Art. 52. $f(x)=x^n+p_1x^{n-1}+p_2x^{n-2}+ \dots \dots +p_{n-1}x+p_n=0$ **is a polynmial and** $a+\sqrt{2}b$ **be a root of** $f(x)=0$ (a and b are integers) then $a-\sqrt{2}b$ **will also be a root of** $f(x)=0$

Let $a+\sqrt{2}b=x$, $f(x)=0$

Then $f(a+\sqrt{2}b)=0$. But $f(a+\sqrt{2}b)=A+\sqrt{2}B$.

or, $A+\sqrt{2}B=0$ i. e., $A=0$ and $B=0$

Otherwise $\sqrt{2}=-A/B$. Since $\sqrt{2}$ is an irrational number, so the ratio is impossible.

Hence $A=0$ and $B=0$

In the same way $A-\sqrt{2}B=0=f(a-\sqrt{2}b)$.

Hence $x=a-\sqrt{2}b$ is the other root of $f(x)=0$.

Art. 52. (b) In an equation with real Co-efficients, imaginary roots occur in pairs.

Let $f(x)=0$ be an equation with real co-efficients. Let $a+ib$; be an imaginay root of $f(x)=0$. It is now required to show that $a-ib$ is also a root of $f(x)=0$.

The product factors of $f(x)$ corresponding to these roots

$\{x-(a+ib)\}\{x-(a-ib)\}=(x-a)^2+b^2$

Let $f(x)$ be divided by $(x-a)^2+b^2$; then we have

$f(x)=Q\{(x-a)^2+b^2\}+Rx+R'$ ... ... (1)

where Q is the quotient of degree $(n-2)$ in x and the remainder, if any is $Rx+R'$

If we put $x=a+ib$, in (1), then $f(x)=0$ by hypothesis, also

$(x+a)^2+b^2=0$; hence $(a+ib)+R'=0$

or, $Ra+R'+i\,Rb=0$ ... ... (2)

Equating to zero the real and imaginary parts of (2)

we have $Ra+R'=0$, $R.b=0$ ... ... (3)

by hypothesis, $b\neq0$. Therefore, $R=0$ Hence (3) we have $R'=0$

Hence $f(x)$ is exactly divisible by $(x-a)^2+b^2$ i. e. $\{x-(a+ib)\}\{x-(a-ib)\}$

Thus $x=a-ib$ is also a root.

Art. 53. Relation between roots and Co-efficients. C. H. 199

Let a_1, a_2 .... a_n be the roots of $f(x)=0$, where

$f(x)\equiv x^n+p_1x^{n-1}+p_2x^{n-2}+ \dots +p_{n-1}x+p_n$

Then

$(x-a_1)(x-a_2)\dots(x-a_n)=x^n-x^{n-1}\sum a_1+x^{n-2}\sum a_1a_2+\dots\dots+(-1)^n a_1a_2 \dots\dots a_n$

Equating the co-efficients of like powers from two sides we have

$p_1=\sum a_1=-$(sum of the roots)

$p_2=\sum a_1a_2=$ sum of the products of the roots taken two at a time.

$p_3=-\sum a_1a_2a_3=-$ (Sum of the products of the roots taken three at a time) and so on.

$p_n=(-1)^n a_1a_2a_3 \dots\dots a_n$, products of roots.

If the given equation is of the form.

$p_0x^n+p_1x^{n-1}+\dots\dots+p_{n-1}x+p_n=0$, $p_0\neq0$

The co-efficient of x is obtained by putting the three x's equal to zero except x in the 1st column and the 1st row : The determinant is now $\begin{vmatrix} 0 & d & e \\ -d & 0 & f \\ -e & -f & 0 \end{vmatrix}$ which is skew symmetric determinant of odd order and so vanishes. There are $^4c_1$ i. e. 4 such determinants and all of them vanish.

... x^2 and they are a^2, b^2, c^2, d^2, e^2. Thus the total co- ... $+f^2$) ie $^4c_2$ terms = 6 terms.

Clearly, the co-efficient of x^4 is unity.

Hence $D = x^4+x^2(a^2+b^2+c^2+d^2+e^2+f^2)+(af-bc+cd)^2$

Ex. 8 Solve the equation $\equiv \begin{vmatrix} x & -6 & -1 \\ 2 & -3x & x-3 \\ -3 & 2x & x+2 \end{vmatrix} = 0$ ✓

Applying R_1-R_2 we get

$$\begin{vmatrix} x-2 & 3(x-2) & -(x-2) \\ 2 & -3x & x-3 \\ -3 & 2x & x+2 \end{vmatrix} = (x-2)\begin{vmatrix} 1 & 3 & 1 \\ 2 & -3x & x-3 \\ -3 & 3x & x+2 \end{vmatrix}$$

