Solution to HW 4, Problem 6.6

Al-Naji, Nader

What proportion of the forests with N nodes have no trees consisting of a single node? For N=1,2,3, and 4, the answers are 0,1/2,2/5, and 3/7, respectively.

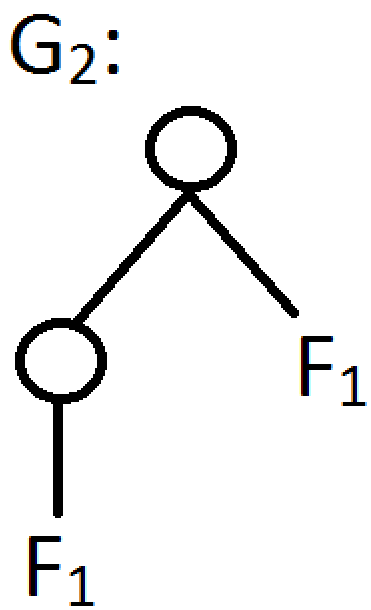
Solution:
Let $\mathcal{F}_1$ denote a forest that *can* contain trees of size 1.
Let $\mathcal{G}_1$ denote a general tree that *can* consist of only one node.
Let $\mathcal{F}_2$ denote a forest that *cannot* contain contain trees of size 1.
Let $\mathcal{G}_2$ denote a tree that *cannot* consist of only one node.
Let $\mathcal{Z}$ denote a single atom.

We are trying to find an expression for $\mathcal{F}_2$. Using this notation, it follows that:

$$\begin{aligned}
&\mathcal{F}_1 = SEQ(\mathcal{G}_1)\\
&\mathcal{G}_1 = \mathcal{Z} \times \mathcal{F}_1\\
&\mathcal{F}_2 = SEQ(\mathcal{G}_2)\\
&\mathcal{G}_2 = (\mathcal{Z} \times \mathcal{F}_1) \times (\mathcal{Z} \times \mathcal{F}_1)
\end{aligned}$$

The first equality holds because a regular forest is defined as a sequence of trees, including trees with only one internal node. The second equality holds because a regular tree is a node connected to a forest. The third equality holds because a forest without trees containing only one internal node is simply a sequence of such trees. Finally, the last equality holds because a tree with two or more internal nodes can be recursively defined as two internal nodes connected to each other, with a (regular) forest connected to each of the two nodes (see diagram).

Using the basic transfer theorems from chapter 3, we immediately get the generating func-



tions:

$$
\begin{aligned}
F_1(z) &= \frac{1}{1-G_1(z)} \\
G_1(z) &= zF_1(z) \\
F_2(z) &= \frac{1}{1-G_2(z)} \\
G_2(z) &= z^2 \cdot F_1(z)^2 \\
\Rightarrow F_1(z) &= \frac{1-\sqrt{1-4z}}{2z} && \text{(plugging in and solving)} \\
\Rightarrow F_2(z) &= \frac{1}{1-z^2(\frac{1-\sqrt{1-4z}}{2z})^2} && \text{(plugging in)} \\
&= \frac{2}{1+\sqrt{1-4z}+2z} && \text{(solving and simplifying)} \\
&= \frac{1-\sqrt{1-4z}+2z}{2z(2+z)} && \text{(multiply by conjugate)} \\
&= \frac{1}{2z(2+z)} - \frac{\sqrt{1-4z}}{2z(2+z)} + \frac{1}{(2+z)} && \text{(expand)}
\end{aligned}
$$

Note that we multiply by the top and bottom by $(1-\sqrt{1-4z}+2z)$ in the last step to get the square root term out of the denominator. Next, we simplify terms using the binomial expansion theorem: $(1+z)^n = \sum_{k\geq 0}\binom{n}{k}z^k$ and the geometric expansion theorem: $\frac{1}{1-z} =$

$\sum_{k\geq 0} z^k$.

