12. $\int \frac{x+\sin x}{1+\cos x}\,dx$

let $I = \int \frac{x+\sin x}{1+\cos x}\,dx$

$= \int \frac{x}{1+\cos x}\,dx + \int \frac{\sin x}{1+\cos x}\,dx$

$= \int \frac{x}{2\cos^2 x/2}\,dx + \int \frac{\sin x}{1+\cos x}\,dx$

$= \frac{1}{2}\int x \sec^2\frac{x}{2}\,dx - \int \frac{-\sin x}{1+\cos x}\,dx$

$= \frac{1}{2}\left\{x\int \sec^2\frac{x}{2}\,dx - \int\left\{1.\int \sec^2\frac{x}{2}\,dx\right\}dx\right\} - \log(1+\cos x)$

$= \frac{1}{2}\left\{x \tan\frac{x}{2}.2 - \int \tan\frac{x}{2}.2\,dx\right\} - \log(1+\cos x)$

$= x\tan\frac{x}{2} - \int \tan\frac{x}{2}\,dx - \log(1+\cos x)$

Put $\frac{x}{2} = t$

$dx = 2dt$

$= x\tan\frac{x}{2} - \int \tan t.2\,dt - \log(1+\cos x)$

$= x\tan\frac{x}{2} + 2\log(\cos\frac{x}{2}) - \log(1+\cos x) + c$ (Answer)

$= x\tan\frac{x}{2} + \log(\cos^2\frac{x}{2}) - \log(1+\cos x) + c$

$= x\tan\frac{x}{2} + \log(1+\cos x) - \log(1+\cos x) + c'$

$= x\tan\frac{x}{2}$ Answer.

13. $\int \sin^{-1}\sqrt{\frac{x}{x+a}}\,dx$

let $I = \int \sin^{-1}\sqrt{\frac{x}{x+a}}\,dx$

Put $x = a\tan^2\theta$

$dx = 2a\tan\theta\sec^2\theta\,d\theta$

$$\therefore I = \int \sin^{-1}\sqrt{\frac{a\tan^2\theta}{a(1+\tan^2\theta)}}\, 2a\tan\theta\sec^2\theta\, d\theta$$

$$= \int \sin^{-1}\sqrt{\left(\frac{\tan^2\theta}{\sec^2\theta}\right)} \cdot 2a\tan\theta\sec^2\theta\, d\theta$$

$$= \int \sin^{-1}(\sin\theta)\, 2a\tan\theta\sec^2\theta\, d\theta$$

$$= 2a\int \theta\tan\theta\sec^2\theta\, d\theta$$

$$= 2a\left[\theta\cdot\int\tan\theta\sec^2\theta\, d\theta - \int\left\{1\cdot\int\tan\theta\sec^2\theta\, d\theta\right\}d\theta\right.$$

$$= 2a\left[\theta\cdot \cancel{\frac{\tan^2\theta}{2}} \int\tan\theta\, d(\tan\theta) - \int\left\{\int\tan\theta\, d(\tan\theta)\right\}\right.$$

$$= 2a\left[\theta\,\frac{\tan^2\theta}{2} - \int\frac{\tan^2\theta}{2}\, d\theta\right]$$

$$= 2a\left[\frac{\theta}{2}\tan^2\theta - \frac{1}{2}\int(\sec^2\theta - 1)\, d\theta\right]$$

$$= a\left[\theta\tan^2\theta - (\tan\theta - \theta)\right] + c$$

$$= a\left[\frac{x}{a}\tan^{-1}\sqrt{\frac{x}{a}} - \sqrt{\frac{x}{a}} + \tan^{-1}\sqrt{\frac{x}{a}}\right] + c$$

$$= x\tan^{-1}\sqrt{\frac{x}{a}} - a\sqrt{\frac{x}{a}} + a\tan^{-1}\sqrt{\frac{x}{a}} + c$$

