Interpolation with Equal Intervals

Interpolation

- Interpolation is a process of computing intermediate values of an unknown function $f(x)$ from a set of given values of that function.
- Let $y = f(x)$ be a function of given by the values of $y_0, y_1, y_2, \ldots, y_n$ which it takes for the values $x_0, x_1, x_2, \ldots x_n$ of the independent variable x.
- If the given function $f(x)$ is totally unknown or complicated, it is desirable to replace the given function by another which can be easily handled.
- Let $\phi(x)$ denotes an arbitrary simpler function so constructed that it takes the same values as $f(x)$ for the values $x_0, x_1, x_2, \ldots, x_n$.
- Then if $f(x)$ is replaced by $\phi(x)$ over a given interval, the process constitutes interpolation, and the function $\phi(x)$ is a formula of interpolation.

Interpolation

- The $\phi(x)$ can take a variety of forms.
- When $\phi(x)$ is a polynomial, the process of representing $f(x)$ by $\phi(x)$ is called parabolic or polynomial interpolation.
- When $\phi(x)$ is a finite trigonometric series, the process is trigonometric interpolation.
- Similarly $\phi(x)$ may be a series of exponential function, Legendre polynomials, Bessel function, etc.
- In practical problems we always choose for $\phi(x)$ the simplest function which will represent the given function over the interval in question.

Interpolation: Justification

- The justification for replacing a given function by a polynomial rest on Weierstrass's [1885] theorem stated below:
  - Every function which is continuous in an interval (a, b) can be represented in that interval, to any desired degree of accuracy, by a polynomial.
  - That is, it is possible to find a polynomial $P(x)$ such that $|f(x) - P(x)| < \varepsilon$ for every value of x in the interval (a, b), where ε is the desired accuracy and $\varepsilon > 0$.

Interpolation: Justification

- To justify the replacement of a given trigonometric function Weierstrass's [1885] theorem states that:
  - Every continuous trigonometric function of period 2π can be represented by a finite trigonometric series of the form

$$g(x) = a_0 + a_1\sin(x) + a_2\sin(2x) + \ldots + a_n\sin(nx) + b_1\cos(x) + b_2\cos(2x) + \ldots + b_n\cos(nx)$$

  - That is, it is possible to find a trigonometric function $g(x)$ such that $|f(x) - g(x)| < \varepsilon$ for every value of x in the interval 2π where ε is the desired accuracy and $\varepsilon > 0$.

Forward Differences

- If $y_0, y_1, y_2, ..., y_n$ denote a set of values of any function $y = f(x)$, then $y_1 - y_0, y_2 - y_1, y_3 - y_2, ..., y_n - y_{n-1}$ are called the differences of the function y.
- We denote these differences by $\Delta y_0, \Delta y_1, \Delta y_2$ etc., where $\Delta y_0 = y_1 - y_0,\ \Delta y_1 = y_2 - y_1, ..., \Delta y_{n-1} = y_n - y_{n-1}, \Delta y_n = y_{n+1} - y_n$.
- Here, Δ is called the forward difference operator and $\Delta y_0, \Delta y_1, \Delta y_2, ..., \Delta y_n$ are called first forward differences.

Forward Differences

- The differences of these first forward differences are called second forward differences and are denoted by $\Delta^2 y_0 = \Delta y_1 - \Delta y_0$, $\Delta^2 y_1 = \Delta y_2 - \Delta y_1$, etc.

- Similarly, one can define third forward differences, fourth forward differences, etc.

Forward Differences

Thus,

$$\Delta^2 y_0 = \Delta y_1 - \Delta y_0 = y_2 - y_1 - (y_1 - y_0) = y_2 - 2y_1 + y_0$$

$$\begin{aligned}\Delta^3 y_0 = \Delta^2 y_1 - \Delta^2 y_0 &= y_3 - 2y_2 + y_1 - (y_2 - 2y_1 + y_0) \\ &= y_3 - 3y_2 + 3y_1 - y_0,\end{aligned}$$

and

$$\begin{aligned}\Delta^4 y_1 = \Delta^3 y_1 - \Delta^3 y_0 &= y_4 - 3y_3 + 3y_2 - y_1 - (y_3 - 3y_2 + 3y_1 - y_0) \\ &= y_4 - 4y_3 + 6y_2 - 4y_1 + y_0\end{aligned}$$

