DEFINITE INTEGRALS (REC)

Ramisha

01. Show that $\int_0^{\frac{1}{2}} \frac{dx}{(1-2x^2)\sqrt{(1-x^2)}} = \frac{1}{2}\log(2+\sqrt{3})$

Let, $I = \int_0^{1/2} \frac{dx}{(1-2x^2)\sqrt{1-x^2}}$

$$I = \int_0^{\pi/6} \frac{\cos\theta\, d\theta}{(1-2\sin^2\theta)\sqrt{1-\sin^2\theta}}$$

Put $x = \sin\theta$
$dx = \cos\theta\, d\theta$
when $x=0$ then $\theta =$
& $x=\frac{1}{2}$ then $\theta =$

$$= \int_0^{\pi/6} \frac{\cos\theta\, d\theta}{\cos 2\theta \cos\theta}$$

$$= \int_0^{\pi/6} \sec 2\theta\, d\theta$$

$$= \frac{1}{2}\left[\log\left\{\tan\left(\frac{2\theta}{2}+\frac{\pi}{4}\right)\right\}\right]_0^{\pi/6}$$

or,

Again put $2\theta = z$

$2d\theta = dz$

$d\theta = \frac{1}{2}dz$

θ	0	$\pi/6$
z	0	$\pi/3$

$$\therefore I = \int_0^{\pi/3} \sec z \cdot \frac{1}{2}dz = \frac{1}{2}\int_0^{\pi/3} \sec z\, dz$$

$$= \frac{1}{2}\left[\log\left\{\tan\left(\frac{z}{2}+\frac{\pi}{4}\right)\right\}\right]_0^{\pi/3}$$

$$= \frac{1}{2}\log\left\{\tan\left(\frac{\pi}{6}+\pi/4\right)\right\} - \frac{1}{2}\log\left(\tan\frac{\pi}{4}\right)$$

$$= \frac{1}{2}\log\left\{\frac{\tan\pi/6 + \tan\frac{\pi}{4}}{1-\tan\pi/6\tan\pi/4}\right\} - \frac{1}{2}\log 1$$

$$= \frac{1}{2}\log\left(\frac{\frac{1}{\sqrt{3}}+1}{1-\frac{1}{\sqrt{3}}}\right) = \frac{1}{2}\log\left(\frac{\frac{1+\sqrt{3}}{\sqrt{3}}}{\frac{\sqrt{3}-\sqrt{3}}{\sqrt{3}}}\right.$$

$$I = \frac{1}{2}\log\left(\frac{1+\sqrt{3}}{\sqrt{3}-1}\right) = \frac{1}{2}\log\left\{\frac{(1+\sqrt{3})(1+\sqrt{3})}{(\sqrt{3}-1)(\sqrt{3}+1)}\right\}$$

$$= \frac{1}{2}\log\left(\frac{1+3+2\sqrt{3}}{3-1}\right)$$

$$= \frac{1}{2}\log\left(\frac{4+2\sqrt{3}}{2}\right)$$

$$= \frac{1}{2}\log(2+\sqrt{3})$$

(02) Prove that $\displaystyle\int_0^{\pi/2} \frac{dx}{a^2\cos^2 x + b^2\sin^2 x} = \frac{\pi}{2ab}$ $[a, b > 0]$

$$L.H.S = \int_0^{\pi/2} \frac{dx}{a^2\cos^2 x + b^2\sin^2 x}$$

$$= \int_0^{\pi/2} \frac{\sec^2 x\, dx}{a^2 + b^2\tan^2 x}$$

Put $\tan x = t$

$\sec^2 x\, dx = dt$

when $x = 0$ then $t = 0$

& $x = \pi/2$ then $t = \infty$

$$= \int_0^{\infty} \frac{dt}{a^2 + b^2 t^2}$$

$$= \int_0^{\infty} \frac{dt}{b^2\left(t^2 + \frac{a^2}{b^2}\right)}$$

$$= \frac{1}{b^2}\int_0^{\infty} \frac{dt}{t^2 + \left(\frac{a}{b}\right)^2}$$

$$= \frac{1}{b^2}\cdot\frac{1}{\left(\frac{a}{b}\right)}\left[\tan^{-1}\frac{t}{a/b}\right]_0^{\infty}$$

