Maxima and Minima

Define maxima and minima of a function of a single variable.

Answer: Maxima : A function $f(x)$ is said to have a maximum value for $x=c$, provided we can get a positive quantity δ such that for all values of x in the interval

$$c-\delta < x < c+\delta \ (x \neq c) \ f(c) > f(x)$$

i.e if $f(c+h) - f(c) < 0$, for $|h|$ sufficiently small.

Minimum : A function $f(x)$ has a minimum value for $x=c$ provided we can get ~~an interval $c-\delta<x<c+\delta$~~ a positive quantity δ such that for all values of x in the interval

$$c-\delta < x < c+\delta \quad (x \neq c), \ f(c) < f(x)$$

i.e if $f(c+h) - f(c) > 0$, for $|h|$ sufficiently small.

1. theorem : what is the necessary condition for maximum and minimum.

Answer : If $f(x)$ be a maximum or a minimum at $x=c$ and if $f'(c)$ exists, then $f'(c) = 0$.

By definition $f(x)$ is a maximum at $x=c$ if we can find a positive number δ, such that

$$f(c+h) - f(c) < 0 \quad \text{whenever } -\delta < h < \delta, (h \neq 0)$$

$$\therefore \quad \frac{f(c+h) - f(c)}{h} < 0$$

If h be positive and sufficiently small, and $h > 0$

$$\text{i.e} \quad \lim_{h \to 0^+} \frac{f(c+h) - f(c)}{h} \leq 0$$

If h be negative and sufficieciently small

$$\text{Thus} \quad \lim_{h \to 0^-} \frac{f(c+h) - f(c)}{h} \geq 0$$

Now if $f'(c)$ exists, the above two limits, which represent the right-hand and left-hand derivatives respectively of $f(x)$ at $x=c$, must be equal. Hence, the only common value of the limit is zero. Thus $f'(c)=0$.

Exactly similar is the proof when $f(c)$ is a minimum.

2. Find the maximum and minimum value of $2x^3-21x^2+36x-20$.

Solⁿ: Let $f(x)=2x^3-21x^2+36x-20 \longrightarrow (1)$

$\therefore f'(x)=6x^2-42x+36 \longrightarrow (2)$

$f''(x)=12x-42 \longrightarrow (3)$

For maximum and minimum

$f'(x)=0$

or, $6x^2-42x+36=0$ [Using (2)]

or, $x^2-7x+6=0$

or, $x^2-(6+1)x+6=0$

or, $x^2-6x-x+6=0$

or, $x(x-6)-1(x-6)=0$

or $(x-1)(x-6)=0$, $x-1=0$ or, $x-6=0$

i.e $x=1, 6$

Now, when $x=1$, $f''(1)=12-42=-30$ which is negative

when $x=6$, $f''(6)=72-42=30$ which is positive.

Hence the given expression is maximum for $x=1$, and minimum for $x=6$.

The maximum value is $f(1)=-3$

and the minimum value is $f(6)=-128$ Answer

03. Examine $f(x) = x^3 - 9x^2 + 24x - 12$ for maximum or minimum values.

Solution: Given that $f(x) = x^3 - 9x^2 + 24x - 12$

Differentiation with respect to x we have

$$f'(x) = 3x^2 - 18x + 24 \longrightarrow (1)$$

Again $f''(x) = 6x - 18$

For maximum or minimum, $f'(x) = 0$

or, $3x^2 - 18x + 24 = 0$

or, $x^2 - 6x + 8 = 0$

or, $x^2 - 4x - 2x + 8 = 0$

or, $x(x-4) - 2(x-4) = 0$

or, $(x-2)(x-4) = 0$

$\therefore x = 2, 4$

Now when $x = 2$, $f''(x) = 6.2 - 18 = -6$ which is (-ve)

and when $x = 4$, $f''(x) = 24 - 18 = 6$ which is positive

Hence the given function is maximum for $x = 2$ and minimum for $x = 4$

Therefore The maximum value is $f(2) = 2^3 - 9.2^2 + 24.2 - 12$
$= 8$

and the minimum value is $f(4) = 4^3 - 9.4^2 + 24.4 - 12 = 4$

04. Investigate for what values of x

$f(x) = 5x^6 - 18x^5 + 15x^4 - 10$ is a maximum or minimum.

