42 Filter networks

At the end of this chapter you should be able to:

- appreciate the purpose of a filter network
- understand basic types of filter sections, i.e., low-pass, high-pass, band-pass and band-stop filters
- understand characteristic impedance and attenuation of filter sections
- understand low and high pass ladder networks
- design a low and high pass filter section
- calculate propagation coefficient and time delay in filter sections
- understand and design '*m*-derived' filter sections
- understand and design practical composite filters

42.1 Introduction

A **filter** is a network designed to pass signals having frequencies within certain bands (called **passbands**) with little attenuation, but greatly attenuates signals within other bands (called **attenuation bands** or **stopbands**).

As explained in the previous chapter, an attenuator network pad is composed of resistances only, the attenuation resulting being constant and independant of frequency. However, a filter is frequency sensitive and is thus composed of reactive elements. Since certain frequencies are to be passed with minimal loss, ideally the inductors and capacitors need to be pure components since the presence of resistance results in some attenuation at all frequencies.

Between the pass band of a filter, where ideally the attenuation is zero, and the attenuation band, where ideally the attenuation is infinite, is the **cut-off frequency**, this being the frequency at which the attenuation changes from zero to some finite value.

A filter network containing no source of power is termed **passive**, and one containing one or more power sources is known as an **active** filter network.

The filters considered in this chapter are symmetrical unbalanced T and π sections, the reactances used being considered as ideal.

Filters are used for a variety of purposes in nearly every type of electronic communications and control equipment. The bandwidths of filters used in communications systems vary from a fraction of a hertz to many megahertz, depending on the application.

42.2 Basic types of filter sections

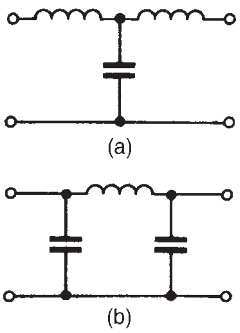

Figure 42.1

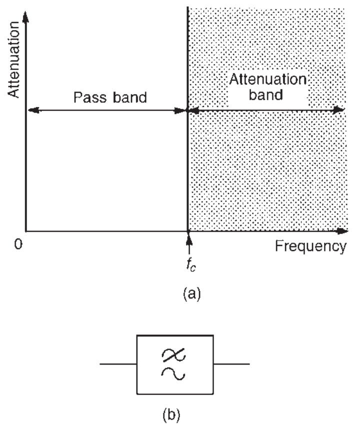

Figure 42.2

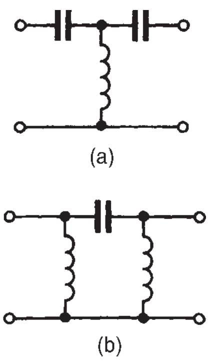

Figure 42.3

(a) Low-pass filters

Figure 42.1 shows simple unbalanced T and π section filters using series inductors and shunt capacitors. If either section is connected into a network and a continuously increasing frequency is applied, each would have a frequency-attenuation characteristic as shown in Figure 42.2(a). This is an ideal characteristic and assumes pure reactive elements. All frequencies are seen to be passed from zero up to a certain value without attenuation, this value being shown as f_c, the cut-off frequency; all values of frequency above f_c are attenuated. It is for this reason that the networks shown in Figures 42.1(a) and (b) are known as **low-pass filters**. The electrical circuit diagram symbol for a low-pass filter is shown in Figure 42.2(b).

Summarizing, a low-pass filter is one designed to pass signals at frequencies below a specified cut-off frequency.

When rectifiers are used to produce the d.c. supplies of electronic systems, a large ripple introduces undesirable noise and may even mask the effect of the signal voltage. Low-pass filters are added to smooth the output voltage waveform, this being one of the most common applications of filters in electrical circuits.

Filters are employed to isolate various sections of a complete system and thus to prevent undesired interactions. For example, the insertion of low-pass decoupling filters between each of several amplifier stages and a common power supply reduces interaction due to the common power supply impedance.

(b) High-pass filters

Figure 42.3 shows simple unbalanced T and π section filters using series capacitors and shunt inductors. If either section is connected into a network and a continuously increasing frequency is applied, each would have a frequency-attenuation characteristic as shown in Figure 42.4(a).

Once again this is an ideal characteristic assuming pure reactive elements. All frequencies below the cut-off frequency f_c are seen to be attenuated and all frequencies above f_c are passed without loss. It is for this reason that the networks shown in Figures 42.3(a) and (b) are known as **high-pass filters**. The electrical circuit-diagram symbol for a high-pass filter is shown in Figure 42.4(b).

Summarizing, a high-pass filter is one designed to pass signals at frequencies above a specified cut-off frequency.

The characteristics shown in Figures 42.2(a) and 42.4(a) are ideal in that they have assumed that there is no attenuation at all in the pass-bands and infinite attenuation in the attenuation bands. Both of these conditions are impossible to achieve in practice. Due to resistance, mainly in the inductive elements the attenuation in the pass-band will not be zero, and in a practical filter section the attenuation in the attenuation band will have a finite value. Practical characteristics for low-pass and high-pass filters are discussed in Sections 42.5 and 42.6. In addition to the resistive loss there is often an added loss due to mismatching. Ideally when a filter is inserted into a network it is matched to the impedance of that network.

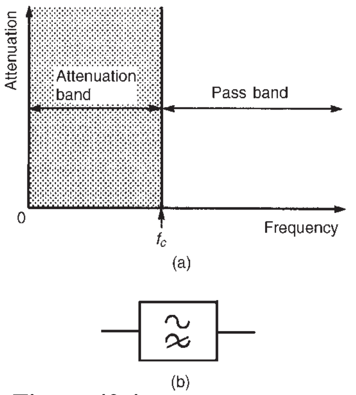

Figure 42.4

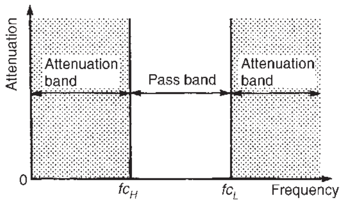

Figure 42.5

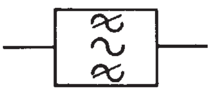

Figure 42.6

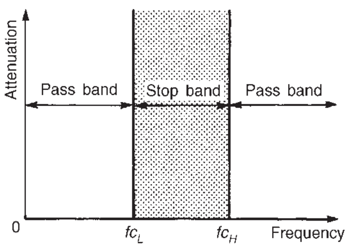

Figure 42.7

However the characteristic impedance of a filter section will vary with frequency and the termination of the section may be an impedance that does not vary with frequency in the same way. To minimize losses due to resistance and mismatching, filters are used under image impedance conditions as far as possible (see Chapter 41).

(c) Band-pass filters

A band-pass filter is one designed to pass signals with frequencies between two specified cut-off frequencies. The characteristic of an ideal band-pass filter is shown in Figure 42.5.

