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"THEORY OF EQUATIONS" Exam

Example – 4 XXV.

4 (3)

Theorem :- Every equation of the nth degree has n roots, and no more.

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or

state and prove the fundamental theorem of Algebra.

solution: - statement :- Every equation of the nth degree has at least one root.

Proof :- suppose,

$p_0x^n + p_1x^{n-1} + p_2x^{n-2} + \dots + p_n = 0$ be the nth degree equation.

Let the equation denoted by $f(x) = 0$,

where,

$$f(x) = p_0x^n + p_1x^{n-1} + p_2x^{n-2} + \dots + p_n$$

The equation $f(x)=0$ has a root, real or imaginary.

Let this root be a_1, then $f(x)$ is divisible by $x-a_1$, so that

$$f(x) = (x-a_1)\,\varphi_1(x) \longrightarrow (a)$$

where, $\varphi_1(x)$ is a rotational function of degree $(n-1)$.

Again the equation $\varphi_1(x)=0$ has a root, real or imaginary. Let this root be a_2. then $\varphi_1(x)$ is divisible by $x-a_2$, so that

$$\varphi_1(x) = (x-a_2)\,\varphi_2(x) \longrightarrow (b)$$

where, $\varphi_2(x)$ is a rotational function of degree $(n-2)$

putting the value of $\varphi_1(x)$ in (a), we get

$$f(x) = (x-a_1)(x-a_2)\,\varphi_2(x)$$

Proceeding in this way, we obtain

$$f(x) = (x-a_1)(x-a_2) \dots (x-a_n)$$

Hence the equation $f(x)=0$ has n roots. since $f(x)$ vanishes when x has any of the values $a_1, a_2, a_3, \dots, a_n$.

Investigate the relations between the roots and the coefficients in any equation.

Solution: Let us denote the equation by

$x^n + p_1 x^{n-1} + p_2 x^{n-2} + \cdots + p_{n-1} x + p_n = 0 \longrightarrow (1)$

Let, $a_1, a_2, \cdots, a_n$ be the roots of the above equation

$\therefore x^n + p_1 x^{n-1} + p_2 x^{n-2} + \cdots + p_{n-1} x + p_n = (x-a_1)(x-a_2)\cdots(x-a_n) \longrightarrow (2)$

Now,

$(x-a_1)(x-a_2)\cdots(x-a_n) = \{x^2 - x(a_1+a_2) + a_1 a_2\}(x-a_3)\cdots(x-a_n)$

$= \{x^3 - x^2 a_3 - x^2(a_1+a_2) + x a_3 (a_1+a_2) + a_1 a_2 x - a_1 a_2 a_3\}(x-a_4)\cdots(x-a_n)$

$= \{x^3 - x^2(a_1+a_2+a_3) + x(a_1a_2 + a_2a_3 + a_3a_1) - a_1a_2a_3\}(x-a_4)\cdots(x-a_n)$

$= \{x^3 - x^2 \Sigma a_1 + x \Sigma a_1 a_2 - a_1 a_2 a_3\}(x-a_4)\cdots(x-a_n)$

Proceeding in this way we get

$(x-a_1)(x-a_2)\cdots(x-a_n) = x^n - x^{n-1}\Sigma a_1 + x^{n-2}\Sigma a_1 a_2 - x^{n-3}\Sigma a_1 a_2 a_3 + \cdots + (-1)^n a_1 a_2 a_3 \cdots a_n$

from (2),

$x^n + p_1 x^{n-1} + p_2 x^{n-2} + \cdots + p_{n-1}x + p_n = x^n - x^{n-1}\Sigma a_1 + x^{n-2}\Sigma a_1 a_2 - x^{n-3}\Sigma a_1 a_2 a_3 + \cdots + (-1)^n a_1 a_2 \cdots a_n$

Equating the coefficient of like powers of x from both sides we get

$\Sigma a_1 = -p_1$; $\Sigma a_1 a_2 = p_2$; $\Sigma a_1 a_2 a_3 = -p_3$

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$a_1 a_2 a_3 \cdots a_n = (-1)^n p_n$ which is the required relation.

THEORY OF EQUATIONS.

Examples. XXXV. 6

Solve the equation:

$x^4 - 16x^3 + 86x^2 - 176x + 105 = 0$ two roots being 1 and 7.

Solⁿ: The given equation

$x^4 - 16x^3 + 86x^2 - 176x + 105 = 0$

and given roots 1 and 7.

