Tanv

Q1. Integrate $\int \frac{dx}{\sqrt{x+a}+\sqrt{x+b}}$

Soln: Let $I = \int \frac{dx}{\sqrt{x+a}+\sqrt{x+b}}$

Multiplying the numerator and denominator by

$\sqrt{x+a} - \sqrt{x+b}$

$$\therefore I = \int \frac{\sqrt{x+a}-\sqrt{x+b}}{(\sqrt{x+a})^2-(\sqrt{x+b})^2}\,dx = \int \frac{\sqrt{x+a}-\sqrt{x+b}}{a-b}\,dx$$

$$I = \frac{1}{(a-b)} \int \left(\sqrt{x+a}-\sqrt{x+b}\right) dx$$

~~Let~~ $$I = \frac{1}{(a-b)}\left\{\frac{(x+a)^{3/2}}{\frac{3}{2}} \cancel{+} \frac{(x+b)^{3/2}}{3/2}\right\} + c$$

$$\therefore I = \frac{2}{3}\cdot\frac{1}{(a-b)}\left\{(x+a)^{3/2} - (x+b)^{3/2}\right\} + c$$

where c is an integrating constant **Answer.**

Q2. $$I = \int e^{a\sin^{-1}x}\cdot\frac{1}{\sqrt{1-x^2}}\,dx$$

Let $\sin^{-1}x = z$

$$\therefore \frac{1}{\sqrt{1-x^2}}\,dx = dz$$

$$\therefore I = \int e^{az}\cdot dz$$

$$= \frac{e^{az}}{a} + c$$

$$= \frac{1}{a}e^{a\sin^{-1}x} + c$$ **Answer.**

3 COPY

3. (i) $\int \frac{e^x - 1}{e^x + 1}\,dx$ (ii) $\int \frac{dx}{e^x + 1}$ (iii) $\int \frac{e^{2x}}{e^x + 1}\,dx$

Solution : let $I = \int \frac{e^x - 1}{e^x + 1}\,dx$

$$I = \int \frac{e^{x/2}\left(e^{x/2} - e^{-x/2}\right)dx}{e^{x/2}\left(e^{x/2} + e^{-x/2}\right)} = \int \frac{e^{x/2} - e^{-x/2}}{e^{x/2} + e^{-x/2}}\,dx$$

let $e^{x/2} + e^{-x/2} = t$

$$e^{x/2} \cdot \frac{1}{2}dx + e^{-x/2} \cdot \left(-\frac{1}{2}\right)dx = dt$$

$$\therefore \left(e^{x/2} - e^{-x/2}\right)dx = 2\,dt$$

$$I = \int \frac{2dt}{t} = 2\log t + c$$

$$\therefore I = 2\log\left(e^{x/2} + e^{-x/2}\right) + c \quad \underline{\text{Ans.}}$$

(ii) let $I = \int \frac{dx}{e^x + 1}$

$$= \int \frac{e^{-x}dx}{1 + e^{-x}}$$

Put $1 + e^{-x} = t \quad \therefore 0 + e^{-x}(-dx) = dt$

$$\therefore e^{-x}dx = -dt$$

$$I = \int \frac{-dt}{t} = -\log t + c$$

$$= -\log\left(1 + e^{-x}\right) + c$$

(iii) let $I = \int \frac{e^{2x}}{e^x+1}\,dx$

$\therefore I = \int \frac{e^x \cdot e^x}{e^x+1}\,dx$ let, $e^x+1 = t$

$e^x dx = dt$

$\therefore e^x = t-1$

$= \int \frac{(t-1)\,dt}{t}$

$= \int \left(1-\frac{1}{t}\right)dt = \int dt - \int \frac{1}{t}\,dt$

$= t - \log t + c$

Therefore, $I = e^x+1 - \log(e^x+1) + c$. Ans.

