First Order Ordinary Linear Differential Equations

- Ordinary Differential equations does not include partial derivatives.
- A linear first order equation is an equation that can be expressed in the form

$$\frac{dy}{dx} + p(x)y = q(x).$$

Where p and q are functions of x

Types Of Linear DE:

1. Separable Variable
2. Homogeneous Equation
3. Exact Equation
4. Linear Equation

Separable Variable

The first-order differential equation

$$\frac{dy}{dx} = f(x, y) \qquad (1)$$

is called separable provided that f(x,y) can be written as the product of a function of x and a function of y.

Suppose we can write the above equation as

$$\frac{dy}{dx} = g(x)h(y)$$

We then say we have " separated " the variables. By taking $h(y)$ to the LHS, the equation becomes

$$\frac{1}{h(y)}dy = g(x)dx$$

Integrating, we get the solution as

$$\int \frac{1}{h(y)}dy = \int g(x)\,dx + c$$

where c is an arbitrary constant.

Example 1. Consider the DE $\dfrac{dy}{dx} = y$

Separating the variables, we get

$$\frac{1}{y}dy = dx$$

Integrating we get the solution as

$$\ln|y| = x + k$$

or $y = ce^{x}, c$ an arbitrary constant.

Homogeneous equations

Definition A function $f(x, y)$ is said to be homogeneous of degree n in x, y if

$$f(tx, ty) = t^n f(x, y) \text{ for all } t, x, y$$

Examples $f(x, y) = x^2 - 2xy - y^2$

is homogeneous of degree 2.

$$f(x, y) = \frac{y}{x} + \sin\left(\frac{y - x}{x}\right)$$

is homogeneous of degree 0.

A first order DE $M(x,y)\,dx + N(x,y)\,dy = 0$

is called **homogeneous** if $M(x,y)$, $N(x,y)$ are homogeneous functions of x and y of the **same degree**.

This DE can be written in the form

$$\frac{dy}{dx} = f(x,y)$$

The substitution $\boldsymbol{y = vx}$ converts the given equation into **"variables separable"** form and hence can be solved. (Note that v is also a new variable)

Working Rule to solve a HDE:

1. Put the given equation in the form

$$M(x,y)dx + N(x,y)dy = 0 \quad (1)$$

2. Check M and N are Homogeneous function of the same degree.

3. Let $y = zx$.

4. Differentiate y = z x to get

$$\frac{dy}{dx} = z + x\frac{dz}{dx}$$

5. Put this value of dy/dx into (1) and solve the equation for z by separating the variables.

6. Replace z by y/x and simplify.

EXACT DIFFERENTIAL EQUATIONS

A first order DE $M(x, y)\,dx + N(x, y)\,dy = 0$

is called an exact DE if

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

The solution id given by:

$$\int M dx + \int N dy = c$$

(Terms free from 'x')

Solution is:

$$\int M dx + \int N dy = c$$

$$\int y dx + \int \frac{2}{y} dy = c$$

$$xy + 2 \ln y = c$$

Integrating Factors

Definition: If on multiplying by $\mu(x, y)$, the DE

$$M\,dx + N\,dy = 0$$

becomes an exact DE, we say that $\mu(x, y)$ is an Integrating Factor of the above DE

$\dfrac{1}{xy}, \dfrac{1}{x^2}, \dfrac{1}{y^2}$ are all integrating factors of the non-exact DE $y\,dx - x\,dy = 0$

Rule 1: I.F is a function of 'x' alone.

$$\frac{\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}}{N}=g(x)$$

$$\mu = e^{\int g(x)\,dx}$$

is an integrating factor of the given DE

$$M\,dx + N\,dy = 0$$

Rule 2: I.F is a function of 'y' alone.

If $$\frac{\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x}}{M} = h(y)$$, a function of y alone,

then $$\mu = e^{\int h(y)\,dy}$$

is an integrating factor of the given DE.

Rule 3: Given DE is homogeneous.

$$\frac{1}{Mx + Ny} = \mu$$

is an integrating factor of the given DE

$$M\,dx + N\,dy = 0$$

Rule 4: Equation is of the form of

$$f1(xy)\,y\,dx + f2(xy)\,x\,dy = 0$$

Then,

$$\mu = \frac{1}{Mx - Ny}$$

Linear Equations

A linear first order equation is an equation that can be expressed in the form

$$a_1(x)\frac{dy}{dx}+a_0(x)y=b(x), \qquad (1)$$

where $a_1(x)$, $a_0(x)$, and $b(x)$ depend only on the independent variable x, not on y.

We assume that the function $a_1(x)$, $a_0(x)$, and $b(x)$ are continuous on an interval and that $a_1(x) \neq 0$on that interval. Then, on dividing by $a_1(x)$, we can rewrite equation (1) in the standard form

$$\frac{dy}{dx} + P(x)y = Q(x) \quad (2)$$

where $P(x)$, $Q(x)$ are continuous functions on the interval.

Rules to solve a Linear DE:

1. Write the equation in the standard form

$$\frac{dy}{dx}+P(x)y=Q(x)$$

2. Calculate the IF $\mu(x)$ by the formula

$$\mu(x)=\exp\left(\int P(x)dx\right)$$

3. Multiply the equation by $\mu(x)$.

4. Integrate the last equation.

Applications

- **Cooling/Warming law**

We have seen in Section 1.4 that the mathematical formulation of Newton's empirical law of cooling of an object in given by the linear first-order differential equation

$$\frac{dT}{dt} = \alpha(T - T_m)$$

This is a separable differential equation. We have

$$\frac{dT}{(T - T_m)} = \alpha dt$$

or $\ln|T-T_m|=\alpha t+c_1$

or $T(t) = T_m+c_2e^{\alpha t}$

In Series Circuits

i(t), is the solution of the differential equation.

$$\mathrm{I}\frac{di}{dt} + Ri = E(t)$$

Since $i = \frac{dq}{dt}$ It can be written as

$$R\frac{dq}{dt} + \frac{1}{C}q = E(t)$$