Such a filter may be formed by cascading a high-pass and a low-pass filter. f_{C_H} is the cut-off frequency of the high-pass filter and f_{C_L} is the cut-off frequency of the low-pass filter. As can be seen, $f_{C_L} > f_{C_H}$ for a band-pass filter, the pass-band being given by the difference between these values. The electrical circuit diagram symbol for a band-pass filter is shown in Figure 42.6.

Crystal and ceramic devices are used extensively as band-pass filters. They are common in the intermediate-frequency amplifiers of vhf radios where a precisely-defined bandwidth must be maintained for good performance.

(d) Band-stop filters

A band-stop filter is one designed to pass signals with all frequencies except those between two specified cut-off frequencies. The characteristic of an ideal band-stop filter is shown in Figure 42.7. Such a filter may be formed by connecting a high-pass and a low-pass filter in parallel. As can be seen, for a band-stop filter $f_{C_H} > f_{C_L}$, the stop-band being given by the difference between these values. The electrical circuit diagram symbol for a band-stop filter is shown in Figure 42.8.

Sometimes, as in the case of interference from 50 Hz power lines in an audio system, the exact frequency of a spurious noise signal is known. Usually such interference is from an odd harmonic of 50 Hz, for example, 250 Hz. A sharply tuned band-stop filter, designed to attenuate the 250 Hz noise signal, is used to minimize the effect of the output. A high-pass filter with cut-off frequency greater than 250 Hz would also remove the interference, but some of the lower frequency components of the audio signal would be lost as well.

42.3 The characteristic impedance and the attenuation of filter sections

Nature of the input impedance

Let a symmetrical filter section be terminated in an impedance Z_O. If the input impedance also has a value of Z_O, then Z_O is the characteristic impedance of the section.

Figure 42.9 shows a T section composed of reactive elements X_A and X_B. If the reactances are of opposite kind, then the input impedance of the section, shown as Z_O, when the output port is open or short-circuited

Figure 42.8

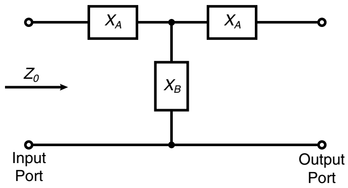

Figure 42.9

can be either inductive or capacitive depending on the frequency of the input signal.

For example, if X_A is inductive, say jX_L, and X_B is capacitive, say, $-jX_C$, then from Figure 42.9,

$$Z_{OC} = jX_L - jX_C = j(X_L - X_C)$$

and
$$Z_{SC} = jX_L + \frac{(jX_L)(-jX_C)}{(jX_L) + (-jX_C)} = jX_L + \frac{(X_L X_C)}{j(X_L - X_C)}$$
$$= jX_L - j\left(\frac{X_L X_C}{X_L - X_C}\right) = \boldsymbol{j\left(X_L - \frac{X_L X_C}{X_L - X_C}\right)}$$

Since $X_L = 2\pi f L$ and $X_C = (1/2\pi f C)$ then Z_{OC} and Z_{SC} can be inductive, (i.e., positive reactance) or capacitive (i.e., negative reactance) depending on the value of frequency, f.

Let the magnitude of the reactance on open-circuit be X_{OC} and the magnitude of the reactance on short-circuit be X_{SC}. Since the filter elements are all purely reactive they may be expressed as jX_{OC} or jX_{SC}, where X_{OC} and X_{SC} are real, being positive or negative in sign. Four combinations of Z_{OC} and Z_{SC} are possible, these being:

(i) $Z_{OC} = +jX_{OC}$ and $Z_{SC} = -jX_{SC}$

(ii) $Z_{OC} = -jX_{OC}$ and $Z_{SC} = +jX_{SC}$

(iii) $Z_{OC} = +jX_{OC}$ and $Z_{SC} = +jX_{SC}$

and (iv) $Z_{OC} = -jX_{OC}$ and $Z_{SC} = -jX_{SC}$

From general circuit theory, input impedance Z_O is given by:

$$Z_O = \sqrt{(Z_{OC} Z_{SC})}$$

Taking either of combinations (i) and (ii) above gives:

$$Z_O = \sqrt{(-j^2 X_{OC} X_{SC})} = \sqrt{(X_{OC} X_{SC})},$$

which is real, thus the input impedance will be **purely resistive**.

Taking either of combinations (iii) and (iv) above gives:

$$Z_O = \sqrt{(j^2 X_{OC} X_{SC})} = +j\sqrt{(X_{OC} X_{SC})},$$

which is imaginary, thus the input impedance will be **purely reactive**.

Thus since the magnitude and nature of Z_{OC} and Z_{SC} depend upon frequency then so also will the magnitude and nature of the input impedance Z_O depend upon frequency.

Characteristic impedance

Figure 42.10 shows a low-pass T section terminated in its characteristic impedance, Z_O.

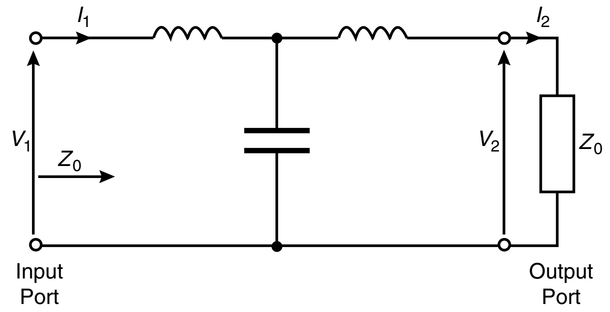

Figure 42.10

From equation (41.2), page 760, the characteristic impedance is given by $Z_O = \sqrt{(Z_{OC}Z_{SC})}$.

The following statements may be demonstrated to be true for any filter:

(a) *The attenuation is zero throughout the frequency range for which the characteristic impedance is purely resistive.*

(b) *The attenuation is finite throughout the frequency range for which the characteristic impedance is purely reactive.*

To demonstrate statement (a) above:

Let the filter shown in Figure 42.10 be operating over a range of frequencies such that Z_O is purely resistive.

From Figure 42.10, $Z_O = \dfrac{V_1}{I_1} = \dfrac{V_2}{I_2}$

Power dissipated in the output termination, $P_2 = V_2 I_2 \cos\phi_2 = V_2 I_2$ (since $\phi_2 = 0$ with a purely resistive load).

Power delivered at the input terminals,

$$P_1 = V_1 I_1 \cos\phi_1 = V_1 I_1 (\text{since } \phi_1 = 0)$$

No power is absorbed by the filter elements since they are purely reactive.

Hence $\quad P_2 = P_1,\ V_2 = V_1$ and $I_2 = I_1$.

Thus if the filter is terminated in Z_O and operating in a frequency range such that Z_O is purely resistive, then all the power delivered to the input is passed to the output and there is therefore no attenuation.

To demonstrate statement (b) above:

Let the filter be operating over a range of frequencies such that Z_O is purely reactive.

