The Z Transform

- **Introduction to Z Transform**
- Z Transform and Examples
- Properties of the Region of Convergence of the Z Transform
- Inverse Z Transform and Examples
- Properties of Z Transform and Examples
- Analysis and characterization of LTI systems using the ZT
- Geometric evaluation of the Fourier transform from the pole-zero plot
- Types of filters
- Numerical calculation of the FT

M. Amer
Concordia University
Electrical and Computer Engineering

Figures and examples in these course slides are taken from the following sources:

- A. Oppenheim, A.S. Willsky and S.H. Nawab, Signals and Systems, Prentice-Hall
- M.J. Roberts, Signals and Systems, McGraw Hill, 2004
- J. McClellan, R. Schafer, M. Yoder, Signal Processing First, Prentice Hall, 2003

DT-FT ?

- DTFT represents a DT <u>aperiodic</u> signal as a sum of infinitely many complex exponentials, with the frequency varying continuously in (-π, π)

$$\mathrm{H}(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h[n]e^{-j\omega n};\ h[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} H(e^{j\omega})e^{j\omega n}d\omega \qquad \omega \in (-\pi, \pi)$$

- What if the DTFT of *x[n]* does not exist, i.e., $\sum_{n=-\infty}^{\infty} |h[n]| \to \infty$?
  - We can use the z−transform (ZT) as an extension of the DTFT

- Let *h[n]* be the impulse response of a LTI system
- Its response to a complex exponential input of the form z^n is $y[n] = H(z)z^n$ where $H(z) = \sum_{n=-\infty}^{\infty} h[n]z^{-n}$; z = $re^{j\omega}$; r, ω real

- *H(z)*, the ZT of h[n], is a complex function (since z is a complex quantity)

(Why do we deal with complex signals? They are often analytically simpler to deal with than real signals)

The Z Transform

- *H(z)* is called the system function: $H(z) = \sum_{n=-\infty}^{\infty} h[n] z^{-n}$
- Since the ZT is a power series, it converge when $h[n]z^{-n}$ is absolutely summable, i.e.,

$$\sum_{n=-\infty}^{\infty} | h[n] z^{-n} | < \infty$$

➔H(z) is defined for a region in z, called the region of convergence (ROC), for which the sum exists

- The FT of *h[n]* can be obtained by evaluating the ZT at $z = e^{j\omega}$, i.e., r=1

$$H(e^{jw}) = \sum_{n=-\infty}^{\infty} h[n] e^{-jwn}$$

➔For LTI systems we have:

- h[n] is the **impulse response**
- $H(e^{j\omega})$ is the **frequency response**
- H(z) is the **system function**

The Z Transform:
Rational function/ Poles and Zeros

- Often, the ZT is a ratio of polynomials (rational function of *z*):

$$H(z) = \frac{P(z)}{Q(z)}; \quad z = re^{j\omega}$$

- For any two polynomials, their ratio is called a rational function

$$H(z) = \frac{b_0 + b_1 z^{-1} + \cdots + b_M z^{-M}}{a_0 + a_1 z^{-1} + \cdots + a_N z^{-N}} \qquad \text{Example: } H(z) = \frac{z^2 - 4}{2z^2 + z - 3} = \frac{(z+2)(z-2)}{(2z+3)(z-1)}$$

- The ZT is undefined when the denominator equals zero, i.e., we have discontinuities (poles) in the function
- The ZT is zero when the numerator equals zero, i.e., we have zeros in the function

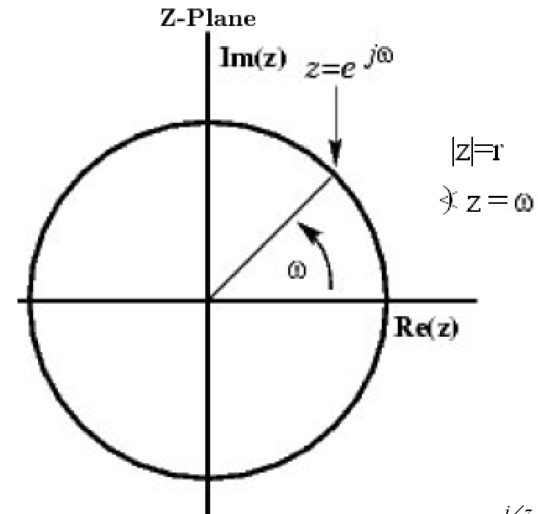
- ***Zeros:*** All value(s) for z where P (z) = 0, i.e., all *z* that make *H(z)* zero
- ***Poles:*** All value(s) for z where Q(z) = 0, i.e., all z that make H(z) infinite

➔ The rational function (the ZT) is characterized by its zeros and poles

The Z Transform:
The Z plane (complex plane)

- The z-plane is a complex plane of the complex-valued variable $z = re^{j\omega}$, with a real and imaginary axis
- The position on the complex plane is given in a polar form by $re^{j\omega}$
  - ω: the angle from the positive real axis around the plane
- Once the poles and zeros are found, they can be plotted into the z plane

- $H(z)$ is defined everywhere on this plane
- $H(e^{j\omega})$ is defined only where $|z| = 1$, which is referred to as the unit circle
  - This is useful because by representing the FT as the ZT on the unit circle, the periodicity of FT is easily seen



$$z = x + jy = |z| e^{j\angle z}$$

$$|z| = \sqrt{x^2 + y^2}$$

$$\omega = \angle z = \arctan \frac{y}{x}$$

$$\omega \in [-\pi, \pi]$$

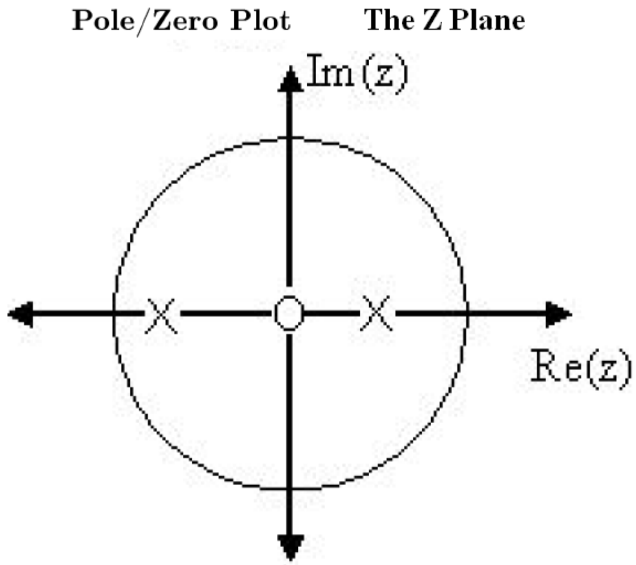
Example 1

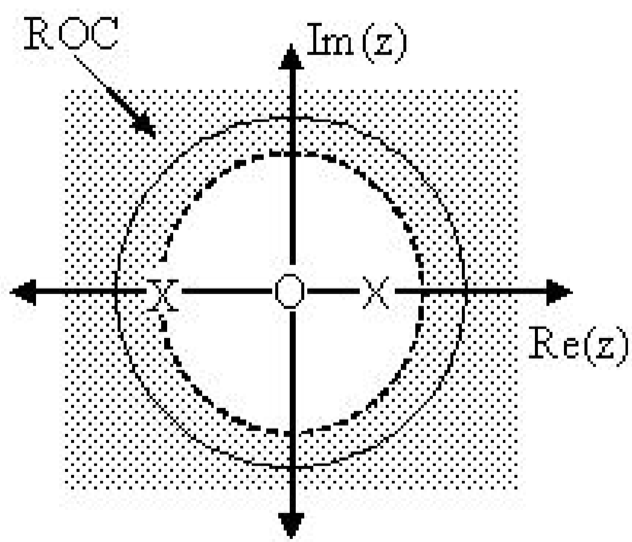
$$H(z) = \frac{z}{\left(z - \frac{1}{2}\right)\left(z + \frac{3}{4}\right)}$$

The zeros are: $\{0\}$

The poles are: $\left\{\frac{1}{2}, -\left(\frac{3}{4}\right)\right\}$

- Poles are denoted by "x" and zeros by "o"
- We use shaded regions to indicates the Region of Convergence (ROC) for the z transform





Example 2

$$H(z) = \frac{(z-j)(z+j)}{(z+1)(z-(\frac{1}{2}-\frac{1}{2}j))(z-(\frac{1}{2}+\frac{1}{2}j))}$$

The zeros are: {j, -j}

The poles are: $\{-1, \frac{1}{2}-\frac{1}{2}j, \frac{1}{2}+\frac{1}{2}j\}$

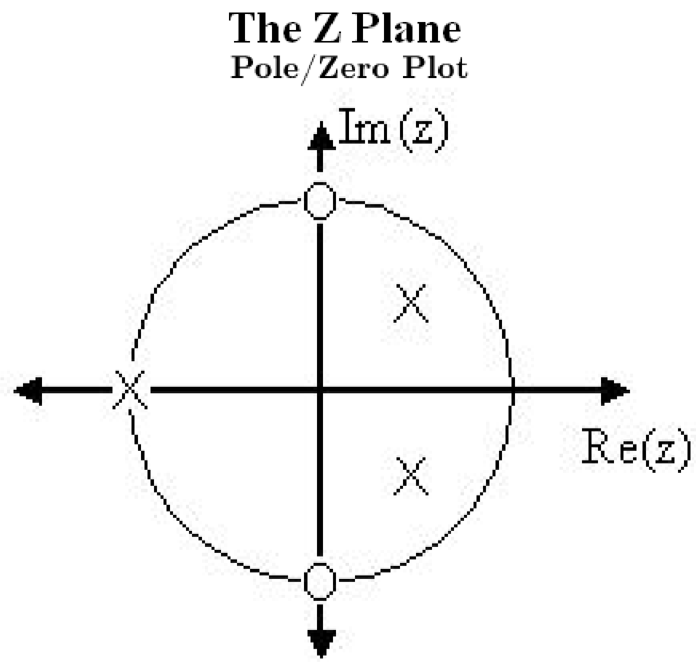
Recall:

$$z = x + jy = |z| e^{j\angle z}$$

$$|z| = \sqrt{x^2 + y^2}$$

$$\omega = \angle z = \arctan \frac{y}{x}$$

$$\omega \in [-\pi, \pi]$$



- In MATLAB you can create pole/zero plots:

```
% Set up vector for zeros
z = [j ; -j];
% Set up vector for poles
p = [-1 ; .5+.5j ; .5-.5j];
figure(1);
zplane(z,p);
title('Pole/Zero Plot');
```

Outline

- Introduction to Z Transform
- **Z Transform and Examples**
- Properties of the Region of Convergence of the Z Transform
- Inverse Z Transform and Examples
- Properties of Z Transform and Examples
- Analysis and characterization of LTI systems using z-transforms
- Geometric evaluation of the Fourier transform from the pole-zero plot
- Summary

The Z Transform: right-sided x[n]

- Consider the signal $x[n] = a^n u[n]$, a is real
- The $z-$transform of $x[n]$ is $X(z) = \sum_{n=-\infty}^{\infty} a^n u[n] z^{-n} = \sum_{n=0}^{\infty} (az^{-1})^n$
- For $X(z)$ to converge, we require that $\sum_{n=0}^{\infty} |az^{-1}|^n < \infty$
- The ROC is the range of values of z for which $|az^{-1}| < 1$, or equivalently, $|z| > |a|$
- Then $X(z) = \sum_{n=0}^{\infty} (az^{-1})^n = \dfrac{1}{1 - az^{-1}} = \dfrac{z}{z-a}$, $|z| > |a|$
- Therefore, the $z-$transform exists for any value of a, with a ROC determined by the magnitude of a according to the above equation
- When $a = 1$, $x[n]$ is the unit step sequence with $z-$transform

$$X(z) = \frac{1}{1 - 1z^{-1}}, \quad |z| > 1$$

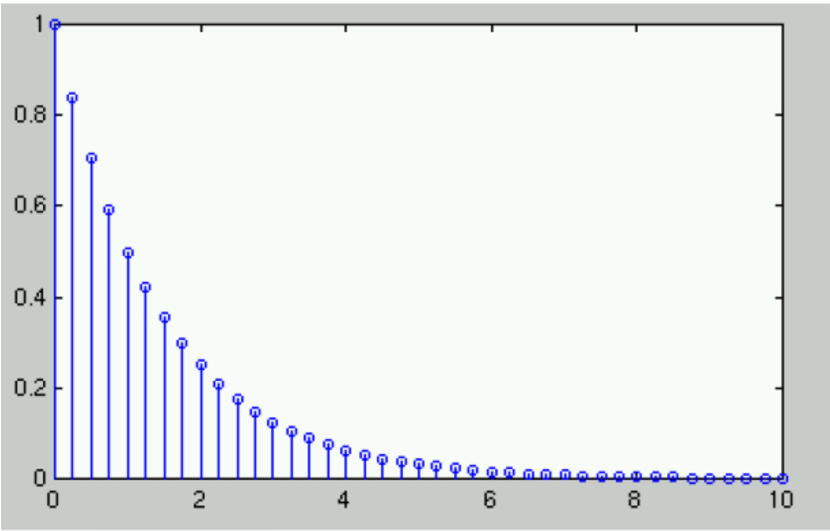
Power series

Infinite Geometric Series

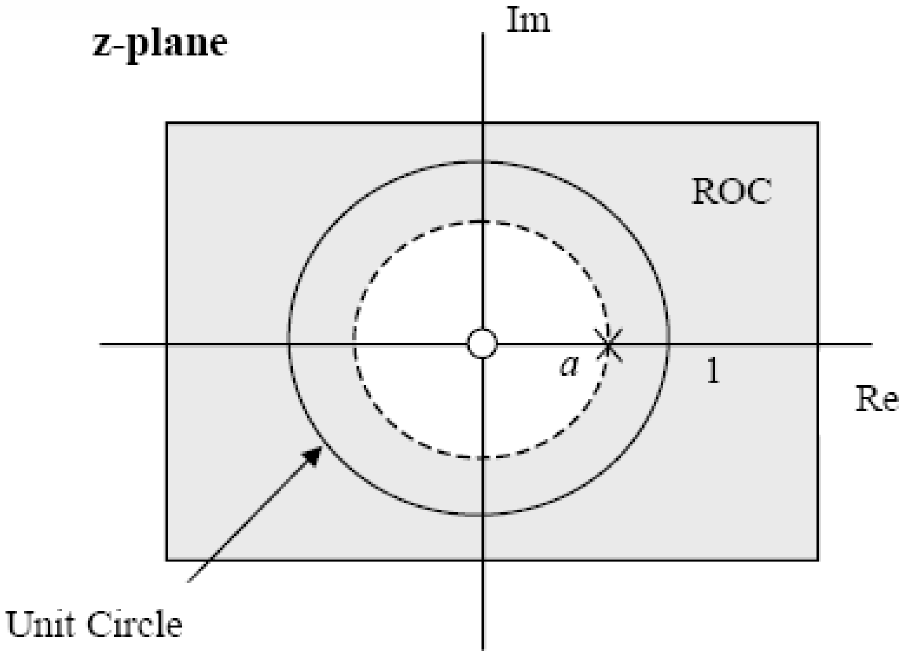
$$\sum_{n=m}^{\infty} cx^n = \frac{cx^m}{1-x}$$