$$= \frac{1}{b^2}\cdot\frac{b}{a}\left[\tan^{-1}\infty - \tan^{-1}0\right]$$

$$= \frac{1}{ab}\left[\tan^{-1}\tan\frac{\pi}{2} - \tan^{-1}\tan 0^\circ\right]$$

$$= \frac{\pi}{2ab} = R.H.S \quad \text{(Proved)}$$

03. Integrate $\int_0^{\pi/2} \frac{dx}{(a^2\cos^2 x + b^2 \sin^2 x)^2} = \frac{\pi}{4}\, \frac{a^2+b^2}{a^3 b^3}$

Let $I = \int_0^{\pi/2} \frac{dx}{(a^2\cos^2 x + b^2\sin^2 x)^2}$

$$= \int_0^{\pi/2} \frac{\sec^4 x\, dx}{(a^2 + b^2\tan^2 x)^2} = \int_0^{\pi/4} \frac{\sec^2 x \sec^2 x\, dx}{(a^2 + b^2 \tan^2 x)^2}$$

$$= \int_0^{\pi/2} \frac{(1+\tan^2 x)\sec^2 x\, dx}{(a^2 + b^2\tan^2 x)^2}$$

$$= \int_0^{\pi/2} \frac{\left(1 + \frac{a^2}{b^2}\tan^2\theta\right)\frac{a}{b}\sec^2\theta\, d\theta}{(a^2 + a^2\tan^2\theta)^2}$$

Put $b\tan x = a\tan\theta$
$b\sec^2 x\, dx = a\sec^2\theta\, d$
$\sec^2 x\, dx = \frac{a}{b}\sec^2\theta\, d$
when $x = 0$, then $\theta =$
,, $x = \pi/2$ then $\theta =$

$$= \int_0^{\pi/2} \frac{\frac{a}{b}\frac{1}{b^2}(b^2 + a^2\tan^2\theta)}{(a^2\sec^2\theta)^2} \sec^2\theta\, d\theta$$

$$= \frac{a}{b^3} \cdot \int_0^{\pi/2} \frac{b^2 + a^2\tan^2\theta}{a^4 \sec^2\theta}\, d\theta$$

$$= \frac{1}{a^3 b^3} \int_0^{\pi/2} \left(b^2\cos^2\theta + a^2 \frac{\sin^2\theta}{\cos^2\theta}\cdot\cos^2\theta\right) d\theta$$

$$= \frac{1}{a^3 b^3}\left\{ b^2 \int_0^{\pi/2} \cos^2\theta\, d\theta + a^2 \int_0^{\pi/2} \sin^2\theta\, d\theta \right\}$$

$$= \frac{1}{a^3 b^3}\left[b^2\, \frac{1}{2}\frac{\pi}{2} + a^2\, \frac{1}{2}\frac{\pi}{2}\right]$$

$$= \frac{\pi}{4}\, \frac{a^2+b^2}{a^3 b^3}$$

v.v.Imp. Show that $\displaystyle\int_0^{\pi/2} \frac{x\sin x\cos x}{(a^2\cos^2 x+b^2\sin^2 x)^2}\,dx=\frac{\pi}{4ab^2(a+b)}$, $[a,b>0]$

Solution:

Let $\displaystyle I=\int\frac{x\sin x\cos x}{(a^2\cos^2 x+b^2\sin^2 x)^2}\,dx$

Integrating by Parts

$$I=x\int\frac{\sin x\cos x\,dx}{(a^2\cos^2 x+b^2\sin^2 x)^2}-\int 1\cdot\left\{\int\frac{\sin x\cos x}{(a^2\cos^2 x+b^2\sin^2 x)^2}dx\right\}dx \longrightarrow (1)$$