Forward (Diagonal) Difference Table

x	y	Δ	Δ^2	Δ^3	Δ^4	Δ^5	Δ^6
x_0	y_0						
		Δy_0					
x_1	y_1		$\Delta^2 y_0$				
		Δy_1		$\Delta^3 y_0$			
x_2	y_2		$\Delta^2 y_1$		$\Delta^4 y_0$		
		Δy_2		$\Delta^3 y_1$		$\Delta^5 y_0$	
x_3	y_3		$\Delta^2 y_2$		$\Delta^4 y_1$		$\Delta^6 y_0$
		Δy_3		$\Delta^3 y_2$		$\Delta^5 y_1$	
x_4	y_4		$\Delta^2 y_3$		$\Delta^4 y_2$		
		Δy_4		$\Delta^3 y_3$			
x_5	y_5		$\Delta^2 y_4$				
		Δy_5					
x_6	y_6						

Problem

Given the set of values

x	10	15	20	25	30	35
y	19.97	21.51	22.47	23.52	24.65	25.89

form the difference table and write down the values of

$$\Delta y_{10}, \qquad \Delta^2 y_{20}, \qquad \Delta^3 y_{15} \qquad and \qquad \Delta^5 y_{10}$$

$$\Delta y_{10} = y_{15} - y_{10} = 21.51 - 19.97 = 1.54$$

$$\Delta^2 y_{20} = \Delta^2 y_{25} - \Delta^2 y_{20} = (\Delta\, y_{30} - \Delta\, y_{25}) - (\Delta\, y_{25} - \Delta\, y_{20})$$

$$= \Delta\, y_{30} - 2\Delta\, y_{25} + \Delta\, y_{20} = 24.65 - 2 * 23.52 + 21.51 = -0.08$$

Problem

Given the set of values, find the followings

$$\Delta y_{10}, \qquad \Delta^2 y_{20}, \qquad \Delta^3 y_{15} \qquad and \qquad \Delta^5 y_{10}$$

x	y	D y	D^2 y	D^3 y	D^4 y	D^5 y
10	19.97					
		1.54				
15	21.51		-0.58			
		0.96		0.67		
20	22.47		0.09		-0.68	
		1.05		-0.01		0.72
25	23.52		0.08		0.04	
		1.13		0.03		
30	24.65		0.11			
		1.24				
35	25.89					

Backward (Horizontal) Differences

- The differences $y_1 - y_0, y_2 - y_1, \ldots, y_n - y_{n-1}$ are called Backward or Horizontal Differences, if they are denoted by $\nabla y_1, \nabla y_2, \ldots, \nabla y_n$
- Here, $\nabla y_1 = y_1 - y_0, \nabla y_2 = y_2 - y_1, \ldots \nabla y_n = y_n - y_{n-1},$
- ∇ is called the backward difference operator.
- In a similar way, one can define backward differences of higher orders.
- Thus we obtain,

$$\nabla^2 y_2 = \nabla y_2 - \nabla y_1 = y_2 - y_1 - (y_1 - y_0) = y_2 - 2y_1 + y_0,$$

$$\nabla^3 y_3 = \nabla^2 y_3 - \nabla^2 y_2 = y_3 - 3y_2 + 3y_1 - y_0, \quad etc$$

Backward (Horizontal) Difference Table

x	y	∇	∇^2	∇^3	∇^4	∇^5	∇^6
x_0	y_0						
x_1	y_1	∇y_1					
x_2	y_2	∇y_2	$\nabla^2 y_2$				
x_3	y_3	∇y_3	$\nabla^2 y_3$	$\nabla^3 y_3$			
x_4	y_4	∇y_4	$\nabla^2 y_4$	$\nabla^3 y_4$	$\nabla^4 y_4$		
x_5	y_5	∇y_5	$\nabla^2 y_5$	$\nabla^3 y_5$	$\nabla^4 y_5$	$\nabla^5 y_5$	
x_6	y_6	∇y_6	$\nabla^2 y_6$	$\nabla^3 y_6$	$\nabla^4 y_6$	$\nabla^5 y_6$	$\nabla^6 y_6$

Problem

Given the set of values

x	10	15	20	25	30	35
y	19.97	21.51	22.47	23.52	24.65	25.89

form the difference table and write down the values of

$$\nabla y_{20}, \qquad \nabla^2 y_{25}, \qquad \nabla^3 y_{30} \qquad and \qquad \nabla^5 y_{35}$$

Problem

Given the set of values, find the followings

$$\nabla y_{10}, \qquad \nabla^2 y_{20}, \qquad \nabla^3 y_{15} \qquad and \qquad \nabla^5 y_{10}$$

x	**y**	∇**y**	∇^2 **y**	∇^3 **y**	∇^4 **y**	∇^5 **y**
10	19.97					
15	21.51	1.54				
20	22.47	0.96	-0.58			
25	23.52	1.05	0.09	0.67		
30	24.65	1.13	0.08	-0.01	-0.68	
35	25.89	1.24	0.11	0.03	0.04	0.72