Now applying c_2-3c_1, c_3+c_1 we get

$$\Delta = (x-2)\begin{vmatrix} 1 & 0 & 0 \\ 2 & -3x-6 & x-1 \\ -3 & 2x+9 & x-1 \end{vmatrix} = (x-2)(x-1)\begin{vmatrix} -3x-6 & 1 \\ 2x+9 & 1 \end{vmatrix}$$

$=(x-2)(x-1)(-5x-15)=0 \therefore x= 1, 2, -3$

Ex. 9. Solve the equation $2x-y-z=4$, $x-2y+z=5$, $x-2y+2z-1=0$

$$\therefore \frac{x}{\begin{vmatrix} -1 & -1 & 4 \\ -2 & 1 & 5 \\ -2 & 2 & 1 \end{vmatrix}} = \frac{-y}{\begin{vmatrix} 2 & -1 & 4 \\ 1 & 1 & 5 \\ 1 & 2 & 1 \end{vmatrix}} = \frac{z}{\begin{vmatrix} 2 & -1 & 4 \\ 1 & -2 & 5 \\ 1 & -2 & 1 \end{vmatrix}} = \frac{1}{\begin{vmatrix} 2 & -1 & -1 \\ 1 & -2 & 1 \\ 1 & -2 & 2 \end{vmatrix}}$$

or, $x = \frac{\Delta_1}{\Delta}, y = -\frac{\Delta_2}{\Delta}, z = \frac{\Delta_3}{\Delta}$ or; $x = \frac{9}{-3} = -3, y = \frac{-18}{-3} = -6, z = \frac{12}{-3} = -4$

$\therefore x = 3, y = -6, z = -4$

Ex. 10. Solve the equation $\begin{vmatrix} 1 & 1 & 1 \\ x & a & b \\ x^3 & a^3 & b^3 \end{vmatrix} = 0$

Applying c_2-c_1; c_3-c_2 we get

$$\begin{vmatrix} 1 & 0 & 0 \\ x & a-x & b-a \\ x^3 & a^3-x^3 & b^3-a^3 \end{vmatrix} = 0 \quad \text{or;} \begin{vmatrix} a-x & b-a \\ a^3-x^3 & b^3-a^3 \end{vmatrix} = 0$$

$$\begin{vmatrix} a+6b & a+7b & a+8b \end{vmatrix}$$

50. Show that

$$\begin{vmatrix} -a^2 & ab & ac \\ ab & -b^2 & bc \\ ab & bc & -c^2 \end{vmatrix}$$ **Ans.** $4a^2b^2c^2$

51. (a) Prove that

$$\begin{vmatrix} x+a & a & a \\ b & x+b & b \\ c & c & x+c \end{vmatrix} = x^2(x+a+b+c)$$ C. U. 1988

Prove that

$$\begin{vmatrix} -2a & a+b & c+a \\ b+a & -2b & b+c \\ c+a & c+b & -2c \end{vmatrix} = 4(b+$$

51. Show that

$$\begin{vmatrix} 1+x & 2 \\ 1 & 2+x \\ 1 & 2 \\ 1 & 2 \end{vmatrix}$$

52. If x, y, z are differe

$$\begin{vmatrix} x & x^2 & 1+ \\ y & y^2 & 1+ \\ z & z^2 & 1+ \end{vmatrix}$$

Show that xyz +

Apply Cramer's Rule to solve the equation. (ক্রেমারের নিয়মে সমাধা

53. $3x+2y-z=20$ ✓
$2x+3y+6z=70$
$x-y+6z=41$
Ans. $x=5, y=6, z=7$

54. $x+2y+3z=14$ ✓
$2x+3y+4z=20$
$3x+4y+6z=33$
Ans. $x=5, y=-6, z=7$

55. $2x+$
$3x+$
$4x+$
Ans

56. $x-2y+3z=11$
$2x+y+2z=10$
$3x+2y+z=9$
Ans. $x=2, y=0, z=3$

57. $3x-2y=5$ ✓
$4y-z=4$
$2z+3y=14$
Ans. $y=3, y=2, z=4$

57. (a) $x+$
$x+2$
$x+y$
Ans. $x=1,$

57. (b) Solve the following equation with the help of determi
$x+y+z=0, 3x+2y+2z+1, x-y-2z=1$
Ans. $x=0, y=1, z=-1$

58. Find the vlues of λ for which the following equation
equations for the values of λ also (সমীকরণগুলির বৈধতার জন্য λ-এর মান
(i) $3x+\lambda y=5, \lambda x-3y=-4, 3x-y=-1$ **Ans.** $\lambda=3, \lambda=14$ $x=1/6, y=3/2$
$3x-2y+1=0, 4x-\lambda y+2=0$ **Ans.** $\lambda=3, x=1, y=$
uations (সমীকরণগুলি বৈধতার পরী
$10z=23$ **Ans.** Co