Now $\displaystyle\int\frac{\sin x\cos x\,dx}{(a^2\cos^2 x+b^2\sin^2 x)^2}=\int\frac{\sin x\cos x}{\{a^2+(b^2-a^2)\sin^2 x\}^2}dx$

$$=\int\frac{\frac{1}{2(b^2-a^2)}dt}{t^2}=\frac{1}{2(b^2-a^2)}\int\frac{dt}{t^2}$$

$$=-\frac{1}{2(b^2-a^2)}\cdot\frac{1}{t}$$

Put, $a^2+(b^2-a^2)\sin^2 x=t$

$2(b^2-a^2)\sin x\cos x\,dx=dt$

$\sin x\cos x\,dx=\frac{dt}{2(b^2-a^2)}$

$$\therefore\int\frac{\sin x\cos x\,dx}{(a^2\cos^2 x+b^2\sin^2 x)^2}=-\frac{1}{2(b^2-a^2)}\cdot\frac{1}{a^2+(b^2-a^2)\sin^2 x} \longrightarrow (2)$$

Again $\cancel{-\frac{1}{2(b^2-a^2)}\int\frac{dx}{a^2+(b^2-a^2)\sin^2 x}=\int}$

$$\int\left\{\int\frac{\sin x\cos x\,dx}{(a^2\cos^2 x+b^2\sin^2 x)^2}\right\}dx=-\frac{1}{2(b^2-a^2)}\int\frac{dx}{a^2+(b^2-a^2)\sin^2 x}$$

$$=-\frac{1}{2(b^2-a^2)}\int\frac{dx}{a^2\cos^2 x+b^2\sin^2 x}$$

$$= -\frac{1}{2(b^2-a^2)}\int \frac{\sec^2 x\, dx}{a^2 + b^2 \tan^2 x}$$

$$= -\frac{1}{2(b^2-a^2)}\int \frac{dt}{a^2+b^2t^2}$$

Put, $\tan x = t$
$\sec^2 x\, dx = dt$

$$= -\frac{1}{2(b^2-a^2)} \cdot \int \frac{dt}{b^2(a^2/b^2 + t^2)}$$

$$= -\frac{1}{2(b^2-a^2)} \cdot \frac{1}{b^2} \int \frac{dt}{\left(\frac{a}{b}\right)^2 + t^2}$$

$$= -\frac{1}{2b^2(b^2-a^2)} \cdot \frac{1}{a/b} \tan^{-1} \frac{t}{a/b}$$

$$= -\frac{1}{2ab(b^2-a^2)} \tan^{-1}\left(\frac{b}{a}\tan x\right)$$

Therefore

$$\int_0^{\pi/2} \left\{ \int \frac{\sin x \cos x\, dx}{(a^2\cos^2 x + b^2 \sin^2 x)^2} \right\} dx = -\frac{1}{2ab(b^2-a^2)} \left[\tan^{-1}\left(\frac{b}{a}\tan x\right)\right]$$

$$= \frac{-1}{2ab(b^2-a^2)} \cdot \frac{\pi}{2} = -\frac{\pi}{4ab(b^2-a^2)}$$

and

$$x \int_0^{\pi/2} \frac{\sin x \cos x\, dx}{(a^2 \cos^2 x + b^2 \sin^2 x)^2} = -\frac{1}{2(b^2-a^2)} \left[\frac{x}{a^2 + (b^2-a^2)\sin^2 x}\right]_0^{\pi/2}$$

$$= -\frac{1}{2(b^2-a^2)} \left(\frac{\pi/2}{b^2} - \cancel{\left(\frac{b}{a}\right)}\right)$$

$$= -\frac{1}{2b^2(b^2-a^2)} \frac{\pi}{2} = -\frac{\pi}{4b^2(b^2-a^2)}$$

Hence $\displaystyle\int_0^{\pi/2} \frac{x\cos x \sin x \, dx}{(a^2\cos^2 x + b^2 \sin^2 x)^2} = -\frac{\pi}{4b^2(b^2-a^2)} + \frac{\pi}{4ab(b^2-a^2)}$

$$= \frac{\pi}{4b\,\cancel{4ab}(b^2-a^2)}\left[-\frac{1}{b}+\frac{1}{a}\right]$$

$$= \frac{\pi}{4b(b^2-a^2)}\left(\frac{-a+b}{ab}\right)$$

$$= \frac{\pi}{4ab^2(b-a)(b+a)}(b-a).$$

$$= \frac{\pi}{4ab^2(a+b)}.$$ Answer

or, (Proved)