Solution: Given that $f(x) = 5x^6 - 18x^5 + 15x^4 - 10$

$$f'(x) = 30x^5 - 90x^4 + 60x^3 \longrightarrow (1)$$

Again $f''(x) = 30(5x^4 - 12x^3 + 6x^2) \longrightarrow$ ②

For maximum or minimum $f'(x) = 0$

$30(x^5 - 3x^4 + 2x^3) = 0$ [using ①]

or, $x^3(x^2 - 3x + 2) = 0$

or, $x^3 (x-1)(x-2) = 0$

$\therefore x = 0, 1, 2$

when $x = 1$, $f''(x) = 30(5 - 12 + 6) =$ negative and hence $f(x)$ is a maximum at $x = 1$

when $x = 2$, $f''(x) = 30(5.2^4 - 12.2^3 + 6.2^2)$ is positive and hence $f(x)$ is minimum for $x = 2$

Again when $x = 0$, $f''(x) = 0$ so the test fails

We have to examine higher order derivatives

$f'''(x) = 30(20x^3 - 36x^2 + 12x) = 120(5x^3 - 9x^2 + 3x)$

$\therefore f'''(x) = 0$ for $x = 0$

$f^{iv}(x) = 120(15x^2 - 18x + 3)$

$\therefore f^{iv}(x) = 360$ at $x = 0$

Since even order derivative is positive for $x = 0$.

Therefore for $x = 0$, $f(x)$ is a minimum.

5. Find the maximum and minimum of $1+2\sin x+3\cos^2 x$

Solution: Let $f(x) = 1+2\sin x+3\cos^2 x$

Then $f'(x) = 0 + 2\cos x + 6\cos x(-\sin x)$

$f'(x) = 2\cos x(1-3\sin x) \longrightarrow (1)$

Again $f''(x) = 2\cos x(0-3\cos x) + 2(-\sin x)(1-3\sin x)$

$f''(x) = -6\cos^2 x - 2\sin x + 6\sin^2 x$

For maximum and minimum, $f'(x) = 0$

$\therefore 2\cos x(1-3\sin x) = 0$

or, $2\cos x = 0$ or $1-3\sin x = 0$

or, $\cos x = 0$ or, $\sin x = \frac{1}{3}$

or, $\cos x = \cos\frac{\pi}{2}$

$\therefore x = \frac{\pi}{2}$

Now when $x = \frac{\pi}{2}$, $f''(x) = -6.0 - 2.1 + 6 = 4$ i.e +ve

Hence $f(x)$ is minimum at $x = \pi/2$ and this value is

$f(\pi/2) = 1+2.1+0 = 3$.

$f''(x) = -6(1-\sin^2 x) - 2\sin x + 6\sin^2 x$

$\therefore f''(x) = -6(1-\frac{1}{9}) - 2\frac{1}{3} + 6.\frac{1}{9}$ for $\sin x = \frac{1}{3}$

$=$ negative

Therefore, for $\sin x = \frac{1}{3}$, $f(x)$ is a ~~minimum~~ maximum and the maximum value is $1+2.\frac{1}{3} + 3(1-\frac{1}{9}) = 4\frac{1}{3}$.

6. Find the maximum and minimum values of U where

$$U=\frac{4}{x}+\frac{36}{y} \text{ and } x+y=2.$$

Solution: Given that,

$$U=\frac{4}{x}+\frac{36}{y} \longrightarrow (1)$$

$$x+y=2$$

$$y=2-x \longrightarrow (2)$$

Using equation (2) in eq^n (1) we have

$$U=\frac{4}{x}+\frac{36}{2-x} \longrightarrow (3)$$

Diff. (3) w. r to x we have

$$\frac{dU}{dx}=-\frac{4}{x^2}+\frac{36}{(2-x)^2}$$

Again,

$$\frac{d^2U}{dx^2}=-4\cdot\frac{-2}{x^3}+\frac{36\cdot 2}{(2-x)^3}$$

$$=\frac{8}{x^3}+\frac{72}{(2-x)^3}$$

For maximum and minimum values $\frac{dU}{dx}=0$

or, $-\frac{4}{x^2}+\frac{36}{(2-x)^2}=0$

or, $\frac{-4(2-x)^2+36x^2}{x^2(2-x)^2}=0$

or, $-4(4-4x+x^2)+36x^2=0$

or, $-4+4x-x^2+9x^2=0$

or, $8x^2+4x-4=0$

or, $2x^2+x-1=0$

or, $2x^2+2x-x-1=0$

or, $2x(x+1)-1(x+1)=0$

$(2x-1)(x+1)=0$

$\therefore\ x = \frac{1}{2}$ or -1.

