COS 488 **Collaborator 1** : last name, first name
Spring 2013 **Collaborator 2** : last name, first name

Solution to HW 10, Problem VI.1 Written

Al-Naji, Nader

Use the standard function scale to directly derive an asymptotic expression for the number of strings in the following CGF:

$$\begin{aligned}S &= \epsilon + U \times Z_1 \times S + D \times Z_0 \times S \\ U &= Z_0 + U \times U \times Z_1 \\ D &= Z_1 + D \times D \times Z_0\end{aligned}$$

Solution:
Using the standard transfer theorems, we can immediatly write the GF's for D, U, and S:

$$\begin{aligned}U(z) &= z + zU(z)^2 \\ &= \frac{1-\sqrt{1-4z^2}}{2z} \\ D(z) &= z + zD(z)^2 \\ &= \frac{1-\sqrt{1-4z^2}}{2z} \\ S(z) &= 1 + zU(z)S(z) + zD(z)S(z) \\ &= \frac{1}{1 - zD(z) - zU(z)} \\ &= \frac{1}{\sqrt{1-4z^2}}\end{aligned}$$

Newton's generalized binomial theorem gives us:

$$\begin{aligned}\frac{1}{(1-z)^\alpha} &= \sum_{n\geq 0}\binom{n-1+\alpha}{n}z^n \\ &= \sum_{n\geq 0}\binom{n-1+\alpha}{\alpha-1}z^n\end{aligned}$$

Where we have, by definition:

$$\begin{aligned}\binom{n-1+\alpha}{\alpha-1} &= \frac{(n+1)(n+2)...(n+(\alpha-1))}{(\alpha-1)!} \\ &= n^{\alpha-1}\frac{(1+\frac{1}{n})(1+\frac{2}{n})...(1+\frac{\alpha-1}{n})}{(\alpha-1)!} \\ &\sim \frac{n^{\alpha-1}}{\Gamma(\alpha)}\end{aligned}$$

Now, applying all of this work to $S(z)$ yields:

$$\begin{aligned}
S(z) &= \frac{1}{\sqrt{1-4z^2}} \\
&= \sum_{n\geq 0} \binom{n-1+\frac{1}{2}}{\frac{1}{2}-1} (4z^2)^n && \text{(Applying Newton's formula)} \\
&= \sum_{n\geq 0, n \ even} 4^{\frac{n}{2}} \binom{\frac{n}{2}-1+\frac{1}{2}}{\frac{1}{2}-1} z^n && \text{(Changing indices)} \\
\Rightarrow [z^n]S(z) &= 4^{\frac{n}{2}} \binom{\frac{n}{2}-1+\frac{1}{2}}{\frac{1}{2}-1} \\
&\sim \frac{4^{\frac{n}{2}} \left(\frac{n}{2}\right)^{\frac{1}{2}-1}}{\Gamma(\frac{1}{2})} && \text{(Applying derived approximation)} \\
&\sim 4^{\frac{n}{2}} \sqrt{\frac{2}{\pi n}}
\end{aligned}$$

Note that this only holds for n even. When n is odd, we have $[z^n]S(z) = 0$.

Solution to HW 10, Problem VI.2 Written

Al-Naji, Nader

Give an asymptotic expression for the number of rooted ordered trees for which every node has 0, 2, or 3 children. How many bits are necessary to represent such a tree?

Solution:
The generating function for the number of rooted ordered trees for which every node has 0, 2, or 3 children is given by:

$$M = Z \times SEQ_{0,2,3}(M)$$
$$\Rightarrow M(z) = z(1 + M(z)^2 + M(z)^3)$$

Now, applying the theorem, we have:

$$M(z) = z\phi(M(z))$$
$$\Rightarrow \phi(u) = 1 + u^2 + u^3$$
$$\Rightarrow \lambda = 0.657298$$

where λ is the positive real root of $\phi(u) = u\phi'(u)$. Finally, plugging into the theorem yields:

$$[z^n]M(z) = \frac{n^{-3/2}\,(\phi'(\lambda))^n\,\sqrt{\phi(\lambda)}}{\sqrt{2\pi\phi''(\lambda)}}$$
$$= \frac{0.214358 \cdot 2.61072^n}{n^{3/2}}$$

Thus giving us the desired asymptotic expression for the number of trees of size n.

The number of bits needed to encode each tree can be expressed asymptotically by creating a bijection between trees and bitstrings. We can do this by mapping each tree to a bitstring where the first $\lg\lg([z^n]M(z))$ bits contain the tree size and the next $\lg([z^n]M(z))$ bits contain an index denoting *which* tree of size n we are referring to. This encoding unambiguously maps each tree to a bitstring and requires $\sim \lg([z^n]M(z)) \sim 0.959626n$ bits to encode a tree of size n.