Central Differences and Central Difference Table

The central difference operator δ is defined by the relations

$$y_1 - y_0 = \delta y_{1/2}, \qquad y_2 - y_1 = \delta y_{3/2}, \cdots\cdots, y_n - y_{n-1} = \delta y_{n-1/2}$$

x	y	δ	δ^2	δ^3	δ^4	δ^5	δ^6
x_0	y_0						
		$\delta y_{1/2}$					
x_1	y_1		$\delta^2 y_1$				
		$\delta y_{3/2}$		$\delta^3 y_{3/2}$			
x_2	y_2		$\delta^2 y_2$		$\delta^4 y_2$		
		$\delta y_{5/2}$		$\delta^3 y_{5/2}$		$\delta^5 y_{5/2}$	
x_3	y_3		$\delta^2 y_3$		$\delta^4 y_3$		$\delta^6 y_3$
		$\delta y_{7/2}$		$\delta^3 y_{7/2}$		$\delta^5 y_{7/2}$	
x_4	y_4		$\delta^2 y_4$		$\delta^4 y_4$		
		$\delta y_{9/2}$		$\delta^3 y_{9/2}$			
x_5	y_5		$\delta^2 y_5$				
		$\delta y_{11/2}$					
x_6	y_6						

Problem

Given the set of values

x	10	15	20	25	30	35
y	19.97	21.51	22.47	23.52	24.65	25.89

form the difference table and write down the values of

$$\delta y_{10}, \quad \delta^2 y_{20}, \quad \delta^3 y_{15} \quad and \quad \delta^5 y_{10}$$

Relations in the Three Difference Table

- From the three tables we can see that

$$\Delta y_0 = y_1 - y_0$$
$$\nabla y_1 = y_1 - y_0$$
$$\delta_{1/2} = y_1 - y_0$$

$$\begin{aligned}\Delta^3 y_2 &= \Delta^2 y_3 - \Delta^2 y_2 = (\Delta\, y_4 - \Delta\, y_3) - (\Delta\, y_3 - \Delta\, y_2) \\ &= \Delta\, y_4 - 2\Delta\, y_3 + \Delta\, y_2 = (y_5 - y_4) - 2(y_4 - y_3) + (y_3 - y_2) \\ &= y_5 - 3y_4 + 3y_3 - y_2\end{aligned}$$

$$\begin{aligned}\nabla^3 y_5 &= \nabla^2 y_5 - \nabla^2 y_4 = (\nabla\, y_5 - \nabla\, y_4) - (\nabla\, y_4 - \nabla\, y_3) \\ &= \nabla\, y_5 - 2\nabla\, y_4 + \nabla\, y_3 = (y_5 - y_4) - 2(y_4 - y_3) + (y_3 - y_2) \\ &= y_5 - 3y_4 + 3y_3 - y_2\end{aligned}$$

$$\Delta y_0 = \nabla y_1 = \delta_{1/2}$$

$$\Delta^3 y_2 = \nabla^3 y_5 = \delta^3 y_{7/2}$$

Relations in the Three Difference Table

Thus we obtain

$$\Delta^m y_k = \nabla^m y_{k+m} = \delta^m y_{(2k+m)/2}$$

Effect of an Error in a Tabular Value

Let y_0, y_1, y_2, ...y_n be the true values of a function, and suppose the value y_3 to be effected with an error ε, so that its erroneous value is $y_3 + \varepsilon$. Then the successive differences of the y's are as shown below:

y	Δ	Δ^2	Δ^3
y_0			
	Δy_0		
y_1		$\Delta^2 y_0$	
	Δy_1		$\Delta^3 y_0 + \varepsilon$
y_2		$\Delta^2 y_1 + \varepsilon$	
	$\Delta y_2 + \varepsilon$		$\Delta^3 y_1 - 3\varepsilon$
$y_3 + \varepsilon$		$\Delta^2 y_2 - 2\varepsilon$	
	$\Delta y_3 - \varepsilon$		$\Delta^3 y_2 + 3\varepsilon$
y_4		$\Delta^2 y_3 + \varepsilon$	
	Δy_4		$\Delta^3 y_3 - \varepsilon$
y_5		$\Delta^2 y_4$	
	Δy_5		
y_6			

Cont.

