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Solution to HW 2, Problem 2.17

Al-Naji, Nader

Solve the recurrence:

$$a_n = a_{n-1} - \frac{2a_{n-1}}{n} + 2(1 - \frac{2a_{n-1}}{n}) \text{ for } N > 0 \text{ with } a_0 = 0.$$

This recurrence describes the following random process: A set of N elements is collected into two-nodes and 3-nodes. At each step a 2-node is likely to turn into a 3-node with probability 2/n and each 3-node is likely to turn into two 2-nodes with probability 3/n. What is the average number of 2-nodes after n steps?

Solution:

First, note that when $n = 6$, the recurrence has no dependence on a_{n-1}, since all of the a_{n-1} terms cancel, leaving us with just $a_6 = 2$. Thus, in what follows we can (and will) define our base case to instead be $a_6 = 2$, rather than a_0.

$$a_n = a_{n-1} - \frac{2a_{n-1}}{n} + 2(1 - \frac{2a_{n-1}}{n}) \qquad \text{(defined)}$$

$$a_n = \frac{n-6}{n}a_{n-1} + 2 \qquad \text{(simplify; let } x_n = \frac{n-6}{n}\text{)}$$

$$\frac{a_n}{\frac{n-6}{n}\frac{n-7}{n-1}\cdots\frac{2}{8}\frac{1}{7}} = \frac{a_{n-1}}{\frac{n-7}{n-1}\cdots\frac{2}{8}\frac{1}{7}} + \frac{2}{\frac{n-6}{n}\frac{n-7}{n-1}\cdots\frac{2}{8}\frac{1}{7}} \qquad \text{(divide both sides by } x_n ... x_7\text{)}$$

$$a_n\binom{n}{6} = a_{n-1}\binom{n-1}{6} + 2\binom{n}{6} \qquad \text{(simplify)}$$

$$a_n\binom{n}{6} = a_6\binom{6}{6} + 2\binom{7}{6} + ... + 2\binom{n}{6} \qquad \text{(iterating)}$$

$$a_n\binom{n}{6} = 2\sum_{6\le i\le n}\binom{i}{6} = 2\binom{n+1}{7} \qquad \text{(simplify)}$$

$$a_n = 2\frac{\binom{n+1}{7}}{\binom{n}{6}} \qquad \text{(divide both sides by } \binom{n}{6}\text{)}$$

$$a_n = 2\frac{n+1}{7} \qquad \text{(simplify; true for } n > 6\text{)}$$

Now that we have an expression for a_n when $n > 6$, we simply define a piece-wise function to capture the behavior for a_n where $n \leq 6$.

$$a_n = \begin{cases} 0 & \text{if } n = 0 \\ 2 & \text{if } n = 1 \\ -2 & \text{if } n = 2 \\ 4 & \text{if } n = 3 \\ 0 & \text{if } n = 4 \\ 2 & \text{if } n = 5 \\ 2 & \text{if } n = 6 \\ 2^{\frac{n+1}{7}} & \text{if } n > 6 \end{cases}$$