The following is Mathematica that can be used to compute asymptotics of this type in general:

```
\[Phi][u_] := 1 + u^2 + u^3

M[z_] := m /. Solve[m == z \[Phi][m], m][[3]]

\[Lambda] :=
 Select[u /. Solve[\[Phi][u] == u D[\[Phi][u], u], u],
   Element[#, Reals] &][[1]]

coeff[n_] :=
 Sqrt[\[Phi][u]]/(Sqrt[2 Pi D[\[Phi][u], u, u] ]) (D[\[Phi][u], u])^
    n n^(-3/2) //. u -> \[Lambda]

ratio[n_] := Re[SeriesCoefficient[N[M[z]], {z, 0, n}]/N[coeff[n]]]

ratio[100]
```

Solution to HW 10, Problem VI.1 Programming

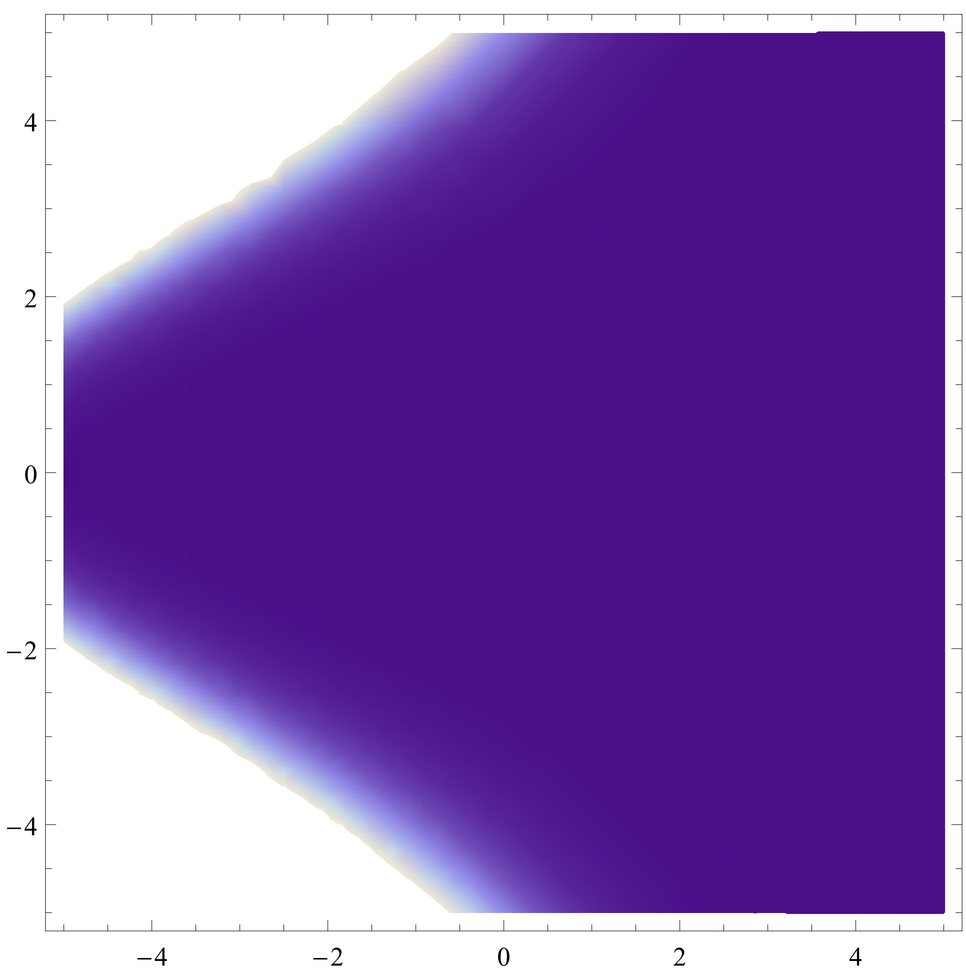
Al-Naji, Nader

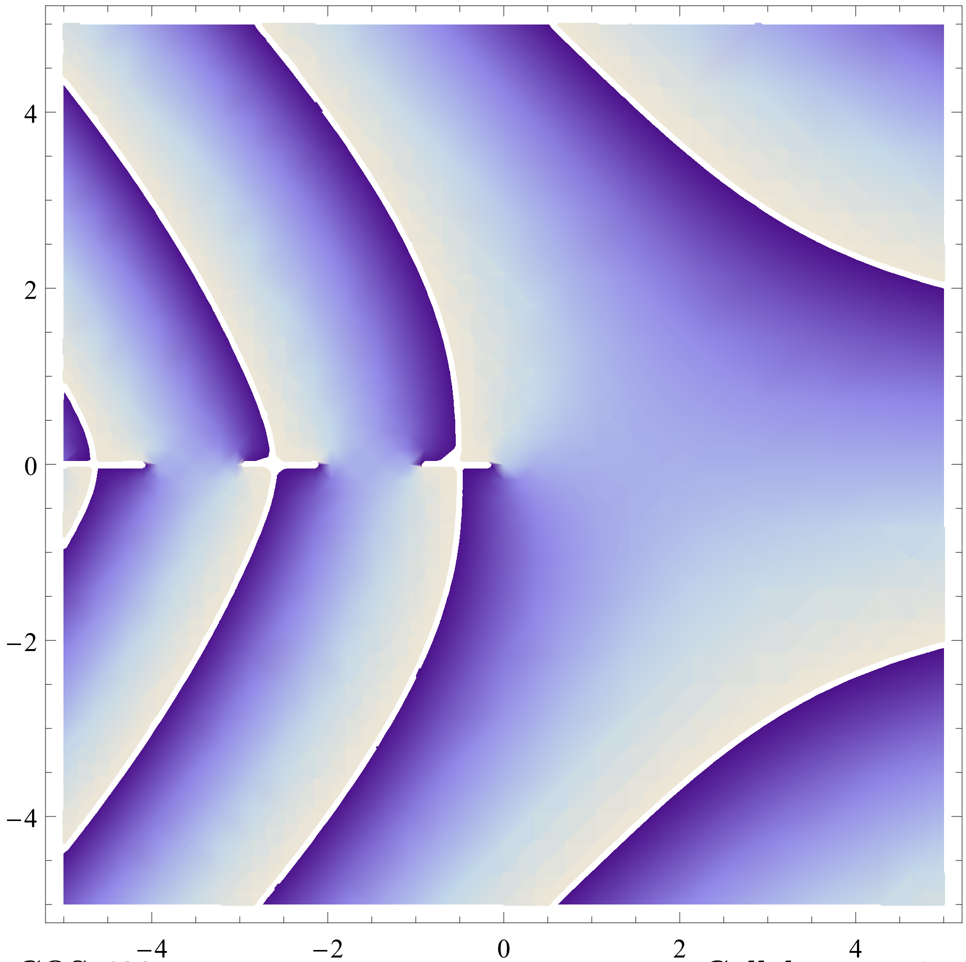
Do r and θ plots of $\frac{1}{\Gamma(z)}$ in the unit square of size 10 centered at the origin.

Solution: The following is Mathematica code used to produce the desired plots:

```
DensityPlot[Abs[1/Gamma[x + I y]],{x,-5,5},{y,-5,5}]
DensityPlot[Arg[1/Gamma[x + I y]],{x,-5,5},{y,-5,5}]
```

The following are the plots themselves, r first then θ:





COS 488
Spring 2013

Collaborator 1 : last name, first name
Collaborator 2 : last name, first name

Solution to HW 10, Problem VII.1 Written

Al-Naji, Nader