Then, from Figure 42.10, $\dfrac{V_1}{I_1} = jZ_O = \dfrac{V_2}{I_2}$.

Thus voltage and current are at 90° to each other which means that the circuit can neither accept nor deliver any active power from the source to the load ($P = VI\cos\phi = VI\cos 90° = VI(0) = 0$). There is therefore infinite attenuation, theoretically. (In practise, the attenuation is finite, for the condition $(V_1/I_1) = (V_2/I_2)$ can hold for $V_2 < V_1$ and $I_2 < I_1$, since the voltage and current are 90° out of phase.)

Statements (a) and (b) above are important because they can be applied to determine the cut-off frequency point of any filter section simply from a knowledge of the nature of Z_O. **In the pass band, Z_O is real, and in the attenuation band, Z_O is imaginary**. The cut-off frequency is therefore at the point on the frequency scale at which Z_O changes from a real quantity to an imaginary one, or vice versa (see Sections 42.5 and 42.6).

42.4 Ladder networks

Low-pass networks

Figure 42.11 shows a low-pass network arranged as a ladder or repetitive network. Such a network may be considered as a number of T or π sections in cascade. In Figure 42.12(a), a T section may be taken from the ladder by removing ABED, producing the low-pass filter section shown in Figure 42.13(a). The ladder has been cut in the centre of each of its inductive elements hence giving $L/2$ as the series arm elements in Figure 42.13(a).

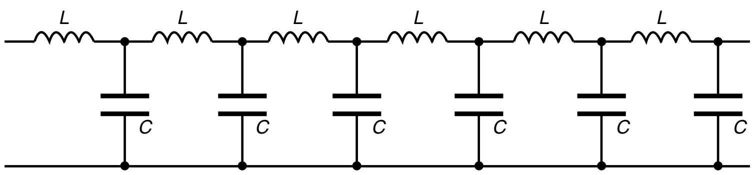

Figure 42.11

Similarly, a π section may be taken from the ladder shown in Figure 42.12(a) by removing FGJH, producing the low-pass filter section

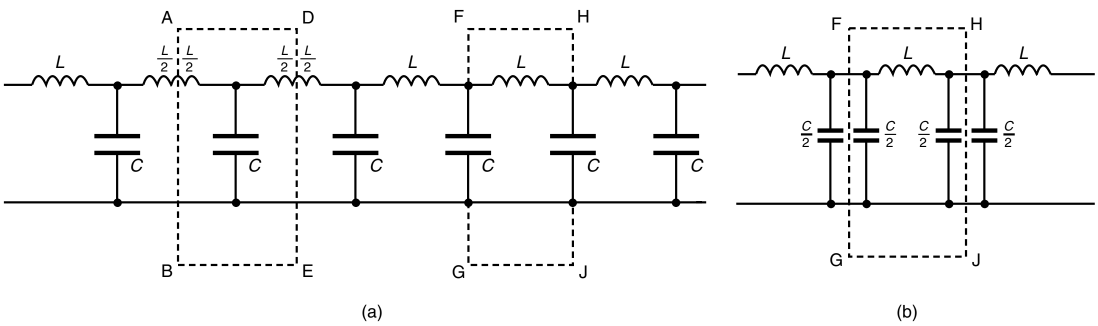

Figure 42.12

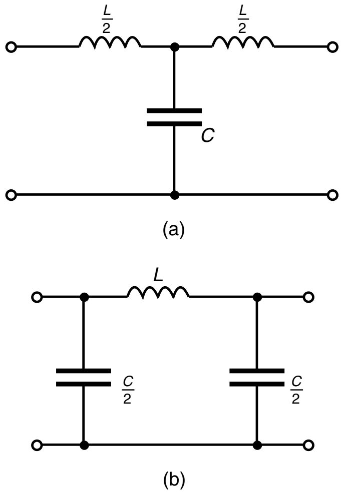

Figure 42.13

shown in Figure 42.13(b). The shunt element C in Figure 42.12(a) may be regarded as two capacitors in parallel, each of value $C/2$ as shown in the part of the ladder redrawn in Figure 42.12(b). (Note that for parallel capacitors, the total capacitance C_T is given by

$$C_T = C_1 + C_2 + \cdots. \text{ In this case } \frac{C}{2} + \frac{C}{2} = C).$$

The ladder network of Figure 42.11 can thus either be considered to be a number of the T networks shown in Figure 42.13(a) connected in cascade, or a number of the π networks shown in Figure 42.13(b) connected in cascade.

It is shown in Section 44.3, page 871, that an infinite transmission line may be reduced to a repetitive low-pass filter network.

High-pass networks

Figure 42.14 shows a high-pass network arranged as a ladder. As above, the repetitive network may be considered as a number of T or π sections in cascade.

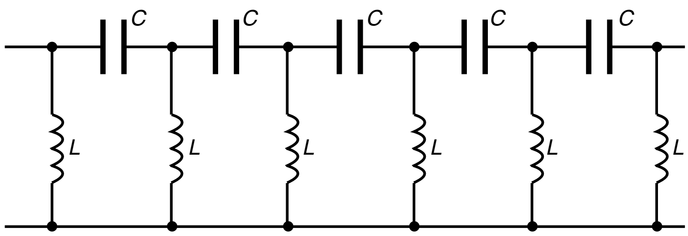

Figure 42.14

In Figure 42.15, a T section may be taken from the ladder by removing ABED, producing the high-pass filter section shown in Figure 42.16(a).

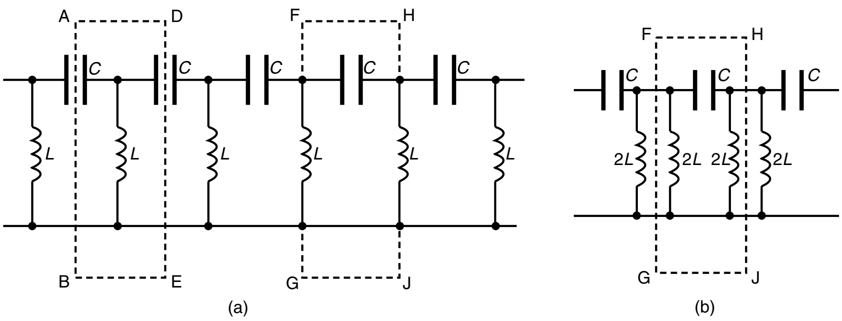

Figure 42.15

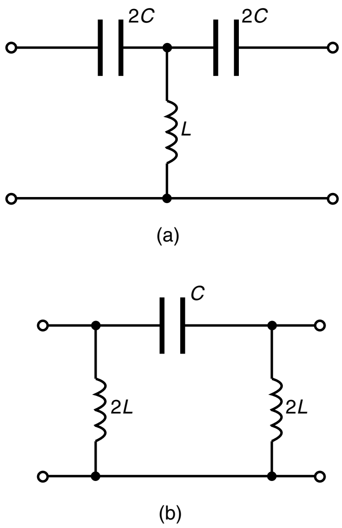

Figure 42.16

Note that the series arm elements are each $2C$. This is because two capacitors each of value $2C$ connected in series gives a total equivalent value of C, (i.e., for series capacitors, the total capacitance C_T is given by

$$\frac{1}{C_T} = \frac{1}{C_1} + \frac{1}{C_2} + \cdots)$$

Similarly, a π section may be taken from the ladder shown in Figure 42.15 by removing FGJH, producing the high-pass filter section shown in Figure 42.16(b). The shunt element L in Figure 42.15(a) may be regarded as two inductors in parallel, each of value $2L$ as shown in the part of the ladder redrawn in Figure 42.15(b). (Note that for parallel inductance, the total inductance L_T is given by

$$\frac{1}{L_T} = \frac{1}{L_1} + \frac{1}{L_2} + \cdots. \text{ In this case, } \frac{1}{2L} + \frac{1}{2L} = \frac{1}{L}.)$$

