DSP: System analysis:

Magnitude-Phase Representation of systems

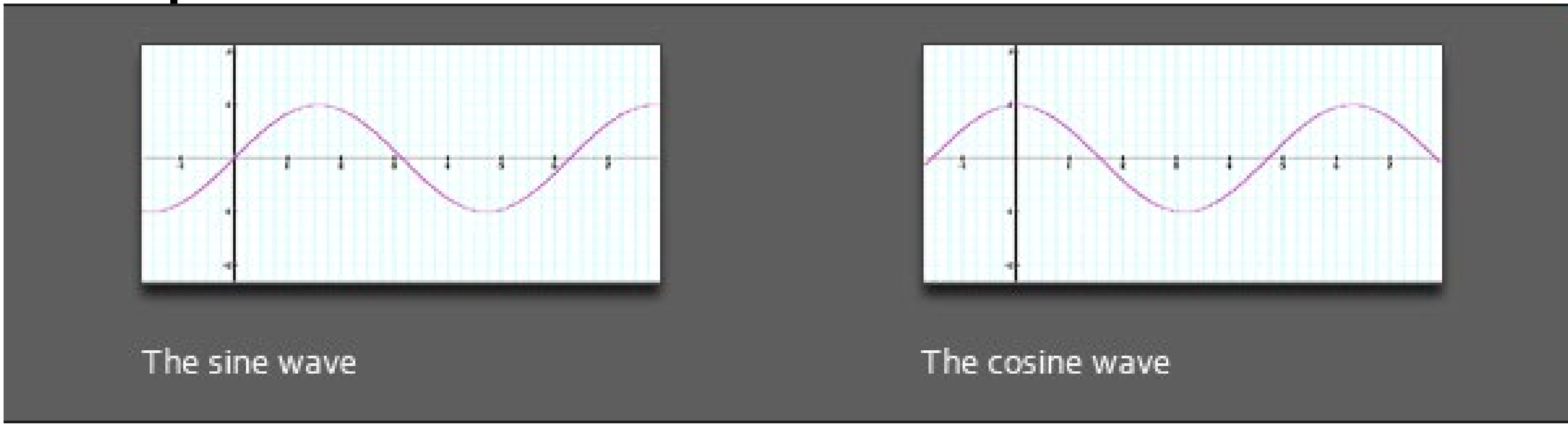
- Introduction
- Magnitude and Phase Representation of signals
- Amplitude-Phase Representation of signals
- System analysis:
  - System function
  - Group Delay of LTI Systems
  - IIR and FIR Systems
  - Frequency Response of Rational Systems
  - Relation between Magnitude/Phase & poles/zeros
  - All-pass systems
  - Minimum-phase systems
  - Linear systems with generalized linear phase

M. Amer
Concordia University
Electrical and Computer Engineering

Content and figures are based on/from

- A. Oppenheim, A.S. Willsky and S.H. Nawab, Signals and Systems, 2nd Edition, Prentice-Hall, 1997
- http://metalab.uniten.edu.my/~zainul/images/Signals&Systems
- *Slides by Robert Akl, http://www.cse.unt.edu/~rakl/*
- Dr. Güner Arslan, 351M Digital Signal Processing, http://signal.ece.utexas.edu/~arslan/courses/dsp
- Dr. Zheng-Hua Tan, Digital Signal Processing III, 2009, http://kom.aau.dk/~zt/cources/DSP/

Introduction $A\cos(\omega_o n+\theta)$



The sine wave

The cosine wave

- Sinusoids: both waves are periodic, i.e., after a certain time, called the period, they look the same again
- Both waves look alike, but the cosine starts at its maximum, while the sine its minimum (zero)

- An important property of sinusoids is “frequency”:
  - tells us how many peaks and valleys we can count in a given period of time
  - High frequency means many peaks and valleys
  - Low frequency means few peaks and valleys
- "frequency" can mean Hz (cycles per second), or radians per second

Introduction: Phase $A\cos(\omega_o n+\theta)$

- The cosine wave starts (with respect to a reference *n=0*) out later *than the sine wave* in its period ➔It has an *offset*
  - It is common to measure this offset in degree or radians
  - One complete period equals 360° or 2π radian
- The cosine wave thus has an offset of 90° or $\pi/2$
- This offset is called the *phase* of a sinusoid

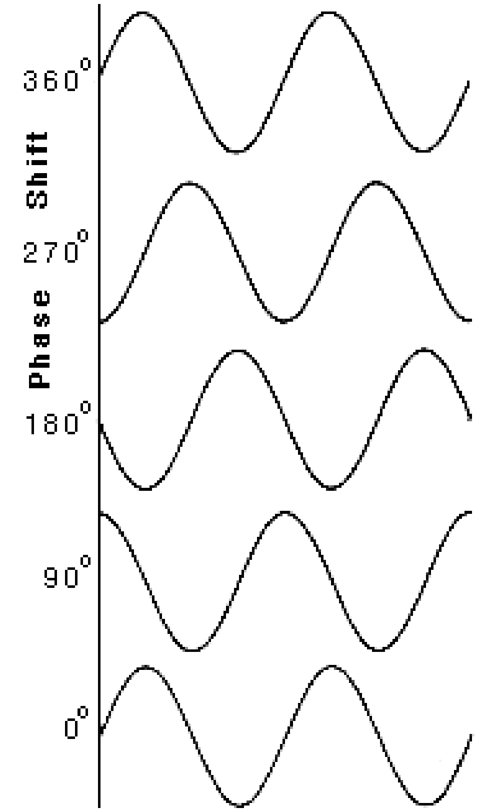
- We cannot restrict a signal *x[n]* to start out at zero phase or 90° phase all the time

➔**Must determine signal amplitude, frequency, and phase to <u>uniquely describe</u> that signal at any time instant**

Introduction:

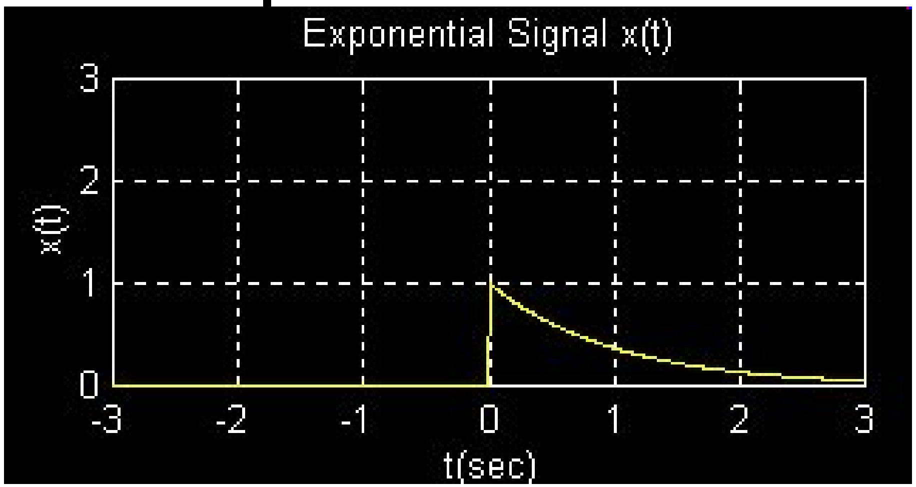
Phase shift $A\cos(\omega_o n+\theta)$

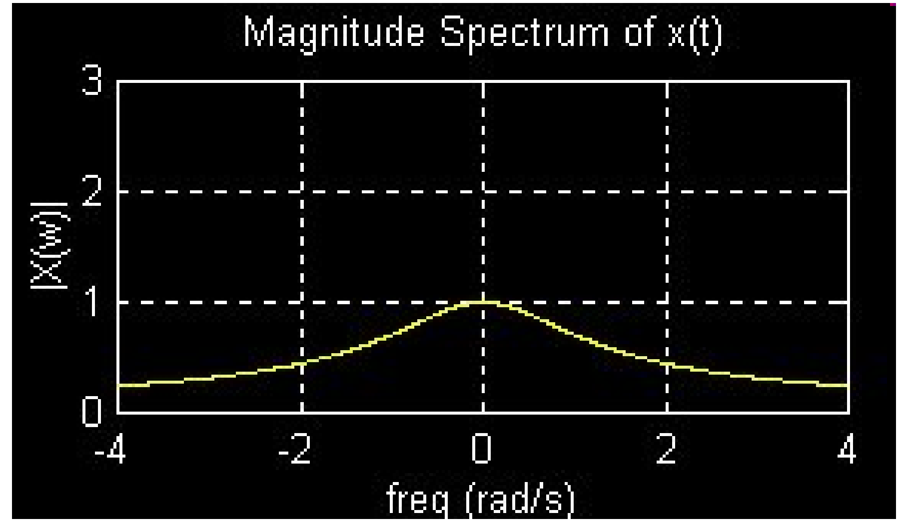
- The **phase θ of the signal** is the offset in the displacement from a specified reference point at n=0
- θ represents a "shift" from zero phase
- A change in θ is also referred to as a **phase-shift**
- For infinitely long sinusoids, a change in θ is the same as a shift in time.

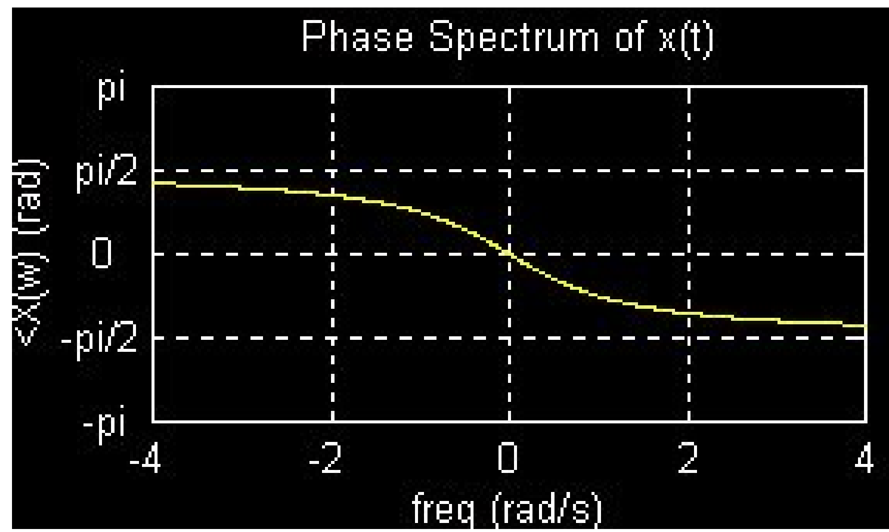


Signals with different phase shift compared to the bottom signal

Introduction: Importance of Phase



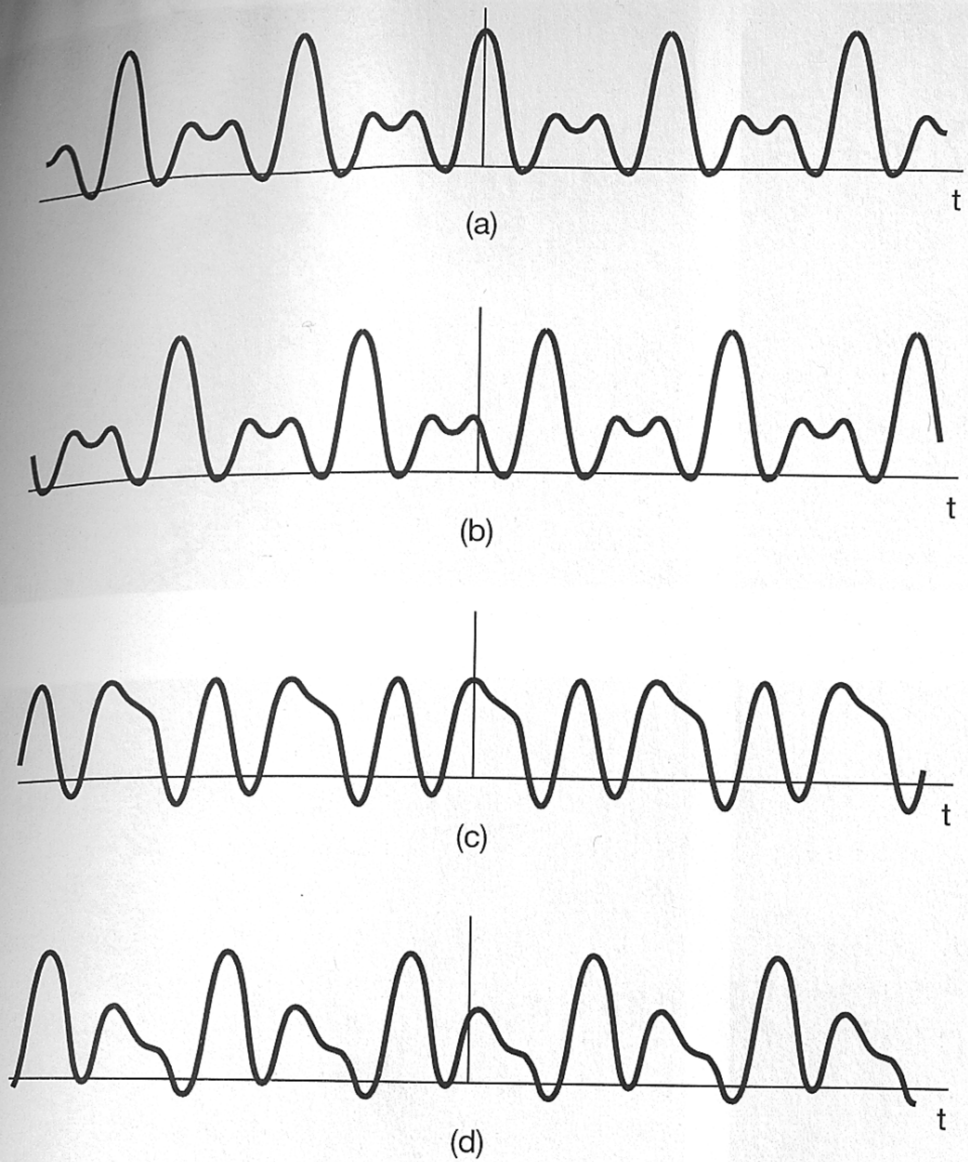




- Magnitude tells "how much" of a certain frequency component is present in x[n]
- Phase tells "where" the frequency component is in the signal
- Typically, it is hard to infer much from a phase plot (a group delay plot is more useful; see later)

Introduction: **Importance of Phase**

$$x(t) = 1 + \frac{1}{2}\cos(2\pi t + \phi_1) + \cos(4\pi t + \phi_2) + \frac{2}{3}\cos(6\pi t + \phi_3).$$



➔ Phase causes signal shift and/or signal shape distortion

Figure 6.1 The signal $x(t)$ given in eq. (6.3) for several different choices of the phase angles ϕ_1, ϕ_2, and ϕ_3: (a) $\phi_1 = \phi_2 = \phi_3 = 0$; (b) $\phi_1 = 4$ rad, $\phi_2 = 8$ rad, $\phi_3 = 12$ rad (c) $\phi_1 = 6$ rad, $\phi_2 = -2.7$ rad, $\phi_3 = 0.93$ rad; (d) $\phi_1 = 1.2$ rad, $\phi_2 = 4.1$ rad, $\phi_3 = -7.02$ rad.

Introduction: Importance of Phase

- In images, phase carries considerable information
  - The phase data dominates our perception in images
  - A random distortion of the phases can dramatically distort the signal

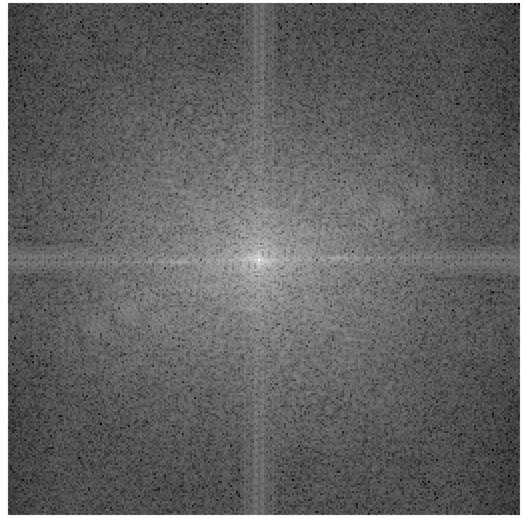
- Signal delay corresponds to phase change:

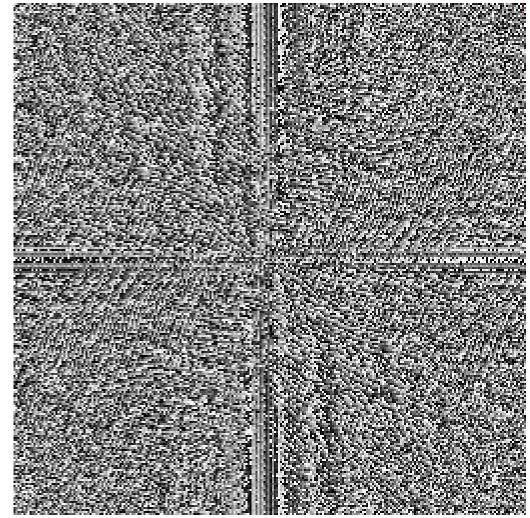
$$x[n] \xleftrightarrow{Z} X(z)$$
$$\Rightarrow x[n-n_0] \xleftrightarrow{Z} z^{-n_0} X(z),$$

➔Signal delay, in the frequency domain, alters only the phase of the signal

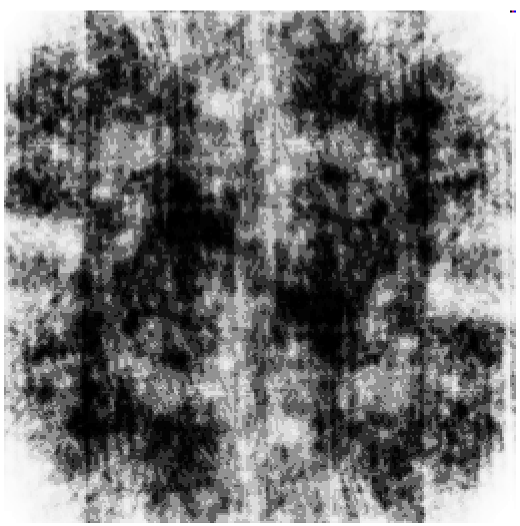
➔No difference will be observed when viewing the FT magnitude

Introduction: Importance of Phase





- The value of each point determines the phase of the corresponding frequency



🡐 The inverse FT to the above magnitude image while ignoring the phase, gives

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Magnitude and Phase Representation of signals

- An LTI system is <u>completely</u> characterized by …

1. Time domain: impulse response $y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$

2. Frequency domain: frequency response $Y(e^{j\omega}) = X(e^{j\omega})H(e^{j\omega})$

3. z-transform: system function $Y(z) = X(z)H(z)$

- Often x[n] is real but its FT is, in general, complex

- The FT is generally represented by its magnitude and phase
  - **Magnitude tells** "how much" of a certain frequency component is present in x[n]
  - **Phase tells** "where" the frequency component is in the signal

- Two signals may have the same magnitudes but the phase differs

Magnitude and Phase Representation of systems

- Input and output relation: $$Y(e^{j\omega}) = X(e^{j\omega})H(e^{j\omega})$$
- In polar form
  - Magnitude $$|Y(e^{j\omega})| = |X(e^{j\omega})| \cdot |H(e^{j\omega})|$$

    ➔ ***magnitude response, gain, attenuation, distortion***
  - Phase $$\angle Y(e^{j\omega}) = \angle X(e^{j\omega}) + \angle H(e^{j\omega})$$

    ➔***phase response, phase shift, attenuation , distortion***

➔Magnitude does not uniquely characterize the system

Magnitude and Phase Representation of systems

- Consider the transfer function: $H(z)=\dfrac{\prod_{i=1}^{m}(z-z_i)}{\prod_{j=1}^{n}(z-p_j)}$

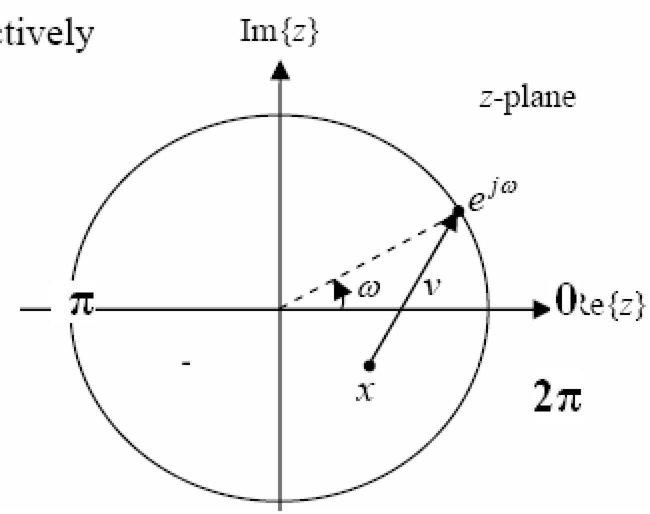
  where z_i's and p_j's represent the zeros and poles of the transfer function, respectively

  We have:

$$\left|H(e^{j\omega})\right|=\frac{\prod_{i=1}^{m}\left|e^{j\omega}-z_i\right|}{\prod_{j=1}^{n}\left|e^{j\omega}-p_j\right|}=\frac{\prod \text{"distancesfromzeros"}}{\prod \text{"distancesfrompoles"}}$$

$$\angle H(e^{j\omega})=\sum_{i=1}^{m}\angle(e^{j\omega}-z_i)-\sum_{j=1}^{n}\angle(e^{j\omega}-p_j)$$

- Note that $\left|e^{j\omega}-x\right|$ and $\angle(e^{j\omega}-x)$ represent the magnitude and phase of the vector v from the point x to the point $e^{j\omega}$ (which is a point on the unit circle with the phase ω) in the complex plane, respectively



Magnitude and Phase Representation of systems

$$H(z)=\frac{\prod_{i=1}^{m}(z-z_i)}{\prod_{j=1}^{n}(z-p_j)}$$

- Given H(z) and its poles *p* and zeros *z*
- At any frequency ω , find the magnitude and phase of the vectors drawn from the poles and zeros to the point $e^{j\omega}$ (a point on the UC with angle ω)

1. The magnitude of H ($e^{j\omega}$) at ω is equal to the product of the magnitudes of all vectors associated with the zeros divided by the product of the magnitudes of all vectors associated with the poles

$$\left|H(e^{j\omega})\right|=\frac{\prod_{i=1}^{m}\left|e^{j\omega}-z_i\right|}{\prod_{j=1}^{n}\left|e^{j\omega}-p_j\right|}=\frac{\prod \text{"distancesfromzeros"}}{\prod \text{"distancesfrompoles"}}$$

2. The phase of H ($e^{j\omega}$) ω is equal to the summation of the angles of all vectors associated with the zeros minus the summation of the angles of all vectors associated with the poles

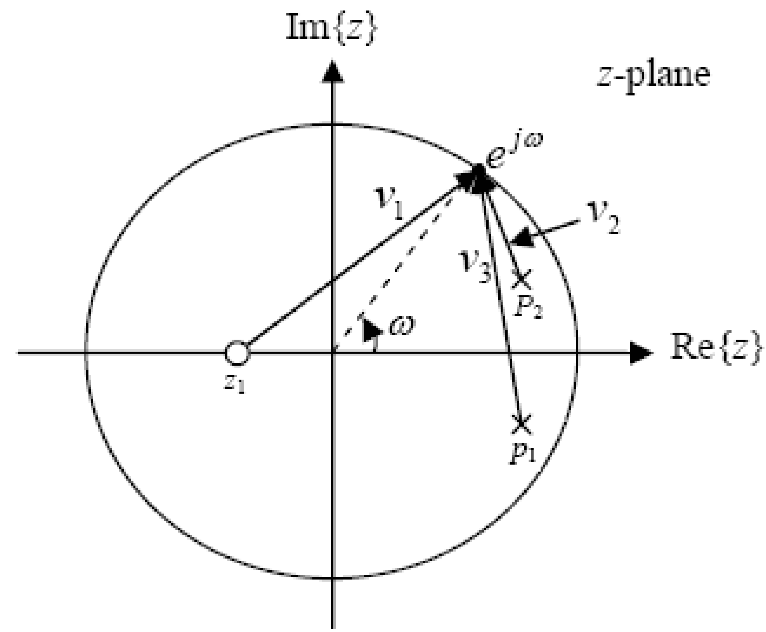
$$\angle H(e^{j\omega})=\sum_{i=1}^{m}\angle(e^{j\omega}-z_i)-\sum_{j=1}^{n}\angle(e^{j\omega}-p_j)$$