Use the tree-like schema to develop an asymptotic expression for the number of bracketings with n leaves. See example I.15 on page 69.

Solution:
The generating function for the number of bracketings of size n is given in the textbook on page 69 as:

$$S = Z + SEQ_{\geq 2}(S)$$
$$\Rightarrow S(z) = z + \frac{S(z)^2}{1 - S(z)}$$

To each bracketing of size n is associated with a tree whose external nodes correspond the variable being bracketed (and determine size), with internal nodes corresponding to bracketings and having degree at least 2 (while not contributing to size). Because of this correspondence, if we wish to count the number of bracketings with n leaves, all we need to do is count the number of bracketings of size n. We do this according to lecture slide 40:

$$\begin{aligned}
S(z) &= z + \frac{S(z)^2}{1-S(z)} \\
\Rightarrow \phi(z,w) &= z + \frac{w^2}{1-w}
\end{aligned}$$

Defining ϕ as such, we get the following characteristic equations:

$$\begin{aligned}
&\phi(r,s) = s \\
&\frac{\partial \varphi \phi(z,w)}{\partial w}(r,s) = 1 \\
&\Rightarrow r + \frac{s^2}{1-s} = s \\
&\Rightarrow \frac{s^2}{(1-s)^2} + \frac{2s}{1-s} = 1
\end{aligned}$$