The ladder network of Figure 42.14 can thus be considered to be either a number of T networks shown in Figure 42.16(a) connected in cascade, or a number of the π networks shown in Figure 42.16(b) connected in cascade.

42.5 Low-pass filter sections

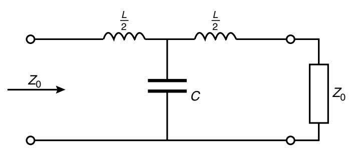

Figure 42.17

(a) The cut-off frequency

From equation (41.1), the characteristic impedance Z_0 for a symmetrical T network is given by: $Z_0 = \sqrt{(Z_A^2 + 2Z_AZ_B)}$. Applying this to the low-pass T section shown in Figure 42.17,

$$Z_A = \frac{j\omega L}{2} \text{ and } Z_B = \frac{1}{j\omega C}$$

Thus $$Z_0 = \sqrt{\left[\frac{j^2\omega^2L^2}{4} + 2\left(\frac{j\omega L}{2}\right)\left(\frac{1}{j\omega C}\right)\right]}$$

$$= \sqrt{\left(\frac{-\omega^2L^2}{4} + \frac{L}{C}\right)}$$

i.e., $$Z_0 = \sqrt{\left(\frac{L}{C} - \frac{\omega^2L^2}{4}\right)} \qquad (42.1)$$

Z_0 will be real if $\dfrac{L}{C} > \dfrac{\omega^2L^2}{4}$

Thus attenuation will commence when $\dfrac{L}{C} = \dfrac{\omega^2L^2}{4}$

i.e., $$\text{when } \omega_c^2 = \frac{4}{LC} \qquad (42.2)$$

where $\omega_c = 2\pi f_c$ and f_c is the cut-off frequency.

Thus $\quad (2\pi f_c)^2 = \dfrac{4}{LC}$

$$2\pi f_c = \sqrt{\left(\frac{4}{LC}\right)} = \frac{2}{\sqrt{(LC)}}$$

and $\quad f_c = \dfrac{2}{2\pi\sqrt{(LC)}} = \dfrac{1}{\pi\sqrt{(LC)}}$

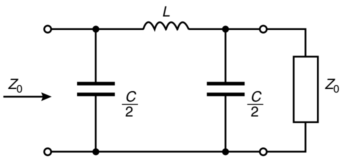

Figure 42.18

i.e., **the cut-off frequency,** $\boxed{\boldsymbol{f_c = \dfrac{1}{\pi\sqrt{(LC)}}}}$ (42.3)

The same equation for the cut-off frequency is obtained for the low-pass π network shown in Figure 42.18 as follows:
From equation (41.3), for a symmetrical π network,

$$Z_0 = \sqrt{\left(\frac{Z_1 Z_2^2}{Z_1 + 2Z_2}\right)}$$

Applying this to Figure 42.18 $Z_1 = j\omega L$ and $Z_2 = \dfrac{1}{j\omega\dfrac{C}{2}} = \dfrac{2}{j\omega C}$

$$\text{Thus} \quad Z_0 = \sqrt{\left\{\frac{(j\omega L)\left(\dfrac{2}{j\omega C}\right)^2}{j\omega L + 2\left(\dfrac{2}{j\omega C}\right)}\right\}} = \sqrt{\left\{\frac{(j\omega L)\left(\dfrac{4}{-\omega^2 C^2}\right)}{j\omega L - j\left(\dfrac{4}{\omega C}\right)}\right\}}$$

$$= \sqrt{\left\{\frac{-j\dfrac{4L}{\omega C^2}}{j\left(\omega L - \dfrac{4}{\omega C}\right)}\right\}} = \sqrt{\left\{\frac{\dfrac{4L}{\omega C^2}}{\dfrac{4}{\omega C} - \omega L}\right\}}$$

$$= \sqrt{\left\{\frac{4L}{\omega C^2\left(\dfrac{4}{\omega C} - \omega L\right)}\right\}} = \sqrt{\left(\frac{4L}{4C - \omega^2 LC^2}\right)}$$

i.e., $\quad Z_0 = \sqrt{\left(\dfrac{1}{\dfrac{C}{L} - \dfrac{\omega^2 C^2}{4}}\right)}$ (42.4)

Z_0 will be real if $\dfrac{C}{L} > \dfrac{\omega^2 C^2}{4}$

Thus attenuation will commence when $\dfrac{C}{L} = \dfrac{\omega^2 C^2}{4}$

i.e., when $\omega_c^2 = \dfrac{4}{LC}$

from which, **cut-off frequency,** $\boxed{\boldsymbol{f_c = \dfrac{1}{\pi\sqrt{(LC)}}}}$ as in equation (42.3)).

(b) Nominal impedance

When the frequency is very low, ω is small and the term $(\omega^2 L^2/4)$ in equation (42.1) (or the term $(\omega^2 C^2/4)$ in equation (42.4)) may be neglected. The characteristic impedance then becomes equal to $\sqrt{(L/C)}$, which is purely resistive. This value of the characteristic impedance is known as the **design impedance** or the **nominal impedance** of the section and is often given the symbol R_0,

i.e., $$\boxed{\boldsymbol{R_0 = \sqrt{\frac{L}{C}}}} \qquad (42.5)$$

> Problem 1. Determine the cut-off frequency and the nominal impedance of each of the low-pass filter sections shown in Figure 42.19.

