Conditional Probability

Example

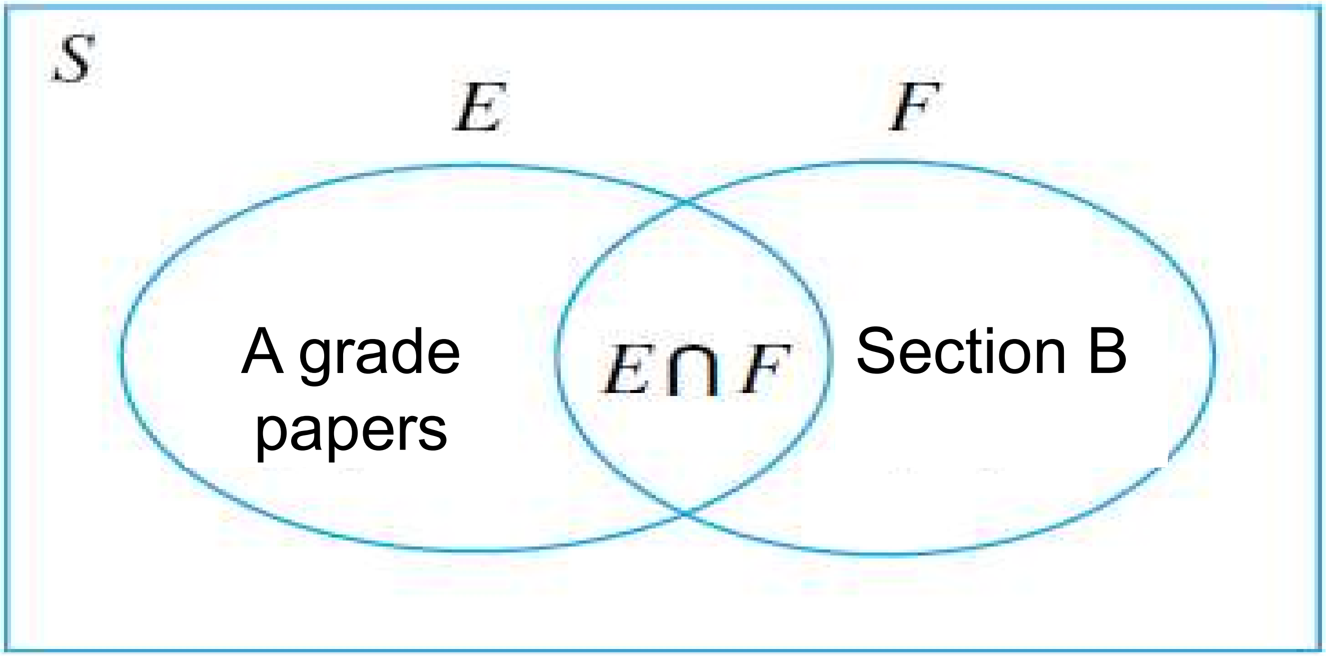
- Professor teaches two sections the same subject
- Section A has 35 students and section B has 25 students
- The professor gave both sections the same test
- 14 students got A grade
  - 5 in section A and 9 in section B

Question#1

- If a test paper is selected at random from all papers, what is the probability that it is an A grade paper?

Question#2

- A test paper is selected at random
- If it is known that the paper is from the section B
- What is the probability the paper has A grade?