$$= (x+a)\tan^{-1}\sqrt{\frac{x}{a}} - \sqrt{ax} + c$$ <u>Answer</u>

13. (1) Integrate $\int\frac{e^x}{x}(1+x\log x)\, dx$

$$I = \int e^x\left(\frac{1}{x} + \log x\right) dx$$

$$= \int e^x\left(\log x + \frac{1}{x}\right) dx$$

$$= e^x\log x + c$$ <u>Answer</u>

$$\int \frac{x+\sin x}{1+\cos x}\,dx$$

let $I = \int \frac{x+\sin x}{1+\cos x}\,dx$

$= \int \frac{x}{1+\cos x}\,dx + \int \frac{\sin x}{1+\cos x}\,dx$

$= \int \frac{x}{2\cos^2 x/2}\,dx + \int \frac{\sin x}{1+\cos x}\,dx$

$= \frac{1}{2}\int x \sec^2\frac{x}{2}\,dx - \int \frac{-\sin x}{1+\cos x}\,dx$

$= \frac{1}{2}\left\{x\int \sec^2\frac{x}{2}\,dx - \int\left\{1.\int \sec^2\frac{x}{2}\,dx\right\}dx\right\} - \log(1+\cos x)$

$= \frac{1}{2}\left\{x \tan\frac{x}{2}\cdot 2 - \int \tan\frac{x}{2}\cdot 2\,dx\right\} - \log(1+\cos x)$

$= x\tan\frac{x}{2} - \int \tan\left(\frac{x}{2}\right)dx - \log(1+\cos x)$

Put $\frac{x}{2} = t$

$dx = 2dt$

$= x\tan\frac{x}{2} - \int \tan t \cdot 2\,dt - \log(1+\cos x)$

$= x\tan\frac{x}{2} + 2\log(\cos\frac{x}{2}) - \log(1+\cos x) + c$ (Answer)

$= x\tan\frac{x}{2} + \log(\cos^2\frac{x}{2}) - \log(1+\cos x) + c$

$= x\tan\frac{x}{2} + \log(1+\cos x) - \log(1+\cos x) + c'$

$= x\tan\frac{x}{2}$ Answer.

13. $\int \sin^{-1}\sqrt{\frac{x}{x+a}}\,dx$

let $I = \int \sin^{-1}\sqrt{\frac{x}{x+a}}\,dx$

Put $x = a\tan^2\theta$

$dx = 2a\tan\theta\sec^2\theta\,d\theta$

$\therefore I = \int \sin^{-1}\sqrt{\frac{a\tan^2\theta}{a(1+\tan^2\theta)}} \cdot 2a\tan\theta\sec^2\theta\, d\theta$

$= \int \sin^{-1}\sqrt{\left(\frac{\tan^2\theta}{\sec^2\theta}\right)} \cdot 2a\tan\theta\sec^2\theta\, d\theta$

$= \int \sin^{-1}(\sin\theta)\, 2a\tan\theta\sec^2\theta\, d\theta$

$= 2a\int \theta\tan\theta\sec^2\theta\, d\theta$

$= 2a\left[\theta\cdot\int \tan\theta\sec^2\theta\, d\theta - \int\left\{1\cdot\int \tan\theta\sec^2\theta\, d\theta\right\}d\theta\right.$

$= 2a\left[\theta\cdot \cancel{\frac{\tan^2\theta}{2}} \int \tan\theta\, d(\tan\theta) - \int\left\{\int \tan\theta\, d(\tan\theta)\right\}\right.$

$= 2a\left[\theta\frac{\tan^2\theta}{2} - \int \frac{\tan^2\theta}{2}\, d\theta\right]$

$= 2a\left[\frac{\theta}{2}\tan^2\theta - \frac{1}{2}\int(\sec^2\theta - 1)\, d\theta\right]$

$= a\left[\theta\tan^2\theta - (\tan\theta - \theta)\right] + c$

$= a\left[\frac{x}{a}\tan^{-1}\sqrt{\frac{x}{a}} - \sqrt{\frac{x}{a}} + \tan^{-1}\sqrt{\frac{x}{a}}\right] + c$

$= x\tan^{-1}\sqrt{\frac{x}{a}} - a\sqrt{\frac{x}{a}} + a\tan^{-1}\sqrt{\frac{x}{a}} + c$

$= (x+a)\tan^{-1}\sqrt{\frac{x}{a}} - \sqrt{ax} + c$ Answer

13. ① Integrate $\int \frac{e^x}{x}(1 + x\log x)\, dx$

$I = \int e^x\left(\frac{1}{x} + \log x\right) dx$

$= \int e^x\left(\log x + \frac{1}{x}\right) dx$

$= e^x\log x + c$ Answer

13. If $U=\int e^{ax}\cos bx\,dx$, $V=\int e^{ax}\sin bx\,dx$. Prove that

(a) $\tan^{-1}\frac{V}{U}+\tan^{-1}\frac{b}{a}=bx$ (b) $(a^2+b^2)(U^2+V^2)=e^{2ax}$.

