CS1504 Artificial Intelligence

A^* Search

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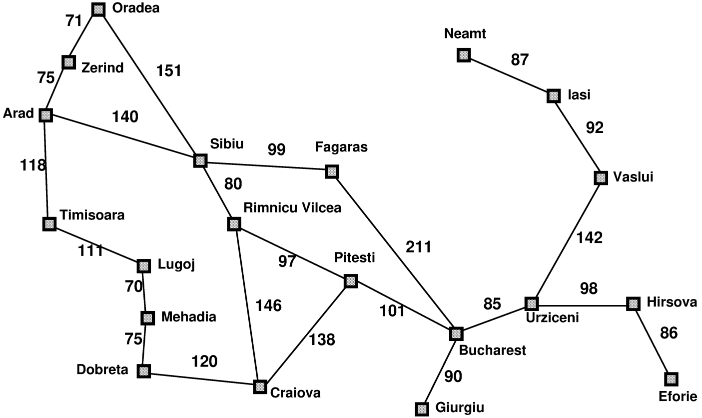
Evaluating a state

- Sometimes, it may be possible to design an evaluation function $f(s)$ that evaluates the "badness" (to be minimized) or "goodness" (to be maximized) of a state s
- In such cases, the most desirable state may be chosen from the working set
- Working set is maintained as a priority queue based on the evaluation function f
- Obviously, the quality of search depends on the evaluation function f

Heuristics

- Usually, such an evaluation function $f(s)$ is designed based on some heuristics $h(s)$ — estimation of cost of reaching a goal state from state s
- For example, can you think of a heuristics for the route finding problem in a map? — Straight line distance (SLD) from the current city to the destination city
- Heuristics should be an easy function to compute!
- $h(s^*)$ should be 0 for any goal state s^*

Example: Route finding problem



Straight−line distance to Bucharest

Arad	366
Bucharest	0
Craiova	160
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Find route from Arad to Bucharest

Example: Sliding puzzle

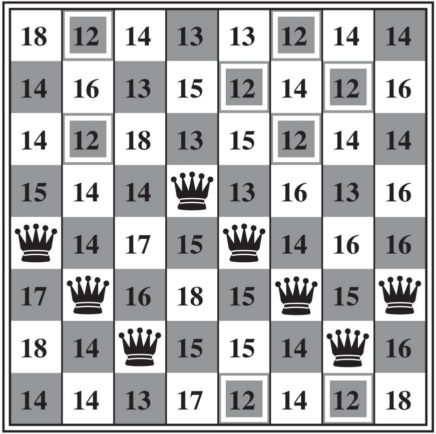
- Consider the sliding puzzle, such as

3	2	7
5	8	
1	4	6

- What may be a good heuristics for this state space?
- $h_1(s)$: Number of misplaced tiles — for the above state $h_1(s) = 7$
- $h_2(s)$: Sum of Manhattan distances of tiles from their goal positions — for the above state $h_2(s) = 2 + 0 + 2 + 2 + 1 + 1 + 4 + 1 = 13$
- Which heuristics is better? — an estimate which is closer to the actual is always better!
- We say that h_2 dominates h_1
- An admissible heuristics is one which does not overestimate — in our example, both $h_1(x)$ and $h_2(x)$ are admissible

Example: *n*-queens problem

- What may a good heuristics for *n*-queens problem?
- Cost estimate: Number of pairs of queens that are attacking each other, either directly or indirectly



Best-first greedy search

- As a simple strategy, we may let the evaluation function f to be the same as the heuristics function h
- Nodes in the working set (priority queue) are organized based on estimated cost and the one with the least cost is given preference
- This is a generalization of greedy design strategy, that you have learnt in the previous semester

Best-First Greedy: Complexities

- Is Greedy strategy complete? — No! — not in general
- Is it optimal? — No!
- Time complexity? — $O(b^m)$
- Space complexity? — $O(b^m)$

Questions?