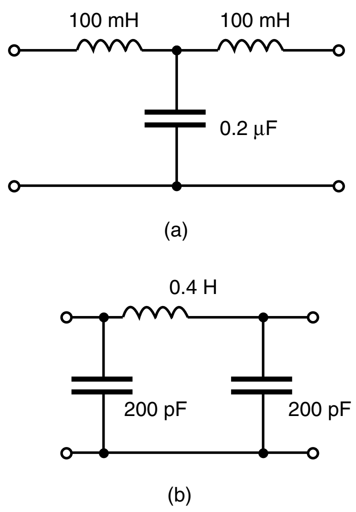

Figure 42.19

(a) Comparing Figure 42.19(a) with the low-pass T section in Figure 42.17 shows that $(L/2) = 100$ mH, i.e., inductance, $L = 200$ mH $= 0.2$ H and capacitance, $C = 0.2$ μF $= 0.2 \times 10^{-6}$ F

From equation (42.3), cut-off frequency,

$$f_c = \frac{1}{\pi\sqrt{(LC)}} = \frac{1}{\pi\sqrt{(0.2 \times 0.2 \times 10^{-6})}} = \frac{10^3}{\pi(0.2)}$$

i.e., $\boldsymbol{f_c}$**= 1592 Hz** or **1.592 kHz**

From equation (42.5), **nominal impedance,**

$$\boldsymbol{R_0} = \sqrt{\left(\frac{L}{C}\right)} = \sqrt{\left(\frac{0.2}{0.2 \times 10^{-6}}\right)} = \mathbf{1000\ \Omega} \text{ or } \mathbf{1\ k\Omega}$$

(b) Comparing Figure 42.19(b) with the low-pass π section shown in Figure 42.18 shows that $(C/2) = 200$ pF, i.e., capacitance, $C = 400$ pF $= 400 \times 10^{-12}$ F and inductance, $L = 0.4$ H,

From equation (42.3), **cut-off frequency**,

$$\boldsymbol{f_c} = \frac{1}{\pi\sqrt{(LC)}} = \frac{1}{\pi\sqrt{(0.4 \times 400 \times 10^{-12})}} = \mathbf{25.16\ kHz}$$

From equation (42.5), **nominal impedance**,

$$\boldsymbol{R_0} = \sqrt{\left(\frac{L}{C}\right)} = \sqrt{\left(\frac{0.4}{400 \times 10^{-12}}\right)} = \mathbf{31.62\ k\Omega}$$

From equations (42.1) and (42.4) it is seen that the characteristic impedance Z_0 varies with ω, i.e., Z_0 varies with frequency. Thus if the nominal impedance is made to equal the load impedance into which the filter feeds then the matching deteriorates as the frequency increases from zero towards f_c. It is however convention to make the terminating impedance equal to the value of Z_0 well within the pass-band, i.e., to take the limiting value of Z_0 as the frequency approaches zero. This limit is obviously $\sqrt{(L/C)}$. This means that the filter is properly terminated at very low frequency but as the cut-off frequency is approached becomes increasingly mismatched. This is shown for a low-pass section in Figure 42.20 by curve (a). It is seen that an increasing loss is introduced into the pass band. Curve (b) shows the attenuation due to the same low-pass section being correctly terminated at all frequencies. A curve lying somewhere between curves (a) and (b) will usually result for each section if several sections are cascaded and terminated in R_0, or if a matching section is inserted between the low pass section and the load.

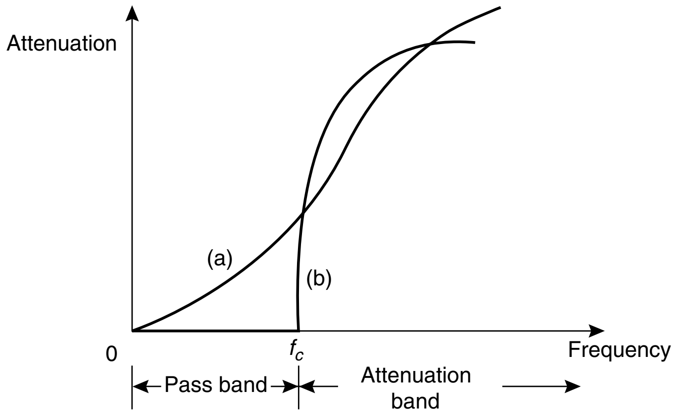

Figure 42.20

(c) To determine values of L and C given R_0 and f_c

If the values of the nominal impedance R_0 and the cut-off frequency f_c are known for a low pass T or π section it is possible to determine the values of inductance and capacitance required to form the section.

From equation (42.5), $R_0 = \sqrt{\dfrac{L}{C}} = \dfrac{\sqrt{L}}{\sqrt{C}}$ from which, $\sqrt{L} = R_0\sqrt{C}$

Substituting in equation (42.3) gives:

$$f_c = \frac{1}{\pi\sqrt{L}\sqrt{C}} = \frac{1}{\pi(R_0\sqrt{C})\sqrt{C}} = \frac{1}{\pi R_0 C}$$

from which, $\boxed{\textbf{capacitance } \boldsymbol{C = \frac{1}{\pi R_0 f_c}}}$ (42.6)

Similarly from equation (42.5), $\sqrt{C} = \dfrac{\sqrt{L}}{R_0}$

Substituting in equation (42.3) gives: $f_c = \dfrac{1}{\pi\sqrt{L}\left(\dfrac{\sqrt{L}}{R_0}\right)} = \dfrac{R_0}{\pi L}$

from which, $\boxed{\textbf{inductance, } \boldsymbol{L = \frac{R_0}{\pi f_c}}}$ (42.7)

> Problem 2. A filter section is to have a characteristic impedance at zero frequency of 600 Ω and a cut-off frequency at 5 MHz. Design (a) a low-pass T section filter, and (b) a low-pass π section filter to meet these requirements.

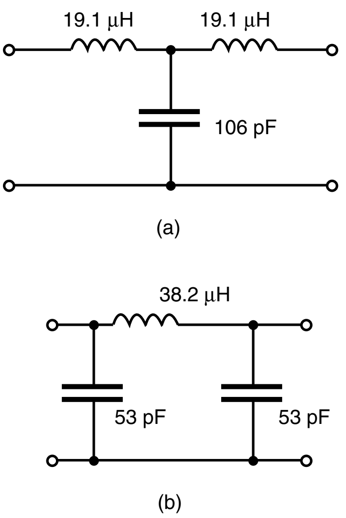

Figure 42.21

The characteristic impedance at zero frequency is the nominal impedance R_0, i.e., $R_0 = 600\ \Omega$; cut-off frequency, $f_c = 5$ MHz $= 5 \times 10^6$ Hz.

