UIT2504 Artificial Intelligence

Propositional Logic

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Knowledge-Based Agents

- Knowledge base = set of sentences in a formal language
- Knowledge-based agent comprises of domain specific knowledge base and domain independent inference mechanism to process knowledge
- Knowledge is in declarative form
- Suitable for partially observable environments where hidden information can be inferred

Interpretations, Models, and Entailment

- Sentences written in logic must be well-formed formula and follow a grammar
- There are several possible interpretations for a set of sentences KB
- Interpretations in which KB evaluates to true are called as models of KB
- Given a new sentence α, KB logically entails α (written as $KB \models \alpha$) iff every model of KB is also a model of α
- We write $KB \vdash \alpha$ if α can be derived from KB using syntactic derivation rules
- Sentences in logic are usually written in a normal form
- There may be several strategies for effective application of the derivation rules

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- Complex sentences can be formed using connectives
  - Negation ($\neg$) : $\neg W$
  - Disjunction ($\vee$) : $\neg W \vee H$
  - Conjunction ($\wedge$) : $(\neg W \vee H) \wedge W$
  - Implication ($\Rightarrow$) : $P \Rightarrow W$
  - Bi-conditional ($\Leftrightarrow$) : $P \Leftrightarrow W$

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- Sometimes, we write $\neg p_2$ as $\overline{p_2}$

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P	Q	$\neg P$	$P \wedge Q$	$P \vee Q$	$P \Rightarrow Q$	$P \Leftrightarrow Q$
F	F	T	F	F	T	T
F	T	T	F	T	T	F
T	F	F	F	T	F	F
T	T	F	T	T	T	T

Well-Formed Formula

- An atom by itself is a well-formed formula (WFF)
- If F is a WFF, then $\neg F$ is a WFF
- If F is a WFF, then (F) is a WFF
- If F_1 and F_2 are WFF, then
  - $F_1 \vee F_2$ is a WFF
  - $F_1 \wedge F_2$ is a WFF
  - $F_1 \Rightarrow F_2$ is a WFF
  - $F_1 \Leftrightarrow F_2$ is a WFF
- Nothing else if a WFF

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- The above sentence is same as

$$(((\neg P) \vee (Q \wedge R)) \Rightarrow S)$$

Grammar of sentences in propositional logic

$$
\begin{array}{rcl}
Sentence & \rightarrow & AtomicSentence \mid ComplexSentence \\
AtomicSentence & \rightarrow & True \mid False \mid P \mid Q \mid R \mid \ldots \\
ComplexSentence & \rightarrow & (\ Sentence\ ) \\
& \mid & \neg\ Sentence \\
& \mid & Sentence \wedge Sentence \\
& \mid & Sentence \vee Sentence \\
& \mid & Sentence \;\Rightarrow\; Sentence \\
& \mid & Sentence \;\Leftrightarrow\; Sentence \\
\\
\text{OPERATOR PRECEDENCE} & : & \neg, \wedge, \vee, \Rightarrow, \Leftrightarrow
\end{array}
$$

Example

- Represent the following in propositional logic

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- Student can not pass the exam unless she works hard. Student can not work hard unless she is healthy. Student works hard.

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- Let S be $(P \Rightarrow Q) \wedge P \Rightarrow Q$. Is it a WFF?

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- A model of a formula S is an interpretation in which F evaluates to true
- Example: Is I_1 a model of S?

Semantics and Truth Tables

- Each row in the truth table corresponds to an interpretation (the first two columns)
- Rows in which the formula S evaluates to “True” (last column is “True”) corresponds to a model
- In this example, all the four interpretations are models!

P	Q	$P \Rightarrow Q$	$(P \Rightarrow Q) \land P$	$(P \Rightarrow Q) \land P \Rightarrow Q$
F	F	T	F	T
F	T	T	F	T
T	F	F	F	T
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Another Example

- Consider the following formula S

$$(P_{13} \vee P_{22}) \wedge \neg P_{22} \wedge \neg P_{31}$$

Another Example

- Consider the following formula S

$$(P_{13} \lor P_{22}) \land \neg P_{22} \land \neg P_{31}$$

- Interpretations and models of S can be captured through a truth table

P_{13}	P_{22}	P_{31}	$\neg P_{22}$	$\neg P_{31}$	$P_{13} \lor P_{22}$	S
F	F	F	T	T	F	F
F	F	T	T	F	F	F
F	T	F	F	T	T	F
F	T	T	F	F	T	F
T	F	F	T	T	T	T
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  - Is there a pit in cell $(2, 2)$? (Is P_{22} true?)
  - Is there a pit in cell $(1, 3)$?
- Check in the previous example:
  - Is the student healthy?
  - Does the student pass the exam?
- Given KB and α, we need to answer the question: Does KB entail α? ($KB \models^{?} \alpha$)

Properties of Sentences

- A sentence S is satisfiable if it has a model (unsatisfiable if it has no model)

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- A sentence S is satisfiable if it has a model (unsatisfiable if it has no model)
- S is a valid sentence (tautology) if it is true in all the interpretations (invalid if it is false in at least one interpretation)
- Two sentences S_1 and S_2 are equivalent if they have same set of models

Some Popular Equivalences

$$\begin{array}{lcll}
(\alpha \wedge \beta) & \equiv & (\beta \wedge \alpha) & \text{commutativity of } \wedge \\
(\alpha \vee \beta) & \equiv & (\beta \vee \alpha) & \text{commutativity of } \vee
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((\alpha \wedge \beta) \wedge \gamma) & \equiv & (\alpha \wedge (\beta \wedge \gamma)) & \text{associativity of } \wedge \\
((\alpha \vee \beta) \vee \gamma) & \equiv & (\alpha \vee (\beta \vee \gamma)) & \text{associativity of } \vee
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\neg(\neg\alpha) &\equiv \alpha \quad \text{double-negation elimination}
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Deduction Theorems

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Deduction Theorem II

α is a logical consequence of KB if and only if $KB \wedge \neg\alpha$ is unsatisfiable

The Satisfiability Problem

SAT

Given a well formed formula S over a set of propositions $X = \{x_1, \ldots, x_n\}$, decide whether S is satisfiable (that is, decide if there exists an assignment of truth values to the propositions such that S evaluates to true)

Derivations

- SAT is a *NP*-Complete problem and algorithms based on model checking do not scale up
- Alternatively, we can check if α can be derived from *KB*, using some syntactic inference rules
- Inference procedure is sound if every α derivable is entailed by *KB*
- Inference procedure is complete if every α that is entailed by *KB* can be derived from *KB*
- When we have a sound and complete inference procedure, $KB \models^{?} \alpha$ can be reduced to $KB \vdash^{?} \alpha$

Questions?