$$\frac{1}{(2+z)} = \sum_{k=0}^{\infty} -\left(-\frac{1}{2}\right)^{k+1} z^k \qquad \text{(geometric)}$$

$$\frac{1}{2z(2+z)} = \sum_{k=-1}^{\infty} (-1)^{k+1}\left(\frac{1}{2}\right)^{k+3} z^k \qquad \text{(geometric)}$$

$$\sqrt{1-4z} = \sum_{k=0}^{\infty} (-1)^k 4^k \binom{\frac{1}{2}}{k} z^k \qquad \text{(binomial)}$$

$$= \sum_{k=0}^{\infty} \frac{\binom{2k}{k}}{1-2k} z^k \qquad \text{(simplifying and collecting)}$$

$$\frac{\sqrt{1-4z}}{2z(2+z)} = \sum_{n=-1}^{\infty} z^n \left(\sum_{k=0}^{n+1} \frac{\binom{2k}{k}\left(-\frac{1}{2}\right)^{-k+n+3}}{1-2k}\right) \qquad \text{(binomial convolution)}$$

Thus, we have that the total number of forests of size n that have the property is expressable exactly as:

$$-\sum_{k=0}^{n+1} \frac{\binom{2k}{k}\left(-\frac{1}{2}\right)^{-k+n+3}}{1-2k} - \left(-\frac{1}{2}\right)^{n+1} + \left(-\frac{1}{2}\right)^{n+3}$$

To get the proportion and our final answer, we simply divide by the total number of trees of size n and get:

$$\frac{\left(-\frac{1}{2}\right)^{n+3} - \sum_{k=0}^{n+1} \frac{\binom{2k}{k}\left(-\frac{1}{2}\right)^{-k+n+3}}{1-2k} - \left(-\frac{1}{2}\right)^{n+1}}{\frac{\binom{2n}{n}}{n+1}}$$

Solution to HW 4, Problem 6.27

Al-Naji, Nader

For $N = 2^n - 1$, what is the probability that a perfectly balanced tree structure (all 2n external nodes on level n) will be built, if all N! key insertion sequences are equally likely?

Solution:
First, we note that if we have a permutation of size N with N odd, in order for it to map to a perfectly balanced binary search tree, the $\left(\frac{N+1}{2}\right)^{th}$ element *must* be at the beginning. Then, the order in which we insert the elements smaller than this middle value must map to a balanced binary search tree of size $\left(\frac{N-1}{2}\right)$ and the same must be true for the order in which we insert all values larger than this middle element. Then, because there are $\binom{N-1}{\left(\frac{N-1}{2}\right)}$ ways to interleave these two ordered sets of elements, we get the following recurrence as our answer:

$$P_N = \binom{N-1}{\frac{N-1}{2}} P^2_{\frac{N-1}{2}}$$

Now we must unpack this recurrence and plug in $N = 2^n - 1$:

$$\begin{aligned}
P_N &= \binom{N-1}{\frac{N-1}{2}} P^2_{\frac{N-1}{2}} \\
&= \binom{N-1}{\frac{N-1}{2}} \binom{\frac{N-1}{2} - 1}{\frac{\frac{N-1}{2}-1}{2}}^2 \binom{\frac{\frac{N-1}{2}-1}{2} - 1}{\frac{\frac{\frac{N-1}{2}-1}{2}-1}{2}}^4 \cdots && \text{(unpack)} \\
&= \prod_{i=0}^{n-1} \binom{2(2^i - 1)}{2^i - 1}^{2^{n-1-i}} && \text{(notice pattern)} \\
&= (2^n - 2)! \prod_{i=1}^{n-1} \frac{1}{(2^i - 1)^{2^{n-i}}} && \text{(telescope)} \\
\Rightarrow Pr_N &= \frac{1}{(2^n - 1)!} (2^n - 2)! \prod_{i=1}^{n-1} \frac{1}{(2^i - 1)^{2^{n-i}}} && \text{(divide by } N! \text{ to get probability)} \\
&= \frac{1}{(2^n - 1)} \prod_{i=1}^{n-1} \frac{1}{(2^i - 1)^{2^{n-i}}} && \text{(exact answer)}
\end{aligned}$$

Solution to HW 4, Problem 6.43

Al-Naji, Nader

Internal nodes in binary trees fall into one of three classes: they have either two, one, or zero external children. What fraction of the nodes are of each type, in a random binary search tree of N nodes?