[From ③]

When $x = \frac{1}{2}$: $\frac{d^2U}{dx^3} = \frac{8}{(\frac{1}{2})^3} + \frac{72}{(\frac{3}{2})^3}$ which is positive

i.e For $x = \frac{1}{2}$, U is a minimum

Therefore minimum value of $U = \frac{4}{\frac{1}{2}} + \frac{36}{2-\frac{1}{2}} = 32$

When $x = -1$ then $\frac{d^2U}{dx^2} = -8 + \frac{72}{27}$, which is negative

For $x = -1$, U is maximum.

Hence maximum value of $U = \frac{4}{-1} + \frac{36}{3} = 8$.

7. (a) Find the maximum or minimum value of $x^{\frac{1}{x}}$.

(b) Show that the maximum value of $\left(\frac{1}{x}\right)^x$ is $e^{\frac{1}{e}}$

(c) Show that the minimum value of $4e^{2x} + 9e^{-2x}$ is 12.

(d) Show that $x^2 \log(1/x)$ is a maximum for $x = \frac{1}{\sqrt{e}}$ (Do yourself)

Solution: (a) Let $y = f(x) = x^{\frac{1}{x}} \longrightarrow$ ①

Taking Log on both sides we get

$$\log y = \log(x^{\frac{1}{x}}) = \frac{1}{x}\log x$$

$$\therefore \frac{1}{y}\frac{dy}{dx} = \frac{1}{x}\cdot\frac{1}{x} + \left(-\frac{1}{x^2}\right)\log x$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{x^2}(1 - \log x) \longrightarrow ②$$

Again Diff. ② w.r.to x we have

$$\frac{1}{y}\frac{d^2y}{dx^2} - \frac{1}{y^2}\left(\frac{dy}{dx}\right)^2 = \frac{1}{x^2}\left(0 - \frac{1}{x}\right) + \left(-\frac{2}{x^3}\right)(1 - \log x)$$

$$\text{or, } \frac{1}{y}\frac{d^2y}{dx^2} - \frac{1}{y^2}\left(\frac{dy}{dx}\right)^2 = -\frac{1}{x^3} - \frac{2}{x^3}(1 - \log x) \longrightarrow ③$$

$\log e$ — — ~~Answer~~

For maximum and minimum $\frac{dy}{dx} = 0$ then from (2)

$$\frac{1}{x^2}(1-\log x) = 0$$

$$\text{or, } \log x = 1$$

$$\text{or } \log x = \log e$$

$$\therefore x = e$$

For $x=e$, $\frac{d^2y}{dx^2} = e^{\frac{1}{e}}\left\{-\frac{1}{e^3} - \frac{2}{e^3}(1-1)\right\} = -\frac{e^{\frac{1}{e}}}{e^3}$.

which is negative

Therefore y is maximum for $x=e$. Hence the maximum value is $e^{\frac{1}{e}}$.

(b) Let $f(x) = \left(\frac{1}{x}\right)^x$

Taking Log on both sides we have

$$\log f(x) = \log\left(\frac{1}{x}\right)^x = x\log\left(\frac{1}{x}\right) \longrightarrow (1)$$

Differentiating (1) with respect to x we get

$$\text{or } \frac{1}{f(x)}\cdot\frac{df}{dx} = x\,\frac{1}{\frac{1}{x}}\left(-\frac{1}{x^2}\right) + 1\cdot\log\left(\frac{1}{x}\right)$$

$$= x\cdot x\cdot\left(-\frac{1}{x^2}\right) + \log\left(\frac{1}{x}\right)$$

$$\frac{1}{f}\frac{df}{dx} = -1 + \log\left(\frac{1}{x}\right) \longrightarrow (2)$$