নির্ণায়কের বিপরীত একটি প্রতিসাম্য নির্ণায়ক।)

6. Prove that the square of any determinant is symmetric d
একটি প্রতিসাম্য হয়।)

7. Show that the reciprocal adjoint of a skew-symmetric
symmetric. (বিজোড় প্রতিসাম্য নির্ণায়কের বিপরীত একটি প্রতিসাম্য হয়।)

7. (a) Adjugate of a skew symmetric determinant of even order
অবসাম্য নির্ণায়কের অনুবন্ধ নির্ণায়ক একটি অবসাম্য নির্ণায়ক হইবে।)

8. Prove that the square of any determinant of even order be e
determinant. (যুগ্ম পর্যায়ের নির্ণায়ক অবসাম্য নির্ণায়ক।)

9. Prove the following indentities. (অভেদগুলি প্রমাণ কর।)

$$\begin{vmatrix} a^2 & bc & ac+c^2 \\ a^2+ab & b^2 & ac \\ ab & b^2+bc & c^2 \end{vmatrix} = 4a^2b^2c^2$$

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9. (a) $\begin{vmatrix} a \\ a-b \\ b+c \end{vmatrix}$

10. $$\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = (b-c)(c-a)(a-b)$$

11. $\begin{vmatrix} a-b-c \\ 2b \\ 2c \end{vmatrix}$

12. $$\begin{vmatrix} bc & a & a^2 \\ ca & b & b^2 \\ ab & c & c^2 \end{vmatrix} = \begin{vmatrix} 1 & a^2 & a^3 \\ 1 & b^2 & b^3 \\ 1 & c^2 & c^3 \end{vmatrix} = (c-a)(c-b)(b$$

13. $$\begin{vmatrix} b^2c^2 & bc & b+c \\ c^2a^2 & ca & c+a \\ a^2b^2 & ab & a+b \end{vmatrix} = 0$$

13. (a) $\begin{vmatrix} 1 \\ \cos(\alpha-\beta) \\ \cos(\alpha-\gamma) \end{vmatrix}$

14. $$\begin{vmatrix} (b+c)^2 & a^2 & a^2 \\ b^2 & (c+a)^2 & b^2 \\ c^2 & c^2 & (a+b)^2 \end{vmatrix} = \begin{vmatrix} (a+b)^2 & ca & bc \\ ca & (b+c)^2 & ab \\ bc & ab & (c+a)^2 \end{vmatrix}$$

or, $u+v, -\frac{1}{2}(u+v) \pm \frac{1}{2}(u-v)\sqrt{(-3)}$; where $u^3+v^3=-G,\ uv=-H$

In the equation $x^3+3Hx+G=0$

G^2+4H^3 is called discriminant of the cubic equation.

Case 1. If $G^2+4H^3=0$, then $u=v$, therefore $x_2=x_3$

Two roots of cubic equation are equal.

Case 2. If $G=0, H=0$, then three roots of cubic equations are equal.

Case 3. $G^2+4H^3>0, u$ and v are real i. e. x_1 is the real root of the cubic, x_2 and x_3 are imaginary but conjugate to each other.

Case 4. If $G^2+4H^3<0$, In this u^3 and v^3 conjugate imaginaries and so u and v are conjugate imaginaries .

If $u=a+ib$, then $v=a-ib$

$\therefore u+v=2a=x_1,\ u-v=2ib$

$x_1=-a-b\sqrt{3},\ x_2=-a+b\sqrt{3}$, all are real. Hence all roots of the cubic equatin $x^2+3Hx+G=0$, are real, If $G^2+4H^3<0$

Ex. Solve the equations $x^3-12x^2-6x-10=0$ by the Cardon's method.

Remove the 2nd term from the equation.

[by the Art, 66 and 67] $\therefore na_0h+a_1=0$ or, $3.1h-12=0$

or, $h=4$. Diminish the root by 4 and the transfered equation is $z^3-54z-162=0$ ...... (2)

Let $x=u+v$, cube it, $z^3-3uv(u+v)-(u^3+v^3)=0$

or, $z-3uvz-(u^3+v^3)=0$ ... ... (3)

Compare it with (2), then $uv=18,\ u^3+v^3=162$

Then a new equation whose roots are u^3 and v^3 is

$t^2-162t+(18)^3=0$ or, $(t-54)(t-108)=0 \therefore t=54, 108$

We have $u^3=54$ or, $u=3\sqrt[3]{2}$ ... ... (3)

$v^3=108$ or, $v=3\sqrt[3]{4}$ ... ... (4)

$\therefore z=u+v,\ uw+uw^2,\ uw^2+vw$ But $z=x-4$ or, $x=z+4$

$\therefore x=u+v+4,\ (uw+uw^3)+4,\ (uw^3+uw)+4$

where u and v are available from [3] and [4]

Ex. 2. Solve the equation $28x^3-9x^2+1=0$ by Cardan's method ... ... (1)

Put $z=1/x$, then (1) becomes $z^3-9z+28=0$ ... . (2)

Put $z=u+v$, and cube it and then compare it with (2)

Then $uv=3,\ u^3+v^3=-28$ or, $u^3v^3=27$.

