2$$

$$x = 11.4y - 205.2 + 970$$

$$x = 11.4y + 764.8$$

$$20 = 0.0315x - 12.63$$

$$20 + 12.63 = 0.0315x$$

$$\frac{32.63}{0.0315} = x$$

$x = 1035.87$ I is not a Regression line

The line of regression of x on y

$x = 11.4 \times 20 + 764.8$

$= 228 + 764.8$

$x = 992.8$

Q. A statistical Investigation obtains the following recursion equations in survey

$x - y - 6 = 0$ & $0.64x - 4.08 = 0$.

Find (i) Mean of x & y (ii) coeff of correlation between x & y (iii) σ_y if

$\sigma_x = 4$

Soln:

W.K.T the regression lines passes through $(\bar{x}, \bar{y})$

$\bar{x} - \bar{y} = 6$

$0.64\bar{x} + 0\bar{y} = 4.08$

$\bar{x} = 6.375$, $\bar{y} = 0.375$

Let eqn (2) be x on y.

$0.64x = 4.08$

$x = \frac{4.08}{0.64}$

$x = 0y + 6.375$

y

$b_{xy} = 0$

Let eqn ① be y on x

$-y = -x + 6$

$y = x - 6$

$b_{yx} = 1$

s the

$r = \pm\sqrt{b_{xy} \times b_{yx}}$

survey

$= \pm\sqrt{0} = 0.$

$r = 0.$

(iii) Find σ_y :

if

w.k.T $b_{yx} = r\dfrac{\sigma_y}{\sigma_x}$

$1 = 0 \times \dfrac{\sigma_y}{4}$

through

$\sigma_y = 0.$