Gregory's series

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Problem: To expand θ in powers of $\tan\theta$ $(-\frac{1}{4}\pi \le \theta \le +\frac{1}{4}\pi)$.

Solⁿ we have,

$$i\tan\theta = \frac{i\sin\theta}{\cos\theta}$$

$$= \frac{2i\sin\theta}{2\cos\theta}$$

$$= \frac{e^{i\theta} - e^{-i\theta}}{e^{i\theta} + e^{-i\theta}}$$

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$$

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

Therefore by componendo and dividendo, we have

$$\frac{1 + i\tan\theta}{1 - i\tan\theta} = \frac{e^{i\theta} + e^{-i\theta} + e^{i\theta} - e^{-i\theta}}{e^{i\theta} + e^{-i\theta} - e^{i\theta} + e^{-i\theta}}$$

$$= \frac{2e^{i\theta}}{2e^{-i\theta}}$$

$$= \frac{e^{i\theta}}{e^{-i\theta}}$$

$$= e^{i\theta + i\theta} = e^{2i\theta}$$

$$\therefore e^{2i\theta} = \frac{1 + i\tan\theta}{1 - i\tan\theta}$$

Taking logarithms of both sides, we get

$$\log e^{2i\theta} = \log \frac{1 + i\tan\theta}{1 - i\tan\theta}$$

$$\therefore 2i\theta = \log(1 + i\tan\theta) - \log(1 - i\tan\theta)$$

$$\therefore 2i\theta = i\tan\theta - \frac{1}{2} i^2 \tan^2\theta + \frac{1}{3} i^3 \tan^3\theta - \cdots\cdots$$
$$-\left(-i\tan\theta - \frac{1}{2} i^2 \tan^2\theta - \frac{1}{3} i^3 \tan^3\theta - \cdots\cdots\right)$$

$$= i\tan\theta - \frac{1}{2} i^2 \tan^2\theta + \frac{1}{3} i^3 \tan^3\theta - \cdots\cdots$$
$$+ i\tan\theta + \frac{1}{2} i^2 \tan^2\theta + \frac{1}{3} i^3 \tan^3\theta + \cdots\cdots$$
$$= 2i\tan\theta + \frac{2}{3} i^3 \tan^3\theta + 2\cdot\frac{1}{5} i^5 \tan^5\theta + \cdots$$
$$= 2i\tan\theta - \frac{2}{3} i \tan^3\theta + 2\cdot\frac{1}{5} i \tan^5\theta - \cdots\cdots$$
$$\therefore 2i\theta = 2i\left(\tan\theta - \frac{1}{3}\tan^3\theta + \frac{1}{5}\tan^5\theta - \cdots\cdots\right)$$

Therefore $\theta = \tan\theta - \frac{1}{3}\tan^3\theta + \frac{1}{5}\tan^5\theta - \cdots$ to infinity

where $-\frac{1}{4}\pi \leq \theta \leq +\frac{1}{4}\pi$

This is called Gregory's series.

putting $\tan\theta = x$, so that $\theta = \tan^{-1}x$ we have

$$\tan^{-1}x = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \cdots\cdots$$

where $-1 \leq x \leq +1$

putting, $\theta = \frac{\pi}{4}$, in Gregory's series, we have

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots\cdots$$

$\therefore \cancel{\pi = 4 - \frac{4}{3}}$ [Explain a technique to find the nume value of π using Gregory's series]

Find the numerical value of π to 5 places of decimals by Machin's series.

Sol^n we know, by Machin's series,

$$\frac{\pi}{4} = 4\tan^{-1}\frac{1}{5} - \tan^{-1}\frac{1}{239}$$

$$= 4\left(\frac{1}{5} - \frac{1}{3}\cdot\frac{1}{5^3} + \frac{1}{5}\cdot\frac{1}{5^5} - \cdots\cdots\right)$$

$$-\left(\frac{1}{239} - \frac{1}{3}\cdot\frac{1}{(239)^3} + \cdots\cdots\right)$$

$$= 4(0.2 - 0.0026666 + 0.000064 - \cdots\cdots)$$

$$- 0.0041841 + \cdots\cdots$$

$$= 0.7853983$$

$$\therefore \pi = 4 \times 0.7853983$$

$$= 3.14159$$

Prove that, $\dfrac{\pi}{8} = \dfrac{1}{1.3} + \dfrac{1}{5.7} + \dfrac{1}{9.11} + \cdots$

using Gregory's series.