The equation whose roots are u^3 and v^3 is

$t^2+28t+27=0$ or, $(t+1)(t+27)=0$

$\therefore t=-27$ or, -1 i. e., $u^3=-27,\ v^3=-1$

$\therefore u=-3;\ v=-1 \therefore z=u+v=-3-1=-4$

Now $(z^3-9z+28)=(z+4)(z^2-4z+7)=0$

$\therefore z=-4, \frac{1}{2}[4 \pm \sqrt{(16-28)}] = 2 \pm \sqrt{(-3)} = 2 \pm 3i$

$x=\frac{1}{z}=-1/4, \frac{1}{2+i\sqrt{3}}, \frac{1}{2-i\sqrt{3}}$

$\therefore x=-\frac{1}{4}, \frac{2-i\sqrt{3}}{7}, \frac{2+i\sqrt{3}}{7}$ are the roots of equation (i)

$$\begin{vmatrix} x^2-xy & x^2-yz & y^2-xz \\ y^2-zx & z^2-xy & y^2-yz \end{vmatrix}$$

If $A_1, B_1, C_1,$ are the co-factors of a_1, b_1, c_1 (i= 1, 2, 3) in the determin

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

23. (b) $\begin{vmatrix} B_1+C_1 & C_1+A_1 & A_1+B_1 \\ B_2+C_2 & C_2+A_2 & A_2+B_2 \\ B_3+C_3 & C_3+A_3 & A_3+B_3 \end{vmatrix} = 2 \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

24. **Solve the following equations.** (সমীকরণগুলি সমাধান কর)

$\begin{vmatrix} x-2 & 2x-3 & 3x-4 \\ x-4 & 2x-9 & 3x-16 \\ x-8 & 2x-27 & 3x-64 \end{vmatrix} = 0,$

Ans. $x = 4$

25. $\begin{vmatrix} x+4 & 3 & 3 \\ 3 & x+4 & 5 \\ 5 & 5 & x+4 \end{vmatrix}$

Ans. $x = 0, 1, -12$

26. $\begin{vmatrix} a & a & x \\ m & m & m \\ b & x & b \end{vmatrix} = 0$ **Ans.** $x = a, b$

27. $\begin{vmatrix} x+2 & 2x+3 \\ 2x+3 & 3x+4 \\ 3x+4 & 5x+8 & 1 \end{vmatrix}$

28. $3x+5y-7z = 13$ (i)
$4x+y-12z = 6$ (ii)
$2x+9y-3z = 20$
Ans. $x = 1, y = 2, z = 0$

29. $x+2y+3z=6$
$2x+4y+z=7$
$3x+2y+9z=14$
Ans. $x=y=z=1$

30. If $A \equiv \begin{vmatrix} 2 & 3 & 4 \\ 1 & 3 & 4 \\ 7 & 2 & 5 \end{vmatrix}$ and $B \equiv \begin{vmatrix} 3 & 2 & 7 \\ 4 & -4 & -2 \\ 5 & -5 & -3 \end{vmatrix}$

Show that the product of the value of the determinants A and B i
eterminants of $(A.B)$ R.U. 1958

31. Factorise the determinant (নির্ণায়ককে
ৎপাদকে বিশ্লেষণ কর ।) $\begin{vmatrix} 1+x & 1-x-2x^2 \\ 1+y & 1-y-2y^2 \end{vmatrix}$

...ms = 6 terms. ...are a^2, b^2, c^2, d^2, e^2. Thus the total co-
...ed by putting the three x's equal to zero except x in the 1st column and the 1st row : The determinant is now $\begin{vmatrix} 0 & d & e \\ -d & 0 & f \\ -e & -f & 0 \end{vmatrix}$ which is skew symmetric determinant of odd order and so vanishes. There are ${}^4C_1$ i. e. 4 such determinants and all of them vanish.

Clearly, the co-efficient of x^4 is unity.