Magnitude and phase of systems from H(z): Example One zero, two poles

Consider an LTI system with an $h[n]$

- Assume that H(z) is a rational function of *z* whose pole-zero configuration is given in the plot
- From this plot, it can be concluded that the *z*-transform of the impulse response is:

$$H(z) = K\frac{(z - z_1)}{(z - p_1)(z - p_2)}$$



Magnitude and phase of systems from H(z): Example One zero, two poles

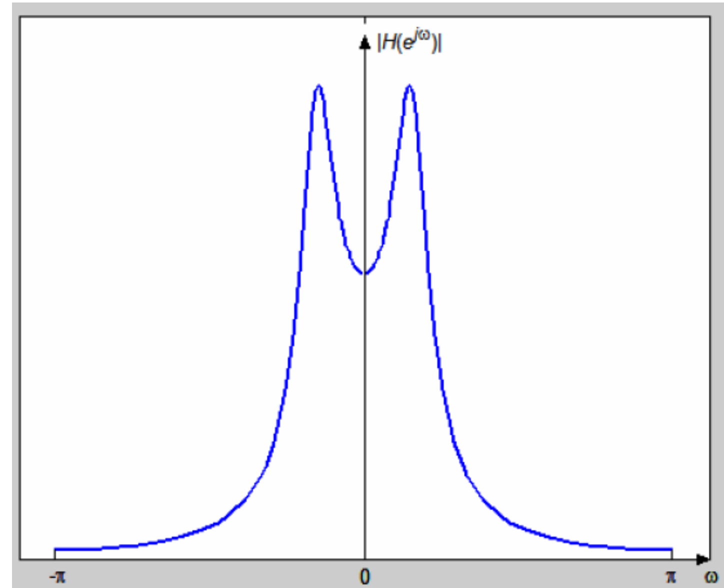
- The amplitude and angle of the vectors v_1, v_2 and v_3 depend on the frequency ω
- The magnitude of the frequency response of this system is proportional to

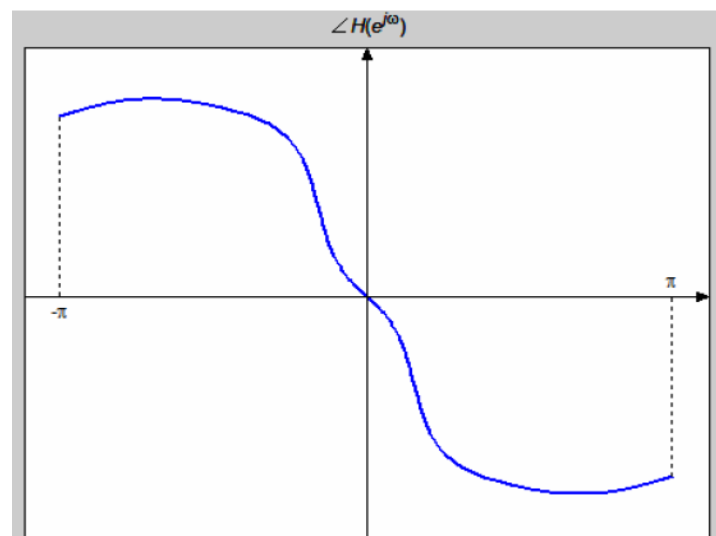
$$\frac{|v_1|}{|v_2||v_3|}$$

- The phase of the frequency response of this system is equal to

$$\angle v_1 - \angle v_2 - \angle v_3$$

- The magnitude of the frequency response is large at those frequencies that correspond to the points on the unit circle which are close to the poles and far from the zeros
- Similarly, the magnitude of the frequency response is small at those frequencies that correspond to the points on the unit circle which are close to the zeros and far from the poles





Magnitude-Phase Representation of systems from H(z)

$$H(e^{j\omega}) = H(z)|_{z=e^{j\omega}}$$

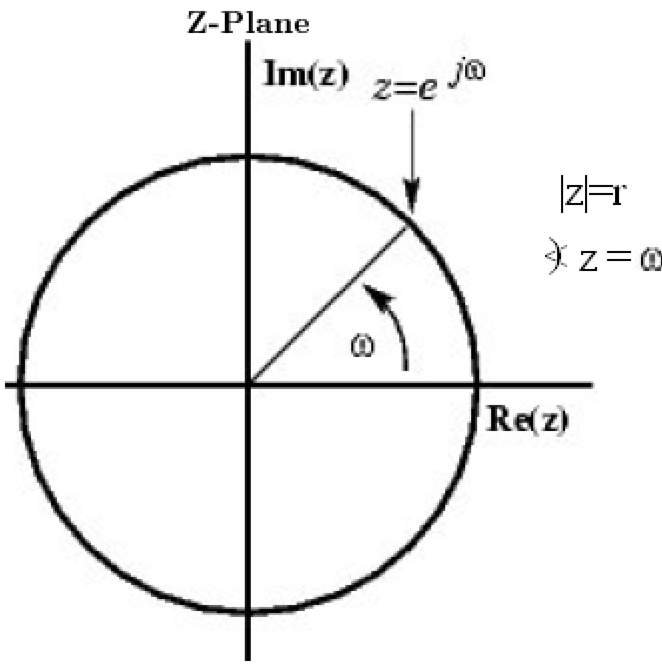
$$H(e^{j\omega}) = Hr(e^{j\omega}) + Hi(e^{j\omega})$$

$$= |H(e^{j\omega})| e^{j\theta_1(\omega)}$$

$$\theta_1(\omega) = \angle H(e^{j\omega})$$

$$Magnitude = |H(e^{j\omega})|$$

$$Phase\ (\theta) = \angle H(e^{j\omega})$$



- In magnitude and phase plots when H(z) has a zero on the UC $z=e^{j\omega}$, as ω goes through that zero on the UC
    1. the magnitude will go to zero and
    2. the phase will flip by π (phase reversal)

Example 1:

$$h[n] = -\delta[n] \qquad -\infty < n < \infty$$

$$\Longrightarrow |H(e^{j\omega})| = 1, \quad \angle H(e^{j\omega}) = \pi$$

$$H(e^{j\omega}) = -1 = e^{j\pi}$$

➔In the magnitude-phase representation, a real-valued frequency response does not necessarily mean that the system is zero-phase

Example 2: ideal lowpass filter

- Frequency response $H(e^{j\omega}) = \begin{cases} 1, & |\omega| < \omega_c, \\ 0, & \omega_c < |\omega| < \pi \end{cases}$
- Impulse response

$$h_{lp}[n] = \frac{\sin \omega_c n}{\pi n}, \qquad -\infty < n < \infty \qquad h[n] \stackrel{?}{=} 0, \quad n < 0$$

➔Non-causal, cannot be implemented!

Example 3: ideal delay system

- Ideal delay system $h_{id}[n] = \delta[n - n_d]$

$$H_{id}(e^{j\omega}) = e^{-j\omega n_d}$$

$$|H_{id}(e^{j\omega})| = 1$$

Delay distortion

$$\angle H_{id}(e^{j\omega}) = -\omega n_d, \quad |\omega| < \pi$$

Linear phase distortion

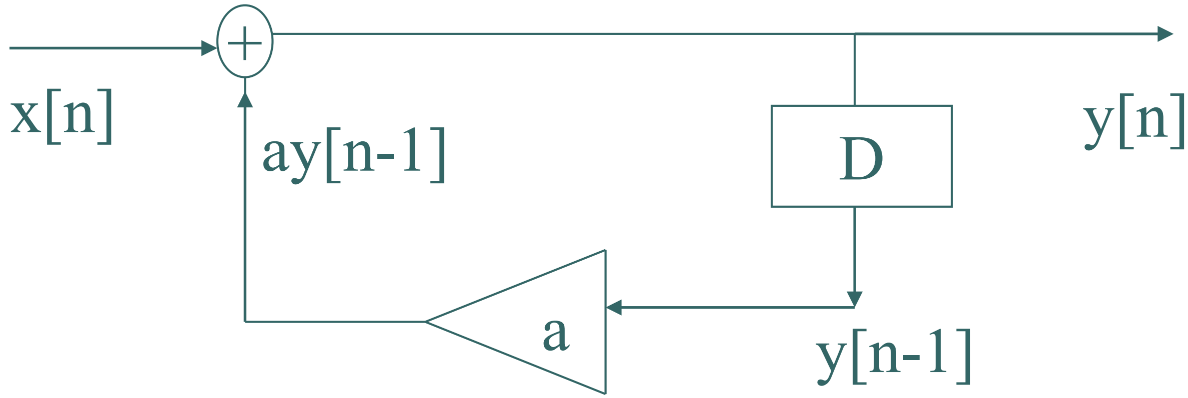
- Ideal lowpass filter with linear phase

$$H_{lp}(e^{j\omega}) = \begin{cases} e^{-j\omega n_d}, & |\omega| < \omega_c, \\ 0, & \omega_c < |\omega| < \pi \end{cases}$$

Ideal lowpass filter is always non-causal !

$$h_{lp}[n] = \frac{\sin \omega_c (n - n_d)}{\pi (n - n_d)}, \qquad -\infty < n < \infty$$

Example 4: First-order Recursive DT Filter



$$y[n] - ay[n-1] = x[n]$$

1) Taking the DTFT : $\mathrm{Y}(\mathrm{e}^{j\omega}) - a\mathrm{e}^{-j\omega} Y(\mathrm{e}^{j\omega}) = X(\mathrm{e}^{j\omega})$

$$==> \frac{\mathrm{Y}(\mathrm{e}^{j\omega})}{X(\mathrm{e}^{j\omega})} = \mathrm{H}(\mathrm{e}^{j\omega}) = \frac{1}{1 - a\mathrm{e}^{-j\omega}}$$

2) Eigenfunction property : $\mathrm{x}[\mathrm{n}] = \mathrm{e}^{j\omega n}$, then $\mathrm{y}[\mathrm{n}] = \mathrm{H}(\mathrm{e}^{j\omega})\mathrm{e}^{j\omega n}$

Substituting in the diff. eq. : $\mathrm{H}(\mathrm{e}^{j\omega})\mathrm{e}^{j\omega n} - a\mathrm{H}(\mathrm{e}^{j\omega})\mathrm{e}^{j\omega(n-1)} = \mathrm{e}^{j\omega n}$

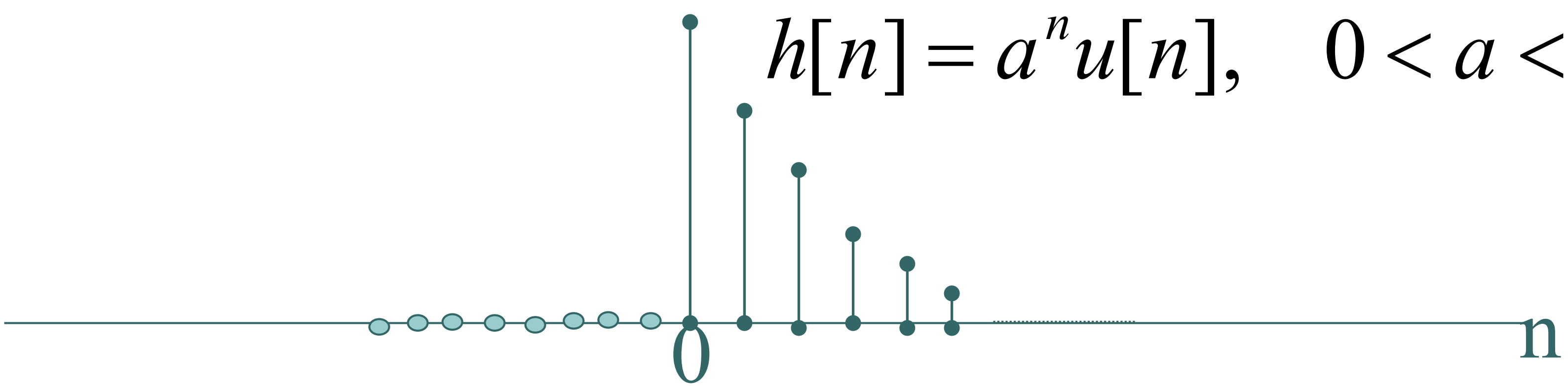
$$\mathrm{H}(\mathrm{e}^{j\omega}) = \frac{1}{1 - a\mathrm{e}^{-j\omega}}$$

Example 4: first order recursive DT filter

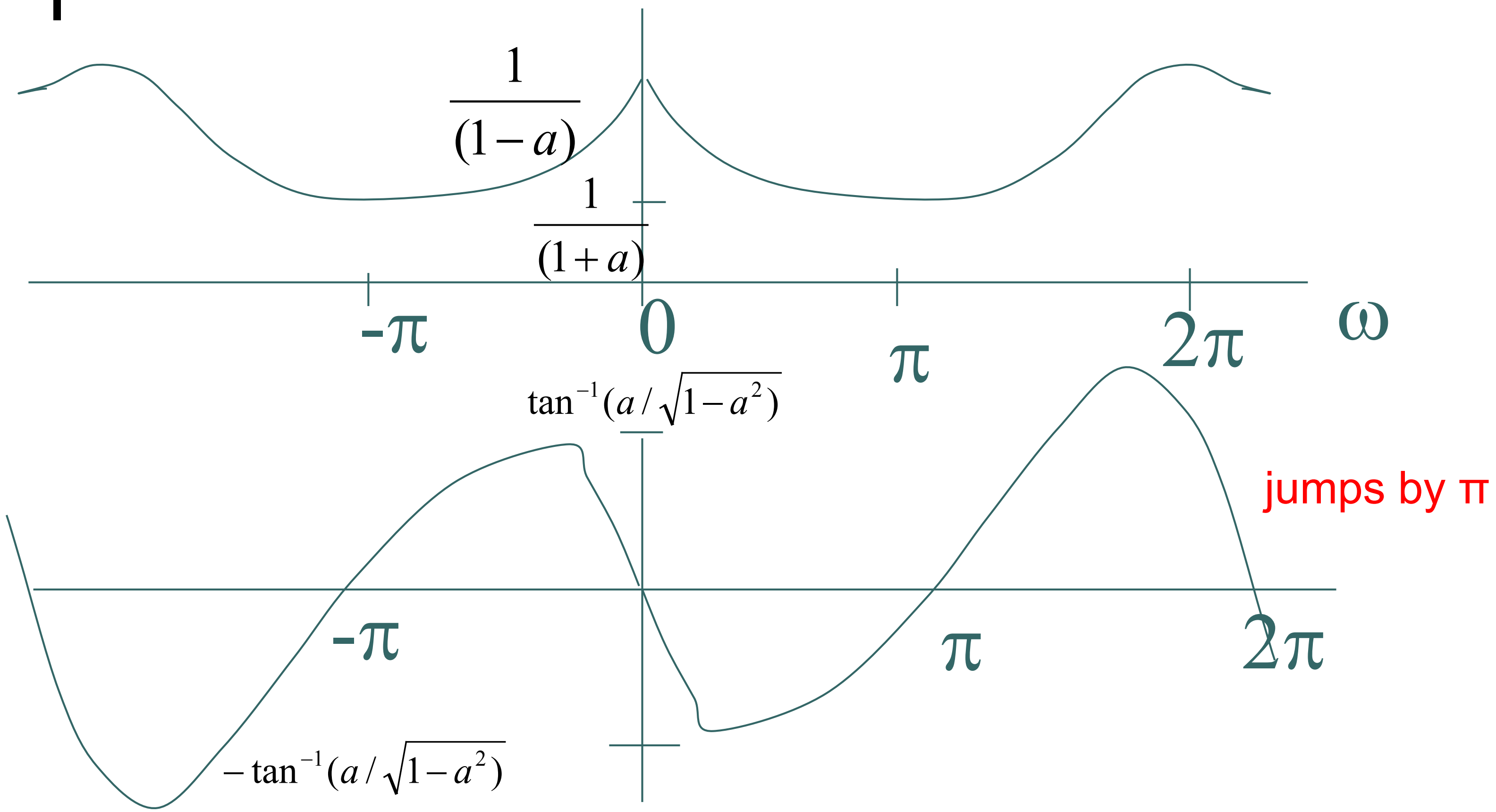
$$h[n] \overset{F.T}{\leftrightarrow} \mathrm{H}(\mathrm{e}^{j\omega})$$

$$a^n u[n] \overset{F.T}{\leftrightarrow} \frac{1}{1 - a\mathrm{e}^{-j\omega}}$$

$$h[n] = a^n u[n], \quad 0 < a < 1$$



Example 4: First order recursive filter $H(e^{j\omega}) = \dfrac{1}{1 - ae^{-j\omega}}$



- Typically, it is hard to infer much from a phase plot (a group delay plot is more useful; see later)

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Amplitude-Phase Representation of signals

- Recall: In magnitude and phase plots:
  - as ω goes through a zero on the unit circle of H(z)
    - the magnitude will go to zero and
    - the phase will flip by π **(phase reversal)**

- If $H(e^{j\omega})$ is real but bipolar, an alternative representation is to remove these jumps of π in a phase plot
  - A **bipolar signal** may assume two polarities and is usually symmetrical with respect to zero amplitude
  - An audio waveform is an example of a bipolar signal

Amplitude-Phase Representation of signals

- A signal can be expressed in terms of complex polar coordinates or as Amplitude/Phase Representation

$$H(e^{j\omega}) = A(e^{j\omega})e^{j\theta_2(\omega)}$$

$$\theta_2(\omega) = \angle H(e^{j\omega})$$

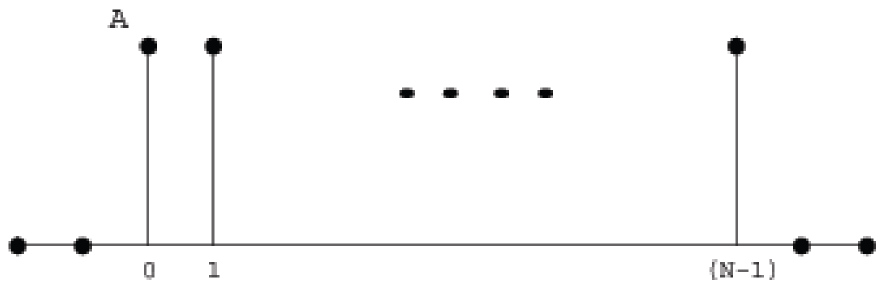
$$H(e^{j\omega}) = |H(e^{j\omega})| e^{j\theta_1(\omega)}$$

$$\theta_1(\omega) = \angle H(e^{j\omega})$$

- $A(e^{j\omega})$ is real but not restricted to be positive, thus $\theta_2(\omega)$ is different from $\theta_1(\omega)$
  - The sign change is contained in $A(e^{j\omega})$, and the jumps of π do not exist in $\theta_2(\omega)$

Amplitude-Phase Representation of signals

- Consider *h(n)*



$$H(e^{j\omega}) = Ae^{-j\omega(N-1)/2}\sin(\omega N/2)/\sin(\omega/2)$$

- In the magnitude-phase representation: θ1(ω) has jumps of π at the sign change
- In the amplitude-phase representation: θ2(ω)= -ω(N-1)/2, a straight line with the slope –(N-1)/2
  - Moreover, here phase would be the same whether A is positive or negative

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System function

- Linear constant-coefficient difference equation

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{m=0}^{M} b_m x[n-m]$$

- ZT format

$$\sum_{k=0}^{N} a_k z^{-k} Y(z) = \sum_{m=0}^{M} b_m z^{-m} X(z)$$

$$\Rightarrow H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{m=0}^{M} b_m z^{-m}}{\sum_{k=0}^{N} a_k z^{-k}}$$

$$= \left(\frac{b_0}{a_0}\right) \frac{\prod_{m=1}^{M} (1 - c_m z^{-1})}{\prod_{k=1}^{N} (1 - d_k z^{-1})}$$

$(1 - c_m z^{-1})$ in the numerator

a zero at $z = c_m$; a pole at $z = 0$

$(1 - d_k z^{-1})$ in the denominator

a zero at $z = 0$; a pole at $z = d_k$

System function:
Frequency Response

- DTFT of a LTI rational system function

$$H\left(e^{j\omega}\right) = \frac{\sum_{k=0}^{M} b_k e^{-j\omega k}}{\sum_{k=0}^{N} a_k e^{-j\omega k}} = \left(\frac{b_0}{a_0}\right) \frac{\prod_{k=1}^{M}\left(1 - c_k e^{-j\omega}\right)}{\prod_{k=1}^{N}\left(1 - d_k e^{-j\omega}\right)}$$

- Magnitude Response

$$\left|H\left(e^{j\omega}\right)\right| = \left|\frac{b_0}{a_0}\right| \frac{\prod_{k=1}^{M}\left|1 - c_k e^{-j\omega}\right|}{\prod_{k=1}^{N}\left|1 - d_k e^{-j\omega}\right|}$$

System function: Log-Magnitude Response

- Log-Magnitude in decibels (dB)

$$20\log_{10}\left|H\left(e^{j\omega}\right)\right| = 20\log_{10}\left|\frac{b_0}{a_0}\right| + \sum_{k=1}^{M} 20\log_{10}\left|1 - c_k e^{-j\omega}\right| - \sum_{k=1}^{N} 20\log_{10}\left|1 - d_k e^{-j\omega}\right|$$

➔ Frequency response is a sum of the contributions from each pole and each zero

- System gain in dB = $20\log_{10}\left|H\left(e^{j\omega}\right)\right|$
- System attenuation in dB $= -20\log_{10}\left|H\left(e^{j\omega}\right)\right| = -\text{Gain in dB}$
- Examples:
  - $|H(e^{j\omega})|=0.001$ translates into –60dB gain or 60dB attenuation
  - $|H(e^{j\omega})|=0.5$ translates into -6dB gain
  - $|H(e^{j\omega})|=1$ translates into 0dB gain
  - $|H(e^{j\omega})| > 1$ translates into positive gain

System function: Phase response Instantaneous, wrapped, Unwrapped Phase

- Phase shift is in general ambiguous:

$$e^{j(\theta+2\prod k)} = e^{j\theta}, \text{any integer } k$$

- The **instantaneous (or local) phase** of a complex-valued function is the real-valued function

$$\theta(\omega) = \angle H(e^{j\omega})$$

- When the phase is constrained to an interval, e.g., [- π,π], it is called the **wrapped** phase *ARG(H)*
  ➔it does <u>not</u> distinguish between multiples of 2π

- An **unwrapped** phase *arg(H)* is a continuous function of ω
  ➔The unwrapped phase is not restricted to the *wrapped* range, and has no jumps

System function: Phase response
Unwrapped (Continuous) Phase

- Phase is ambiguous: when calculating the arctan(.) function on a computer
  - Values between $-\pi$ and $+\pi$
  - Denoted in the book as ARG(.)