S = all exam papers = 60

F = all section B papers = 25

$\overline{F}$ = all section A papers = 35

E = all A grade papers = 14

$E \cap F$

A grade and section B = 9

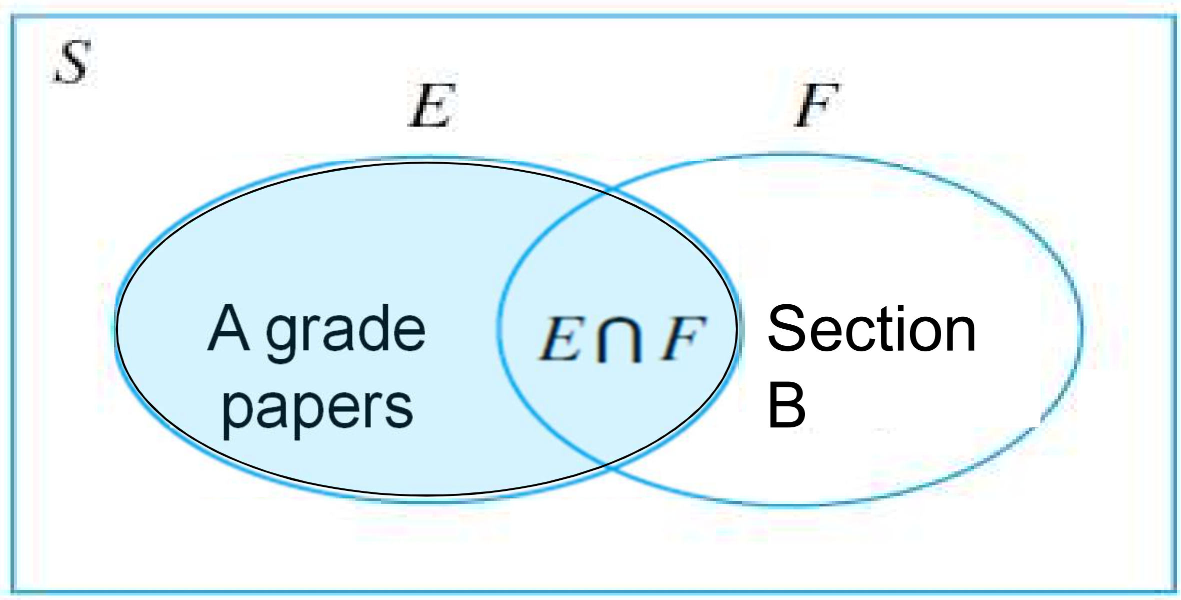
$E \cap \overline{F}$

A grade and section A = 5

Solution to Question#1

Sample space = S

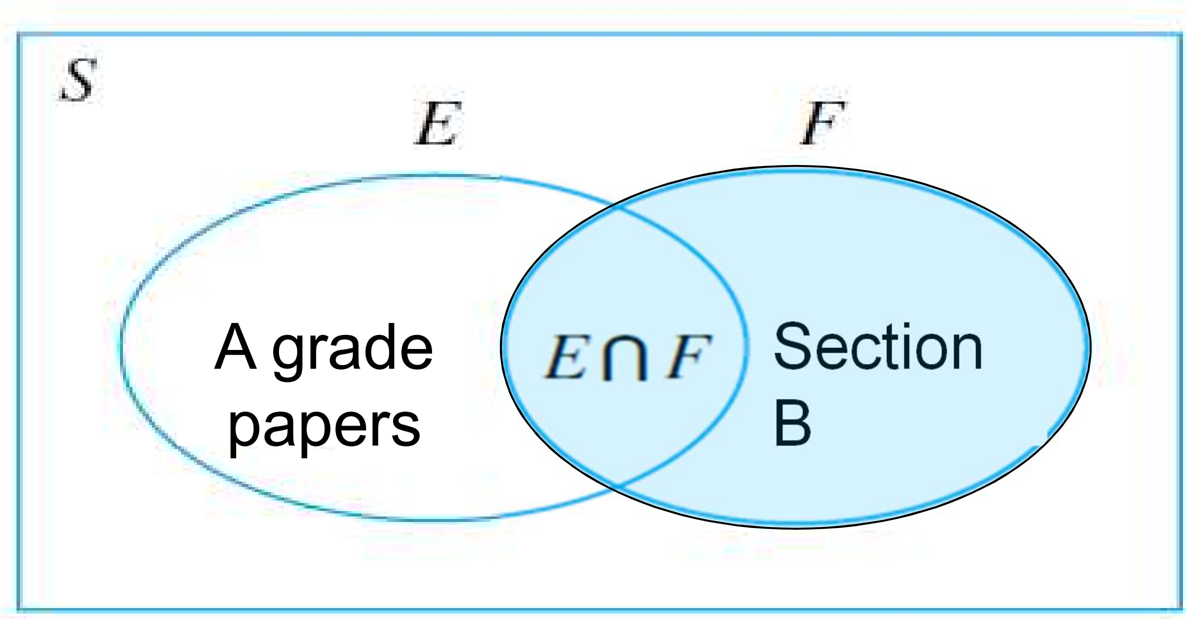
- We want P(E)
- n(E)/n(S) =14/60 = 0.233