Let the another roots be α and β

sum of the roots $\alpha + \beta + 1 + 7 = 16$

$\Rightarrow \alpha + \beta = 8$ —— (1)

product of the roots $7\alpha\beta = 105$

$\Rightarrow \alpha\beta = 15$ —— (2)

$\therefore (\alpha-\beta)^2 = (\alpha+\beta)^2 - 4\alpha\beta$

$\therefore \alpha - \beta = 2$ —— (3)

Adding (1) and (3) we get $2\alpha = 10$

$\therefore \alpha = 5$.

and subtracting (3) from (1) we get

$\beta = 3$

Thus the roots are 1, 7, 3, 5. or 1.3.5.7

(Ans)

$4x^3 + 16x^2 - 9x - 36 = 0$, the sum of two of the roots being zero.

Solⁿ: The given equation

$4x^3 + 16x^2 - 9x - 36 = 0$

Let $a, -a$ and b be the roots of the given equation.

sum of the roots

$$a-a+b = \frac{-16}{4} = -4$$

$$\therefore b = -4.$$

sum of the product of the roots taken two at a time

$$-a^2-ab+ab = \frac{-9}{4}$$

$$\Rightarrow a^2 = 9/4$$

$$\therefore a = \frac{3}{2} \; [+ve]$$

product of the roots

$$-a^2b = \frac{36}{4}$$

$$\Rightarrow -a^2 = \frac{9}{b} \Rightarrow -a^2 = \frac{9}{-4}$$

$$\Rightarrow a = \pm 3/2$$

Hence the roots of the given equation are $3/2, -3/2, -4$

(Ans)

7. $4x^3+20x^2-23x+6=0$, two roots being equal

solution: The given equation

$$4x^3+20x^2-23x+6=0 \quad \text{—} (1)$$

Let a, a and b be the roots of (1)

Now, sum of the roots

$$a+a+b = \frac{-20}{4}$$

$$\therefore b = -5-2a \quad \text{—} (2)$$

sum of the product of the roots taken two at a time

$$a^2+ab+ab = \frac{-23}{4}$$

$$\Rightarrow a^2+2ab = -23/4 \quad \text{—} (3)$$

Product of the roots $a^2b = \frac{-6}{4} = -3/2$

putting $b = -5-2a$ in (3) we get

$a^2 + 2a(-5-2a) = -23/4 \Rightarrow 12a^2 + 40a \cdot 23$; $\therefore a = \frac{-40 \pm \sqrt{(40)^2 - 4 \cdot 12 \cdot 23}}{2 \cdot 12}$ (taking +ve)

$\Rightarrow a = 1/2$ and $a \neq -23/6$; $\frac{-40+46}{24} = \frac{12}{24} = \frac{1}{2}$.

From (2) when $a = \frac{1}{2}$, $b = -6$ or, $12a^2 + 46a - 6a - 23 = 0$

when $a = \frac{-23}{6}$, $b = 2/3$ $\Rightarrow 2a(6a + 23) - 1(6a+23) = 0$

$\Rightarrow (6a+23) \neq 0$ $(2a-1) = 0$

But $\frac{-23}{6}$ and $2/3$ do not satisfy the 3rd equation $\therefore a \neq \frac{-23}{6}$ $\therefore a = \frac{1}{2}$

$\therefore$ The roots are $\frac{1}{2}, \frac{1}{2}, -6$ (Ans).

3. $3x^3 - 26x^2 + 52x - 24 = 0$, the roots being in geometrical Progression.

Solution: The given equation

$$3x^3 - 26x^2 + 52x - 24 = 0 \quad \text{—(1)}$$

Let $\frac{a}{r}$, a and ar be the roots of (1)

Now sum of the roots

$$a + ar + \frac{a}{r} = \frac{26}{3}$$

$$\Rightarrow a\left(1 + r + \frac{1}{r}\right) = \frac{26}{3} \quad \text{—(2)}$$

Sum of the product of the roots taken two at a time

$$\frac{a^2}{r} + a^2r + a^2 = \frac{52}{3}$$

$$\Rightarrow a^2\left(\frac{1}{r} + r + 1\right) = 52/3 \quad \text{—— (3)}$$

product of the roots

$$\frac{a}{r} \cdot a \cdot ar = \frac{24}{3} = 8$$

$$\therefore a = 2$$

putting $x = 2$ in ⑥ we get,

$$\Rightarrow \left(1 + r + \frac{1}{r}\right) = 26/3$$

$$\Rightarrow \frac{1 + r + r^2}{r} = 13/3$$

$$\Rightarrow 3r^2 - 10r + 3 = 0$$

$$r = 3, 1/3$$

Hence the roots of ① are $2/3, 2, 6$.