Solution: We can write

$$U=\int e^{ax}\cos bx\,dx=\frac{e^{ax}}{\sqrt{a^2+b^2}}\cos\left(bx-\tan^{-1}\frac{b}{a}\right)\longrightarrow ①$$

and $$V=\int e^{ax}\sin bx\,dx=\frac{e^{ax}}{\sqrt{a^2+b^2}}\sin\left(bx-\tan^{-1}\frac{b}{a}\right)\longrightarrow ②$$

Dividing ② by ① we get

$$\frac{V}{U}=\tan\left(bx-\tan^{-1}\frac{b}{a}\right)$$

$$\Rightarrow \tan^{-1}\frac{V}{U}=bx-\tan^{-1}\frac{b}{a}$$

$$\therefore \tan^{-1}\frac{V}{U}+\tan^{-1}\frac{b}{a}=bx \quad \text{(Proved)}$$

Squaring equation ① and ②, then Adding we get

$$U^2+V^2=\left\{\frac{e^{ax}}{\sqrt{a^2+b^2}}\right\}^2\left\{\cos^2\left(bx-\tan^{-1}\frac{b}{a}\right)+\sin^2\left(bx-\tan^{-1}\frac{b}{a}\right)\right\}$$

or $$U^2+V^2=\frac{e^{2ax}}{a^2+b^2}\cdot 1$$

$$\therefore (a^2+b^2)(U^2+V^2)=e^{2ax}$$

(Proved)

14. Integrate $\int \frac{xe^x}{(x+1)^2}dx$

Let $$I=\int\frac{xe^x}{(x+1)^2}dx=\int\frac{(x+1)-1}{(x+1)^2}\cdot e^x\,dx=\int\frac{e^x dx}{(x+1)}-\int\frac{e^x dx}{(x+1)^2}$$

$$=\int e^x\left\{\frac{1}{x+1}-\frac{1}{(x+1)^2}\right\}dx$$

$$=e^x\frac{1}{x+1}+c \qquad \because \int e^x\{f(x)+f'(x)\}=e^x f(x)$$

15. Integrate $\int \log(x+\sqrt{x^2+a^2})\,dx$

Let, $I = \int \log(x+\sqrt{x^2+a^2})\,dx$

$$= \log(x+\sqrt{x^2+a^2})\int dx - \int\left[\frac{d}{dx}\left\{\log(x+\sqrt{x^2+a^2})\right\}\int dx\right]dx$$

$$= \log(x+\sqrt{x^2+a^2})\cdot x - \int\left\{\frac{1+\frac{1}{2\sqrt{x^2+a^2}}\cdot(2x+0)}{x+\sqrt{x^2+a^2}}\right\}\cdot x\,dx$$

$$= x\log(x+\sqrt{x^2+a^2}) - \int \frac{\frac{\sqrt{x^2+a^2}+x}{\sqrt{x^2+a^2}}}{x+\sqrt{x^2+a^2}}\cdot x\,dx$$

$$= x\log(x+\sqrt{x^2+a^2}) - \int \frac{x\,dx}{\sqrt{x^2+a^2}}$$

$$= x\log(x+\sqrt{x^2+a^2}) - \frac{1}{2}\int \frac{2x\,dx}{\sqrt{x^2+a^2}}$$

$$= x\log(x+\sqrt{x^2+a^2}) - \frac{1}{2}\cdot 2\sqrt{x^2+a^2} + c$$

$$= x\log(x+\sqrt{x^2+a^2}) - \sqrt{x^2+a^2} + c$$

Answer

16. Integrate $\int (3x-2)\sqrt{x^2-x+1}\,dx$

Let $I = \int (3x-2)\sqrt{x^2-x+1}\,dx$

Since $3x-2 = \frac{3}{2}(2x-1)+\frac{3}{2}-2$

$$= \frac{3}{2}(2x-1)+\frac{3-4}{2}$$

$$= \frac{3}{2}(2x-1)-\frac{1}{2}$$

$\therefore\ I = \int\left\{\frac{3}{2}(2x-1)-\frac{1}{2}\right\}\sqrt{x^2-x+1}\,dx$

$$= \frac{3}{2}\int(2x-1)\sqrt{x^2-x+1}\,dx - \frac{1}{2}\int\sqrt{x^2-x+1}\,dx$$