From equation (42.6),

$$\text{capacitance, } C = \frac{1}{\pi R_0 f_c} = \frac{1}{\pi(600)(5 \times 10^6)} \text{ F} = 106 \text{ pF}$$

and from equation (42.7),

$$\text{inductance, } L = \frac{R_0}{\pi f_c} = \frac{600}{\pi(5 \times 10^6)} \text{ H} = 38.2\ \mu\text{H}$$

(a) A low-pass T section filter is shown in Figure 42.21(a), where the series arm inductances are each L/2 (see Figure 42.17), i.e., $(38.2/2) = 19.1\ \mu$H

(b) A low-pass π section filter is shown in Figure 42.21(b), where the shunt arm capacitances are each $(C/2)$ (see Figure 42.18), i.e., $(106/2) = 53$ pF

(d) 'Constant-k' prototype low-pass filter

A ladder network is shown in Figure 42.22, the elements being expressed in terms of impedances Z_1 and Z_2. The network shown in Figure 42.22(b)

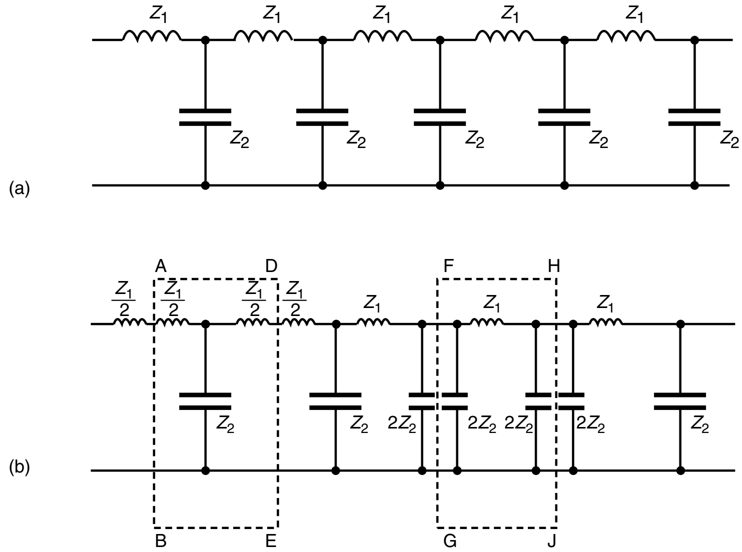

Figure 42.22

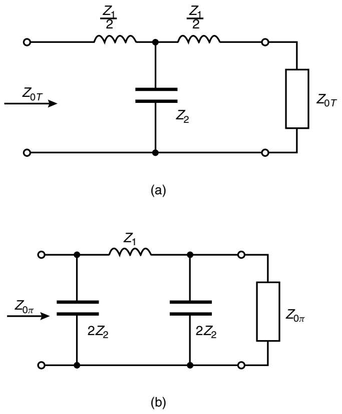

Figure 42.23

is equivalent to the network shown in Figure 42.22(a), where $(Z_1/2)$ in series with $(Z_1/2)$ equals Z_1 and $2Z_2$ in parallel with $2Z_2$ equals Z_2. Removing sections ABED and FGJH from Figure 42.22(b) gives the T section shown in Figure 42.23(a), which is terminated in its characteristic impedance Z_{OT}, and the π section shown in Figure 42.23(b), which is terminated in its characteristic impedance $Z_{0\pi}$.

From equation (41.1), page 760,

$$Z_{OT} = \sqrt{\left[\left(\frac{Z_1}{2}\right)^2 + 2\left(\frac{Z_1}{2}\right)Z_2\right]}$$

i.e., $$Z_{OT} = \sqrt{\left(\frac{Z_1^2}{4} + Z_1 Z_2\right)} \qquad (42.8)$$

From equation (41.3), page 760

$$Z_{0\pi} = \sqrt{\left[\frac{(Z_1)(2Z_2)^2}{Z_1 + 2(2Z_2)}\right]} = \sqrt{\left[\frac{Z_1(Z_1)(4Z_2^2)}{Z_1(Z_1 + 4Z_2)}\right]}$$

$$= \frac{2Z_1Z_2}{\sqrt{(Z_1^2 + 4Z_1Z_2)}} = \frac{Z_1Z_2}{\sqrt{\left(\frac{Z_1^2}{4} + Z_1Z_2\right)}}$$

i.e., $Z_{0\pi} = \dfrac{Z_1 Z_2}{Z_{OT}}$ from equation (42.8)

Thus $\boxed{\mathbf{Z_{0T}Z_{0\pi} = Z_1 Z_2}}$ (42.9)

This is a general expression relating the characteristic impedances of T and π sections made up of equivalent series and shunt impedances.

From the low-pass sections shown in Figures 42.17 and 42.18,

$$Z_1 = j\omega L \text{ and } Z_2 = \frac{1}{j\omega C}.$$

Hence $\quad Z_{0T}Z_{0\pi} = (j\omega L)\left(\frac{1}{j\omega C}\right) = \frac{L}{C}$

Thus, from equation (42.5), $\boxed{\mathbf{Z_{0T}Z_{0\pi} = R_0^2}}$ (42.10)

From equations (42.9) and (42.10),

$$Z_{0T}Z_{0\pi} = Z_1 Z_2 = R_0^2 = \text{constant (k)}.$$

A ladder network composed of reactances, the series reactances being of opposite sign to the shunt reactances (as in Figure 42.23) are called **'constant-k' filter sections**. Positive (i.e., inductive) reactance is directly proportional to frequency, and negative (i.e., capacitive) reactance is inversely proportional to frequency. Thus the product of the series and shunt reactances is independent of frequency (see equations (42.9) and (42.10)). The constancy of this product has given this type of filter its name.

From equation (42.10), it is seen that Z_{0T} and $Z_{0\pi}$ will either be both real or both imaginary together (since $j^2 = -1$). Also, when Z_{0T} changes from real to imaginary at the cut-off frequency, so will $Z_{0\pi}$. The two sections shown in Figures 42.17 and 42.18 will thus have identical cut-off frequencies and thus identical pass bands. Constant-k sections of any kind of filter are known as **prototypes**.

(e) Practical low-pass filter characteristics

From equation (42.1), the characteristic impedance Z_{0T} of a low-pass T section is given by:

$$Z_{0T} = \sqrt{\left(\frac{L}{C} - \frac{\omega^2 L^2}{4}\right)}$$

Rearranging gives:

$$Z_{0T} = \sqrt{\left[\frac{L}{C}\left(1 - \frac{\omega^2 LC}{4}\right)\right]} = \sqrt{\left(\frac{L}{C}\right)}\sqrt{\left(1 - \frac{\omega^2 LC}{4}\right)}$$

$$= R_0\sqrt{\left(1 - \frac{\omega^2 LC}{4}\right)} \text{ from equation (42.5)}$$

From equation (42.2), $\omega_c^2 = \dfrac{4}{LC}$, hence $Z_{0T} = R_0\sqrt{\left(1 - \dfrac{\omega^2}{\omega_c^2}\right)}$

i.e., $$\boxed{\boldsymbol{Z_{0T} = R_0\sqrt{\left[1 - \left(\frac{\omega}{\omega_c}\right)^2\right]}}} \tag{42.11}$$

Also, from equation (42.10), $Z_{0\pi} = \dfrac{R_0^2}{Z_{0T}} = \dfrac{R_0^2}{R_0\sqrt{\left[1 - \left(\dfrac{\omega}{\omega_c}\right)^2\right]}}$

i.e., $$\boxed{\boldsymbol{Z_{0\pi} = \frac{R_0}{\sqrt{\left[1 - \left(\frac{\omega}{\omega_c}\right)^2\right]}}}} \tag{42.12}$$

