UIT2504 Artificial Intelligence

Derivations in Propositional Logic



Logic in General

- Sentences written in logic must be well-formed formula and follow a grammar
- There are several possible interpretations for a set of sentences KB
- Interpretations in which KB evaluates to true are called as models of KB
- Given a new sentence α , KB logically entails α (written as KB $\models \alpha$) iff every model of KB is also a model of α
- We write $KB \vdash \alpha$ if α can be derived from KB using syntactic derivation rules
- Sentences in logic are usually written in a normal form
- There may be several strategies for effective application of the derivation rules

Derivations in Logic

• KB $\vdash \alpha$ if α can be derived from KB using syntactic inference rules

$$x + y = 4$$

 $x + y - y = 4 - y$
 $x = 4 - y$

- Inference procedure is sound if every α derivable is entailed by KB
- Inference procedure is complete if every α that is entailed by KB can be derived from KB
- When we have sound and complete inference procedure, $KB \stackrel{?}{=} \alpha$ can be reduced to $KB \stackrel{?}{=} \alpha$



Properties of Sentences

- A sentence KB is satisfiable if it has a model
 - (unsatisfiable if it has no model)
- KB is a valid sentence (tautology) if it is true in all the interpretations (invalid if it is false in at least one interpretation)
- Two sentences KB₁ and KB₂ are equivalent if they have same set of models



Some Popular Equivalences

$$(\alpha \wedge \beta) \equiv (\beta \wedge \alpha) \quad \text{commutativity of } \wedge \\ (\alpha \vee \beta) \equiv (\beta \vee \alpha) \quad \text{commutativity of } \vee \\ ((\alpha \wedge \beta) \wedge \gamma) \equiv (\alpha \wedge (\beta \wedge \gamma)) \quad \text{associativity of } \wedge \\ ((\alpha \vee \beta) \vee \gamma) \equiv (\alpha \vee (\beta \vee \gamma)) \quad \text{associativity of } \vee \\ \neg(\neg \alpha) \equiv \alpha \quad \text{double-negation elimination} \\ (\alpha \Rightarrow \beta) \equiv (\neg \beta \Rightarrow \neg \alpha) \quad \text{contraposition} \\ (\alpha \Rightarrow \beta) \equiv (\neg \alpha \vee \beta) \quad \text{implication elimination} \\ (\alpha \Leftrightarrow \beta) \equiv ((\alpha \Rightarrow \beta) \wedge (\beta \Rightarrow \alpha)) \quad \text{biconditional elimination} \\ \neg(\alpha \wedge \beta) \equiv (\neg \alpha \vee \neg \beta) \quad \text{de Morgan} \\ \neg(\alpha \vee \beta) \equiv (\neg \alpha \wedge \neg \beta) \quad \text{de Morgan} \\ (\alpha \wedge (\beta \vee \gamma)) \equiv ((\alpha \wedge \beta) \vee (\alpha \wedge \gamma)) \quad \text{distributivity of } \wedge \text{ over } \vee \\ (\alpha \vee (\beta \wedge \gamma)) \equiv ((\alpha \vee \beta) \wedge (\alpha \vee \gamma)) \quad \text{distributivity of } \vee \text{ over } \wedge \\ \end{pmatrix}$$

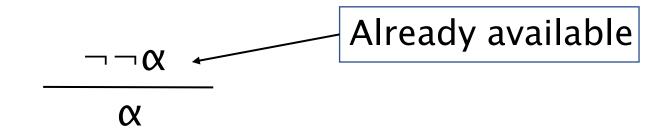


- Given a set of sentences KB, new sentences can be derived using inference rules
- Equivalences shown before can be used as inference rules
- Example: Double Negation Elimination

$$\frac{\neg\neg\alpha}{\alpha}$$

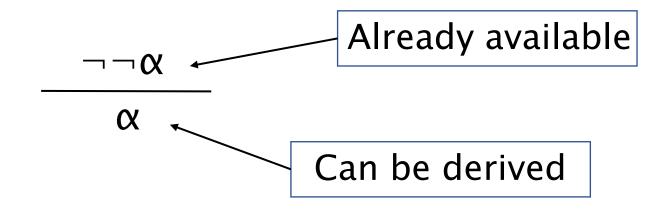


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Modus Ponens (⇒-Elimination)

$$\frac{P \Rightarrow Q}{Q}$$



Modus Ponens (⇒-Elimination)

$$\frac{P \Rightarrow Q}{O}$$

• ∧ – elimination and ∧ – intro

$$\frac{\mathsf{P} \wedge \mathsf{Q}}{\mathsf{P}}$$

$$\frac{P \wedge Q}{Q}$$

$$\frac{P}{P \wedge Q}$$



• ⇔-elim and ⇔-intro

$$\frac{\mathsf{P} \Leftrightarrow \mathsf{Q}}{\mathsf{P} \Rightarrow \mathsf{Q}}$$

$$\frac{\mathsf{P} \Leftrightarrow \mathsf{Q}}{\mathsf{Q} \Rightarrow \mathsf{P}}$$

$$\begin{array}{ccc} P \Rightarrow Q & Q \Rightarrow P \\ \hline P \Leftrightarrow Q & \end{array}$$