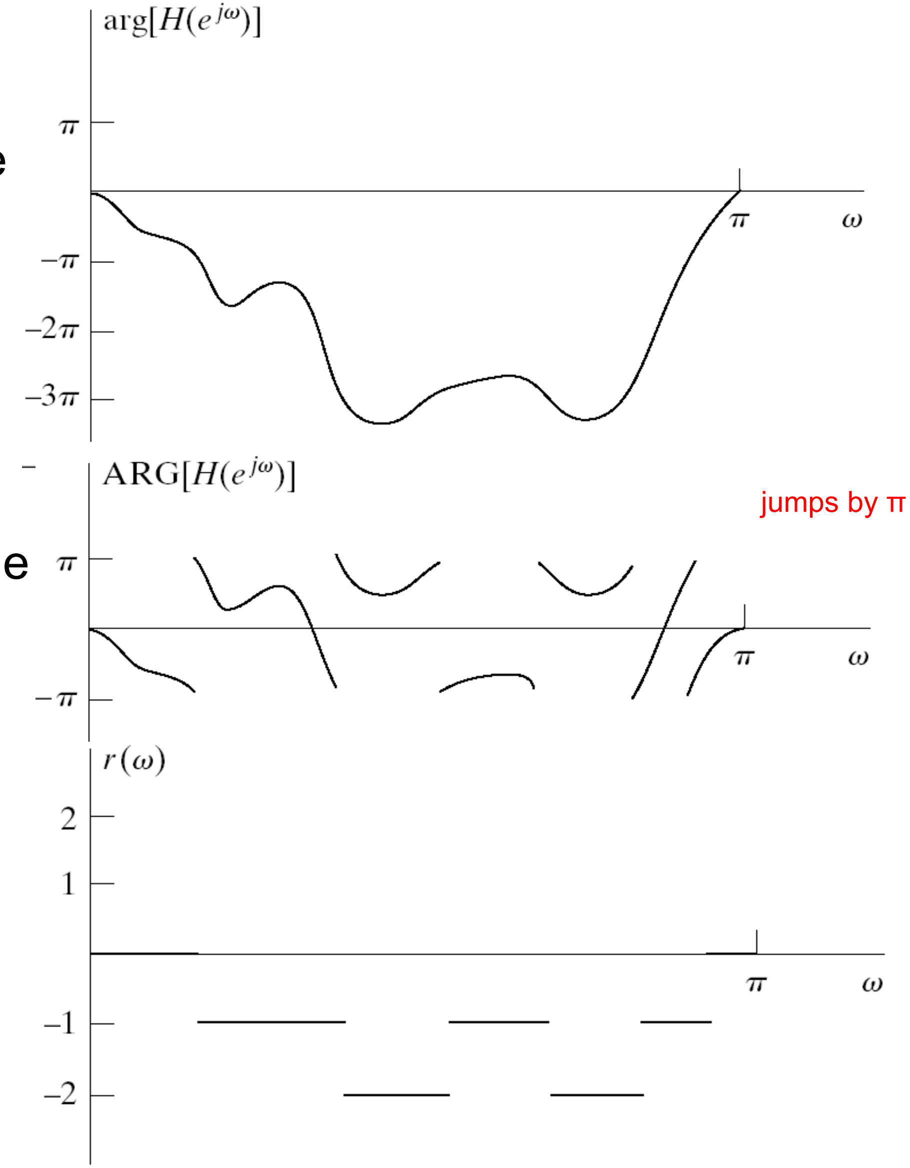
$$-\pi < \mathrm{ARG}\left[H\left(e^{j\omega}\right)\right] \leq \pi$$

  - Any multiple of 2π would give the same result

$$\angle H\left(e^{j\omega}\right) = \mathrm{ARG}\left[H\left(e^{j\omega}\right)\right] + 2\pi r(\omega)$$

  - Here $r(\omega)$ is an integer for any given value of ω
- Group delay is the derivative of the unwrapped phase

$$\mathrm{grd}\left[H\left(e^{j\omega}\right)\right] = -\frac{d}{d\omega}\left[\arg\left[H\left(e^{j\omega}\right)\right]\right]$$



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Group Delay of LTI Systems

- Typically, it is hard to infer much from a phase plot
  - A group delay plot gives more useful information
- Group delay is defined as $$\tau(\omega) = -\frac{d}{d\omega}\{\angle H(e^{j\omega})\}$$
- A group delay is about the delay of the system
  - It measures the linearity of the phase
  - It is about delay of samples on the signal
- Group delay is a measure of the transit time of a signal through a system/device versus frequency
  - Group delay is a useful measure of phase distortion
  - The variations in group delay cause signal distortion
- In Matlab, group delay is calculated using the FT rather than differentiation

Group Delay of LTI Systems

$$Y(e^{j\omega}) = X(e^{j\omega})H(e^{j\omega})$$

$$|Y(e^{j\omega})| = |X(e^{j\omega})||H(e^{j\omega})|$$

$$\angle Y(e^{j\omega}) = \angle H(e^{j\omega}) + \angle X(e^{j\omega})$$

$If\ H(e^{j\omega}) = e^{-j\omega n_d}$ (an ideal lowpass filter wit h linear phase)

$Then\ |H(e^{j\omega})| = 1 \ \ and \ \ \angle H(e^{j\omega}) = -\omega n_d$

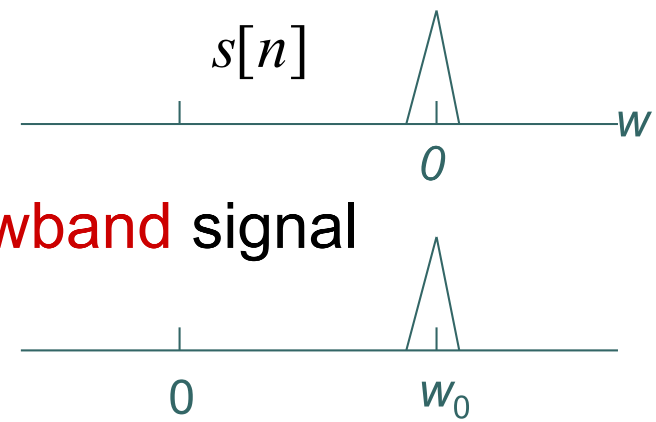
==> The phase is linear, i.e., the system has a linear phase

What this system does, is to shift (delay) the input by n_d, i.e., $y[n] = x(n - n_d)$

Group delay is define as $\tau(\omega) = -\dfrac{d}{d\omega}\{\angle H(e^{j\omega})\}$

➔It is a measure of the linearity of the phase

Group delay $\tau(\omega)$



- Effect of phase distortion on a narrowband signal

$$x[n] = s[n]\cos(\omega_0 n)$$

- For this input x[n] with spectrum only around w_0
  - phase effect can be approximated around w_0 as linear (though in reality maybe nonlinear)

$$\angle H(e^{j\omega}) \approx -\omega n_d - \phi_0$$

  - The output y[n] = x[n]*h[n] is approximately

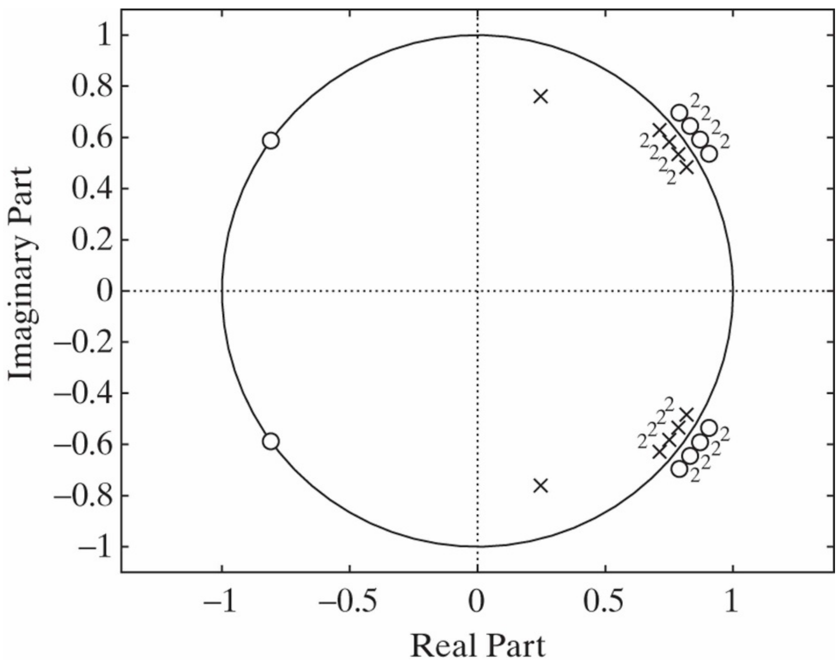
$$y[n] \approx |H(e^{j\omega_0})|\, s[n-n_d]\cos(\omega_0(n-n_d)-\phi_0)$$

- Group delay

$$\tau = grd[H(e^{j\omega})] = -\frac{d}{d\omega}\{\arg[H(e^{j\omega})]\}$$

Example: Frequency response of system
Pole and zero plot

- Consider the pole-zero plot of a filter
  - The number 2 indicates double-order poles and zeroes)
- What are the magnitude, phase, group delay responses of this filter?

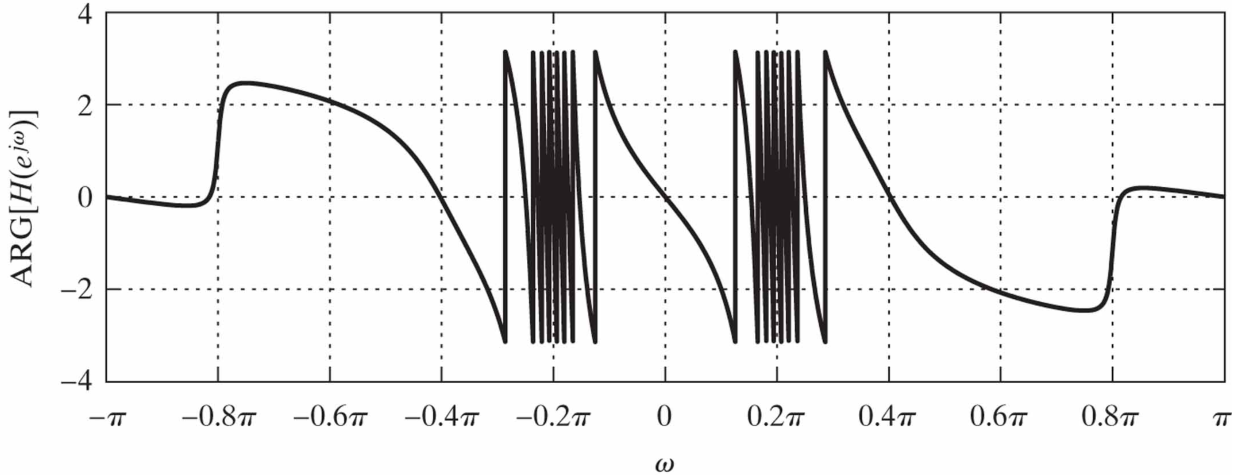


Example: Frequency response of system

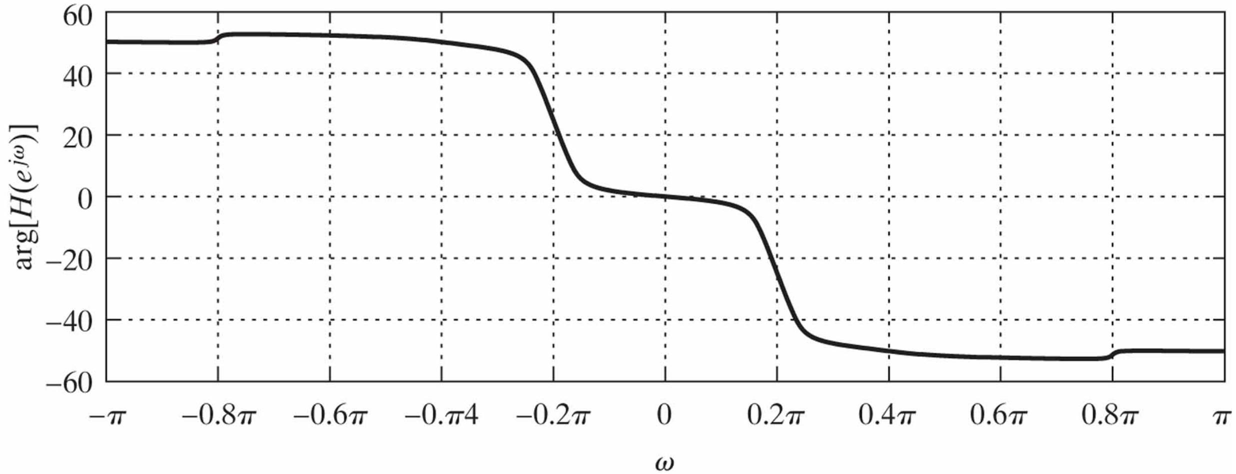
Phase response functions

(a) Principal value phase ARG[$H(e^{j\omega})$]

(b) Continuous phase arg [$H(e^{j\omega})$]



(a) Principle Value of Phase Response

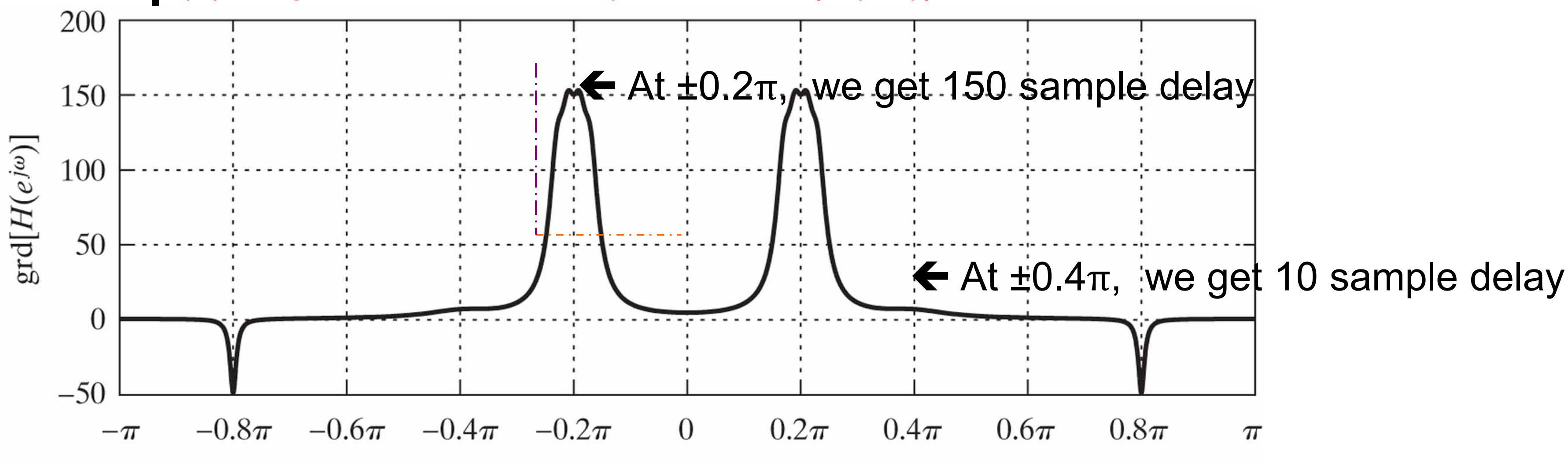


(b) Unwrapped Phase Response

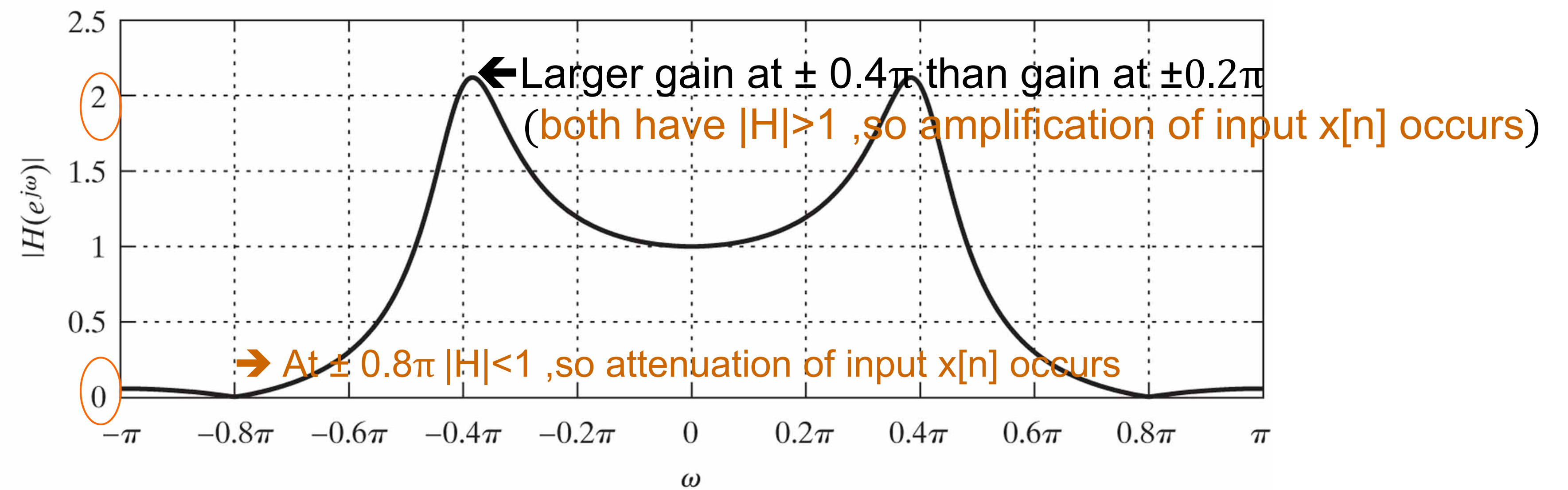
Example: Frequency response of system

(a) Group delay function, grd[$H(e^{j\omega})$]

(b) Magnitude of frequency response, $|H(e^{j\omega})|$



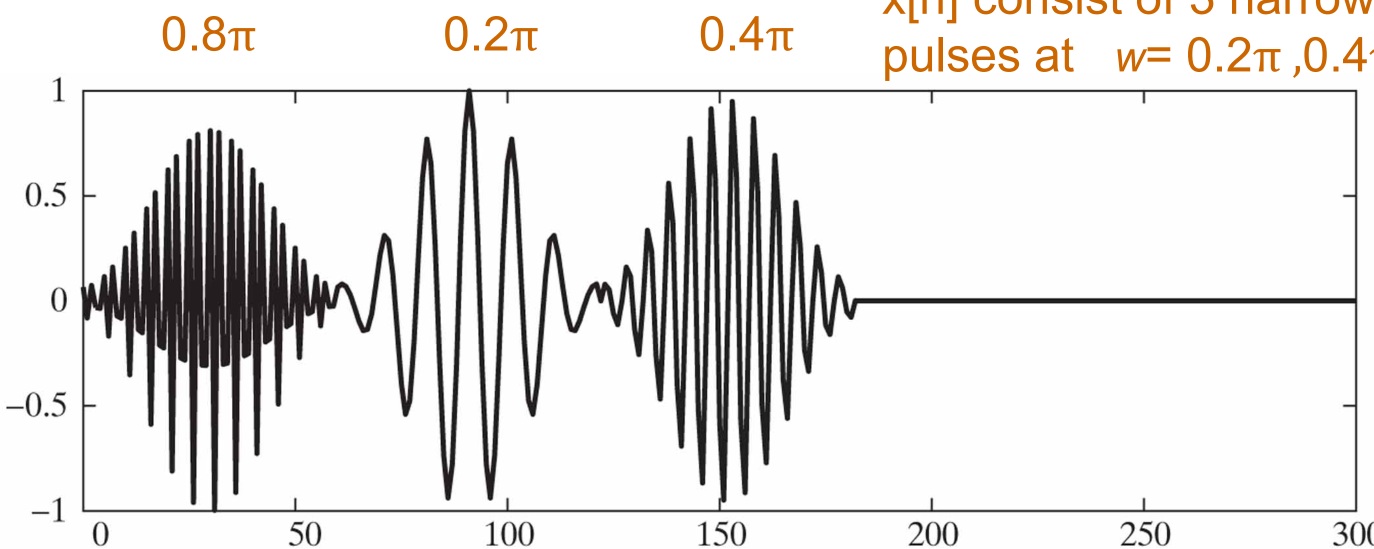
(a) Group delay of $H(z)$



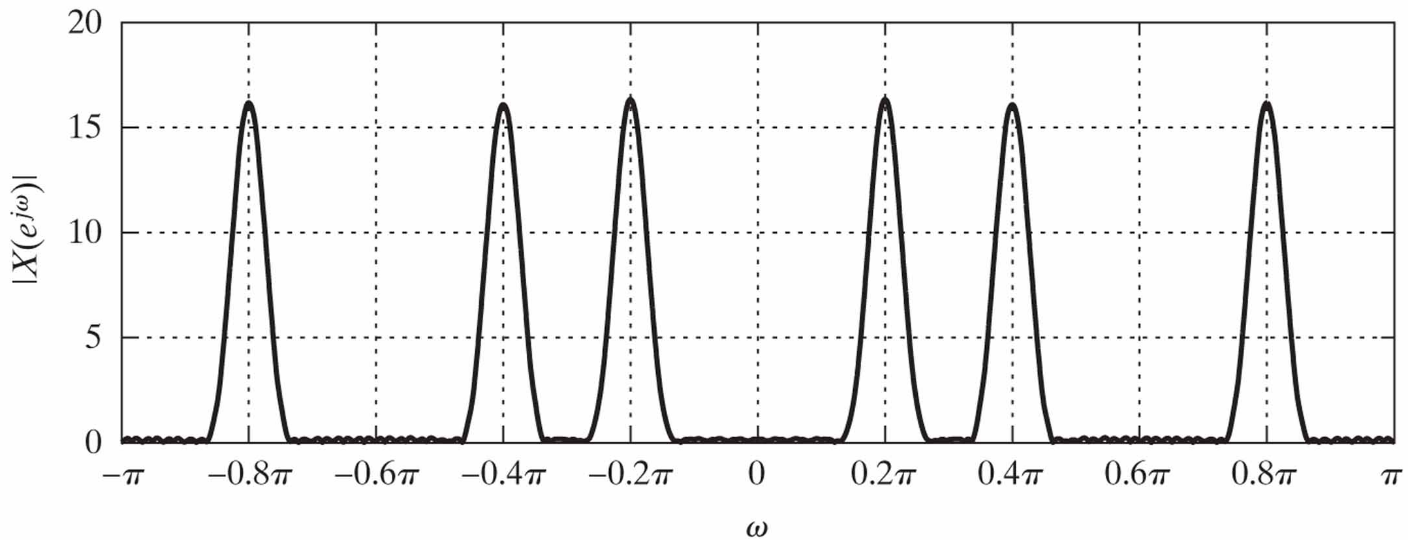
(b) Magnitude of Frequency Response

Example: (a) Input signal $x[n]$, (b) Corresponding DTFT magnitude $|X(e^{j\omega})|$

x[n] consist of 3 narrowband pulses at $w = 0.2\pi, 0.4\pi, 0.8\pi$



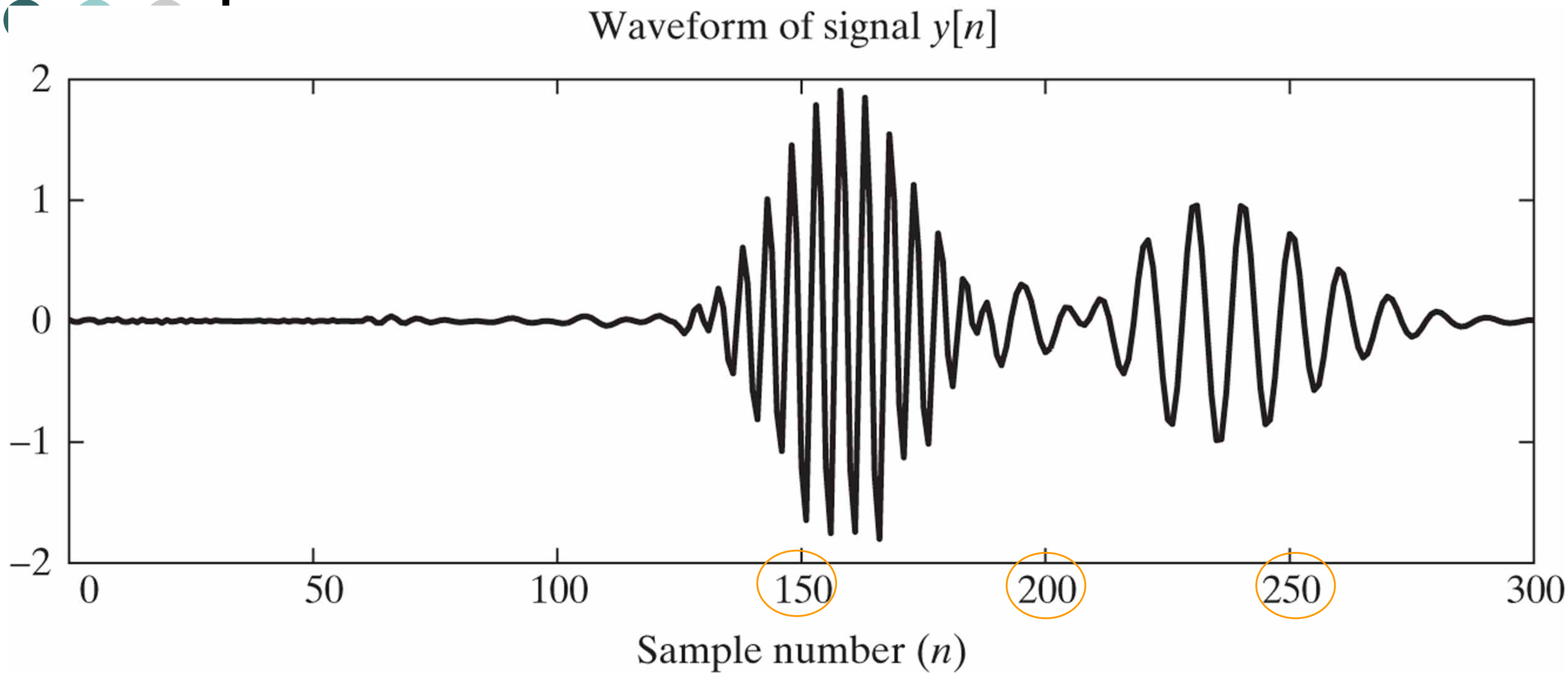
(a) Waveform of signal $x[n]$



(b) Magnitude of DTFT of $x[n]$

Example: Output signal

$$|Y(e^{j\omega})| = |X(e^{j\omega})||H(e^{j\omega})|$$



Since the filter has considerable attenuation at $\omega = 0.8\pi$, the pulse at that frequency is not clearly present in the output. Also, since the group delay at $\omega = 0.2\pi$ is approximately 150 samples and at $\omega = 0.5\pi$ is approximately 10 samples, the second pulse in $x[n]$ will be delayed by about 150 samples and the third pulse by 10 samples,