Again Differentiating (2) w.r to x we get

$$\frac{1}{f}\frac{d^2f}{dx^2} - \frac{1}{f^2}\frac{df}{dx}\frac{df}{dx} = 0 - 1 + \log(1/x)$$

For maximum and minimum $\frac{df}{dx} = 0$

then from ② becomes $-1+\log\left(\frac{1}{x}\right)=0$

or, $-1+\log 1-\log x=0$

or $\log x = 1 = \log e$

$\therefore x = e$

For $x=e$, $\frac{d^2f}{dx^2} = \left(\frac{1}{e}\right)^e\left\{-1+\log(1/e)\right\}$

$= \left(\frac{1}{e}\right)^e(-1+\log 1-\log e)$

$= -2\left(\frac{1}{e}\right)^e$ which is negative

Hence the maximum value of the given function is $\left(\frac{1}{e}\right)^e$ for $x=e$.

Extra ⊛ let $y = x/\log x = x\cdot\frac{1}{\log x}$

Differentiating w. r to x

$\frac{dy}{dx} = \frac{\log x\cdot 1 - x\cdot\frac{1}{x}}{(\log x)^2} = \frac{\log x - 1}{(\log x)^2}$

$\therefore \frac{d^2y}{dx^2} = \frac{(\log x)^2\cdot\left(\frac{1}{x}-0\right) - 2\log x\,\frac{1}{x}\,(\log x-1)}{(\log x)^4}$

$= \frac{\frac{1}{x}(\log x)^2 - \frac{2}{x}\log x(\log x-1)}{(\log x)^4}$

For maximum or minimum $\frac{dy}{dx}=0 \Rightarrow \log x = 1 \Rightarrow x=e$

For $x=e$, $\frac{d^2y}{dx^2}$ is positive, y is minimum for $x=e$

Hence the minimum value $\frac{e}{\log e} = e$ Answer.

(c) let $y = 4e^{2x} + 9e^{-2x} \longrightarrow (1)$

Differentiating (1) w.r to x we have

$$\frac{dy}{dx} = 4e^{2x}.2 + 9e^{-2x}(-2)$$

$$= 8e^{2x} - 18e^{-2x}$$

Again $\frac{d^2y}{dx^2} = 8e^{2x}.2 - 18e^{-2x}.(-2)$

$$= 16e^{2x} + 36e^{-2x}$$

For maximum or minimum we know $\frac{dy}{dx} = 0$

$$8e^{2x} - 18e^{-2x} = 0$$

$$\Rightarrow e^{2x} = \frac{18}{8}e^{-2x}$$

$$\text{or, } e^{2x}.e^{2x} = \frac{18}{8} = \frac{9}{4}$$

$$\text{or } (e^{2x})^2 = \left(\frac{\pm 3}{2}\right)^2$$

$$\therefore e^{2x} = -\frac{3}{2} \text{ or, } e^{2x} = \frac{3}{2} \quad \text{or, } 2x = \log(3/2)$$

$$\text{and } e^{-2x} = \frac{2}{3}. \quad x = \pm\log(3/2)$$

When $e^{2x} = \frac{3}{2}$ or, $e^{-2x} = \frac{2}{3}$ then $\frac{d^2y}{dx^2} = 16\frac{3}{2} + 36\frac{2}{3} = 48$

which is positive.

Therefore y is minimum when $e^{2x} = \frac{3}{2}$

Hence the minimum value of $y = 4.\frac{3}{2} + 9.\frac{2}{3} = 12.$ (Showed)

Extra $\Big\{$ when $e^{2x} = -\frac{3}{2}$ then $\frac{d^2y}{dx^2} = 16.\left(\frac{-3}{2}\right) + 36\left(\frac{-2}{3}\right) = -48$ (-ve)

$\therefore y$ is maximum for $e^{2x} = -\frac{3}{2}.$

Maximum Value $= 4.\left(\frac{-3}{2}\right) + 9\left(\frac{-2}{3}\right) = -12.$

That is maximum value less than minimum value.