(Alternatively, the expression for $Z_{0\pi}$ could have been obtained from equation (42.4), where

$$Z_{0\pi} = \sqrt{\left(\frac{1}{\dfrac{C}{L} - \dfrac{\omega^2 C^2}{4}}\right)} = \sqrt{\left[\frac{\dfrac{L}{C}}{\dfrac{L}{C}\left(\dfrac{C}{L} - \dfrac{\omega^2 C^2}{4}\right)}\right]}$$

$$= \frac{\sqrt{\dfrac{L}{C}}}{\sqrt{\left(1 - \dfrac{\omega^2 LC}{4}\right)}} = \frac{R_0}{\sqrt{\left[1 - \left(\dfrac{\omega}{\omega_c}\right)^2\right]}} \text{ as above).}$$

From equations (42.11) and (42.12), when $\omega = 0$ (i.e., when the frequency is zero),

$$Z_{0T} = Z_{0\pi} = R_0.$$

At the cut-off frequency, f_c, $\omega = \omega_c$
and from equation (42.11), Z_{0T} falls to zero,
and from equation (42.12), $Z_{0\pi}$ rises to infinity.

These results are shown graphically in Figure 42.24, where it is seen that Z_{0T} decreases from R_0 at zero frequency to zero at the cut-off frequency; $Z_{0\pi}$ rises from its initial value of R_0 to infinity at f_c.

(At a frequency, $f = 0.95 f_c$, for example, $Z_{0\pi} = \dfrac{R_0}{\sqrt{(1 - 0.95^2)}} = 3.2R_0$ from equation (42.12)).

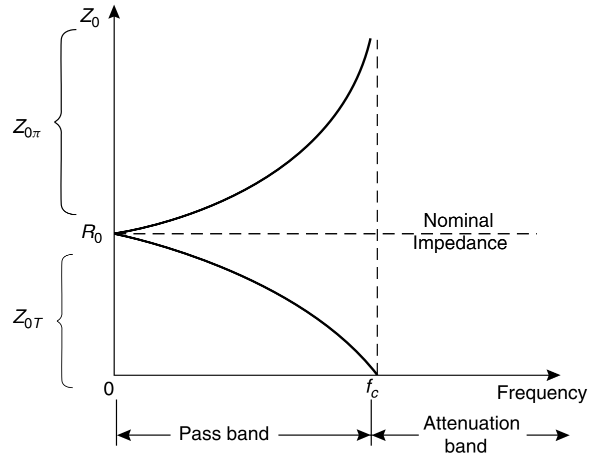

Figure 42.24

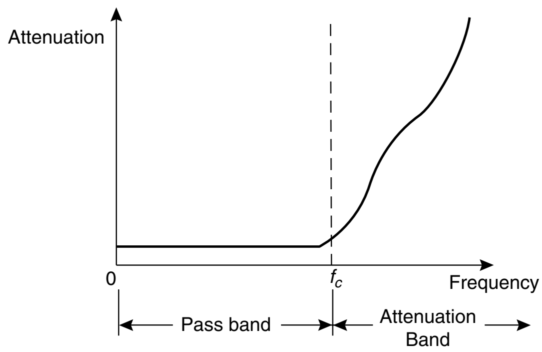

Figure 42.25

Note that since Z_0 becomes purely reactive in the attenuation band, it is not shown in this range in Figure 42.24.

Figure 42.2(a), on page 791, showed an ideal low-pass filter section characteristic. In practise, the characteristic curve of a low-pass prototype filter section looks more like that shown in Figure 42.25. The characteristic may be improved somewhat closer to the ideal by connecting two or more identical sections in cascade. This produces a much sharper cut-off characteristic, although the attenuation in the pass band is increased a little.

Problem 3. The nominal impedance of a low-pass π section filter is 500 Ω and its cut-off frequency is at 100 kHz. Determine (a) the value of the characteristic impedance of the section at a frequency of 90 kHz, and (b) the value of the characteristic impedance of the equivalent low-pass T section filter.

At zero frequency the characteristic impedance of the π and T section filters will be equal to the nominal impedance of 500 Ω.

(a) From equation (42.12), the characteristic impedance of the π section at 90 kHz is given by:

$$Z_{0\pi} = \frac{R_0}{\sqrt{\left[1 - \left(\dfrac{\omega}{\omega_c}\right)^2\right]}} = \frac{500}{\sqrt{\left[1 - \left(\dfrac{2\pi 90 \times 10^3}{2\pi 100 \times 10^3}\right)^2\right]}}$$

$$= \frac{500}{\sqrt{[1 - (0.9)^2]}} = \mathbf{1147\ \Omega}$$

(b) From equation (42.11), the characteristic impedance of the T section at 90 kHz is given by:

$$Z_{0T} = R_0\sqrt{\left[1 - \left(\frac{\omega}{\omega_c}\right)^2\right]} = 500\sqrt{[1 - (0.9)^2]} = \mathbf{218\ \Omega}$$

(Check: From equation (42.10),

$$Z_{0T}Z_{0\pi} = (218)(1147) = 250\,000 = 500^2 = R_0^2)$$

Typical low-pass characteristics of characteristic impedance against frequency are shown in Figure 42.24.

> Problem 4. A low-pass π section filter has a nominal impedance of 600 Ω and a cut-off frequency of 2 MHz. Determine the frequency at which the characteristic impedance of the section is (a) 600 Ω (b) 1 kΩ (c) 10 kΩ

From equation (42.12), $Z_{0\pi} = \dfrac{R_0}{\sqrt{\left[1 - \left(\dfrac{\omega}{\omega_c}\right)^2\right]}}$

(a) When $Z_{0\pi} = 600\ \Omega$ and $R_0 = 600\ \Omega$, then $\omega = 0$, i.e., **the frequency is zero**

(b) When $Z_{0\pi} = 1000\ \Omega$, $R_0 = 600\ \Omega$ and $f_c = 2 \times 10^6$ Hz

$$\text{then} \quad 1000 = \frac{600}{\sqrt{\left[1 - \left(\dfrac{2\pi f}{2\pi 2 \times 10^6}\right)^2\right]}}$$

from which, $1 - \left(\frac{f}{2 \times 10^6}\right)^2 = \left(\frac{600}{1000}\right)^2 = 0.36$

and $\left(\frac{f}{2 \times 10^6}\right) = \sqrt{(1 - 0.36)} = 0.8$

Thus when $Z_{0\pi} = 1000\ \Omega$,

$$\textbf{frequency, } \boldsymbol{f} = (0.8)(2 \times 10^6) = \mathbf{1.6\ MHz}$$

(c) When $Z_{0\pi} = 10\ \text{k}\Omega$, then

$$10\,000 = \frac{600}{\sqrt{\left[1 - \left(\frac{f}{2}\right)^2\right]}},$$ where frequency, f is in megahertz.