• ⇔-elim and ⇔-intro

$$\frac{\mathsf{P} \Leftrightarrow \mathsf{Q}}{\mathsf{P} \Rightarrow \mathsf{Q}}$$

$$\frac{P \Leftrightarrow Q}{Q \Rightarrow P}$$

$$\begin{array}{ccc} P \Rightarrow Q & Q \Rightarrow P \\ \hline P \Leftrightarrow Q & \end{array}$$

•∨-intro and ∨-elim

$$\frac{\mathsf{P}}{\mathsf{P} \vee \mathsf{Q}}$$

$$P \vee Q$$

 $P \vee Q$ Assume P, Derive R Assume Q, Derive R

R



• ⇒-intro

$$P \Rightarrow Q$$



• ⇒-intro

Assume P and derive Q
$$P \Rightarrow Q$$

• ¬-intro

Assume P and derive contradiction

$$\neg P$$



• ⇒-intro

Assume P and derive Q
$$P \Rightarrow Q$$

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• ¬-elim

Assume ¬P and derive contradiction

P



• ⇒-intro

Assume P and derive Q
$$P \Rightarrow Q$$

• ¬-intro

Assume P and derive contradiction

$$\neg P$$

• ¬-elim

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Assume ¬P and derive contradiction

P

Contradiction



- Given $(P \Rightarrow W)$, $(W \Rightarrow H)$, W
- Derive H



- Given $(P \Rightarrow W)$, $(W \Rightarrow H)$, W
- Derive H

1.
$$P \Rightarrow W$$

 $2. W \Rightarrow H$

3. W

Premise

Premise

Premise



- Given $(P \Rightarrow W)$, $(W \Rightarrow H)$, W
- Derive H

1.
$$P \Rightarrow W$$

 $2. W \Rightarrow H$

3. W

4. H

Premise

Premise

Premise

Modus Ponens, 2, 3



- Given $(P \Rightarrow Q)$, $(Q \Rightarrow R)$
- Derive $P \Rightarrow R$



- Given $(P \Rightarrow Q)$, $(Q \Rightarrow R)$
- Derive $P \Rightarrow R$

1.
$$P \Rightarrow Q$$

$$2. Q \Rightarrow R$$

Premise

Premise



- Given $(P \Rightarrow Q)$, $(Q \Rightarrow R)$
- Derive $P \Rightarrow R$

1.
$$P \Rightarrow Q$$

Premise

$$2. Q \Rightarrow R$$

Premise

3. Assume P, Derive R

4.
$$P \Rightarrow R$$

$$\Rightarrow$$
-intro, 3



- Given $(P \Rightarrow Q)$, $(Q \Rightarrow R)$
- Derive $P \Rightarrow R$

1.
$$P \Rightarrow Q$$

Premise

$$2. Q \Rightarrow R$$

Premise

3. Assume P, Derive R

3.1 P

Assumption

4.
$$P \Rightarrow R$$



- Given $(P \Rightarrow Q)$, $(Q \Rightarrow R)$
- Derive $P \Rightarrow R$

1.
$$P \Rightarrow Q$$
 Premise

2.
$$Q \Rightarrow R$$
 Premise

4.
$$P \Rightarrow R$$

$$\Rightarrow$$
-intro, 3



- Given $(P \Rightarrow Q)$, $(Q \Rightarrow R)$
- Derive $P \Rightarrow R$

1.
$$P \Rightarrow Q$$
 Premise

2.
$$Q \Rightarrow R$$
 Premise

4.
$$P \Rightarrow R \Rightarrow -intro, 3$$



- Given $(P \Rightarrow Q)$
- Derive $\neg (P \land \neg Q)$



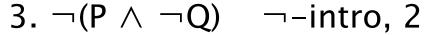
- Given $(P \Rightarrow Q)$
- Derive $\neg (P \land \neg Q)$

1.
$$P \Rightarrow Q$$

Premise

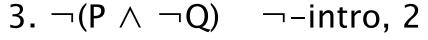


- Given $(P \Rightarrow Q)$
- Derive $\neg (P \land \neg Q)$
- 1. $P \Rightarrow Q$ Premise
- 2. Assume (P \wedge \neg Q), Derive Contradiction





