Solution to HW 9, Problem V.1

Al-Naji, Nader

Give an asymptotic expression for the number of strings that do not contain the pattern 0000000001. Do the same for 0101010101.

Solution:
Every pattern, p, has an autocorrelation polynomial. To get it, we slide the pattern to the left over itself one step at a time. Then, for each match of the i trailing bits with the leading bits, include a term $z^{|p|-i}$. This is illustrated on slide 15 of the applications lecture on the book site. We illustrate it again here for completeness for the two strings given in the problem:

$$
\begin{aligned}
&0000000001\\
z^0 &:0000000001\\
z^1 &:000000001\\
z^2 &:00000001\\
z^3 &:0000001\\
z^4 &:000001\\
z^5 &:00001\\
z^6 &:0001\\
z^7 &:001\\
z^8 &:01\\
z^9 &:1
\end{aligned}
$$

Of all of these substrings, we have a match only on the z^0 term and so the autocorrelation polynomial for 0000000001 is $c_{0000000001} = 1$.

For 0101010101 we have:

$$
\begin{aligned}
&0101010101\\
z^0 &: 0101010101\\
z^1 &: 101010101\\
z^2 &: 01010101\\
z^3 &: 1010101\\
z^4 &: 010101\\
z^5 &: 10101\\
z^6 &: 0101\\
z^7 &: 101\\
z^8 &: 01\\
z^9 &: 1
\end{aligned}
$$

Of all these substrings, we have a match only on the even powers of z, and so the autocorrelation polynomial for 0101010101 is $c_{0101010101} = 1 + z^2 + z^4 + z^6 + z^8$.

Now that we have the autocorrelation polynomials, we can compute the generating function and extract coefficients using complex asymptotics. From slide 17, we know the generating function for strings that do not contain a specified pattern p is given by:

$$
\begin{aligned}
S_p(z) &= \frac{c_p(z)}{z^{|p|} + (1-2z)c_p(z)}\\
\Rightarrow S_{0000000001}(z) &= \frac{1}{z^{10} + 1 - 2z}\\
\Rightarrow S_{0101010101}(z) &= \frac{1 + z^2 + z^4 + z^6 + z^8}{z^{10} + (1-2z)(1 + z^2 + z^4 + z^6 + z^8)}\\
&= \frac{1 + z^2 + z^4 + z^6 + z^8}{z^{10} - 2z^9 + z^8 - 2z^7 + z^6 - 2z^5 + z^4 - 2z^3 + z^2 - 2z + 1}
\end{aligned}
$$

Now, in order to extract coefficients, we simply use the transfer theorem for meromorphic GF's given on slide 3 of the applications lecture. Taking $\frac{f(z)}{g(z)}$ to be our generating function, we start by computing the dominant pole (smallest root of $g(z)$ in terms of absolute value). Then, we simply apply the straightforward transfer theorem. We do this below using

Mathematica to compute everything:

$$S_{0000000001}(z) = \frac{1}{z^{10}+1-2z}$$
$$f[z\_] := 1$$
$$g[z\_] := z^{10}+1-2z$$
$$alpha = Min[Abs[(z/.NSolve[g[z] == 0, z, 30])]]$$
$$h_{-1} = -f[x]/D[g[x], x]//.x- > \alpha$$
$$c = h_{-1}/\alpha$$
$$\beta = 1/\alpha$$
$$asymp[n\_] := c\beta^n$$

$$\begin{aligned}
\Rightarrow \alpha &= 0.500493118286552256059268459994\\
h_{-1} &= 0.504975344667395635312844243517 0\\
c &= 1.00895562040131214356311963833\\
\beta &= 1.99802947026228669866233146697\\
asymp_{0000000001}(n) &= 1.008955 \cdot 1.998029^n\\
[z^n]S_{0000000001}(z) &= 1.008955 \cdot 1.998029^n
\end{aligned}$$

