Counting

The Basics of counting

※ A counting problem:

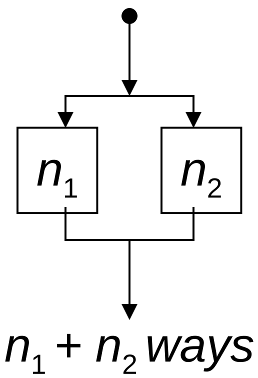
Each user on a computer system has a password, which is six to eight characters long, where each characters is an uppercase letter or a digit. Each password must contain at least one digit. How many possible passwords are there?

- This section introduces
  - a variety of other counting problems
  - the basic techniques of counting.

Basic counting principles

※**The sum rule:**

If a first task can be done in n_1 ways and a second task in n_2 ways, and if these tasks cannot be done at the same time. then there are n_1+n_2 ways to do either task.



Example 1

- Suppose that either a member of the mathematics faculty or a student who is a mathematics major is chosen as a representative to a university committee. How many different choices are there for this representative if there are 37 members of the mathematics faculty and 83 mathematics majors and no one is both a faculty member and a student?

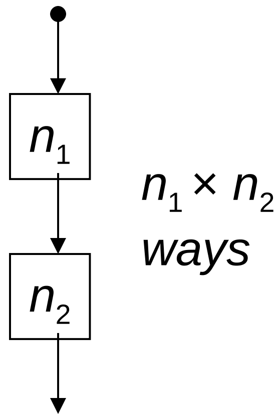
Sol: There are 37 ways to choose a member of the mathematics faculty and there are 83 ways to choose a student who is a mathematics major. Choosing a member of the mathematics faculty is never the same as choosing a student who is a mathematics major because no one is both a faculty member and a student.

By the sum rule it follows that there are 37 + 83 = 120 possible ways to pick this representative.

Example 2 A student can choose a computer project from one of three lists. The three lists contain 23, 15 and 19 possible projects respectively.
How many possible projects are there to choose from?

Sol: $23+15+19=57$ projects.

※ **The product rule:**



Suppose that a procedure can be broken down into two tasks. If there are n_1 ways to do the first task and n_2 ways to do the second task after the first task has been done, then there are $n_1\ n_2$ ways to do the procedure.

Example 3 The chair of an auditorium (The great Hall) is to be labeled with <u>a letter</u> and <u>a positive integer not exceeding 100.</u> What is the largest number of chairs that can be labeled differently?

Sol: $\underbrace{26}_{\text{letter}} \times \underbrace{100}_{\boxed{\begin{array}{l}1 \le x \le 100 \\ x \in N\end{array}}} = 2600$ ways to label chairs.

Example 4 How many different bit strings are there of length seven?

Sol: $\begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ \square & \square & \square & \square & \square & \square & \square \\ \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow \\ 0,1 & 0,1 & 0,1 & & \ldots\ldots & & 0,1 \end{array} \rightarrow 2^7$ ways

Example 4

How many different <u>license plates</u> are available if each plate contains a sequence of <u>3 letters followed by 3 digits</u> ?

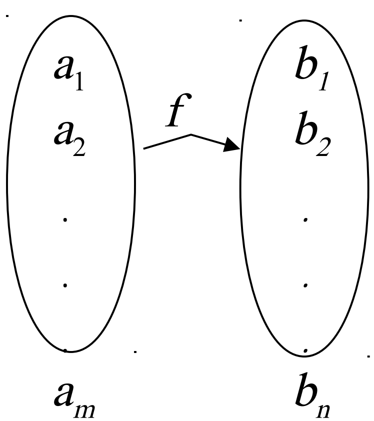
Sol: □ □ □ □ □ □ $\rightarrow 26^3 \cdot 10^3$

letter digit

Example 6

How many functions are there from a set with m elements to one with n elements?