Hence $D = x^4+x^2(a^2+b^2+c^2+d^2+e^2+f^2)+(af-bc+cd)^2$

Ex. 8 Solve the equation ≡ $\begin{vmatrix} x & -6 & -1 \\ 2 & -3x & x-3 \\ -3 & 2x & x+2 \end{vmatrix} = 0$

Applying R_1-R_2 we get

$$\begin{vmatrix} x-2 & 3(x-2) & -(x-2) \\ 2 & -3x & x-3 \\ -3 & 2x & x+2 \end{vmatrix} = (x-2)\begin{vmatrix} 1 & 3 & 1 \\ 2 & -3x & x-3 \\ -3 & 3x & x+2 \end{vmatrix}$$

Now applying c_2-3c_1, c_3+c_1 we get

$$\Delta = (x-2)\begin{vmatrix} 1 & 0 & 0 \\ 2 & -3x-6 & x-1 \\ -3 & 2x+9 & x-1 \end{vmatrix} = (x-2)(x-1)\begin{vmatrix} -3x-6 & 1 \\ 2x+9 & 1 \end{vmatrix}$$

$=(x-2)(x-1)(-5x-15)=0 \therefore x = 1, 2, -3$

Ex. 9. Solve the equation $2x-y-z=4, x-2y+z=5, x-2y+2z-1=0$

$$\therefore \frac{x}{\begin{vmatrix} -1 & -1 & 4 \\ -2 & 1 & 5 \\ -2 & 2 & 1 \end{vmatrix}} = \frac{-y}{\begin{vmatrix} 2 & -1 & 4 \\ 1 & 1 & 5 \\ 1 & 2 & 1 \end{vmatrix}} = \frac{z}{\begin{vmatrix} 2 & -1 & 4 \\ 1 & -2 & 5 \\ 1 & -2 & 1 \end{vmatrix}} = \frac{1}{\begin{vmatrix} 2 & -1 & -1 \\ 1 & -2 & 1 \\ 1 & -2 & 2 \end{vmatrix}}$$

or, $x = \dfrac{\Delta_1}{\Delta}, y = -\dfrac{\Delta_2}{\Delta}, z = \dfrac{\Delta_3}{\Delta}$ or; $x = \dfrac{9}{-3} = -3, y = \dfrac{-18}{-3} = -6, z = \dfrac{12}{-3} = -4$

$\therefore x = 3, y = -6, z = -4$

Ex. 10. Solve the equation $\begin{vmatrix} 1 & 1 & 1 \\ x & a & b \\ x^3 & a^3 & b^3 \end{vmatrix} = 0$

Applying $c_2-c_1; c_3-c_2$ we get

$$\begin{vmatrix} 1 & 0 & 0 \\ x & a-x & b-a \\ x^3 & a^3-x^3 & b^3-a^3 \end{vmatrix} = 0 \quad \text{or;} \begin{vmatrix} a-x & b-a \\ a^3-x^3 & b^3-a^3 \end{vmatrix} = 0$$

15. $$\begin{vmatrix} 1 & a & b+c \\ 1 & b & c+a \\ 1 & c & a+b \end{vmatrix} = 0$$ ✓

15. (a) $$\begin{vmatrix} a+b+c & -c & - \\ -c & a+b+c & - \\ -b & -a & a+b \end{vmatrix}$$

16. $$\begin{vmatrix} 1^2 & 2^2 & 3^2 & 4^2 \\ 2^2 & 3^2 & 4^2 & 5^2 \\ 3^2 & 4^2 & 5^2 & 6^2 \\ 4^2 & 5^2 & 6^2 & 7^2 \end{vmatrix} = 0$$

17. $$\begin{vmatrix} a & b & c \\ x & y & z \\ p & q & r \end{vmatrix} = \begin{vmatrix} y & b & q \\ x & a & p \\ z & c & r \end{vmatrix} = \begin{vmatrix} x & y & z \\ p & q & r \\ a & b & c \end{vmatrix}$$

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18. $$\begin{vmatrix} 1 & a & a^2 & 0 \\ 0 & 1 & a & a^2 \\ a^2 & 0 & 1 & a \\ a & a^2 & 0 & 1 \end{vmatrix} = 1+a^4+a^8$$

19. $$\begin{vmatrix} a & b & c & d \\ -b & a & -d & c \\ -c & d & a & -b \\ -d & -c & b & a \end{vmatrix} = (a^2+b^2+c^2+d^2)^2$$

$$\begin{vmatrix} 1 & 1 & 1 \end{vmatrix} = (x-y)(y-z)(z-$$

The derivative of 3rd order determinant $\Delta(x)$ is equal to the sum of three determinats, ch determinant is obtained by differentiating one row of $\Delta(x)$ or $\Delta'(x)$ will be the sum of the ree determinants each obtained by differentiating one column of the determinant leaving the other columns unchanged. In the same way the nth order determinant when differentiated the derivative will be the sum of the n determinants each obtained by differentiating either a row of a column leaving the other rows or columns unchanged.