Admissible search

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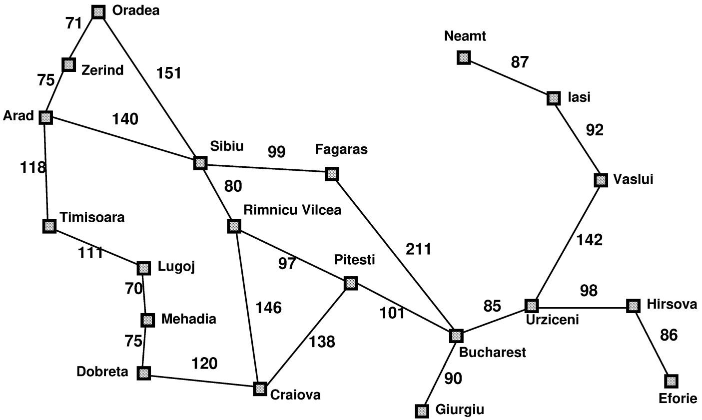
- It may be prudent to consider evaluating a state s by the sum of cost of reaching that state s from the start state and the estimated cost of reaching a goal state from s
- In other words $f(s) = g(s) + h(s)$

Best-first A^* search



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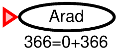
Find the best route from Arad to Bucharest

A^* search



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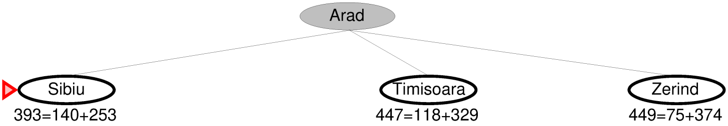

Arad
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393=140+253
Timisoara
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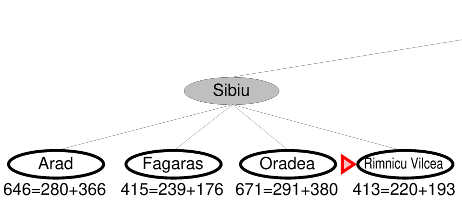
449=75+374

A^* search



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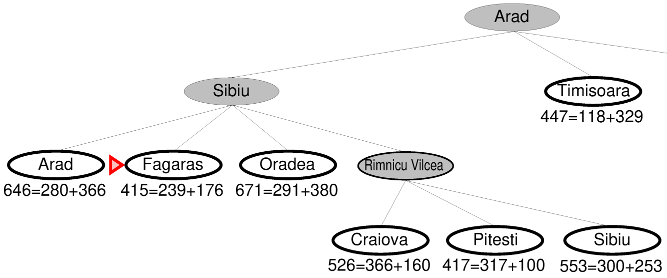

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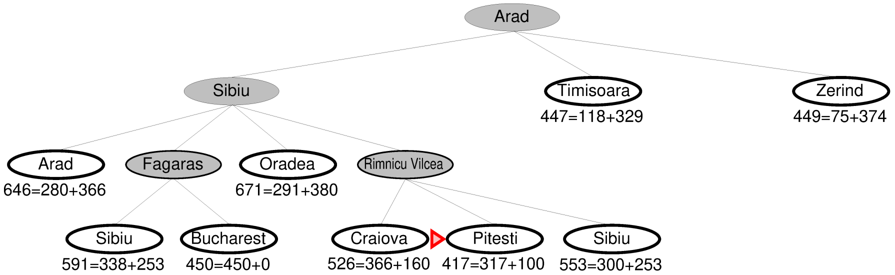
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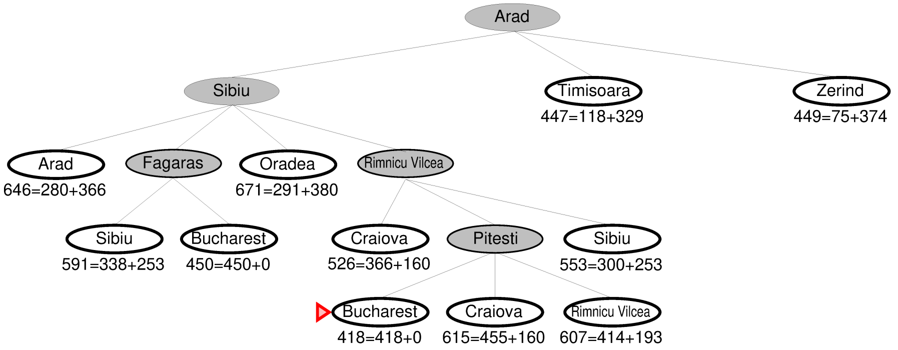