Sol:



$f(a_1)$=? Can map to $b_1, b_2, \ldots b_n$, Totalling ***n*** ways

$f(a_2)$=? Can map to $b_1, b_2, \ldots b_n$, Totalling ***n*** ways

$\vdots$

$f(a_m)$=? Can map to $b_1, b_2, \ldots b_n$, Totalling ***n*** ways

$\therefore n^m$

Example 5 How many one-to-one functions are there from a set with m elements to one with n element? $(m \le n)$

Sol: $f(a_1) = ?$ Can map to $b_1, b_2, \dots b_n$, a total of n functions

$f(a_2) = ?$ Can map to $b_1, b_2, \dots b_n$, but not $= f(a_1)$, a total of $n-1$ functions

$f(a_3) = ?$ Can map to $b_1, b_2, \dots b_n$, but not $= f(a_1)$, or $= f(a_2)$, a total of $n-2$ functions

⋮

⋮

$f(a_m) = ?$ Not $= f(a_1), f(a_2), \dots, f(a_{m-1})$, a total of $n-(m-1)$ functions

$\therefore$ there are a total of $n \times (n-1) \times (n-2) \times \dots \times (n-m+1)$ 1-1 functions #

Example 6 Each user on a computer system has a password which is 6 to 8 characters long, where each character is an <u>uppercase letter</u> or a <u>digit</u>. Each password must <u>contain at least one digit</u>.
How many possible passwords are there?

Sol: P_i: # of possible passwords of length i, i=6,7,8

$P_6 = 36^6 - 26^6$

$P_7 = 36^7 - 26^7$

$P_8 = 36^8 - 26^8$

$\therefore P_6 + P_7 + P_8 = 36^6 + 36^7 + 36^8 - 26^6 - 26^7 - 26^8$

possible passwords

Example 7

In a version of Basic, the name of a variable is a string of one or two alphanumeric characters, where uppercase and lowercase letters are not distinguished. Moreover, a variable name must begin with a letter and must be different from the five strings of two characters that are reserved for programming use. How many different variable names are there in this version of Basic?

Sol:

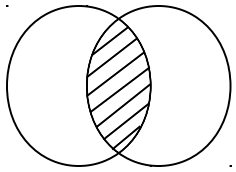
Let V_i be the number of variable names of length i.

$$V_1 = 26$$

$$V_2 = 26 \cdot 36 - 5$$

$\therefore 26 + 26 \cdot 36 - 5$ different names.

※ The Inclusion-Exclusion Principle



$$|A \cup B| = |A| + |B| - |A \cap B|$$

Example 8 How many bit strings of length eight either start with a 1 bit or end with the two bits 00 ?

Sol:

1 2 3 4 5 6 7 8

□□□□□□□□

↑ ↑ . . . .

① **1** 0,1 0,1 → A total of 2^7 bit strings

② . . . . 0 0 → A total of 2^6 bit strings

③ **1** . . . . 0 0 → A total of 2^5 bit strings

$\therefore\ 2^7 + 2^6 - 2^5$ bit strings

Example 9 A computer company receives 350 applications from computer graduates for a job planning a line of new Web servers. Suppose that 220 of these applicants majored in computer science, 147 majored in business, and 51 majored both in computer science and in business. How many of these applicants majored neither in computer science nor in business?

Sol:

34 of the applicants majored neither in computer science nor in business

- The subtraction rule, or the principle of inclusion–exclusion, can be generalized to find the number of ways to do one of *n different tasks or, equivalently, to find the number of elements* in the union of *n sets, whenever n is a positive integer.*
- For 3 sets

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

Example 10

A total of 1232 students have taken a course in Spanish, 879 have taken a course in French, and 114 have taken a course in Russian. Further, 103 have taken courses in both Spanish and French, 23 have taken courses in both Spanish and Russian, and 14 have taken courses in both French and Russian. If 2092 students have taken at least one of Spanish, French, and Russian, how many students have taken a course in all three languages?

Sol:

$|S \cap F \cap R| = 7.$

The Principle of Inclusion–Exclusion

- Let $A_1, A_2, \ldots, A_n$ *be finite sets.* Then

$$|A_1 \cup A_2 \cup \cdots \cup A_n| = \sum_{1 \le i \le n} |A_i| - \sum_{1 \le i < j \le n} |A_i \cap A_j| + \sum_{1 \le i < j < k \le n} |A_i \cap A_j \cap A_k| - \cdots + (-1)^{n+1} |A_1 \cap A_2 \cap \cdots \cap A_n|.$$

- The inclusion–exclusion principle gives a formula for the number of elements in the union of *n sets for every positive integer n. There are terms in this formula for the number of elements* in the intersection of every nonempty subset of the collection of the *n sets is. Hence, there* are $2^n - 1$ *terms in this formula.*