Thus $1 - \left(\frac{f}{2}\right)^2 = \left(\frac{600}{10\,000}\right)^2 = (0.06)^2$

and $\frac{f}{2} = \sqrt{[1 - (0.06)^2]} = 0.9982$

Hence when $Z_{0\pi} = 10\ \text{k}\Omega$, **frequency f** $= (2)(0.9982)$

$$= \mathbf{1.996\ MHz}$$

The above three results are seen to be borne out in the characteristic of $Z_{0\pi}$ against frequency shown in Figure 42.24.

Further problems on low-pass filter sections may be found in Section 42.10, problems 1 to 6, page 837.

42.6 High-pass filter sections

(a) The cut-off frequency

High-pass T and π sections are shown in Figure 42.26, (as derived in Section (42.4)), each being terminated in their characteristic impedance.

From equation (41.1), page 760, the characteristic impedance of a T section is given by:

$$Z_{0T} = \sqrt{(Z_A^2 + 2Z_A Z_B)}$$

From Figure 42.26(a), $Z_A = \frac{1}{j\omega 2C}$ and $Z_B = j\omega L$

Thus $$Z_{0T} = \sqrt{\left[\left(\frac{1}{j\omega 2C}\right)^2 + 2\left(\frac{1}{j\omega 2C}\right)(j\omega L)\right]}$$

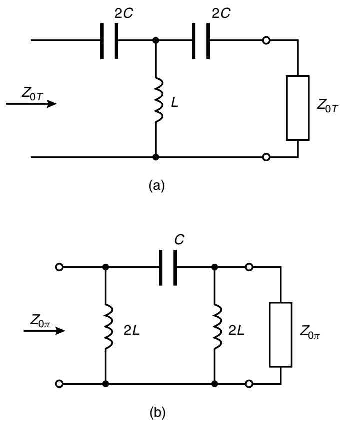

Figure 42.26

$$= \sqrt{\left[\frac{1}{-4\omega^2C^2} + \frac{L}{C}\right]}$$

$$\text{i.e.,}\quad Z_{0T} = \sqrt{\left(\frac{L}{C} - \frac{1}{4\omega^2C^2}\right)} \qquad (42.13)$$

Z_{0T} will be real when $\dfrac{L}{C} > \dfrac{1}{4\omega^2C^2}$

Thus the filter will pass all frequencies above the point

$$\text{where } \frac{L}{C} = \frac{1}{4\omega^2C^2}$$

$$\text{i.e.,}\quad \text{where } \omega_c^2 = \frac{1}{4LC} \qquad (42.14)$$

where $\omega_c = 2\pi f_c$, and f_c is the cut-off frequency.

$$\text{Hence}\quad (2\pi f_c)^2 = \frac{1}{4LC}$$

and **the cut-off frequency**, $$\boxed{\boldsymbol{f_c = \frac{1}{4\pi\sqrt{(LC)}}}} \qquad (42.15)$$

The same equation for the cut-off frequency is obtained for the high-pass π network shown in Figure 42.26(b) as follows:

From equation (41.3), page 760, the characteristic impedance of a symmetrical π section is given by:

$$Z_{0\pi} = \sqrt{\left(\frac{Z_1Z_2^2}{Z_1 + 2Z_2}\right)}$$

From Figure 42.26(b), $Z_1 = \dfrac{1}{j\omega C}$ and $Z_2 = j2\omega L$

$$\text{Hence } Z_{0\pi} = \sqrt{\left\{\frac{\left(\frac{1}{j\omega C}\right)(j2\omega L)^2}{\frac{1}{j\omega C} + 2j2\omega L}\right\}}$$

$$= \sqrt{\left\{\frac{j4\frac{\omega L^2}{C}}{j\left(4\omega L - \frac{1}{\omega C}\right)}\right\}} = \sqrt{\left(\frac{\frac{4L^2}{C}}{4L - \frac{1}{\omega^2 C}}\right)}$$

$$\text{i.e.,}\quad Z_{0\pi} = \sqrt{\left(\frac{1}{\frac{C}{L} - \frac{1}{4\omega^2L^2}}\right)} \qquad (42.16)$$

$Z_{0\pi}$ will be real when $\dfrac{C}{L} > \dfrac{1}{4\omega^2 L^2}$ and the filter will pass all frequencies above the point where $\dfrac{C}{L} = \dfrac{1}{4\omega^2 L^2}$, i.e., where $\omega_c^2 = \dfrac{1}{4LC}$ as above.
Thus the cut-off frequency for a high-pass π network is also given by

$$\boxed{f_c = \frac{1}{4\pi\sqrt{(LC)}}} \quad \text{(as in equation (42.15))} \qquad (42.15')$$

(b) Nominal impedance

When the frequency is very high, ω is a very large value and the term $(1/4\omega^2 C^2)$ in equations (42.13) and (42.16) are extremely small and may be neglected.

The characteristic impedance then becomes equal to $\sqrt{(L/C)}$, this being the nominal impedance. Thus for a high-pass filter section the nominal impedance R_0 is given by:

$$\boxed{R_0 = \sqrt{\left(\frac{L}{C}\right)}} \qquad (42.17)$$

the same as for the low-pass filter sections.

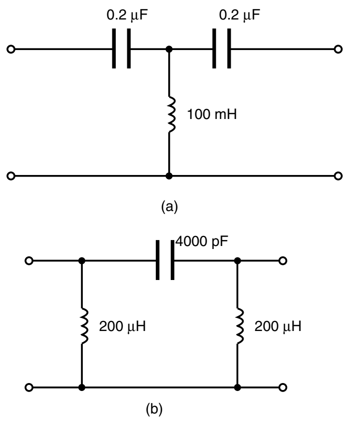

Figure 42.27

> Problem 5. Determine for each of the high-pass filter sections shown in Figure 42.27 (i) the cut-off frequency, and (ii) the nominal impedance.

(a) Comparing Figure 42.27(a) with Figure 42.26(a) shows that:

$$2C = 0.2\ \mu\text{F}, \text{ i.e., capacitance, } C = 0.1\ \mu\text{F} = 0.1 \times 10^{-6}\ \text{F}$$
$$\text{and inductance, } L = 100\ \text{mH} = 0.1\ \text{H}$$

(i) From equation (42.15),

$$\text{cut-off frequency, } f_c = \frac{1}{4\pi\sqrt{(LC)}} = \frac{1}{4\pi\sqrt{[(0.1)(0.1 \times 10^{-6}]}}$$

i.e., $$f_c = \frac{10^3}{4\pi(0.1)} = \mathbf{796\ Hz}$$

(ii) From equation (42.17),

$$\textbf{nominal impedance, } \boldsymbol{R_0} = \sqrt{\left(\frac{L}{C}\right)} = \sqrt{\left(\frac{0.1}{0.1 \times 10^{-6}}\right)}$$
$$= \mathbf{1000\ \Omega \text{ or } 1\ k\Omega}$$