Solution to Question#2

It is known that the paper is from section B

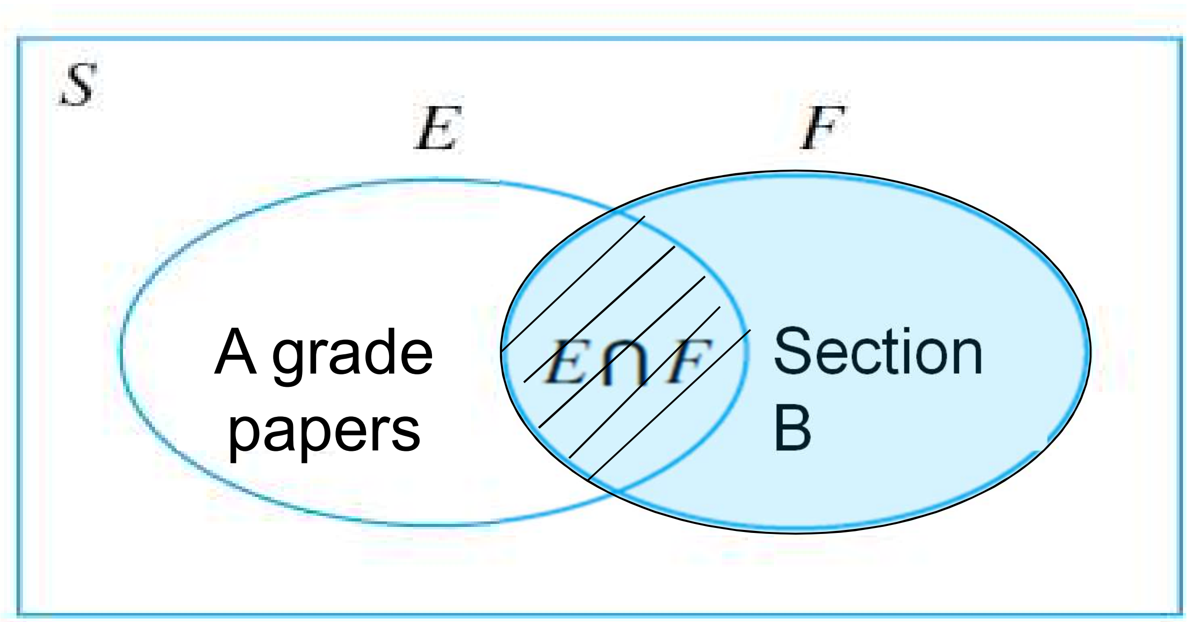
Reduced Sample space = F



Reduced Sample space = F

Known that the paper is from section B and having A grade,

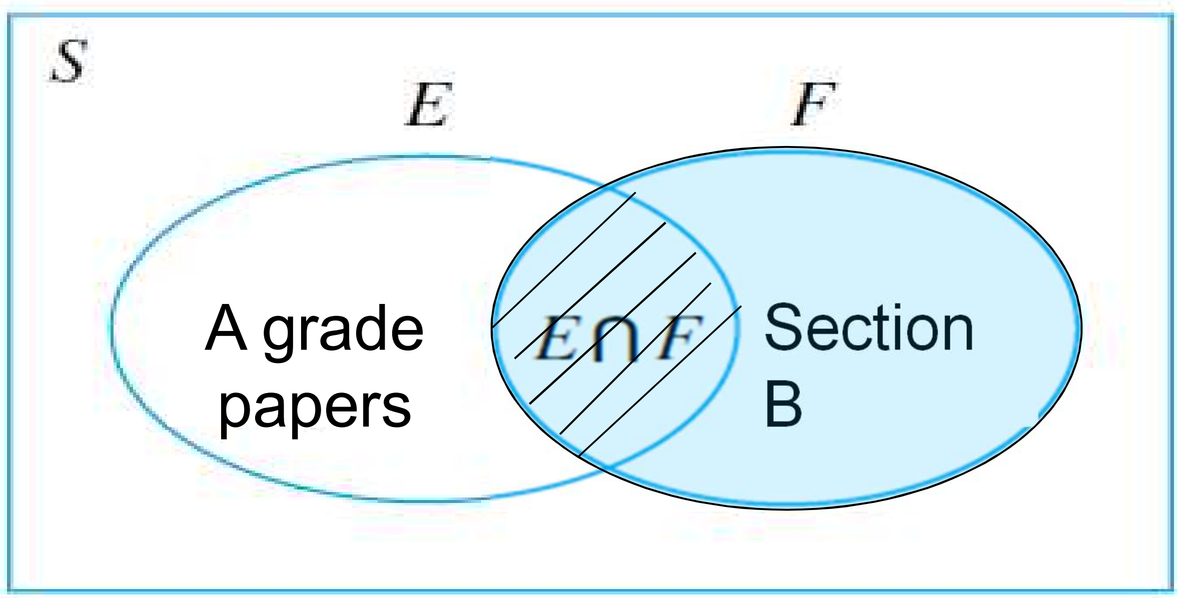
$P(E|F) = n(E \cap F)/n(F) = 9/25$



Conditional probability ≡ Reduced sample space

$P(E|F) = n(E \cap F)/n(F)$

$[n(E \cap F)/n(S)]/[n(F)/n(S)] = P(E \cap F)/P(F)$



Question#2

- A test paper is selected at random
- If it is known that the paper is from the section B
- What is the probability the paper has A grade?

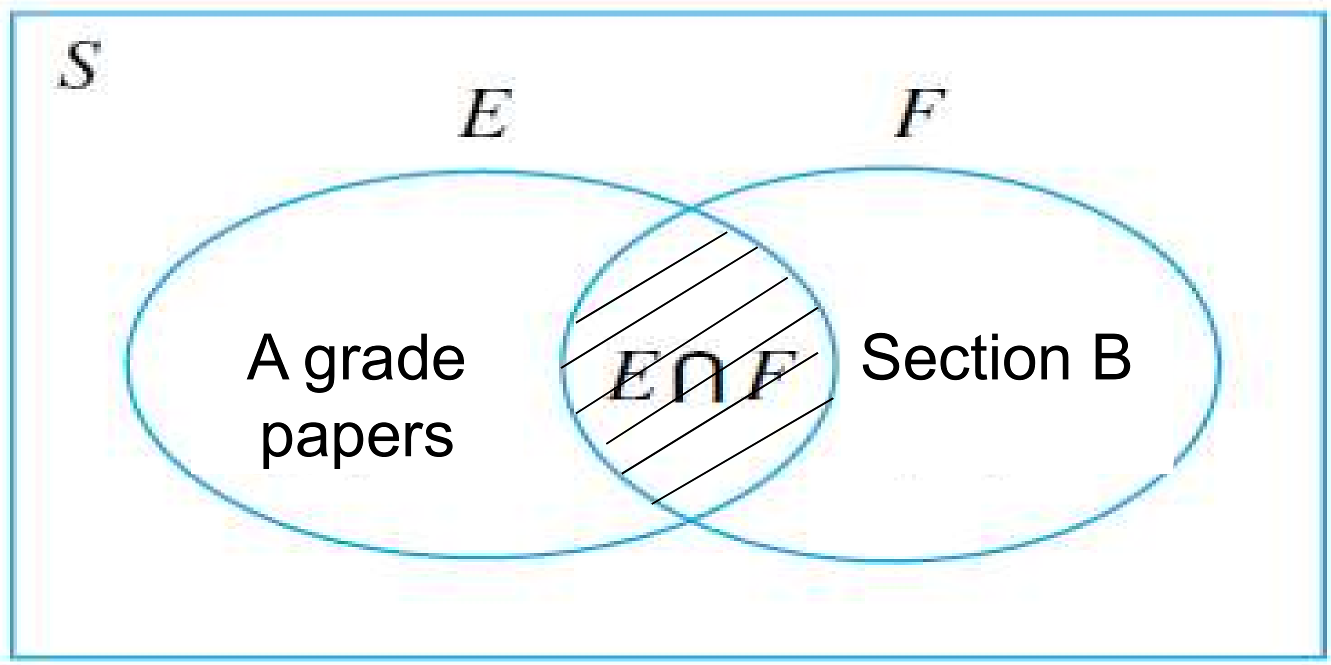
Question#3

- A test paper is selected at random
- What is the probability the paper has from section B and having A grade?

Solution to question 3

Probability (paper is section B and paper has A grade) = n($E \cap F$)/n(S) = 9/60

Sample space = S



Understand the difference

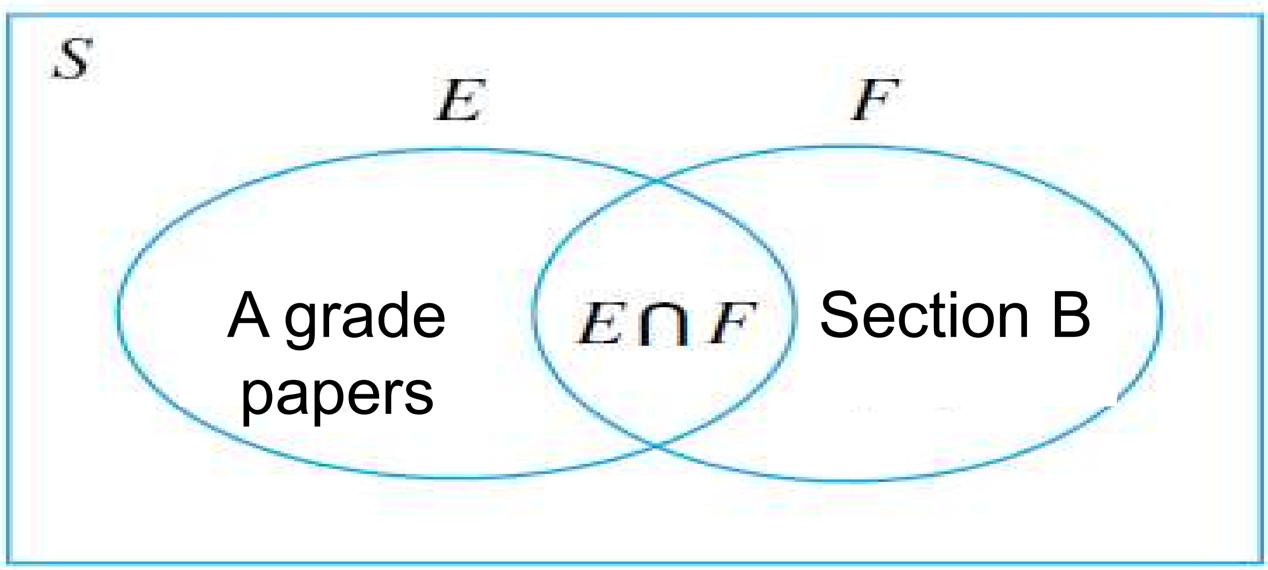
It is known that the paper is from section B

What is the probability of having A grade?