(Ans)

9. $2x^3 - x^2 - 22x - 24 = 0$, two of the roots being in the ratio of $3:4$.

solution: The given equation

$$2x^3 - x^2 - 22x - 24 = 0 \longrightarrow ①$$

Let 3α, 4α and β be the roots of ①

sum of the roots

$$3\alpha + 4\alpha + \beta = \frac{1}{2}$$

$$\Rightarrow \beta = \frac{1}{2} - 7\alpha \longrightarrow ②$$

sum of the product of the roots taken two at a time.

$$12\alpha^2 + 4\alpha\beta + 3\alpha\beta = \frac{-22}{2} = -11$$

$$\Rightarrow 12\alpha^2 + 7\alpha\left(\frac{1}{2} - 7\alpha\right) = -11 \Rightarrow 24\alpha^2 + 7\alpha - 98\alpha^2 = -22$$

$$\Rightarrow 74\alpha^2 - 7\alpha - 22 = 0$$

$$\Rightarrow \alpha = -\frac{1}{2} \text{ and } 37\alpha - 22 \neq 0 \Rightarrow 74\alpha^2 - 44\alpha + 37\alpha - 22 =$$

$$\Rightarrow 2\alpha(37\alpha - 22) + 1(37\alpha$$

$$\Rightarrow (37\alpha - 22) \neq 0, \quad 2\alpha = - \quad \therefore \alpha =$$

$$\text{or } \alpha = \frac{7 \pm \sqrt{6561}}{148} = -\frac{1}{2}$$

From ② $\Rightarrow \beta = \frac{1}{2} + 7/2 = 4$

Thus the roots of ① are $-3/2, -2$ and 4

Ans.

10. $24x^3+46x^2+9x-9=0$; one roots being double another of the roots.

sol^n: The given equation

$$24x^3+46x^2+9x-9=0 \longrightarrow (1)$$

Let $\alpha, 2\alpha$ and β be the roots of (1)

Now, sum of the roots

$$\alpha+2\alpha+\beta = \frac{-46}{24}$$

$$\Rightarrow \beta = -\frac{23}{12} - 3\alpha \longrightarrow (2)$$

sum of the product of the roots taken two at a time

$$2\alpha^2+2\alpha\beta+\alpha\beta = \frac{9}{24}$$

$$\Rightarrow 2\alpha^2+3\alpha\left(-\frac{23}{12}-3\alpha\right) = \frac{9}{24}$$

$$\Rightarrow \quad \alpha = -3/4, \quad 14\alpha+1 \neq 0$$

$\Rightarrow 56\alpha^2+46\alpha+3=0$

$\Rightarrow 56\alpha^2+42\alpha+4\alpha+3=0$

$\Rightarrow 14\alpha(4\alpha+3)+1(4\alpha$

$\Rightarrow (4\alpha+3)(14\alpha+1)=$

$\Rightarrow 4\alpha+3=0$

$\therefore \alpha = -\frac{3}{4}$, 14α

$\therefore \alpha$

From (2), $\beta = \frac{1}{3}$

Thus the roots of (1) are $-3/4, -3/2, 1/3$ (Ans)

11. $8x^4-2x^3-27x^2+6x+9=0$; two of the roots being equal but opposite in sign.

sol^n: The given equation

$$8x^4-2x^3-27x^2+6x+9=0 \longrightarrow (1)$$

Let $\alpha, -\alpha, \beta, \gamma$ be the roots of (1)

sum of the roots,

$$\beta+\gamma = 1/4 \longrightarrow (2)$$

$\alpha, -\alpha, \beta, \gamma$

sum of the product of the roots taken two at a time

$$-\alpha^2 + \alpha\beta - \alpha\beta + \gamma\beta + \alpha\gamma - \alpha\gamma = -27/8$$

$$\Rightarrow \alpha^2 = \frac{27}{8} + \gamma\beta \longrightarrow (3)$$

sum of the product of the roots taken three at a time.