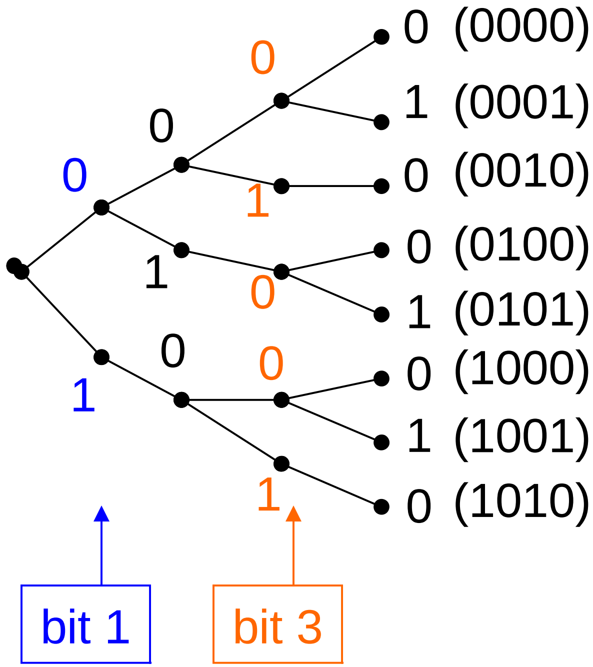
Example 11

Give a formula for the number of elements in the union of four sets.

※ Tree Diagrams

Example10 How many bit strings of length four do not have two consecutive 1s ?

Sol:



∴ 8 bit strings

Exercise

- How many bit strings of length ten both begin and end with a 1?
- How many strings of five ASCII characters contain the character @ (“at” sign) at least once? [*Note:* There are 128 different ASCII characters.
- How many strings of three decimal digits
  a) do not contain the same digit three times?
  b) begin with an odd digit?
  c) have exactly two digits that are 4s?
- How many license plates can be made using either two uppercase English letters followed by four digits or two digits followed by four uppercase English letters?
- How many subsets of a set with 100 elements have more than one element?
- A **palindrome** is a string whose reversal is identical to the string. How many bit strings of length *n* are palindromes?
- How many positive integers not exceeding 100 are divisible either by 4 or by 6?
- How many ways are there to arrange the letters a, b, c, and d such that a is not followed immediately by b?

Ex 40. How many subsets of a set with 100 elements have more than one element ?

Sol: ?

Ex 41. A palindrome is a string whose reversal is identical to the string. How many bit strings of length n are palindromes ?
(abcdcba is a palindrome, abcd is not)

Sol: ?

The Pigeonhole Principle

Theorem 1 (The Pigeonhole Principle)

If $k+1$ or more objects are placed into k boxes, then there is at least one box containing two or more of the objects.

Proof

Suppose that none of the k boxes contains more than one object. Then the total number of objects would be at most k. This is a contradiction.

Example 1. Among any 367 people, there must be at least two with the same birthday, because there are only 366 possible birthdays.

Example 2 In any group of 27 English words, there must be at least two that begin with the same letter.

Example 3 How many students must be in a class to guarantee that at least two students receive the same score on the final exam ? (0~100 points)

Sol: $102.$ $(101+1)$

Theorem 2. (The generalized pigeon hole principle)
If N objects are placed into k boxes, then there is at least one box containing at least $\lceil \frac{N}{k} \rceil$ objects.

e.g. 21 objects, 10 boxes $\Rightarrow$ there must be one box containing at least $\lceil \frac{21}{10} \rceil = 3$ objects.

Example 4 Among 100 people there are at least $\lceil \frac{100}{12} \rceil = 9$ who were born in the same month. (100 objects, 12 boxes)

Example 5: During a month with 30 days a baseball team plays at least 1 game a day, but no more than 45 games. Show that there must be a period of some number of consecutive days during which the team must play exactly 14 games.

day	1	2	3	4	5	... 15		30
# of game	3	2	1					2

$\Rightarrow \text{sum} \leq 45$

Number of games for some time =14

Sol: ?