$$S_{0101010101}(z) = \frac{1+z^2+z^4+z^6+z^8}{z^{10}-2z^9+z^8-2z^7+z^6-2z^5+z^4-2z^3+z^2-2z+1}$$

$$f[z\_] := 1+z^2+z^4+z^6+z^8$$
$$g[z\_] := z^{10}-2z^9+z^8-2z^7+z^6-2z^5+z^4-2z^3+z^2-2z+1$$
$$alpha = Min[Abs[(z/.NSolve[g[z] == 0, z, 30])]]$$
$$h_{-1} = -f[x]/D[g[x], x]//.x- > \alpha$$
$$c = h_{-1}/\alpha$$
$$\beta = 1/\alpha$$
$$asymp[n\_] := c\beta^n$$

$$
\begin{aligned}
\Rightarrow \alpha &= 0.500369104835149419780578378279 \\
h_{-1} &= 0.50346949376019440664350466737 \\
c &= 1.00619620375256069146534072410 \\
\beta &= 1.99852466976245056415537078735 \\
asymp_{0101010101}(n) &= 1.0061962 \cdot 1.99852466^n \\
[z^n]S_{0101010101}(z) &\sim 1.0061962 \cdot 1.99852466^n
\end{aligned}
$$

Solution to HW 9, Problem V.2

Al-Naji, Nader

Give asymptotic expressions for the number of objects of size N and the number of parts in a random object of size N for the following classes: compositions of 1s, 2s, and 3s, triple surjections, and alignments with no singleton cycles.

Solution:
A composition is the number of ways to express n as a sum of integers. When dealing with all integers, from slide 39, this class is clearly expressed by $C = SEQ(I)$ where $I = SEQ_{>0}(Z)$. Restricting ourselves to the numbers 1, 2, and 3 then trivially yields $C_3 = SEQ(G)$ where $G = Z + Z^2 + Z^3$. We can then use the theorem from slide 51 to get coefficients:

$$[z^n]F(z) \sim \frac{1}{G'(\lambda)} \frac{1}{\lambda^{n+1}}$$

where λ is the root of $G(\lambda) = 1$. The following Mathematica code can be used to compute the values for any such function:

$$g(\text{z\_}) := z^3 + z^2 + z$$
$$f(\text{z\_}) := \frac{1}{1 - g(z)}$$
$$\lambda := \min(|z/.\,\text{NSolve}[g(z) = 1, z, 30]|)$$
$$\text{asympCoeff}(\text{n\_}) := \frac{1}{\lambda^{n+1} \frac{\partial g(z)}{\partial z}} //.\, z \to \lambda$$
$$\text{asympParts}(\text{n\_}) := \frac{\frac{\partial^2 g(z)}{\partial z\, \partial z}}{\left(\frac{\partial g(z)}{\partial z}\right)^2} + \frac{n+1}{\lambda \frac{\partial g(z)}{\partial z}} - 1 //.\, z \to \lambda$$
$$\text{estCoeff}(\text{n\_}) := \frac{\text{asymp}(n)}{\text{SeriesCoefficient}[f(z), \{z, 0, n\}]} //.\, z \to 0$$
$$\text{FullSimplify}[\text{asympCoeff}(n)]$$
$$\text{FullSimplify}[\text{asympParts}(n)]$$

For compositions of 1s, 2s, and 3s, we have $G(z) = z + z^2 + z^3$ and therefore, using the code above:

$$\lambda = 0.5436890126920763615708559718 02$$
$$[z^n]C_3(z) = 0.6184199223193925509453304380 7e^{0.60937786343600623153680337117n}$$
$$\mu_n = 0.2133007416550358565426809809 6 + 0.6184199223193925509453304380 7n$$

We can apply this code to triple surjections. With triple surjections, from slide 29, we have $R_{>2} = SEQ(SET_{>2}(Z)) \Rightarrow G = SET_{>2}(Z) \Rightarrow G(z) = e^z - \frac{z^2}{2} - z - 1$. Then, plugging this into the transfer theorem directly yields:

$$\begin{aligned} \lambda &= 1.5681199923933759610150673324824310839176177978516 \\ [z^n]R_{>2}(z) &= 0.28603106376e^{-0.44987744476n} \\ \mu_n &= 0.0500366058163079053874 + 0.2860310637668502412n \end{aligned}$$