Outline

- Introduction
- Magnitude and Phase Representation of signals
- Amplitude-Phase Representation of signals
- System analysis:
  - System function
  - Group Delay of LTI Systems
  - IIR and FIR Systems: impulse response of rational systems
  - Frequency Response of Rational Systems
  - Relation between Magnitude/Phase & poles/zeros
  - All-pass systems
  - Minimum-phase systems
  - Linear systems with generalized linear phase

Infinite Impulse Response (IIR) Systems

- Rational system function

$$H(z) = \left(\frac{b_0}{a_0}\right)\frac{\prod_{k=1}^{M}\left(1 - c_k z^{-1}\right)}{\prod_{k=1}^{N}\left(1 - d_k z^{-1}\right)}$$

- If **at least one nonzero pole** does not cancel with a zero, in h[n] there will at least one term of the form

$$a^n u[n] \quad \text{OR} \quad -a^n u[-n-1]$$

- Therefore, h[n], the impulse response, will be <u>infinite length</u>
  ➔IIR system $\quad y[n] - ay[n-1] = x[n]$

- <u>Example:</u> Causal system of the form

$$H(z) = \frac{1}{1 - az^{-1}} \quad ; ROC : |z| > \text{a} \ \ \text{from causality}$$

$$h[n] = a^n u[n]$$

Finite Impulse Response (FIR) Systems

- The rational system function

$$H(z)=\frac{Y(z)}{X(z)}=\frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}}$$

- If H(z) does not have any poles except at z=0, then N=0 ➔the rational system function

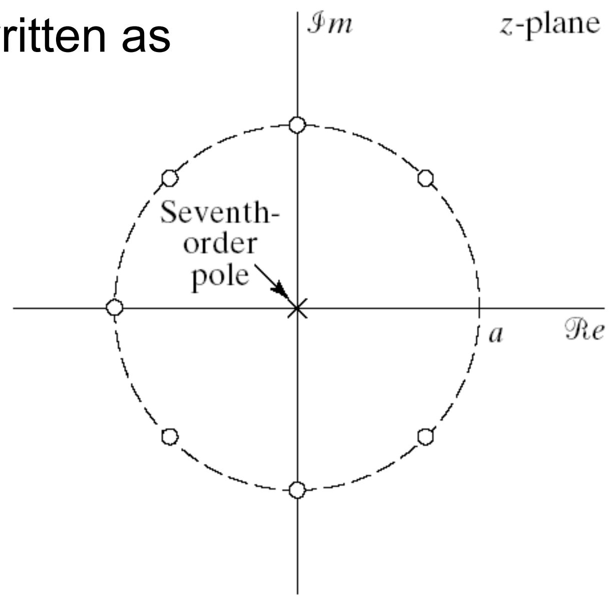
$$H(z)=\sum_{k=0}^{M} b_k z^{-k}$$

- No partial fraction expansion possible (or needed)
- The impulse response can be seen to be $$h[n]=\sum_{k=0}^{M} b_k \delta[n-k]$$
- Impulse response is of finite length ➔ FIR system

Example 5.5: FIR System with system function

$$h[n] = \begin{cases} a^n & 0 \le n \le M \\ 0 & \text{else} \end{cases} \qquad H(z) = \sum_{n=-\infty}^{\infty} h[n] z^{-n} = \sum_{n=0}^{M} a^n z^{-n} = \frac{1 - a^{M+1} z^{-M-1}}{1 - a z^{-1}}$$

← finite length



- Assuming *a* is both real and positive, the zeros can be written as

$$z_k = a e^{j2\pi k/(M+1)} \quad \text{for} \quad k = 0,1,\ldots,M$$

- For k=0 we have a zero at $z_0=a$
  - (The zero cancels the pole at z=a)
- From convolution → $y[n] = \sum_{k=0}^{M} a^k x[n-k]$
- From $H(z) = \dfrac{1 - a^{M+1} z^{-M-1}}{1 - a z^{-1}}$

→ $y[n] - a y[n-1] = x[n] - a^{M+1} x[n-M-1]$ ←Need memory

→ The more the zero cancels the pole at ***a*** ,
the more equal the two equations become

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    - Frequency response
    - Phase response
    - Group delay
    - Squared magnitude
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  - All-pass systems
  - Minimum-phase systems
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Frequency Response of Rational System Functions

- DTFT of a stable and LTI rational system function

$$H\left(e^{j\omega}\right) = \frac{\sum_{k=0}^{M} b_k e^{-j\omega k}}{\sum_{k=0}^{N} a_k e^{-j\omega k}} = \left(\frac{b_0}{a_0}\right) \frac{\prod_{k=1}^{M}\left(1 - c_k e^{-j\omega}\right)}{\prod_{k=1}^{N}\left(1 - d_k e^{-j\omega}\right)}$$

- Magnitude Response

$$\left|H\left(e^{j\omega}\right)\right| = \left|\frac{b_0}{a_0}\right| \frac{\prod_{k=1}^{M}\left|1 - c_k e^{-j\omega}\right|}{\prod_{k=1}^{N}\left|1 - d_k e^{-j\omega}\right|}$$

- Magnitude Squared

$$\left|H\left(e^{j\omega}\right)\right|^2 = H\left(e^{j\omega}\right)^* H\left(e^{j\omega}\right) = \left(\frac{b_0}{a_0}\right)^2 \frac{\prod_{k=1}^{M}\left(1 - c_k e^{-j\omega}\right)\left(1 - c_k^* e^{j\omega}\right)}{\prod_{k=1}^{N}\left(1 - d_k e^{-j\omega}\right)\left(1 - d_k^* e^{j\omega}\right)}$$

Frequency Response of Rational System Functions

- Log Magnitude in decibels (dB)

$$20\log_{10}\left|H\left(e^{j\omega}\right)\right| = 20\log_{10}\left|\frac{b_0}{a_0}\right| + \sum_{k=1}^{M} 20\log_{10}\left|1 - c_k e^{-j\omega}\right| - \sum_{k=1}^{N} 20\log_{10}\left|1 - d_k e^{-j\omega}\right|$$

➔Frequency response is a sum of the contributions from each pole and zero

$$\text{Gain in dB} = 20\log_{10}\left|H\left(e^{j\omega}\right)\right|$$

$$\text{Attenuation in dB} = -\,20\log_{10}\left|H\left(e^{j\omega}\right)\right| = -\text{Gain in dB}$$

- Output of system

$$20\log_{10}\left|Y\left(e^{j\omega}\right)\right| = 20\log_{10}\left|H\left(e^{j\omega}\right)\right| + 20\log_{10}\left|X\left(e^{j\omega}\right)\right|$$

Phase Response of Rational System Functions

- Phase response of a rational system function

$$\angle\left|H\left(e^{j\omega}\right)\right| = \angle\left(\frac{b_0}{a_0}\right) + \sum_{k=1}^{M} \angle\left(1 - c_k e^{-j\omega}\right) - \sum_{k=1}^{N} \angle\left(1 - d_k e^{-j\omega}\right)$$

➔Phase response is a sum of the contributions from each pole and zero

- Group delay

$$grd\left|H\left(e^{j\omega}\right)\right| = -\sum_{k=1}^{M} \frac{d}{d\omega} \arg\left(1 - c_k e^{-j\omega}\right) + \sum_{k=1}^{N} \frac{d}{d\omega} \arg\left(1 - d_k e^{-j\omega}\right)$$

  - Here arg[.] represents the continuous (unwrapped) phase

- Equivalently, we get

$$\mathrm{grd}\left|H\left(e^{j\omega}\right)\right| = \sum_{k=1}^{M} \frac{\left|d_k\right|^2 - \mathrm{Re}\left\{d_k e^{-j\omega}\right\}}{1 + \left|d_k\right|^2 - 2\,\mathrm{Re}\left\{d_k e^{-j\omega}\right\}} - \sum_{k=1}^{M} \frac{\left|c_k\right|^2 - \mathrm{Re}\left\{c_k e^{-j\omega}\right\}}{1 + \left|c_k\right|^2 - 2\,\mathrm{Re}\left\{c_k e^{-j\omega}\right\}}$$

Summary: Frequency, magnitude, phase, and group responses

$$\Rightarrow H(e^{j\omega}) = \frac{\sum_{k=0}^{M} b_k e^{-j\omega k}}{\sum_{k=0}^{N} a_k e^{-j\omega k}} = \frac{b_0}{a_0} \frac{\prod_{k=1}^{M}(1-c_k e^{-j\omega})}{\prod_{k=1}^{N}(1-d_k e^{-j\omega})}$$

$$\Rightarrow |H(e^{j\omega})| = \left|\frac{b_0}{a_0}\right| \frac{\prod |1-c_k e^{-j\omega}|}{\prod |1-d_k e^{-j\omega}|}$$

$$|H(e^{j\omega})|^2$$

$$\Rightarrow \measuredangle H(e^{j\omega}) = \measuredangle\left[\frac{b_0}{a_0}\right] + \sum_{k=1}^{M} \measuredangle[1-c_k e^{-j\omega}] - \sum_{k=1}^{N} \measuredangle[1-d_k e^{-j\omega}]$$

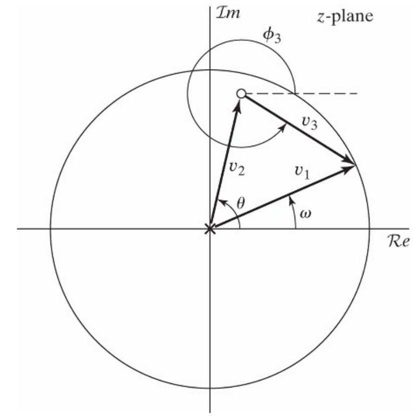
$$\Rightarrow \mathrm{grd}[H(e^{j\omega})] = \sum_{k=1}^{N} \frac{d}{d\omega}[\arg(1-d_k e^{-j\omega})] - \sum_{k=1}^{M} \frac{d}{d\omega}[\arg(1-c_k e^{-j\omega})]$$

$$\Rightarrow 20\log_{10}\left|H\left(e^{j\omega}\right)\right| = 20\log_{10}\left|\frac{b_0}{a_0}\right| + \sum_{k=1}^{M} 20\log_{10}\left|1-c_k e^{-j\omega}\right| - \sum_{k=1}^{N} 20\log_{10}\left|1-d_k e^{-j\omega}\right|$$

➔Frequency/magnitude/phase/delay responses:
a sum of the contributions from each pole and zero

Example 1: 1st order (single zero or pole) IIR system

$$H(z) = \frac{z - re^{j\theta}}{z} \qquad \text{zero at } re^{j\theta}$$



- Let's analyze the effect of a single term

$$\left|1 - c_k e^{-j\omega}\right|^2 = \left|1 - re^{j\theta}e^{-j\omega}\right|^2 = 1 + r^2 - 2r\cos(\omega - \theta)$$

- If we represent it in dB

$$20\log_{10}\left|1 - re^{j\theta}e^{-j\omega}\right| = 10\log_{10}\left[1 + r^2 - 2r\cos(\omega - \theta)\right]$$

- The phase term is written as

$$\mathrm{ARG}\left[1 - re^{j\theta}e^{-j\omega}\right] = \arctan\left[\frac{r\sin(\omega - \theta)}{1 - r\cos(\omega - \theta)}\right]$$

- And the group delay obtained by differentiating the phase

$$\mathrm{grd}\left[1 - re^{j\theta}e^{-j\omega}\right] = \frac{r^2 - r\cos(\omega - \theta)}{1 + r^2 - 2r\cos(\omega - \theta)} = \frac{r^2 - r\cos(\omega - \theta)}{\left|1 - re^{j\theta}e^{-j\omega}\right|^2}$$

- Minimum and maximum values of magnitude

$$10\log_{10}\left[1 + r^2 - 2r\cos(\omega - \theta)\right] = 10\log_{10}\left[1 + r^2 + 2r\right] = 20\log_{10}\left[1 + r\right]$$

$$10\log_{10}\left[1 + r^2 - 2r\cos(\omega - \theta)\right] = 10\log_{10}\left[1 + r^2 - 2r\right] = 20\log_{10}\left|1 - r\right|$$

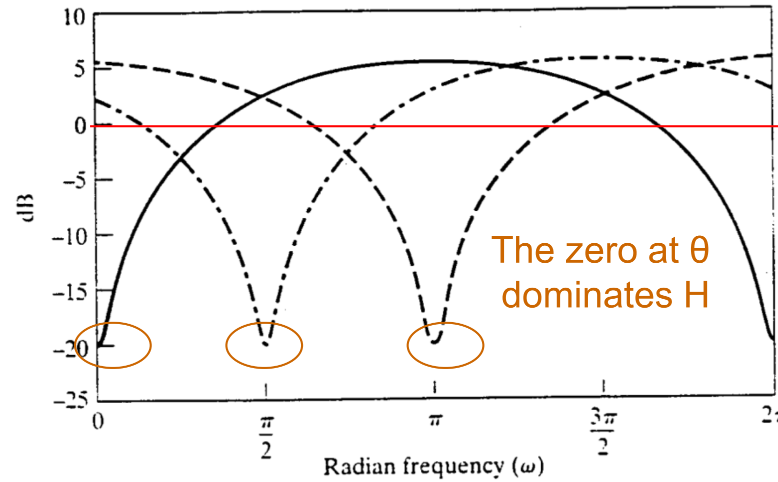
A single zero or pole

Consider $H(z) = (1 - re^{j\theta} z^{-1}) = \dfrac{z - re^{j\theta}}{z}$

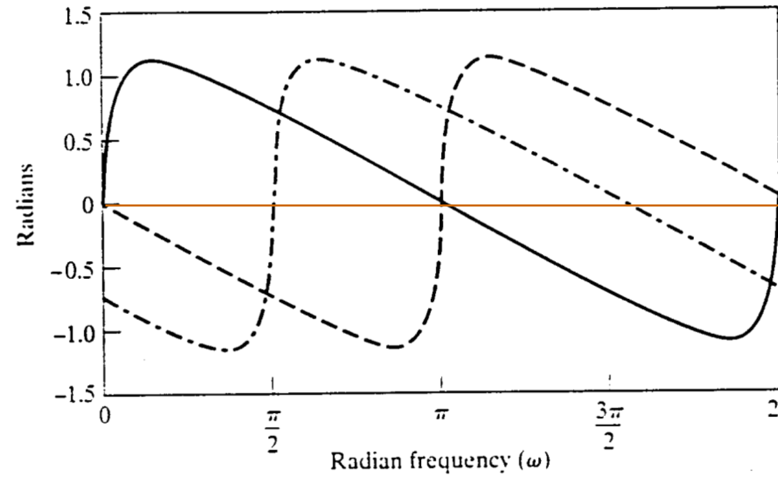
$$|H(e^{j\omega})|^2 = 1 + r^2 - 2r\cos(\omega - \theta)$$

$$\measuredangle H(e^{j\omega}) = \arctan\left[\frac{r\cos(\omega-\theta)}{1 - r\cos(\omega-\theta)}\right]$$

$$\text{grd}[H(e^{j\omega})] = \frac{r^2 - r\cos(\omega-\theta)}{|1 - re^{j\theta}e^{-j\omega}|^2}$$

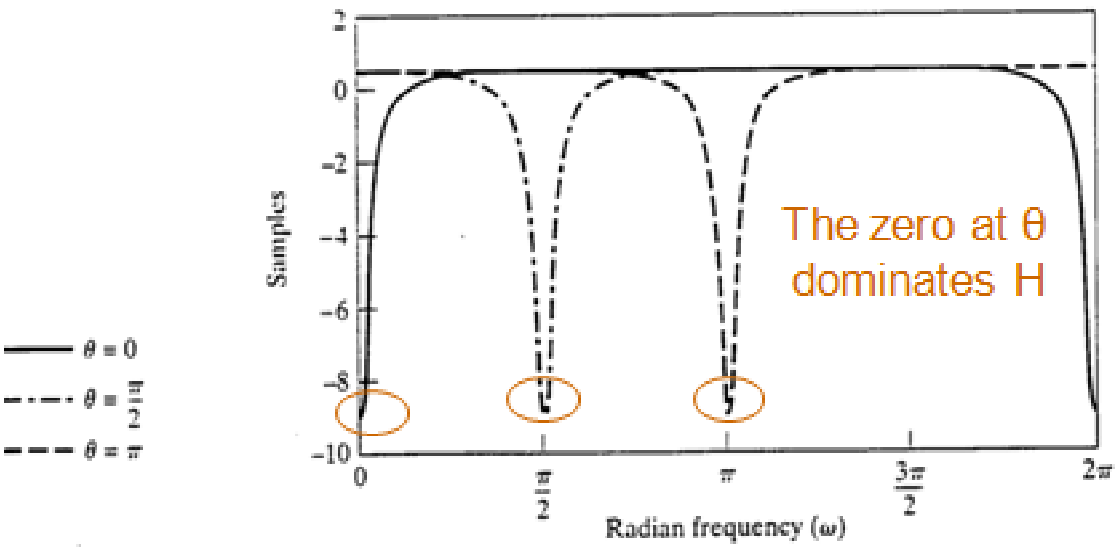


(a)



(b)

Phase(w=θ) =0

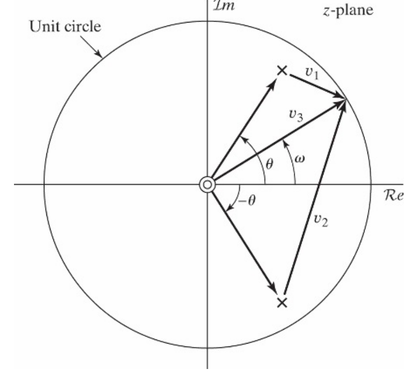


(c)

Figure 5.9 Frequency response for a single zero, with $r = 0.9$ and the three values of θ shown. (a) Log magnitude. (b) Phase. (c) Group delay.

Multiple poles and zeros

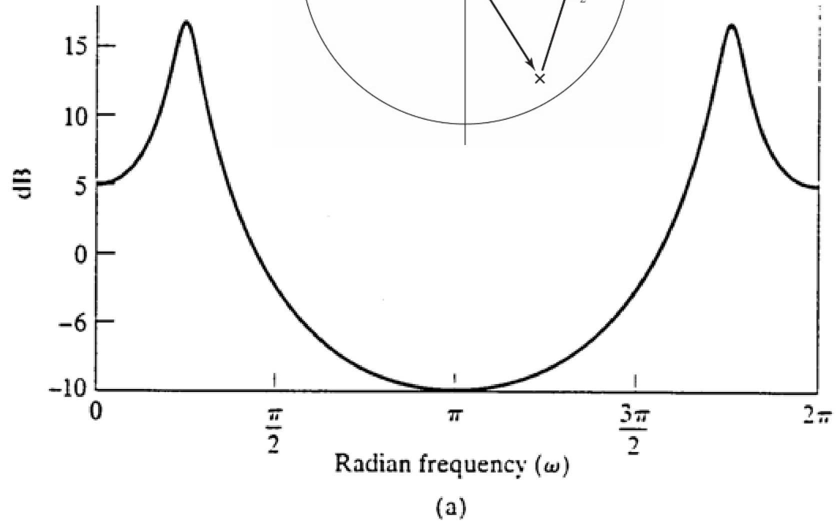
Example 2: 2nd order IIR filter

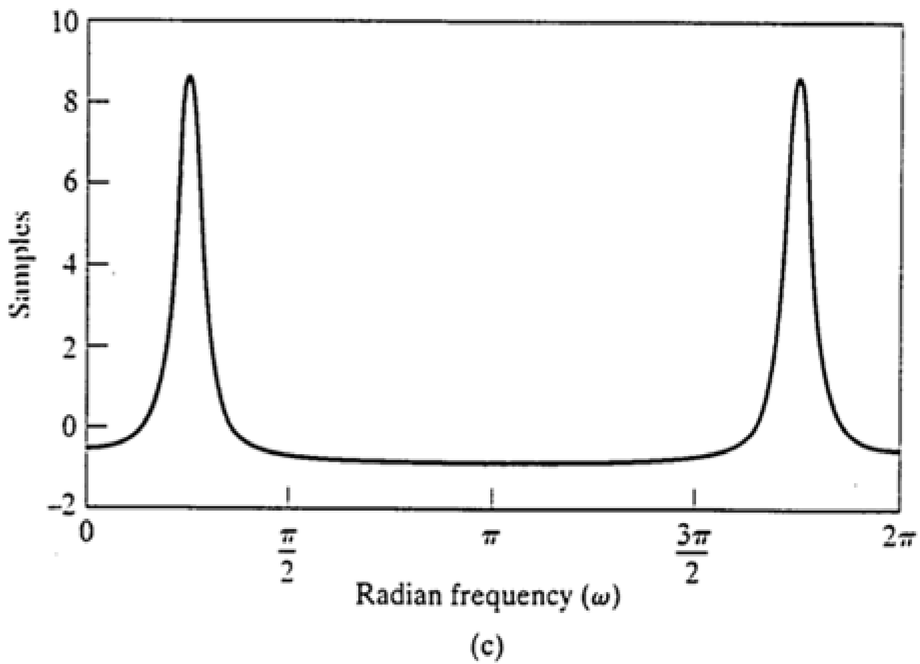


$$H(z) = \frac{1}{(1-re^{j\theta}z^{-1})(1-re^{-j\theta}z^{-1})} = \frac{1}{1-2r\cos\theta\, z^{-1}+r^2 z^{-2}}$$

$$20\log_{10}|H(e^{j\omega})| = -10\log_{10}[1+r^2-2r\cos(\omega-\theta)]$$
$$-10\log_{10}[1+r^2-2r\cos(\omega+\theta)]$$

➔2 poles at $re^{j\theta}$ & $re^{-j\theta}$ and 2 zeros at z=0





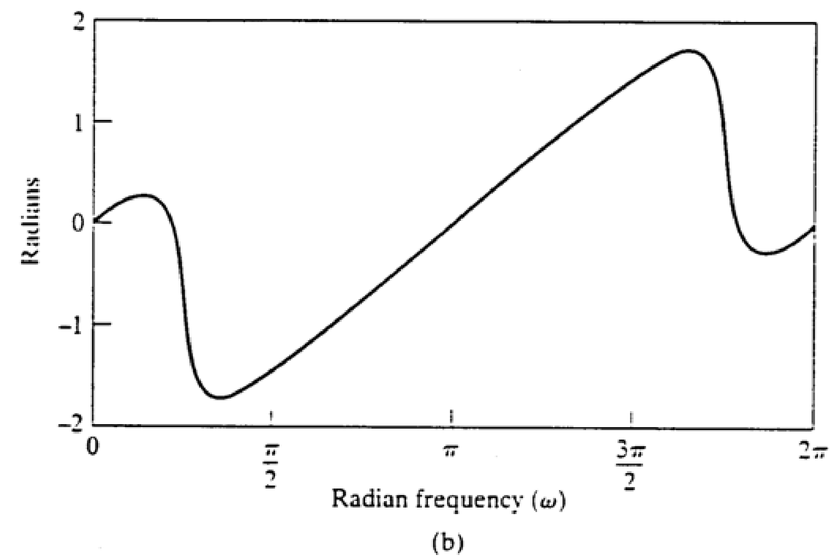


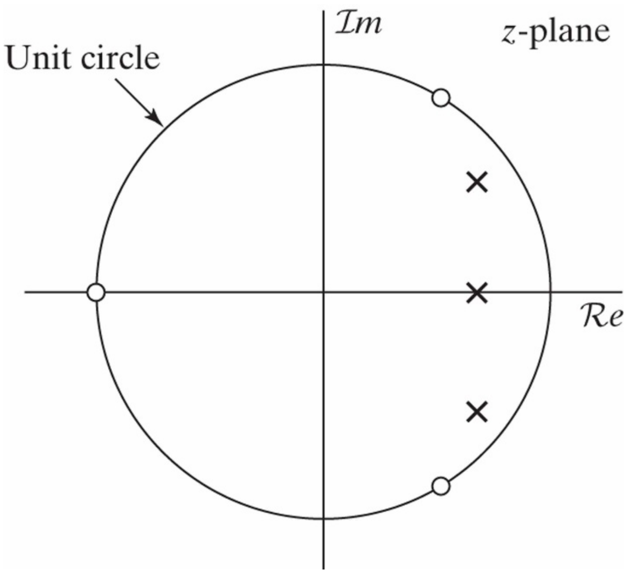
Figure 5.13 Frequency response for a complex-conjugate pair of poles as i Example 5.6, with $r = 0.9$, $\pi/4$. (a) Log magnitude. (b) Phase. (c) Group delay.

