Dept. of Computer Science and Engineering
CSE2131 (Discrete Mathematics)-2018
Class Test 2 Time: 1 (One) Hour
NB: Answer any three Questions. (3x10=30 Marks)

1. For the following questions, give a proof using set laws:
   i. $A \cap (B - C) = (A \cap B) - (A \cap C)$

$(A \cap B) - (A \cap C)$
$= (A \cap B) \cap (A \cap C')$
$= (A \cap B) \cap (A' \cup C')$
$= ((A \cap B) \cap A') \cup ((A \cap B) \cap C')$
$= (A \cap B \cap A') \cup (A \cap B \cap C')$
$= \Phi \cup (A \cap B \cap C')$
$= A \cap B \cap C'$
$= A \cap (B - C)$

   ii. $A - (B \cup C) = (A - B) \cap (A - C)$

$A - (B \cup C)$
$= A \cap (B \cup C)'$
$= A \cap (B' \cap C')$
$= A \cap A \cap B' \cap C'$
$= (A \cap B') \cap (A \cap C')$
$= (A - B) \cap (A - C)$

   iii. $(A - B') \cup (A - C') = A \cap (B \cap C)$

$(A - B') \cup (A - C')$
$= (A \cap B'') \cup (A \cap C'')$
$= (A \cap B) \cup (A \cap C)$
$= A \cap (B \cup C)$

2. a) Prove the associative law of set-union, $(A \cup B) \cup C = A \cup (B \cup C)$.

$(A \cup B) \cup C$
$= \{ \text{defn} \cup \}$
$\{x \mid (x \in A \lor x \in B) \lor x \in C\}$
$= \{ \lor \text{ associative} \}$
$\{x \mid x \in A \lor (x \in B \lor x \in C)\}$
$= \{ \text{defn} \cup \}$
$A \cup (B \cup C)$

   b) Prove DeMorgan's law for set intersection, $(A \cap B)' = A' \cup B'$.

$(A \cap B)'$
$= \{ \text{defn. complement} \}$
$\{x \mid x \in U \land x \notin (A \cap B)\}$
$= \{ \text{defn.} \notin, \cap \}$
$\{x \mid x \in U \land \neg(x \in A \land x \in B)\}$
$= \{ \text{DM} \}$
$\{x \mid x \in U \land (\neg(x \in A) \lor \neg(x \in B))\}$
$= \{ \text{defn.} \notin \}$
$\{x \mid x \in U \land (x \notin A \lor x \notin B)\}$
$= \{ \land \text{ over } \lor \}$

$\{x|(x \in U \wedge x \notin A) \vee (x \in U \wedge x \notin B)\}$
= { defn. − }
$\{x|(x \in U - A) \vee (x \in U - B)\}$
= { defn. complement, ∪ }
$A' \cup B'$

3.
$A = \{1, 2, 3, 4, 5\}$; $B = \{6, 7, 8, 9, 10\}$; $D = \{7, 8, 9, 10\}$; $C = \{a, b, c, d , e\}$
$f :: A \rightarrow B$, $f = \{(1, 7), (2, 6), (3, 9), (4, 7), (5, 10)\}$
$g = \{(6, b), (7, a), (6, d), (8, c), (10, b)\}$
i. Is f a function? Why or why not?   ii. Is f injective (that is, one-to-one)? Why or why not?
iii. Is f surjective (that is, onto)? Why or why not?   iv. Is g a function? Why or why not?

We will use the bullet/target metaphor of functions. In this metaphor, the domain of the function is regarded as the set of bullets. The codomain (also known as the range) is regarded as the set of targets. When x comes from the domain and y comes from the codomain and y = f (x), the target y is said to be hit by the bullet x .
Following this metaphor, we can define some of the terminology as follows:

• Function. A function is a subset of the Cartesian product of the set of bullets with the set of targets that does not use the same bullet for more than one target.

• Surjective. A function is surjective if it hits all of the targets.

• Injective. A function is injective if it hits no targets more than once.

i. Yes, f is a function because it is a subset of A × B with the property that whenever both (a, b) and (a, c) are elements of f , it is the case that b = c. That's what it means to be a function of type A → B. Or, using the metaphor, f is a function because no single bullet hits two or more targets.

ii. No, f is not injective because f (1) = 7 and f (4) = 7. So, f −1 ({7}) contains two elements (1 and 4), and inverse images of singleton sets never have more than one element for injective functions. Or, using the metaphor, f is not injective because it hits the target 7 with bullet 1 and bullet 4.

iii. No, f is not surjective because f (A) is {6, 7, 9, 10}, which is lacking an element of B (8). The image of the domain is the entire range when a function is surjective. Or, using the metaphor, f is not surjective because it fails to hit the target 8.

iv. No, g is not a function because it contains the pairs (6,b) and (6,d), and functions are not allowed to contain two pairs with the same first element and different second elements. Or, using the metaphor, g is not a function because the bullet 6 hits two targets (b and d).

4. Let *A = 1, 2, . . . , n*. Suppose *f :: A → P (A)*, where *P (A)* is the power set of *A*.
i. Prove that *f* is not surjective.
ii. Suppose *g :: X → Y* and the set *(g.f)(A)* is a subset of *A* (same *A* and same *f* as before), where the dot operator (.) stands for function composition. Can g be injective? Why or why not?
iii. Define f and g with the above domains and ranges such that (f .g) is bijective (again, the dot operator (.) stands for function composition).

i. The codomain of f is P (A), which contains 2 n elements (where 2 n stands for 2 to the power n). That is, the function f has 2 n targets. The domain of f contains n elements. That is, the function f has n bullets. Since a function must use at least one bullet per target, and since 2 n always exceeds n, f doesn't have enough bullets to hit all the targets, so it can't be surjective.

ii. The composition $g.f$ is defined when g is defined on the image $f(A)$. So, X is a subset of $P(A)$. Furthermore, since $(g.f)(A)$ is a subset of A, and since $(g.f)(A) = g(f(A))$, Y, the codomain of g, must contain at least those elements of A that are in $(g.f)(A)$. That's one way to look at it. Here's another way: The composition $(g.f)$ is defined when the codomain of f is the domain of g. Taking this position, $X = P(A)$ and Y, the codomain of g contains at least $g(f(A))$, which, by hypothesis, is a subset of A. Either position is ok. It depends on how the term "composition" is defined. Because we were not precise about when to consider the composition defined, either answer is ok.

iii. Again, it depends on the details of the definition of composition. Taking the first view, the domain of g must contain, at least, the set $f(A)$. It may contain only this set, in which case it can be injective because the number of targets f can hit cannot exceed the number of bullets in A, so g will not have more bullets than targets and can be defined to hit each potential target at most once. On the other hand, if $P(A)$ is considered to be the domain of g, then g will have too many bullets for A. So, if g is to be injective, its codomain must contain elements other than those of A. The intersection of A and the codomain of g will contain, at least, $(g.f)(A)$.

iv. Define $g : f(A) \to A$ as follows. For each s in $f(A)$, let $g(s)$ be one of the elements in $f^{-1}(s)$. Any element will do. Just pick one, and let that be $g(s)$. There must be at least one such element because s is in $f(A)$. Then $(f.g) : f(A) \to f(A)$ will be bijective.