Solution to HW 2, Problem 2.69

Al-Naji, Nader

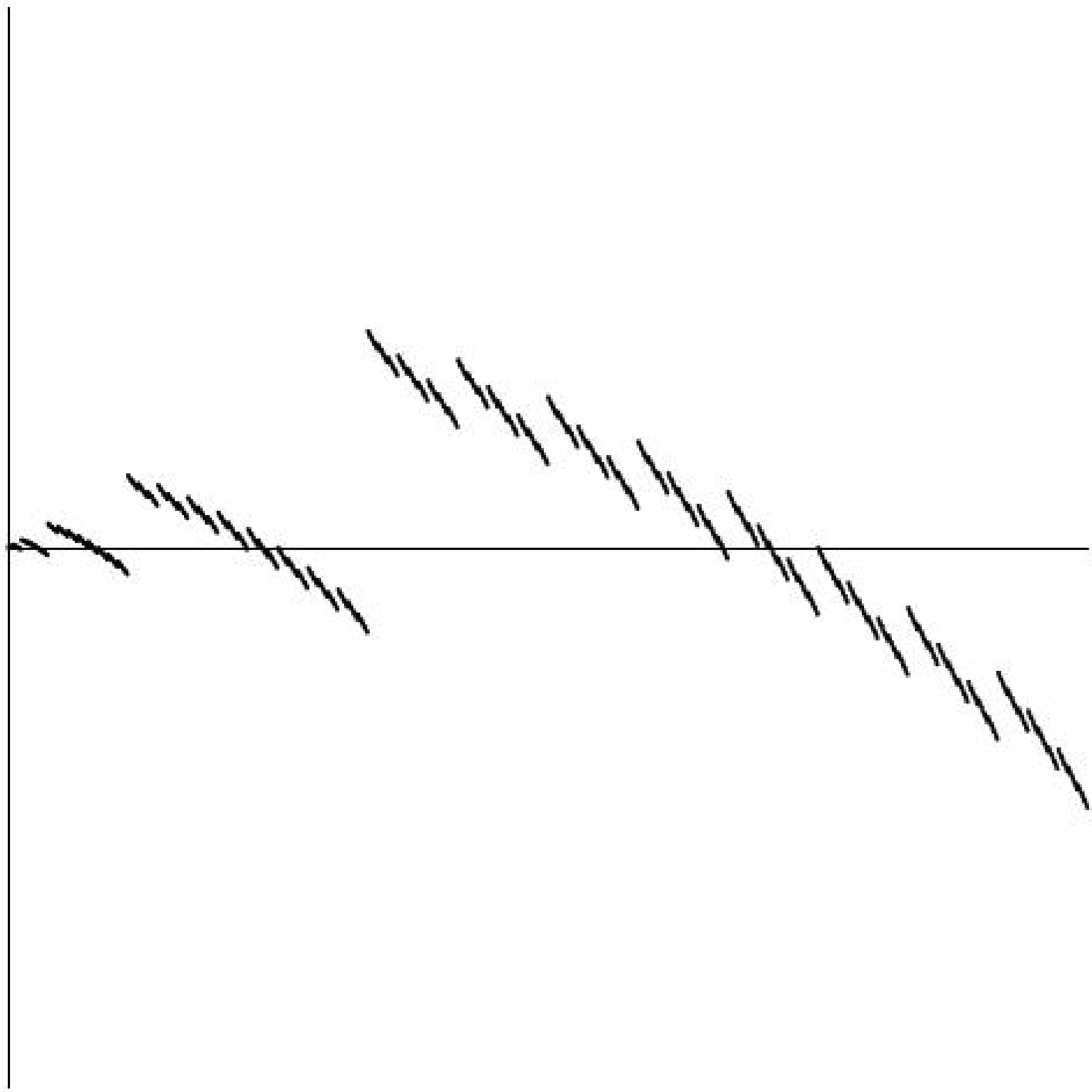
Plot the periodic part of the following recurrence for $1 \leq N \leq 972$:

$$a_N = 3a_{\lfloor N/3 \rfloor} + N \text{ for } N > 3 \text{ with } a_1 = a_2 = a_3 = 1$$

Solution:
We want to isolate the periodic part of our function and plot it without any drift, such that it oscillates about the x-axis. In order to figure out exactly how to subtract off the drift, we simply solve the recurrence in the naive case where $N = 3^n$ for some $n \geq 0$:

$$\begin{aligned}
a_{3^n} &= 3a_{3^{n-1}} + 3^n && \text{(by definition)}\\
a_{3^n} &= 3^2 a_{3^{n-2}} + 2 \cdot 3^n && \text{(iterate)}\\
a_{3^n} &= 3^k a_{3^{n-k}} + k \cdot 3^n && \text{(notice the pattern)}\\
a_{3^n} &= 3^{n-1} a_{3^1} + (n-1) \cdot 3^n && \text{(complete iteration)}\\
a_{3^n} &= \frac{N}{3} + Nlog_3 N - N && \text{(sub } n = log_3 N\text{)}\\
a_{3^n} &= Nlog_3 N - \frac{2N}{3} && \text{(simplify)}
\end{aligned}$$

Now, if we subtract this from the value of the recurrence for all n, we isolate the periodic part of our process. The graph of this is shown below with the horizontal line denoting the x-axis:



Solution to HW 2, Problem 3.20

Al-Naji, Nader

Solve the recurrence:

$$a_n = 3a_{n-1} - 3a_{n-2} + a_{n-3} \text{ for } n > 2 \text{ with } a_0 = a_1 = 0 \text{ and } a_2 = 1$$

Solve the same recurrence with the initial condition on a_1 changed to $a_1 = 1$

Solution:

$$\begin{aligned}
a_n &= 3a_{n-1} - 3a_{n-2} + a_{n-3} && \text{(definition)}\\
a_0 &= 3a_{-1} - 3a_{-2} + a_{-3} = 0 \rightarrow \delta_0 = 0 && \text{(follows since } a_0 = 0\text{)}\\
a_1 &= 3a_0 - 3a_{-1} + a_{-2} = 0 \rightarrow \delta_1 = 0 && \text{(follows since } a_1 = 0\text{)}\\
a_2 &= 3a_1 - 3a_0 + a_{-1} = 0 \rightarrow \delta_2 = 1 && \text{(follows since } a_2 = 1\text{)}\\
a_n z^n &= 3a_{n-1}z^n - 3a_{n-2}z^n + a_{n-3}z^n + \delta_2 z^n && \text{(multiply by } z^n\text{)}\\
\sum_{n\geq 0} a_n z^n &= 3z\sum_{n\geq 0} a_{n-1}z^{n-1} - 3z^2\sum_{n\geq 0} a_{n-2}z^{n-2} + z^3\sum_{n\geq 0} a_{n-3}z^{n-3} + \delta_2 z^2 && \text{(sum on both sides)}\\
A(z) &= 3zA(z) - 3z^2A(z) + z^3A(z) + z^2 && \text{(def of } A(z)\text{)}\\
A(z) &= \frac{z^2}{(1-z)^3} && \text{(simplify)}\\
\frac{z^2}{(1-z)^3} &= \sum_{n\geq 2}\binom{n}{2}z^n && \text{(table 3.1)}\\
a_n &= \binom{n}{2} \text{ for all } n > 2 && \text{(follows)}
\end{aligned}$$

Thus, as our final answer we have:

$$a_n = \begin{cases} 0 & \text{for } n = 0 \text{ and } n = 1\\ 1 & \text{for } n = 2\\ \binom{n}{2} & \text{for } n > 2 \end{cases}$$

We now perform the same analysis but with the base-case $a_1 = 1$:

$$\begin{aligned}
a_n &= 3a_{n-1} - 3a_{n-2} + a_{n-3} && \text{(definition)}\\
a_0 &= 3a_{-1} - 3a_{-2} + a_{-3} = 0 \rightarrow \delta_0 = 0 && \text{(follows since } a_0 = 0)\\
a_1 &= 3a_0 - 3a_{-1} + a_{-2} = 0 \rightarrow \delta_1 = 1 && \text{(follows since } a_1 = 1)\\
a_2 &= 3a_1 - 3a_0 + a_{-1} = 3(1) \rightarrow \delta_2 = -2 && \text{(follows since } a_2 = 1)\\
a_n z^n &= 3a_{n-1}z^n - 3a_{n-2}z^n + a_{n-3}z^n + \delta_1 z^n \delta_2 z^n && \text{(multiply by } z^n)\\
\sum_{n\geq 0} a_n z^n &= 3z \sum_{n\geq 0} a_{n-1}z^{n-1} - 3z^2 \sum_{n\geq 0} a_{n-2}z^{n-2} + z^3 \sum_{n\geq 0} a_{n-3}z^{n-3} + \delta_1 z + \delta_2 z^2 && \text{(sum on both sides)}\\
A(z) &= 3zA(z) - 3z^2 A(z) + z^3 A(z) + z - 2z^2 && \text{(def of } A(z))\\
A(z) &= \frac{z}{(1-z)^3} - \frac{2z^3}{(1-z)^3} && \text{(simplify)}\\
A(z) &= \frac{z}{(1-z)^2}\frac{1}{(1-z)} - \frac{2z^3}{(1-z)^3} && \text{(separating terms)}\\
A(z) &= \sum_{n\geq 0}(\sum_{0\leq k\leq n} k)z^n - 2\sum_{n\geq 0}\binom{n}{2} && \text{(tables 3.1 and 3.2)}\\
A(z) &= \sum_{n\geq 0}(\frac{n(n+1)}{2} - (n)(n-1))z^n && \text{(expanding)}\\
A(z) &= \sum_{n\geq 0}(\frac{n(3-n)}{2})z^n && \text{(simplifying)}\\
a_n &= \frac{n(3-n)}{2} \text{ for all } n > 2 && \text{(follows)}
\end{aligned}$$

Thus, as our final answer we have:

$$a_n = \begin{cases} 0 & \text{for } n = 0 \\ 1 & \text{for } n = 1 \text{ and } n = 2 \\ \frac{n(3-n)}{2} & \text{for } n > 2 \end{cases}$$

Solution to HW 2, Problem 3.28

Al-Naji, Nader

Find an expansion for:

$$[z^n]\frac{1}{\sqrt{1-z}}\ln\frac{1}{1-z}$$

Solution:
We begin by figuring out a general expression for the k^{th} derivative of $(1-z)^{-\alpha}$. We then use this to compute the Taylor expansion of this expression. Once we have this expansion (generating function), we differentiate both sides (using logarithmic differentiation to differentiate

the choose function) and arrive at our answer.