Q5. Prove that $\displaystyle\int_2^e \left\{\frac{1}{\log x} - \frac{1}{(\log x)^2}\right\} dx = e - \frac{2}{\log 2}$

$$\text{L.H.S} = \int_2^e \frac{1}{\log x}\,dx - \int_2^e \frac{1}{(\log x)^2}\,dx$$

$$= \left[\frac{1}{\log x}\int dx\right]_2^e - \int_2^e\left\{-\frac{1}{(\log x)^2}\cdot\frac{1}{x}\int dx\right\}dx - \int_2^e \frac{1}{(\log x)^2}$$

$$= \left[\frac{x}{\log x}\right]_2^e + \int_2^e \frac{1}{(\log x)^2}\cdot\frac{1}{x}\cdot x\,dx - \int_2^e \frac{dx}{(\log x)^2}$$

$$= \frac{e}{1} - \frac{2}{\log 2} + \int_2^e \frac{dx}{(\log x)^2} - \int_2^e \frac{dx}{(\log x)^2}$$

$$= e - \frac{2}{\log 2} = \text{R. Hand side}$$

(Proved)

Ramisha

6. Show that, $\int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}\,dx = \frac{\pi}{4}$

let $I = \int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}\,dx \longrightarrow (1)$

$I = \int_0^{\pi/2} \frac{\sqrt{\sin(\pi/2 - x)}}{\sqrt{\sin(\pi/2-x)}+\sqrt{\cos(\pi/2-x)}}\,dx$ [using Property of integration]

$I = \int_0^{\pi/2} \frac{\sqrt{\cos x}}{\sqrt{\cos x}+\sqrt{\sin x}}\,dx \longrightarrow (2)$

Adding equation (1) and (2) we get

$2I = \int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}\,dx + \int_0^{\pi/2} \frac{\sqrt{\cos x}}{\sqrt{\cos x}+\sqrt{\sin x}}\,dx$

$= \int_0^{\pi/2} \left(\frac{\sqrt{\sin x}+\sqrt{\cos x}}{\sqrt{\sin x}+\sqrt{\cos x}}\right) dx$

$= \int_0^{\pi/2} dx = x\Big|_0^{\pi/2} = \frac{\pi}{2}$

$\therefore 2I = \frac{\pi}{2}$

$I = \frac{\pi}{4}$

Hence $\int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}\,dx = \frac{\pi}{4}$ showed

7. Show that $\int_0^{\pi/2} \log \sin x\,dx = \int_0^{\pi/2} \log \cos x\,dx = \frac{\pi}{2}\log\frac{1}{2}$

Soln:

let, $I = \int_0^{\pi/2} \log \sin x\,dx \longrightarrow (1)$

$= \int_0^{\pi/2} \log \sin(\pi/2 - x)\,dx$ {Using Property of Integration}

$$I = \int_0^{\pi/2} \log\cos x \, dx \longrightarrow ②$$

$$\therefore \int_0^{\pi/2} \log\sin x \, dx = \int_0^{\pi/2} \log\cos x \, dx \quad \text{(Proved)}$$

Now Adding equation ① & ②

$$2I = \int_0^{\pi/2} (\log\sin x + \log\cos x) \, dx$$

$$= \int_0^{\pi/2} \log(\sin x\cos x) \, dx$$

$$= \int_0^{\pi/2} \log\{\tfrac{1}{2}(\sin 2x)\} dx$$

$$2I = \int_0^{\pi/2} \log(\sin 2x) \, dx - \int_0^{\pi/2} \log 2 \, dx$$

$$2I = \int_0^{\pi/2} \log(\sin 2x) \, dx - \log 2 [x]_0^{\pi/2}$$

$$= \int_0^{\pi/2} \log(\sin 2x) \, dx - \log 2 \cdot \frac{\pi}{2}$$

Put $2x = t$

$dx = \frac{dt}{2}$

x	0	$\pi/2$
t	0	π

$$= \int_0^{\pi} \log\sin t \, \frac{dt}{2} + \frac{\pi}{2} \log 2^{-1}$$

$$2I = \frac{1}{2}\int_0^{\pi} \log(\sin t) \, dt + \frac{\pi}{2}\log\frac{1}{2}$$