Examples

Ex. 1. Evaluate the following determinant by rule of Sarrus

$$\Delta = \begin{vmatrix} 3 & 1 & 2 \\ 1 & 2 & 3 \\ 2 & 3 & 1 \end{vmatrix}$$

$$\begin{array}{ccccc} & & -8 & -27 & -1 \\ 3 & 1 & 2 & 3 & 3 \\ 1 & 2 & 3 & 1 & 1 \\ 2 & 3 & 1 & 2 & 2 \\ & & 6 & 6 & 6 \end{array}$$

$\therefore \Delta = 6+6+6-8-27-1 = -18$

Ex.2. Prove that $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix} = (b-c)(c-a)(a-b)(a+b+c)$

$$\text{Let } \Delta = \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix} \overset{c_2-c_1,\ c_3-c_1}{=} \begin{vmatrix} 1 & 0 & 0 \\ a & b-a & c-b \\ a^3 & b^3-a^3 & c^3-b^3 \end{vmatrix} = (b-a)(c-b) \begin{vmatrix} 1 & 0 & 0 \\ a & 1 & 1 \\ a^2 & b^2+ab+a^2 & c^2+bc+c^2 \end{vmatrix}$$

$$= (b-a)(c-b) \begin{vmatrix} 1 & 1 \\ b^2+ab+a^2 & c^2+bc+b^2 \end{vmatrix}$$

$$= (b-a)(b-c)(a^2+bc-a^2-ab) = (a-b)(b-c)(c-a)(a+b+c)$$

Ex.3. Factorise the determinat $\Delta = \begin{vmatrix} 1 & 1 & 1 \\ a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \end{vmatrix}$

If $a = b$ two columns are indentical. The determinant Δ vanishes, hence $a-b$ is factor of the determinant. Similarly $b-c$, and $c-a$ are factors of Δ.

The degree of the determinant is five, hence we require another factor of second degree. This factor is either of the form $(a^2+b^2+c^2)$ or $(ab+bc+ca)$. The factor of the type $(a^2+b^2+c^2)$ is easily cancelled as a^3, b^3, c^3 are they highest power of a, b, c, respectively in Δ. Therefore, the factor is of the type $k(ab+bc+ca)$ where k is a constant quantity independent of a, b, c.

$$\therefore \Delta = \begin{vmatrix} 1 & 1 & 1 \\ a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \end{vmatrix} = k(a-b)(b-c)(c-a)(ab+bc+ca)$$

Now we have to find out value of k.

Let $a = 0, b = 1, c = -1$.

$$\begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & -1 \end{vmatrix} = k(-1)2(-1)(-1)$$

If we put k = 6 in (2), then $S_6=p^6+3p^3q+3q^2+6pt$.

To find S_{-3}, put k = 4,3,2, 1 in eq. (2) in succession, then

$S_4+pS_3+qS_1+tS_{-1}=0$, or, $S_{-1}=0$

$S_3+pS_2+tS_{-2}=0$ or, $S_{-2}=-2q/t$

$S_2+pS_1+qS_{-1}+tS_{-3}=0$ or, $S_{-3}=0$

Art. 58. Descartes' Rule of signs

The number of real positive roots of the equation $f(x)=0$ cannot exceed the number of changes in the signs of the co-efficients of the terms in $f(x)$ and the number of real negative roots cannot exceed the unmber of changes in the signs of the co-efficients of $f(-x)$.

Let us consider the following equations.

$2x^3-5x^2+3x-9=0$ ... . (1)

$x^3+x^2+7x-5=0$ ... . (2)

$2x^4-3x^3-4x^2+5x-6=0$ ... ... (3)

$x^7-2x^5+7x^4+x^3-9=0$ ... ... (4)

In the eq. (1) first term has the sign+, the next term–, the next, one after that+ last one–, If we write these sign consecutively, we have + – + –. There are three changes of signs + to–, –to +and+to–. In the same way we can show that eq. (2) has one change of sign, the eq. (3) has three changes of signs and the last eq. has three changes of sings.

Let us consider the case of any equation $f(x)=0$ where none of the co-efficients of $f(x)$ is zero.

Let the sequences of signs of $f(x)=0$ be

$+ + + - + - - - - + + - -$ ... . (5)

There are five changes of sign.

Multiply the equation by (x–a) where a is any positive number. The signs of the term in the multiplication will be as shown in the following scheme. The signs of the co-efficients of x–a are + and–

$$
\begin{array}{l}
+ + + - + - - - - + + - - \\
\underline{+ - \qquad\qquad\qquad\qquad\qquad\qquad} \\
+ + + - + - - - - + + - - \\
\underline{\quad - - - + - + + + - - - + +} \\
+ \pm \pm - + - \mp \mp \mp + \pm - \mp +
\end{array}
$$
... . (6)

Hence we see that in the product

(1) an ambiguity replaces each continuation of sign in the original equation.

(2) The signs before and after an ambiguity or a set of ambiguities aré unlike.

(3) a change of sign is introduced at the end.

Now in the product let us take the most unfavourable case (ii) and suppose that all the ambiguities are replaces by continuations, the upper signs may be adopted for the ambiguities Then the changes of sign in (6) and 6 in.