$P(E|F)=n(E\cap F)/n(F)$

What is the probability the paper is from section B and having A grade?

$P(E\cap F)=n(E\cap F)/n(S)$



Start with this example

scenario

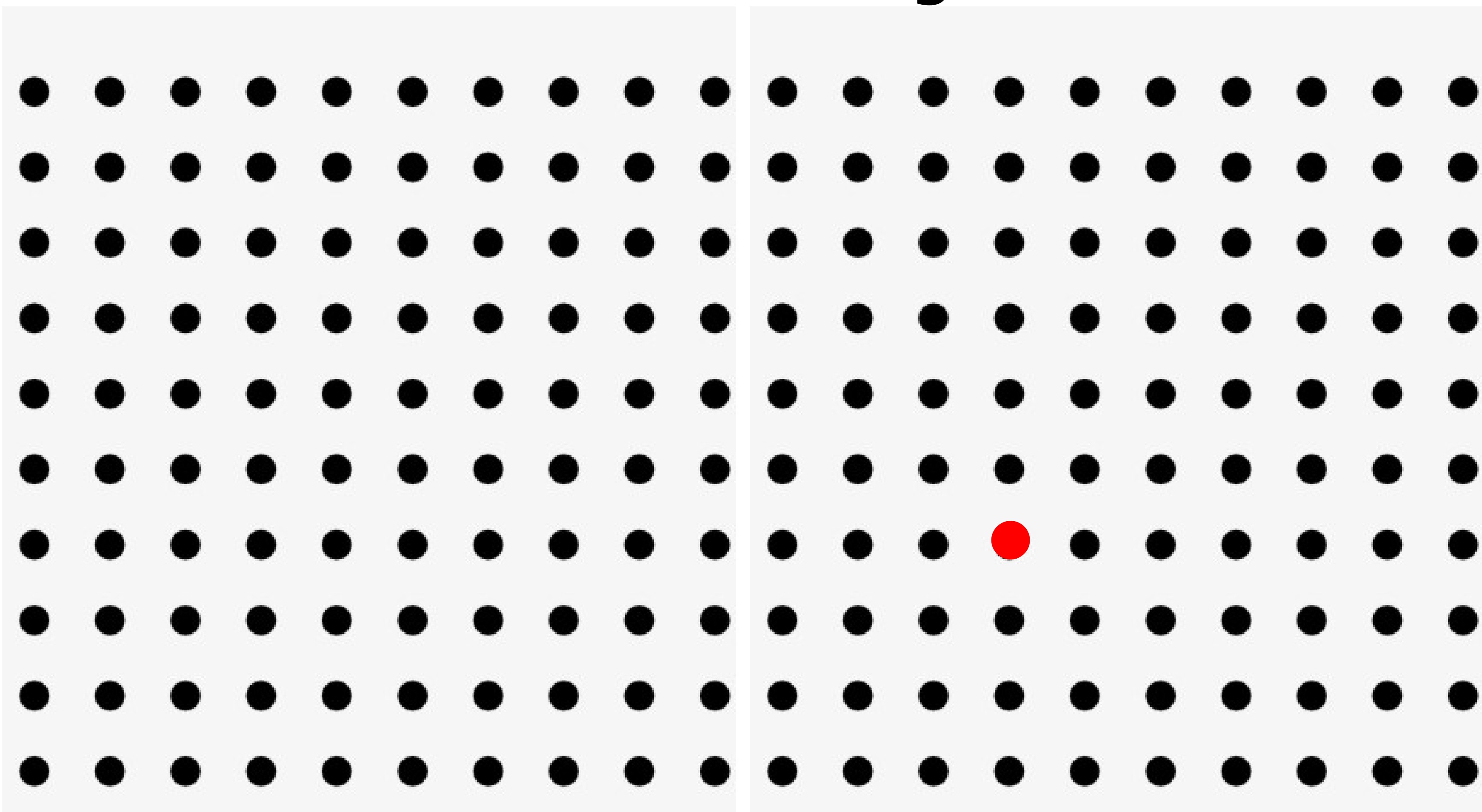
- A person is sick
- Goes to the doctor
- Doctor does many tests and one test shows positive – it's a very rare disease