Suppose that there is an error of +1 unit in a certain tabular value. As higher differences are formed, the error spreads out fanwise, and is at the same time, considerably magnified as shown below:

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k.$$

y	Δ	Δ^2	Δ^3	Δ^4	Δ^5
0					
	0				
0		0			
	0		0		
0		0		0	
	0		0		1
0		0		1	
	0		1		− 5
0		1		− 4	
	1		− 3		10
1		− 2		6	
	− 1		3		− 10
0		1		− 4	
	0		− 1		5
0		0		1	
	0		0		− 1
0		0		0	
	0		0		
0		0			
	0				
0					

Cont.

The table shows the following characteristics:

•The effect of an error increases with the successive differences.

•The coefficients of the ε's are the binomial coefficients with alternating signs.

•The algebraic sum of the errors in any difference column is zero.

•The maximum error in the differences is in the same horizontal line as the erroneous value.

The table in the next slide shows the effect of horizontal difference table:

Cont.

y	Δ	Δ^2	Δ^3	Δ^4	Δ^5
3010					
	414				
3424		-36			
	378		-39		
3802		-75		178	
	303		139		-449
4105		64		-271	
	367		-132		452
4472		-68		181	
	299		49		-227
4771		-19		-46	
	280		3		
5051		-16			
	264				
5315					

$-4\,\varepsilon = 178$

$\varepsilon = -45$ (approximate)

$6\,\varepsilon = -271$

$\varepsilon = -45$ (approximate)

Therefore, the actual entry is $4105 - \varepsilon = 4105 - (-45) = 4150$

Pascal's Triangle

0: 1

1: 1 1

2: 1 2 1

3: 1 3 3 1

4: 1 4 6 4 1

5: 1 5 10 10 5 1

6: 1 6 15 20 15 6 1

7: 1 7 21 35 35 21 7 1

8: 1 8 28 56 70 56 28 8 1

Effect of an Error in a Tabular Value of Backward Interpolation

x	y	∇	∇^2	∇^3	∇^4
x_0	y_0				
x_1	y_1	∇y_1			
x_2	y_2	∇y_2	$\nabla^2 y_2$		
x_3	y_3	∇y_3	$\nabla^2 y_3$	$\nabla^3 y_3$	
x_4	y_4	∇y_4	$\nabla^2 y_4$	$\nabla^3 y_4$	$\nabla^4 y_4$
x_5	$y_5+\varepsilon$	$\nabla y_5+\varepsilon$	$\nabla^2 y_5+\varepsilon$	$\nabla^3 y_5+\varepsilon$	$\nabla^4 y_5+\varepsilon$
x_6	y_6	$\nabla y_6-\varepsilon$	$\nabla^2 y_6-2\varepsilon$	$\nabla^3 y_6-3\varepsilon$	$\nabla^4 y_6-4\varepsilon$
x_7	y_7	∇y_7	$\nabla^2 y_7+\varepsilon$	$\nabla^3 y_7+3\varepsilon$	$\nabla^4 y_7+6\varepsilon$
x_8	y_8	∇y_8	$\nabla^2 y_8$	$\nabla^3 y_8-\varepsilon$	$\nabla^4 y_8-4\varepsilon$
x_9	y_9	∇y_9	$\nabla^2 y_9$	$\nabla^3 y_9$	$\nabla^4 y_9+\varepsilon$
x_{10}	y_{10}	∇y_{10}	$\nabla^2 y_{10}$	$\nabla^3 y_{10}$	$\nabla^4 y_{10}$

- The effect of the error is the same as in the preceding table
- But in this table the first erroneous of any order is in the same horizontal line as the erroneous tabular value.

Newton’s formula for Forward Interpolation

- Let $y = f(x)$ denote a function which takes the values $y_0, y_1, y_2, \ldots, y_n$ for the equidistant values $x_0, x_1, x_2, \ldots, x_n$ of the independent variable x.
- It is required to find $\phi(x)$, a polynomial of the n-th degree such that y and $\phi(x)$ agree at the tabulated points (i.e., they have the same values).
- Let $\phi(x)$ denote a polynomial of the n-th degree.
- This polynomial can be written in the form

$$\phi(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1) + a_3(x - x_0)(x - x_1)(x - x_2) + \\ + \ldots\ldots + a_n(x - x_0)(x - x_1)\ldots\ldots(x - x_{n-1}) \qquad (1)$$

Newton's formula for Forward Interpolation

- We shall now determine the coefficients $a_0, a_1, a_2, ..., a_n$, so that we can get $\phi(x_0) = y_0$, $\phi(x_1) = y_1$, $\phi(x_2) = y_2$, ..., $\phi(x_n) = y_n$.
- We know that

$$\phi(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1) + a_3(x - x_0)(x - x_1)(x - x_2) + \\ + ... + a_n(x - x_0)(x - x_1)....(x - x_{n-1}) \qquad (1)$$

- We can substitute the given successive values $x_0, x_1, x_2, ..., x_n$ in equation (1).
- At the same time we can put $\phi(x_0) = y_0$, $\phi(x_1) = y_1$, $\phi(x_2) = y_2$, ..., $\phi(x_n) = y_n$.
- And let $x_1 - x_0 = h$. Then, $x_2 - x_0 = 2h$, etc, (since the values of x are equidistance).