Def. Suppose that $a_1, a_2, \ldots, a_N$ is a sequence of numbers. A subsequence of this sequence is a sequence of the form $a_{i_1}, a_{i_2}, \ldots, a_{i_m}$ where $1 \le i_1 < i_2 < \ldots < i_m \le N$

$(i.e.,\ \text{A subsequence is a sequence obtained from the original sequence})$

e.g. sequence: 8, 11, 9, 1, 4, 6, 12, 10, 5, 7

subsequence: 8, 9, 12 (✓)

9, 11, 4, 6 (✘)

Def. A sequence is called increasing if $a_i \le a_{i+1}$

A sequence is called decreasing if $a_i \ge a_{i+1}$

A sequence is called strictly increasing if $a_i < a_{i+1}$

A sequence is called strictly decreasing if $a_i > a_{i+1}$

Theorem 3. Every sequence of $\underline{n^2+1}$ distinct real numbers contains a subsequence of length $\underline{n+1}$ that is either strictly increasing or strictly decreasing.

Example 6. The sequence 8, 11, 9, 1, 4, 6, 12, 10, 5, 7 contains $10=3^2+1$ terms (i.e., n=3).
There is a strictly increasing subsequence of length four, namely, 1, 4, 5, 7. There is also a decreasing subsequence of length 4, namely, 11, 9, 6, 5.

Exercise 7 Construct a sequence of 16 positive integers that has no increasing or decreasing subsequence of 5 terms.

Sol: $$\| 4,3,2,1 \| 8,7,6,5 \| 12,11,10,9 \| 16,15,14,13$$

Exercise

- Show that among any group of five (not necessarily consecutive) integers, there are two with the same remainder when divided by 4.
-

  a) Show that if five integers are selected from the first eight positive integers, there must be a pair of these integers with a sum equal to 9.
  b) Is the conclusion in part (a) true if four integers are selected rather than five?

a) How many numbers must be selected from the set { I , 2, 3, 4, 5, 6} to guarantee that at least one pair of these numbers add up to 7?

b) There are 38 different time periods during which classes at a university can be scheduled. If there are 677 different classes, how many different rooms will be needed?

Permutations and Combinations

Example 1. In how many ways can we select three students from a group of five students to stand in line for a picture? In how many ways can we arrange all five of these students in a line for a picture?

Sol: There are five ways to select the first student to stand at the start of the line. Once this student has been selected, there are four ways to select the second student in the line. After the first and second students have been selected, there are three ways to select the third student in the line. By the product rule, there are $5 \times 4 \times 3 = 60$ ways to select three students from a group of five students to stand in line for a picture.

To arrange all five students in a line for a picture, we select the first student in five ways, the second in four ways, the third in three ways, the fourth in two ways, and the fifth in one way. Consequently, there are $5 \times 4 \times 3 \times 2 \times 1 = 120$ ways to arrange all five students in a line for a picture.

This example illustrates how ordered arrangements of distinct objects can be counted. This leads to some terminology.

Permutations and Combinations

Def. A permutation of a set of distinct objects is an ordered arrangement of these objects. An ordered arrangement of r elements of a set is called an r-permutation.

Example 1. Let $S = \{1, 2, 3\}$.

The arrangement <u>3,1,2</u> is a permutation of S.

The arrangement <u>3,2</u> is a 2-permutation of S.

Theorem 1. The number of r-permutation of a set with n distinct elements is

$$P(n,r) = n \cdot (n-1) \cdot (n-2) \ldots (n-r+1) = \frac{n!}{(n-r)!}$$

position : $1 \quad 2 \quad 3 \quad \ldots \quad r$

□ □ □ … □

insert : $n \quad n-1 \quad n-2 \quad \ldots \quad n-r+1$

Example 2. How many different ways are there to select a first-prize winner, a second-prize winner, and a third-prize winner from 100 different people who have entered a contest ?

Sol:

$$P(100,3)=100\times 99\times 98$$

Example 3. Suppose that a saleswoman has to visit 8 different cities. She must begin her trip in a specified city, but she can visit the other cities in any order she wishes. How many possible orders can the saleswoman use when visiting these cities ?

Sol:

Def. An *r*-combination of elements of a set is an unordered selection of r elements from the set.

Example 4: Let S be the set $\{1, 2, 3, 4\}$. Then $\{1, 3, 4\}$ is a 3-combination from S.