Solution:
Number all of the elements in each permutation $1, ..., N$. Now, we consider any three consecutive elements a, b, c such that $a \neq 1$ and $c \neq N$ (for example: $3, 4, 5$) and show that the relative order in which these elements appear in any particular permutation of the numbers $1, ..., N$ exactly determines what kind of node b will be. Let us start by listing all possible permutations of these three elements. We use dots below to indicate that other elements can appear *between* a, b, c (because we only care about *relative* order):

$a...b...c$ one right child
$a...c...b$ leaf node
$b...a...c$ two internal children
$b...c...a$ two internal children
$c...a...b$ leaf node
$c...b...a$ one left child

We now argue that the relative orderings form a direct correspondence with what kind of node b will be.

First, if b appears *after* a and c in the permutation, it will be a leaf node. To see this, consider the case where a is inserted first and then c. Wherever a ends up in the tree, its parent will be either larger than both a and c OR smaller than both a and c because there are no elements in between a and c other than b. In this case, when c is inserted, because its relative order with respect to all the inserted elements will be identical to a, it will therefore traverse to the same position that a is at in the tree and it will end up hanging to the right of a. When b is inserted, since there are no nodes between c and b or between a and b, by the same logic, b will traverse to the same position that c is at in the tree and end up hanging to the left of c. After this point, b will remain a leaf since no other node will be larger than a and less than c and, therefore, no other node will make it to b's position in the tree. The same argument holds for the case where c is inserted first since this case is symmetric to the previous case.

Next, if b appears *before* a and c in the permutation, it will have two internal children. To see this, consider the case where b is inserted, then a, then c. When b is inserted, its parent will be either larger than both a and c OR smaller than both a and c. Because of this, when it comes time for a to be inserted, it will have the same relative order as b with respect to all the other elements in the tree, which will cause it to traverse to b's position in the tree. When a reaches b's position, it will hang to the left of b because it is smaller than b. c, by the same logic, will reach b's position in the tree and hang to the right of b, making b a node with two internal children. The same argument holds for the case where c is inserted first since this case is symmetric to the previous case.

Finally, if b is inserted in between a and c, it will have exactly one child. To see this, again consider the case where a is inserted, then b, then c. When a is inserted its parent will be either larger than both b and c OR smaller than both b and c. Because of this, when it comes time for b to be inserted, it will have the same relative order with respect to all the other elements in the tree as a, and this will cause it to traverse to a's position in the tree. When b reaches a's position in the tree, it will hang to the right of a because it is larger. When c is inserted, by the same logic, it will traverse to b's position in the tree and hang to the right of b. After this point, b will not gain any other children because no nodes are larger than a and smaller than b and, therefore, no nodes can traverse to b's left. Thus b will be will have exactly one right child in this case. The same argument holds for the case where c is inserted first, except that a will hang off of b and b will have exactly one *left* child.

Having reasoned through all these cases, we now use linearity of expectation to compute the expected number of each kind of node. We will start with leaf nodes. Create an indicator variable that is one if element k (that is, the element of rank k) is a leaf node and zero otherwise. This random variable is one if and only if the element of rank k shows up after its neighbors in the permutation that defines the tree. Because we are sampling from all permutations uniformly, this probability is therefore 1/3. Thus, the expected number of leaf nodes is simply:

$$\begin{aligned}
E[\text{leaf nodes}] &= E[\sum_{2 \le i \le N-1} \mathbf{1}_i \text{ is after neighbors}] \\
&= \sum_{2 \le i \le N-1} E[\mathbf{1}_i \text{ is after neighbors}] \\
&= \sum_{2 \le i \le N-1} 1/3 \\
&= \frac{N-2}{3}
\end{aligned}$$

We are almost done; we have considered all elements except the highest and lowest ranked elements. These are special because they only have one neighbor each. For the smallest element, the probability it is a leaf node is the probability it shows up after its neighbor (by the same logic as before) and for the largest element, the probability it is a leaf node is the probability it shows up after its neighbor as well. Since there are only two possibilities for each of these elements (before or after their neighbor) and both are equally likely (because all permutations are equally likely), the probability each is a leaf node is simply 1/2. Thus, we have that the expected number of leaf nodes is: $\frac{N-2}{3} + 1 = \frac{N+1}{3}$.

