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Spring 2013 | **Collaborator 2** : last name, first name

Solution to HW 8, Problem IV.28

Al-Naji, Nader

A "supernecklace" of the 3rd type is a labelled cycle of cycles (see page 125). Draw all the supernecklaces of the 3rd type of size n for $n = 1, 2, 3$, and 4. Then develop an asymptotic estimate of the number of supernecklaces of size n by showing that

$$[z^n] \ln \left(\frac{1}{1 - \ln\left(\frac{1}{1-z}\right)} \right)$$

Solution:
First, we note a simple fact. If we have $A(z) = \sum_{n\geq 0} a_n z^n$ and, therefore, $[z^n]A(z) = a_n$, when we take a derivative, we have: $\frac{\delta A(z)}{\delta z} = \sum_{n\geq 0} n a_n z^{n-1} \Rightarrow z\frac{\delta A(z)}{\delta z} = \sum_{n\geq 0} n a_n z^n \Rightarrow \frac{1}{n}[z^n]z\frac{\delta A(z)}{\delta z} = a_n = [z^n]A(z)$.

For this problem, we have: $A(z) = \ln\left(\frac{1}{1-\ln\left(\frac{1}{1-z}\right)}\right) \Rightarrow z\frac{\delta A(z)}{\delta z} = \frac{z}{(1-z)(1-\ln(\frac{1}{1-z}))}$. Then, we can use the transfer theorem for Meromorphic GF's to extract coefficients for the derivative. Using the theorem, we find that $\alpha = 1 - \frac{1}{e}$ and that its multiplicity $M = 1$. Then, plugging into our theorem yields:

$$\begin{aligned}
\text{Let } z\frac{\delta A(z)}{\delta z} &= \frac{f(z)}{g(z)} \\
f(z) &= z \\
g(z) &= (1-z)(1-\ln(\frac{1}{1-z})) \\
\Rightarrow \alpha &= 1-\frac{1}{e},\ M=1 \\
c &= (-1)^M\frac{Mf(\alpha)}{\alpha^M g^{(M)}(\alpha)} \\
&= (-1)\frac{\alpha}{\alpha(-2-\log\left(1-\frac{e-1}{e}\right))} \\
&= 1 \\
\beta &= \frac{1}{\alpha} \\
&= (1-e^{-1})^{-1} \\
[z^n]\frac{z}{(1-z)(1-\ln(\frac{1}{1-z}))} &\sim c\beta^n n^{M-1} \\
&\sim (1-e^{-1})^{-n} \\
\text{So, letting } A(z) &= \ln\left(\frac{1}{1-\ln\left(\frac{1}{1-z}\right)}\right) \\
\Rightarrow [z^n]z\frac{\delta A(z)}{\delta z} &= (1-e^{-1})^{-n} \\
\Rightarrow \frac{1}{n}[z^n]\frac{\delta A(z)}{\delta z} &= \frac{1}{n}(1-e^{-1})^{-n} \\
\Rightarrow [z^n]A(z) &= \frac{1}{n}(1-e^{-1})^{-n}
\end{aligned}$$

Then, because this is an EGF, we multiply by $n!$ and get that the number of supernecklaces of size n is asymptotic to $\frac{n!}{n}(1-e^{-1})^{-n}$.

Solution to HW 8, Problem Programming 1

Al-Naji, Nader

Compute the percentage of permutations having no singleton or doubleton cycles and compare with AC asymptotic estimate for $N = 10$ and $N = 20$.

Solution:
A permutation is a set of cycles denoted symbolically by $P = SET(CYC_{>0}(Z))$. If we want to restrict our analysis only to permutations that have no singleton or doubleton cycles, we therefore need to work with the class denoted by $P_{>2} = SET(CYC_{>2}(Z))$. Using the transfer theorems for sets and cycles yields:

$$P_{>2}(z) = \frac{e^{-z-\frac{z^2}{2}}}{1-z}.$$

We can then use the transfer theorem for meromorphic GF's (used on slide 63 of the complex asymptotics lecture) to get an asymptotic for the coefficients:

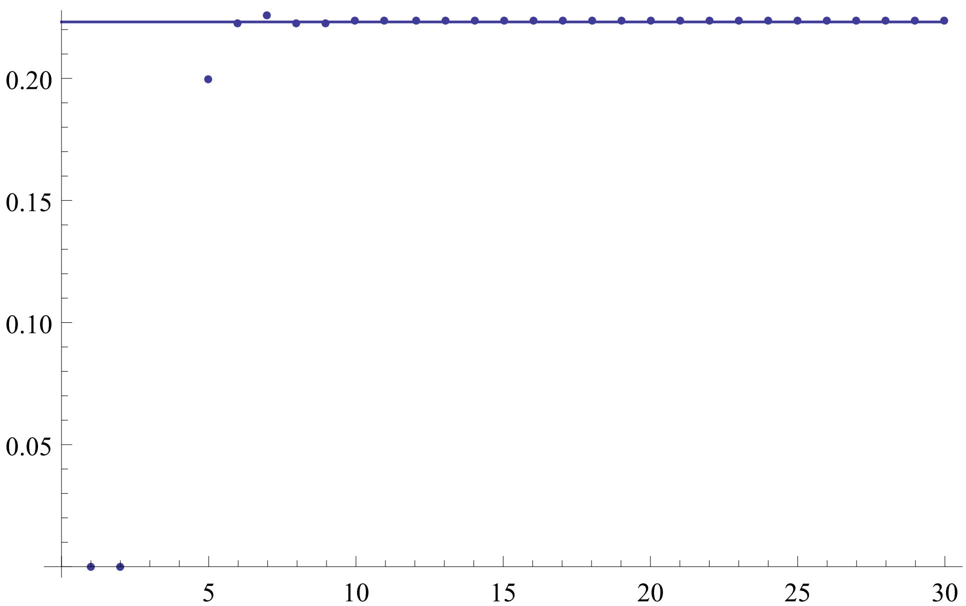
$$\begin{aligned}[z^n]P_{>2}(z) &\sim e^{-H_2} \\ &\sim e^{-\frac{3}{2}}\end{aligned}$$

Since this is the EGF for the desired class, we have that the proportion of permutations without singleton or doubleton cycles is $\sim e^{-\frac{3}{2}}$.