$I = I_1 + I_2$ (say) $\longrightarrow$ (1)

Where $I_1 = \frac{3}{2}\int(2x-1)\sqrt{x^2-x+1}\,dx$, & $I_2 = -\frac{1}{2}\int\sqrt{x^2-x+1}\,dx$

$$\therefore \quad I_1 = \frac{3}{2}\int (2x-1)\sqrt{x^2-x+1}\,dx$$

$$I_1 = \frac{3}{2}\int 2z\,dz\; z\,dz \qquad \Bigg| \quad \text{Put } x^2-x+1 = z^2$$

$$(2x-1)\,dx = 2z\,dz$$

$$= 3\int z^2\,dz$$

$$= 3\,\frac{z^3}{3} + c_1$$

$$I_1 = \left(\sqrt{x^2-x+1}\right)^3 + c_1 = (x^2-x+1)^{3/2} + c_1$$

And $$I_2 = -\frac{1}{2}\int\sqrt{x^2-x+1}\,dx$$

$$= -\frac{1}{2}\int\sqrt{x^2-2\cdot\tfrac{1}{2}x+\left(\tfrac{1}{2}\right)^2-\tfrac{1}{4}+1}\,dx$$

$$= -\frac{1}{2}\int\sqrt{\left(x-\tfrac{1}{2}\right)^2+\tfrac{3}{4}}\,dx$$

Rule! $\int\sqrt{x^2+a^2}\,dx = \frac{x\sqrt{x^2+a^2}}{2} + \frac{a^2}{2}\log\left(x+\sqrt{x^2+a^2}\right)$

$$= -\frac{1}{2}\int\sqrt{\left(x-\tfrac{1}{2}\right)^2+\left(\tfrac{\sqrt{3}}{2}\right)^2}\,dx$$

$$= -\frac{1}{2}\left[\frac{\left(x-\frac{1}{2}\right)\sqrt{x^2-x+1}}{2} + \frac{3}{4\cdot 2}\log\left\{\left(x-\tfrac{1}{2}\right)+\sqrt{x^2-x+1}\right\}\right] + c_2$$

$$= -\frac{1}{2}\left[\frac{(2x-1)\sqrt{x^2-x+1}}{4} + \frac{3}{8}\log\left\{\left(x-\tfrac{1}{2}\right)+\sqrt{x^2-x+1}\right\}\right] + c_2$$

$$I_2 = -\frac{1}{8}(2x-1)\sqrt{x^2-x+1} - \frac{3}{16}\log\left\{\left(x-\tfrac{1}{2}\right)+\sqrt{x^2-x+1}\right\} + c_2$$

Then from eq^n ① becomes

$$I = (x^2-x+1)^{3/2} - \frac{1}{8}(2x-1)\sqrt{x^2-x+1} - \frac{3}{16}\log\left(x-\tfrac{1}{2}+\sqrt{x^2-x+1}\right) + A$$

where $A = c_1 + c_2$ is another constant.

17. Integrate (a) $\displaystyle\int\frac{2\sin x+3\cos x}{3\sin x+4\cos x}\,dx$ (b) $\displaystyle\int\frac{\cos x\,dx}{2\sin x+3\cos x}$ (do Yourself)

(c) $\displaystyle\int\frac{11\cos x-16\sin x}{2\cos x+5\sin x}\,dx$

(e) let $I = \int \frac{11\cos x - 16\sin x}{2\cos x + 5\sin x}\,dx$

Again let, $11\cos x - 16\sin x = l$(denominator) + m(differential coefficient of denominator)

$11\cos x - 16\sin x = l(2\cos x + 5\sin x) + m(-2\sin x + 5\cos x) \longrightarrow (*)$

or, $11\cos x - 16\sin x = (2l + 5m)\cos x + (5l - 2m)\sin x$

Equating the coefficient of cosx and sinx

$2l + 5m = 11 \longrightarrow (1)$

$5l - 2m = -16 \longrightarrow (2)$

Multiple eqn (1) by 2 and equation (2) by 5, then adding

$$\begin{aligned} 4l + 10m &= 22 \\ 25l - 10m &= -80 \\ \hline 29l + 0 &= -58 \end{aligned}$$