The argument for the expected number of elements that appear as nodes with exactly one internal child is identical to that above. We create an indicator that is one if the element has exactly one child and run it over all of the elements of rank $2, ..., N-1$. The leftmost and rightmost elements again have probability 1/2 of being nodes of this kind and so we get as our final answer $\frac{N+1}{3}$ again.

Finally, to compute the expected number of nodes with two internal children, we create an indicator once again and run it over the middle elements. This time, however, the leftmost and rightmost elements *can't* have two internal children (since they are the smallest and largest respectively) and so we get as our final answer $\frac{N-2}{3}$.

To summarize:

$$\frac{N+1}{3} \text{ leaf nodes}$$
$$\frac{N+1}{6} \text{ one left child}$$
$$\frac{N+1}{6} \text{ one right child}$$
$$\frac{N-2}{3} \text{ two children}$$

Solution to HW 4, Problem 7.29

Al-Naji, Nader

An arrangement of N elements is a sequence formed from a subset of the elements. Prove that the EGF for arrangements is $e^z/(1-z)$. Express the coefficients as a simple sum and interpret that sum and interpret that sum combinatorially.

Solution:
An arrangement is a particular permutation of any subset of N elements. The number of subsets of size k of N elements can be expressed as $\binom{N}{k}$. Each of these subsets has $k!$ different possible permutations. Thus, the total number of arrangements of N elements is $\sum_{k=0}^{N}\binom{N}{k}k!$. Expressing this as coefficients of an EGF, we get: $\sum_{n=0}^{N}\left(\sum_{k=0}^{n}\binom{n}{k}k!\right)\frac{z^n}{n!}$. We can relate this to the EGF $\frac{e^z}{1-z}$ using the Binomial sum in table 3.4. By this identity, $e^zA(z) = \sum_{n=0}^{N}\left(\sum_{k=0}^{n}\binom{n}{k}a_k\right)\frac{z^n}{n!}$. Plugging in $A(z) = \frac{1}{1-z}$ gives us $\frac{e^z}{1-z} = \sum_{n=0}^{N}\left(\sum_{k=0}^{n}\binom{n}{k}k!\right)\frac{z^n}{n!}$. This is the same as the sum we derived for arrangements combinatorially and thus we conclude that $\frac{e^z}{1-z}$ is the EGF for the number of arrangements.

Solution to HW 4, Problem 7.45

Al-Naji, Nader

Find the CGF for the total number of inversions in all involutions of length N. Use this to find the average number of inversions in an involution.

Solution:
Every involution of length $|p|$ corresponds to (i) one involution of length $|p|+1$ formed by appending the singleton cycle consisting of length $|p|+1$; and (ii) $|p|+1$ involutions of length $|p|+2$, formed by, for each k from 1 to $|p|+1$, adding 1 to permutation elements greater than k, then appending the doubleton cycle consisting of k and $|p|+2$. Both of these cases are illustrated in the figure.