The Z Transform: right-sided x[n]

$$x[n] = a^n u[n] \longleftrightarrow \frac{1}{1 - az^{-1}} = \frac{z}{z-a}, \quad |z| > |a|$$



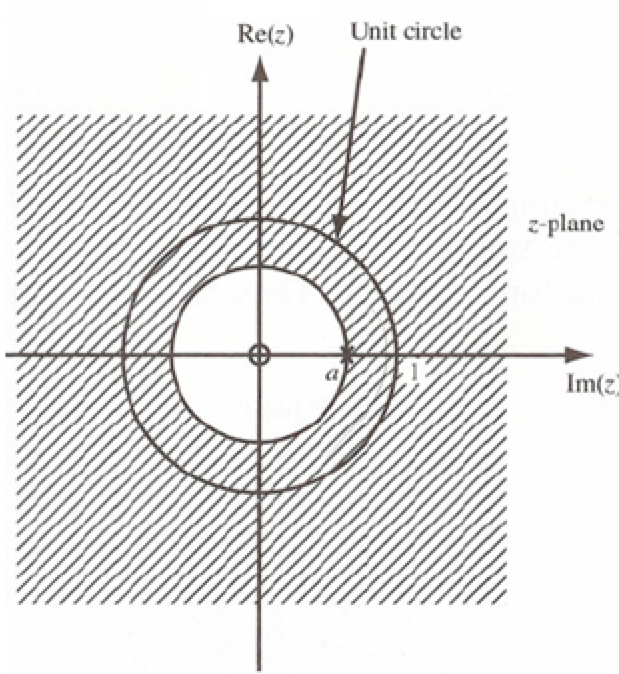
$x[n] = a^n u[n]$ where $a = 0.5$

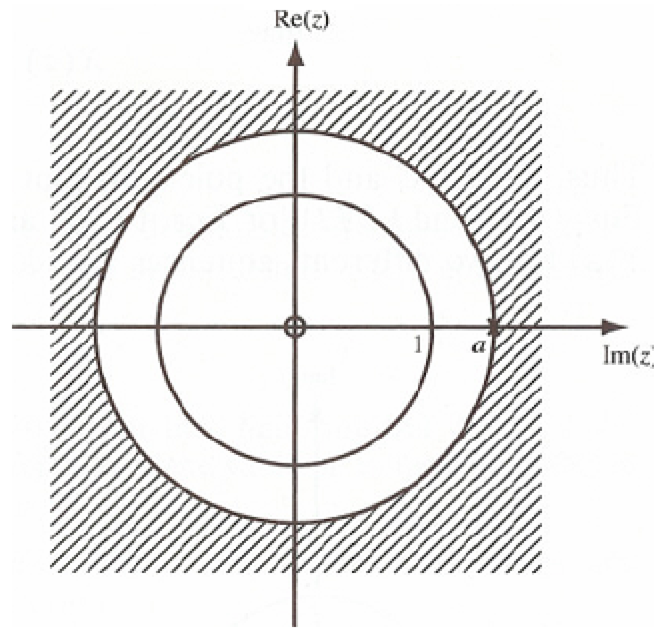


- x[n] is right-sided; it decays when a<1 (e.g., a=0.5)
- Its ZT is a rational function with **one zero at z = 0 and one pole at z = a**

The Z Transform: ROC in the form |z| > |a|

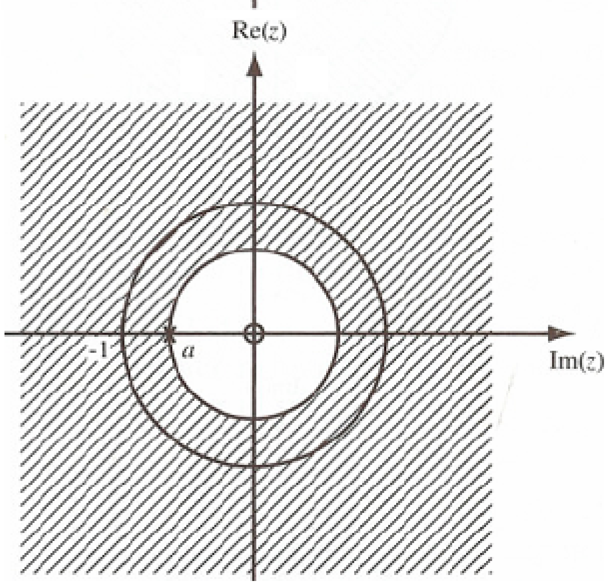
0 < a < 1

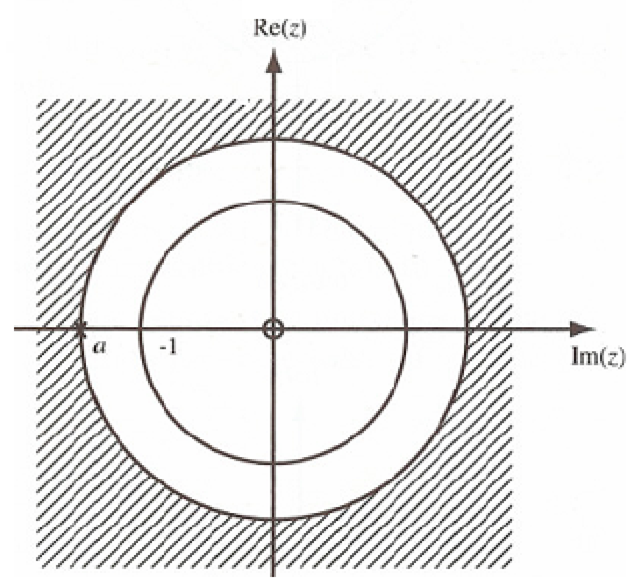




a > 1

-1 < a < 0





a < -1

The Z Transform: Left-sides *x[n]*

- Let $x[n] = -a^n u[-n-1]$

Then $X(z) = -\sum_{n=-\infty}^{\infty} a^n u[-n-1] z^{-n}$ Note that $u[-n-1] = 0 \quad \forall n : \underbrace{-n-1 < 0}_{n>-1}$,

$$X(z) = -\sum_{n=-\infty}^{\infty} a^n u[-n-1] z^{-n} = -\sum_{n=-\infty}^{-1} a^n \underbrace{u[-n-1]}_{1} z^{-n} - \sum_{n=0}^{\infty} a^n \underbrace{u[-n-1]}_{0} z^{-n}$$

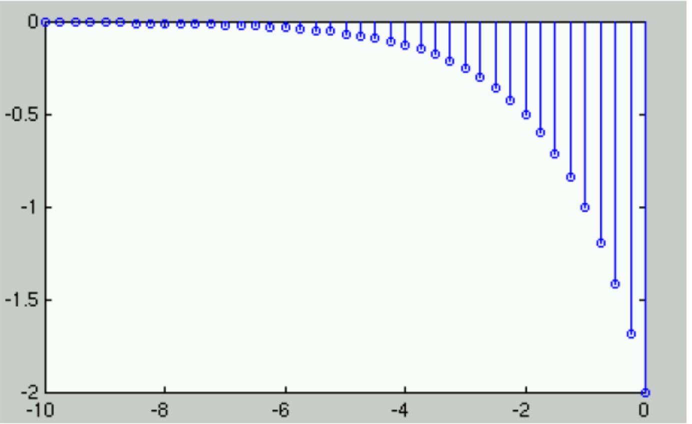
$$= -\sum_{n=-\infty}^{-1} a^n z^{-n} = -\sum_{n=-\infty}^{-1} (a^{-1} z)^{-n} \qquad \text{(by combining the power n)}$$

$$= -\sum_{n=1}^{\infty} (az^{-1})^n \quad \text{(by multiplying every n by -1)}$$

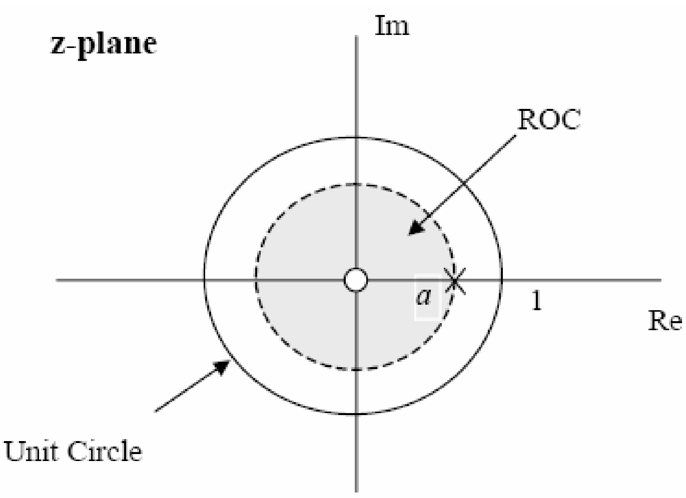
$$\Rightarrow -\sum_{n=1}^{\infty} (az^{-1})^n = 1 - \sum_{n=0}^{\infty} (az^{-1})^n \quad \left(\text{because } 1 - (az^{-1})^0 - \sum_{n=1}^{\infty} (az^{-1})^n \right)$$

- This sum converges if $|a^{-1} z| < 1$ or $|z| > |a^{-1}|$, consequently

$$X(z) = 1 - \frac{1}{1 - a^{-1} z} = \frac{1}{1 - az^{-1}} = \frac{z}{z-a}, \qquad |z| < |a|$$

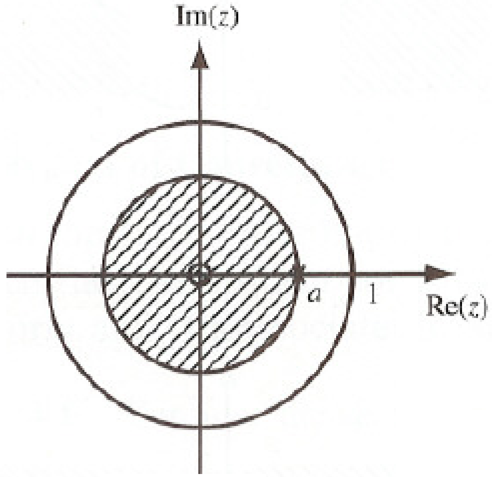


$x[n] = -a^n u[-n-1]$ where $a = 0.5$.

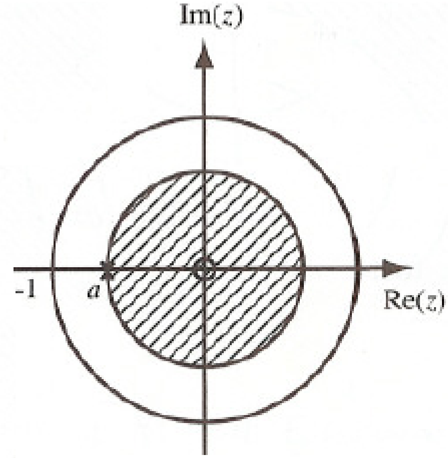


We note: The algebraic expression of X(z) for *x[n]=a^n u[n]* and *x[n]=-a^n u[-n-1]* are identical but the ROCs of their ZT is different

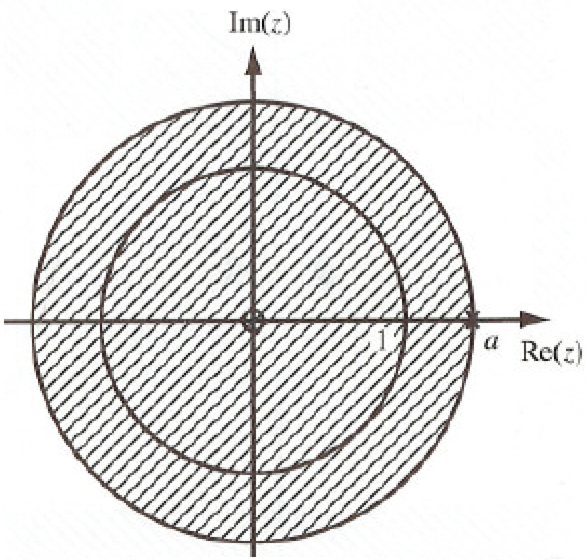
The Z Transform: ROC in the form |z| < |a|



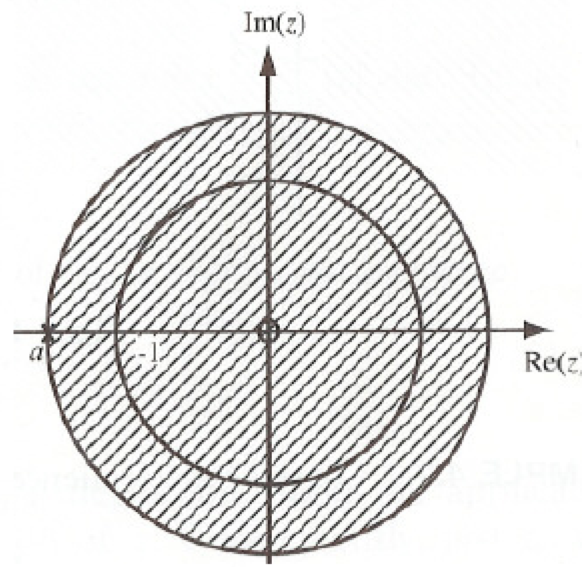
0 < a < 1



-1 < a < 0



a > 1



a <- 1

Example: Multiple Poles

- let

$$x[n] = 7\left(\frac{1}{3}\right)^n u[n] - 6\left(\frac{1}{2}\right)^n u[n].$$

- The z–transform is given by

$$\begin{aligned}
X(z) &= \sum_{n=-\infty}^{\infty}\left(7\left(\frac{1}{3}\right)^n u[n] - 6\left(\frac{1}{2}\right)^n u[n]\right) z^{-n} \\
&= 7\sum_{n=0}^{\infty}\left(\frac{1}{3}z^{-1}\right)^n - 6\sum_{n=0}^{\infty}\left(\frac{1}{2}z^{-1}\right)^n \\
&= \frac{7}{1-\frac{1}{3}z^{-1}} - \frac{6}{1-\frac{1}{2}z^{-1}} \\
&= \frac{z\left(z-\frac{3}{2}\right)}{\left(z-\frac{1}{3}\right)\left(z-\frac{1}{2}\right)}
\end{aligned}$$

Example: Multiple Poles

- We first find the ROC for each term individually and then find the ROC of both terms combined