Newton's formula for Forward Interpolation

- In equation (1) we have

$$\phi(x)=a_0+a_1(x-x_0)+a_2(x-x_0)(x-x_1)+a_3(x-x_0)(x-x_1)(x-x_2)+$$
$$+\ldots\ldots+a_n(x-x_0)(x-x_1)\ldots(x-x_{n-1}) \qquad (1)$$

- That is, at $x = x_0$ (substituting x with x_0 in equation (1)) we have

$$\phi(x_0)=a_0+a_1(x_0-x_0)+a_2(x_0-x_0)(x_0-x_1)+\ldots\ldots+a_n(x_0-x_0)(x_0-x_1)\ldots(x_0-x_{n-1})$$

$$or, \phi(x_0) = a_0 = y_0$$

- Therefore we get, $a_0 = y_0$

- Similarly, substituting x_1 in the eq(1) we get

$$\phi(x_1) = y_1 = a_0 + a_1(x_1 - x_0) = y_0 + a_1 h$$

$$or, a_1 = \frac{y_1 - y_0}{h} = \frac{\Delta y_0}{h}$$

Newton's formula for Forward Interpolation

- Substituting x_2 in equation (1) we get,

$$y_2 = a_0 + a_1(x_2 - x_0) + a_2(x_2 - x_0)(x_2 - x_1)$$

$$= y_0 + \frac{y_1 - y_0}{h}(2h) + a_2(2h)(h)$$

$$\Rightarrow a_2 = \frac{y_2 - 2y_1 + y_0}{2h^2} = \frac{\Delta^2 y_0}{2h^2}$$

Newton's formula for Forward Interpolation

- Substituting x_3 in equation (1) we get,

$$y_3 = a_0 + a_1(x_3 - x_0) + a_2(x_3 - x_0)(x_3 - x_1) + a_3(x_3 - x_0)(x_3 - x_1)(x_3 - x_2)$$

$$= y_0 + \frac{y_1 - y_0}{h}(3h) + \frac{y_2 - 2y_1 + y_0}{2h^2}(3h)(2h) + a_3(3h)(2h)(h)$$

$$\Rightarrow a_3 = \frac{y_3 - 3y_2 + 3y_1 - y_0}{6h^3} = \frac{\Delta^3 y_0}{3!\,h^3}$$

Newton’s formula for Forward Interpolation

- Similarly,

$$a_4 = \frac{\Delta^4 y_0}{4!h^4} \quad (Class \quad Work)$$

$$a_5 = \frac{\Delta^5 y_0}{5!h^5}, \quad a_6 = \frac{\Delta^6 y_0}{6!h^6}, \ldots, a_n = \frac{\Delta^n y_0}{n!h^n}$$

Newton's formula for Forward Interpolation

Substituting in equation (1) the values a_0, a_1, a_2,...a_n , we have,

$$\phi(x) = y_0 + \frac{\Delta y_0}{h}(x - x_0) + \frac{\Delta^2 y_0}{2h^2}(x - x_0)(x - x_1) + \frac{\Delta^3 y_0}{3!h^3}(x - x_0)(x - x_1)(x - x_2) +$$

$$\ldots\ldots + \frac{\Delta^n y_0}{n!h^n}(x - x_0)(x - x_1)\ldots.(x - x_{n-1}) \qquad (2)$$

- This is Newton's formula for forward interpolation, written in term of *x*.
- This formula can be simplified by a change of variable.

Newton's formula for Forward Interpolation

Now, we can rewrite eq(2) in the following equivalent form

$$\phi(x) = y_0 + \Delta y_0 \left(\frac{x - x_0}{1!\,h}\right) + \frac{\Delta^2 y_0}{2!}\left(\frac{x - x_0}{h}\right)\left(\frac{x - x_1}{h}\right) +$$

$$\ldots\ldots + \frac{\Delta^n y_0}{n!}\left(\frac{x - x_0}{h}\right)\left(\frac{x - x_1}{h}\right)\ldots\left(\frac{x - x_{n-1}}{h}\right) \qquad (3)$$