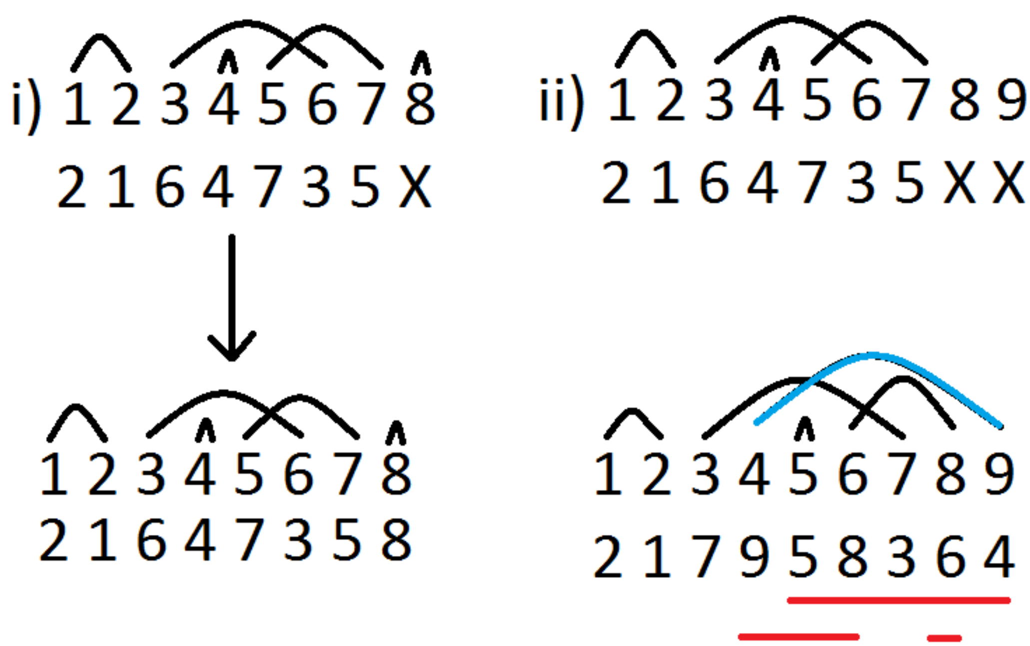


Figure 1: Red marks inversions created by augmenting the involution. In general, case (i) adds 0 inversions and case (ii) adds $(|p|+2-k)+(|p|+1-k) = 2|p|-2k+3$ inversions, the first quantity from moving the largest element into position k (illustrated by the top red line in the figure) and the second quantity from moving the element that was at position k to the end.

Noting these two cases will help us count the number of involutions, but in order to count the total number of inversions, we must also keep track of how many inversions we have in each case. In the first, we are just adding a larger element to the end so we don't increase the number of inversions. In the second, however, we move the largest element into position k and we move the element that was once at position k to the end. This results in the creation of $2|p| - 2k + 3$ inversions, as shown in the figure. Thus, putting this all together, we start crunching. In what follows $B(z)$ be the generating function for the number of inversions, let $B_z(z)$ denote the first derivative of $B(z)$ with respect to z, and let $G(z) = e^{z+\frac{z^2}{2}}$ be the generating function for just the number of involutions. Let P_i be the set of all involutions:

$$
\begin{aligned}
B(z) &= \sum_{p\in P_i} inv(p)\frac{z^{|p|}}{|p|!} \\
&= \sum_{p\in P_i} inv(p)\frac{z^{|p|+1}}{(|p|+1)!} + \sum_{p\in P_i}\sum_{k=1}^{|p|+1}(inv(p)+2|p|-2k+3)\frac{z^{|p|+2}}{(|p|+2)!} \\
&= \sum_{p\in P_i} inv(p)\frac{z^{|p|+1}}{(|p|+1)!} + \sum_{p\in P_i}((|p|+1)inv(p)+(|p|+1)^2)\frac{z^{|p|+2}}{(|p|+2)!} \\
&= \sum_{p\in P_i} inv(p)\frac{z^{|p|+1}}{(|p|+1)!} + \sum_{p\in P_i}(|p|+1)(inv(p)+|p|+1)\frac{z^{|p|+2}}{(|p|+2)!} \\
B_z(z) &= \sum_{p\in P_i}(|p|+1)inv(p)\frac{z^{|p|}}{(|p|+1)!} + \sum_{p\in P_i}(|p|+1)(|p|+2)(inv(p)+|p|+1)\frac{z^{|p|+1}}{(|p|+2)!} \\
&= \sum_{p\in P_i} inv(p)\frac{z^{|p|}}{|p|!} + \sum_{p\in P_i}(inv(p)+|p|+1)\frac{z^{|p|+1}}{|p|!} \\
&= \sum_{p\in P_i} inv(p)\frac{z^{|p|}}{|p|!} + \sum_{p\in P_i} inv(p)\frac{z^{|p|+1}}{|p|!} + \sum_{p\in P_i}|p|\frac{z^{|p|+1}}{|p|!} + \sum_{p\in P_i}\frac{z^{|p|+1}}{|p|!} \\
&= \sum_{p\in P_i} inv(p)\frac{z^{|p|}}{|p|!} + z\sum_{p\in P_i} inv(p)\frac{z^{|p|}}{|p|!} + z^2\sum_{p\in P_i}\frac{z^{|p|-1}}{(|p|-1)!} + z\sum_{p\in P_i}\frac{z^{|p|}}{|p|!} \\
&= B(z) + zB(z) + z^2G_z(z) + zG(z) \\
\Rightarrow B(z) &= e^{z+\frac{z^2}{2}}\left(\frac{z^4}{4}+\frac{z^3}{3}+\frac{z^2}{2}\right) \\
\Rightarrow [z^n]B(z) &\sim \left(\frac{e^{\sqrt{n+2}}\left(\frac{n+2}{e}\right)^{\frac{n+2}{2}}}{2\sqrt{2\sqrt{e}}} + \frac{\left(\frac{n+3}{e}\right)^{\frac{n+3}{2}}e^{\sqrt{n+3}}}{3\sqrt{2\sqrt{e}}} + \frac{\left(\frac{n+4}{e}\right)^{\frac{n+4}{2}}e^{\sqrt{n+4}}}{4\sqrt{2\sqrt{e}}}\right) \\
\Rightarrow \frac{[z^n]B(z)}{[z^n]G(z)} &\sim \left(\frac{\frac{e^{\sqrt{n+2}}\left(\frac{n+2}{e}\right)^{\frac{n+2}{2}}}{2\sqrt{2\sqrt{e}}} + \frac{\left(\frac{n+3}{e}\right)^{\frac{n+3}{2}}e^{\sqrt{n+3}}}{3\sqrt{2\sqrt{e}}} + \frac{\left(\frac{n+4}{e}\right)^{\frac{n+4}{2}}e^{\sqrt{n+4}}}{4\sqrt{2\sqrt{e}}}}{\frac{e^{\sqrt{n}}\left(\frac{n}{e}\right)^{n/2}}{\sqrt{2\sqrt{e}}}}\right)
\end{aligned}
$$