Theorem 2 The number of r-combinations of a set with n elements, where n is a positive integer and r is an integer with $0 \le r \le n$, equals

$$C_r^n = C(n,r) = \binom{n}{r} = \frac{p(n,r)}{r!} = \frac{n!}{r!(n-r)!}$$

Known as the **binomial coefficient**

pf **:**

$$P(n,r) = C(n,r) \times r!$$

Example 10. We see that $C(4,2)=6$, since the 2-combinations of $\{a,b,c,d\}$ are the six subsets $\{a,b\}$, $\{a,c\}$, $\{a,d\}$, $\{b,c\}$, $\{b,d\}$ and $\{c,d\}$

Corollary 2. Let n and r be nonnegative integers with $r \le n$. Then $C(n,r) = C(n,n-r)$

***pf* :** From Thm 2.

$$C(n,r)=\frac{n!}{r!(n-r)!}=\frac{n!}{(n-r)!(n-(n-r))!}=C(n,n-r)$$

Meaning: Selecting r elements from n elements, is equivalent to taking out r elements leaving n-r elements remaining

Example 6. How many ways are there to select 5 players from a 10-member tennis team to make a trip to a match at another school ?

Sol: $C(10,5)=252$

Example 7. Suppose there are 9 faculty members in the math department and 11 in the computer science department. How many ways are there to select a committee if the committee is to consist of 3 faculty members from the math department and 4 from the computer science department?

Sol: $C\left(9,3\right)\times C\left(11,4\right)$

Exercise

- How many permutations of {a , b, e, d, e , t, g} end with a?
- How many bit strings of length 10 contain
  - exactly four 1s?
  - at most four 1s?
  - at least four 1s?
  - an equal number of 0s and 1s?

a) A group contains n men and n women. How many ways are there to arrange these people in a row if the men and women alternate?

b) How many permutations of the letters ABCDEFG contain

a) the string BCD? **b)** the string CFGA?

c) the strings BA and GF? **d)** the strings ABC and DE?

e) the strings ABC and CDE? **f)** the strings CBA and BED?

a) Suppose that a department contains 1 0 men and 1 5 women. How many ways are there to form a committee with six members if it must have the same number of men and women?

b) Suppose that a department contains 1 0 men and 1 5 women. How many ways are there to form a committee with six members if it must have more women than men?

Binomial Coefficients

Example 1. $(x+y)^3=(x+y)(x+y)(x+y)=?x^3+?x^2y+?xy^2+y^3$

To obtain a term of the form xy^2 ,

a y must be chosen from two of the brackets, and an x must be chosen from one of the brackets

(Note ： x and y in the same bracket cannot be multiplied) $\Rightarrow xy^2$ The coefficient $=\binom{3}{2}$

∴There are different ways to get xy^2

$$(x+y)^3=\binom{3}{0}x^3+\binom{3}{1}x^2y+\binom{3}{2}xy^2+\binom{3}{3}y^3$$

∴

Theorem 1. (The Binomial Theorem)

Let x,y be variables, and let n be a positive integer, then

$$(x+y)^n=\binom{n}{0}x^n+\binom{n}{1}x^{n-1}y+\ldots+\binom{n}{n-1}xy^{n-1}+\binom{n}{n}y^n=\sum_{j=0}^{n}\binom{n}{j}x^{n-j}y^j$$

Example 2. What is the coefficient of $x^{12}y^{13}$ in the expansion of $(2x-3y)^{25}$?

Sol: $(2x-3y)^{25} = (2x+(-3y))^{25}$

$$\therefore \quad \binom{25}{13} \times 2^{12} \times (-3)^{13}$$

Cor 1. Let n be a positive integer. Then

$$\sum_{k=0}^{n} \binom{n}{k} = \binom{n}{0} + \binom{n}{1} + \ldots + \binom{n}{n} = 2^n$$

pf **:** By Thm 1, let $x = y = 1$

$$(1+1)^n = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \ldots + \binom{n}{n}$$

Cor 2. Let n be a positive integer. Then $\sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0$

pf **:** by Thm 1. $(1-1)^n = 0$

Theorem 2. (Pascal's identity)