$+ + + - + - - - + + - - +$ .... (7)

So, $f(x) = (4)\,\Pi(x-\alpha) = (2x^2-5x+3)\,(2x^3+4x^2+7x-1) - 14x - 6$

Put $x = 1$ and $3/2$

$\therefore 4\,\Pi\,(1-\alpha) = 0 - 14 - 6 = -20$ and $4\,\Pi(3/2-\alpha) = 0 - 14.3/2 - 6 = -27$

Now take the product.

Exercise iv

If a, b, c are the roots of $x^3+px^2+qx+r=0$, find the values of (a,b,c সমীকরণের মূল হইলে মান নির্ণয় কর।–

0 1. $(b+c)\,(c+a)\,(a+b)$ ✓ 2. $\Sigma\dfrac{b^2+c^2}{bc}$

3. Σa^3. 0 4. $\Sigma\dfrac{b^2+c^2}{b+c}$ [D. U. 1991, C. U.1983]

0 3 (i) Σa^2b, Σa^2b^2 ✓

0 4. (a) যদি $x^3+3x^2+5=0$ এর বীজগুলি a, b, c হয় তবে $\Sigma\dfrac{b^2+c^2}{bc}$, $\Sigma\,(b-c)^2$ এর মান নির্ণয় কর। [N. U. 1995, C. U.1989]

If a, b, c be the roots of $x^3+px+q=0$, find the values of (a, b, c সমীকরণের মূল হইলে $x^3+px+q=0$ মান নির্ণয় কর।)

5. $\Sigma\dfrac{1}{a+b-c}$ C. U. 1982 6. $\Sigma(b^2-ca)(c^2-ab)$

7. Σa^4 C.U. 1982 8. $\Sigma(b-c)^2$

9. $(a+b-2c)(b+c-2a)(c+a-2b)$ 10. $\Sigma\dfrac{1}{b+c}$, R.U.1980, '81

If a, b, c are the root of $x^3+px^2+qx+r=0$ find the value of (a, b, c সমীকরণের মূল হইলে $x^3+px^2+qx+r=0$ মান নির্ণয় কর।)

11. $(b+c-3a)\,(c+a-3b)(a+b-3c)$

12. $(1/b+1/c-1/a)(1/c+1/a-1/b)(1/a+1/b-1/c)$ C.H. 1977

13. Σa^4

13. (a) যদি $x^3-3x^2-6x+8=0$, এর মূলগুলি a, b, c হয়, তবে $\Sigma\left(\dfrac{b}{c}+\dfrac{c}{b}\right)$ এবং $\Sigma(b-c)^2$ এর মান নির্ণয় কর।

[If a, b, c be the roots of $x^3-3x^2-6x+8=0$, then find the values of $\Sigma\left(\dfrac{b}{c}+\dfrac{c}{b}\right)$ and $\Sigma(b-c)^2$]

If a, b, c be the roots of $x^3-px^2+qx-r=0$. Find the values of (a, b, c সমীকরণের মূল হইলে নিম্নলিখিত রাশিগুলির মান নির্ণয় কর।)

14. $\Sigma(1/a^3)$ (i) $\Sigma\,(\beta\gamma+1/\alpha)$ 15. $\Sigma(b/c+c/b)$

16. $\Sigma\dfrac{1}{a^2b^2}$ (i) $\Sigma(b-c)(c-a)$; (ii) $\Sigma\dfrac{1}{a^2}$

(ii) $(a^2+1)\,(b^2+1)\,(c^2+1)$ C.H. 1994

17. If a, b, c are the roots of $x^3-4x^2+2x+1=0$, find the values of (i) Σa^2b (ii) Σa^3 C.U. 1984 ✓

0 17. (a) $3x^3-2x^2+1=1$ find the values of Σa^2b, Σa^3b C.U. 1999

0 17. (b) If α, β, γ are the roots of $3x^3-5x^2+2x+1=0$, find the values of ...

(অপ্রতিসাম্য নির্ণায়কের বিপরীত একটি প্রতিসাম্য নির্ণায়ক।)

6. Prove that the square of any determinant is symmetric det
একটি প্রতিসাম্য হয়।)

7. Show that the reciprocal adjoint of a skew-symmetric de
symmetric. (বিজোড় প্রতিসাম্য নির্ণায়কের বিপরীত একটি প্রতিসাম্য হয়।)

7. (a) Adjugate of a skew symmetric determinant of even order is
অবসাম্য নির্ণায়কের অনুবন্ধ নির্ণায়ক একটি অবসাম্য নির্ণায়ক হইবে।)

8. Prove that the square of any determinant of even order be ex
determinant. (যুগ্ম পর্যায়ের নির্ণায়ক অবসাম্য নির্ণায়ক।)

