COS 488 **Collaborator 1** : last name, first name
Spring 2013 **Collaborator 2** : last name, first name

Solution to HW 1, Problem 1.14

Al-Naji, Nader

Solve the recurrence:

$$A_N = 1 + \frac{2}{N} \sum_{1 \le j \le N} A_{j-1}$$

Solution:

$$
\begin{aligned}
A_N &= \frac{2}{N} \sum_{1 \le j \le N} A_{j-1} + 1 && \text{Original recurrence} \\
N A_N &= 2 \sum_{1 \le j \le N} A_{j-1} + N && \text{Multiply by N (1)} \\
A_{N-1} &= \frac{2}{N-1} \sum_{1 \le j \le N-1} A_{j-1} + 1 && \text{Recurrence for N-1} \\
(N-1)A_{N-1} &= 2 \sum_{1 \le j \le N-1} A_{j-1} + (N-1) && \text{Multiply by N-1 (2)} \\
N A_N - (N-1)A_{N-1} &= 2\Big(\sum_{1 \le j \le N} A_{j-1} - \sum_{1 \le j \le N-1} A_{j-1}\Big) + N - (N-1) && \text{(1) - (2)} \\
N A_N - (N-1)A_{N-1} &= 2A_{N-1} + 1 && \text{Simplifying} \\
N A_N &= (N+1)A_{N-1} + 1 && \text{Simplifying} \\
\frac{A_N}{N+1} &= \frac{A_{N-1}}{N} + \frac{1}{N(N+1)} && \text{Divide by } N(N+1) \\
\frac{A_N}{N+1} &= \frac{A_{N-2}}{N-1} + \frac{1}{(N-1)(N)} + \frac{1}{N(N+1)} && \text{Expanding} \\
\frac{A_N}{N+1} &= \frac{A_1}{2} + \sum_{2 \le k \le N} \frac{1}{(k)(k+1)} && \text{Notice the pattern} \\
\frac{A_N}{N+1} &= \frac{A_1}{2} + \sum_{2 \le k \le N} \frac{1}{k} - \sum_{2 \le k \le N} \frac{1}{k+1} && \text{Expamding} \\
\frac{A_N}{N+1} &= \frac{A_1}{2} + \sum_{2 \le k \le N} \frac{1}{k} - \sum_{3 \le k \le N+1} \frac{1}{k} && \text{Changing index.} \\
\frac{A_N}{N+1} &= \frac{A_1}{2} + \frac{1}{2} - \frac{1}{N+1} && \text{Simplifying} \\
A_N &= \frac{(N+1)}{2} A_1 + \frac{N-1}{2} && \text{Simplifying} \\
A_N &= \frac{N-1}{2} && \text{Final answer}
\end{aligned}
$$

Solution to HW 1, Problem 1.15

Al-Naji, Nader

Show that the number of exchanges used during the first partition stage (before the pointers cross) is $(N-2)/6$.

Solution:
Suppose the pivot is the $(i+1)^{th}$ largest element in the array, meaning its end position will be at position i if we zero-index our array. This implies that the first partitioning stage will end when the left (and right) pointer hits or just passes position i. Further, an exchange is made exactly when the left pointer passes over an element that is bigger than the pivot. Since the left pointer will pass over and attempt to exchange the elements at positions $[0,i)$, we now use linearity of expectation to compute the total number of elements larger than the pivot in this region and claim that this will be the total number of elements exchanged in this iteration. Let $I_{\{\text{element is bigger than the pivot}\}}[x]$ be one if an element at position x is bigger than the pivot and zero otherwise. But this is identically distributed for all $x \in [0,i)$ since all positions have the same probability of containing an element larger than the pivot by symmetry. Since there are $n-i-1$ possible elements larger than the pivot by assumption, and since there are $n-1$ total positions that aren't the pivot position, this random variable is therefore simply 1 with probability $\frac{n-i-1}{n-1}$ and zero with probability $\frac{i}{n-1}$. By linearity of expectation we therefore have that the number of elements bigger than the pivot in the region $x \in [0,i)$ is:

$$\begin{aligned}
&E[\text{number of elements bigger than the pivot in } [0,i) \mid \text{pivot is at position } i] \\
&= \sum_{0 \le x < i} I_{\{\text{element is bigger than the pivot}\}}[x] \\
&= \sum_{0 \le x < i} Pr[\text{element is bigger than the pivot}] \\
&= i \cdot Pr[\text{element is bigger than the pivot}] \\
&= i \frac{n-i-1}{n-1}.
\end{aligned}$$