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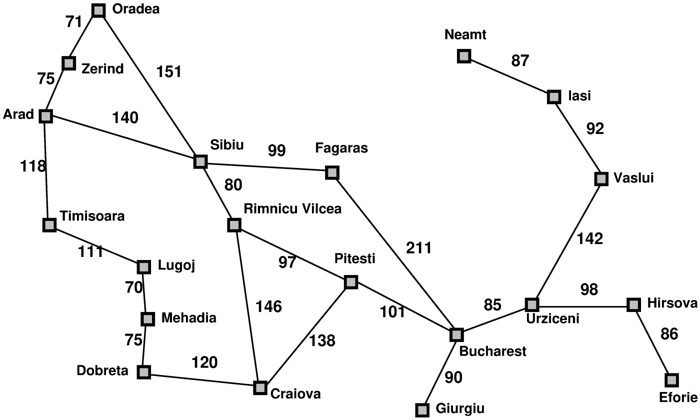

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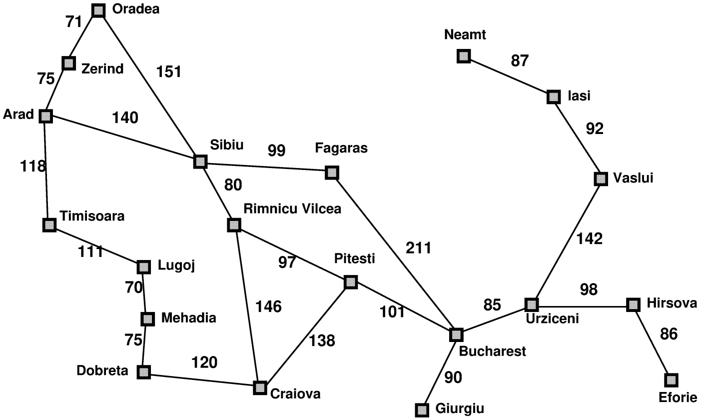

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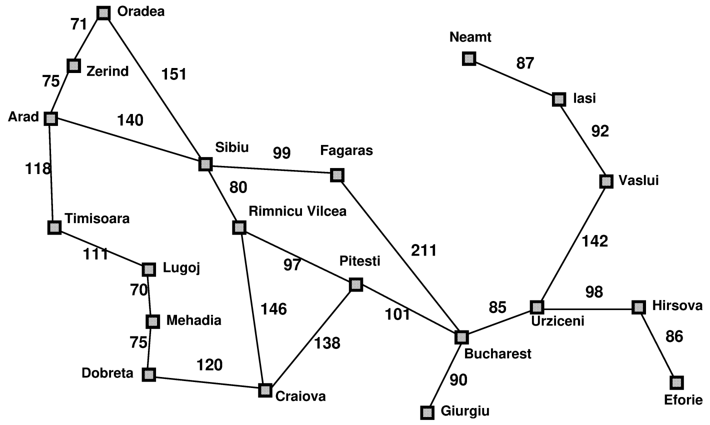