$\therefore l = -2$

from eqn (1) $-4 + 5m = 11$

$\therefore m = 3$

Putting the value of l and m in eqn (*) we get

$11\cos x - 16\sin x = -2(2\cos x + 5\sin x) + 3(-2\sin x + 5\cos x)$

$\therefore I = \int \frac{-2(2\cos x + 5\sin x) + 3(-2\sin x + 5\cos x)}{2\cos x + 5\sin x}\,dx$

$= -2\int dx + 3\int \frac{-2\sin x + 5\cos x}{2\cos x + 5\sin x}\,dx$

$= -2x + 3\log(2\cos x + 5\sin x) + c$

Hence, $I = 3\log(2\cos x + 5\sin x) - 2x + c$

Answer

18. (a) $\int \frac{dx}{a+b\sin x}dx$ (b) $\int \frac{dx}{a+b\cos x}$ (Do Yourself)

let $I = \int \frac{dx}{a+b\sin x}$

$= \int \frac{dx}{a(\sin^2\frac{x}{2}+\cos^2\frac{x}{2})+b\,2\sin x/2\cos x/2}$

$= \int \frac{\sec^2\frac{x}{2}dx}{a(\tan^2\frac{x}{2}+1)+2b\tan x/2}$

$\therefore I = \int \frac{2dt}{a(t^2+1)+2bt}$

Put $\tan\frac{x}{2} = t$

$\sec^2\frac{x}{2}\cdot\frac{1}{2}dx = dt$

$\sec^2\frac{x}{2}dx = 2dt$

$= \int \frac{2dt}{at^2+a+2bt}$

$= \frac{2}{a}\int \frac{dt}{t^2+2\frac{b}{a}\cdot t+1}$

$= \frac{2}{a}\int \frac{dt}{t^2+2\frac{b}{a}\cdot t+(\frac{b}{a})^2-(\frac{b}{a})^2+1}$

$I = \frac{2}{a}\int \frac{dt}{(t+\frac{b}{a})^2+\frac{a^2-b^2}{a^2}}$

Case (i): If $a>b$ then

$I = \frac{2}{a}\int \frac{dt}{(t+\frac{b}{a})^2+(\frac{\sqrt{a^2-b^2}}{a})^2}$

$= \frac{2}{a}\cdot\frac{a}{\sqrt{a^2-b^2}}\tan^{-1}\frac{t+\frac{b}{a}}{\frac{\sqrt{a^2-b^2}}{a}}+C$

$= \frac{2}{\sqrt{a^2-b^2}}\tan^{-1}\left(\frac{ta+b}{a}\cdot\frac{a}{\sqrt{a^2-b^2}}\right)+C$

$= \frac{2}{\sqrt{a^2-b^2}}\tan^{-1}\left(\frac{ta+b}{\sqrt{a^2-b^2}}\right)+C$

$$\therefore \quad I = \frac{2}{\sqrt{a^2-b^2}} \tan^{-1}\left(\frac{a\tan\frac{x}{2}+b}{\sqrt{a^2-b^2}}\right) + C$$

Case (ii) : If $a<b$, then (a^2-b^2) is negative, we have

$$I = \frac{2}{a}\int \frac{dt}{\left(t+\frac{b}{a}\right)^2 - \frac{b^2-a^2}{a^2}}$$

$$= \frac{2}{a}\int \frac{dt}{\left(t+\frac{b}{a}\right)^2 - \left(\frac{\sqrt{b^2-a^2}}{a}\right)^2}$$

$$= \frac{2}{a} \cdot \frac{1}{2\frac{\sqrt{b^2-a^2}}{a}} \cdot \log\left(\frac{t+\frac{b}{a}-\frac{\sqrt{b^2-a^2}}{a}}{t+\frac{b}{a}+\frac{\sqrt{b^2-a^2}}{a}}\right) + C$$

$$= \frac{1}{\sqrt{b^2-a^2}} \log\left(\frac{ta+b-\sqrt{b^2-a^2}}{at+b+\sqrt{b^2-a^2}}\right) + C$$

$$\therefore I = \frac{1}{\sqrt{b^2-a^2}} \log\left(\frac{a\tan x/2+b-\sqrt{b^2-a^2}}{a\tan\frac{x}{2}+b+\sqrt{b^2-a^2}}\right) + C$$

Answer