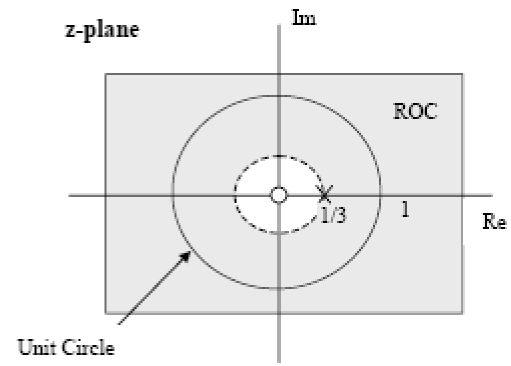
- Provided that

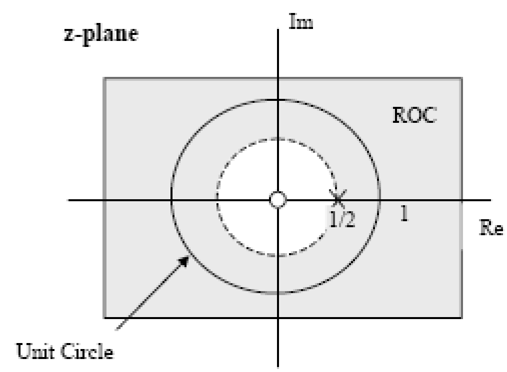
  $|\frac{1}{3}z^{-1}|<1$ and $|\frac{1}{2}z^{-1}|<1$

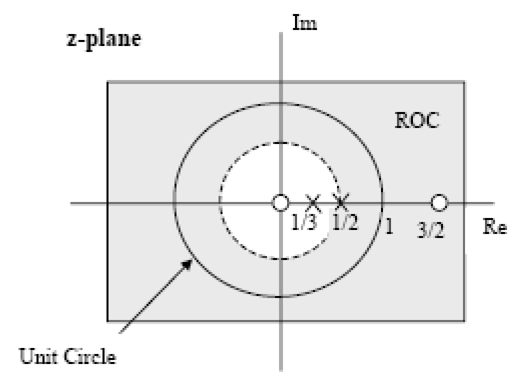
  or equivalently

  $|z| > \frac{1}{3}$ and $|z| > \frac{1}{2}$

$\Rightarrow$ The ROC is $|z| > \frac{1}{2}$







Outline

- Introduction to Z Transform
- Z Transform and Examples
- **Properties of the Region of Convergence**
- Inverse Z Transform and Examples
- Properties of Z Transform and Examples
- Analysis and characterization of LTI systems using z-transforms
- Geometric evaluation of the Fourier transform from the pole-zero plot
- Summary

Properties of the ROC

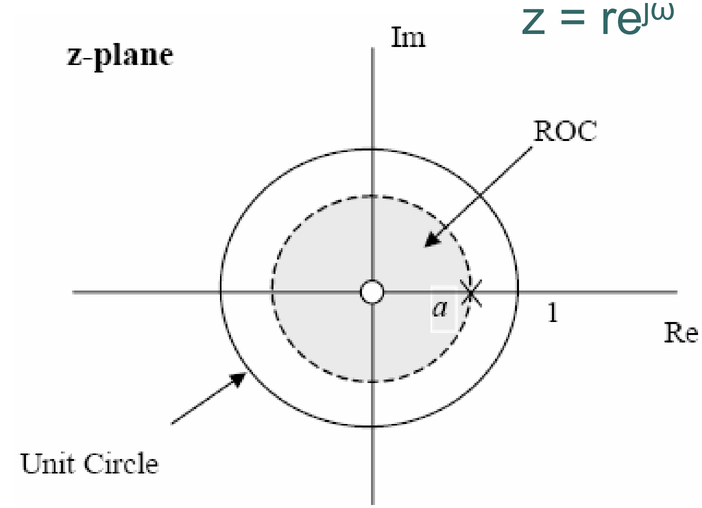
$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$

Important properties of the ROC of the ZT:

1. The ROC of X(z) consists of a ring in the z-plane centered about the origin with $0 \leq r_r < |z| < r_l \leq \infty$
   - Convergence depends only on *r*, not on ω

2. The ROC does not contain ay poles

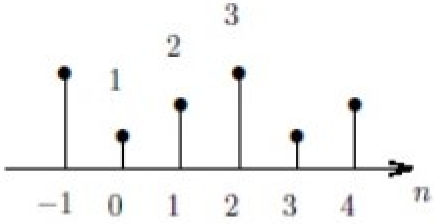
$z = re^{j\omega}$



Properties of the ROC

3. A finite-duration sequence *x[n]* is nonzero in a finite interval $N_1 < \mathrm{n} < N_2$

$$X(z) = \sum_{n=-N_1}^{N_2} x[n] z^{-n}$$



- As long as each value if *x[n]* is finite then the sequence will be absolutely summable

- When N_2>0, there will be a z^{-1} term and thus the ROC will not include z=0
- When $N_2 \leq 0$, the ROC will include z=0
- When $N_1 < 0$, the sum will be finite and thus the ROC will not include |z|=∞
- When $N_1 \geq 0$, the ROC will include |z|= ∞

➔With these constraints, the only signal with ROC as the entire z-plane is $x[n] = c\delta(n)$

Properties of the ROC
Examples

- **Example 1:** Let $x[n] = \delta[n]$, Then

$$X(z) = \sum_{n=-\infty}^{\infty} \delta[n] z^{-n} = 1$$

  which suggests that the ROC is the entire z−plane, including $z = 0$ and $z = \infty$

- **Example 2:** Now consider $x[n] = \delta[n - n_0]$ where $n_0 \neq 0$
  X(z) then becomes

$$X(z) = \sum_{n=-\infty}^{\infty} \delta[n - n_0] z^{-n} = z^{-n_0}$$

  - If $n_0 > 0$ then the ROC contains the entire z−plane except at $z = 0$
  - But if $n_0 < 0$, the ROC contains the entire z−plane except at $z = \infty$

Properties of the ROC
Examples

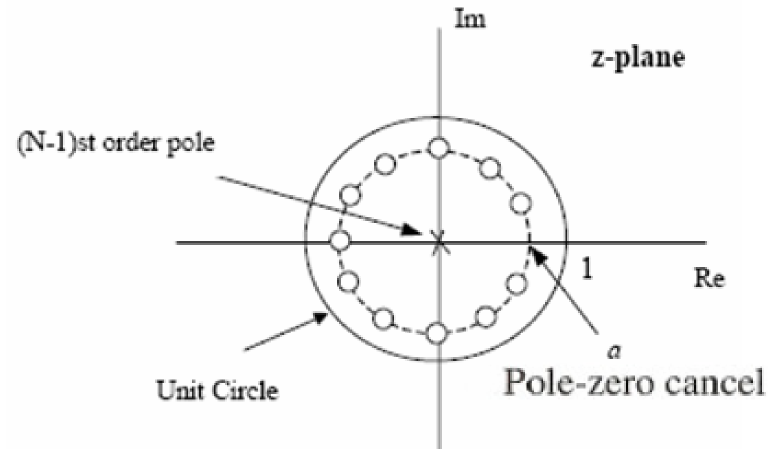
Find X(z) and plot its poles and zeros

- Consider the signal $x[n] = \begin{cases} a^n, & 0 \le n \le N-1, a > 0 \\ 0, & \text{otherwise} \end{cases}$

$$\text{Then } X(z) = \sum_{n=0}^{N-1} a^n z^{-n} = \sum_{n=0}^{N-1} (az^{-1})^n = \frac{1-(az^{-1})^N}{1-az^{-1}} = \frac{1}{z^{N-1}} \frac{z^N - a^N}{z-a}$$

- Since the system has a finite impulse response and is zero for n<0, then we should expect, according to Property 3, the ROC to include the entire z−plane except possibly at z = 0 and/or z = ∞
- X(z) has
  - N zeros at $z_k = ae^{j(2\pi k/N)}, \quad k = 0, \ldots, N-1$
  - a pole at z = 0 of order N−1
  - a pole at z = a but there is also a zero at z=a

  ➔ the pole at z=a and zero at z = a (k=0) cancel out
  - What is left is a polynomial in the numerator of degree N −1, suggesting that there are N−1 zeros $z_k = ae^{j(2\pi k/N)}, \quad k = 1, \ldots, N-1$



N=8

Properties of the ROC

4. If *x[n]* is a **right-sided** sequence, and if the circle $|z| = r_0$ is in the ROC, then all finite values of z for which $|z| > r_0$ will also be in the ROC
   - For a right-sided sequence, the ROC is bounded on the inside by a circle and extending outward to infinity

5. If *x[n]* is a **left-sided** sequence, and if the circle $|z| = r_0$ is in the ROC, then all finite values of z for which $0 < |z| < r_0$ will also be in the ROC
   - For a left-sided sequence, the ROC is bounded on the outside by a circle and extending to the origin

6. If *x[n]* is a **two-sided** sequence, and if the circle $|z| = r_0$ is in the ROC, then all finite values of z for which $|z| = r_0$ will also be in the ROC
   - A two-sides sequence can decomposed into at least one left-sided sequence and one right-sided sequence
   - The ROC for both sequences combined is the intersection of both individual ROCs

Properties of the ROC

7. If the ZT $X(z)$ of *x[n]* is **rational**, then its ROC is bounded by the poles or extends to infinity

8. If the ZT $X(z)$ of $x[n]$ is rational, and if $x[n]$ is **right-sided**, then the ROC is the region in the z−plane outside the outermost pole (i.e., outside the circle of radius equal to the largest magnitude of the poles of $X(z)$
   - if **x[n] is causal** (i.e., if it is right-sided and equal to 0 for $n < 0$), then the ROC also includes $z = \infty$

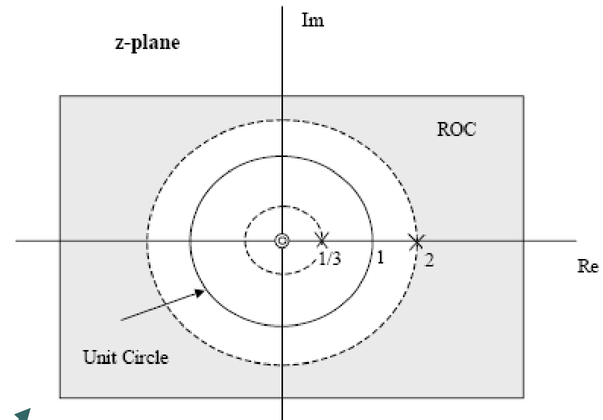
9. If the ZT $X(z)$ of $x[n]$ is rational, and if $x[n]$ is **left-sided**, then the ROC is the region in the z−plane inside the innermost pole (i.e., outside the circle of radius equal to the smallest magnitude of the poles of $X(z)$ other than any at $z = 0$ and extending inward to and possibly including $z = 0$
   - if $x[n]$ is anti-causal (i.e., if it is left-sided and equal to 0 for $n > 0$), then the ROC also includes $z = 0$

Properties of the ROC
Examples

- Consider $X(z) = \dfrac{1}{\left(1 - \frac{1}{3}z^{-1}\right)\left(1 - 2z^{-1}\right)}$
- Since there are two poles; then there are three possibilities for the ROC:

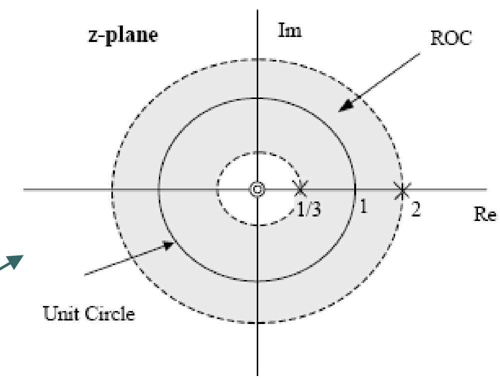
1) ROC: |z| >2

   ➔the ROC is extending outward from the outermost pole, suggesting that the sequence x[n] is right-sided



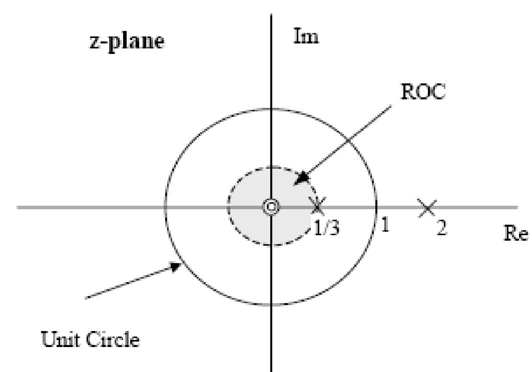
2) ROC: 1/3 < |z| <2

   ➔the ROC is bounded between two poles, suggesting that the sequence x[n] is two-sided



3) ROC: |z| < 1/3

   ➔the ROC is inward from the innermost pole, suggesting that the sequence x[n] is left-sided



Outline

- Introduction to Z Transform
- Z Transform and Examples
- Properties of the Region of Convergence
- **Inverse Z Transform and Examples**
- Properties of Z Transform and Examples
- Analysis and characterization of LTI systems using z-transforms
- Geometric evaluation of the Fourier transform from the pole-zero plot
- Summary

The inverse Z transform

- When using the z-transform $X(z) = \sum_{n=-\infty}^{\infty} \left(x[n] z^{-n}\right)$

  it is often useful to be able to find x [n] given X (z) (inverse transform)

- The equation for the inverse *z*-transform is a contour integration in the *z*-plane (counterclockwise):

$$x[n] = \frac{1}{2\pi j} \oint X(z) z^{n-1} dz$$

The inverse ZT: Common ZT Pairs

	Signal, $x[n]$	Z-transform, $X(z)$	ROC
1	$\delta[n]$	1	all z
2	$\delta[n-n_0]$	$\frac{1}{z^{n_0}}$	$\lvert z\rvert > 0$
3	$u[n]$	$\frac{z}{z-1}$	$\lvert z\rvert > 1$
4	$a^n u[n]$	$\frac{1}{1-az^{-1}}$	$\lvert z\rvert > \lvert a\rvert$
5	$na^n u[n]$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert > \lvert a\rvert$
6	$-a^n u[-n-1]$	$\frac{1}{1-az^{-1}}$	$\lvert z\rvert < \lvert a\rvert$
7	$-na^n u[-n-1]$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert < \lvert a\rvert$
8	$\cos(\omega_0 n)u[n]$	$\frac{1-z^{-1}\cos(\omega_0)}{1-2z^{-1}\cos(\omega_0)+z^{-2}}$	$\lvert z\rvert > 1$
9	$\sin(\omega_0 n)u[n]$	$\frac{z^{-1}\sin(\omega_0)}{1-2z^{-1}\cos(\omega_0)+z^{-2}}$	$\lvert z\rvert > 1$
10	$a^n\cos(\omega_0 n)u[n]$	$\frac{1-az^{-1}\cos(\omega_0)}{1-2az^{-1}\cos(\omega_0)+a^2z^{-2}}$	$\lvert z\rvert > \lvert a\rvert$
11	$a^n\sin(\omega_0 n)u[n]$	$\frac{az^{-1}\sin(\omega_0)}{1-2az^{-1}\cos(\omega_0)+a^2z^{-2}}$	$\lvert z\rvert > \lvert a\rvert$

The inverse Z transform

- There are *at least* 4 different methods :
  1. Inspection
  2. Partial-Fraction Expansion
  3. Power Series Expansion
  4. Long Division
- **Inspection method**: become familiar with the z-transform pair tables and then "reverse engineer"