4. (i) $\int \sqrt{\frac{a+x}{a-x}}\,dx$ (ii) $\int \sqrt{\frac{x}{a-x}}\,dx$

let $I = \int \sqrt{\frac{a+x}{a-x}}\,dx$

$I = \int \sqrt{\frac{(a+x)(a+x)}{(a-x)(a+x)}}\,dx$

$= \int \frac{(a+x)\,dx}{\sqrt{a^2-x^2}} = \int \frac{a}{\sqrt{a^2-x^2}}\,dx + \int \frac{x\,dx}{\sqrt{a^2-x^2}}$

$= a \cdot \sin^{-1}\left(\frac{x}{a}\right) - \frac{1}{2}\int \frac{-2x\,dx}{\sqrt{a^2-x^2}}$

$= a\sin^{-1}\frac{x}{a} - \frac{1}{2}\,2\sqrt{a^2-x^2} + c$

$I = a\sin^{-1}\frac{x}{a} - \sqrt{a^2-x^2} + c$

where c is an Integrating constant.

$$= -\int \cos\left(2\cot^{-1}\tan\frac{\theta}{2}\right) \sin\theta \, d\theta$$

$$= -\int \cos\left\{2\cot^{-1}\cot\left(\frac{\pi}{2}-\frac{\theta}{2}\right)\right\} \sin\theta \, d\theta$$

$$= -\int \cos\left\{2\left(\frac{\pi}{2}-\frac{\theta}{2}\right)\right\} \sin\theta \, d\theta$$

$$= -\int \cos(\pi-\theta) \sin\theta \, d\theta$$

$$I = \int \cos\theta \sin\theta \, d\theta$$

$$= -\int t \, dt = -\frac{t^2}{2} + c$$

Again Put $\cos\theta = t$

$-\sin\theta \, d\theta = dt$

$$= -\frac{\cos^2\theta}{2} + c$$

$$I = -\frac{x^2}{2} + c \quad \underline{\text{Answer}}$$

6. (i) $\int \frac{dx}{\sqrt{x}+x}$ (ii) $\int \sqrt{\frac{1+x}{1-x}}\, dx$ (Do Your self)

let $I = \int \frac{dx}{\sqrt{x}+x} = \int \frac{dx}{\sqrt{x}(1+\sqrt{x})}$

$$\therefore I = \int \frac{2dt}{t}$$

$$= 2\log t + c$$

$$= 2\log(1+\sqrt{x}) + c$$

Put $1+\sqrt{x} = t$

$0 + \frac{1}{2\sqrt{x}} dx = dt$

$\frac{1}{\sqrt{x}} dx = 2dt$

7. (i) $\int \sqrt{\frac{a+x}{x}}\, dx$ (ii) $\int \frac{\sqrt{x-a}}{x}\, dx$

Solution: (i) let $I = \int \sqrt{\frac{a+x}{x}}\, dx$

$$I = \int \frac{a+x}{\sqrt{x(a+x)}}\,dx = \int \frac{a+x}{\sqrt{ax+x^2}}\,dx$$

$$\cancel{= \int \frac{a}{\sqrt{x^2+ax}}\,dx + \frac{1}{2}} \int \frac{(2x+2a)}{\sqrt{ax+x^2}}\,dx$$

$$= \frac{1}{2}\int \frac{2x+a+a}{\sqrt{x^2+ax}}\,dx = \frac{1}{2}\int \frac{2x+a}{\sqrt{x^2+ax}}\,dx + \frac{1}{2}\int \frac{a\,dx}{\sqrt{x^2+a}}$$

$$= \frac{1}{2}\int \frac{2x+a}{\sqrt{x^2+ax}}\,dx + \frac{a}{2}\int \frac{dx}{\sqrt{\left\{x^2+2\cdot\frac{a}{2}x+\left(\frac{a}{2}\right)^2-\left(\frac{a}{2}\right)^2\right\}}}$$

$$= \frac{1}{2}\cdot 2\sqrt{x^2+ax} + \frac{a}{2}\int \frac{dx}{\sqrt{\left(x+\frac{a}{2}\right)^2-\left(\frac{a}{2}\right)^2}}$$

$$= \sqrt{x^2+ax} + \frac{a}{2}\log\left\{\left(x+\frac{a}{2}\right)+\sqrt{\left(x+\frac{a}{2}\right)^2-\left(\frac{a}{2}\right)^2}\right\}+c$$

$$I = \sqrt{x^2+ax} + \frac{a}{2}\log\left\{\left(x+\frac{a}{2}\right)+\sqrt{x(a+x)}\right\}+c$$