Newton’s formula for Forward Interpolation

- Now, put the following in equation (3)

$$\frac{x - x_0}{h} = u, \quad or \qquad x = x_0 + hu$$

- Then, since $x_1 = x_0 + h$, $x_2 = x_0 + 2h$, etc. we have

$$\frac{x - x_1}{h} = \frac{x - (x_0 + h)}{h} = \frac{x - x_0}{h} - \frac{h}{h} = u - 1$$

- Similarly,

$$\frac{x - x_2}{h} = \frac{x - (x_0 + 2h)}{h} = \frac{x - x_0}{h} - \frac{2h}{h} = u - 2,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\frac{x - x_{n-1}}{h} = \frac{x - [x_0 + (n-1)h]}{h} = \frac{x - x_0}{h} - \frac{(n-1)h}{h} = u - n + 1$$

Newton's formula for Forward Interpolation

- Substituting the values of $(x - x_0)/h$, $(x - x_1)/h$ etc. in equation (3)

$$\phi(x) = \phi(x_0 + hu) = g(u)$$

$$= y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \ldots$$

$$+ \frac{u(u-1)(u-2)(u-3)\ldots\ldots(u-n+1)}{n!}\Delta^n y_0 \qquad (4)$$

- This is the form in which Newton's formula for forward interpolation is usually written.

- The reason for the name "forward" interpolation formula since the formula contains values of the tabulated function from y_0 onward to the right (forward from y_0) and none to the left of this value.

- Because of this fact this formula is used mainly for interpolating the values of y near the beginning of a set of tabular values.

Example 1

Find the cubic polynomial which takes the following values

$y(0) = 1,$ $y(1) = 0,$ $y(2) = 1$ and $y(3) = 10$

Hence, or otherwise obtain $y(0.5)$.

Solution.

x	y	Δ	Δ^2	Δ^3
0	1			
		-1		
1	0		2	
		1		6
2	1		8	
		9		
3	10			

Example 1 Cont.

Here, h = 1, and

$$y(x) = y_0 + \frac{\Delta y_0}{h}(x - x_0) + \frac{\Delta^2 y_0}{2h^2}(x - x_0)(x - x_1) + \frac{\Delta^3 y_0}{3!h^3}(x - x_0)(x - x_1)(x - x_2) +$$

$$\ldots\ldots + \frac{\Delta^n y_0}{n!h^n}(x - x_0)(x - x_1)\ldots\ldots(x - x_{n-1})$$

x	y	Δ	Δ^2	Δ^3
0	**1**			
		-1		
1	**0**		**2**	
		1		**6**
2	**1**		**8**	
		9		
3	**10**			

$$= 1 + \frac{(-1)(x-0)}{1} + \frac{(x-0)(x-1)}{2(1)^2}(2) + \frac{(x-0)(x-1)(x-2)}{6(1)^3}(6)$$

$$= 1 - x + x(x-1) + x(x-1)(x-2)$$

$$= 1 - x + x^2 - x + x^3 - 3x^2 + 2x$$

$$= x^3 - 2x^2 + 1$$

Example 1 Cont.

Therefore, the polynomial we obtained for the given tabular values is.

$$y = x^3 - 2x^2 + 1$$

Now,

$$y(0.5) = (0.5)^3 - 2*(0.5)^2 + 1 = 0.625$$

(which is the same value as that obtained by substituting $x = 0.5$ in the cubic polynomial above.)

Problem

The table below gives the values of tan (x) for $0.10 \leq x \leq 0.30$.

Find tan(0.12)

x	0.10	0.15	0.20	0.25	0.30
$y = \tan x$	0.1003	0.1511	0.2027	0.2553	0.3093

Answer is 0.120537

Problem

The population of a town is given below for a range of years. Estimate the population for the year 1895.

Year : x	1891	1901	1911	1921	1931
Population: y (in thousands)	46	66	81	93	101

Answer: 54.85 Thousands

Newton's formula for Backward Interpolation

- Let $y = f(x)$ denote a function which takes the values $y_0, y_1, y_2, ..., y_n$ for the equidistant values $x_0, x_1, x_2, ..., x_n$ of the independent variable x.
- It is required to find $\phi(x)$, a polynomial of the n-th degree such that y and $\phi(x)$ agree at the tabulated points (i.e., they have the same values).
- Let $\phi(x)$ denote a polynomial of the n-th degree.
- This polynomial can be written in the form

$$\phi(x) = a_0 + a_1(x - x_n) + a_2(x - x_n)(x - x_{n-1}) + a_3(x - x_n)(x - x_{n-1})(x - x_{n-2}) + \\ + ... + a_n(x - x_n)(x - x_{n-1}) .... (x - x_1) \qquad (1)$$