$$-\alpha^2\beta - \alpha^2\gamma + \alpha\beta\gamma - \alpha\beta\gamma = -6/8$$

$$\Rightarrow -\alpha^2(\beta+\gamma) = -3/4$$

$$\therefore -\alpha^2 \cdot \frac{1}{4} = -3/4 \Rightarrow \alpha = \pm\sqrt{3}.$$

From (3) we get,

$$3 - \frac{27}{8} = \gamma\beta$$

$$\Rightarrow \beta\gamma = -3/8$$

$$\gamma - \beta = \left\{(\gamma+\beta)^2 - 4\gamma\beta\right\}^{1/2}$$

$$= \frac{5}{4} \longrightarrow (4)$$

Adding (2) and (4) we get

$$\gamma = 3/4$$

subtracting (2) from (4) we get

$$\beta = -\frac{1}{2}$$

Thus the roots of (1) are $\pm\sqrt{3}, 3/4, -\frac{1}{2}$ (Ans).

12. $54x^3 - 39x^2 - 26x + 16 = 0$; the roots being in geometrical progression.

soln: The given equation

$$54x^3 - 39x^2 - 26x + 16 = 0 \longrightarrow (1)$$

Let $\frac{a}{r}, a, ar$ be the roots of (1) sum of the roots

$a\left(1+r+\frac{1}{r}\right) = \frac{39}{54} = \frac{13}{12} \rightarrow (2)$

sum of the product of the roots taken two at a time

$a^2\left(1+r+\frac{1}{r}\right) = \frac{-26}{54} = -13/27 \rightarrow (3)$

product of the roots

$\frac{a}{r} \cdot a \cdot ar = -16/54$

$\therefore a = -2/3$

putting the value $a = -2/3$ in (2) we get

$-\frac{2}{3}\left(1+r+\frac{1}{r}\right) = \frac{13}{12}$

$\Rightarrow \frac{1+r+r^2}{r} = -13/12 \Rightarrow 12r^2+25r+12=0$

$\Rightarrow \therefore r = \frac{-25 \pm \sqrt{(25)^2 - 4 \cdot 12 \cdot 12}}{2 \cdot 12}$

$\Rightarrow r = -4/3$ but $r = -3/4$

$= \frac{-25+7}{24}$ (+ve) $= -\frac{18}{24}$

$= -\frac{3}{4}$

thus the roots of (1) are

$1/2, -2/3, 8/9$ (Ans).

(-ve) $r = -\frac{32}{24} = -\frac{4}{3}$

Ques. 07: $32x^3 - 48x^2 + 22x - 3 = 0$ the roots being in arithmatical progression.

soln: The given equation

$32x^3 - 48x^2 + 22x - 3 = 0 \rightarrow (1)$

Let, $a-d$, a and $a+d$ be the roots of (1)

sum of the roots

$a-d+a+a+d = 48/32$

$\therefore a = 1/2$

sum of the product of the roots taken two at a time

$(a-d)a + a(a+d) + (a+d)(a-d) = 22/32 \Rightarrow 3a^2 - d^2 = 11/16$

$\Rightarrow \frac{3}{4} - d^2 = 11/16$ Thus the roots of (1) are

$\therefore d = \pm 1/4$ $-\frac{1}{4}, \frac{1}{2}, 3/4$ (Ans.)

$d = -1/4$

14. $6x^4 - 29x^3 + 40x^2 - 7x - 12 = 0$ the product of two of the roots being 2.

Sol^n: The given equation

$$6x^4 - 29x^3 + 40x^2 - 7x - 12 = 0 \longrightarrow (1)$$

Let a, b, c, d be the roots of (1)

sum of the roots

$$a+b+c+d = 29/6 \longrightarrow (2)$$

sum of the product of the roots taken two at a time

$$ab + ac + ad + bc + bd + cd = \frac{40}{6} = \frac{20}{3} \longrightarrow (3)$$

sum of the product of the the roots taken three at a time,

$$abc + acd + abd + bcd = 7/6 \longrightarrow (4)$$

product of the roots

$$abcd = -2 \longrightarrow (5)$$

Since product of two roots being 2

Let $ab = 2 \longrightarrow (6)$

From (5) and (6) we get

$$cd = -1 \longrightarrow (7)$$

using (6) and (7) in (4) we obtain

$$2c - b - a + 2d = 7/6 \longrightarrow (8)$$

Adding (2) and (8) we get

$$3(c+d) = 6 \Rightarrow c+d = 2 \longrightarrow (9)$$

From ② we get

$a+b = \frac{-??}{6} - 2 = 17/6$

$a-b = 1/6$ and $c-d = 2\sqrt{2}$

$\therefore \; a = 3/2, \; b = 4/3$

$c = 1+\sqrt{2}, \; d = 1-\sqrt{2}$

Hence the roots of ④ are $3/2, 4/3, 1 \pm \sqrt{2}$ (Ans).