Let n and k be positive integers with $n \geq k$

Then

$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$$

PASCAL's triangle

$$\begin{array}{ccccccc}
 & & & \binom{0}{0} & & & \\
 & & \binom{1}{0} & & \binom{1}{1} & & \\
 & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & \\
\binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3} \\
 & & \vdots & & & \binom{4}{3} &
\end{array}$$

$$\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 1 & & 1 & & & \\
 & & 1 & & 2 & & 1 & & \\
 & 1 & & 3 & & 3 & & 1 & \\
1 & & 4 & & 6 & & 4 & & 1 \\
 & & & & \vdots & & & &
\end{array}$$

$$\binom{n+1}{k}=\binom{n}{k-1}+\binom{n}{k}$$

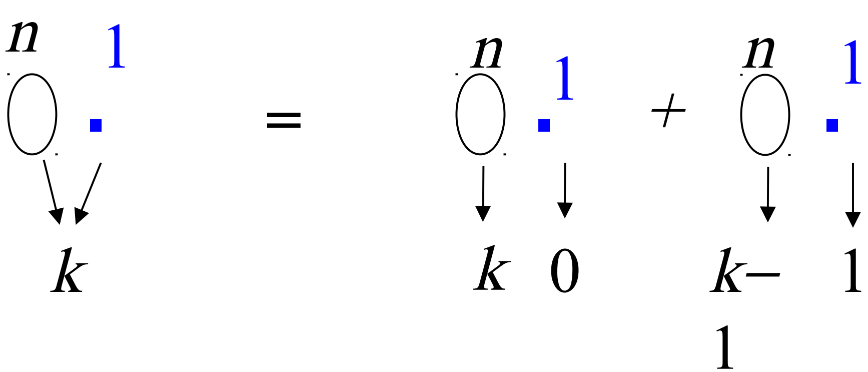
pf : ①(algebraic proof)

$$\binom{n+1}{k}=\frac{(n+1)!}{k!(n+1-k)!}$$

$$\binom{n}{k-1}+\binom{n}{k}=\frac{n!}{(k-1)!(n-k+1)!}+\frac{n!}{k!(n-k)!}$$

$$=\frac{k\times n!}{k!(n-k+1)!}+\frac{(n-k+1)\times n!}{k!(n-k+1)!}=\frac{(n+1)\times n!}{k!(n-k+1)!}$$

②(combinatorial proof):



$$\Rightarrow \binom{n+1}{k}=\binom{n}{k-1}\binom{1}{0}+\binom{n}{k}\binom{1}{1}$$

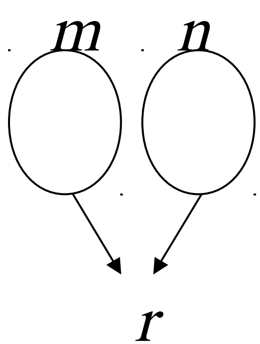
Theorem 3. (Vandermode's Identity)

$$m, n, r \in Z^{+}, \quad 0 \le r \le m, n$$

$$C(m+n, r) = \sum_{k=0}^{r} C(m, r-k) \cdot C(n, k)$$

pf :

$C(m+n, r) =$



$$= \begin{matrix} m & n \\ \downarrow & \downarrow \\ r, & 0 \end{matrix} + \begin{matrix} m & n \\ \downarrow & \downarrow \\ r-1, & 0 \end{matrix} + \ldots + \begin{matrix} m & n \\ \downarrow & \downarrow \\ 0, & r \end{matrix}$$

$$= \binom{m}{r}\binom{n}{0} + \binom{m}{r-1}\binom{n}{1} + \ldots + \binom{m}{0}\binom{n}{r}$$

$$= \sum_{k=0}^{r} C(m, r-k) \cdot C(n, k)$$

Exercise

- What is the coefficient of x^9 in $(2 - x)^{19}$?
- Prove that if n and k are integers with $1 \leq k \leq n$, then $k\binom{n}{k} = n\binom{n-1}{k-1}$
  a) using a combinatorial proof. [Hint: Show that the two sides of the identity count the number of ways to select a subset with k elements from a set with n elements and then an element of this subset.]
  b) using an algebraic proof based on the formula for $\binom{n}{k}$ given in Theorem 2.
- Show that if n is a positive integer, then $\binom{2n}{2} = 2\binom{n}{2} + n^2$
  a) using a combinatorial argument.

Generalized Permutations and Combinations

Permutations with Repetition

Example 1 How many strings of length r can be formed from the English alphabet?

Sol. 26^r

Thm 1. The number of r-permutations of a set of n objects with repetition allowed is n^r.