Newton's formula for Backward Interpolation

- We shall now determine the coefficients $a_0, a_1, a_2, \ldots, a_n$, so that we can get $\phi(x_n) = y_n$, $\phi(x_{n-1}) = y_{n-1}$, $\phi(x_{n-2}) = y_{n-2}$, ..., $\phi(x_0) = y_0$.
- We know that

$$\phi(x) = a_0 + a_1(x - x_n) + a_2(x - x_n)(x - x_{n-1}) + a_3(x - x_n)(x - x_{n-1})(x - x_{n-2}) + \ldots\ldots + a_n(x - x_n)(x - x_{n-1})\ldots(x - x_1) \qquad (1)$$

- We can substitute the given successive values x_n, x_{n-1}, x_{n-2}, ..., x_0 in equation (1).
- At the same time we can put $\phi(x_n) = y_n$, $\phi(x_{n-1}) = y_{n-1}$, $\phi(x_{n-2}) = y_{n-2}$, ..., $\phi(x_0)=y_0$.
- And let $x_{n-1} - x_n = -h$. Then, $x_{n-2} - x_n = -2h$, etc, (since the values of x are equidistance).

Newton’s formula for Backward Interpolation

- In equation (1) we have

$$\phi(x)=a_0+a_1(x-x_n)+a_2(x-x_n)(x-x_{n-1})+a_3(x-x_n)(x-x_{n-1})(x-x_{n-2})+$$
$$\ldots\ldots+a_n(x-x_n)(x-x_{n-1})\ldots.(x-x_1) \qquad (1)$$

- That is, at $x = x_n$ (substituting x with x_n in equation (1)) we have

$$\phi(x_n)=a_0+a_1(x_n-x_n)+a_2(x_n-x_n)(x_n-x_{n-1})+\ldots\ldots+a_n(x_n-x_n)(x_n-x_{n-1})\ldots.(x_n-x_1)$$

$$or, \phi(x_n) = a_0 = y_n$$

- Therefore we get, $a_0 = y_n$
- Similarly,

$$y_{n-1} = a_0 + a_1(x_{n-1} - x_n) = y_n - a_1 h$$

$$or, a_1 = \frac{y_n - y_{n-1}}{h} = \frac{\nabla y_n}{h}$$

Newton's formula for Backward Interpolation

- Substituting x_2 in equation (1) we get,

$$y_{n-2} = a_0 + a_1(x_{n-2} - x_n) + a_2(x_{n-2} - x_n)(x_{n-2} - x_{n-1})$$

$$= y_n + \frac{y_n - y_{n-1}}{h}(-2h) + a_2(-2h)(-h)$$

$$\Rightarrow a_2 = \frac{y_n - 2y_{n-1} + y_{n-2}}{2h^2} = \frac{\nabla^2 y_n}{2h^2}$$

Newton's formula for Backward Interpolation

- Substituting x_{n-3} in equation (1) we get,

$$y_{n-3} = a_0 + a_1(x_{n-3} - x_n) + a_2(x_{n-3} - x_n)(x_{n-3} - x_{n-1})$$
$$+ a_3(x_{n-3} - x_n)(x_{n-3} - x_{n-1})(x_{n-3} - x_{n-2})$$

$$a_3 = \frac{\nabla^3 y_n}{3!\,h^3}$$ (Class Work)

- Similarly,

$$a_4 = \frac{\nabla^4 y_n}{4!\,h^6}, \ldots, a_n = \frac{\nabla^n y_n}{n!\,h^n}$$

Newton's formula for Backward Interpolation

Substituting in equation (1) the values a_0, a_1, a_2,...a_n , we have,

$$\phi(x) = y_n + \frac{\nabla y_n}{h}(x - x_n) + \frac{\nabla^2 y_n}{2h^2}(x - x_n)(x - x_{n-1}) + \frac{\nabla^3 y_n}{3!h^3}(x - x_n)(x - x_{n-1})(x - x_{n-2}) +$$

$$\ldots\ldots + \frac{\nabla^n y_n}{n!h^n}(x - x_n)(x - x_{n-1})\ldots.(x - x_1) \qquad (2)$$

- This is Newton's formula for backward interpolation, written in term of x.
- This formula can be simplified by a change of variable.

