ELEC 442/6601: Digital Signal Processing

1. (12 marks) When the input to an LTI system is

$$x(n) = (\tfrac{1}{2})^n u(n) + 2^n u(-n-1),$$

the output is

$$y(n) = 6(\tfrac{1}{2})^n u(n) - 6(\tfrac{3}{4})^n u(n)$$

a) Find the system function H(z) of the system. Plot the poles and zeros of H(z), and indicate the ROC.
b) Find the impulse response h(n) of the system.
c) Write the difference equation that characterizes the system.

2. (16 marks) For each of the following systems, determine whether it is (1) linear, (2) time-invariant, (3) causal, and (4) stable. Briefly explain your answer.

(a) y(n)=x(n)+nx(n+1)

(b) $y(n) = \begin{cases} x(n) & if\ x(n) \geq 0 \\ 0 & if\ x(n) < 0 \end{cases}$

3. (12 marks) A band-limited continuous-time signal has the Fourier transform shown in Fig.1. It is used as the input to the system shown in Fig.2. In this system, $H(j\Omega)$ is an ideal low-pass filter with the cutoff frequency $\frac{\pi}{T}$ and gain T. It is known that $f_M = 4\times 10^3$ Hz and $f_0 = 16\times 10^3$ Hz. Sketch $Y(j\Omega)$, $Y_s(j\Omega)$ and $Y_r(j\Omega)$ as well as express $y_r(t)$ in terms of $x(t)$ when the sampling duration is $T = \frac{1}{24}\times 10^{-3}$ seconds. (You must indicate all the interesting points in both x- and y-axes.

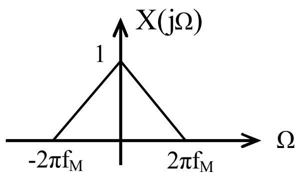

Fig. 1

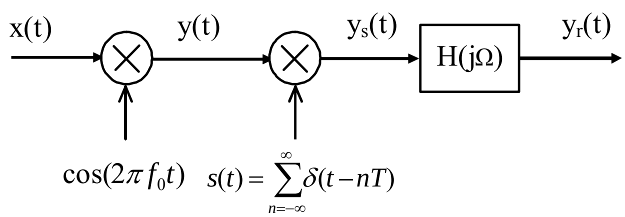

Fig. 2

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Q.1 (a)

$$x(n) = \left(\tfrac{1}{2}\right)^n u(n) + 2^n u(-n-1)$$

$$X(z) = \frac{1}{1-\frac{1}{2}z^{-1}} - \frac{z}{z-2} \qquad \tfrac{1}{2} < |z| < 2$$

$$= \frac{-\frac{3}{2}z^{-1}}{\left(1-\frac{1}{2}z^{-1}\right)\left(1-2z^{-1}\right)}$$

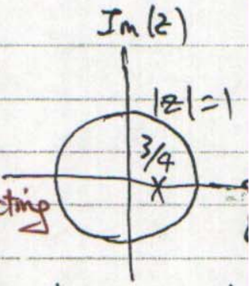
$$y(n) = 6\left(\tfrac{1}{2}\right)^n u(n) - 6\left(\tfrac{3}{4}\right)^n u(n)$$

$$Y(z) = \frac{6}{1-\frac{1}{2}z^{-1}} - \frac{6}{1-\frac{3}{4}z^{-1}}, \qquad |z| > \tfrac{3}{4}$$

$$= \frac{-\frac{3}{2}z^{-1}}{\left(1-\frac{1}{2}z^{-1}\right)\left(1-\frac{3}{4}z^{-1}\right)}$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1-2z^{-1}}{1-\frac{3}{4}z^{-1}}$$

ROC: $|z| > \frac{3}{4}$
(otherwise, contradicting with ROC of $Y(z)$)



(b) Note that $|z| > \frac{3}{4}$, i.e., the system is causal and stabl[e]

$$H(z) = \frac{1}{1-\frac{3}{4}z^{-1}} - \frac{2z^{-1}}{1-\frac{3}{4}z^{-1}}$$

$$h(n) = \left(\tfrac{3}{4}\right)^n u(n) - 2\left(\tfrac{3}{4}\right)^{n-1} u(n-1)$$

(c) From $H(z) = \dfrac{Y(z)}{X(z)} = \dfrac{1-2z^{-1}}{1-\frac{3}{4}z^{-1}}$,

$$Y(z) - \tfrac{3}{4}z^{-1}Y(z) = X(z) - 2z^{-1}X(z)$$

we obtain $\quad y(n) - \frac{3}{4}y(n-1) = x(n) - 2x(n-1)$

Q2. (a) (i)

$y_1(n) = x_1(n) + n\,x_1(n+1)$

$y_2(n) = x_2(n) + n\,x_2(n+1)$

$y_3(n) = x_3(n) + n x_3(n+1)$, $x_3(n) = a x_1(n) + b x_2(n)$

$= a x_1(n) + b x_2(n) + n\left[a x_1(n+1) + b x_2(n+1)\right]$

$= a x_1(n) + n \cdot a x_1(n+1) + b x_2(n) + n \cdot b x_2(n+1)$

$= a \cdot y_1(n) + b \cdot y_2(n)$

Linear.

(b) Let $x_1(n) = x(n-n_0)$

$y_1(n) = x_1(n) + n \cdot x_1(n+1) = x[n-n_0] + n \cdot x(n-n_0+1)$

$y(n-n_0) = x_1(n-n_0) + (n-n_0)\,x_1(n-n_0+1) \neq y_1(n)$

Not T.I.