We can then solve this system of equations for r and s using Mathematica. After doing so, all that's left is to compute α so that we can use the theorem from lecture slide 41:

$$\begin{aligned}
r &= 3 - 2\sqrt{2} \\
s &= \frac{1}{4}\left(4 - 2\sqrt{2}\right) \\
\alpha &= \sqrt{\frac{2r\frac{\partial \varphi(z,w)}{\partial z}(r,s)}{\frac{\partial^2 \varphi(z,w)}{\partial w\, \partial w}(r,s)}} \\
&= 0.246293
\end{aligned}$$

Finally, plugging these values into the theorem yields:

$$\begin{aligned}
[z^n]S(z) &= \frac{\alpha n^{-3/2}\left(\frac{1}{\mathrm{r}}\right)^n}{2\sqrt{\pi}} \\
&= \frac{0.06947793242304366 \cdot 5.82843^n}{n^{3/2}}
\end{aligned}$$

By the argument before, this asymptotic gives the number of bracketings with n leaves as desired.

The following is the Mathematica code that can be used to produce asymptotics for general tree-like schema:

```
(*Equation given on page 69 of AC and slide 46 of lecture*)
\[CurlyPhi][z_,s_]:=z+s^2/(1-s)

(*Solve for (r,s) according to slide 40 of lecture*)
Reduce[{\[CurlyPhi][r,s]==s,D[\[CurlyPhi][r,s],s]==1},{r,s}]

(*From solving for (r,s) above*)
r1=3-2 Sqrt[2]
s1=(1+r1)/4

(*Compute \[Alpha] from slide 41 of lecture*)
\[Alpha]:=Sqrt[2 r1 D[\[CurlyPhi][r,s],r]/D[\[CurlyPhi][r,s],s,s]]//.r->r1//.s->s1

(*Asymptotic coefficient*)
coeff[n_]:=\[Alpha]/(2 Sqrt[Pi]) (1/r1)^n n^(-3/2)
FullSimplify[coeff[n]]

(*Solve for GF explicitly and plot*)
S[z_]:=s/.Solve[\[CurlyPhi][z,s]-s==0,s][[1]]
DensityPlot[Abs[S[x + I y]],{x,-5,10},{y,-5,5}]
DensityPlot[Arg[S[x + I y]],{x,-5,10},{y,-5,5}]

(*Check correctness*)
ratio[n_]:=coeff[N[n]]/SeriesCoefficient[1/4 (1+z-Sqrt[1-6 z+z^2]),{z,0.0,n}]
ratio[100001]
```

