UIT2504
Artificial Intelligence

Introduction to
First-order Logic

Outline

- Why first-order logic?
- FOL: Syntax
- FOL: Semantics
- Using FOL
- Inferences in first-order logic

Propositional Logic is good …

- Propositional logic is **declarative**
- PL allows **partial / disjunctive / negated** information
- PL is **compositional**
- Unlike natural languages, meaning in PL is **context-independent**

But . . .

- PL lacks expressive power – can not express property of a “class” and reason with “objects” of that class
- P: Birds fly
- Q: Tweety is a bird
- R: Tweety flies
- **Can we infer R from P $\wedge$ Q ???**

FOL: Syntax

- PL assumes that a proposition is "atomic", represented by a symbol
- FOL provides different mechanisms for building **objects** (referred to as **terms**) and uses **predicates** to make statements about relationship among objects

FOL: Building blocks

- **Constants:** Symbols that directly refer to an object: India, Tweety, Tom, Jerry
- **Variables:** Symbols that can bind to different objects from a domain: x, y, z, cat, mouse
- **Functions:** Symbols that indirectly refer to an object through some other objects: f(x,y), father_of(Tom)

FOL: Terms

- **Terms** can be built using these symbols
  – Constant by itself is a term
  – Variable by itself is a term
  – If $t_1, t_2, \ldots, t_n$ are terms and f is a n-place function symbol, then $f(t_1, t_2, \ldots, t_n)$ is a term
  – Nothing else is a term
- Note that a term refers to an object and does not have any truth value!
- Examples: Zero, s(Zero), s(s(Zero)), sum(s(Zero),s(s(s(Zero))))

FOL: Predicates

- **Predicates** are statements that bring out relationship among objects

  ge(s(x), x); likes(y, Aravindan)

- Note that propositions are predicates with zero arguments
- Predicates are referred to as **atoms** (and a **literal** is negation of an atom)
- A **ground atom** (or literal) is one with no variables appearing in it
- Ground atoms can assume truth values, either ***True*** or ***False***

FOL: Connectives

- **Connectives:** $\neg$, $\Rightarrow$, $\wedge$, $\vee$, $\Leftrightarrow$
- **Quantifiers:** $\forall$, $\exists$
- ($\forall$x F) is True iff for every possible binding for x in F, F evaluates to True
- ($\exists$x F) is True iff there exists some binding for x in F s.t. F evaluates to True

FOL: Quantifiers

- $\forall x(country(x) \Rightarrow \exists y(capital(y, x)))$

- $\forall s(student(s) \wedge pass(s) \Rightarrow work(s))$

- $\forall x(student(x) \wedge at(x, SSN) \Rightarrow smart(x))$

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- Note that this is very different from
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- $\exists y(\text{student}(y) \wedge \text{at}(y, \text{SSN}) \wedge \text{genius}(y))$

FOL: Quantifiers

- Represent the following using the predicate loves(x, y), which stands for “x loves y” and suitable quantifiers:
  - Everybody loves everybody
  - Everybody loves somebody
  - Someone is loved by everyone
  - Somebody loves everybody

FOL: Quantifiers

- $\exists x \exists y$ is same as $\exists y \exists x$
- $\forall x \forall y$ is same as $\forall y \forall x$
- $\exists x \forall y$ is NOT the same as $\forall y \exists x$
- $\exists x$ F can also be expressed as $\neg \forall x \neg$ F
- $\forall x$ F can also be expressed as $\neg \exists x \neg$ F

FOL: Well-formed formula

- An atom is a wff
- If F is a wff then $\neg$F and (F) are wff
- If F1 and F2 are wff, then
  - F1 $\vee$ F2, F1 $\wedge$ F2, F1 $\Rightarrow$ F2, F1 $\Leftrightarrow$ F2 are wff
- If F is a wff with a free variable x in it, then $\forall$x F and $\exists$x F are wff
- Nothing else is a wff

FOL: Example

All employees earning Rs. 1,50,000 or more pay taxes. Some employees are sick today. No employee earns more than the President.