- Given $(P \Rightarrow Q)$
- Derive $\neg (P \land \neg Q)$
- 1. $P \Rightarrow Q$ Premise
- 2. Assume (P \wedge \neg Q), Derive Contradiction
- 2.1 P \wedge \neg Q Assumption





- Given $(P \Rightarrow Q)$
- Derive $\neg (P \land \neg Q)$
- 1. $P \Rightarrow Q$

Premise

- 2. Assume (P \wedge \neg Q), Derive Contradiction
- 2.1 P \wedge \neg Q Assumption
- 2.2 P

∧-elim, 2.1





- Given $(P \Rightarrow Q)$
- Derive $\neg (P \land \neg Q)$
- 1. $P \Rightarrow Q$

Premise

- 2. Assume (P \wedge \neg Q), Derive Contradiction
- 2.1 P \wedge \neg Q Assumption
- 2.2 P ∧-elim, 2.1
- 2.3 ¬Q ∧-elim, 2.1



- Given $(P \Rightarrow Q)$
- Derive $\neg (P \land \neg Q)$
- 1. $P \Rightarrow Q$

Premise

- 2. Assume (P \wedge \neg Q), Derive Contradiction
- 2.1 P \wedge \neg Q Assumption
- 2.2 P ∧-elim, 2.1
- 2.3 ¬Q ∧-elim, 2.1
- 2.4 Q Modus Ponens, 1, 2.2

3.
$$\neg (P \land \neg Q) \quad \neg -intro, 2$$



- Given $(P \Rightarrow Q)$
- Derive $\neg (P \land \neg Q)$
- 1. $P \Rightarrow Q$

Premise

- 2. Assume (P \wedge \neg Q), Derive Contradiction
- 2.1 P \wedge \neg Q Assumption
- 2.2 P

 \wedge -elim, 2.1

2.3 ¬Q

∧-elim, 2.1

2.4 Q

Modus Ponens, 1, 2.2

2.5

Contradiction, 2.3, 2.4

3. $\neg (P \land \neg Q) \quad \neg \text{-intro}, 2$

Normal Forms

- Disjunctive normal form
 - Disjunction of conjunction of literals

$$(P \wedge Q) \vee (\neg Q \wedge R) \vee (\neg S \wedge T)$$



Normal Forms

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$$(P \wedge Q) \vee (\neg Q \wedge R) \vee (\neg S \wedge T)$$

- Conjunctive normal form
 - Conjunction of disjunction of literals

$$(\neg P \lor Q) \land (P \lor R) \land (S \lor Q)$$



Normal Forms

- Disjunctive normal form
 - Disjunction of conjunction of literals

$$(P \wedge Q) \vee (\neg Q \wedge R) \vee (\neg S \wedge T)$$

- Conjunctive normal form
 - Conjunction of disjunction of literals

$$(\neg P \lor Q) \land (P \lor R) \land (S \lor Q)$$

 Any well-formed sentence can be transformed into its equivalent CNF



Conversion to CNF

- Eliminate ⇔
 - $P \Leftrightarrow Q \text{ is same as } (P \Rightarrow Q) \land (Q \Rightarrow P)$
- Eliminate ⇒
 - $P \Rightarrow Q$ is same as $\neg P \lor Q$
- Move ¬ close to atoms
 - Use double-negation elimination and de Morgan laws
- Use the distributive laws to rearrange the sentences in CNF



CNF: Example

Example: $B_{1,1} \Leftrightarrow (P_{1,2} \lor P_{2,1})$



CNF: Example

Example:
$$B_{1,1} \Leftrightarrow (P_{1,2} \vee P_{2,1})$$

 $(B_{1,1} \Rightarrow (P_{1,2} \vee P_{2,1})) \wedge ((P_{1,2} \vee P_{2,1}) \Rightarrow B_{1,1})$
 $(\neg B_{1,1} \vee P_{1,2} \vee P_{2,1}) \wedge (\neg (P_{1,2} \vee P_{2,1}) \vee B_{1,1})$

$$(\neg B_{1,1} \lor P_{1,2} \lor P_{2,1}) \land ((\neg P_{1,2} \land \neg P_{2,1}) \lor B_{1,1})$$

$$(\neg B_{1,1} \lor P_{1,2} \lor P_{2,1}) \land (\neg P_{1,2} \lor B_{1,1}) \land (\neg P_{2,1} \lor B_{1,1})$$



Clauses

- A clause is a disjunction of literals
- A sentence in conjunctive normal form (conjunction of clauses) can be expressed as a set of clauses

$$KB = {\neg P \lor Q, P}$$

- A Horn Clause is a clause with at most one positive literal
- A Horn clause with exactly one positive literal is a definite clause
- A clause may also be written in implicative form. For example,

$$KB = \{ P \Rightarrow Q, P \}$$



Questions?



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