In order to check this, we can compute the coefficients directly using Mathematica compare our result to the asymptotic. The following code does this:

$$f(\text{n\_}):=\text{SeriesCoefficient}\left[\frac{e^{-\frac{z^2}{2.}-z}}{1-z}, \{z, 0, n\}\right];$$
$$\text{numpoints} = 30;$$
$$\text{l1} = \text{ListPlot}[\text{Table}[f(n), \{n, \text{numpoints}\}]];$$
$$\text{p1} = \text{Plot}\left[e^{\left\{-\frac{3}{2}\right\}}, \{z, 0, \text{numpoints}\}\right];$$
$$\text{Show}[\text{l1}, \text{p1}]$$

Thus, the approximation is fairly close for n reasonably large. Comparing specifically with



the asymptotic estimate for $n = 10, 20$, we have:

$$\text{SetPrecision}\left[\frac{f(10)}{e^{\left\{-\frac{3}{2}\right\}}}, 100\right]$$
$$\text{SetPrecision}\left[\frac{f(20)}{e^{\left\{-\frac{3}{2}\right\}}}, 100\right]$$
$$\{1.000199179824653894499419948260765522718429565429687 50\}$$
$$\{0.99999999968115260706724711781134828925132751464843750\}$$

This helps confirm the closeness of the asymptotic as well.

Solution to HW 8, Problem Programming 2

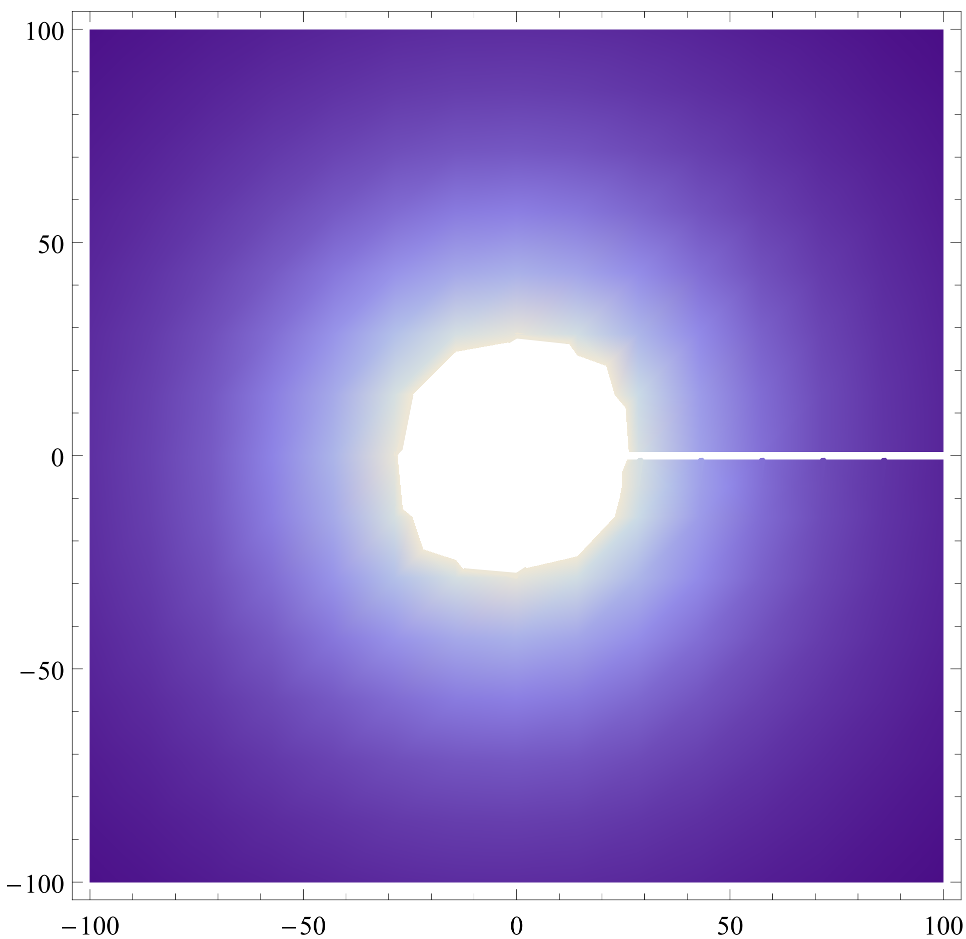
Al-Naji, Nader

Plot the derivative of the supernecklace GF in the style of the plots in the lecture.

Solution:
The following code produces a plot in the style of the lecture slides using Mathematica:

$$f(\text{z\_}):=\log\left(\frac{1}{1-\log\left(\frac{1}{1-z}\right)}\right)$$
$$\text{DensityPlot}[|\text{df}(x,y)|,\{x,-100,100\},\{y,-100,100\}]$$
$$\text{df}(\text{x\_},\text{y\_}):=\frac{\partial f(z)}{\partial z}//.\,z\to x+iy$$

This is the plot produced. Lighter implies higher values:



100
50
0
−50
−100
−100
−50
0
50
100