Given

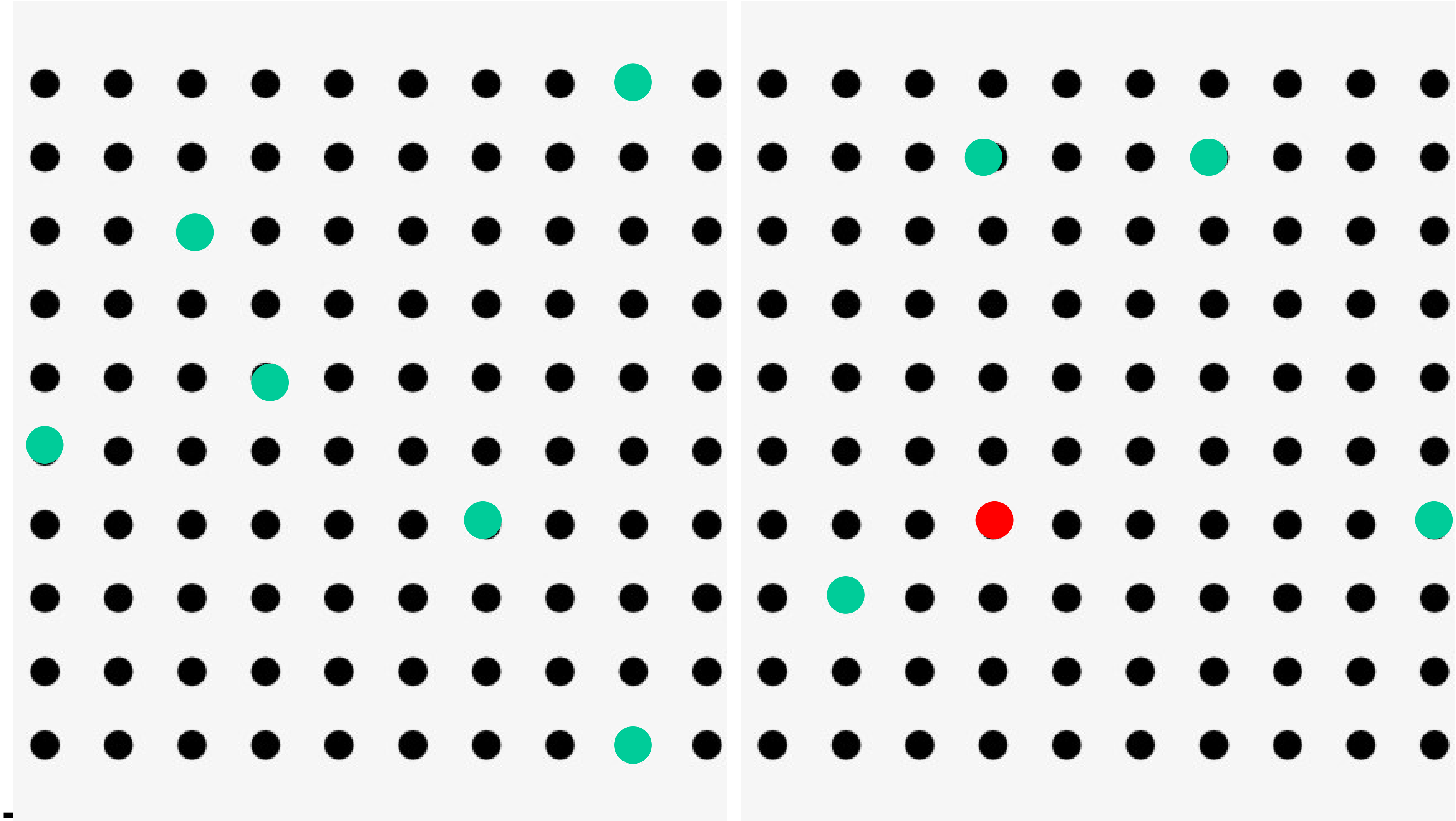
- Sometimes that test (which shows positive) gives wrong answer i.e. false positive (1%)
- Since it's a rare disease - only one in 1000 gets affected (i.e. 0.1%)

What is the chance that the person has the disease?

99% is the wrong answer



1 in 1000 has the disease
999 do not have disease

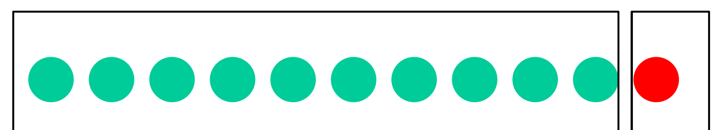


Test errs with 1%

999 do not have disease – If the test is run on them it’ll wrongly identify 9.9 people to have the disease ~ say 10

Logically arrive @ conclusion

- 1 in 1000 has the disease $\rightarrow$ one person is true positive in 1000 people
- Test errs with 1% $\rightarrow$ gives 10 positive in 999 people
- Totally how many positive are there?
- 10 positive from 999 + one person tested positive who had been to doctor



You may be here OR may be here

What is the chance of being “True +”?
1/11 = 0.9 = 9%

Interpretation

- Before testing all (i.e. all 1000) have the probability of 1% being positive
- After testing, two things can happen
  - If it is negative do not worry
  - If it is positive what is the chance of being of positive?
    - It increases from 1% to 9%.

Hypothesis and evidence

1. We have an hypothesis (H)
    - There is a chance that H could be true
    - Also there is a chance that H could be false
    - Attach some probability values
2. We receive an evidence
3. Evidence is either going to increase or decrease the true and false probability values

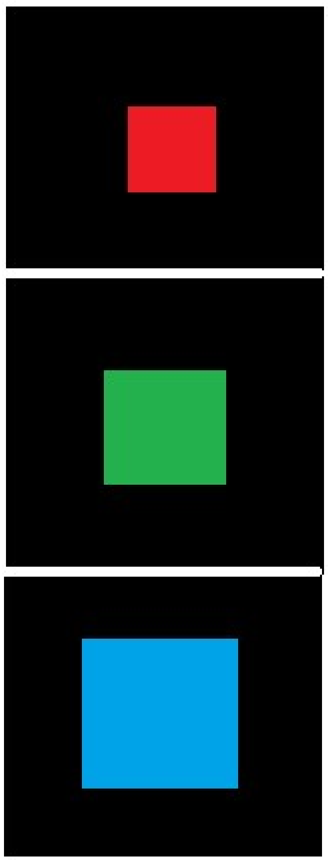
Example

- Say we an object found near a apple garden
- **Hypothesis:** It’s an apple
- With some probability value
- **Evidence Received:** Shape of the object is cube
- What’s going to happen to hypothesis?
- The probability value decreases

Another example

- A crime happens
- I am suspected by police and court
- With some probability value (P)
- Evidences are received
- This is either going to increase or decrease P
- If P increases but not sufficiently then the court demands some more evidence
- This happens till the P hits 100%
- If the evidence decreases P then some relaxation is given to me
- I am acquitted when P becomes 0

e.g.