Thm 2. There are $C(r+n-1, r)$ r-combinations from a set with n elements when repetition of elements is allowed.

pf : (Each r-combination of a set with n elements when repetition is allowed can be represented by a list of $n-1$ bars and r stars. The $n-1$ bars are used to mark off n different cells, with the ith cell containing a star for each time the ith element of the set occurs in the combination)

Example : $n = 4$, set is $\{a_1, a_2, a_3, a_4\}$, $r = 6$

A_2 appear 1 time a_3 does not appear

$$\underbrace{*\ *}_{a_1 \text{ appears 2 times}} \mid * \mid \mid \underbrace{*\ *\ *}_{a_4 \text{ appears 3 times}} \Longrightarrow \{a_1, a_1, a_2, a_4, a_4, a_4\}$$

$$\therefore \binom{r+n-1}{r} \text{ combinations } _{\#}$$

Example 3. Suppose that a cookie shop has 4 different kinds of cookies. How many different ways can 6 cookies be chosen?

Sol: The number of ways to choose six cookies is the number of 6-combinations of a set with four elements. From Theorem 2 this equals $C(4 + 6 - 1, 6) = C(9, 6)$ *ways*

Example 4. How many solutions does the equation

$$x_1 + x_2 + x_3 = 11$$

have, where x_1, x_2, x_3 are nonnegative integers?'

Sol: 11 stars to be inserted between 2 bars

By theorem 2 ➡ $\binom{11+2}{11}$ solutions

※ since x_1, x_2, x_3 are nonnegative integers, $x_1 \geq 1, x_2 \geq 2, x_3 \geq 3,$

Original equation is $x_1 + x_2 + x_3 = 11$

can be changed to $(x_1 - 1) + (x_2 - 2) + (x_3 - 3) = 11 - 1 - 2 - 3 = 5$

At this point $y_1 + y_2 + y_3 = 5$

Where $y_1 = x_1 - 1,\ y_2 = x_2 - 2,\ y_3 = x_3 - 3$ and $y_1, y_2, y_3 \in \boldsymbol{N}$

∴5 stars to be inserted between 2 bars $\Rightarrow \binom{5+2}{5}$ solutions

(Note : case $y_1 = 1, y_2 = 3, y_3 = 1$ is equal to $x_1 = 2, x_2 = 5, x_3 = 4$)

※if the problem is $1 \leq x_1 \leq 3, x_2 \geq 2, x_3 \geq 3$, you need to exclude

$x_1 > 3$ case

(i.e $x_1 \geq 4$ case)

because $(x_1 - 4) + (x_2 - 2) + (x_3 - 3) = 11 - 9 = 2$

∴ A total of $\binom{5+2}{5} - \binom{2+2}{2}$ combinations

Exercise

- How many solutions are there to the equation $x_1 + x_2 + x_3 + x_4 + x_5 = 21$, where x_i , i = 1 , 2, 3, 4, 5, is a nonnegative integer such that
  a) $\mathrm{x}_1 \geq 1$?
  b) $\mathrm{x}_1 \geq 2$ for i = 1,2,3,4,5?
  c) $0 \leq x_1 \leq 10$?
  d) $0 \leq x_1 \leq 3, 1 \leq x_2 < 4, and\ x_3 \geq 15$?
- How many solutions are there to the inequality $x_1 + x_2 + x_3 \leq 11,$ where $x_1\ x_2,\ \ and\ x_3$ are nonnegative integers? [*Hint:* Introduce an auxiliary variable x_4 such that $x_1 + x_2 + x_3 + x_4 = 11$].
- How many positive integers less than 1,000,000 have the

※ Permutations with indistinguishable objects

Example 5. How many different strings can be made by reordering the letters of the word SUCCESS ?

Sol:

There are 3 *S*s, 2 Cs, 1 U and 1 E,

The three *S*s can be placed among the seven positions in **C(7,3)** different ways leaving four positions free.

Then the two Cs can be placed in **C(4, 2)** ways, leaving two free positions.

Then the two Cs can be placed in **C(2, 1)** ways, leaving two free positions.