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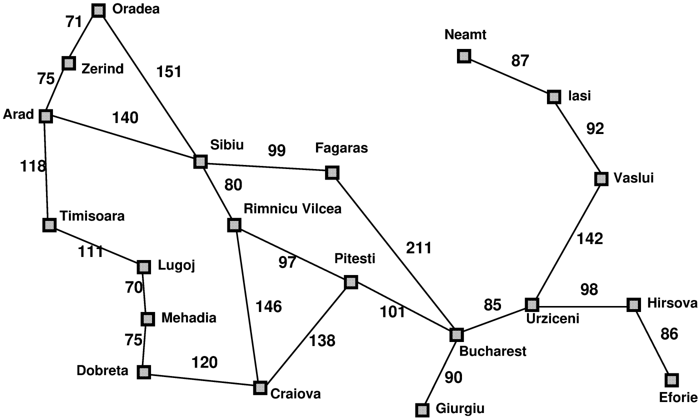
Solution found: Arad → Sibiu → Rimnicu Vilcea → Pitesti → Bucharest with total cost 418



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Does A^* work?

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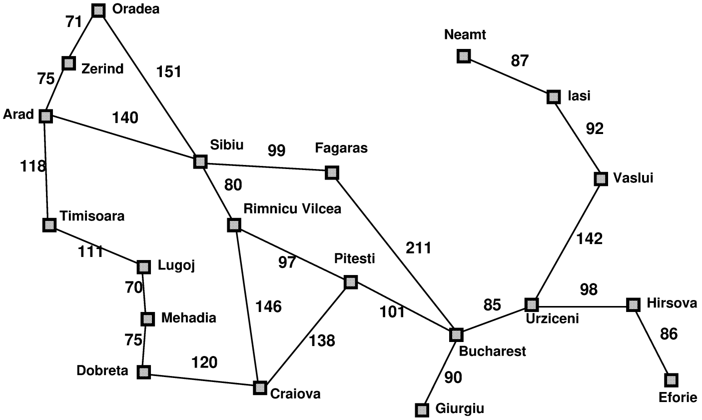

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- An admissible heuristic **never overestimates** the cost to reach the goal, i.e., it is always **optimistic**
- The SLD heuristics and the two heuristics for sliding puzzle problem are examples of admissible heuristics

Consistent Heuristics

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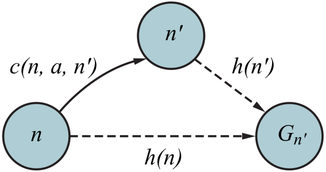

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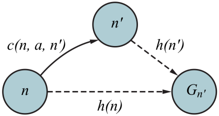

- When h is consistent, we can infer the following

$$f(n') = g(n') + h(n')$$
$$= g(n) + c(n, a, n') + h(n')$$
$$\geq g(n) + h(n)$$
$$\geq f(n)$$

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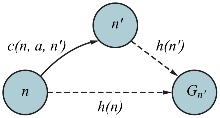

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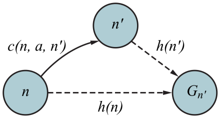

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$$

- That means, evaluation function f is monotonic — it is non-decreasing along any path
- Every consistent heuristics is also admissible

A^* is optimal

Theorem

When h is an admissible heuristics, A^* using TREE-SEARCH is optimal

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- Suppose a sub-optimal goal G_2 is generated and is in the frontier
- Let G be an optimal goal. There must be some node n in the frontier that leads to G
- We need to prove that A^* selects n ahead of G_2 from the frontier

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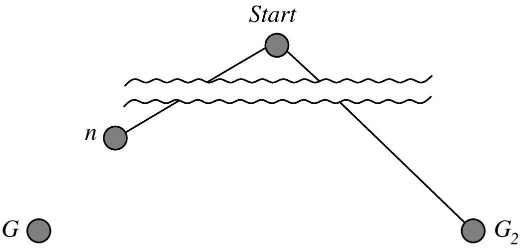

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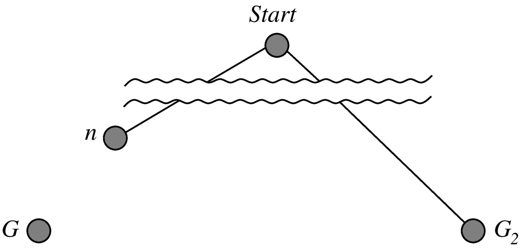