Finally, for alignments with no singleton cycles, from slide 33, we have $O_{>0} = SEQ(CYC_{>0}(Z)) \Rightarrow G = CYC_{>0}(Z) \Rightarrow G(z) = \ln \frac{1}{1-z} - 1$. Once again plugging into the code above yields:

$$\begin{aligned} \lambda &= 0.86466471676338730810600050503 \\ [z^n]O_{>0}(z) &= 0.15651764274966565181808062347e^{0.14541345786885905697264815010n} \\ \mu_n &= 0.15651764274966565181808062347(n+1) \end{aligned}$$

Solution to HW 9, Problem Programming 1

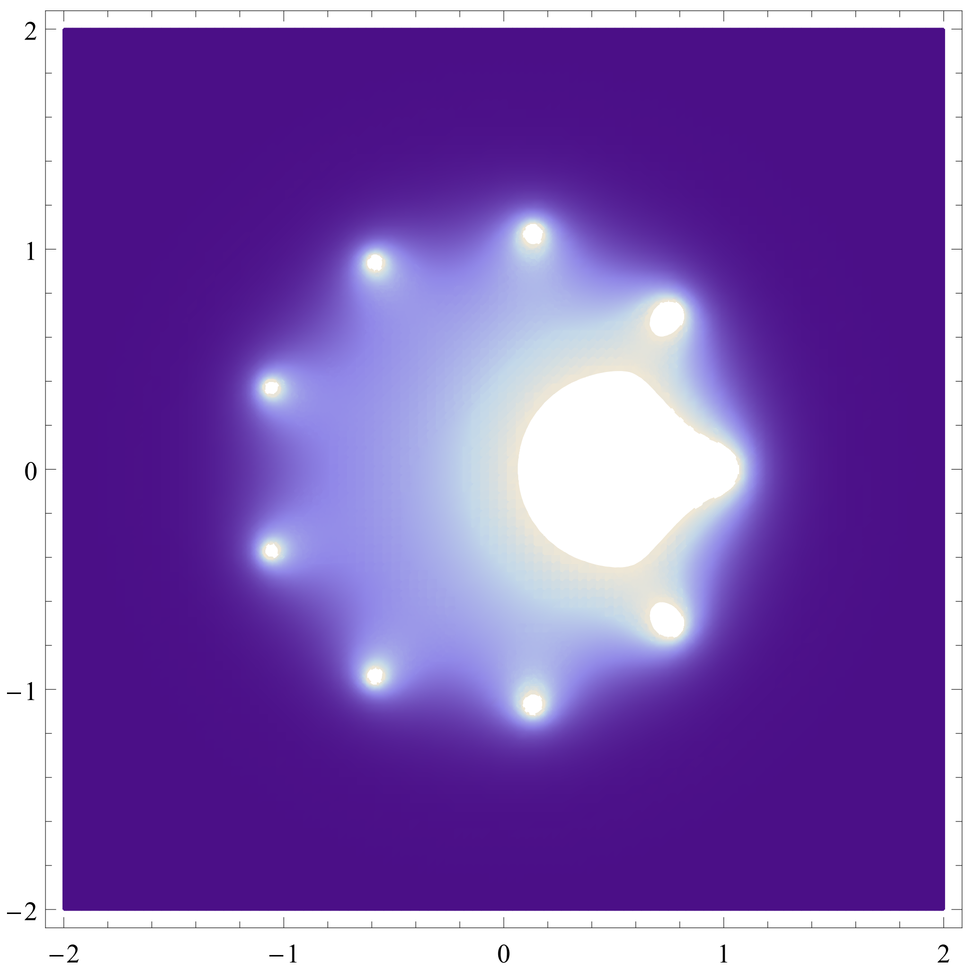
Al-Naji, Nader

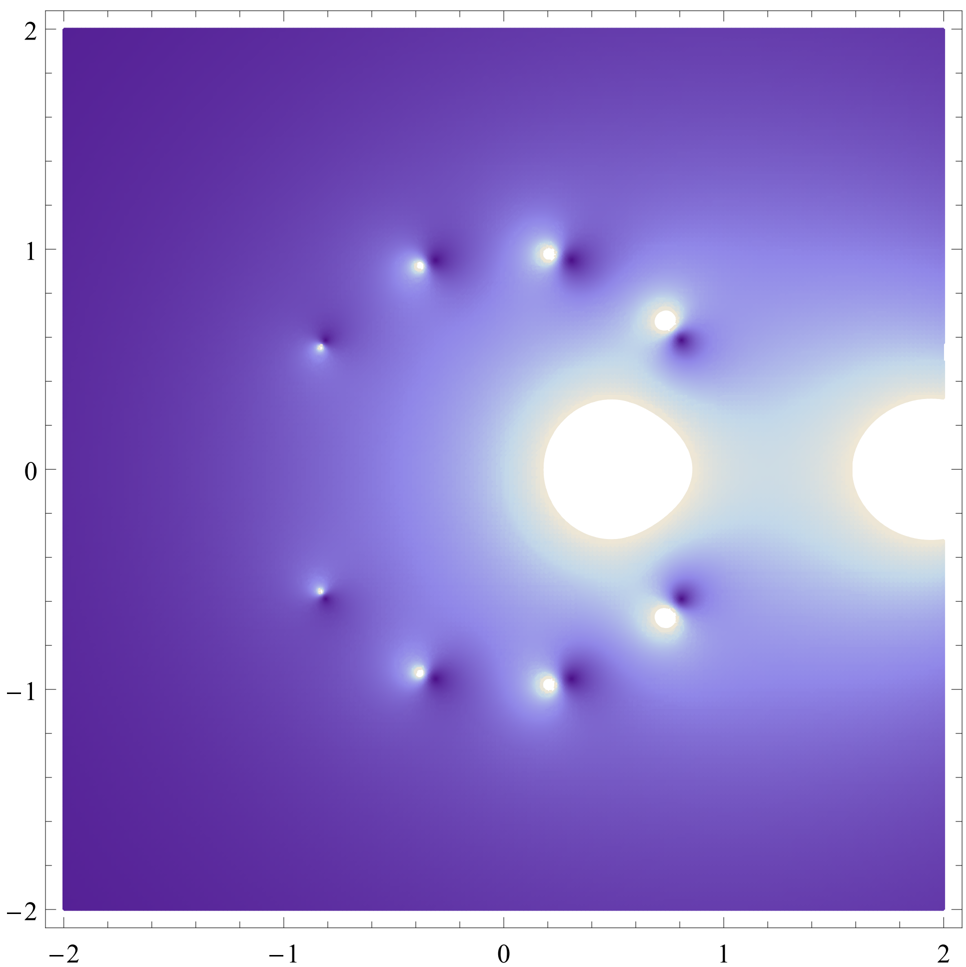
In the style of the plots in the lecture slides, plot the GF's for the set of bitstrings having no occurrence of the pattern 0000000001**. Do the same for** 0101010101**.**

Solution:
The following Mathematica code produces plots over the complex plane:

$$f[z\_] := SetPrecision[(1 + z^2 + z^4 + z^6 + z^8)/(z^(10) - 2z^9 + z^8$$
$$- 2z^7 + z^6 - 2z^5 + z^4 - 2z^3 + z^2 - 2z + 1), 60]$$
$$DensityPlot[Abs[f[x + Iy]], x, -2, 2, y, -2, 2, PlotPoints- > 200]$$

The following is the plot of the GF of the pattern 0000000001: The following is the plot of the GF of the pattern 0101010101:


2
1
0
−1
−2
−2
−1
0
1
2



2
1
0
−1
−2
−2
−1
0
1
2