Hence *E* can be placed in **C(1, 1)** way

Consequently, from the product rule, the number of different strings that can be made is

C(7, 3)C(4, 2)C(2, 1)C(1, 1) = 420

Thm 3. The number of different permutations of n objects, where

$$\begin{cases} \text{type } 1 : & n_1 \text{ ways} \\ \text{type } 2 : & n_2 \text{ ways} \\ \quad \vdots & \\ \text{type } k : & n_k \text{ ways} \end{cases}$$

is $\dfrac{n!}{n_1!\,n_2!\cdots n_k!}$

pf :

$$\binom{n}{n_1}\binom{n-n_1}{n_2}\binom{n-n_1-n_2}{n_3}\cdots\binom{n-n_1-n_2-\cdots-n_{k-1}}{n_k} = \frac{n!}{n_1!\,n_2!\cdots n_k!}$$

Exercise

- Solve the following
  a) How many ways are there to deal hands of five cards to six players from a standard 52-card deck?
  b) How many ways are there to distribute ***n*** distinguishable objects into ***k*** distinguishable boxes so that n_i objects are placed in box ***i***?
- How many ways are there to choose a dozen apples from a bushel containing 20 indistinguishable Delicious apples, 20 indistinguishable Macintosh apples, and 20 indistinguishable Granny Smith apples, if at least three of each kind must be chosen?
- How many ways are there to pack eight identical DVDs into five indistinguishable boxes so that each box contains at least one DVD?
- How many ways are there to distribute six indistinguishable objects into four indistinguishable boxes so that each of the boxes contains at least one object?

Distributing objects into Boxes

Distinguishable Objects and distinguishable boxes

Example 8. How many ways are there to distribute hands of 5 cards to each of four players from the standard deck of 52 cards?

Sol: player 1 : $\binom{52}{5}$ ways

player 2 : from the remaining 5 $\Rightarrow \binom{47}{5}$

$$\therefore \binom{52}{5}\binom{47}{5}\binom{42}{5}\binom{37}{5} = \frac{52!}{5!\,47!}\cdot\frac{47!}{5!\,42!}\cdot\frac{42!}{5!\,37!}\cdot\frac{37!}{5!\,32!} = \frac{52!}{5!\,5!\,5!\,5!\,32!}$$

Note: the above question is equivalent to 52 cards in 5 different box method,

that is box 1 for player 1,

box 2 for player 2

box 3 for player 3

box 4 for player 4

while box 5 is for the cards left.

Exercise

- How many ways can ***n*** books be placed on ***k*** distinguishable shelves
  a) if the books are indistinguishable copies of the same title?
  b) if no two books are the same, and the positions of the books on the shelves matter?

Thm 4. The number of ways to distribute n distinguishable objects into k distinguishable boxes so that n_i objects are placed into box i, i=1, 2, …, k, equals

$$\frac{n!}{n_1!n_2!\cdots n_k!}$$ (the same as Thm 3)

Indistinguishable Objects and distinguishable boxes

Example 9. How many ways are there to place 10 indistinguishable balls into eight distinguishable bins?

Sol: $C(8+10-1, 10)$

$\Rightarrow$There are $C(n+r-1, n-1)$ ways to place r indistinguishable objects into n distinguishable boxes.

Distinguishable Objects and indistinguishable boxes

Example 6. How many ways are there to put four different employees into three indistinguishable offices, when each office can contain any number of employees?

Sol: employees: A, B, C, D

4 people in the same office: {{A, B, C, D}} $\Rightarrow$ 1 way

3 people in the same office, 1 person in another office
: {{A, B, C}, {D}}, {{A, B, D}, {C}},… $\Rightarrow$ 4 ways

2 people in the same office, 2 people in another office
: {{A, B}, {C, D}}, {{A, D}, {B, C}},… $\Rightarrow$ 3 ways

2 people in the same office, The other 2 people in the remaining two Office:
{{A, B}, {C}, {D}}, {{A, D}, {B}, {C}}, … $\Rightarrow$ 6 ways

$\therefore$ Total: 14 ways

Note. There is no simple closed formula. You may refer to Stirling numbers of the second kind. (p. 378)

Indistinguishable Objects and indistinguishable boxes

Example 7. How many ways are there to pack six copies of the same book into four identical boxes, where a box can obtain as many as six books?

Sol:

6	3, 3	2, 2, 1, 1
5, 1	3, 2, 1	$\therefore$ Total 9
4, 2	3, 1, 1, 1	
4, 1, 1	2, 2, 2	

Note. This problem is the same as writing n as the sum of at most k positive integers in nonincreasing order.

That is, $a_1+a_2+\ldots+a_j = n$, where $a_1, a_2, \ldots, a_j$ are positive integers with $a_1 \geq a_2 \geq \ldots \geq a_j$ and $j \leq k$.

No simple closed formula exists.