Example

When given

$$X(z) = \frac{z}{z - \alpha}$$

with an ROC of

$$|z| > \alpha$$

we could determine "by inspection" that

$$x[n] = \alpha^n u[n]$$

The inverse Z transform

- **Partial-Fraction Expansion**: Find the *partial fraction expansion* for the *ZT* expression that is a rational function of *z*
  - We get

$$X(z) = \sum_{i=1}^{m} \frac{A_i}{1 - a_i z^{-1}}$$

  - Then we take the inverse ZT of *each individual term* easily

- Other forms:

$$H(z) = \frac{\sum_{m=0}^{M} b_m z^{-m}}{\sum_{k=0}^{N} a_k z^{-k}} = \left(\frac{b_0}{a_0}\right) \frac{\prod_{m=1}^{M} (1 - c_m z^{-1})}{\prod_{k=1}^{N} (1 - d_k z^{-1})};$$

zeros at c_m and poles at d_k

The inverse Z transform
Partial Fraction Expansion: Examples

- Find the inverse z-transform of

$$X(z) = \frac{1 + 2z^{-1} + z^{-2}}{1 + (-3z^{-1}) + 2z^{-2}}$$

where the ROC is $|z| > 2$. In this case $M = N = 2$, so we have to use long division to get

$$X(z) = \frac{1}{2} + \frac{\frac{1}{2} + \frac{7}{2}z^{-1}}{1 + (-3z^{-1}) + 2z^{-2}}$$

- Next factor the denominator

$$X(z) = \frac{1}{2} + \frac{\frac{1}{2} + \frac{7}{2}z^{-1}}{(1 - 2z^{-1})(1 - z^{-1})}$$

- Now do partial-fraction expansion

$$X(z) = \frac{1}{2} + \frac{A_1}{1 - 2z^{-1}} + \frac{A_2}{1 - z^{-1}} = \frac{1}{2} + \frac{\frac{9}{2}}{1 - 2z^{-1}} + \frac{-4}{1 - z^{-1}}$$

- Now each term can be inverted using the inspection method and the z-transform table. Thus, since the ROC is $|z| > 2$,

$$x[n] = \frac{1}{2}\delta[n] + \frac{9}{2}2^n u[n] + (-4u[n])$$

The inverse Z transform
Partial Fraction Expansion: Examples

- Find the inverse z-transform of the following expression: $X(z)=\dfrac{3-\frac{5}{6}z^{-1}}{(1-\frac{1}{4}z^{-1})(1-\frac{1}{3}z^{-1})}$

  with the following regions of convergence:

  (a) $|z|>\dfrac{1}{3}$ (b) $\dfrac{1}{4}<|z|<\dfrac{1}{3}$

- **Solution:**

  The partial fraction expansion for this expression in z^{-1} is $X(z)=\dfrac{1}{1-\frac{1}{4}z^{-1}}+\dfrac{2}{1-\frac{1}{3}z^{-1}}$

  (a) Since the ROC of $X(z)$ is outside the outermost pole, the ROC corresponding to each term must be outside the pole associated with each one. This means that the signal corresponding to each term must be right-sided. Therefore, we will have:

  $x[n]=x_1[n]+x_2[n]$ $\quad x_1[n]\xleftrightarrow{z}\dfrac{1}{1-\frac{1}{4}z^{-1}},\quad |z|>\dfrac{1}{4}$ $\quad x_2[n]\xleftrightarrow{z}\dfrac{2}{1-\frac{1}{3}z^{-1}},\quad |z|>\dfrac{1}{3}$

  So: $x[n]=\left(\dfrac{1}{4}\right)^n u[n]+2\left(\dfrac{1}{3}\right)^n u[n]$

The inverse Z transform
Partial Fraction Expansion: Examples

- $$X(z) = \frac{3 - \frac{5}{6}z^{-1}}{(1 - \frac{1}{4}z^{-1})(1 - \frac{1}{3}z^{-1})} = \frac{1}{1 - \frac{1}{4}z^{-1}} + \frac{2}{1 - \frac{1}{3}z^{-1}} \qquad \text{(b) } \frac{1}{4} < |z| < \frac{1}{3}$$

(b) Since the ROC of $X(z)$ is the intersection of $|z| < \frac{1}{3}$ and $|z| > \frac{1}{4}$ (in the form of a ring), the signal corresponding to the pole $z = \frac{1}{3}$ is a left-sided signal and the signal corresponding to the pole $z = \frac{1}{4}$ is a right-sided signal. Therefore, we will have:

$$x[n] = x_1[n] + x_2[n] \qquad x_1[n] \xleftrightarrow{Z} \frac{1}{1 - \frac{1}{4}z^{-1}}, \quad |z| > \frac{1}{4} \qquad x_2[n] \xleftrightarrow{Z} \frac{2}{1 - \frac{1}{3}z^{-1}}, \quad |z| < \frac{1}{3}$$

So: $$x[n] = \left(\frac{1}{4}\right)^n u[n] - 2\left(\frac{1}{3}\right)^n u[-n-1]$$

The inverse Z transform: Long Division Method

- In algebra, ***polynomial long division*** is an algorithm for dividing a polynomial by another polynomial of lower degree
- A generalized version of the familiar arithmetic technique called ***long division***
- It can be done by hand, because it separates an otherwise complex division problem into smaller ones
- For any polynomials *F(z)* and *G(z)*, where the degree of F*(z)* is greater than or equal to the degree of *G(z)*, there exist unique polynomials *Q(z)* and *R(z)* such that

$$\frac{F(z)}{G(z)} = Q(z) + \frac{R(z)}{G(z)} \quad \Leftrightarrow \quad G(z)\,\overset{\displaystyle Q(z)}{\overline{\big)\,F(z)}} \atop \overline{R(z)}$$

with *R(z)* having smaller degree than *G(z)*

The inverse Z transform: Long Division Method: Examples

- Using long division, find the inverse z-transform of $X(z)=\dfrac{1}{1-az^{-1}}$ with the regions of convergence: (a) $|z|>|a|$ (b) $|z|<|a|$

- **Solution:**

(a) We have:

$$
\begin{array}{r|l}
 & 1+az^{-1}+a^2z^{-2}+\cdots \\ \hline
1-az^{-1} & 1 \\
 & \underline{1-az^{-1}} \\
 & \quad az^{-1} \\
 & \underline{az^{-1}-a^2z^{-2}} \\
 & \qquad a^2z^{-2}
\end{array}
$$

This means that: $\dfrac{1}{1-az^{-1}}=1+az^{-1}+a^2z^{-2}+\cdot$

Since $|z|>|a|$, we have $|az^{-1}|<1$ and the series $1+az^{-1}+a^2z^{-2}+\cdots$ converges. So, by comparing this series with the general equation for the z-transform of a signal we will have: $x[n]=0$ for $n<0$, $x[0]=1$, $x[1]=a$, $x[2]=a^2,\cdots$

and in general: $x[n]=a^n u[n]$

The inverse Z transform: Long Division Method: Examples

(b) For $|z| < |a|$ the series given by $\dfrac{1}{1-az^{-1}} = 1 + az^{-1} + a^2 z^{-2} + \cdots$ does not converge as $|az^{-1}| > 1$

Therefore, one can use the following form of long division:

$$
\begin{array}{r l}
 & -a^{-1}z - a^{-2}z^{2} - \cdots \\
\cline{2-2}
-az^{-1} + 1 \big) & 1 \\
 & \underline{1 - a^{-1}z} \\
 & \quad a^{-1}z
\end{array}
$$

This means that: $\dfrac{1}{1-az^{-1}} = -a^{-1}z - a^{-2}z^{2} - \cdots$

Since $|z| < |a|$, we have $|a^{-1}z| < 1$ and the series converges

So, by comparing this series with the general equation for the z-transform of a signal we will have: $x[n] = 0$ for $n \geq 0$, $x[-1] = -a^{-1}$, $x[-2] = -a^{-2}$, $\cdots$ and in general: $x[n] = -a^{n}u[-n-1]$

The inverse Z transform: Power Series Expansion

- One can find the inverse z-transform of non-rational expressions of z, by writing that expression as a power series (for example using Taylor expansion)
- The z-transform is defined as a power series in the form

$$X(z) = \sum_{n=-\infty}^{\infty} \left(x[n] z^{-n} \right)$$

- Then each term of the sequence x [n] can be determined by looking at the coefficients of the respective power of z^{-n}

Example Now look at the z-transform of a **finite-length sequence.**

$$\begin{aligned} X(z) &= z^2 \left(1 + 2z^{-1}\right) \left(1 - \tfrac{1}{2} z^{-1}\right) \left(1 + z^{-1}\right) \\ &= z^2 + \tfrac{5}{2} z + \tfrac{1}{2} + \left(-\left(z^{-1}\right)\right) \end{aligned}$$

In this case, since there were no poles, we multiplied the factors of $X(z)$. Now, by inspection, it is clear that

$$x[n] = \delta[n+2] + \frac{5}{2}\delta[n+1] + \frac{1}{2}\delta[n] + \left(-\left(\delta[n-1]\right)\right)$$

- One of the advantages of the power series expansion method is that many functions encountered in engineering problems have their power series' tabulated

➔Thus functions such as log, sin, exponent, sinh, etc, can be easily inverted

The inverse Z transform: Power Series Expansion: Examples

Example Suppose

$$X(z) = \log_n\left(1 + \alpha z^{-1}\right)$$

Noting that

$$\log_n(1+x) = \sum_{n=1}^{\infty}\left(\frac{-1^{n+1}x^n}{n}\right)$$

Then

$$X(z) = \sum_{n=1}^{\infty}\left(\frac{-1^{n+1}\alpha^n z^{-n}}{n}\right)$$

Therefore

$$x[n] = \begin{cases} \frac{-1^{n+1}\alpha^n}{n} \text{ if } n \geq 1 \\ 0 \text{ if } n \leq 0 \end{cases}$$

The inverse Z transform: Power Series Expansion: Examples

- Using the power series expansion, find the inverse z-transform of the following expression:

$$X(z) = e^{(a/z)}, \quad |z| > |a|$$

- **Solution:** With $|z| > |a|$ or equivalently $|az^{-1}| < 1$, we can use the Taylor series expansion for $e^{(a/z)}$ as follows:

$$e^{(az^{-1})} = 1 + az^{-1} + \frac{a^2 z^{-2}}{2!} + \cdots, \quad |az^{-1}| < 1$$

This implies that:

$$x[n] = \begin{cases} \dfrac{a^n}{n!}, & n \geq 0 \\ 0, & n < 0 \end{cases}$$

Outline

- Introduction to Z Transform
- Z Transform and Examples
- Properties of the Region of Convergence
- Inverse Z Transform and Examples
- **Properties of Z Transform and Examples**
- Analysis and characterization of LTI systems using z-transforms
- Geometric evaluation of the Fourier transform from the pole-zero plot
- Summary

Properties of the Z Transform

- Linearity

$$x[n] \xleftrightarrow{Z} X(z), \quad \mathrm{ROC} = \mathrm{R}_1$$

$$y[n] \xleftrightarrow{Z} Y(z), \quad \mathrm{ROC} = \mathrm{R}_2$$

$$\Rightarrow ax[n] + by[n] \xleftrightarrow{Z} aX(z) + bY(z), \quad \mathrm{ROC\ contains\ R}_1 \cap \mathrm{R}_2$$

- Time shift

$$x[n] \xleftrightarrow{Z} X(z), \quad \mathrm{ROC} = \mathrm{R}$$

$$\Rightarrow x[n - n_0] \xleftrightarrow{Z} z^{-n_0} X(z), \quad \mathrm{ROC} = \mathrm{R},\ \text{except for the possible addition or deletion of the origin or infinity}$$

- Examples:
  - The ZT of δ [*n*] is equal to 1 and the ROC is the entire *z*-plane
  - The ZT of δ [*n* −1] is equal to *z* and the ROC is the entire *z*-plane except for the infinity
  - The ZT of δ [*n* +1] is equal to z^{-1} and the ROC is the entire *z*-plane except for the origin

Properties of the Z Transform

- Scaling in the z-domain: For any complex number z_0 we have:

$$x[n] \xleftrightarrow{Z} X(z), \quad \mathrm{ROC} = \mathrm{R}$$
$$\Rightarrow z_0^n x[n] \xleftrightarrow{Z} X\left(\frac{z}{z_0}\right), \quad \mathrm{ROC} = |z_0|\mathrm{R}$$

- Time reversal:

$$x[n] \xleftrightarrow{Z} X(z), \quad \mathrm{ROC} = \mathrm{R}$$
$$\Rightarrow x[-n] \xleftrightarrow{Z} X\left(\frac{1}{z}\right), \quad \mathrm{ROC} = \frac{1}{R}$$

- Conjugation:

$$x[n] \xleftrightarrow{Z} X(z), \quad \mathrm{ROC} = \mathrm{R}$$
$$\Rightarrow x^*[n] \xleftrightarrow{Z} X^*(z^*), \quad \mathrm{ROC} = R$$

- Time Expansion (Up-sampling)

$$x_{(k)}[n] = \begin{cases} x[n/k], & \text{if } n \text{ is multiple of } k. \\ 0, & \text{if } n \text{ is not a multiple of } k. \end{cases}$$

$$x_{(k)}[n] \Leftrightarrow X(z^k) \quad \text{with ROC} = R^{1/k}$$

Properties of the Z Transform

- Convolution

$$x_1[n] \longleftrightarrow X_1(z) \ \ \mathrm{ROC}_1 \qquad x_2[n] \longleftrightarrow X_2(z) \ \ \mathrm{ROC}_2$$

$$x_1[n] * x_2[n] \longleftrightarrow X_1(z) X_2(z)$$

ROC at least the intersction of ROC_1 and ROC_2

- Accumlation (DT counterpart to "integration")

$$\mathrm{x[n]} \longleftrightarrow \mathrm{X(z)} \qquad \mathrm{ROC = R}$$

$$\sum_{\mathrm{k}=-\infty}^{n} x[n] \longleftrightarrow \frac{1}{1-z^{-1}} X(z) = \frac{z}{z-1} X(z)$$

$$ROC = R \bigcap \{|\, z \,| > 1\}$$

- Differentiation in the z-domain:

$$x[n] \stackrel{z}{\longleftrightarrow} X(z), \quad \mathrm{ROC = R}$$

$$\Rightarrow nx[n] \stackrel{z}{\longleftrightarrow} -z \frac{dX(z)}{dz}, \quad \mathrm{ROC} = R$$

Properties of the Z Transform

- The initial-value theorem: If $x[n]=0$ for $n<0$, then:

$$x[0]=\lim_{z\to\infty} X(z)$$

  This property can be easily seen from the definition of the z-transform.