FOL: Example

All employees earning Rs. 150,000 or more pay taxes. Some employees are sick today. No employee earns more than the President.

$\forall x(\text{ emp}(x) \wedge \text{ge}(s(x),150000) \Rightarrow \text{tax}(x))$

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FOL: Example

All employees earning Rs. 150,000 or more pay taxes. Some employees are sick today. No employee earns more than the President.

$\forall x(emp(x) \wedge ge(s(x),150000) \Rightarrow tax(x))$

$\exists y(emp(y) \wedge sick(y))$

$\forall x \forall y(emp(x) \wedge pre(y) \Rightarrow ge(s(y),s(x)))$

FOL: Example

$\forall s\ (\ Set(s) \Leftrightarrow (s = \{\}\ ) \vee \exists x,s_2\ (Set(s_2) \wedge s = \{x|s_2\})\ )$

$\neg\exists x,s\ (\ \{x|s\} = \{\}\ )$

$\forall x,s\ \ (\ x \in s \Leftrightarrow \exists s'\ (s = \{x|s'\})\ )$

$\forall x,s\ (\ x \in s \Leftrightarrow [\ \exists y,s_2\ (s = \{y|s_2\} \wedge (x = y \vee x \in s_2))]\ )$

$\forall s_1,s_2\ \ (\ s_1 \subseteq s_2 \Leftrightarrow (\forall x\ \ x \in s_1 \Rightarrow x \in s_2)\ )$

$\forall s_1,s_2\ \ (\ (s_1 = s_2) \Leftrightarrow (s_1 \subseteq s_2 \wedge s_2 \subseteq s_1)\ )$

$\forall x,s_1,s_2\ (\ x \in (s_1 \cap s_2) \Leftrightarrow (x \in s_1 \wedge x \in s_2)\ )$

$\forall x,s_1,s_2\ \ (\ x \in (s_1 \cup s_2) \Leftrightarrow (x \in s_1 \vee x \in s_2)\ )$

FOL: Interpretations

$n(Zero) \wedge \forall x\, (n(x) \Rightarrow \exists y\, (p(y,s(x))))$

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- Fix a **domain**

  $D = \{0, 1, 2, 3\}$

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- Fix a **domain**

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- Assign values to constants from the domain

  $Zero \mapsto 0$

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  $Zero \mapsto 0$

- Assign values to function symbols

  $s(0) \mapsto 1,\ s(1) \mapsto 2,\ s(2) \mapsto 3,\ s(3) \mapsto 0$

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- Assign truth values to ground atoms

  $\{n(0),n(1),n(2),n(3),p(3,2),p(3,1),p(3,0)\}$

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- This completes one **interpretation**

FOL: Interpretations

$n(Zero) \wedge \forall x \; (n(x) \Rightarrow \exists y \; (p(y,s(x))))$

- Fix a **domain**

  $D = \{0, 1, 2, 3\}$
- Assign values to constants from the domain

  $Zero \mapsto 0$
- Assign values to function symbols

  $s(0) \mapsto 1,\; s(1) \mapsto 2,\; s(2) \mapsto 3,\; s(3) \mapsto 0$
- Assign truth values to ground atoms

  $\{n(0), n(1), n(2), n(3), p(3,2), p(3,1), p(3,0)\}$
- This completes one **interpretation**
- Is this a **model** for the given sentence?

FOL: Interpretations

$n(Zero) \wedge \forall x \, (n(x) \Rightarrow \exists y \, (p(y, s(x))))$

- Herbrand Domain

  HD = {Zero, s(Zero), s(s(Zero)), . . .}

- Herbrand Base

  HB = {n(Zero), n(s(Zero)), p(Zero, s(Zero)), . . .}

- Herbrand Interpretation

  (Assign truth values to ground atoms)

  {n(Zero), p(s(s(Zero)), s(Zero))}

- Herbrand Model

FOL: Models

- Model of a sentence is an interpretation in which the sentence evaluates to true
- Recall definitions of satisfiable, valid, and equivalent sentences
- Recall definition of logical entailment
- Design algorithm for KB $\stackrel{?}{\models}$ α

Questions?