15. $x^4 - 2x^3 - 21x^2 + 22x + 40 = 0$, the roots being arithmatical progression.

Soln: The given equation

$$x^4 - 2x^3 - 21x^2 + 22x + 40 = 0 \longrightarrow ①$$

Let $a-3d$, $a-d$, $a+d$ and $a+3d$ be the roots of ①

Now sum of the roots

$$a-3d + a-d + a+d + a+3d = 2$$

$$\Rightarrow 4a = 2 \Rightarrow a = \tfrac{1}{2}$$

sum of the product of the roots taken two at a time

$$(a-3d)(a-d) + (a-3d)(a+d) + (a-3d)(a+3d) + (a-d)(a+d) + (a-d)(a+3d) + (a+d)(a+3d) = -21$$

(X) $\Rightarrow a^2 - 4ad + 3d^2 + a^2 - 2ad - 3d^2 + 2a^2 - 10d^2 + a^2 - 2ad - 3d^2 + a^2 + 4ad + 3d^2 = -21$

$\Rightarrow 6a^2 - 10d^2 = -21$

$\Rightarrow 6 \cdot \frac{1}{4} - 10d^2 = -21$

$\therefore d = \pm 3/2$

$\therefore d = 3/2$

Then the roots are

$1/2 - 9/2$; $\frac{1}{2} - 3/2$; $\frac{1}{2} + 9/2$; $1/2 + 3/2$

i.e. $-4; -1, 2, 5$ (Ans).

(16) $27x^4-195x^3+494x^2-520x+192=0$; the roots being in geometrical progression.

sol^n: The given equation

$$27x^4-195x^3+494x^2-520x+192=0 \longrightarrow (1)$$

Let the roots be $\frac{a}{r^3}, \frac{a}{r}, ar, ar^3$

sum of the roots, $\frac{a}{r^3}+\frac{a}{r}+ar+ar^3 = \frac{195}{27} \longrightarrow (2)$

sum of the products of the roots taken two at a time

$$\frac{a}{r^3}\cdot\frac{a}{r}+\frac{a}{r^3}ar+\frac{a}{r^3}\cdot ar^3+\frac{a}{r}ar+\frac{a}{r}ar^3+ar\cdot ar^3=\frac{494}{27}$$

$$\Rightarrow \frac{a^2}{r^4}+\frac{a^2}{r^2}+a^2+a^2+a^2r^2+a^2r^4=\frac{494}{27}$$

$$\Rightarrow a^2\left(2+r^2+r^4+\frac{1}{r^2}+\frac{1}{r^4}\right)=\frac{494}{27}$$ and form.

$$\Rightarrow \left(r^2+\frac{1}{r^2}\right)^2+\left(r^2+\frac{1}{r^2}\right)=\frac{494}{27a^2} \longrightarrow (3)$$

product of the roots

$$\frac{a}{r^3}\cdot\frac{a}{r}\cdot ar\cdot ar^3=\frac{192}{27}$$

$$a^2=8/3 \quad \text{Let } p=r^2+\frac{1}{r^2} \longrightarrow (4)$$

From (3) we get

$$p^2+p=\frac{494}{27\cdot 8/3}$$

$$\Rightarrow p^2+p=\frac{247}{36}$$

$$\Rightarrow 36p^2+36p-247=0$$

$$\Rightarrow (6p)^2+2\cdot 6p\cdot 3+(3)^2-256=0$$

$\Rightarrow (6p+3)^2 = (16)^2$

$\Rightarrow 6p = 16-3 \quad \therefore p = \frac{13}{6}$

putting $p = \frac{13}{6}$ in (4) we get

$r^2 + \frac{1}{r^2} = 13/6$

$\Rightarrow 6r^4 - 13r^2 + 6 = 0$

$\Rightarrow 6r^4 - 9r^2 - 4r^2 + 6 = 0$

$\Rightarrow 3r^2(2r^2-3) - 2(2r^2-3) = 0$

$\Rightarrow (3r^2-2)(2r^2-3) = 0$

$\therefore r^2 = 3/2, \; r^2 = 2/3$

Now $\alpha^2 r^2 = 8/3$.