Newton's formula for Backward Interpolation

Now, we can rewrite eq(2) in the following equivalent form

$$\phi(x) = y_n + \nabla y_n\left(\frac{x - x_n}{h}\right) + \frac{\nabla^2 y_n}{2}\left(\frac{x - x_n}{h}\right)\left(\frac{x - x_{n-1}}{h}\right)$$
$$+ \frac{\nabla^3 y_n}{3!}\left(\frac{x - x_n}{h}\right)\left(\frac{x - x_{n-1}}{h}\right)\left(\frac{x - x_{n-2}}{h}\right) +$$
$$\ldots\ldots + \frac{\nabla^n y_n}{n!}\left(\frac{x - x_n}{h}\right)\left(\frac{x - x_{n-1}}{h}\right)\left(\frac{x - x_{n-2}}{h}\right)\left(\frac{x - x_{n-3}}{h}\right)\ldots\left(\frac{x - x_1}{h}\right) \qquad (3)$$

Newton's formula for Backward Interpolation

- Now, put the following in equation (3)

$$\frac{x - x_n}{h} = u, \quad or \qquad x = x_n + hu$$

- Then, since $x_{n-1} = x_n - h$, $x_{n-2} = x_n - 2h$, etc. we have

$$\frac{x - x_{n-1}}{h} = \frac{x - (x_n - h)}{h} = \frac{x - x_n}{h} + \frac{h}{h} = u + 1$$

- Similarly,

$$\frac{x - x_{n-2}}{h} = \frac{x - (x_n - 2h)}{h} = \frac{x - x_n}{h} + \frac{2h}{h} = u + 2,$$

....

$$\frac{x - x_1}{h} = \frac{x - [x_n - (n-1)h]}{h} = \frac{x - x_n}{h} + \frac{(n-1)h}{h} = u + n - 1$$

Newton's formula for Backward Interpolation

- Substituting the values of $(x - x_n)/h$, $(x - x_{n-1})/h$ etc. in equation (3)

$$\phi(x) = \phi(x_n + hu) = g(u)$$

$$= y_n + u\nabla y_n + \frac{u(u+1)}{2!}\nabla^2 y_n + \frac{u(u+1)(u+2)}{3!}\nabla^3 y_n +$$

$$\ldots + \frac{u(u+1)(u+2)(u+3)\ldots.(u+n-1)}{n!}\nabla^n y_n \qquad (2)$$

- This is the form in which Newton's formula for backward interpolation is usually written.

- The reason for the name "backward" interpolation formula since the formula contains values of the tabulated function from y_n onward to the left (backward from y_n) and none to the right of this value.

- Because of this fact this formula is used mainly for interpolating the values of y near the end of a set of tabular values.

Problem

The table below gives the values of tan(x) for $0.10 \le x \le 0.30$.

Find tan (0.26)

x	0.10	0.15	0.20	0.25	0.30
$y = \tan x$	0.1003	0.1511	0.2027	0.2553	0.3093

Answer is 0.265952

Problem

The population of a town is given below for a range of years. Estimate the population for the year 1925.

Year : x	1891	1901	1911	1921	1931
Population: y (in thousands)	46	66	81	93	101

Answer: 98.837 Thousands

Extrapolation

- If the n^{th} differences of a tabulated function are constant when the values of the independent variable are taken in arithmetic progression, the function is a polynomial of degree n.

- The process of finding the value of y for some value of x outside the given range is called extrapolation.

- If a tabulated value is a polynomial, then interpolation and extrapolation would give exact values.

- Newton's forward difference formula is used to extrapolate values to the right of y_n.

- Newton's Backward difference formula is used to extrapolate values to the left of y_0.

Problem

The table below gives the values of tan x for $0.10 \leq x \leq 0.30$.

Find tan (0.05) and tan (0.50)

x	0.10	0.15	0.20	0.25	0.30
$y = \tan x$	0.1003	0.1511	0.2027	0.2553	0.3093

Solution tan (0.5) = 0.050048 and tan (5.0) =0.545836

Example

The table below gives the values of y are consecutive terms of a series of which the number 21.6 is the 6^{th} term.

Find the first and tenth terms of the series.

x	3	4	5	6	7	8	9
y	2.7	6.4	12.5	21.6	34.3	51.2	72.9

Solution

The difference table is shown in the next page

Example

x	y	Δ	Δ^2	Δ^3	Δ^4
3	**2.7**				
		3.7			
4	**6.4**		**2.4**		
		6.1		**0.6**	
5	**12.5**		**3.0**		**0**
		9.1		**0.6**	
6	**21.6**		**3.6**		**0**
		12.7		**0.6**	
7	**34.3**		**4.2**		**0**
		16.9		**0.6**	
8	**51.2**		**4.8**		
		21.7			
9	**72.9**				

- From the difference table, it will be seen that the third differences are constant
- Hence, the tabulated function represents a polynomial of the third degree.

Solution

$$y(1) = 0.1$$
$$y(10) = 100$$