- The final-value theorem: If $x[n]=0$ for $n<0$ and if $x[n]$ has a finite value as $n\to\infty$, then:

$$\lim_{n\to\infty} x[n]=\lim_{z\to 1}[(1-z^{-1})X(z)]$$

Properties of the Z Transform: Summary

	Time domain	Z-domain	ROC
Notation	$x[n] = \mathcal{Z}^{-1}\{X(z)\}$	$X(z) = \mathcal{Z}\{x[n]\}$	ROC: $r_2 < \|z\| < r_1$
Linearity	$a_1 x_1[n] + a_2 x_2[n]$	$a_1 X_1(z) + a_2 X_2(z)$	At least the intersection of ROC_1 and ROC_2
Time shifting	$x[n-k]$	$z^{-k} X(z)$	ROC, except $z=0$ if $k>0$ and $z=\infty$ if $k<0$
Scaling in the z-domain	$a^n x[n]$	$X(a^{-1}z)$	$\|a\| r_2 < \|z\| < \|a\| r_1$
Time reversal	$x[-n]$	$X(z^{-1})$	$\frac{1}{r_2} < \|z\| < \frac{1}{r_1}$
Conjugation	$x^*[n]$	$X^*(z^*)$	ROC
Real part	$\mathrm{Re}\{x[n]\}$	$\frac{1}{2}[X(z) + X^*(z^*)]$	ROC
Imaginary part	$\mathrm{Im}\{x[n]\}$	$\frac{1}{2j}[X(z) - X^*(z^*)]$	ROC
Differentiation	$nx[n]$	$-z\frac{\mathrm{d}X(z)}{\mathrm{d}z}$	ROC
Convolution	$x_1[n] * x_2[n]$	$X_1(z)X_2(z)$	At least the intersection of ROC_1 and ROC_2
Multiplication	$x_1[n]x_2[n]$	$\frac{1}{j2\pi}\oint_C X_1(v)X_2(\frac{z}{v})v^{-1}\mathrm{d}v$	At least $r_{1l}r_{2l} < \|z\| < r_{1u}r_{2u}$
Parseval's relation	$\sum_{n=-\infty}^{\infty} x_1[n]x_2^*[n]$	$\frac{1}{j2\pi}\oint_C X_1(v)X_2^*(\frac{1}{v^*})v^{-1}\mathrm{d}v$	

Outline

- Introduction to Z Transform
- Z Transform and Examples
- Properties of the Region of Convergence
- Inverse Z Transform and Examples
- Properties of Z Transform and Examples
- **Analysis and characterization of LTI systems using the ZT**
- Geometric evaluation of the Fourier transform from the pole-zero plot
- Summary

Analysis and characterization of LTI systems using the ZT

- Linear constant-coefficient difference (LCCD) equation

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{m=0}^{M} b_m x[n-m]$$

- LCCD in the ZT domain

$$\sum_{k=0}^{N} a_k z^{-k} Y(z) = \sum_{m=0}^{M} b_m z^{-m} X(z)$$

$$\Rightarrow H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{m=0}^{M} b_m z^{-m}}{\sum_{k=0}^{N} a_k z^{-k}}$$

- The ROC of H(z) is not specified but must be inferred with additional requirements on the system (e.g., stability)

Analysis and characterization of LTI systems using the ZT

Partial fraction expansion gives:

$$H(z) = \left(\frac{b_0}{a_0}\right)\frac{\prod_{m=1}^{M}(1-c_m z^{-1})}{\prod_{k=1}^{N}(1-d_k z^{-1})};$$

$(1-c_m z^{-1})$ in the numerator

$\Rightarrow$ a zero at $z = c_m$ a pole at $z = 0$

$(1-d_k z^{-1})$ in the denominator

$\Rightarrow$ a zero at $z = 0$ a pole at $z = d_k$

Inverse systems

$$G(z) = H(z)H_i(z) = 1$$

$$H_i(z) = \frac{1}{H(z)}$$

$$g[n] = h[n] * h_i[n] = \delta[n]$$

- Many systems have inverses
- Specially systems with rational system functions

$$H(z) = \left(\frac{b_0}{a_0}\right)\frac{\prod_{m=1}^{M}(1-c_m z^{-1})}{\prod_{k=1}^{N}(1-d_k z^{-1})}$$

$$H_i(z) = \left(\frac{a_0}{b_0}\right)\frac{\prod_{k=1}^{N}(1-d_k z^{-1})}{\prod_{m=1}^{M}(1-c_m z^{-1})}$$

- Poles become zeros and vice versa
- ROC: must have overlap between the two ZT functions for the sake of G(z)

Example

$$H(z) = \frac{1-0.5z^{-1}}{1-0.9z^{-1}}, |z| > 0.9$$

$$H_i(z) = \frac{1-0.9z^{-1}}{1-0.5z^{-1}} = \frac{1}{1-0.5z^{-1}} - \frac{0.9z^{-1}}{1-0.5z^{-1}}$$

So,

$$|z| > 0.5$$

$$h_i[n] = (0.5)^n u[n] - 0.9(0.5)^{n-1} u[n-1]$$

Analysis and characterization of LTI systems using *ZT*

- Consider a DT LTI system with the impulse response $h[n]$ and the system function $H(z)$
- From the convolution property we have:

$$y[n] = h[n] * x[n] \Rightarrow Y(z) = H(z)X(z)$$

Causality:

- For a causal DT-LTI system, h[n]=0 for n <0 ➔h[n] right-sided
- For an anticausal DT-LTI system, h[n]=0 for n≥0 ➔h[n] left-sided

- A DT LTI system with rational system function $H(z)$ is causal if and only if:
  a) The ROC is the exterior of a circle outside the outermost pole
  b) With $H(z)$ expressed as a rational function, the order of the numerator cannot be greater than the order of the denominator
- A DT LTI system is causal if and only if the ROC of its system function is the exterior of a circle, including infinity

Analysis and characterization of LTI systems using *z*-transforms

Stability:

- A DT-LTI system is BIBO stable iff $\sum_{n=-\infty}^{\infty} |h[n]|^2 < \infty$
- A discrete-time LTI system is stable if and only if the ROC of its system function $H(z)$ includes the unit circle $r = 1$

Causality & Stability:

- A causal discrete-time LTI system with rational system function $H(z)$ is stable iff all of the poles of $H(z)$ lie inside the unit circle (i.e., they must all have magnitude smaller than 1)

Stability and causality of systems

- Stable
  - h[n] absolutely summable
  - H(z) has a ROC including the unit circle
- Causal
  - h[n] right side sequence
  - H(z) has a ROC being outside the outermost pole

- Stable and Causal:
  - h[n] absolutely summable & right-sided
  - H(z) has a ROC including the unit circle & being outside the outermost pole

LTI systems and LCCDE: Example

- Given a transfer function one can easily calculate the systems difference equation

$$H(z) = \frac{(z+1)^2}{\left(z-\frac{1}{2}\right)\left(z+\frac{3}{4}\right)}$$

- Given this transfer function of a time-domain filter, we want to find the difference equation. To begin with, expand both polynomials and divide them by the highest order z

$$H(z) = \frac{(z+1)(z+1)}{(z-\frac{1}{2})(z+\frac{3}{4})} = \frac{z^2+2z+1}{z^2+\frac{1}{4}z-\frac{3}{8}} = \frac{1+2z^{-1}+z^{-2}}{1+\frac{1}{4}z^{-1}-\frac{3}{8}z^{-2}} = \frac{Y(z)}{X(z)}$$

- From this transfer function, the coefficients of the two polynomials will be our a_k and b_k values found in the general difference equation formula

  Using these coefficients and the above form of the transfer function, we can easily write the difference equation:

$$H(z) = \frac{Y(z)}{X(z)} \Rightarrow (1+2z^{-1}+z^{-2})X(z) = (1+\frac{1}{4}z^{-1}-\frac{3}{8}z^{-2})Y(z)$$

$$x[n]+2x[n-1]+x[n-2] = y[n]+\frac{1}{4}y[n-1]-\frac{3}{8}y[n-2]$$

- In our final step, we can rewrite the difference equation in its more common form showing the recursive nature of the system.

$$y[n] = x[n]+2x[n-1]+x[n-2]+\frac{-1}{4}y[n-1]+\frac{3}{8}y[n-2]$$

LTI systems characterized by Difference Equations: Example

- Find the z-transform of a discrete-time system whose input $x[n]$ and output $y[n]$ are related through the following difference equation:

$$y[n]-\frac{1}{2}y[n-1]=x[n]+\frac{1}{3}x[n-1]$$

- **Solution:** We have:

$$Y(z)-\frac{1}{2}z^{-1}Y(z)=X(z)+\frac{1}{3}z^{-1}X(z) \qquad \Rightarrow H(z)=\frac{Y(z)}{X(z)}=\frac{1+\frac{1}{3}z^{-1}}{1-\frac{1}{2}z^{-1}}$$

$$(1-\frac{1}{2}z^{-1})Y(z)=(1+\frac{1}{3}z^{-1})X(z)$$

There are two different choices for the impulse response of the system with the given difference equation, as we have two different choices for the ROC: $|z|>\frac{1}{2}$ and $|z|<\frac{1}{2}$

(a) For $|z|>\frac{1}{2}$ we have: $$\frac{1+\frac{1}{3}z^{1}}{1-\frac{1}{2}z^{-1}}=\frac{1}{1-\frac{1}{2}z^{-1}}+\frac{1}{3}z^{-1}\frac{1}{1-\frac{1}{2}z^{-1}}\Rightarrow h[n]=(\frac{1}{2})^{n}u[n]+\frac{1}{3}(\frac{1}{2})^{n-1}u[n-1]$$

(b) For $|z|<\frac{1}{2}$ we have: $$h[n]=-(\frac{1}{2})^{n}u[-n-1]-\frac{1}{3}(\frac{1}{2})^{n-1}u[-(n-1)-1]=-(\frac{1}{2})^{n}u[-n-1]-\frac{1}{3}(\frac{1}{2})^{n-1}u[-n]$$

Outline

- Introduction to Z Transform
- Z Transform and Examples
- Properties of the Region of Convergence
- Inverse Z Transform and Examples
- Properties of Z Transform and Examples
- Analysis and characterization of LTI systems using z-transforms
- **Geometric evaluation of the Fourier transform from the pole-zero plot**
- Summary

DTFT and the ZT

Even if $x(n)$ doesn't have FT, its Z-T might exist

Ex: $x(n) = u(n)$, not absolutely summable

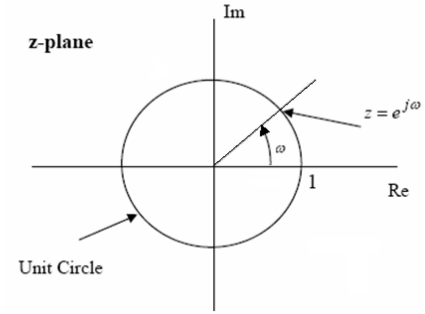
FT doesn't converge absolutely

However, $r^{-n}u(n)$ is absolutely summable if $r>1$

$\therefore$ $|X(z)| < \infty$ when $|z|>1$ ~ ROC.

If ROC includes unit circle $|z|=1$, FT converges.
otherwise FT does not converge absolutely.

DTFT and the ZT



- $H(z) = \sum_{n=-\infty}^{\infty} h[n] z^{-n}$ can be expressed as $H(re^{jw}) = \sum_{n=-\infty}^{\infty} h[n] \left(re^{jw}\right)^{-n}$

  or equivalently $H(re^{jw}) = \sum_{n=-\infty}^{\infty} \left(h[n] r^{-n}\right) e^{-jwn}$
- We can see that $H(z)=H(re^{j\omega})$ is essentially the FT of $h[n]r^{-n}$
- For convergence of the ZT, we require that the FT of h**[n]r^{-n}** converge
- For any specific sequence h[n], we would expect this convergence for some values of *r* and not for others
  - The exponential **r^{-n}** may be decaying or growing with increasing *n* depending on whether *r* is > or < 1
- The range of values for which the ZT converges is the ROC
- If the ROC includes the unit circle, then the FT of *h[n]* converges
  - If we let r = 1, then the ZT reduces to the FT on the **unit circle** $H(z)\big|_{z=e^{j\omega}} = H(e^{j\omega})$

Frequency response vs. System function

- h[n] is the impulse response of an LTI system
- $H(e^{j\omega})$ is the frequency response

$$Y(e^{j\omega}) = H(e^{j\omega})X(e^{j\omega}) \Rightarrow$$

$$|Y(e^{j\omega})| = |X(e^{j\omega})| \cdot |H(e^{j\omega})|; \angle Y(e^{j\omega}) = \angle X(e^{j\omega}) + \angle H(e^{j\omega})$$

- H(z) is the system function; $z=re^{j\omega}$
- H(z) reduces to the FT for |z | = 1, i.e., for the values of the complex variable *z* on the unit circle, **provided that the ROC of the ZT includes the unit circle**
  - As a result, one can use the pole-zero plot of a system function to find the frequency response of the system by evaluating the magnitude and phase of the system **on the unit circle** in the complex plane

Geometric evaluation of the FT from the pole-zero plot

- Consider the transfer function: $H(z) = \dfrac{\prod_{i=1}^{m}(z - z_i)}{\prod_{j=1}^{n}(z - p_j)}$

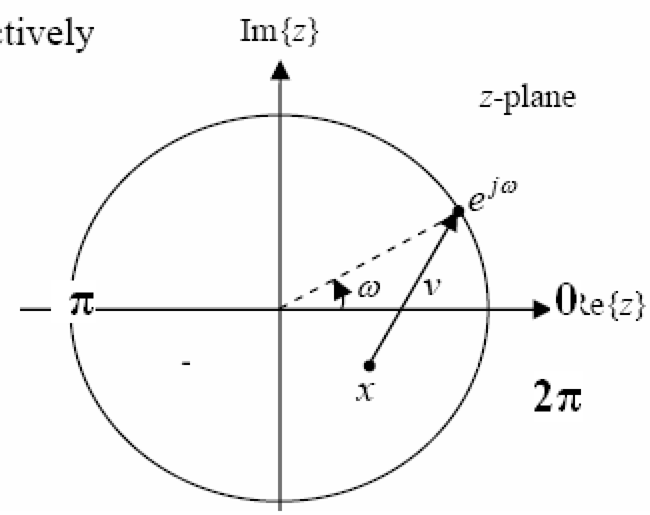
  where z_i's and p_j's represent the zeros and poles of the transfer function, respectively

  We have:

  $$\left|H(e^{j\omega})\right| = \frac{\prod_{i=1}^{m}\left|e^{j\omega} - z_i\right|}{\prod_{j=1}^{n}\left|e^{j\omega} - p_j\right|} = \frac{\prod \text{"distancesfromzeros"}}{\prod \text{"distancesfrompoles"}}$$

  $$\angle H(e^{j\omega}) = \sum_{i=1}^{m} \angle(e^{j\omega} - z_i) - \sum_{j=1}^{n} \angle(e^{j\omega} - p_j)$$

- Note that $\left|e^{j\omega} - x\right|$ and $\angle(e^{j\omega} - x)$ represent the magnitude and phase of the vector v from the point x to the point $e^{j\omega}$ (which is a point on the unit circle with the phase ω) in the complex plane, respectively



Geometric evaluation of the FT from the pole-zero plot

- We will use this result to find the frequency response of any DT LTI system with rational system function given by

$$H(z) = \frac{\prod_{i=1}^{m}(z - z_i)}{\prod_{j=1}^{n}(z - p_j)}$$

- At any frequency ω , find the magnitude and phase of the vectors drawn from the poles and zeros to the point $e^{j\omega}$ (a point on the unit circle with angle ω)