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$\therefore$ Hence the roots are. $\frac{\sqrt{\frac{8}{3}}}{\left(\sqrt{\frac{3}{2}}\right)^3}, \; \frac{2\sqrt{\frac{8}{3}}}{3} \times \frac{\sqrt{3}}{\sqrt{2}}$

$\frac{\sqrt{\frac{8}{3}}}{\left(\frac{\sqrt{3}}{\sqrt{2}}\right)^3}, \; \frac{\sqrt{\frac{8}{3}}}{\sqrt{\frac{3}{2}}}, \; \sqrt{\frac{8}{3}} \times \sqrt{\frac{3}{2}}$, i.e $\frac{8}{9}$, 2

$\sqrt{\frac{8}{3}} \times \left(\sqrt{\frac{3}{2}}\right)^3$

i.e $\frac{8}{9}, \frac{4}{3}, 2, 3$.

17. $18x^3 + 81x^2 + 121x + 60 = 0$; one roots being half the sum of the other two.

soln: The given equation

$18x^3 + 81x^2 + 121x + 60 = 0 \longrightarrow (1)$

Let α, β and γ be the roots of (1)

According to the condition

$\alpha = \frac{\beta+\gamma}{2}$

$\Rightarrow 2\alpha = \beta + \gamma$

$\Rightarrow 3\alpha = \alpha + \beta + \gamma$ — (2)

sum of the roots $\alpha+\beta+\gamma = \frac{-81}{18}$

$\Rightarrow 3\alpha = \frac{-81}{18}$

$\Rightarrow \alpha = -3/2$

$\Rightarrow \beta+\gamma = 2\alpha = -3 \rightarrow (3)$

product of the roots

$\alpha\beta\gamma = -60/18$

$\Rightarrow \beta\gamma = \frac{-60}{18} \times -2/3 = 20/9$

$\therefore \beta-\gamma = \{(\beta+\gamma)^2 - 4\beta\gamma\}^{1/2} = \frac{1}{3} \rightarrow (4)$

Adding (3) and (4) we get,

$2\beta = -3+\frac{1}{3}, \ \beta = -4/3.$

$\therefore \beta+\gamma = -3$

$\Rightarrow \gamma = -5/3$

Hence the roots of (1) are

$-3/2, -4/3, -5/3$ Ans).

18. If a, b, c are the roots of the equation $x^3 - px^2 + qx - r = 0$

find the value of (i) $\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}$; (ii) $\frac{1}{b^2c^2}+\frac{1}{c^2a^2}+\frac{1}{a^2b^2}$

soln: The given equation $x^3 - px^2 + qx - r = 0 \rightarrow (1)$

Hence a, b, c are the roots of (1)

sum of the roots $a+b+c = p$

sum of the product of the roots taken two at a time

$ab+bc+ac = q$

product of the roots $abc = r$.

(i) $\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2} = \frac{b^2c^2+a^2c^2+a^2b^2}{a^2b^2c^2}$

$= \frac{(ab+bc+ca)^2-2abc(a+b+c)}{(abc)^2} = \frac{q^2-2pr}{r^2}$ (Ans)

(ii) $\frac{1}{b^2c^2}+\frac{1}{c^2a^2}+\frac{1}{a^2b^2} = \frac{a^2+b^2+c^2}{a^2b^2c^2}$

$= \frac{(a+b+c)^2-2(ab+bc+ca)}{(abc)^2} = \frac{p^2-2q}{r^2}$ (Ans).

19) If a, b, c are the roots of $x^3+qx+r=0$, find the value of

(i) $(b-c)^2+(c-a)^2+(a-b)^2$

(ii) $(b+c)^{-1}+(c+a)^{-1}+(a+b)^{-1}$

soln: The given equation

$$x^3+qx+r=0 \longrightarrow (1)$$

Here, the roots of (1) are a, b, c

sum of the roots. $a+b+c=0$

Sum of the product of the roots taken two at a time

$ab+bc+ca=q$

Product of the roots $abc=-r$.

(i) $(b-c)^2+(c-a)^2+(a-b)^2$

$= 2(a^2+b^2+c^2)-2(ab+bc+ca)$

$= 2(a+b+c)^2-4(ab+bc+ca)-2(ab+bc+ca)$

$= 2(a+b+c)^2-6(ab+bc+ca)$

$= 2.0-6q = -6q$ (Ans).