- Assume we have an urn with 2 red, 3 green, and 5 blue balls
- The probabilities of picking a red, a green, or a blue ball are:

$$P(red) = 2/10 = 0.2$$

$$P(green) = 3/10 = 0.3$$

$$P(blue) = 5/10 = 0.5$$

Joint probability – e.g

- Assume we have an urn with 2 red, 3 green, and 5 blue balls
- Assume drawing 2 balls without replacing
- The probabilities of picking a red, a green, or a blue ball are:

$$P(red) = 2/10 = 0.2$$

$$P(green) = 3/10 = 0.3$$

$$P(blue) = 5/10 = 0.5$$

P(1st-was-red, 2nd-is-red) = (2/10).(1/9)=2/90

P(1st-was-green, 2nd-is-green) = (2/10).(3/9)=6/90

…

…

…

$$P(X_1, X_2) = \begin{array}{r|ccc} & X_1 = \text{red} & X_1 = \text{green} & X_1 = \text{blue} \\ \hline X_2 = \text{red} & \frac{1}{45} & \frac{1}{15} & \frac{1}{9} \\ X_2 = \text{green} & \frac{1}{15} & \frac{1}{15} & \frac{1}{6} \\ X_2 = \text{blue} & \frac{1}{9} & \frac{1}{6} & \frac{2}{9} \end{array}$$

- How to get P (red)?
- Either add all the elements of “red” row

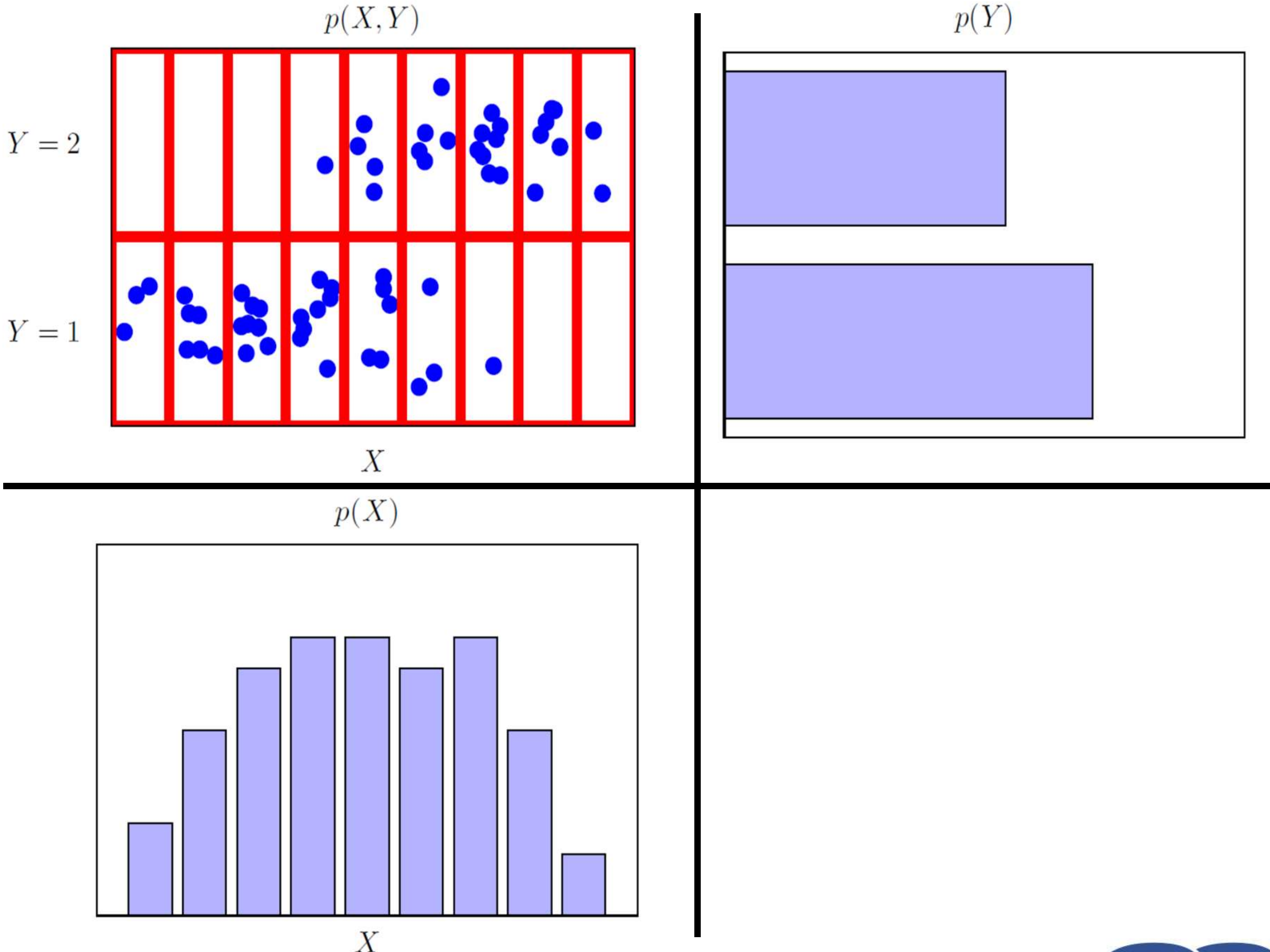
Or

- Add all the elements of “red” column
- Same is true for P(green) and P(blue)

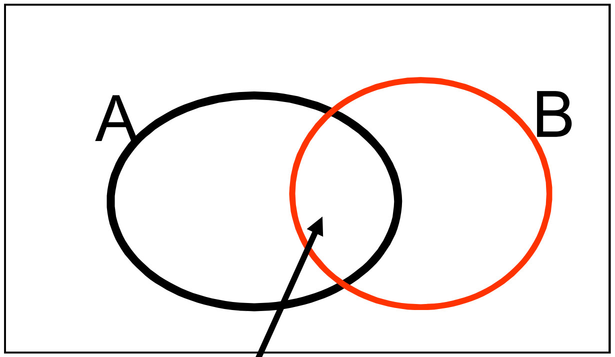
Joint probability to Marginal probability - Generalization

$$P(x = a_i) \equiv \sum_{y \in \mathcal{A}_Y} P(x = a_i, y) \Rightarrow P(x) \equiv \sum_{y \in \mathcal{A}_Y} P(x, y)$$