8. Show that the maximum value of $x+\frac{1}{x}$ is less than its minimum value.

Answer: Let us consider $y = x + \frac{1}{x}$

Differentiating w.r.t. x we have

$$\frac{dy}{dx} = 1 - \frac{1}{x^2} = 1 - x^{-2}$$

$$\therefore \frac{d^2y}{dx^2} = 0 - (-2)x^{-3}, = +\frac{2}{x^3}$$

For maximum or minimum $\frac{dy}{dx} = 0$

$$\therefore 1 - 1/x^2 = 0$$

$$\text{or, } \frac{x^2-1}{x^2} = 0$$

$$\Rightarrow x^2 - 1 = 0 \text{ or, } x^2 = 1 \text{ or } x^2 = (\pm 1)^2$$

$$\therefore x = 1 \text{ or } -1$$

when $x = 1$ then $\frac{d^2y}{dx^2} = \frac{2}{1^3} = 2$ which is +ve

$\therefore y$ is minimum for $x = 1$ and the minimum value is 2.

When $x = -1$ then $\frac{d^2y}{dx^2} = \frac{2}{(-1)^3} = -2$ which is negative

y is maximum for $x = -1$. Hence the maximum value of $y = -1 + \frac{1}{(-1)} = -2$

Therefore $y_{max} = -2$

$y_{min} = 2$

This is show that the maximum value of $x+\frac{1}{x}$ is less than its minimum value.

9. (i) Given $\frac{x}{2}+\frac{y}{3}=1$, find the maximum value of xy and minimum value of x^2+y^2.

(ii) Given $xy=4$, find the maximum and minimum values of $4x+9y$.

Solution: (i) Given $\frac{x}{2}+\frac{y}{3}=1$

or, $\frac{3x+2y}{6}=1$

$\Rightarrow 3x+2y=6 \Rightarrow 2y=6-3x$

$\therefore y=\frac{1}{2}(6-3x)=3-\frac{3}{2}x \longrightarrow (1)$

Let $U=xy$.

$\therefore U=x(3-\frac{3}{2}x)$ [using eq^n (1)]

$U=3x-\frac{3}{2}x^2 \longrightarrow (2)$

Differentiating (2) w. r to x two at a time

$\frac{du}{dx}=3-\frac{3}{2}\cdot 2\cdot x=3-3x$

$\frac{d^2U}{dx^2}=0-3=-3$

For max. or, min. $\frac{dU}{dx}=0 \Rightarrow 3-3x=0$

$\therefore x=1$

When $x=1$ then $\frac{d^2U}{dx^2}$ is negative.

$\therefore$ U is maximum for $x=1$.

Hence the maximum value $=3\cdot 1-\frac{3}{2}\cdot 1^2=\frac{3}{2}$.

Again let $V=x^2+y^2$

$V=x^2+(3-\frac{3}{2}x)^2$ [using eq^n (1)]

$\therefore V = x^2+9-\frac{3}{2}x\cdot 3\cdot 2+\left(\frac{3}{2}x\right)^2$

$= x^2+9-9x+\frac{9}{4}x^2$

$V = \frac{1}{4}13x^2-9x+9 \longrightarrow (3)$

Differentiating (3) w. r to x

$\frac{dV}{dx} = \frac{26}{4}x-9+0$

Again $\frac{d^2V}{dx^2} = \frac{26}{4}$

For max. or, minimum $\frac{dV}{dx} = 0$

$\frac{26}{4}x = 9$

or $x = \frac{9\times 4}{26} = \frac{18}{13}$

when $x = \frac{18}{13}$, $\frac{d^2V}{dx^2}$ is posistive.

Therefore V is minimum for $x = \frac{18}{13}$

The minimum value is $\frac{13}{4}\left(\frac{18}{13}\right)^2-9\cdot\frac{18}{13}+9 = \frac{36}{13}$

10. Find two positive numbers whose sum is 100 and the sum of whose square is minimum.

Solution : Let x and y be the two positive numbers

According to our condition $x+y=100$

$y = 100-x \longrightarrow (1)$

And x^2+y^2 is minimum

say $U = x^2+y^2$

$\therefore U = x^2+(100-x)^2 = x^2+(100)^2-200x+x^2$

$U = 2x^2-200x+(100)^2$

Diff. w. r to x :

$$\frac{dv}{dx} = 4x - 200 + 0$$

For minimum $\frac{dv}{dx} = 0$

$$4x - 200 = 0$$

$$\therefore x = 50$$

From eqn ① $y = 100 - 50 = 50$

Hence the two positive numbers are 50, 50.

Answer