Answer

8. If $a<x<b$, show that

$$\int \frac{dx}{(x-a)\sqrt{\{(x-a)(b-x)\}}} = \frac{2}{a-b}\sqrt{\frac{b-x}{x-a}}$$

$$\text{L.H.S} = \int \frac{dx}{(x-a)\sqrt{(x-a)(b-x)}}$$

Put $x-a = \frac{1}{t}$ $\therefore dx = -\frac{1}{t^2}\,dt$

$\therefore x = \frac{1}{t}+a$

$$\therefore = \int \frac{-\frac{1}{t^2}\,dt}{\frac{1}{t}\sqrt{\frac{1}{t}\left(b-\frac{1}{t}-a\right)}}$$

$$= \int \frac{-\frac{1}{t}\,dt}{\sqrt{\frac{1}{t}\left\{\frac{(b-a)t-1}{t}\right\}}} = \int \frac{-\frac{1}{t}\,dt}{\frac{1}{t}\sqrt{(b-a)t-1}}$$

$$= -\int \frac{dt}{\sqrt{(b-a)t-1}}$$

$$= -\frac{1}{(b-a)}\int \frac{(b-a)\,dt}{\sqrt{(b-a)t-1}}$$

$$= -\frac{1}{(b-a)}\cdot 2\sqrt{(b-a)t-1}$$

$$= -\frac{2}{b-a}\sqrt{(b-a)\cdot\frac{1}{x-a}-1}$$

$\because x-a = \frac{1}{t}$

$t = \frac{1}{x-a}$

$$= \frac{2}{a-b}\sqrt{\frac{b-a-x+a}{x-a}}$$

$$= \frac{2}{a-b}\sqrt{\frac{b-x}{x-a}}$$ (Proved)

9. $\int \frac{dx}{(2x+1)\sqrt{4x+3}}$

Put $4x+3 = t^2$

$4\,dx = 2t\,dt$

$\therefore dx = \frac{t}{2}dt$

$4x = t^2 - 3$

$2x = \frac{1}{2}(t^2-3)$

$2x+1 = \frac{1}{2}t^2 - \frac{3}{2} + 1$

$= \frac{1}{2}t^2 - \frac{1}{2} - \frac{1}{2}(t^2-1)$

$$= \int \frac{\frac{t}{2}dt}{\frac{1}{2}(t^2-1)\cdot t}$$

$$= \int \frac{dt}{t^2-1}$$

$$= \frac{1}{2}\log\frac{t-1}{t+1} + c$$

$$= \frac{1}{2}\log\left\{\frac{\sqrt{(4x+3)}-1}{\sqrt{4x+3}+1}\right\} + c$$

$$= \frac{1}{2}\log\left(\frac{\sqrt{4x+3}-1}{\sqrt{4x+3}+1}\right) + c$$

Answer.

10. $\int \frac{e^{m\tan^{-1}x}}{(1+x^2)^2} dx$

Solution: let $I = \int \frac{e^{m\tan^{-1}x}}{(1+x^2)^2} dx$

$$I = \int \frac{e^{m\theta}}{(1+\tan^2 x)^2} \sec^2\theta \, d\theta \qquad \text{Put } x = \tan\theta$$
$$dx = \sec^2\theta \, d\theta$$

$$= \int \frac{e^{m\theta}}{(\sec^2\theta)^2} \cdot \sec^2\theta \, d\theta$$

$$= \int \frac{e^{m\theta}}{\sec^2\theta} d\theta = \int e^{m\theta} \cos^2\theta \, d\theta$$

$$= \frac{1}{2}\int e^{m\theta} 2\cos^2\theta \, d\theta = \frac{1}{2}\int e^{m\theta}(1+\cos 2\theta)\, d\theta$$

$$= \frac{1}{2}\int \left(e^{m\theta} + e^{m\theta}\cos 2\theta\right) d\theta$$

$$= \frac{1}{2} e^{m\theta} \cdot \frac{1}{m} + \frac{1}{2} \frac{e^{m\theta}(m\cos 2\theta + 2\sin 2\theta)}{m^2+2^2} + c$$

$$= \frac{1}{2m} e^{m\theta} + \frac{1}{2} e^{m\theta} \frac{(m\cos 2\theta + 2\sin 2\theta)}{m^2+2^2} + c$$

$$= \frac{1}{2m} e^{m\tan^{-1}x} + \frac{1}{2} e^{m\tan^{-1}x} \left\{ \frac{m\cos 2(\tan^{-1}x) + 2\sin 2(\tan^{-1}x)}{m^2+4} \right\} + c$$