1. The magnitude of H ($e^{j\omega}$) at ω is equal to the product of the magnitudes of all vectors associated with the zeros divided by the product of the magnitudes of all vectors associated with the poles

$$\left|H(e^{j\omega})\right| = \frac{\prod_{i=1}^{m}\left|e^{j\omega} - z_i\right|}{\prod_{j=1}^{n}\left|e^{j\omega} - p_j\right|} = \frac{\prod \text{"distancesfromzeros"}}{\prod \text{"distancesfrompoles"}}$$

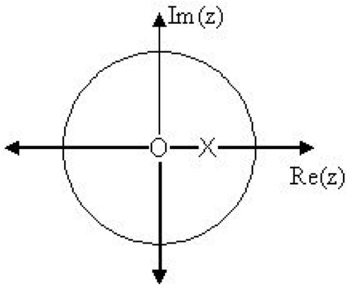
2. The phase of H ($e^{j\omega}$) ω is equal to the summation of the angles of all vectors associated with the zeros minus the summation of the angles of all vectors associated with the poles

$$\angle H(e^{j\omega}) = \sum_{i=1}^{m}\angle(e^{j\omega} - z_i) - \sum_{j=1}^{n}\angle(e^{j\omega} - p_j)$$

Geometric evaluation of the FT from the pole-zero plot

- Using the distances from the unit circle to the poles and zeros, we can plot the frequency response of the system

- As ω goes from $-\pi$ to π, the following two properties specify how one should draw $|H(e^{i\omega})|$



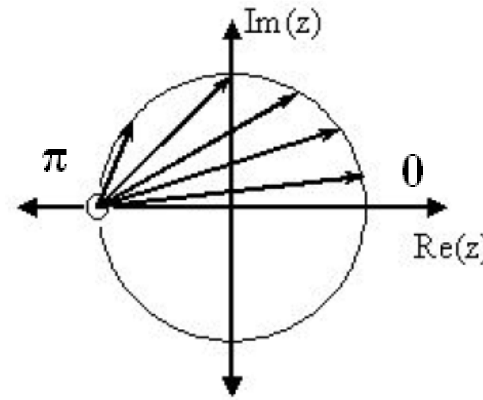
- While moving around the unit circle:
  1. if close to a pole, then the magnitude is large
     - If a pole is on the unit circle, then the frequency response goes to infinity at that point
  2. if close to a zero, then the magnitude is small
     - If a zero is on the unit circle, then the frequency response is zero at that point

Geometric evaluation of the FT from the pole-zero plot: Example 1

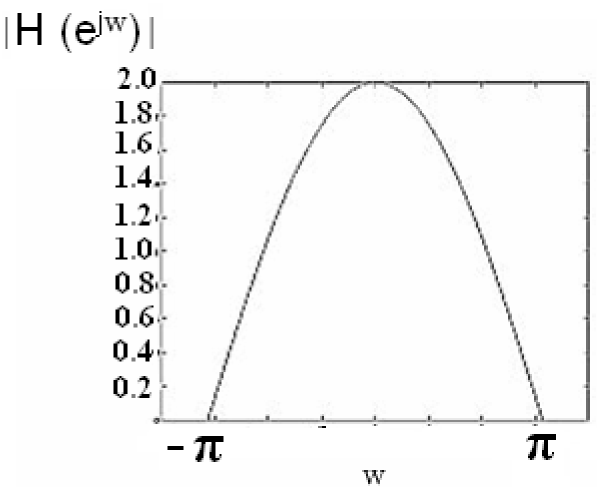
- Consider: $H(z) = z + 1 \rightarrow H(e^{j\omega}) = e^{j\omega}+1$

- Some of the vectors represented by $|e^{j\omega} +1|$, for random values of ω, are explicitly drawn onto the complex plane
- These vectors show how the magnitude of $H(e^{j\omega})$ changes as ω goes from -π to π, and also show the physical meaning of the terms in $H(e^{j\omega})$

- When ω = 0, the vector is the longest and thus the frequency response will have its largest amplitude
- As ω approaches π , the length of the vectors decrease as does $|H(e^{j\omega})|$
- Since there are no poles in the transform, there is only this one vector term



Pole/Zero Plot

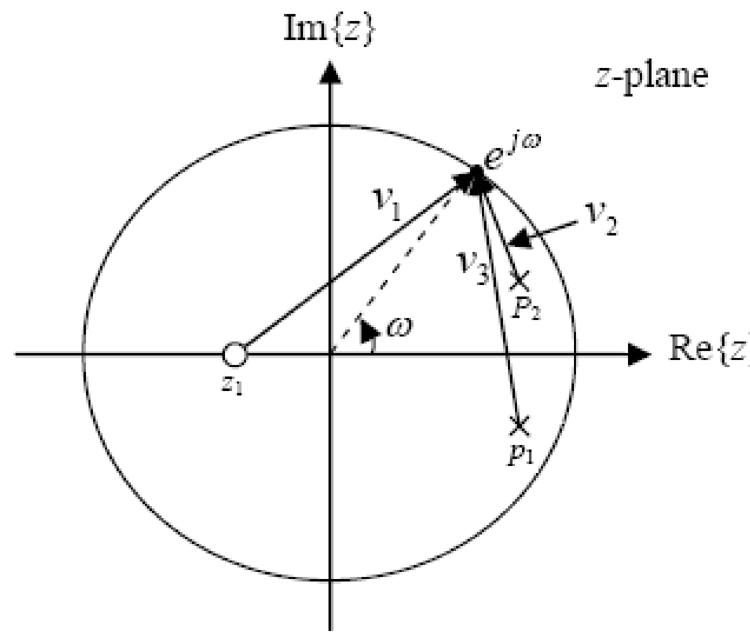


Frequency Response

Geometric evaluation of the FT from the pole-zero plot: Example 2

One zero, two poles

- Consider an LTI system with an impulse response *h*[*n*]
- Assume that H(z) is a rational function of *z* whose pole-zero configuration is given in the plot
- From this plot, it can be concluded that the *z*-transform of the impulse response is:



$$H(z) = K\frac{(z - z_1)}{(z - p_1)(z - p_2)}$$

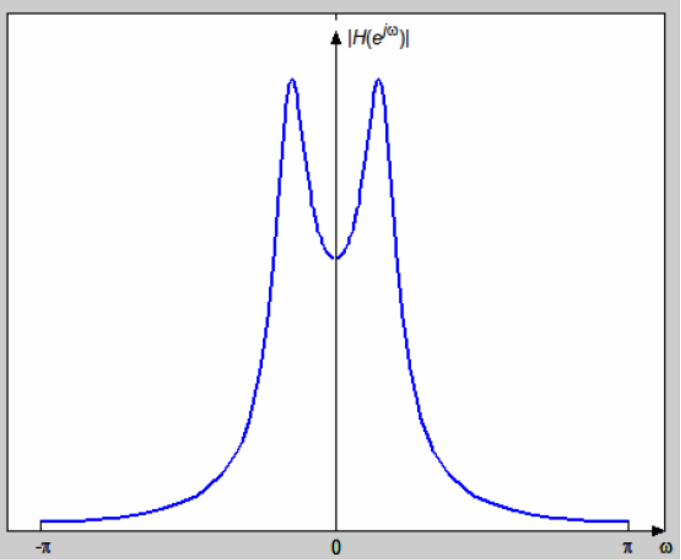
Geometric evaluation of the FT from the pole-zero plot: Example 2

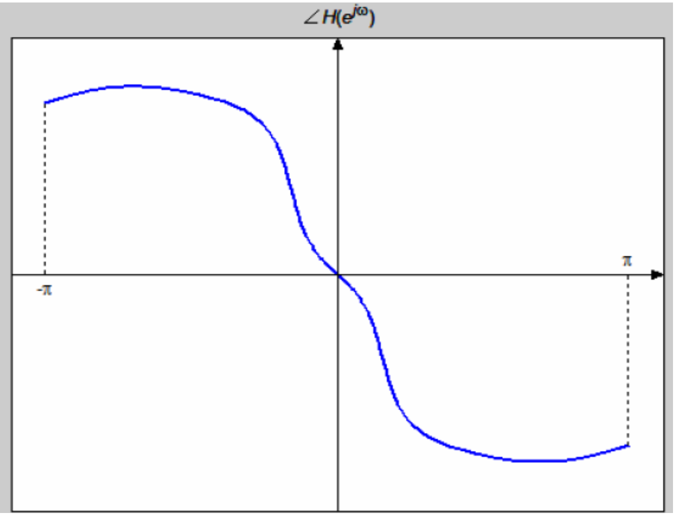
- The magnitude and angle of the vectors v_1, v_2 and v_3 depend on the frequency ω
- The magnitude of the frequency response of this system is proportional to

$$\frac{|v_1|}{|v_2||v_3|}$$

- The phase of the frequency response of this system is equal to $\angle v_1 - \angle v_2 - \angle v_3$

- The magnitude of the frequency response is large at those frequencies that correspond to the points on the unit circle which are close to the poles and far from the zeros
- Similarly, the magnitude of the frequency response is small at those frequencies that correspond to the points on the unit circle which are close to the zeros and far from the poles



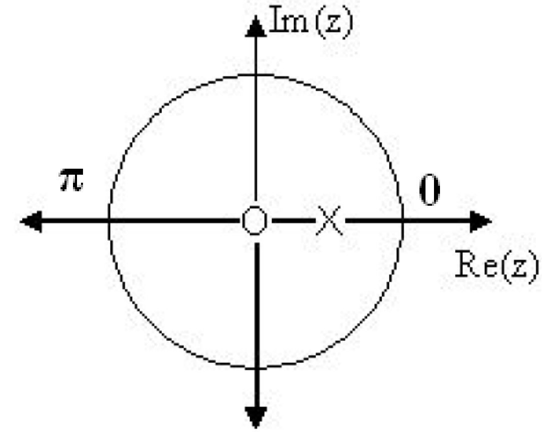


Geometric evaluation of the FT from the pole-zero plot: Example 3

- Analyze the following system function

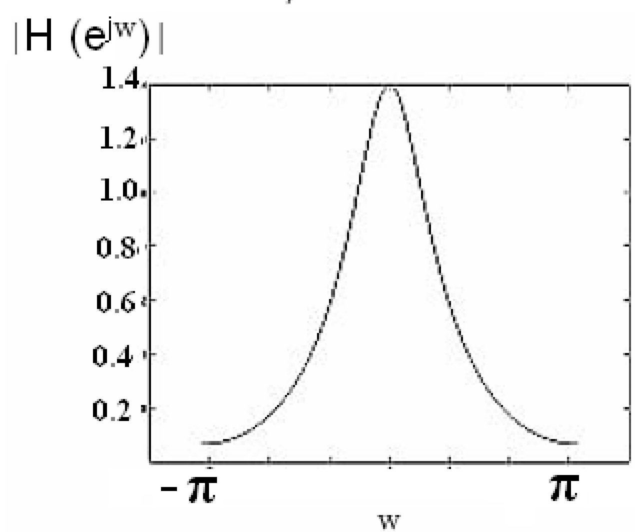
$$H(z)=\frac{z}{z-\frac{1}{2}}=\frac{1}{1-\frac{1}{2}z^{-1}} \Rightarrow H(e^{j\omega})=\frac{1}{1-\frac{1}{2}e^{-j\omega}}$$

in order to represent $|H(e^{j\omega})|$



Pole/Zero Plot

1. We can see that when ω = 0, the frequency response $|H(e^{j\omega})|$ will peak since it is at this value of ω that the pole is closest to the unit circle
2. As ω moves from -π to π, we see how the zero begins to mask the effects of the pole and thus force the frequency response $|H(e^{j\omega})|$ closer to 0



Frequency Response

Types of Filters

A filter remove unwanted signal components and/or enhance wanted ones

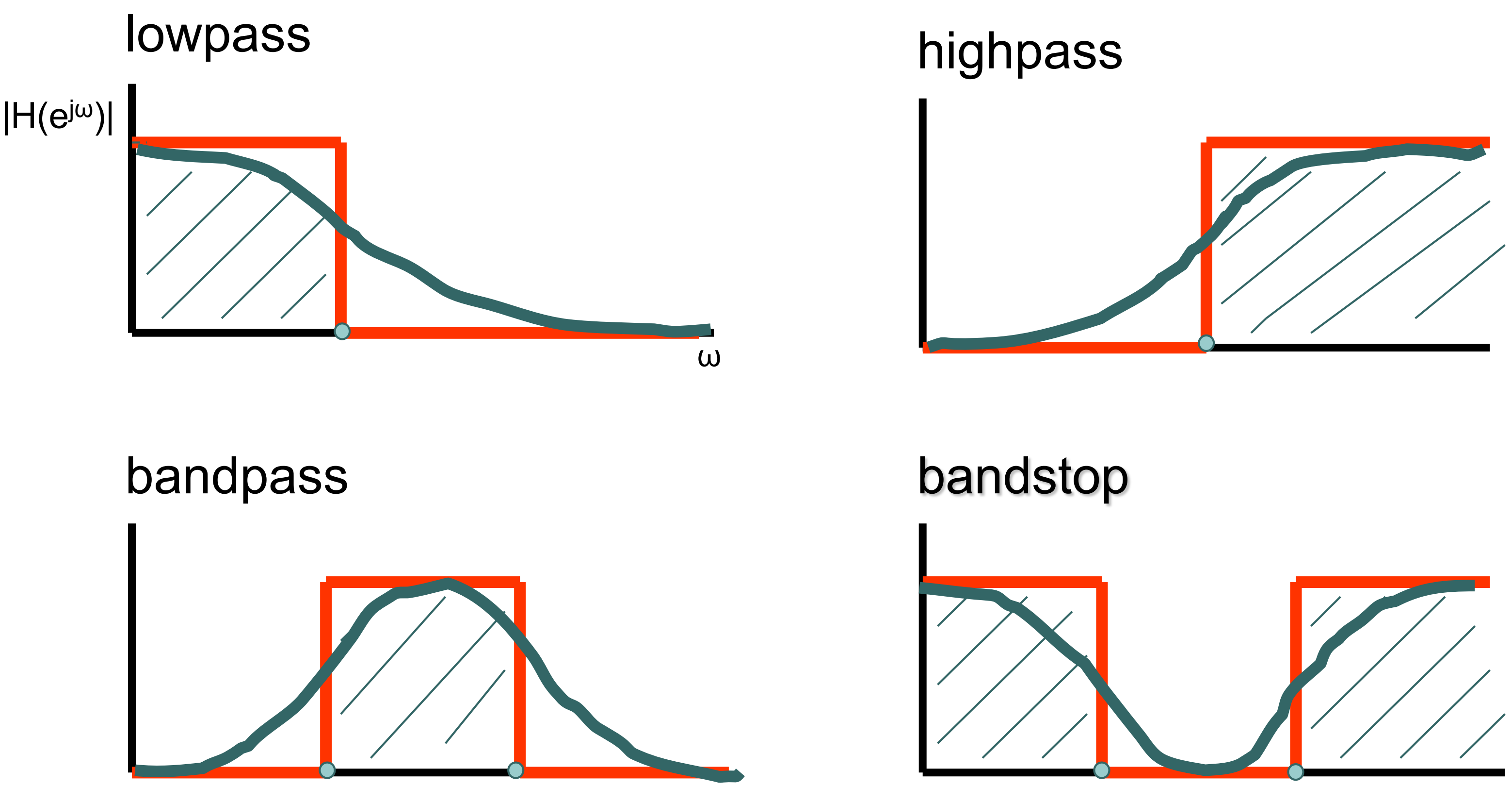
Four Main Filter Types:

- Low-pass: most common
  - Passes low frequencies, attenuates highs
- High-pass:
  - Passes high frequencies, attenuates lows
  - Used to brighten a signal
  - Careful: can also increase noise
- Band-pass:
  - Passes band of frequencies, attenuates those above and below band
  - Most common in implementations of DFF to separate out harmonics
- Band-stop (band-reject):
  - Stops band of frequencies, passes those above and below band

Filter specification through frequency response (magnitude and phase)

Types of Filters

Ideal filters Real filters



Summary

- Generalization of DTFT, with the transform variable defined over the complex z-plane:
  $X(z) = \sum_{n} x[n]z^{-n}$, for certain ROC
  The same X(z) may be associated with different x[n], depending on the ROC
- Relation with DTFT: $X(e^{j\omega}) = X(z)\big|_{z=e^{j\omega}}$ if the ROC contains the unit circle
- Know the shape of ROCs for right sided (outer circle), left sided (inner circle), double sided (ring) functions
- Know how to calculate the Z-transform of simple functions and how to sketch their ROCs and zero-pole plots
- Know how to calculate the inverse Z-transform by partial fraction expansion.
- Know how to solve a difference equation system using Z- transform
- Know how to determine whether a LTI system is causal and stable based on pole locations
- Can graph the Fourier transform magnitude from the zero-pole locations

Summary $Y(z) = H(z)X(z)$

- ZT of impulse response = System function
- Frequency response = ZT on unit circle

$$|Y(e^{j\omega})| = |X(e^{j\omega})| \cdot |H(e^{j\omega})|$$

$$\angle Y(e^{j\omega}) = \angle X(e^{j\omega}) + \angle H(e^{j\omega})$$

- Zeros of a H(z) = frequencies that minimize the system gain
- Poles of a H(z) = frequencies that maximize system gain

➔We can design filters by specifying zeros & poles

- We can analyze systems by looking at H(z) at the unit circle and find the magnitude and phase of $H(e^{j\omega})$

Outline

- Introduction to Z Transform
- Z Transform and Examples
- Properties of the Region of Convergence
- Inverse Z Transform and Examples
- Properties of Z Transform and Examples
- Analysis and characterization of LTI systems using z-transforms
- Geometric evaluation of the Fourier transform from the pole-zero plot
- **Numerical calculation of the FT**

Discrete-time signal transforms

Transform Name	Forward Transform	Inverse Transform	Notes
z-Transform	$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$ $z = re^{j\omega}$	Partial fractions, Power series, Inspection.	has ROC
DTFT * continuous in freq.	$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$	$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}d\omega$	Periodic (2π)
DFS * periodic signal * discrete in freq.	$\tilde{X}[k] = \sum_{n=0}^{N-1} \tilde{x}[n]W_N^{nk}$	$\tilde{x}[n] = \frac{1}{N}\sum_{k=0}^{N-1} \tilde{X}[k]W_N^{-nk}$	$W_N = e^{-j\frac{2\pi}{N}} = e^{-j\omega_o}$ $\tilde{x}[n] = \tilde{x}[n+N]$ N is the period
DFT * discrete in freq. * samples from DTFT.	$X[k] = \sum_{n=0}^{N-1} x[n]W_N^{nk}$	$x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]W_N^{-nk}$	$W_N = e^{-j\frac{2\pi}{N}} = e^{-j\omega_o}$

- DTFT: Discrete-time Fourier tarnsform.
- DFS : Discrete Fourier series.
- DFT : Discrete Fourier transform

70

Numerical Calculation of FT

1. The original signal *x(t)* is digitized to *x[n]*

2. To calculate the FT, a Fast Fourier Transform (FFT) algorithm is applied to a finite duration x[n]

3. The FFT yields *L* samples of the DTFT at equally spaced intervals $2\pi/L$

4. For a signal that is very long, e.g., a speech signal or a music piece, spectrogram is used
   - FT over successive overlapping short intervals

Matlab examples: DTFT

- Suppose that:

$$x[n] = \begin{cases} cos\left(\frac{3\pi n}{8}\right), & \text{if } 0 \le n \le 31 \\ 0, & \text{otherwise} \end{cases}$$

- Analytically, the DTFT is

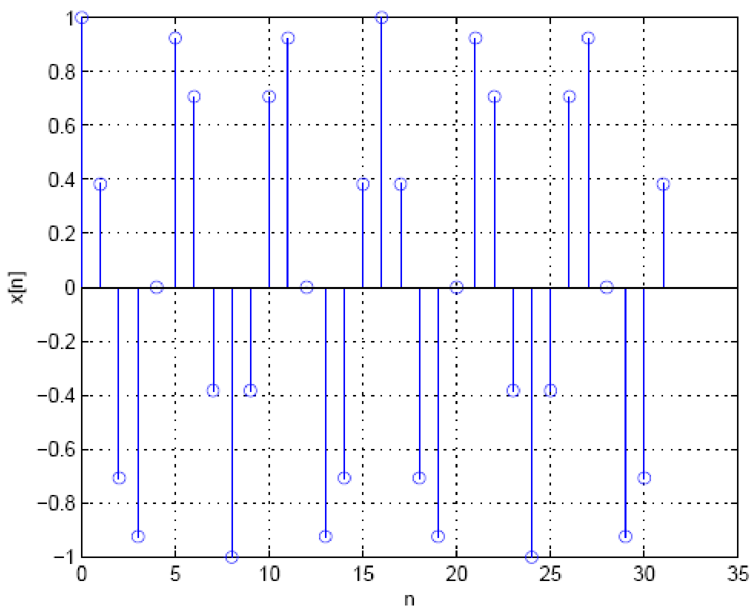
$$X(e^{j\omega}) = \frac{e^{-j\frac{31}{2}(\omega+\frac{3\pi}{8})}}{2}\frac{sin(16(\omega+\frac{3\pi}{8}))}{sin(\frac{1}{2}(\omega+\frac{3\pi}{8}))} + \frac{e^{-j\frac{31}{2}(\omega-\frac{3\pi}{8})}}{2}\frac{sin(16(\omega-\frac{3\pi}{8}))}{sin(\frac{1}{2}(\omega-\frac{3\pi}{8}))}$$

  - $X(e^{j\omega})$: continuous function of ω
  - $X(e^{j\omega})$: periodic with period 2π
- Plot it using

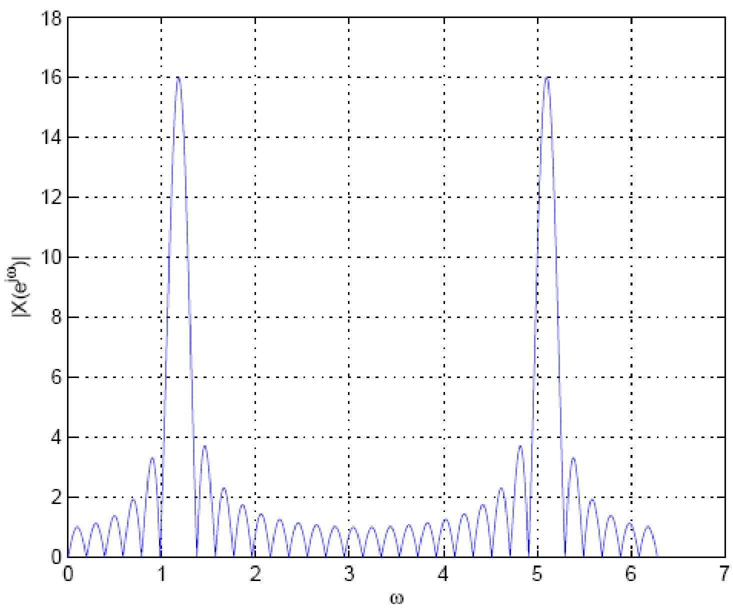
```
N = 32;
n = [0:N-1];
x = cos(3*pi/8*n);
w = 0:0.001:2*pi; % step size is 0.001
X = exp(-i*(31/2)*(w+3*pi/8))/2 .* (sin(16*w+6*pi))./(sin(w/2+3*pi/16))+...
    exp(-i*(31/2)*(w-3*pi/8))/2 .* (sin(16*w-6*pi))./(sin(w/2-3*pi/16));

subplot(1,2,1);
stem(n,x);
xlabel('n'); ylabel('x[n]'); grid on;

subplot(1,2,2);
plot(w,abs(X));
xlabel('\omega'); ylabel('|X(e^{j\omega})|'); grid on;
```

Matlab examples: DTFT

Signal x[n]



DTFT



Matlab examples: DFT

- Close form $X(e^{j\omega})$ not always easy
- To plot $|X(e^{j\omega})|$, we sampled from 0 to 2π
  - In code: w and X are vectors
  - Small step size 0.001 to simulate continuous frequency
- Workaround: DFT
  - Uniform L-samples from DTFT from 0 to 2π
  - Takes discrete values and returns discrete values
  - No need to find $|X(e^{j\omega})|$ analytically
  - Fast implementation using the fast Fourier transform (FFT)
  - Matlab: fft(x,L)
    - L: number of samples to take
    - More L ➔ more resolution
    - Default L is N=length(x)

Matlab examples: DFT

- Calculating the DFT

Example 2: Calculating the DFT coefficients

```
X = fft(x); % Notes:
            % 1. This call is the same as fft(x,L), where L=length(x)=N.
            % 2. The values returned are direct samples of the DTFT
            % 3. The values returned are the DFT coefficients X[k]
```

- Plotting the DFT against k

Example 3: Plotting the DFT against coefficient number

```
k = [0:length(X)-1];
stem(k,abs(X));
grid on;
xlabel('k');
ylabel('|X[k]|');
```

Matlab examples: DFT

Notes:

- Default L=32 gives bad resolution

→ information lost

- x-axis not useful

→Cannot find fundamental frequency 3π/8

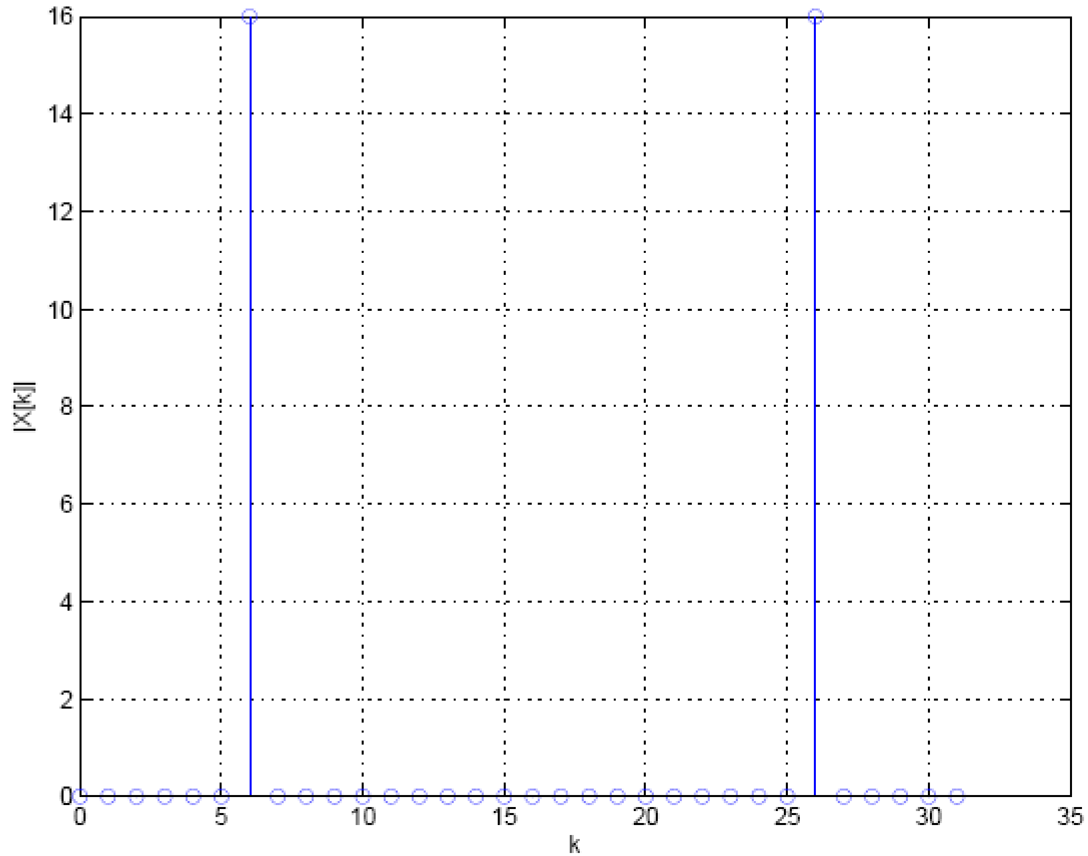


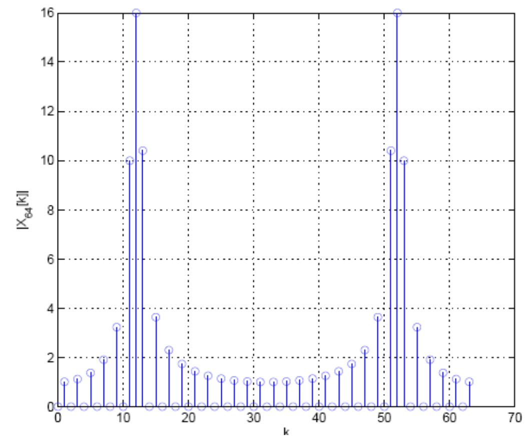
Figure 3: Output of example 3

Matlab examples: DFT

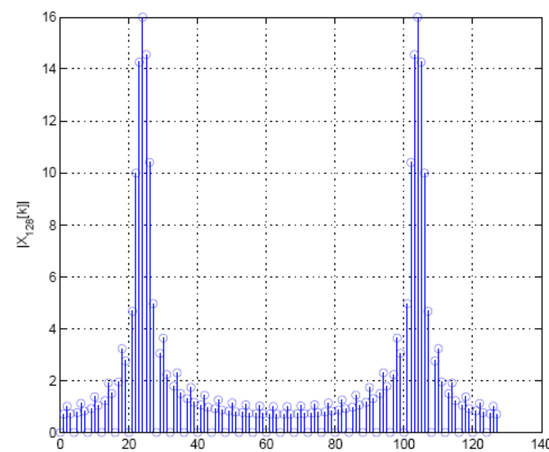
- Effect of increasing L (better resolution)

```
L64  =  64;
L128=  128;
k64  =  0:L64-1;
k128=  0:L128-1;
X64  =  fft(x,L64);
X128=  fft(x,L128);
subplot(1,2,1);
stem(k64,abs(X64));
grid  on;
xlabel('k');
ylabel('|X_{64}[k]|');
subplot(1,2,2);
stem(k128,abs(X128));
grid  on;
xlabel('k');
ylabel('|X_{128}[k]|');
```