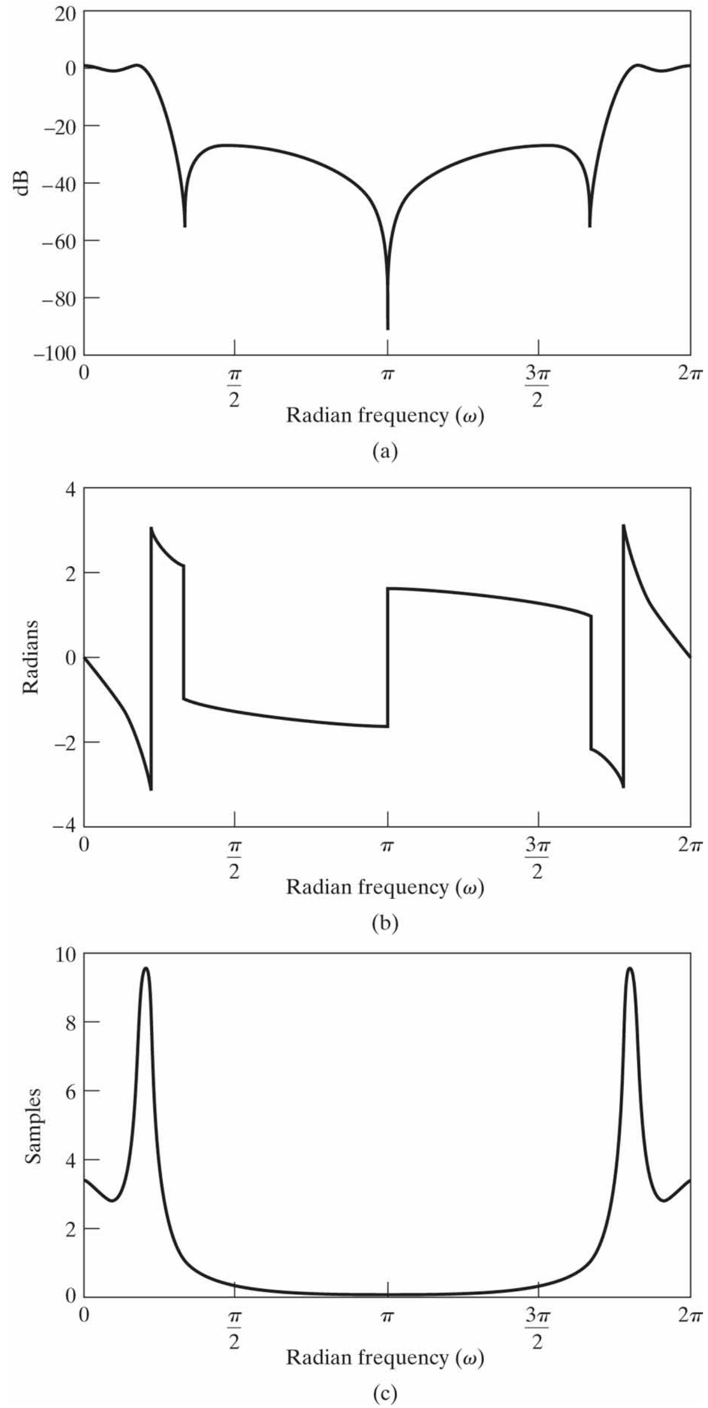
➔Pole at θ dominates H

➔Pole at –θ dominates H

Example 3: 3rd order IIR system

Pole–zero plot for the LPF





Frequency response for the LPF
(a) Log magnitude
(b) Phase
(c) Group delay

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Relation between Magnitude and Phase

- For general LTI system
  - Knowledge about magnitude/phase doesn't provide any information about phase/magnitude
- For LCCDEs, however, there is some constraint between magnitude and phase
  - If magnitude and the number of pole-zeros are known
    - Only a finite number of choices for phase
  - If phase and number of pole-zeros are known
    - Only a finite number of choices for magnitude (ignoring scale)
- A class of systems called minimum-phase
  - Magnitude specifies phase uniquely
  - Phase specifies magnitude uniquely

Relation between Magnitude and Phase

- Stable and causal system:
  ➔ poles inside unit circle, no restriction on zeros

- If zeros are also inside the unit circle
  ➔inverse system is also stable and causal
  (in many situations, we need inverse systems!)

- Minimum-phase systems:
  stable and causal and have stable and causal inverses

Inverse systems

- Many (not all) systems have inverses
  - specially systems with rational system functions

$$G(z) = H(z)H_i(z) = 1$$

$$H(z) = \left(\frac{b_0}{a_0}\right)\frac{\prod_{m=1}^{M}(1 - c_m z^{-1})}{\prod_{k=1}^{N}(1 - d_k z^{-1})}$$

$$H_i(z) = \frac{1}{H(z)}$$

$$g[n] = h[n] * h_i[n] = \delta[n]$$

$$H_i(z) = \left(\frac{a_0}{b_0}\right)\frac{\prod_{k=1}^{N}(1 - d_k z^{-1})}{\prod_{m=1}^{M}(1 - c_m z^{-1})}$$

  - Poles become zeros and vice versa
  - ROC: **must have overlap between** H(z) and Hi(z) for the sake of G(z)

Inverse systems Example

$$H(z)=\frac{1-0.5z^{-1}}{1-0.9z^{-1}},|z|>0.9 \quad H_i(z)=\frac{1-0.9z^{-1}}{1-0.5z^{-1}}$$

$$=\frac{1}{1-0.5z^{-1}}-\frac{0.9z^{-1}}{1-0.5z^{-1}}$$

$\text{If } |z|>0.9, \text{right-sided h[n]}$ ←The only choice for ROC of Hi(z) that overlap with |z|>0.9 is |z|>0.5

$$h_i[n]=(0.5)^n u[n]-0.9(0.5)^{n-1}u[n-1]; \text{stable \& causal}$$

$$\Longrightarrow h[n]*h_i[n]=\delta[n]$$

$\text{If } |z|<0.9, \text{left-sided h[n]}$

$$h_i[n]=\delta[n]+0.8(0.5)^n u[-n]; \text{unstable \& noncausal}$$

$$\text{BUT } h[n]*h_i[n]\,!=\delta[n] \Longrightarrow \text{not inverse}$$

Ex: Inverse for system with zeros in ROC

$$H(z) = \frac{z^{-1} - 0.5}{1 - 0.9z^{-1}} \qquad |z| > 0.9$$

$$H_i(z) = \frac{1 - 0.9z^{-1}}{z^{-1} - 0.5} = \frac{2 - 1.8z^{-1}}{-1 + 2z^{-1}}$$

Two ROCs of $H_i(z)$ (i) $|z| < 2$ both are valid
(ii) $|z| > 2$

$h_{i1}(n) = 2^{n+1}u(-n-1) - 1.8(2)^{n-1}u(-n)$ stable, noncausal

$h_{i2}(n) = -2^{n+1}u(n) + 1.8(2)^{n-1}u(n-1)$ unstable, causal

Questions:

What system has stable inverse system?

" " causal inverse system?

" " both stable and causal system?

Result: A stable & causal system has stable & causal inverse system iff poles and zeros of the system are inside $|z| = 1$

← Minimum phase systems

Square System Function C(z):

useful to construct stable& causal systems

- What possible choices of the system function H(z) from the magnitude-squared $\left|H(e^{j\omega})\right|^2$

$$\left|H(e^{j\omega})\right|^2 = H(e^{j\omega})^* H(e^{j\omega}) = H^*(1/z^*)H(z)\Big|_{z=e^{j\omega}}$$

- Restricting the system to be rational

$$H(z) = \left(\frac{b_0}{a_0}\right)\frac{\prod_{k=1}^{M}\left(1 - c_k z^{-1}\right)}{\prod_{k=1}^{N}\left(1 - d_k z^{-1}\right)} \qquad H^*(1/z^*) = \left(\frac{b_0}{a_0}\right)\frac{\prod_{k=1}^{M}\left(1 - c_k^* z\right)}{\prod_{k=1}^{N}\left(1 - d_k^* z\right)}$$

- The square system function

$$C(z) = H(z)H^*(1/z^*) = \left(\frac{b_0}{a_0}\right)^2 \frac{\prod_{k=1}^{M}\left(1 - c_k z^{-1}\right)\left(1 - c_k^* z\right)}{\prod_{k=1}^{N}\left(1 - d_k z^{-1}\right)\left(1 - d_k^* z\right)}$$

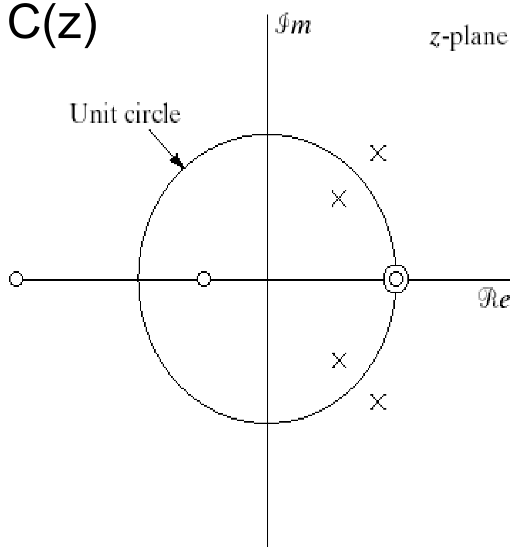
$$d_k^* = 1/d_k$$

Original pole

Newly added

- What information on H(z) can we get from C(z)?

C(z)



Poles and Zeros of Square System Function

$$C(z) = H(z)H^*(1/z^*) = \left(\frac{b_0}{a_0}\right)^2 \frac{\prod_{k=1}^{M}(1 - c_k z^{-1})(1 - c_k^* z)}{\prod_{k=1}^{N}(1 - d_k z^{-1})(1 - d_k^* z)}$$

$$d_k^* = 1/d_k;$$

$$z = 1/d_k^* \text{ or } z = d_k$$

- For every pole d_k in H(z) there is a pole of C(z) at d_k and $(1/d_k)^*$
- For every zero c_k in H(z) there is a zero of C(z) at c_k and $(1/c_k)^*$

➔**Poles and zeros of C(z) occur in conjugate reciprocal pairs**

- C(z): If one of the pole/zero is inside the UC the reciprocal will be outside
  - Unless they are both on the unit circle

- If H(z) is stable all poles have to be inside the UC
  - We can infer which poles of C(z) belong to H(z)
- However, <u>zeros cannot be uniquely determined from C(z)</u>

Example 5.9: Systems with same C(z)

Consider the following stable systems:

$$H_1(z) = \frac{2(1-z^{-1})(1+0.5z^{-1})}{(1-0.8e^{j\pi/4}z^{-1})(1-0.8e^{-j\pi/4}z^{-1})}$$

$$H_2(z) = \frac{(1-z^{-1})(1+2z^{-1})}{(1-0.8e^{j\pi/4}z^{-1})(1-0.8e^{-j\pi/4}z^{-1})}$$

Now

$$C_1(z) = \frac{2(1-z^{-1})(1+0.5z^{-1})2(1-z)(1+0.5z)}{(1-0.8e^{j\pi/4}z^{-1})(1-0.8e^{-j\pi/4}z^{-1})(1-0.8e^{-j\pi/4}z)(1-0.8e^{j\pi/4}z)}$$

$$C_2(z) = \frac{(1-z^{-1})(1+2z^{-1})(1-z)(1+2z)}{(1-0.8e^{j\pi/4}z^{-1})(1-0.8e^{-j\pi/4}z^{-1})(1-0.8e^{-j\pi/4}z)(1-0.8e^{j\pi/4}z)}$$

By using the fact that

$$4(1+0.5z^{-1})(1+0.5z) = (1+2z^{-1})(1+2z)$$

We see that $C_1(z) = C_2(z)$.

- Hence, $H_1(z)$ and $H_2(z)$ have the same $C(z)$ (and thus the same magnitude response).
- The poles of $H_1(z)$ and $H_2(z)$ are the same, but the zeros are different.
- Although the magnitude response are the same, the phase responses of these two systems are different.

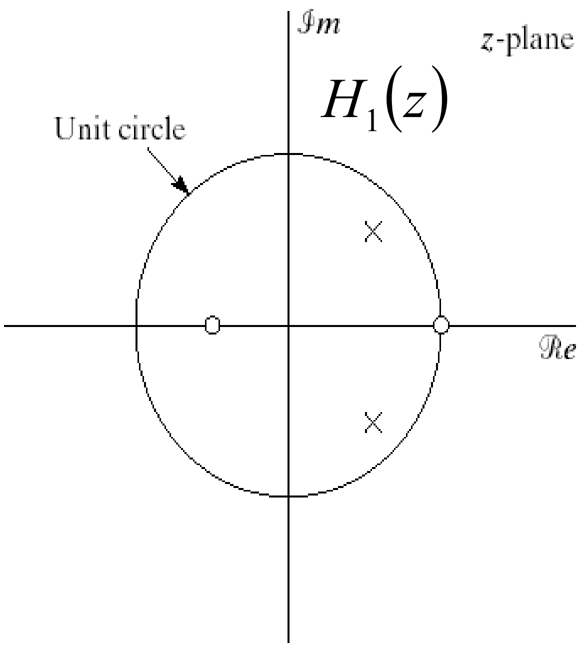
Example 5.9: Systems with same C(z)

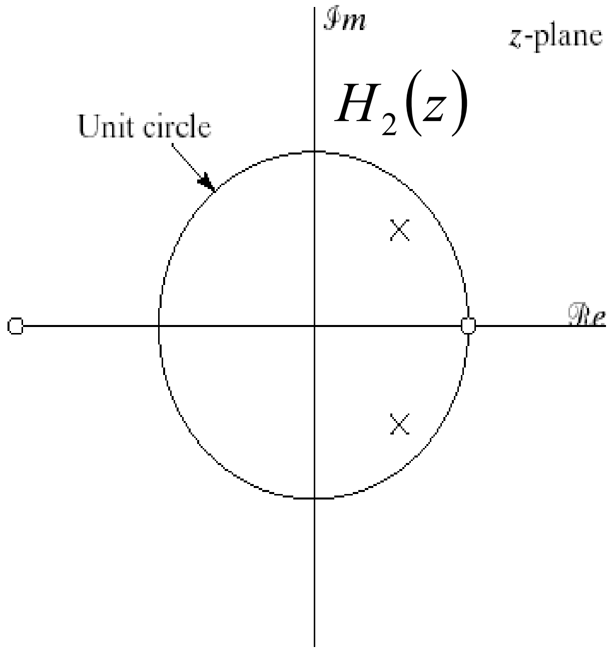
$$H_1(z) = \frac{2(1-z^{-1})(1+0.5z^{-1})}{(1-0.8e^{j\pi/4}z^{-1})(1-0.8e^{-j\pi/4}z^{-1})}$$

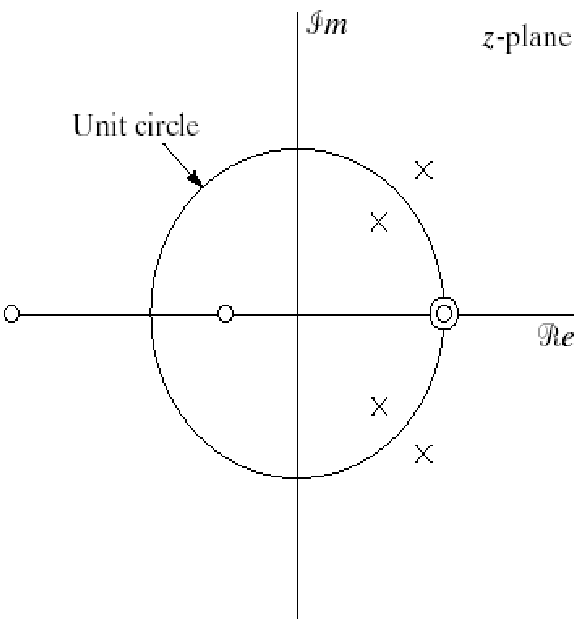
$$H_2(z) = \frac{(1-z^{-1})(1+2z^{-1})}{(1-0.8e^{j\pi/4}z^{-1})(1-0.8e^{-j\pi/4}z^{-1})}$$

- Both differ in zeros
- But both share the same square system function C(z)

$$C(z) = H_1(z)H_1^*(1/z^*)$$
$$= H_2(z)H_2^*(1/z^*)$$







If we are given either of the stable systems H_1 or H_2, we can change the zero position and still have the same magnitude

Poles, zeros and coefficients in H(z)

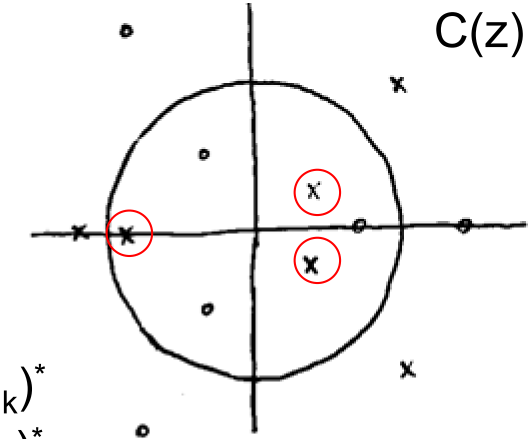
$$H(z)=\frac{Y(z)}{X(z)}=\frac{\sum_{m=0}^{M} b_m z^{-m}}{\sum_{k=0}^{N} a_k z^{-k}}=(\frac{b_0}{a_0})\frac{\prod_{m=1}^{M}(1-c_m z^{-1})}{\prod_{k=1}^{N}(1-d_k z^{-1})}$$

- The complex conjugate root theorem states that if *P* is a polynomial in one variable with real coefficients, and *a* + j*b* is a root of *P* with *a* and *b* real numbers, then its complex conjugate *a* − j*b* is also a root of *P*
  - If the degree of a real-coefficient polynomial is odd, it must have at least one real root

- With H(z) = N(z) / D(z) ratio of two polynomials
  - If the coefficients of polynomials *N*(*z*) and *D*(*z*) in *H(z)* are real, the poles and zeros must be either purely real, or appear in complex conjugate pairs
  - The existence of a single complex pole (or zero) without a corresponding conjugate pole (or zero) would generate complex coefficients in the polynomial *D*(*z*) (or N(z)

Example 5.10: Given C(z); find H(z)

$$H(z)=\frac{Y(z)}{X(z)}=\frac{\sum_{m=0}^{M} b_m z^{-m}}{\sum_{k=0}^{N} a_k z^{-k}}=\left(\frac{b_0}{a_0}\right)\frac{\prod_{m=1}^{M}(1-c_m z^{-1})}{\prod_{k=1}^{N}(1-d_k z^{-1})}$$



- For every pole d_k in H(z) there is a pole of C(z) at d_k and $(1/d_k)^*$
- For every zero c_k in H(z) there is a zero of C(z) at c_k and $(1/c_k)^*$
- If we assume H(z) is stable & causal, poles must be inside the unit circle (zeros can be anywhere) ➔ H(z) has 3 poles
- If we assume ak and bk are real, h[n] has real values, and so the frequency response is an even function and the zeros (and poles) either are real or occur in complex conjugate pairs (i.e., symmetric to the real axis) ➔ H(z) has 3 zeros

➔ There are 4 different stable and causal systems with 3 poles and 3 zeros

- ✓ 3 poles inside UC + 3 zeros inside UC
- ✓ 3 poles inside UC + 2 zeros inside & 1 zero outside UC
- ✓ 3 poles inside UC + 2 zeros outside UC & 1 zero inside UC
- ✓ 3 poles inside UC + 2 zeros outside UC & 1 zero outside
  - These systems have the same magnitude response

Q: how many systems with stable & causal inverse?

- Hence, knowing only the magnitude information gives us finite selections of H(z), and so the phase response is restricted to these finite selections

(If we assume ak and bk are NOT real, the number of choices of stable and causal systems would be greater)

How to construct a stable, causal system?

How many stable, causal systems can be built?

Conclusion:

① Poles and zeros dictate magnitude and phase.

② If number of poles and zeros are known, there are different ways to arrange zeros to change the phase while keeping magnitude the same (choice limite

③ Are there different choices for pole/zero locations such that magnitude can be changed while phase remains the same?

④ Magnitude and phase are associated and in general cannot be determined independently.