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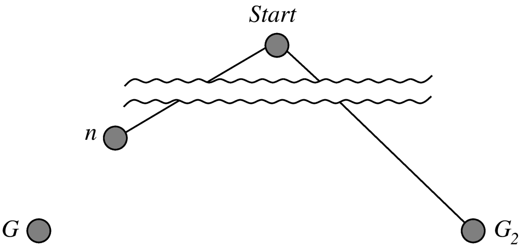

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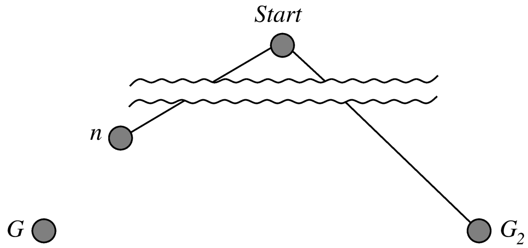

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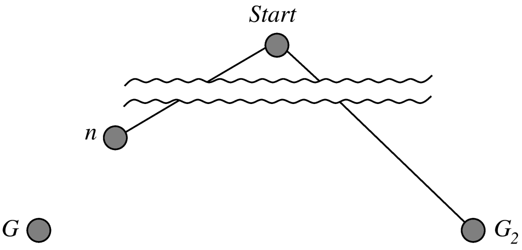

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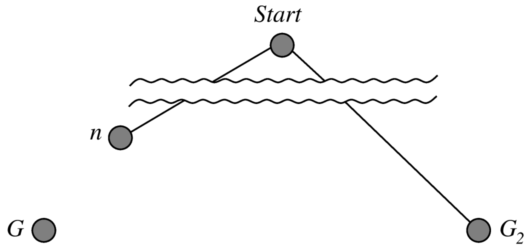

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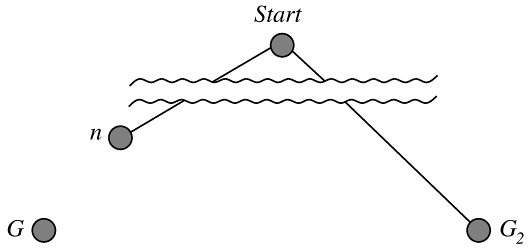

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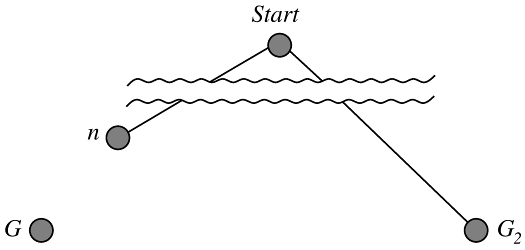

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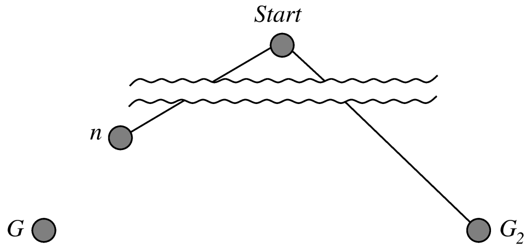

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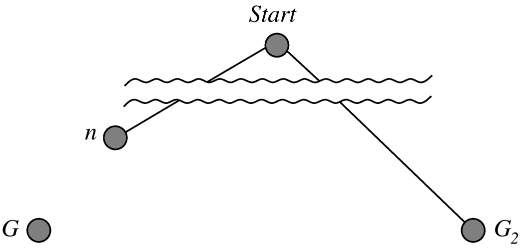

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Hence A^* selects n ahead of G_2