L=64



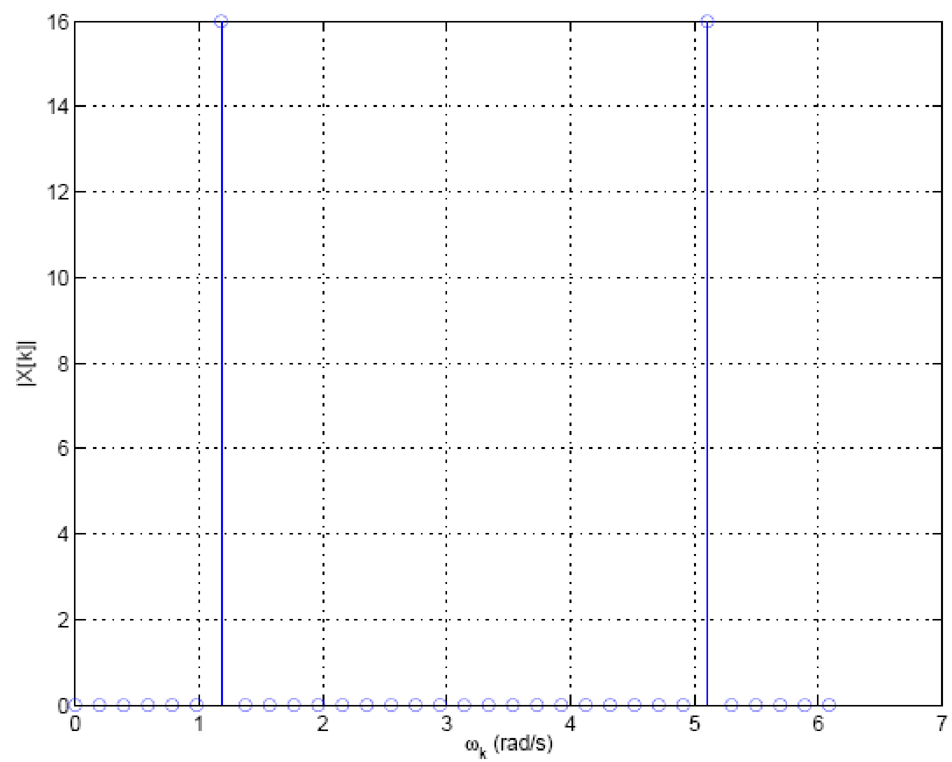
L-128



Matlab examples: DFT

- Obtaining the frequency (x-axis)

```matlab
w = [0:N-1]/N*(2*pi);
stem(w,abs(X));
xlabel('\omega_k (rad/s)');
grid on;
```

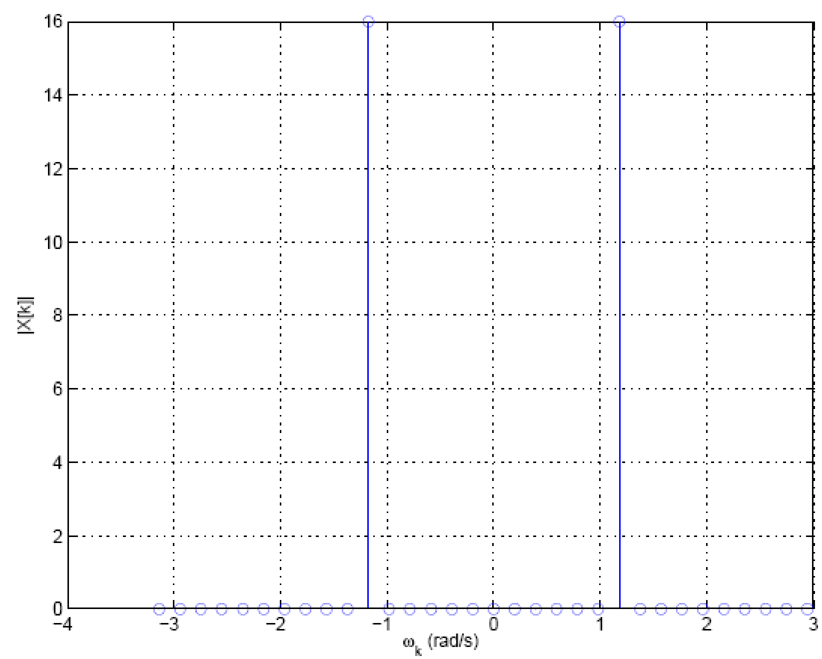


- Spike at 3π/8=1.17
- Spike at 2π-3π/8 = 5.11
- FFT calculates from 0 to 2π
- More familiar to shift using fftshift

Matlab examples: DFT

```
w = [-N/2:N/2-1]/N*(2*pi);
X = fft(x);
X = fftshift(X);
stem(w,abs(X));
xlabel('\omega_k (rad/s)');
grid on;
```

- Spikes at $3\pi/8$ and $-3\pi/8$



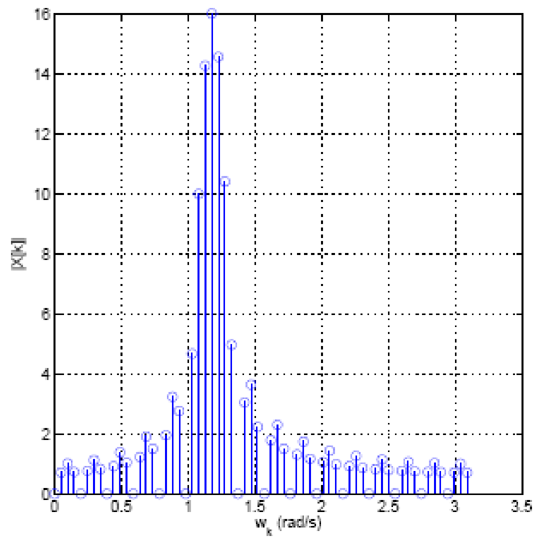
Matlab examples: DFT

Example 7: Changing the frequency axis

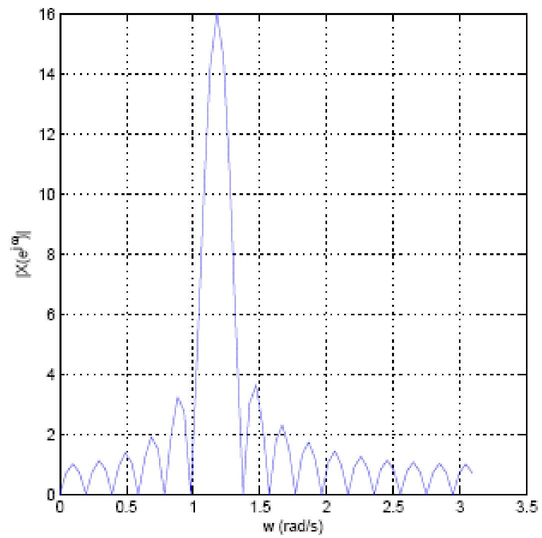
```
N = 32; % length of our signal
n = [0:N-1];
x = cos((3*pi/8)*n); % x[n]

L = 128; % Size of DFT (i.e., L-point DFT)
fs = 1024; % Sampling frequency

w = [0:L-1]/L*(2*pi); % angular freq.
f = w/(2*pi)*fs; % freq. in Hz

X_k    = fft(x,L);
subplot(1,3,1);
stem(w(1:L/2),abs(X_k(1:L/2)));
xlabel('w_k (rad/s)'); ylabel('|X[k]|'); grid on;


X_ejw = fft(x,L);
subplot(1,3,2);
plot(w(1:L/2),abs(X_ejw(1:L/2)));
xlabel('w (rad/s)'); ylabel('|X(e^{j\omega})|'); grid on;

X_c    = fft(x,L)/N; % normalize by length of signal
subplot(1,3,3);
plot(f(1:L/2),abs(X_c(1:L/2)));
xlabel('f (Hz)'); ylabel('|X(f)|'); grid on;
```

- Sometimes we want frequency in Hz

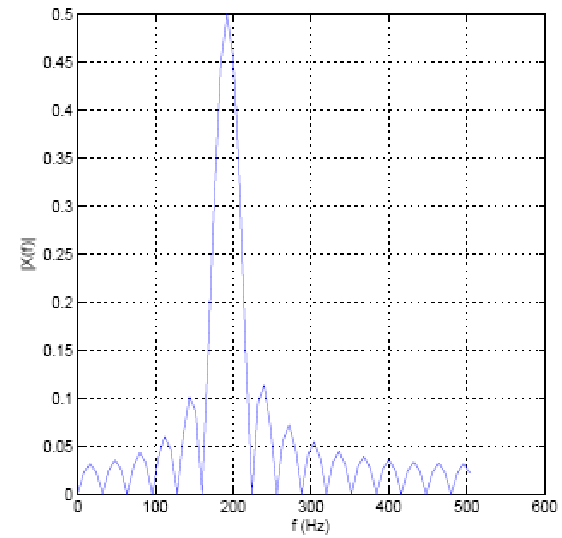
Matlab examples: DFT



- $|X[k]|$ vs. ω_k
- Discrete
- DFT



- $|X(e^{j\omega})|$ vs. ω
- Continuous
- By interpolating DFT



- $|X(f)|$ vs. f
- Continuous
- $f = (\omega/ 2\pi)\ f_s$
- f_s : sampling frequency
- fft values divided by N
- Peak at 0.5 (half our amplitude of 1)

Matlab examples: DFS

- No special function
- Same as DFT
- Provided signal corresponds to 1 period

Matlab examples: z-Transform

- Suppose that:

$$H_1(z) = \frac{1 - 10z^{-1} - 4z^{-2} + 4z^{-3}}{2 - 2z^{-1} - 4z^{-2}} \qquad H_2(z) = \frac{0.094(1 + 4z^{-1} + 6z^{-2} + 4z^{-3} + z^{-4})}{1 + 0.4860z^{-2} + 0.0177z^{-4}}$$

Matlab examples: z-Transform

Partial fraction expansion

```
Num = [1 -10 -4 4]; %numerator polynomial coefficients
Den = [2 -2 -4]; % denominator polynomial coefficients
[r,p,k] = residuez(Num,Den)
```

and the output is

```
r =[1.5000 0.5000]
p =[2 -1]
k =[1.5000   -1.0000]
```

To interpret the answer, check the following equation

$$\frac{B(z)}{A(z)} = \frac{r(1)}{1-p(1)z^{-1}} + ... + \frac{r(n)}{1-p(n)z^{-1}} + k(1) + k(2)z^{-1}..$$

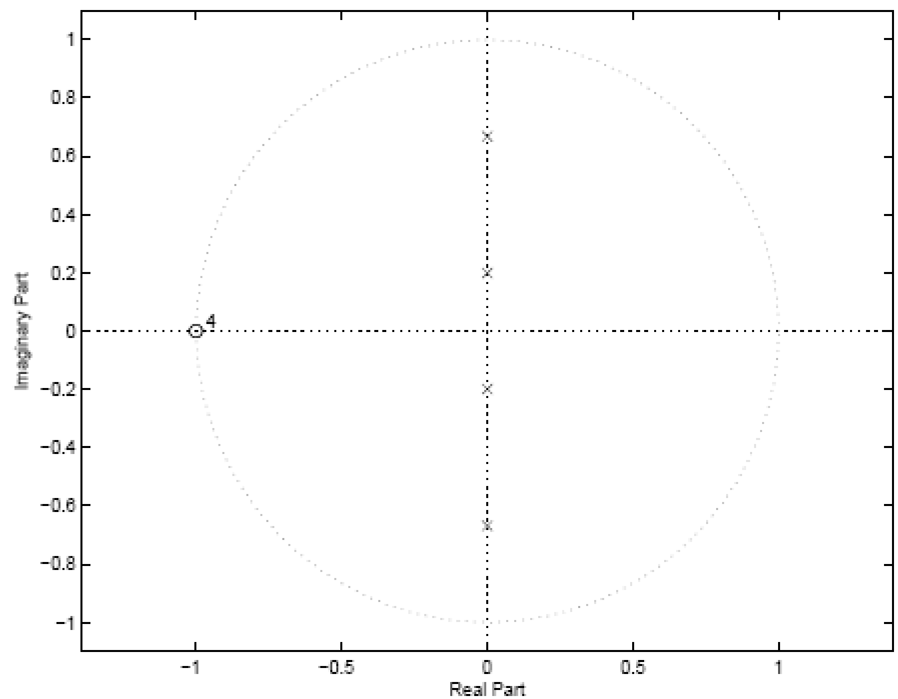
which basically means that the partial fraction expansion of $H_1(z)$ is

$$H_1(z) = \frac{1.5}{1-2z^{-1}} + \frac{0.5}{1+z^{-1}} + 1.5 - z^{-1}$$

Matlab examples: z-Transform

Pole-zero plot

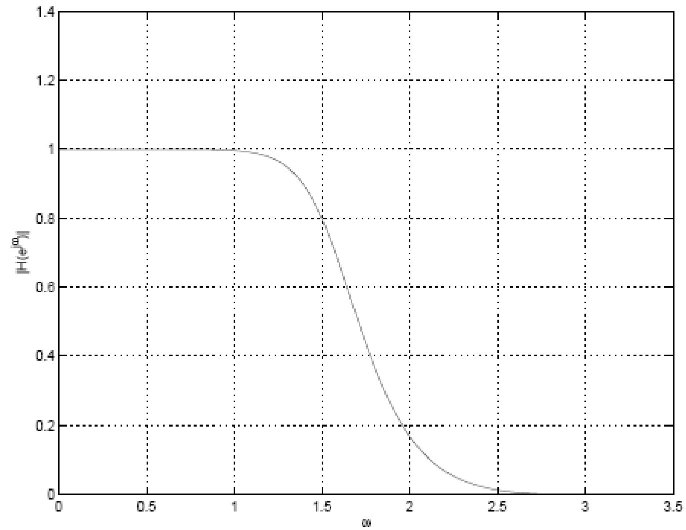
```
b = 0.094*[1 4 6 4 1];
a = [1 0 0.486 0 0.0177];
zplane(b,a);
```



Matlab examples: z-Transform

- Evaluate $H_2(e^{j\omega})$ directly from z-Transform

```
N = 250; % The same as the size of the DFT
[H,w] = freqz(b,a,N); % This command returns the frequency-axis also

Habs = abs(H);
plot(w,Habs); grid on;
xlabel('\omega'); ylabel('|H(e^{j\omega})|');
```



Matlab examples: z-Transform

- Finding z-Transform analytically

```
syms a t n % These now are symbols or variables
H_2 = ztrans(h_2)
```

and the output is

```
%H_2 = -z/(a - z)
```

To get an expression for the DTFT, one can substitute $e^{j\omega}$ in the transfer function.