(c) ~~Not Causal~~ ($y(n)$ depends on $x(n+1)$)

(d) ~~Not stable~~ (when $n \to \infty$. $y(n) \to \infty$)

(ii) $y(n) = \begin{cases} x(n) & x(n) \geq 0 \\ 0 & x(n) < 0 \end{cases}$

Not Linear

counter example: if $x(n) > 0$. $y(n) = x(n)$

However, for $a<0$, $a x(n) < 0$, then $y(n) = 0$

Time Invariant Let $x_1(n) = x(n-n_0)$

$T[x_1(n)] = y_1(n) = x_1(n)$ if $x_1(n) \geq 0$

$= 0$ if $x_1(n) < 0$

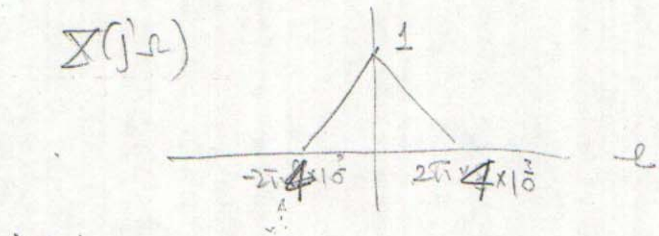
$\Rightarrow \begin{cases} x(n-n_0) & \text{if } x(n-n_0) \geq 0 \\ 0 & \text{if } < 0 \end{cases}$

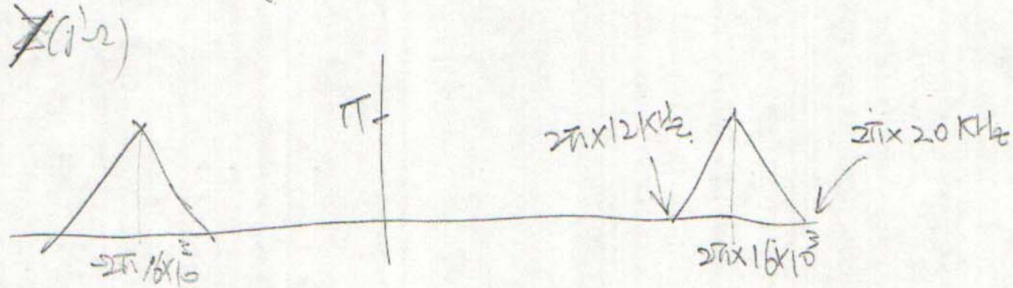
$= y(n-n_0)$

Causal & Stable

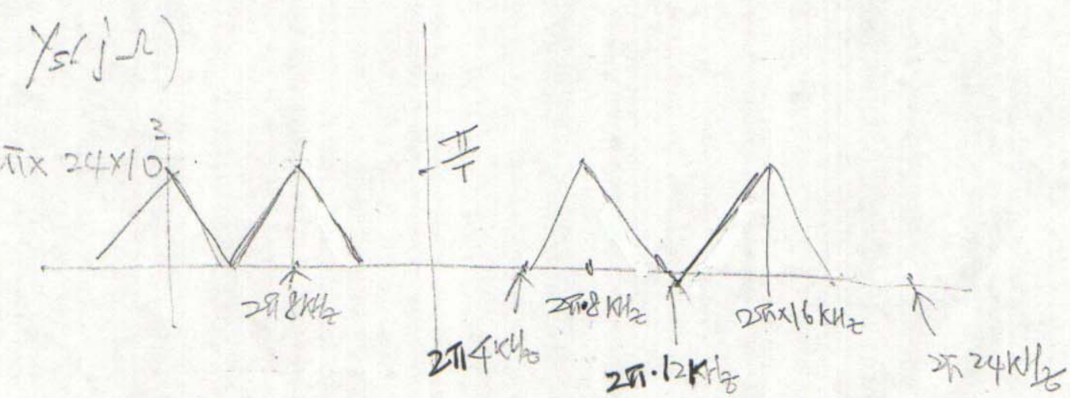
[if $x(n)$ is bounded then $y(n)$ is bounded
Also, $y(n)$ does not depend on $x(n+k)$ $(k>0)$]

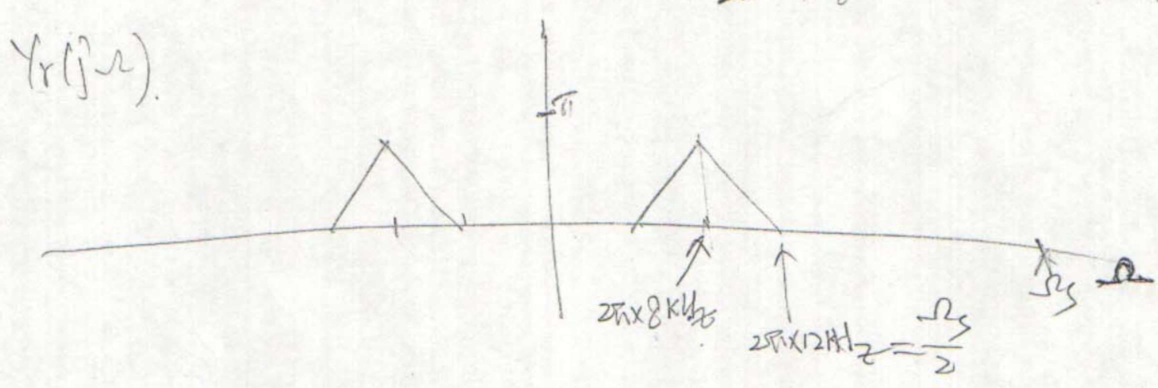
Q.3





$\Omega_s = 2\pi \frac{1}{T} = 2\pi \times 24 \times 10^3$





$y_r(t) = X(A) \cos(2\pi \times 8 \times 10^3 t)$