We now have an expectation conditioned on the pivot being at position i. To complete our computation, we just need to marginalize this expectation over all possible values the position of the pivot could take. $Pr[\text{pivot is at position } i]$ is clearly $1/n$, since the pivot is chosen uniformly. Thus, computing the final expectation we get:

$$
\begin{aligned}
&E[\text{number of exchanges}]\\
&= E[\text{number of elements exchanged by the left pointer}]\\
&= E[\text{number of elements larger than pivot before the pivot's position in the sorted array}]\\
&= \sum_{0\le i<n} E[\#\text{ elements bigger than pivot in } [0,i) \mid \text{pivot is at pos } i] \cdot Pr[\text{pivot at pos } i]\\
&= \sum_{0\le i<n} i\frac{n-i-1}{n-1}\cdot\frac{1}{n}\\
&= \frac{1}{n-1}\sum_{0\le i<n} i - \frac{1}{n(n-1)}\sum_{0\le i<n} i^2 - \frac{1}{n(n-1)}\sum_{0\le i<n} i\\
&= \frac{1}{n}\sum_{0\le i<n} i - \frac{1}{n(n-1)}\sum_{0\le i<n} i^2\\
&= \frac{n-1}{2} - \frac{2n-1}{6)}\\
&= \frac{n-2}{6}.
\end{aligned}
$$

Solution to HW 1, Problem 1.17

Al-Naji, Nader

If we change the first line in the quicksort implementation to cut off to insertion sort when the array size is less than or equal to some constant M, then the total number of compares to sort N elements is described by the recurrence:

$$C_N = \begin{cases} N+1+\frac{1}{N}\sum_{1\le j\le N}(C_{j-1}+C_{N-j}), & \text{for} N > M \\ \frac{1}{4}N(N-1), & \text{for } N \le M \end{cases}$$

Solution:

$$\frac{C_{N>M}}{N+1} = \frac{C_{M+1}}{M+2} + 2\sum_{M+3\le k\le N+1}\frac{1}{k} \qquad \text{Derivation in the book (1)}$$

$$C_{M+1} = (M+1)+1+\frac{1}{M+1}\sum_{1\le j\le M+1}(C_{j-1}+C_{M+1-j}) \qquad \text{Definition of } C_{M+1}$$

$$C_{M+1} = (M+2)+\frac{2}{M+1}\sum_{0\le j\le M}C_j \qquad \text{Simplifying}$$

$$C_{M+1} = (M+2)+\frac{1}{2(M+1)}\sum_{0\le j\le M}j^2-j \qquad \text{Definition of } C_j$$

$$C_{M+1} = 2+\frac{1}{6}\cdot M(M+5) \qquad \text{Simplifying (2)}$$

$$\frac{C_{N>M}}{N+1} = \frac{2+\frac{M(M+5)}{6}}{M+2} + 2\sum_{M+3\le k\le N+1}\frac{1}{k} \qquad \text{Plugging (2) into (1)}$$

$$\frac{C_{N>M}}{N+1} = \frac{M(M+5)}{6(M+2)} + 2\sum_{M+2\le k\le N+1}\frac{1}{k} \qquad \text{Simplifying}$$

$$\frac{C_{N>M}}{N+1} = \frac{M(M+5)}{6(M+2)} + 2H_{N+1} - 2H_{M+1} \qquad \text{Simplifying}$$

$$C_{N>M} = (N+1)\left(\frac{M(M+5)}{6(M+2)} + 2H_N + \frac{2}{N+1} - 2H_M - \frac{2}{M+1}\right) \qquad \text{Simplifying}$$

$$C_{N\le M} = \frac{N(N-1)}{4} \qquad \text{Definition of } C_N$$

Thus, we have as our final answer:

$$C_N = \begin{cases} (N+1)\left(\frac{M(M+5)}{6(M+2)} + 2H_N + \frac{2}{N+1} - 2H_M - \frac{2}{M+1}\right) & \text{for} N > M \\ \frac{1}{4}N(N-1), & \text{for } N \le M \end{cases}$$

Solution to HW 1, Problem 1.18

Al-Naji, Nader

Here since we are discounting small terms relative to N, we asuume $N > M$.

$$C_N = (N+1)(\frac{M(M+5)}{6(M+2)} + 2H_N + \frac{2}{N+1} - 2H_M - \frac{2}{M+1}) \qquad \text{From before}$$

$$C_N = N(\frac{M(M+5)}{6(M+2)} + 2H_N + \frac{2}{N+1} - 2H_M - \frac{2}{M+1})$$
$$+ (\frac{M(M+5)}{6(M+2)} + 2H_N + \frac{2}{N+1} - 2H_M - \frac{2}{M+1}) \qquad \text{Expanding}$$

$$C_N \approx N(\frac{M(M+5)}{6(M+2)} + 2lnN + 2\gamma + \frac{2}{N+1} - 2H_M - \frac{2}{M+1})$$
$$+ (\frac{M(M+5)}{6(M+2)} + 2lnN + 2\gamma + \frac{2}{N+1} - 2H_M - \frac{2}{M+1}) \qquad H_N \text{ approx}$$

$$C_N \approx 2N\ln N + N(\frac{M(M+5)}{6(M+2)} + 2\gamma - \frac{2}{M+1} - 2H_M) + O(\ln N) \qquad \text{Simplifying}$$

$$f(M) = \frac{M(M+5)}{6(M+2)} + 2\gamma - \frac{2}{M+1} - 2H_M \qquad \text{From above (final answer)}$$

The following is a plot of f for varying values of M. The value of M that minimizes $f(M)$ is 10.0138, found using Mathematica.