Outline

- Introduction
- Magnitude and Phase Representation of signals
- Amplitude-Phase Representation of signals
- System analysis
    - System function
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    - All-pass systems
    - Minimum-phase systems
    - Linear systems with generalized linear phase

All-Pass System

- All-pass system is a stable system with <u>a constant</u> magnitude response

➔The system passes all of the frequency components of its input with constant gain

➔The magnitude response of a system remains the same by multiplying (cascading) this system with an all-pass system

$$Any\ H(z), |H(z).H_{ap}(z)| = |H(z)|$$

- 1st-order all-pass system

  ➔Pole: z=a & zero: z=1/a*

$$H_{ap}(z) = \frac{z^{-1} - a^*}{1 - az^{-1}}$$

➔Magnitude response: $|H_{ap}(e^{j\omega})| = 1$

$$H_{ap}(e^{j\omega}) = \frac{e^{-j\omega} - a^*}{1 - ae^{-j\omega}} = e^{-j\omega}\frac{1 - a^* e^{j\omega}}{1 - ae^{-j\omega}}$$

- Num. & denom are complex conjugate of each other ➔ have same magnitude
- And $|e^{-jw}| = 1$

Phase of 1st order All-Pass Systems

$$H_{ap}\left(e^{j\omega}\right)=\frac{e^{-j\omega}-a^{*}}{1-ae^{-j\omega}}=e^{-j\omega}\frac{1-a^{*}e^{j\omega}}{1-ae^{-j\omega}}$$

- The phase

$$a=re^{-j\theta};\angle\left[\frac{e^{-j\omega}-re^{-j\theta}}{1-re^{j\theta}e^{-j\omega}}\right]=-\omega-2\arctan\left[\frac{r\sin\left(\omega-\theta\right)}{1-r\cos\left(\omega-\theta\right)}\right]$$

- For stable and causal system, |r|<1

- For causal all-pass systems:
  Phase between 0 and π is always non-positive

$$\arg\left[H_{ap}\left(e^{j\omega}\right)\right]\leq 0$$

$$\text{for}\quad 0\leq\omega<\pi$$

- The group delay

$$\mathrm{grd}\left[\frac{e^{-j\omega}-re^{-j\theta}}{1-re^{j\theta}e^{-j\omega}}\right]=\frac{1-r^{2}}{1-2r\cos\left(\omega-\theta\right)+r^{2}}=\frac{1-r^{2}}{\left|1-re^{j\theta}e^{-j\omega}\right|^{2}}$$

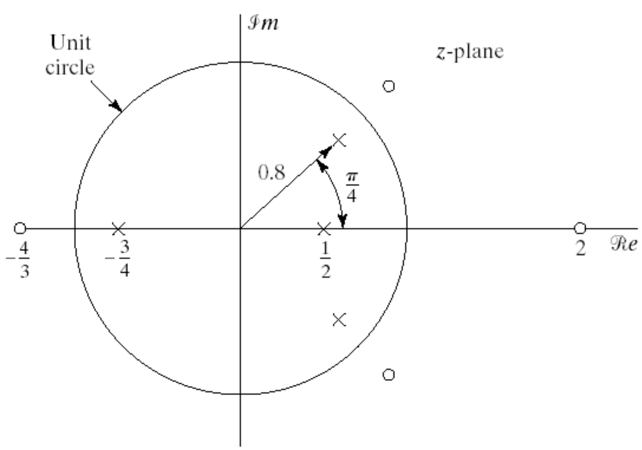
  - Group delay of all-pass stable causal systems is always positive

All-Pass System

- Most general form with real impulse response h[n]: M=N=2Mc+Mr

$$H_{ap}(z) = A\prod_{k=1}^{M_r}\frac{z^{-1}-d_k}{1-d_k z^{-1}}\prod_{k=1}^{M_c}\frac{(z^{-1}-c_k^*)(z^{-1}-c_k)}{(1-c_k z^{-1})(1-c_k^* z^{-1})}$$

- M=N Poles & Zeros
- A: positive constant; d_k: real poles; c_k: complex poles
- For Causal & Stable all-pass: $|d_k| < 1 \;\&\; |c_k| < 1$
- Each pole of $H_{ap}(z)$ is paired with conjugate reciprocal zero
- The poles are all inside the UC, and so the zeros are outside the UC



Typical P&Z plot for an all-pass system

Example 5.13: First-order all-pass system

Pole: z=0.9: θ=0, r=0.9
Pole: z=-0.9: θ=pi, r=0.9
➔Pole at θ dominates H

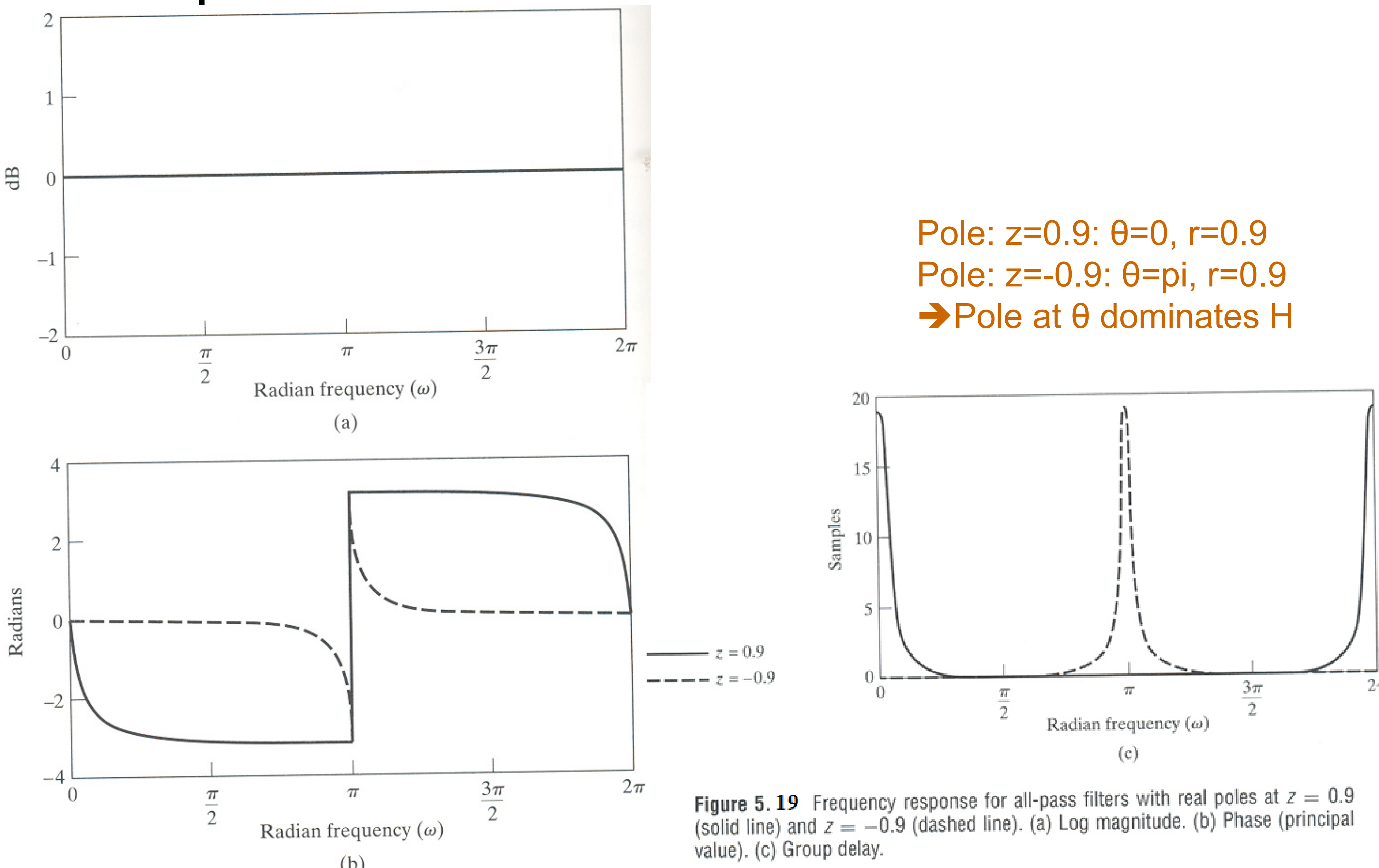


Figure 5.19 Frequency response for all-pass filters with real poles at $z = 0.9$ (solid line) and $z = -0.9$ (dashed line). (a) Log magnitude. (b) Phase (principal value). (c) Group delay.

To avoid the jumps in arg[.], differentiate to get grd[.]

Example 5.13: Second-order all-pass system

➔Pole at θ dominates H

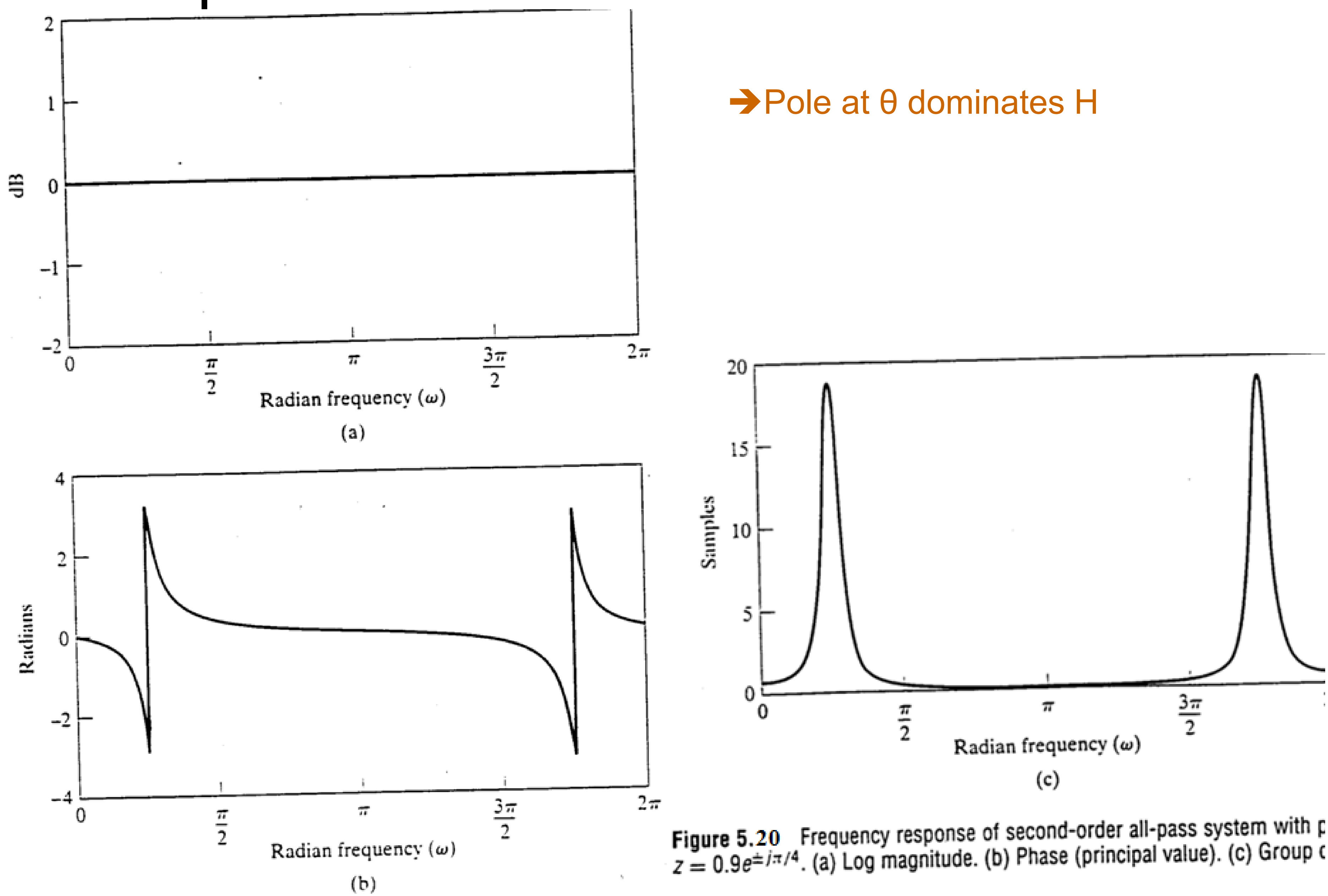


Figure 5.20 Frequency response of second-order all-pass system with p
$z = 0.9e^{\pm j\pi/4}$. (a) Log magnitude. (b) Phase (principal value). (c) Group c

1st order All-Pass System: Summary

$$H(z) = \frac{z^{-1} - a^*}{1 - a z^{-1}}$$ pole: $z = a$, zero: $1/a^*$

$$C(z)\Big|_{z=e^{jw}} = H(z) H^*(1/z^*) = \frac{z^{-1} - a^*}{1 - a z^{-1}} \frac{z - a}{1 - a^* z} = 1$$

$$H_{ap}(e^{jw}) = e^{-jw} \frac{1 - a^* e^{jw}}{1 - a e^{-jw}}$$

$$\angle H_{ap}(e^{jw}) = -w - 2\arctan\left[\frac{r\sin(w-\theta)}{1 - r\cos(w-\theta)}\right]$$ $a = re^{j\theta}$

$$\tau(w) = grd\{\arg[H_{ap}(e^{jw})]\} = \frac{1 - r^2}{|1 - re^{j\theta}e^{-jw}|^2}$$

if $r < 1$, $\tau(w) > 0$ always positive $\tau(w)$

Higher-order All-Pass System: Summary

- High-order all-pass systems with real coefficients

$$H_{ap}(z) = A\prod_{k=1}^{M_r}\frac{z^{-1}-d_k}{1-d_k z^{-1}}\prod_{k=1}^{M_c}\frac{(z^{-1}-c_k^*)(z^{-1}-c_k)}{(1-c_k z^{-1})(1-c_k^* z^{-1})}$$

- Group delay always positive

- All-pass system can be used to adjust system's phase

  For any $H(z)$, $H(z)H_{ap}(z)$ has the same amplitude as $H(z)$

Applications of all-pass systems

$$Any\ H(z), |H(z).H_{ap}(z)| = |H(z)|$$

- Compensating for phase distortion or group delay (Ch7)
  - All pass systems can be used as compensator for phase (or group-delay) distortion
  - When the phase response of an IIR filter is not satisfied in the specified band (e.g., constant group delay), we can cascade it with an all-pass filter to adjust the phase response.

- Transforming a frequency-selective low-pass into other frequency-selective forms (Ch7)

- Obtaining variable-cutoff frequency-selective filters (Ch7)

- Theory of minimum-phase systems (Ch5.6)
  - Stable and causal systems that have stable and causal inverses system

  Any $H(z)=H_{min}(z)\ H_{ap}(z)$

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Minimum-phase systems

- Given
  - the (squared) magnitude response C(z)
  - the corresponding M poles and N zeros for a rational stable and causal H(z)
  - the inverse 1/H(z) is stable and causal

  ➔ then we can uniquely determine the zeros of H(z)
  - Such H(z) is called minimum-phase system

- So if we are given C(z), and know that H(z) is minimum-phase, then H(z) is uniquely determined and will consist of all the poles and zeros of C(z) that lie inside the UC

- Why it is called a minimum-phase system? The group delay of the minimum phase system chosen from C(z) is always less than the group delays of the other systems chosen from C(z)

Minimum-phase and all-pass decomposition

- Minimal-phase system is often followed in filter design when only the magnitude response is specified by the design method used
- Any rational system function can be expressed as: $H(z) = H_{\min}(z) H_{ap}(z)$
- Example:
  - Suppose H(z) has one zero outside the UC at $z = 1/c^*, |c| < 1$
  - Let H1(z) be minimum-phase
  - Then $H(z) = H_1(z)(z^{-1} - c^*)$

$$= H_1(z)(1 - cz^{-1}) \frac{z^{-1} - c^*}{1 - cz^{-1}}$$

Since |c|<1,
also is minimum phase

All-pass

- Generalization: include more zeros outside the UC, and any rational H(z) can be expressed by $H(z) = H_{\min}(z) H_{ap}(z)$

Minimum-phase and all-pass decomposition

$$H(z) = H_{\min}(z) H_{ap}(z)$$

- The minimum-phase component
  contains the poles and zeros of H(z) that lie inside the UC, plus zeros that are conjugate reciprocals of the zeros of H(z) that lie outside the UC $z = 1/c^*, |c| < 1$

- The all-pass component
  comprises all the zeros of H(z) that lie outside the UC, together with poles to cancel the reflected conjugate reciprocal zeros in Hmin(z)

Example: Minimum-Phase/All-pass decomposition

- Consider the following system $H_1(z) = \dfrac{1 + 3z^{-1}}{1 + \frac{1}{2}z^{-1}}$
  - One pole inside the UC at z=−0.5
  - One zero outside the UC at z= −3
- Add an all-pass system to reflect this zero inside the UC

$$H(z) = H_1(z)(z^{-1} - c^*)$$

$$= H_1(z)(1 - cz^{-1})\frac{z^{-1} - c^*}{1 - cz^{-1}}$$

➔ Choose the appropriate all-pass system to reflect this zero inside the UC, i.e., choose c= −(1/3)

$$H_1(z) = \frac{1 + 3z^{-1}}{1 + \frac{1}{2}z^{-1}} = 3\frac{1}{1 + \frac{1}{2}z^{-1}}\left(z^{-1} + \frac{1}{3}\right) = 3\frac{1}{1 + \frac{1}{2}z^{-1}}\left(z^{-1} + \frac{1}{3}\right)\frac{1 - \frac{1}{3}z^{-1}}{1 - \frac{1}{3}z^{-1}}$$

$$H_1(z) = \left(3\frac{1 - \frac{1}{3}z^{-1}}{1 + \frac{1}{2}z^{-1}}\right)\left(\frac{z^{-1} + \frac{1}{3}}{1 - \frac{1}{3}z^{-1}}\right) = H_{min}(z)H_{ap}(z)$$

Properties of minimum-phase systems:

From minimum-phase and all-pass decomposition

$$H(z) = H_{\min}(z) H_{ap}(z)$$

$$\arg[H(e^{j\omega})] = \arg[H_{\min}(e^{j\omega})] + \arg[H_{ap}(e^{j\omega})]$$

1. The Minimum-Phase-Lag Property

- The continuous phase curve of an all-pass system is negative for $0 \le \omega \le \pi$

➔ Change from minimum-phase to non-minimum-phase (+all-pass phase)

- Always decreases the continuous phase or
- Always increases the negative of the phase (called the phase-lag function)
- Minimum-phase is more precisely called minimum phase-lag system

2. The Minimum-Group-Delay Property grd(H) = grd(Hmin) + grd(Hap)

- Group delay of minimum-phase system is less than non-minimum-phase systems

3. Minimum-Energy-Delay Property

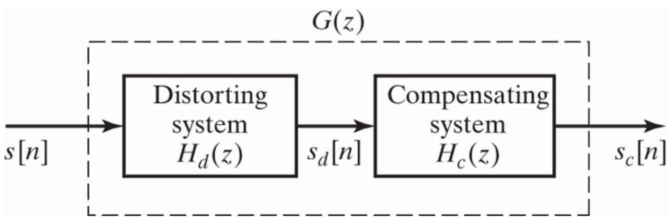
➔The energy of Min. Phase systems is delayed the least of all systems having same magnitude

$$\sum_{m=0}^{n} |h[n]|^2 <= \sum_{m=0}^{n} |h_{\min}[n]|^2$$

Application: Frequency-response Compensation

Distortion compensation by linear filtering

- Assume a signal has been distorted by an LTI system
  - Example: transmitting signals over a communication channel
- Design a system to compensate for this distortion



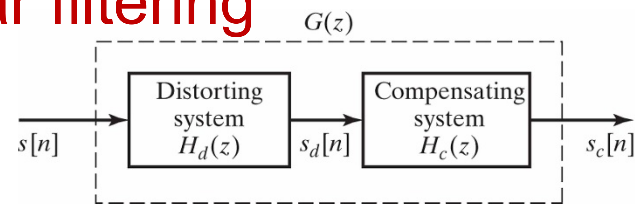
- Perfect compensation: $s_c[n] = s[n]$, i.e., $H_c(z)$ is the inverse of $H_d(z)$

- If we assume that $H_d(z)$ is stable and causal and require $H_c(z)$ to be stable and causal too, then perfect compensation is possible only if $H_d(z)$ is a minimum-phase system

- Solution: $H_d(z) = H_{d\min}(z)H_{ap}(z)$

Application: Frequency-response Compensation

Distortion compensation by linear filtering

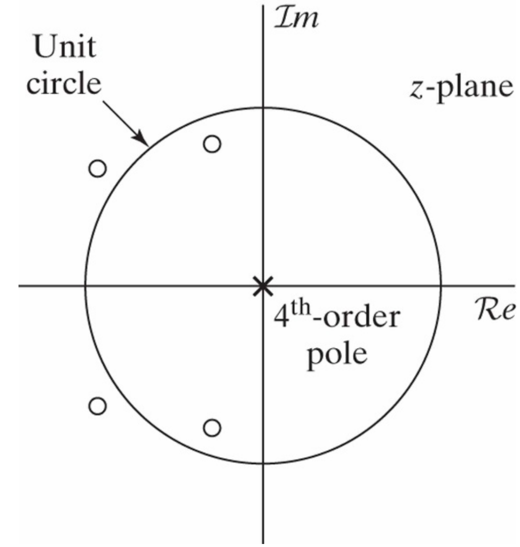


- When the distortion system is not minimum-phase system, use minimum-phase/ all-pass decomposition, i.e., invert minimum phase part
- Decompose $H_d(z)$ into $H_d(z) = H_{d,min}(z) H_{d,ap}(z)$
- Define compensating system as $H_c(z) = \dfrac{1}{H_{d,min}(z)}$
- Cascade of the distorting system and compensating system

$$G(z) = H_c(z) H_d(z) = H_{d,min}(z) H_{d,ap}(z) \frac{1}{H_{d,min}(z)} = H_{d,ap}(z)$$

- By doing so, the overall system is of constant magnitude response, while the phase response is modified to that of all-pass
- Other all-pass filters can be additionally cascaded to transform the phase response to approximate it to the desired one in the specified band

Example 5.13: Compensation of an FIR system



- Consider a distorting system function (FIR)

$$H_d(z) = (1-0.9e^{j0.6\pi}z^{-1})(1-0.9e^{-j0.6\pi}z^{-1})$$
$$\times(1-1.25e^{j0.8\pi}z^{-1})(1-1.25e^{-j0.8\pi}z^{-1})$$

- $H_d(z) = H_{d,min}(z)H_{d,ap}(z)$

The corresponding minimum-phase system is obtained by reflecting the zeros that occur at z=1.25 e(±j0.8π) to their conjugate reciprocal locations inside the UC ➔

$$H_{\min}(z) = (1.25)^2(1-0.9e^{j0.6\pi}z^{-1})(1-0.9e^{-j0.6\pi}z^{-1})$$
$$\times(1-0.8e^{-j0.8\pi}z^{-1})(1-0.8e^{j0.8\pi}z^{-1})$$

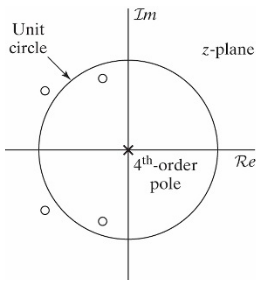
- And the all-pass system that relates $H_{min}(z)$ and $H_d(z)$ is

$$H_{\cancel{\min}\ \mathrm{ap}}(z) = \frac{(z^{-1}-0.8e^{-j0.8\pi})(z^{-1}-0.8e^{j0.8\pi})}{(1-0.8e^{j0.8\pi}z^{-1})(1-0.8e^{-j0.8\pi}z^{-1})}$$

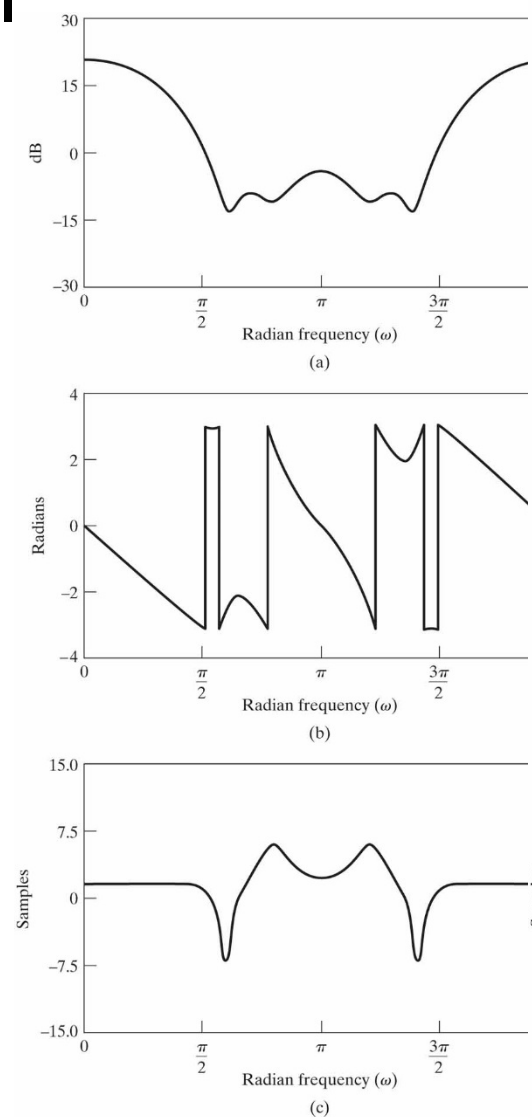
Example 5.13: Compensation of an FIR system

Pole–zero plot of a FIR system:

1. Stable & causal
2. Non-minimum phase

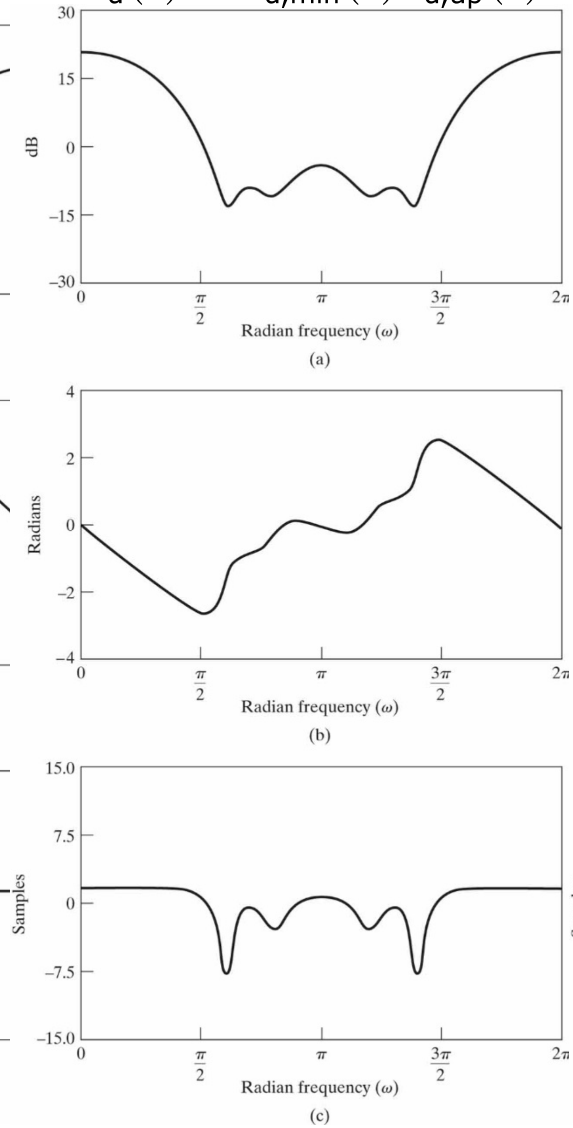


Frequency response for the FIR system (a) Log magnitude. (b) Phase (principal value). (c) Group delay.