(11) $(b+c)^{-1}+(c+a)^{-1}+(a+b)^{-1}$

$= \frac{1}{b+c}+\frac{1}{c+a}+\frac{1}{a+b} =$

$= \frac{(a+b)(c+a)+(a+b)(b+c)+(b+c)(c+a)}{(a+b)(b+c)(c+a)}$

$= \frac{a^2+b^2+c^2+3(ab+bc+ca)}{a^2b+ab^2+abc+ac^2+a^2c+abc+b^2c+bc^2+abc-abc}$

$= \frac{(a+b+c)^2-2(ab+bc+ca)+3(ab+bc+ca)}{ab(a+b+c)+ac(a+b+c)-abc+bc(a+b+c)}$

$= \frac{(0)^2+q}{0(q)+r} = \frac{q}{r}$ (Ans).

$\because a+b+c=0$

$a+b = -c$

$-\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$

$= -\left(\frac{ab+bc+ca}{abc}\right)$

$= -(q/r) = q/r$

Q. Find the sum of the ~~equations~~ square and of the cubes of the roots of $x^4+qx^2+rx+s=0$

Sol^n: The given equation $x^4+qx^2+rx+s=0$

Let the roots of the given equation be a, b, c, d

sum of the roots $a+b+c+d=0$

sum of the products of the roots taken two at a time

$ab+ac+ad+bc+bd+cd=q$

sum of the products of the roots taken three at a time.

$abc+abd+acd+bcd=-r$.

product of the roots $abcd=s$

Now the sum of the squares of the roots

$a^2+b^2+c^2+d^2$

$=(a+b)^2-2ab+(c+d)^2-2cd$

$=(a+b)^2+(c+d)^2-2ab-2cd$

$\sum \alpha^2 = \left(\sum \alpha\right)^2 - 2\sum \alpha\beta$

②

$= (a+b+c+d)^2 - 2(a+b)(c+d) - 2ab - 2cd$

$= (a+b+c+d)^2 - 2(ac+ad+ab+bc+bd+cd)$

$= 0 - 2.q = -2q$ (Ans).

The sum of the cubes of the roots

$a^3+b^3+c^3+d^3$

$= (a+b)^3 - 3(a+b)ab + (c+d)^3 - 3cd(c+d)$

$= (a+b)^3 + (c+d)^3 - 3ab(a+b) - 3cd(c+d)$

$= (a+b+c+d)^3 - 3(a+b)(c+d)(a+b+c+d) - 3ab(a+b) - 3cd(c+d)$

$= 0 - 3(a+b)(c+d).0 - 3[ab(a+b) + cd(c+d)]$

$= -3[ab\{-(c+d)\} + cd\{-(a+b)\}]$

$= -3(-abc - abd - acd - bcd)$

$= 3(-r) = -3r$ Ans.

Q. Find the sum of the fourth powers of the roots of $x^3+qx+r=0$

solⁿ: Given that $x^3+qx+r=0$

Let the roots of the given equation be a, b, c

sum of the roots $a+b+c=0$

sum of the roots taken two at a time

$ab+bc+ac = q$

product of the roots, $abc = -r$.

sum of the fourth powers of the roots

$a^4+b^4+c^4 = (a^2)^2 + (b^2)^2 + (c^2)^2$

$= (a^2+b^2)^2 - 2a^2b^2 + (c^2)^2$

$= (a^2+b^2 ~~+c^2)^2 - 2(a^2+b^2)c^2 - 2~~$

$= (a^2+b^2+c^2)^2 - 2a^2b^2 - 2b^2c^2 - 2a^2c^2$

$= \{(a+b+c)^2 - 2(ab+bc+ca)\}^2 - 2\{a^2b^2+b^2c^2+c^2a^2\}$

$= 4q^2 - 2\{(ab+bc+ca)^2 - 2abc(a+b+c)\}$

$= 4q^2 - 2q^2 = 2q^2$

$= (a^2+b^2)^2 + (c^2)^2 - 2a^2b^2$

$= (a^2+b^2+c^2)^2 - 2(a^2+b^2)c^2 - 2a^2b^2$

$= \{(a+b+c)^2 - 2(ab+bc+ca)\}^2 - 2(a^2c^2+b^2c^2+a^2b^2)$

$= \{(0-2q)\}^2 - 2\{(ab+bc+ca)^2 - 2(abc^2+ab^2c+a^2bc)\}$

$= 4q^2 - 2\{(ab+bc+ca)^2 - 2abc(a+b+c)\}$

$= 4q^2 - 2\{q^2 - 2(-r)\cdot 0\}$

$= 4q^2 - 2q^2$

$= 2q^2$ (Ans)