$$P(y) \equiv \sum_{x \in \mathcal{A}_X} P(x, y)$$


p(X,Y)
Y = 2
Y = 1
X
p(Y)
p(X)
X

Two possible questions



$(A \cap B)$ OR $(B \cap A)$

<u>Question#1</u>

- If A has happened what is the chance that B has happened

<u>Question#2</u>

- If B has happened what is the chance that A has happened

Joint and conditional probability

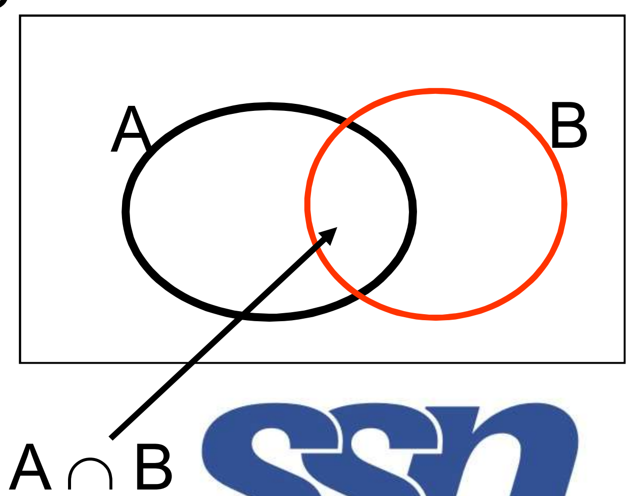
For any two events a and b the probability that both a and b occur is

$$P\left(A \bigcap B\right) = P(B).P(A|B) = P(A).P(B|A)$$

P(A ∩ B) (also written as P(A,B) is called the joint probability of the events a and b

Famous Bayes' theorem:

$$P(B|A) = \frac{P(A|B).P(B)}{P(A)}$$



Conditional probability – e.g

- Assume we have an urn with 2 red, 3 green, and 5 blue balls – Also assume drawing 2 balls
- The probabilities of picking a red, a green, or a blue ball are:

$$P(\text{red}) = 2/10 = 0.2$$

$$P(\text{green}) = 3/10 = 0.3$$

$$P(\text{blue}) = 5/10 = 0.5$$

$$P(\text{2nd-is-red} \mid \text{1st-was-red}) = (2-1)/(10-1) = 1/9$$

$$P(\text{2nd-is-green} \mid \text{1st-was-red}) = 3/(10-1) = 3/9$$

$$P(\text{2nd-is-blue} \mid \text{1st-was-red}) = 5/(10-1) = 5/9$$

$P(\text{1st-was-red}, \text{2nd-is-red}) = P(\text{2nd-is-red} \mid \text{1st-was-red}) P(\text{1st-was-red}) = 1/9 \cdot 1/5 = 1/45$

$P(\text{1st-was-red}, \text{2nd-is-green}) = P(\text{2nd-is-green} \mid \text{1st-was-red}) P(\text{1st-was-red}) = 1/3 \cdot 1/5 = 1/15$

Conditional distribution – e.g.

$$P(X_2 | X_1) = \begin{array}{r|ccc} & X_1 = \text{red} & X_1 = \text{green} & X_1 = \text{blue} \\ \hline X_2 = \text{red} & \frac{1}{9} & \frac{2}{9} & \frac{2}{9} \\ X_2 = \text{green} & \frac{3}{9} & \frac{2}{9} & \frac{3}{9} \\ X_2 = \text{blue} & \frac{5}{9} & \frac{5}{9} & \frac{4}{9} \end{array}$$

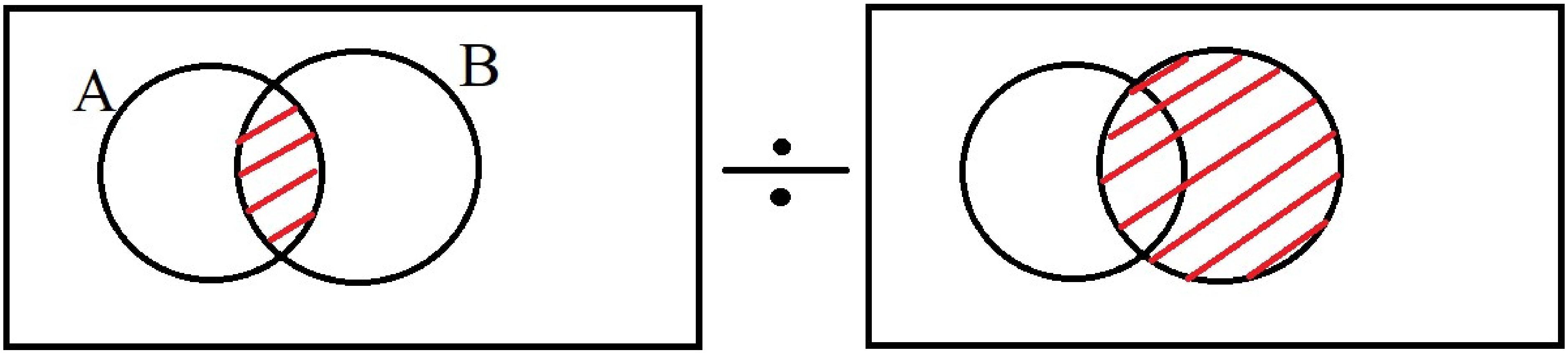
$$P\left(A \bigcap B\right) = P(B).P(A|B) = P(A).P(B|A)$$

$$P(A|B) \propto P\left(A \bigcap B\right)$$ Normalize using P(A)

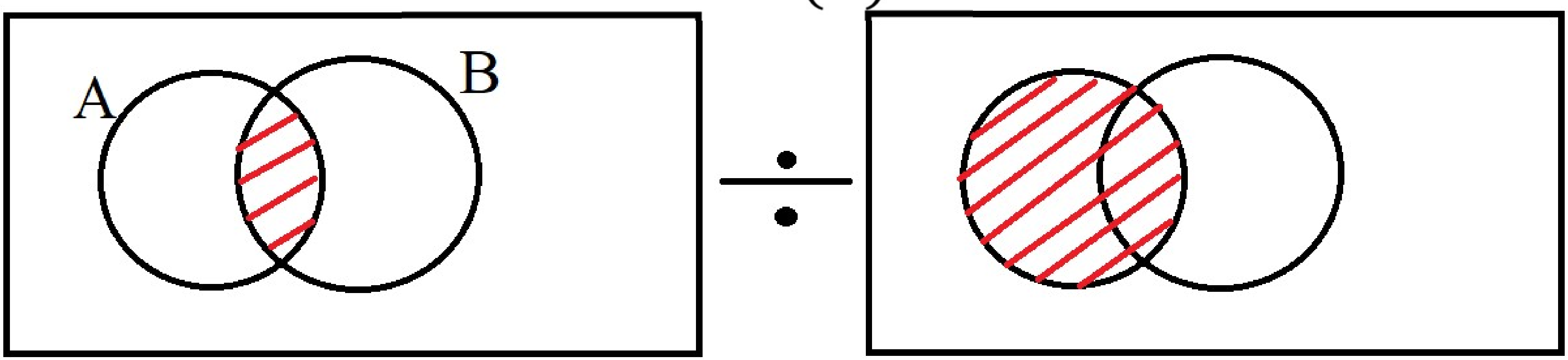
$$P(B|A) \propto P\left(A \bigcap B\right)$$ Normalize using P(B)