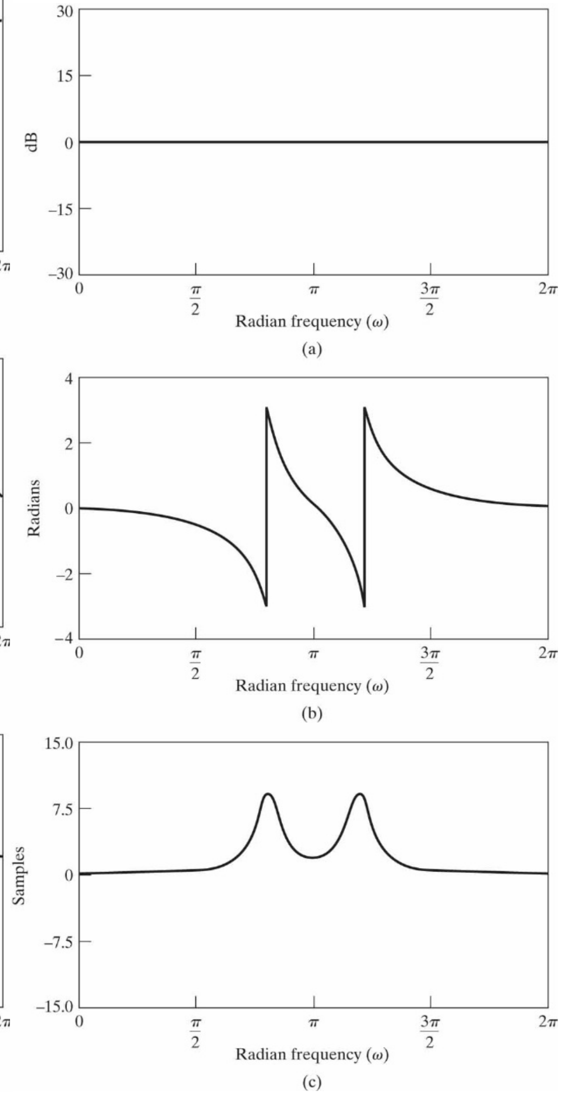


Frequency response for minimum-phase system: reflect the zeros at r=1.25 & 0.8pi inside UC

$$H_d(z) = H_{d,min}(z) H_{d,ap}(z)$$

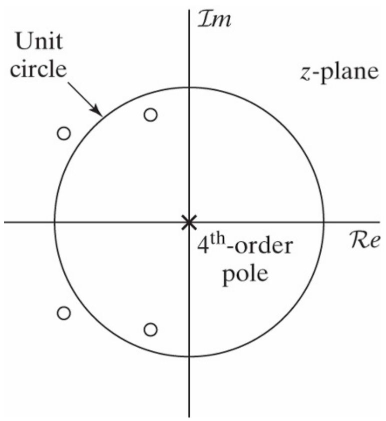


Frequency response of all-pass system: The sum of corresponding curves in FIR and MIN equals the corresponding curve in AP

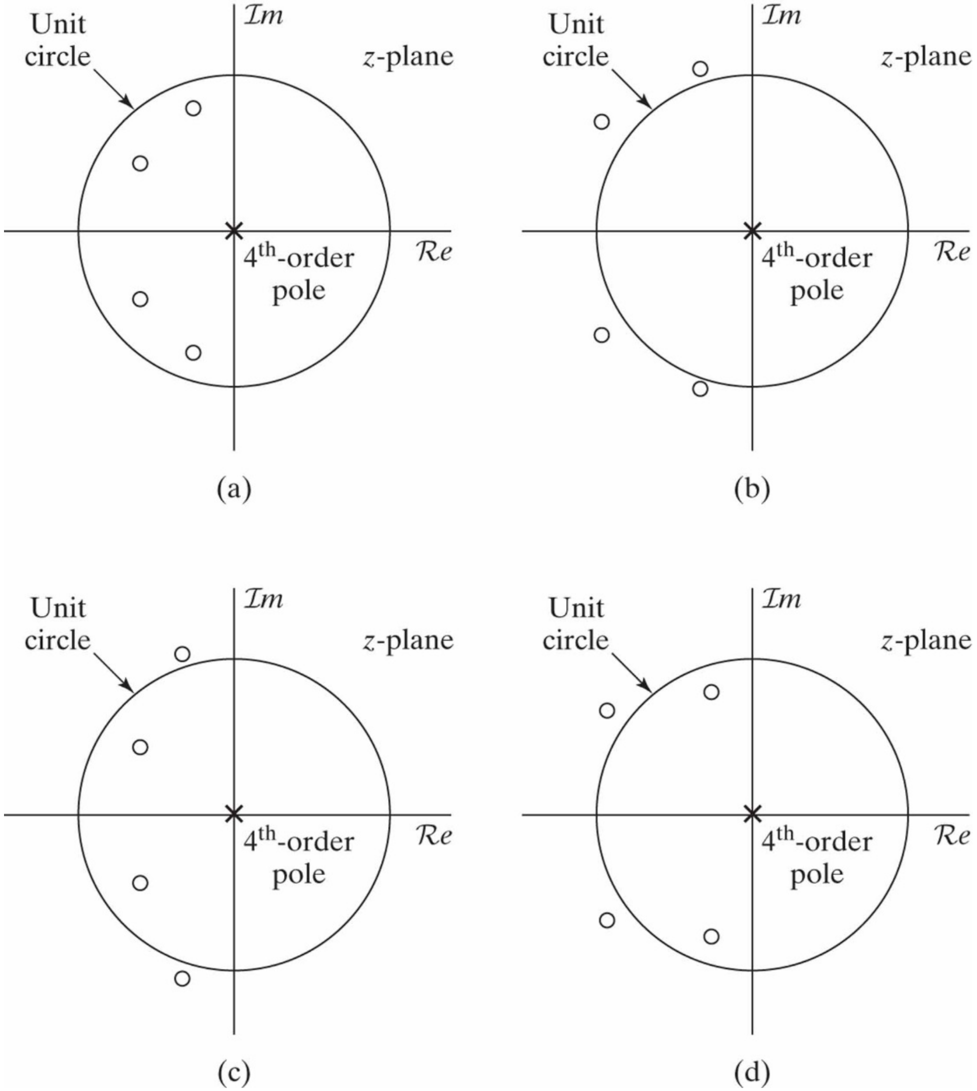


Example 5.13: Compensation of an FIR system

- In the above example



there are a total of four causal FIR systems with real impulse responses that have the same magnitude response



Which one is the minimum-phase?

Zeros are at all combinations of the complex conjugate zero pairs $0.9e^{\pm j0.6\pi}$ and $0.8e^{\pm j0.8\pi}$ and their reciprocals

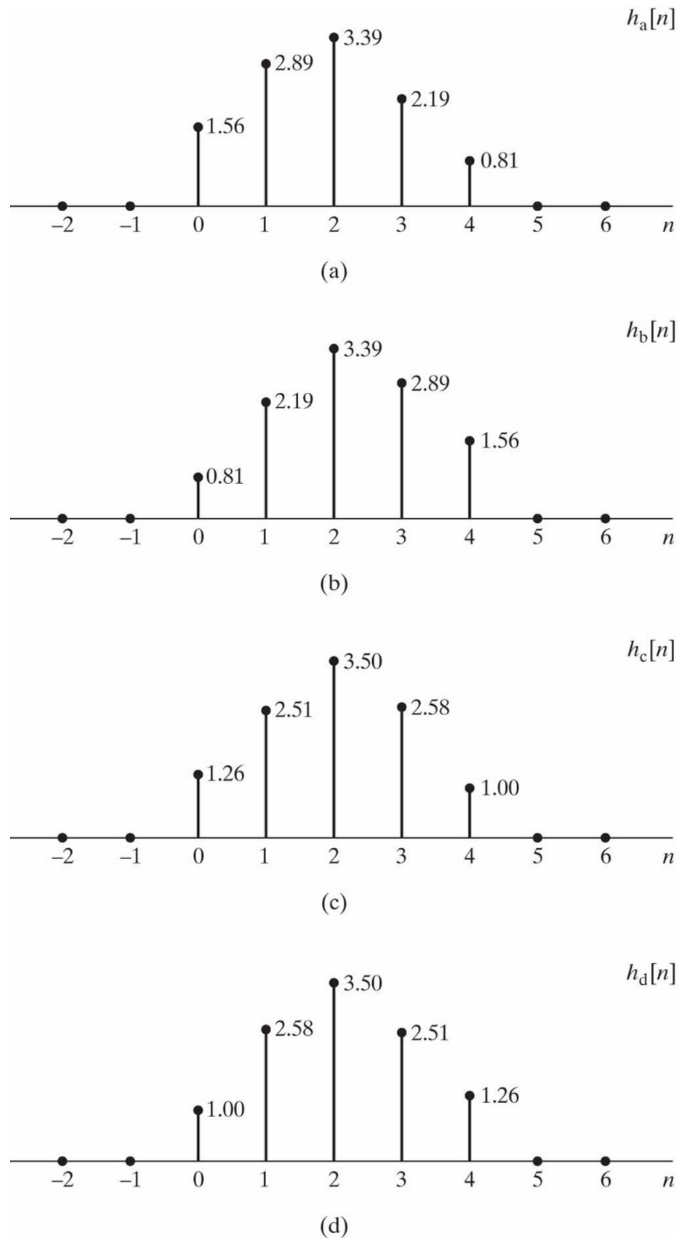
Example 5.13: Compensation of an FIR system

- The four impulse responses of these four FIR systems
- We note $| h[0] | \leq | h_{min}[0] |$

- In general, for any causal stable h[n] impulse response $| h[0] | \leq | h_{min}[0] |$

➔In general, the partial energy of the minimum-phase system is most concentrated around n=0

This means the energy of the minimum-phase system is delayed the least of all systems having the same magnitude response



Summary of all-pass and minimum-phase systems

- Given the magnitude response of a rational system function, the poles can be exactly identified if the system is stable and causal
  - If the number of zeros, M, is fixed, then there are finite choices of the zeros

- If we further restrict the system to be a minimum-phase system, then the zeros are uniquely specified

- We can adjust the phase response of a system by cascading it with an all-pass system

- Any rational system can be decomposed into a minimum-phase system and an all-pass system

Outline

- Introduction
- Magnitude and Phase Representation of signals
- Amplitude-Phase Representation of signals
- System analysis
    - IIR and FIR Systems
    - Group Delay of LTI Systems
    - Frequency Response of Rational Systems
    - Relation between Magnitude/Phase & poles/zeros
    - All-pass systems
    - Minimum-phase systems
    - Linear Phase & Generalized linear-phase systems

Group delay vs. Phase shift?

- Group delay: $\tau = grd[H(e^{j\omega})] = -\frac{d}{d\omega}\{\arg[H(e^{j\omega})]\}$
  - The frequency components of a signal are often delayed when passed through a system (such as an amplifier or a loudspeaker)
  - This signal delay will be different for the various frequencies (non-constant delay)
  - The different delays means that signals consisting of multiple frequency components will suffer distortion because these components are not delayed by the same amount of time at the output of the system
    - This changes the shape of the signal in addition to any constant delay or scale change
- Time delay (phase shift): delay of an input signal *y[n] = x[n-m]*
  - Example: $x[n]=\cos(w_0 n)$; $y[n]=\cos(w_0(n+m))=\cos(w_0 n+\mathbf{w_0 m})$ ← phase shift

- A linear phase system (i.e., with constant group delay): $H(e^{j\omega}) = |H(e^{j\omega})|e^{-j\omega\alpha}$
  time delay and group delay are equal

Linear Phase
Generalized linear-phase systems

- Sometime we desire a system with <u>constant magnitude response</u> ➔thus phase response is not zero ➔we accept phase distortion
- Linear phase filters $H(e^{j\omega}) = |H(e^{j\omega})| e^{-j\omega\alpha}$
- Generalized linear phase filters

$$H(e^{j\omega}) = A(e^{j\omega}) e^{-j\omega\alpha + j\beta}$$

$A(e^{j\omega})$ is a real function of ω,

α and β are real constants

- A linear phase filter has <u>constant group delay</u>
  - All frequency components of a signal have equal delay times
  - There is no distortion due to selected frequencies
- Systems with constant group delay are desirable when we want to *minimize* the distortion on the shape of a signal
- <u>A linear phase shift</u> corresponds to simple delay through the time-shifting property of the FT

$$H(e^{jw}) = e^{-jw\alpha}, \alpha = n_d \text{ integer}$$

Non-linear phase systems

- Non-linear phase will change the shape of the input signal even if the magnitude response is constant

- *Non-linear* phase shift delays some frequency components more than others

- A filter with ***non-linear*** *phase* has a group delay that varies with frequency, resulting in considerable phase distortion

- Many real system functions have a *non-linear* phase shift

Examples: Linear phase systems

- Ex 1: ideal delay $H_{id}(e^{j\omega}) = e^{-j\omega\alpha}, |\omega| < \pi; \alpha \text{ real}$

$$|H_{id}(e^{j\omega})| = 1$$

$$\angle H_{id}(e^{j\omega}) = -\omega\alpha$$ ← linear phase

$$grd[H_{id}(e^{j\omega})] = \alpha$$ ← Constant group delay

$$==> h_{id}[n] = \frac{\sin \pi(n-\alpha)}{\pi(n-\alpha)}$$

- Ex 2: Time-shift:

$$\text{when } \alpha = n_d \text{ with } n_d \text{ integer} ==> h_{id}[n] = \delta[n-n_d]$$

-
- Ex 3: Ideal LPF with linear phase $h_{lp}[n] = \dfrac{\sin \omega_c(n-n_d)}{\pi(n-n_d)}$

Example: Sound reproduction

- Components (such as loudspeakers or filters) of an audio system introduce group delay in the input audio signal

➔ Important to know: the threshold of audibility of group delay with respect to frequency

- For example:
  - 1 kHz ←→ 2 ms
  - 4 kHz ←→1.5 ms

Integer delay

- Ideal Delay System

$$H_{id}\left(e^{j\omega}\right) = e^{-j\omega\alpha} \qquad |\omega| < \pi$$

- Magnitude, phase, and group delay

$$\left|H_{id}\left(e^{j\omega}\right)\right| = 1$$
$$\angle H_{id}\left(e^{j\omega}\right) = -\omega\alpha$$
$$\mathrm{grd}\left[H_{id}\left(e^{j\omega}\right)\right] = \alpha$$

- Impulse response

$$h_{id}[n] = \frac{\sin(\pi(n-\alpha))}{\pi(n-\alpha)}$$

- If **α=n_d is integer**

$$h_{id}[n] = \delta[n - n_d]$$

- For integer α linear phase system delays the input

$$y[n] = x[n] * h_{id}[n] = x[n] * \delta[n-n_d] = x[n-n_d]$$

Symmetry of Linear Phase Impulse Responses

- h1[n] is symmetric about zero: h1[n]= h1[−n]
  - H1(e^{jw}) is real
  - Phase is zero in the amplitude/phase representation

  ➔The group delay is zero, and h1[n] is linear phase

- h2[n] is symmetric about an integer n_0
  - h1[n] delayed by n_0 : h2[n] = h1[n- n_0]
  - H2(e^{jw})= e^{jwn_0} H1(e^{jw})

  ➔ h2[n] is linear phase too

Symmetry of Linear Phase Impulse Responses

- Ideal delay filter:

$$h_{id}[n] = \frac{\sin \pi (n-\alpha)}{\pi (n-\alpha)}$$

  $h_3[n] = h_3[2\alpha - n]$, where 2α is an integer
  - Note: $h_3[n] = h_3[-(n - 2\alpha)]$
  - In amplitude/phase representation $H_3(e^{jw}) = A(e^{jw})\, e^{-jw\alpha}$

  where $A(e^{j\omega})$ is real (possibly bipolar)

  ➔ The phase response of $H_3(e^{jw})$ is linear

  ➔ The group delay is constant

  (see Figure 5.32 of the textbook)
- If α real: low-pass is meaningless

Non-integer delay

- For **non-integer α** , the output is an interpolation of samples
- Easiest way of representing is to think of it in continuous

$$h_c(t) = \delta(t - \alpha T) \qquad \text{and} \qquad H_c(j\Omega) = e^{-j\Omega\alpha T}$$

- This representation can be used even if x[n] was not originally derived from a continuous-time signal

- The output of the system is $\quad y[n] = x(nT - \alpha T)$
  - Samples of a time-shifted, band-limited interpolation of the input x[n]

- A linear phase system $\quad H(e^{j\omega}) = |H(e^{j\omega})| e^{-j\omega\alpha}$

- A zero-phase system is special case of linear-phase
  - It impulse response is real & even

    ➔ real & even & positive FT (in its passband)

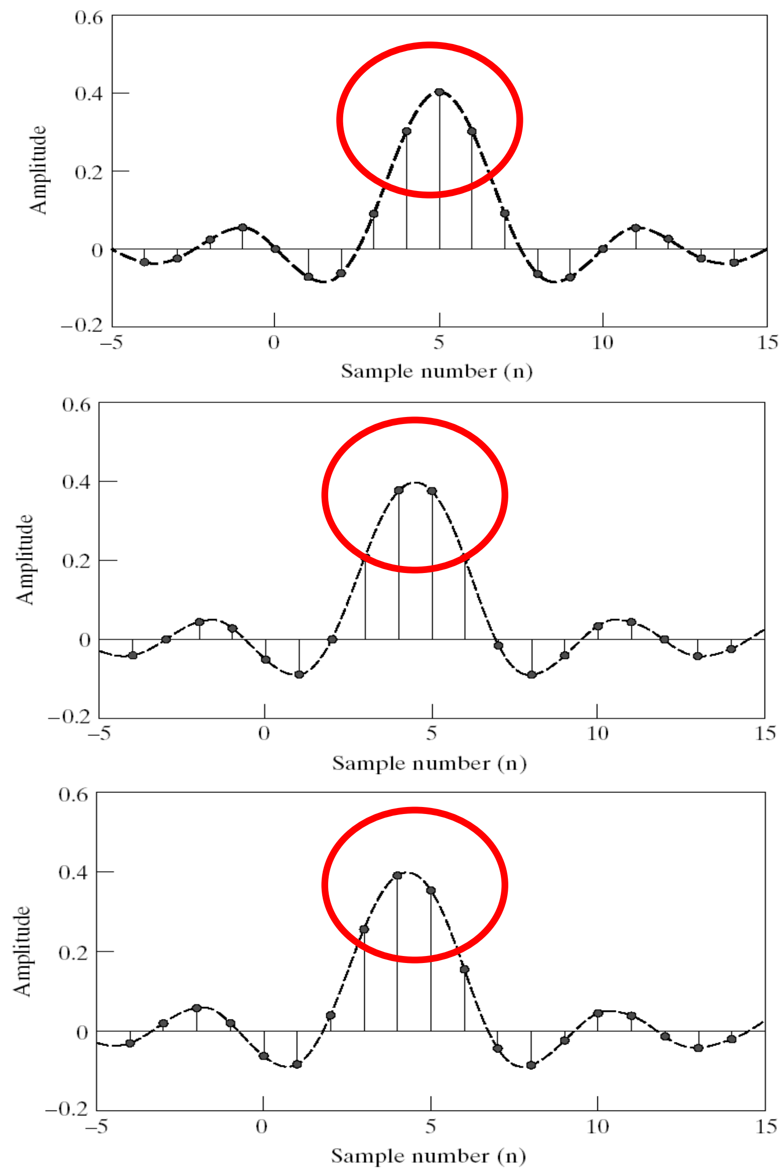
Symmetry of Linear Phase Impulse Responses

- Linear-phase systems

$$H(e^{j\omega}) = |H(e^{j\omega})| e^{-j\omega\alpha}$$

- If 2α is integer
  - Impulse response symmetric

$$h[2\alpha - n] = h[n]$$



$\alpha=5$

$\alpha=4.5$

$\alpha=4.3$

Generalized Linear Phase System

- Generalized Linear Phase

$$H(e^{j\omega}) = A(e^{j\omega})e^{-j\omega\alpha + j\beta}$$

$A(e^{j\omega})$: Real function of ω

α and β constants

- Additive constant in addition to linear term
- Has constant group delay

$$\tau(\omega) = \mathrm{grd}[H(e^{j\omega})] = -\frac{d}{d\omega}(\arg[H(e^{j\omega})]) = \alpha$$

- And linear phase of general form

$$\arg[H(e^{j\omega})] = \beta - \omega\alpha \qquad 0 \le \omega < \pi$$

Condition for Generalized Linear Phase

- A generalized linear phase system response

$$H\left(e^{j\omega}\right) = A\left(e^{j\omega}\right)e^{-j\omega\alpha + j\beta} = A\left(e^{j\omega}\right)\cos\left(\beta - \omega\alpha\right) + jA\left(e^{j\omega}\right)\sin\left(\beta - \omega\alpha\right)$$

- We know

$$H\left(e^{j\omega}\right) = \sum_{n=-\infty}^{\infty} h[n]e^{-j\omega n} = \sum_{n=-\infty}^{\infty} h[n]\cos(\omega n) - j\sum_{n=-\infty}^{\infty} h[n]\sin(\omega n)$$

- The tangent of the phase angle of this system is

$$\tan(\beta - \omega\alpha) = \frac{\sin(\beta - \omega\alpha)}{\cos(\beta - \omega\alpha)} = \frac{-\sum_{n=-\infty}^{\infty} h[n]\sin(\omega n)}{\sum_{n=-\infty}^{\infty} h[n]\cos(\omega n)}$$