Sol^n we know by Gregory's series,

$$\theta = \tan\theta - \frac{1}{3}\tan^3\theta + \frac{1}{5}\tan^5\theta - \frac{1}{7}\tan^7\theta + \cdots$$

putting $\theta = \frac{\pi}{4}$, we get

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \cdots\cdots$$

$$= \left(1 - \frac{1}{3}\right) + \left(\frac{1}{5} - \frac{1}{7}\right) + \left(\frac{1}{9} - \frac{1}{11}\right) + \cdots\cdots$$

$$= \frac{3-1}{1.3} + \frac{7-5}{5.7} + \frac{11-9}{9.11} + \cdots\cdots$$

$$\therefore \frac{\pi}{4} = \frac{2}{1.3} + \frac{2}{5.7} + \frac{2}{9.11} + \cdots\cdots$$

$$\frac{\pi}{4} = 2\left(\frac{1}{1.3} + \frac{1}{5.7} + \frac{1}{9.11} + \cdots\cdots\right)$$

$$\therefore \frac{\pi}{8} = \frac{1}{1.3} + \frac{1}{5.7} + \frac{1}{9.11} + \cdots\cdots$$ (proved)

$$\pi = \frac{8}{1.3} + \frac{8}{5.7} + \frac{8}{9.11} + \cdots\cdots$$ (Proved)

Ex.7 If $\tan x = n\tan y$, then find a series for x.

Sol^n Given that,

$$\tan x = n\tan y$$

$$\Rightarrow \frac{\sin x}{\cos x} = n\frac{\sin y}{\cos y}$$

$$\Rightarrow \frac{2i\sin x}{2\cos x} = n\frac{2i\sin y}{\cos y}$$

$$\Rightarrow \frac{e^{ix}-e^{-ix}}{e^{ix}+e^{-ix}} = n\frac{e^{iy}-e^{-iy}}{e^{iy}+e^{-iy}}$$

$$\Rightarrow \frac{e^{ix}(e^{ix}-e^{-ix})}{e^{ix}(e^{ix}+e^{-ix})} = n\frac{e^{iy}(e^{iy}-e^{-iy})}{e^{iy}(e^{iy}+e^{-iy})}$$

$$\Rightarrow \frac{e^{2ix}-1}{e^{2ix}+1} = n\frac{e^{2iy}-1}{e^{2iy}+1}$$

$$\Rightarrow \frac{e^{2ix}-1}{e^{2ix}+1} = \frac{ne^{2iy}-n}{e^{2iy}+1}$$

$$\Rightarrow \boxed{\frac{e^{2ix}+1+e^{2ix}+1}{e^{2ix}+1-e^{2ix}+1}}$$

$$\Rightarrow \frac{e^{2ix}+1+e^{2ix}-1}{e^{2ix}+1-e^{2ix}+1} = \frac{e^{2iy}+1+ne^{2iy}-n}{e^{2iy}+1-ne^{2iy}+n}$$

$$\Rightarrow \frac{2e^{2ix}}{2} = \frac{(1+n)e^{2iy}+(1-n)}{(1-n)e^{2iy}+(1+n)}$$

$$\therefore e^{2ix} = \frac{\frac{1+n}{1+n} e^{2iy} + \frac{1-n}{1+n}}{\frac{1-n}{1+n} e^{2iy} + \frac{1+n}{1+n}}$$

$$= \frac{e^{2iy} + \frac{1-n}{1+n}}{\frac{1-n}{1+n} e^{2iy} + 1}$$

$$= \frac{e^{2iy} + K}{Ke^{2iy} + 1} \text{, where } K = \frac{1-n}{1+n}$$

$$= \frac{e^{2iy}(1 + Ke^{-2iy})}{1 + Ke^{2iy}}$$

Taking log on both sides, we have

$$\log e^{2ix} = \log e^{2iy} + \log(1 + Ke^{-2iy}) - \log(1 + Ke^{2iy})$$

$$\therefore 2ix = 2iy + \left(Ke^{-2iy} - \frac{1}{2}K^2 e^{-4iy} + \frac{1}{3}K^3 e^{-6iy} \ldots\ldots\right)$$
$$- \left(Ke^{2iy} - \frac{1}{2}K^2 e^{4iy} + \frac{1}{3}K^3 e^{6iy} - \ldots\ldots\right)$$

$$= 2iy - K(e^{2iy} - e^{-2iy}) + \frac{K^2}{2}(e^{4iy} - e^{-4iy}) - \ldots\ldots$$

$$= 2iy - K.2i\sin 2y + \frac{K^2}{2}.2i\sin 4y - \ldots\ldots,$$

$$\therefore x = y - K\sin 2y + \frac{K^2}{2}\sin 4y - \frac{K^3}{3}\sin 6y + \ldots\ldots$$