Solution to HW 10, Problem VII.1 Programming

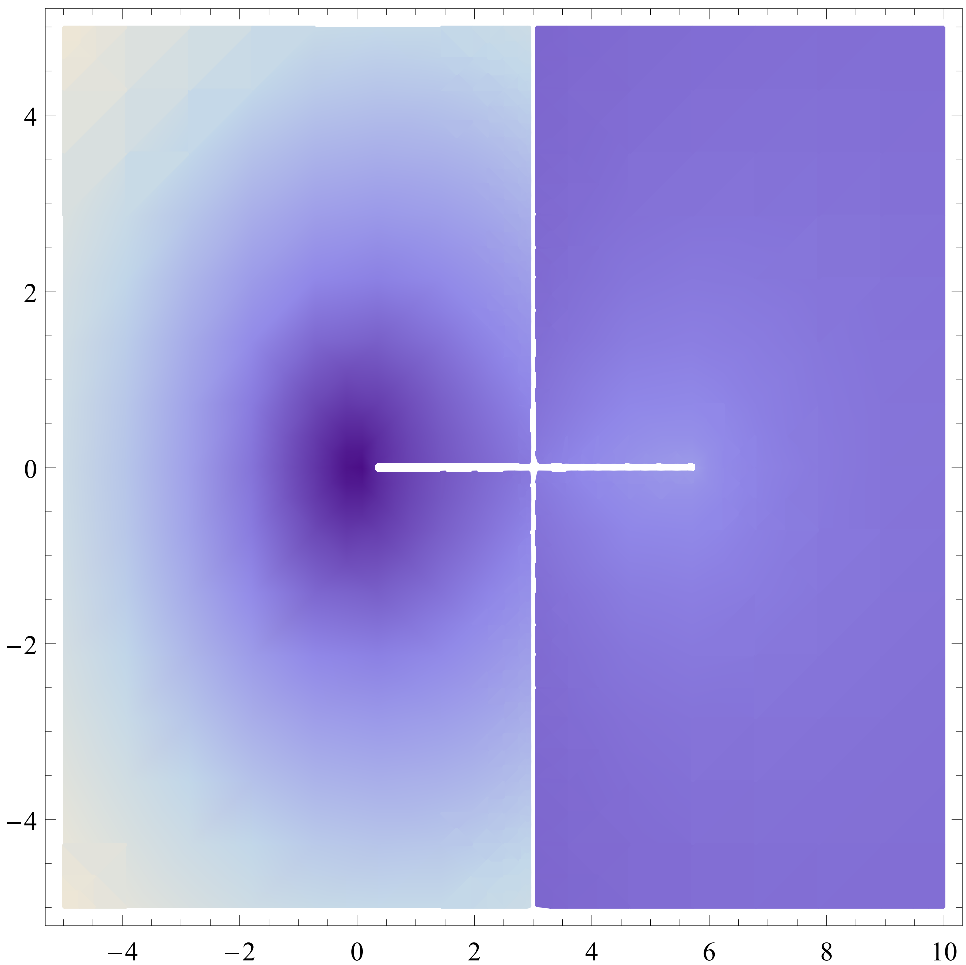
Al-Naji, Nader

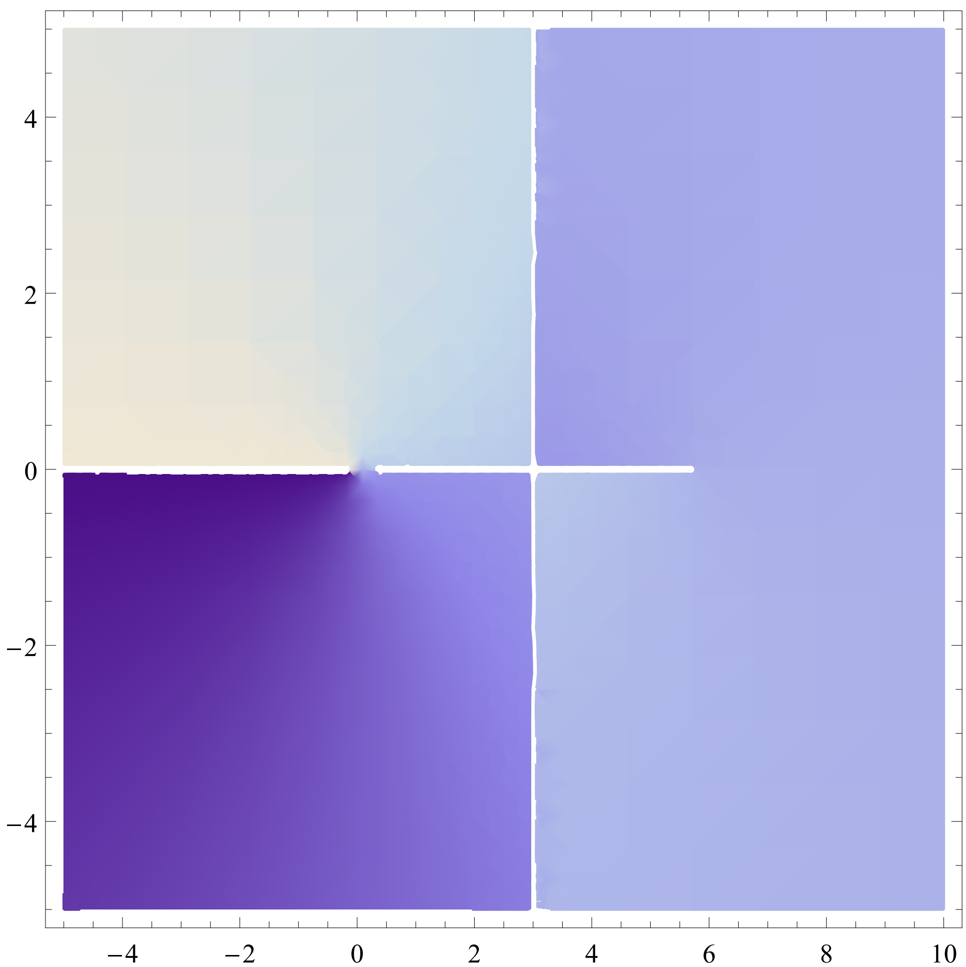
Do r and θ plots of the GF for bracketings (from VII.1 written).

Solution:
First, we simply solve for $S(z)$ explicitly using Mathematica and get:

$$S(z) = \frac{1}{4}\left(-\sqrt{z^2 - 6z + 1} + z + 1\right)$$

The following are r and θ plots of this function produced using the Mathematica code given in the written exercise:




4
2
0
−2
−4
−4
−2
0
2
4
6
8
10