$$\begin{aligned}
\frac{d(1-z)^{-\alpha}}{dz} &= \alpha(1-z)^{-(\alpha+1)} && \text{(differentiate w.r.t. } z\text{)}\\
\frac{d^k(1-z)^{-\alpha}}{dz^k} &= \alpha(\alpha+1)...(\alpha+k-1)(1-z)^{-(\alpha+k)} && \text{(iterate)}\\
(1-z)^{-\alpha} &= \sum_{n\geq 0}\frac{\alpha...(\alpha+n-1)}{n!}z^n && \text{(Taylor expansion)}\\
(1-z)^{-\alpha} &= \sum_{n\geq 0}\binom{\alpha+n-1}{n}z^n && \text{(follows)}\\
[\ln(f)]' &= \frac{f'}{f} && \text{(true in general by chain rule)}\\
f' &= f[\ln(f)]' && \text{(follows; call this (1))}\\
\ln(\binom{\alpha+n-1}{n}) &= \ln(\alpha)+...+\ln(\alpha+n-1)-\ln n! && \text{(follows; call this (2))}\\
\frac{d\binom{\alpha+n-1}{n}}{d\alpha} &= \binom{\alpha+n-1}{n}(\frac{1}{\alpha}+...+\frac{1}{\alpha+n-1}) && \text{(combine (1) and (2))}\\
\frac{d\binom{\alpha+n-1}{n}}{d\alpha} &= \binom{\alpha+n-1}{n}(\sum_{0\leq i<n}\frac{1}{\alpha+i}) && \text{(simplify)}\\
\frac{d(1-z)^{-\alpha}}{d\alpha} &= \frac{1}{(1-z)^{-\alpha}}\ln\frac{1}{1-z} && \text{(take derivative of OGF)}\\
\frac{d(1-z)^{-\alpha}}{d\alpha} &= \sum_{n\geq 0}\binom{\alpha+n-1}{n}(\sum_{0\leq i<n}\frac{1}{\alpha+i})z^n && \text{(take derivative of expansion)}\\
\frac{1}{(1-z)^{-\alpha}}\ln\frac{1}{1-z} &= \sum_{n\geq 0}\binom{\alpha+n-1}{n}(\sum_{0\leq i<n}\frac{1}{\alpha+i})z^n && \text{(follows)}\\
[z^n]\frac{1}{(1-z)^{-\alpha}}\ln\frac{1}{1-z} &= \binom{\alpha+n-1}{n}(\sum_{0\leq i<n}\frac{1}{\alpha+i}) && \text{(by definition)}\\
[z^n]\frac{1}{\sqrt{(1-z)}}\ln\frac{1}{1-z} &= \binom{n-\frac{1}{2}}{n}(\sum_{0\leq i<n}\frac{1}{i+\frac{1}{2}}) && \text{(plug in } \alpha=\tfrac{1}{2}\text{; call this (3))}\\
\binom{n-\frac{1}{2}}{n} &= \frac{(n-\frac{1}{2})(n-\frac{3}{2})...(3/2)(1/2)}{n!} && \text{(def of binomial)}\\
\binom{n-\frac{1}{2}}{n} &= \frac{(2n-1)(2n-3)...(3)(1)}{2^k n!} && \text{(mult by} \tfrac{2^k}{2^k}\text{)}\\
\binom{n-\frac{1}{2}}{n} &= \frac{(2n-1)...(1)\cdot(2)(4)(6)...(2n)}{4^k n!n!} && \text{(mult by } \tfrac{2^k n!}{2^k n!}\text{)}\\
\binom{n-\frac{1}{2}}{n} &= \frac{1}{4^n}\binom{2n}{n} && \text{(simplify; call this (4))}
\end{aligned}$$

$$[z^n]\frac{1}{\sqrt{(1-z)}}\ln\frac{1}{1-z} = \frac{1}{4^n}\binom{2n}{n}\Big(\sum_{0\le i<n}\frac{1}{i+\frac{1}{2}}\Big) \qquad \text{(plug (4) into (3); call this (5))}$$

$$\sum_{0\le i<n}\frac{1}{i+\frac{1}{2}} = 2\sum_{0\le i<n}\frac{1}{2i+1} \qquad \text{(mult by } \tfrac{2}{2}\text{)}$$

$$\sum_{0\le i<n}\frac{1}{i+\frac{1}{2}} = 2\Big(\sum_{1\le i\le 2n}\frac{1}{i} - \frac{1}{2}\sum_{1\le i\le n}\frac{1}{i}\Big) \qquad \text{(expand the sum)}$$

$$\sum_{0\le i<n}\frac{1}{i+\frac{1}{2}} = 2H_{2n} - H_n \qquad \text{(def of Harmonic Numbers; call this (6))}$$

$$[z^n]\frac{1}{\sqrt{(1-z)}}\ln\frac{1}{1-z} = \frac{1}{4^n}\binom{2n}{n}(2H_{2n} - H_n) \qquad \text{(plug (6) into (5))}$$