Conditional probabilities $\propto$ Joint probability

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

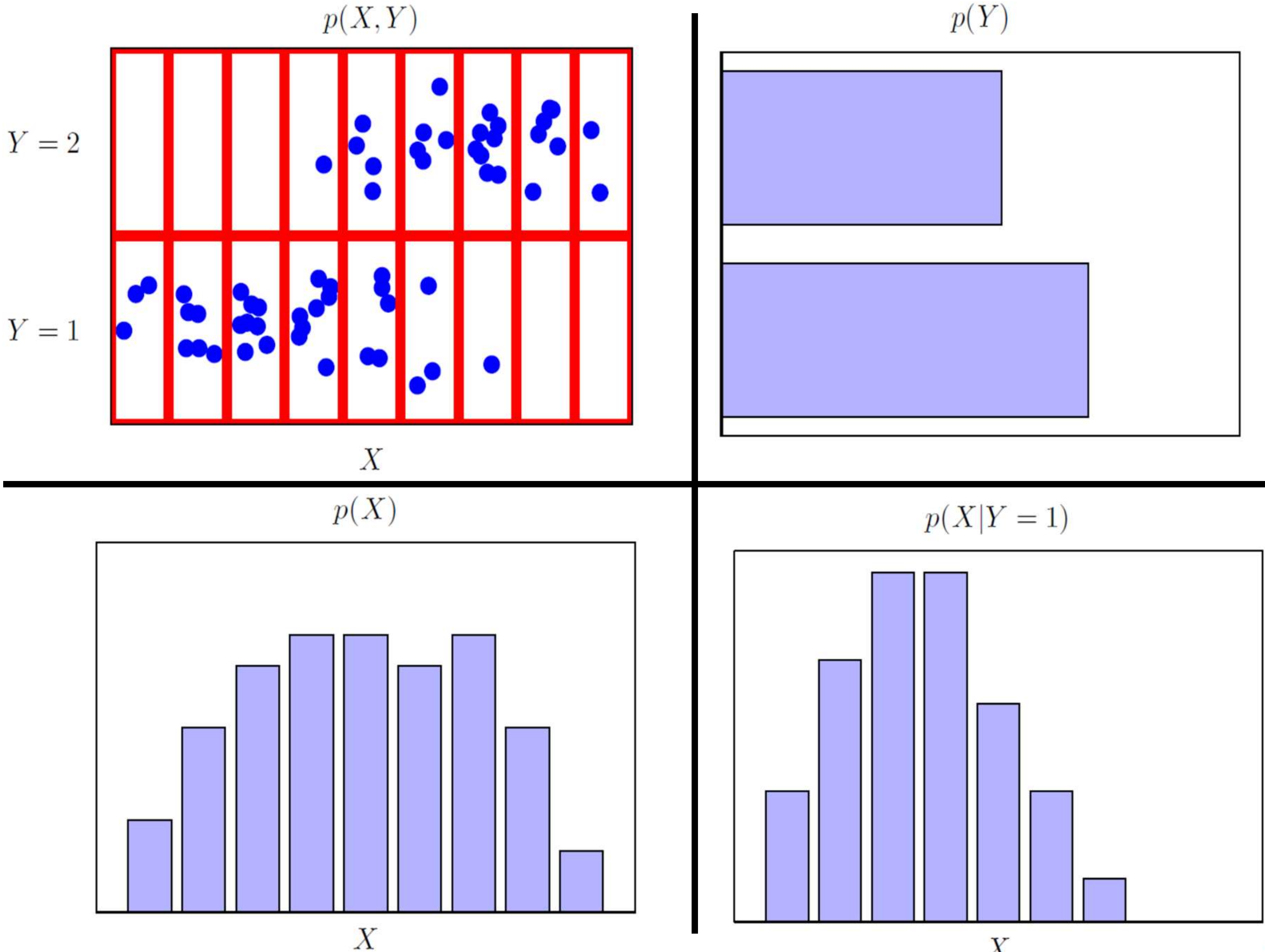


$$P(B|A) = \frac{P(A \cap B)}{P(A)}$$



Conditional probability

- We seek the probability of an event E
- A related event F occurs
- Changing the probability of E
- Denote by P(E|F)



$p(X,Y)$
$Y = 2$
$Y = 1$
X
$p(Y)$
$p(X)$
X
$p(X|Y = 1)$
X

One more example

- Standard deck of cards
- 13 hearts, 13 diamonds, 13 spades and13 clubs – H, D, S and C
  - P(H), P(D), P(S) and P(C)
- Draw a card at random from the deck of 52
- What is P(H)?
- 13/52=0.25

- Draw a card at random from the deck of 52
- It is known that the card is red colour
- What is P(H)?
- Sample space reduces from 52 to 26
- Favourable cards = 13
- 13/26 = 0.5
- It is known that the card is black colour
- What is P(H)?
- Sample space reduces from 52 to 26
- Favourable cards = 0
- 0/26 = 0

Knowing an event calculate the another event’s probability – it may change or may not

- Draw a card at random from the deck of 52
- Consider Face cards (jack, queen and king)
- What is P(F)?
- Sample space 52
- Favourable cards = 12
- 12/52 = 0.23
- It is known that the card is red colour
- What is P(F)?
- Sample space reduces from 52 to 26
- Favourable cards = 6
- 6/26 = 0.23

Knowing an event calculate the another event’s probability – it may change or may not