Kraft-McMillan Inequality (KMI)

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Kraft-McMillan Inequality (1/2)

- If a prefix code C has been constructed for a discrete memoryless source (DMS) with source alphabet $S=\{s_0, s_1, \ldots\ldots\ldots, s_{K-1}\}$ and source statistics $P=\{p_0, p_1, \ldots\ldots\ldots, p_{K-1}\}$ and the codeword for symbol s_k has length l_k, for $k=0,1,\ldots\ldots\ldots,K-1$, then the codeword lengths of the code C always satisfy a certain inequality known as the ***Kraft-McMillan Inequality (KMI)***.

Kraft-McMillan Inequality (2/2)

- The KMI Theorem is shown by,

$$\sum_{k=0}^{K-1} 2^{-l_k} \le 1$$

Where, 2 is the radix (i.e. number of symbols) involved in a binary alphabet {0,1}.

Example : KMI Theorem (1/2)

- Find out the $\sum_{k=0}^{K-1} 2^{-l_k}$ for C_1, C_2 and C_3.

Source Symbol	Probability	C_1	C_2	C_3
S0	0.5	0	0	0
S1	0.25	01	10	1
S2	0.125	011	110	00
S3	0.125	0111	111	11

Example : KMI Theorem (2/2)

- Take C_1, $\sum_{k=0}^{K-1} 2^{-l_k} = 0.9375 < 1$
- Take C_2, $\sum_{k=0}^{K-1} 2^{-l_k} = 1$
- Take C_3, $\sum_{k=0}^{K-1} 2^{-l_k} = 1.5 > 1$
- The coding schemes C_1 and C_2 satisfy the Kraft-McMillan Inequality.
- <u>*Though, C_1 satisfies KMI, but its not a preix code (But fully reversible).*</u>

Important Note on KMI (1/2)

- The Kraft-McMillan Inequality <u>forces the condition on the length of the codewords of a coding scheme (code) rather than controls the code itself</u>.
- Or in other words,

  <u>*Kraft-McMillan Inequality does not make a code as a prefix code*</u>

Important Note on KMI (2/2)

- From the example,
  - Code III violates the Kraft-McMillan Inequality; it can not therefore be a prefix code.
  - Kraft-McMillan Inequality is satisfied by both codes I and II; but only code II is a prefix code.

Prefix Codes as Instantaneous Codes

- Prefix codes are distinguished from other uniquely decodable codes by the fact that the *end of a codeword is always recognizable*.
- Hence, *the decoding of a prefix can be accomplished as soon as the binary sequence representing a source symbol is fully received*.
- For this reason, prefix codes are also referred as *instantaneous codes*.

Average Codeword Length by using Kraft-McMillan Inequality (1/6)

- The average codeword length $\left(\overline{L}\right)$ can be bounded as follows:

$$H\left(S\right) \le \overline{L} < H\left(S\right) + 1$$

- For a prefix code, $p_k = 2^{-l_k}$

Average Codeword Length by using Kraft-McMillan Inequality (2/6)

- The average codeword length is given by,

$$\overline{L} = \sum_{k=0}^{K} p_k l_k$$

- Let us substitute the value of p_k in the formula for $\left(\overline{L}\right)$

Average Codeword Length by using Kraft-McMillan Inequality (3/6)

- Now, the average codeword length is

$$\bar{L} = \sum_{k=0}^{K-1} 2^{-l_k} l_k = \sum_{k=0}^{K-1} \frac{l_k}{2^{l_k}} \quad Eqn.(1)$$

- The entropy of the source is given by,

$$H(S) = \sum_{k=0}^{K-1} p_k \log_2 \frac{1}{p_k}$$

Average Codeword Length by using Kraft-McMillan Inequality (4/6)

- Let us substitute the value of p_k in the formula for H(S)

$$H(S) = \sum_{k=0}^{K-1} 2^{-l_k} \log_2 \frac{1}{2^{-l_k}}$$

$$H(S) = \sum_{k=0}^{K-1} 2^{-l_k} \log_2 2^{l_k}$$

Average Codeword Length by using Kraft-McMillan Inequality (5/6)

- Now, the entropy, H(S)

$$H(S) = \sum_{k=0}^{K-1} l_k 2^{-l_k} \log_2 2$$

$$H(S) = \sum_{k=0}^{K-1} l_k 2^{-l_k}$$

$$H(S) = \sum_{k=0}^{K-1} \frac{l_k}{2^{l_k}} \quad Eqn.(2)$$

Average Codeword Length by using Kraft-McMillan Inequality (6/6)

- By comparing Eqn. (1) and Eqn. (2), For a prefix code that satisfies an efficient Kraft-McMillan Inequality,

$$\overline{L} = H(S)$$

- Kraft-McMillan inequality provides a necessary condition for uniquely decodable codes. That is, if a code is uniquely decodable, the codeword lengths have to satisfy the inequality.