$$2I = \int_0^{\pi/2} \log\sin t \, dt + \frac{\pi}{2}\log\frac{1}{2}$$

$$2I = I + \frac{\pi}{2}\log\frac{1}{2}$$

$$\therefore I = \frac{\pi}{2}\log\frac{1}{2} \quad \text{(Proved)}$$

Q. Show that $\int_0^1 \frac{\log(1+x)}{1+x^2}\,dx = \frac{\pi}{8}\log 2$

Let $I = \int_0^1 \frac{\log(1+x)}{1+x^2}\,dx$

Put $x = \tan\theta$

$dx = \sec^2\theta\, d\theta$

x	0	1
θ	0	$\pi/4$

$\therefore I = \int_0^{\pi/4} \frac{\log(1+\tan\theta)}{1+\tan^2\theta}\sec^2\theta\, d\theta$

$= \int_0^{\pi/4} \frac{\log(1+\tan\theta)}{\sec^2\theta}\sec^2\theta\, d\theta$

$I = \int_0^{\pi/4} \log(1+\tan\theta)\,d\theta = \int_0^{\pi/4} \log\left\{1+\tan\left(\pi/4-\theta\right)\right\} d\theta$

(Using Property of Integral Rule:

$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$

$= \int_0^{\pi/4} \log\left\{1 + \frac{\tan \pi/4 - \tan\theta}{1+\tan\pi/4 \cdot \tan\theta}\right\} d\theta$

$= \int_0^{\pi/4} \log\left(1+\frac{1-\tan\theta}{1+\tan\theta}\right) d\theta$

$= \int_0^{\pi/4} \log\left\{\frac{1+\tan\theta+1-\tan\theta}{1+\tan\theta}\right\} d\theta$

$= \int_0^{\pi/4} \log\left(\frac{2}{1+\tan\theta}\right) d\theta = \int_0^{\pi/4} \log 2\, d\theta - \int_0^{\pi/4} \log(1+\tan$

$I = \log 2 \int_0^{\pi/4} d\theta - I \quad \left[\because I = \int_0^{\pi/4} \log(1+\tan\theta)\,d\theta\right]$

$\therefore 2I = \log 2\ \theta\Big|_0^{\pi/4}$

$2I = \log 2 \cdot \frac{\pi}{4}$

$\therefore I = \frac{\pi}{8}\log 2$

Hence $\int_0^1 \frac{\log(1+x)}{1+x^2}\,dx = \frac{\pi}{8}\log 2$ (Showed)

Q. Show that (i) $\displaystyle\int_0^{\pi} \frac{x\sin x}{1+\cos^2 x}\,dx = \frac{\pi^2}{4}$

(ii) $\displaystyle\int_0^{\pi} \frac{x\tan x}{\sec x+\tan x}\,dx = \frac{1}{2}\pi(\pi-2)$

VVF (iii) $\displaystyle\int_0^{\pi} \frac{x\,dx}{a^2\sin^2 x+b^2\cos^2 x} = \frac{\pi^2}{2ab} \quad (a,b>0)$

VVF/ (iv) $\displaystyle\int_0^{\pi} \frac{x\,dx}{(a^2\cos^2 x+b^2\sin^2 x)^2} = \frac{\pi^2(a^2+b^2)}{4a^3b^3}$

(v) $\displaystyle\int_0^{\pi/2} \frac{\sin^2 x}{\sin x+\cos x}\,dx = \frac{1}{\sqrt{2}}\log(\sqrt{2}+1)$

(vi) $\displaystyle\int_0^{\pi/2} \frac{x\,dx}{\sin x+\cos x} = \frac{\pi}{2\sqrt{2}}\log(\sqrt{2}+1)$

(vii) $\displaystyle\int_0^{1} \cot^{-1}(1-x+x^2)\,dx$

(viii) $\displaystyle\int_0^{\pi/2} \frac{dx}{5+4\sin x} = \frac{2}{3}\tan^{-1}\frac{1}{3}$

(ix) $\displaystyle\int_0^{\pi/2} \frac{dx}{3+5\cos x} = \frac{1}{4}\log 3$