Note: Three roots are in G.P $\rightarrow \alpha, \alpha/\beta, \alpha\beta$

" " " " A.P $\rightarrow \alpha-\beta, \alpha, \alpha+\beta$

4th roots are in G.P $\rightarrow \alpha/\beta^3, \alpha/\beta, \alpha\beta, \alpha\beta^3$

" " " " A.P $\rightarrow \alpha-3\beta, \alpha-\beta, \alpha+\beta, \alpha+3\beta$.

Newtonian formula

$$ax^4+bx^3+cx^2+dx+c=0$$

$P_1 \quad P_2 \quad P_3 \quad P_4$

$$-S_1+P_1=0$$

$$S_2+S_1P_1+2P_2=0$$

$$S_3+S_2P_1+S_1P_2+3P_3=0$$

$$S_4+S_3P_1+S_2P_2+S_1P_3+4P_4=0$$

$S_1=a+b+c$

$S_2=a^2+b^2+c^2$

$S_3=a^3+b^3+c^3$

$S_4=a^4+b^4+c^4$

20. Find the sum of the squares and of the cubes of the roots of $x^4+qx^2+rx+s=0$

Solution: Given the equation

$$x^4+qx^2+rx+s=0$$

$$x^4+0.x^3+qx^2+rx+s=0$$

where, $P_1=0$, $P_2=q$, $P_3=r$, $P_4=s$

$$S_1+P_1=0$$

$$\Rightarrow S_1+0=0 \Rightarrow S_1=0$$

and $S_2+S_1P_1+2P_2=0$

$$S_2+0+2.q=0$$

$$\therefore S_2=-2q$$

Also, $S_3+S_2P_1+S_1P_2+3P_3=0$

$$S_3+0+0+3r=0$$

$$\Rightarrow S_3=-3r$$

Hence the sum of
The square and the cubes roots are $-2q$, $-3r$.

Another method // synthetic division :-

Given that, $x^4+qx^2+rx+s=0$

$$\therefore f(x)=x^4+qx^2+rx+s$$

$$f'(x)=4x^3+2qx+r$$

Now by synthetic division we get

1	4	0	2q	r	-4s		
-0		0	-4q	-4r	0	0	
-q			0	0			
-r				0	$2q^2+4r$	2qr	2qs
-s							
	4	0	-2q	-3r			
		S_1	S_2	S_3			

Hence the sum of the squares and of the cubes of the roots are $-2q$ and $-3r$.

Find the sum of the fourth powers of the roots of $x^3+qx+r=0$

solution: Given equation, $x^3+qx+r=0$

$$\therefore x^3+0.x^2+qx+r=0$$

$P_1 \quad P_2 \quad P_3$

where, $P_1=0$, $P_2=q$, $P_3=r$

$$\therefore S_1+P_1=0$$

$$\Rightarrow S_1+0=0 \Rightarrow S_1=0$$

and $S_2+S_1P_1+2P_2=0$

$$S_2+0+2q=0$$

$$\Rightarrow S_2=-2q$$

and $S_3+S_2P_1+S_1P_2+3P_3=0$

$$\Rightarrow S_3+0+0+3r=0 \Rightarrow S_3=-3r$$

and $S_4+S_3P_1+S_2P_2+S_1P_3+4P_4=0$

$$S_4+0+(-2q)q+0+4.0=0$$

$$\Rightarrow S_4=2q^2$$

Hence the sum of the fourth powers of the roots is $2q^2$.

Alternative method :

Given equ. $f(x) = x^3 + qx + r = 0$

$f'(x) = 3x^2 + q$

$$
\begin{array}{c|cccccc}
1 & 3 & 0 & q & & & \\
0 & & 0 & -3q & -3r & & \\
-q & & & 0 & 0 & 0 & \\
-r & & & & 0 & 2q^2 & 2qr \\
 & & & & & 0 & 3qr & 3r^2 \\
\hline
 & 3 & 0 & -2q & -3r & 2q^2 & \\
 & & s_1 & s_2 & s_3 & s_4 &
\end{array}
$$

Hence the sum of the fourth powers of the roots is $2q^2$.

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