- Cross multiply to get necessary condition for generalized linear phase

$$\sum_{n=-\infty}^{\infty} h[n]\cos(\omega n)\sin(\beta - \omega\alpha) - \sum_{n=-\infty}^{\infty} h[n]\sin(\omega n)\cos(\beta - \omega\alpha) = 0$$

$$\sum_{n=-\infty}^{\infty} h[n][\cos(\omega n)\sin(\beta - \omega\alpha) - \sin(\omega n)\cos(\beta - \omega\alpha)] = 0$$

$$\sum_{n=-\infty}^{\infty} h[n]\sin(\beta - \omega\alpha + \omega n) = \sum_{n=-\infty}^{\infty} h[n]\sin[\beta + \omega(n - \alpha)] = 0$$

Symmetry of Generalized Linear Phase

- **Necessary condition for generalized linear phase**

$$\sum_{n=-\infty}^{\infty} h[n]\sin[\beta + \omega(n-\alpha)] = 0$$

- **For β=0 or π**

$$\sum_{n=-\infty}^{\infty} h[n]\sin[\omega(n-\alpha)] = 0 \longrightarrow h[2\alpha - n] = h[n]$$

$$Symetric\ H(z) = z^{-2\alpha}H(z^{-1})$$

- **For β= π/2 or 3π/2**

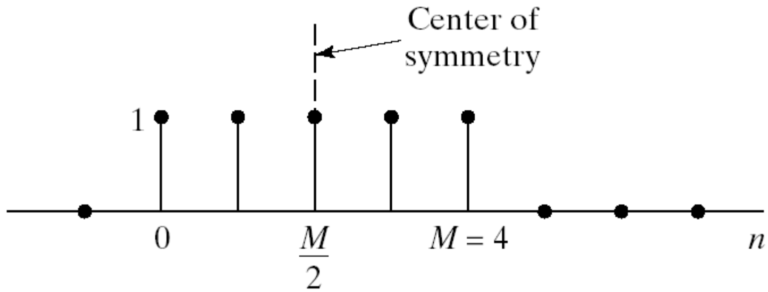
$$\sum_{n=-\infty}^{\infty} h[n]\cos[\omega(n-\alpha)] = 0 \longrightarrow h[2\alpha - n] = -h[n]$$

$$Antisymetric\ H(z) = -z^{-2\alpha}H(z^{-1})$$

Types of Generalized Linear-Phase FIR Filters

Type	impulse response	
I	symmetric	length is odd
II	symmetric	length is even
III	anti-symmetric	length is odd
IV	anti-symmetric	length is even

Type I FIR G. Linear-Phase System



- Type I system is defined with symmetric impulse response

$$h[n] = h[M-n] \qquad \text{for } 0 \le n \le M$$

  - M is an even integer

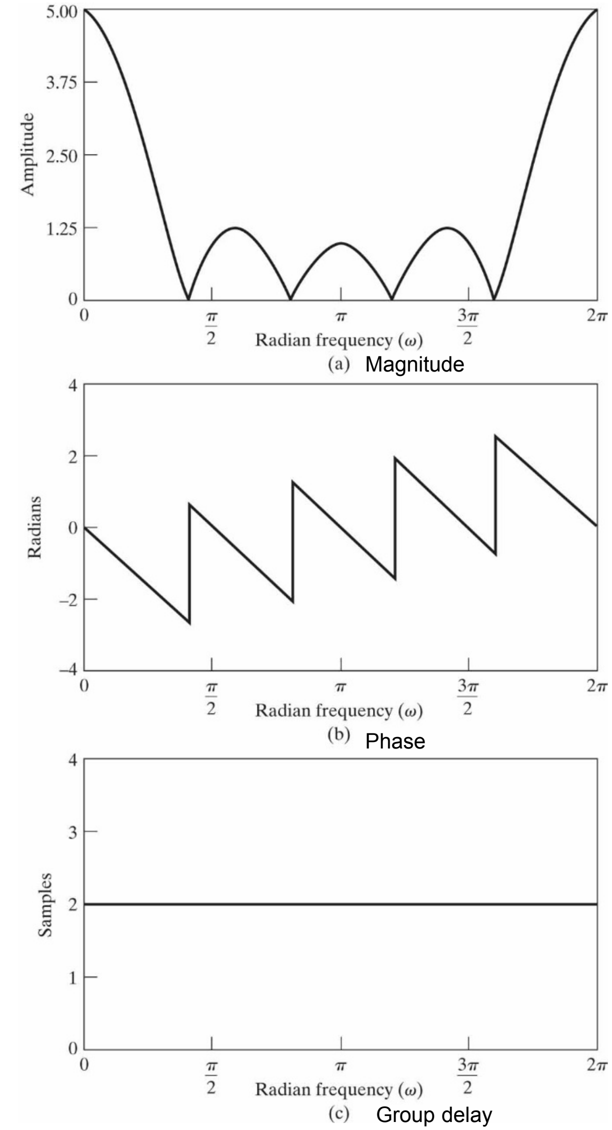
- The frequency response can be written as

$$H(e^{j\omega}) = \sum_{n=0}^{M} h[n] e^{-j\omega n}$$

$$= e^{-j\omega M/2}\left[\sum_{n=0}^{M/2} a[n]\cos(\omega n)\right]$$

- Where

$$a[0] = h[M/2]$$
$$a[k] = 2h[M/2-k] \qquad \text{for } k = 1,2,\ldots,M/2$$



Type II FIR G. Linear-Phase System

- Type II system is defined with symmetric impulse response

$$h[n] = h[M-n] \qquad \text{for } 0 \le n \le M$$

  - M is an odd integer

- The frequency response can be written as

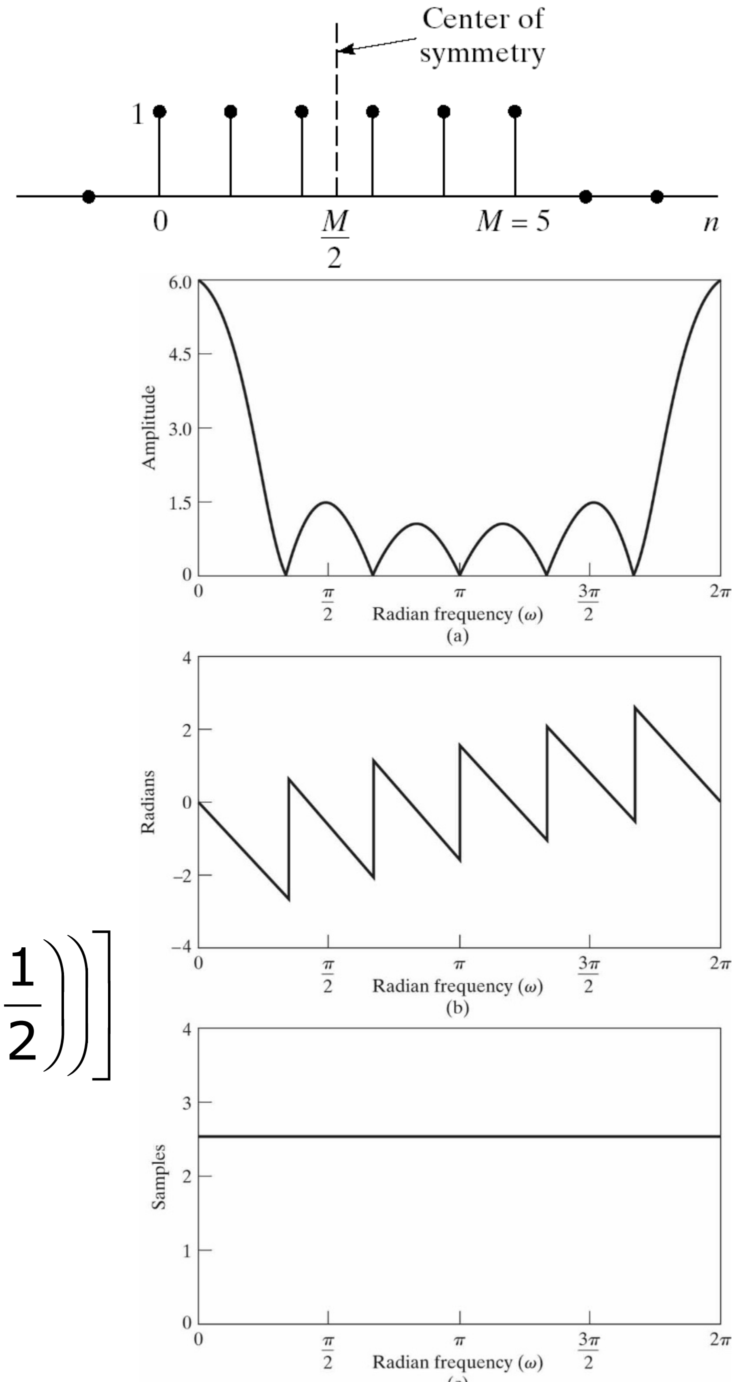
$$H(e^{j\omega}) = \sum_{n=0}^{M} h[n] e^{-j\omega n}$$

$$= e^{-j\omega M/2}\left[\sum_{n=1}^{(M+1)/2} b[n]\cos\left(\omega\left(n - \frac{1}{2}\right)\right)\right]$$

- Where

$$b[k] = 2h[(M+1)/2 - k]$$

$$\text{for } k = 1,2,\ldots,(M+1)/2$$



Type III FIR G. Linear-Phase System

- Type III system is defined with symmetric impulse response

$$h[n] = -h[M-n] \qquad \text{for } 0 \le n \le M$$

  - M is an even integer

- The frequency response can be written as

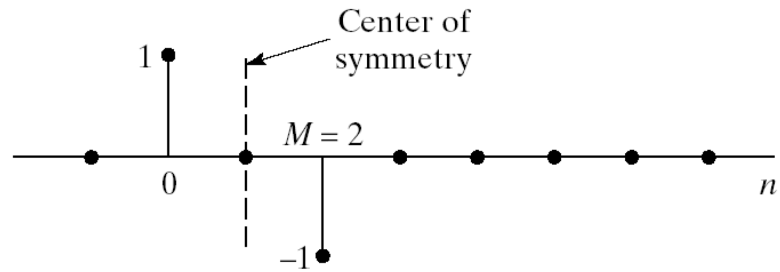
$$H\left(e^{j\omega}\right) = \sum_{n=0}^{M} h[n]e^{-j\omega n}$$

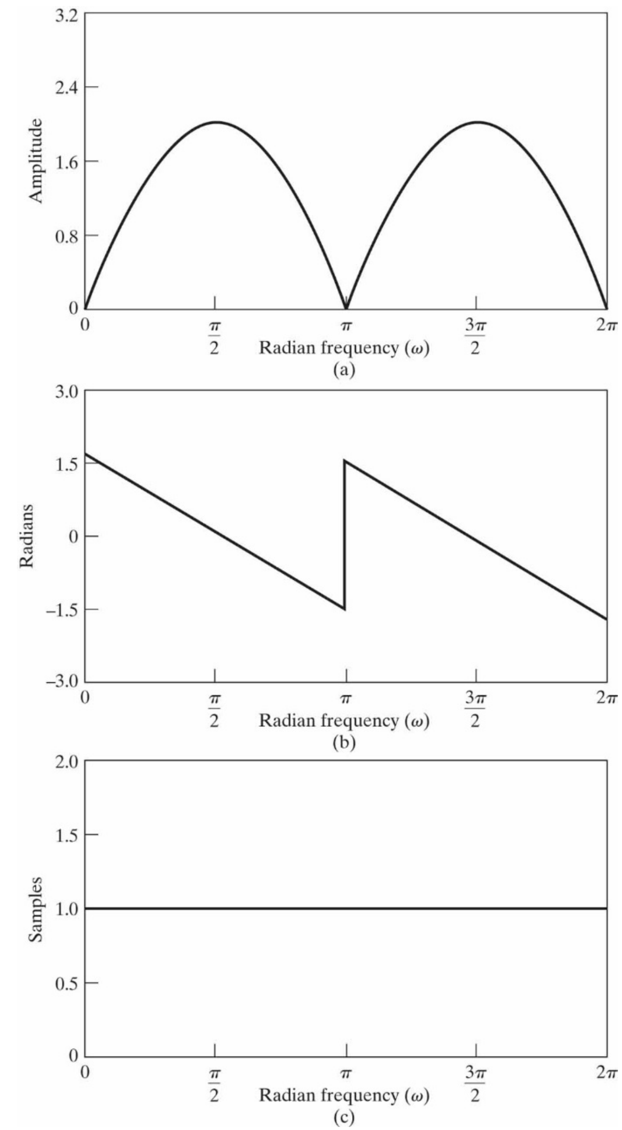
$$= je^{-j\omega M/2}\left[\sum_{n=1}^{M/2} c[n]\sin(\omega n)\right]$$

- Where

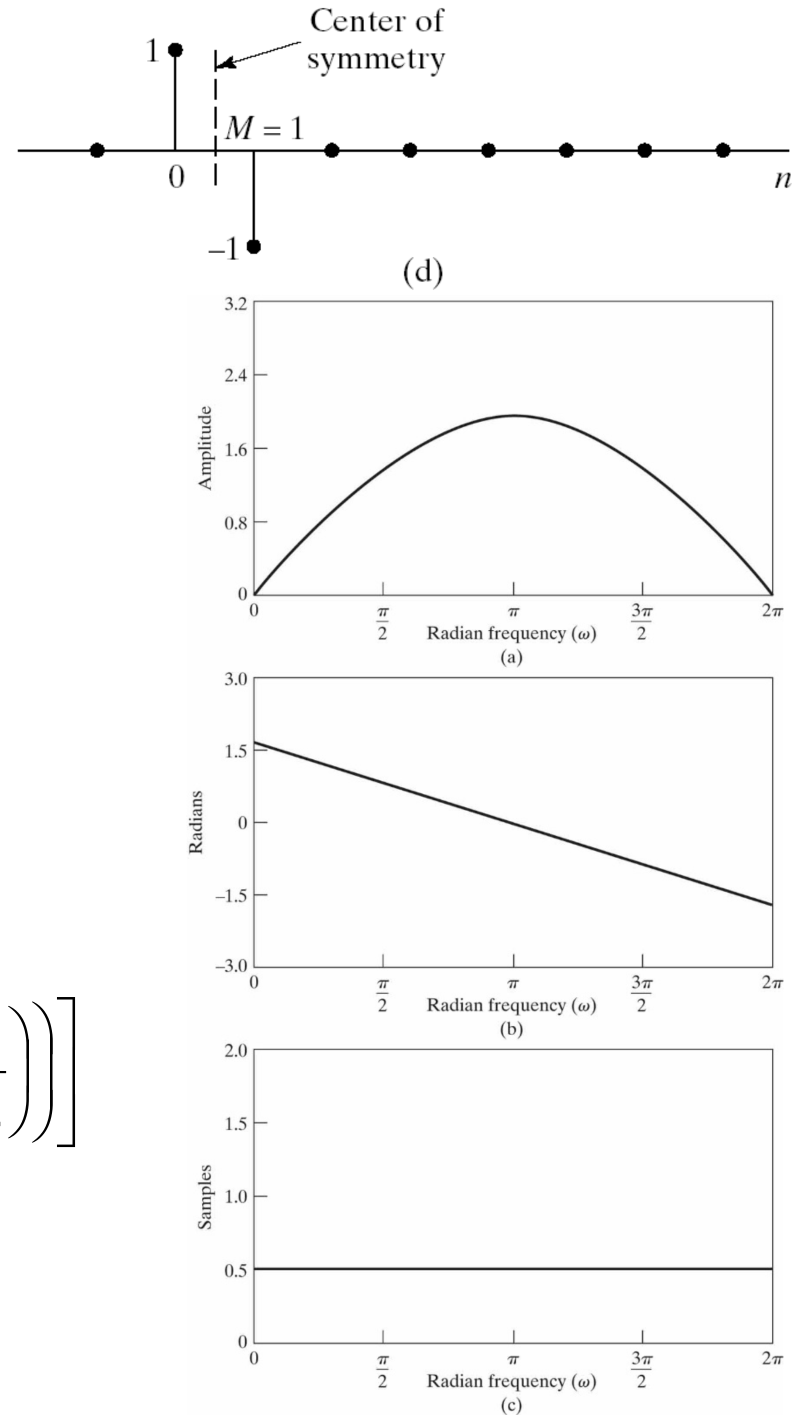
$$c[k] = 2h[M/2-k]$$

$$\text{for } k = 1,2,\ldots,M/2$$





Type IV FIR G. Linear-Phase System



- Type IV system is defined with symmetric impulse response

$$h[n] = -h[M-n] \qquad \text{for } 0 \le n \le M$$

  - M is an odd integer

- The frequency response can be written as

$$H\left(e^{j\omega}\right) = \sum_{n=0}^{M} h[n] e^{-j\omega n}$$

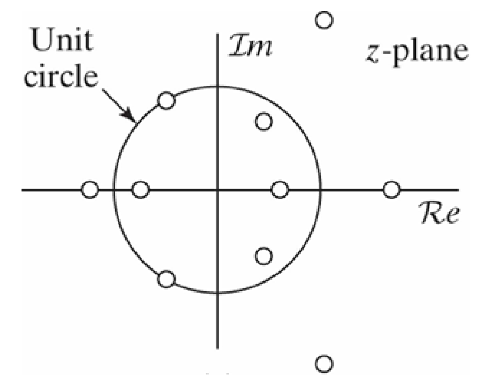
$$= j e^{-j\omega M/2}\left[\sum_{n=1}^{(M+1)/2} d[n] \sin\left(\omega\left(n - \frac{1}{2}\right)\right)\right]$$

- Where $d[k] = 2h[(M+1)/2 - k]$

  for $k = 1,2,\ldots,(M+1)/2$

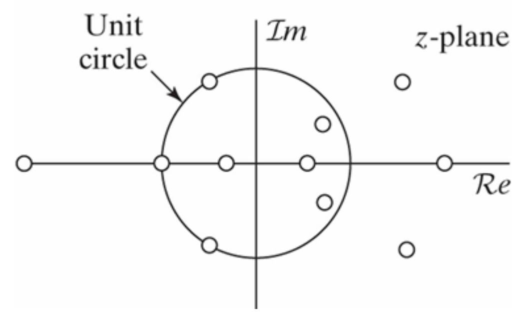
Location of Zeros for Symmetric Cases

For type I and II $h[n]=h[M-n] \xrightarrow{z} H(z)=z^{-M}H\left(z^{-1}\right)$



- So if z_0 is a zero $1/z_0$ is also a zero
- If h[n] is real and z_0 is a zero z_0^* is also a zero

➔ for real and symmetric h[n] zeros come in sets of four



- Special cases where zeros come in pairs
  - If a zero is on the UC, reciprocal is equal to conjugate
  - If a zero is real, conjugate is equal to itself
- Special cases where a zero come by itself
  - If $z=\pm 1$ both the reciprocal and conjugate is itself

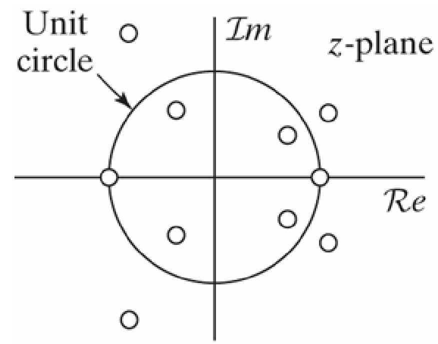
- Particular importance of z=-1
  - If M is odd implies that $H(-1)=(-1)^{M}H(-1)$

➔Cannot design high-pass filter with symmetric FIR filter and M odd

$$H(-1)=0$$

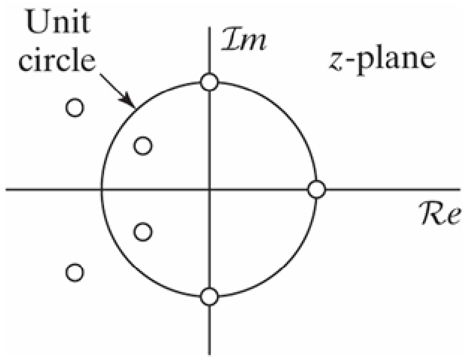
Location of Zeros for Antisymmetric Cases

For type III and IV $h[n] = -h[M-n] \xrightarrow{z} H(z) = -z^{-M}H(z^{-1})$

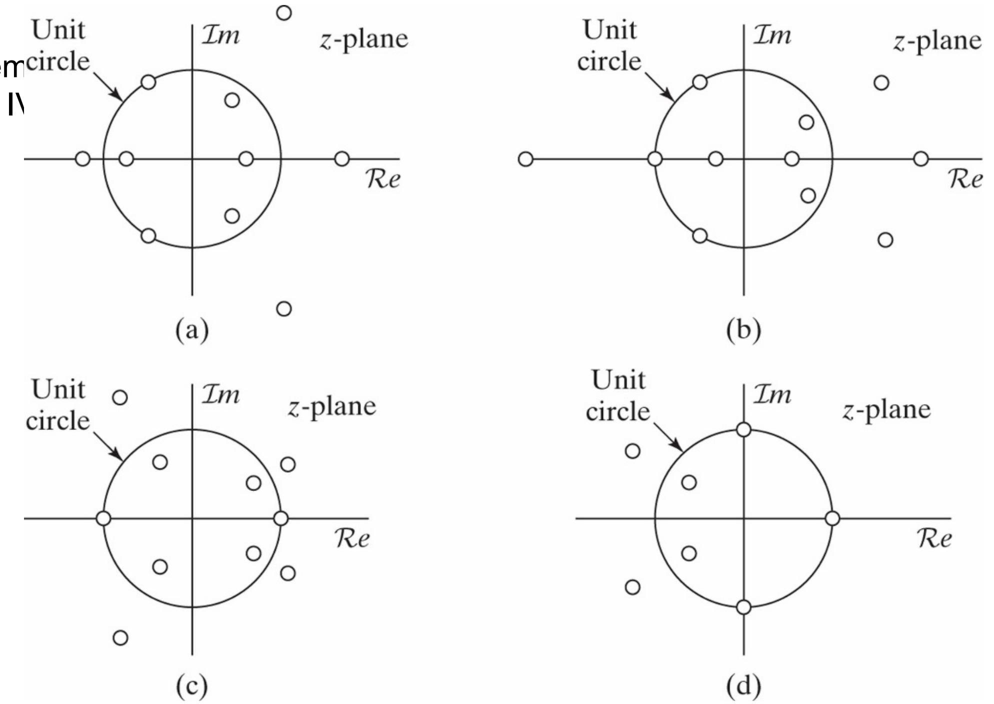


- All properties of symmetric systems holds
- Particular importance of both z=+1 and z=-1
  - If z=1 $H(1) = -H(1) \Rightarrow H(1) = 0$
    - Independent from M odd or even
  - If z=-1 $H(-1) = (-1)^{M+1}H(-1)$
    - If M+1 is odd implies that

$$H(-1) = 0$$



Typical plots of zeros for linear-phase systems.
(a) Type I. (b) Type II. (c) Type III. (d) Type IV



The presence of zeros at $z = \pm 1$ leads to the following limitations on the use:

- A Type 2 FIR filter cannot be used to design a highpass filter since it always has a zero $z = -1$
- A Type 3 FIR filter has zeros at both at both $z = 1$ and $z = -1$, and hence cannot be used to design either a lowpass or a highpass or a bandstop filter
- A Type 4 FIR filter is not appropriate to design lowpass and bandstop filters due to the presence of a zero at $z = 1$
- Type 1 FIR filter has no such restrictions and can be used to design almost any type of filters

Symmetry of Linear Phase Impulse Responses

- When h(n) is nonzero for $0 \leq n < N-1$ (the length of the impulse response h(n) is N),then the symmetry of the impulse response can be written as

$$h(n)=h(N-1-n)$$

and the anti-symmetry can be written as

$$h(n)=-h(N-1-n)$$

- An antisymmetric impulse response is simply a delayed odd impulse response (usually delayed enough to make it causal). The corresponding frequency response is not strictly linear phase, but the phase is instead linear with a constant offset (by $\pm\frac{\pi}{2}$).

- Since an *affine function* is any function of the form $f(w) = \alpha w + \beta$,where α and β are constants , an antisymmetric impulse response can be called a generalized linear phase or an *affine-phase filter*.

- Truly linear-phase filters have both a constant phase delay and a constant group delay.
- Affine-phase filters have a constant group delay, but not a constant phase delay.