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- $f(n) < f(G_2)$

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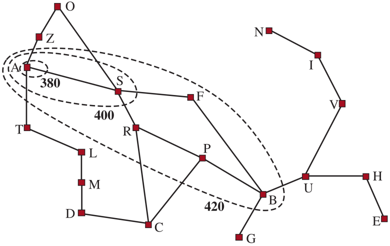
- Suppose A^* selects n for expansion, then optimal path to n has been found — otherwise, there must be another node m in the frontier that leads to n, and $f(m) < f(n)$
- It follows from these two observations that the first goal state found by A^* must be optimal

Properties of A^* search



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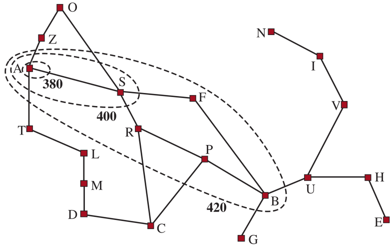
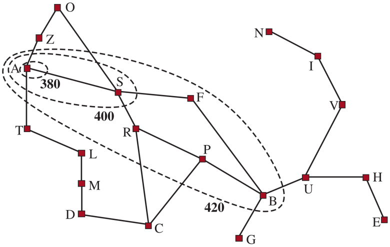


- If C^* is the optimal cost, then
  - A^* expands all nodes with $f(n) \leq C^*$
  - A^* might expand some nodes where $f(n) = C^*$
  - A^* expands no nodes with $f(n) > C^*$

Properties of A^* search



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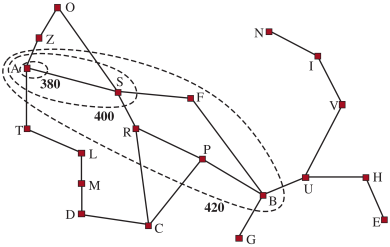

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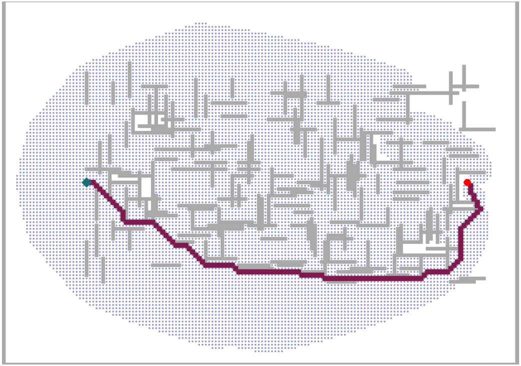
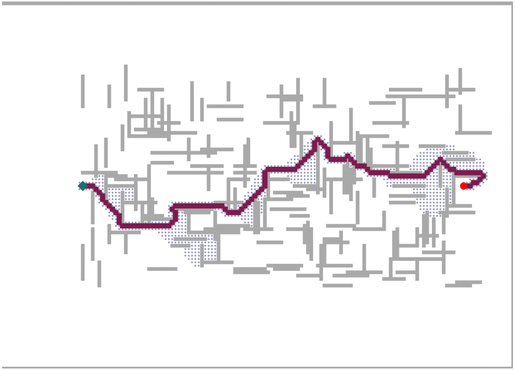
- For example, we may multiply the straight line distance heuristics by what is known as detour index (engineers normally use a detour index in the range of 1.2 and 1.6)
- This idea can be generalized as weighted A^* search, that uses an evaluation function $f(n) = g(n) + W \times h(n)$ for some $W > 1$



- A^* explores several nodes to find an optimal solution in a grid world problem

Satisficing Search — Illustration



- *Weighted* A^*, with $W = 2$, explores comparatively lesser nodes to find a solution that is 5% costlier, in the same instance of grid world problem

Bounded sub-optimal search

- Weighted A^* is an example of a bounded sub-optimal search, where the cost of a solution found is bounded by $W \times C^*$, where C^* is the optimal cost
- Sometimes, it may be required to opt for a bounded cost search, whereby we try to find quickly a solution whose cost does not exceed some prefixed cost C
- If nothing works, sometimes we may for unbounded-cost search whereby we try to quickly find some solution.

Questions?