In the third-to-last line we solve the differential equation for $B(z)$ and in the last, we substitute an approximation for the generating function $e^{z+z^2/2}$ given in chapter 7. This gives us the final answer for the average number of inversions.

Solution to HW 4, Problem 7.61

Al-Naji, Nader

Use asymptotics from generating functions (see Section 5.5) or a direct argument to show that the probability for a random permutation to have j cycles of length k is asymptotic to the Poisson distribution $e^{-\lambda}\lambda^j/j!$ with $\lambda = 1/k$.

Solution:
The probability sought is given by the coefficients of the BGF derived in the discussion on page 404:

$$[u^j z^k]\frac{e^{(u-1)\frac{z^k}{k}}}{1-z}$$

If we treat u as a constant, this is of the form $f(z)/(1-z)^\alpha$ for $\alpha = 1$. Since it satisfies the necessary conditions, we can apply theorem 5.5, which states $[z^n]\frac{f(z)}{(1-z)^\alpha} \sim \frac{f(1)}{\Gamma(\alpha)}n^{\alpha-1}$, and get:

$$\begin{aligned}
&[u^j z^k]\frac{e^{(u-1)\frac{z^k}{k}}}{1-z} && \text{(from BGF)}\\
=&[u^j]([z^k]\frac{e^{(u-1)\frac{z^k}{k}}}{1-z}) \\
\sim&[u^j](\frac{e^{(u-1)\frac{z^k}{k}}|_{z=1}}{\Gamma[1]})n^0 && \text{(by theorem 5.5)}\\
\sim&[u^j](e^{(u-1)\frac{1}{k}}) && \text{(simplify)}\\
\sim&[u^j](e^{\frac{u}{k}}e^{-\frac{1}{k}}) && \text{(separate terms)}\\
\sim&e^{-\frac{1}{k}}[u^j](e^{\frac{u}{k}}) && \text{(pull out constant)}\\
\sim&\frac{e^{-\frac{1}{k}}}{k^j j!} && \text{(coefficeint of Taylor expansion for } e^x)\\
\sim&\frac{e^{-\lambda}\lambda^j}{j!} && \text{(Poisson distribution)}
\end{aligned}$$

Thus, the probability of a random permutation having j cycles of length k is asymptotic to the Poisson distribution.