9. Prove the following indentities. (অভেদগুলি প্রমাণ কর।)

$$\begin{vmatrix} a^2 & bc & ac+c^2 \\ a^2+ab & b^2 & ac \\ ab & b^2+bc & c^2 \end{vmatrix} = 4a^2b^2c^2$$

C. U. 1980, 83

9. (a) $\begin{vmatrix} a \\ a-b \\ b+c \end{vmatrix}$

10. $$\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = (b-c)(c-a)(a-b)$$

11. $\begin{vmatrix} a-b-c \\ 2b & b \\ 2c \end{vmatrix}$

12. $$\begin{vmatrix} bc & a & a^2 \\ ca & b & b^2 \\ ab & c & c^2 \end{vmatrix} = \begin{vmatrix} 1 & a^2 & a^3 \\ 1 & b^2 & b^3 \\ 1 & c^2 & c^3 \end{vmatrix} = (c-a)(c-b)(b-$$

13. $$\begin{vmatrix} b^2c^2 & bc & b+c \\ c^2a^2 & ca & c+a \\ a^2b^2 & ab & a+b \end{vmatrix} = 0$$

13. (a) $\begin{vmatrix} 1 \\ \cos(\alpha-\beta) \\ \cos(\alpha-\gamma) \end{vmatrix}$

14. $$\begin{vmatrix} (b+c)^2 & a^2 & a^2 \\ b^2 & (c+a)^2 & b^2 \\ c^2 & c^2 & (a+b)^2 \end{vmatrix} = \begin{vmatrix} (a+b)^2 & ca & bc \\ ca & (b+c)^2 & ab \\ bc & ab & (c+a)^2 \end{vmatrix} =$$

This series of sign is the same as in the original equation (5) with an additional change of sign of sign by (iii) at the end. Similarly it can be show that number of change of sign in (6) is (6) if we consider lower sign of ambiguities.

Thus in the most unfavourable case one more change of sign occurs in the product (5). If some of the co-efficients are zero, on changes of sign are lost.

Let $f(x) = \psi(x)(x-a)(x-b)(x-c) \ldots (x-k)$, where $\psi(x)$ contains all the factors due to negative and pairs of imaginary roots while all the factors due to negative and pairs of imaginary roots while all the factors x–a, etc. due to positive roots are explicitly given. Now multiply $\psi(x)$ by x–a, x–b, x–c ..... in turn. At each multipliation at least one change of sign is introduced into the product. Therefore, no equation can have more positive roots than it has changes of sign.

Again, the roots of the equation $f(-x)=0$ are equal to the roots of $f(x)=0$ but opposite to them in sign; therefore, the negative roots of $f(x)=0$ are the positive roots of $f(-x)=0$; but the number of the positive roots cannot exceed the number of changes of sign of $f(-x)=0$. Thus the number of negative roots of $f(-x)=0$ cannot exceed the number of changes of sign in $f(-x)=0$.

Cor. 1. If the number of positive and negative roots of an equation of degree n is found by Descartes' Rules to be not more than n_1 where $n_1<n$, we can infer at once that at least $n-n_1$ of the roots $f(x)=0$ are imaginary.

Ex. 7. Find the nature of the roots of the equation $3x^4+12x^2+5x-4=0$

Let $f(x)=3x^4+12x^2+5x-4$

There is only one change of sign. Hence $f(x)=0$ has only one positive root.

Again $f(-x)=3(-x)^4+2(-x)^2-5(-x)-4 = 3x^4+12x^2-5x-4$

There is only one change of sign i.e. from +to–. Hence there is one negative root in $f(x)=0$. As the given equation is of the fourth degree, it must have four roots. Therefore, there are two imaginary roots, one positive root and one negative root.

Ex. 8. Show that $x^6-x^5-10x+7=0$ has two positive and four imaginary roots.

$f(x)=x^6-x^5-10x+7$

There are two changes of sign from $+x^6$ to $-x^5$ and from $-10x$ to 7 i. e. from + to – and – to +

Hence $f(x)=0$ and two positive roots.

Again $f(-x)=x^6+x^5+10x+7$

There is no change of sign. Hence there is no negative root.

As the equation $f(x)=0$ is of 6th degree, it has six roots of which two roots are positive. Hence the equation has four imaginary roots.

parts to be grouped with each of the remaining terms; multiply $f(-x)$ by 4. Then

$$4f(-x)=4x^4-16x^3-24x^2-96x-240$$
$$=x^3(x-16)+x^2(x-24)+x(x^3-96)+(x^4-240)$$

In order to make all the the terms+ve., we require x = 17. Hence 17 is the upper limit of +ve roots of $f(-x)=0$ i. e −17 is the lower limit of − ve roots of $f(x)=0$

By the Theorem (B)

$$\frac{4}{1}+1, \frac{6}{1}+1, \frac{24}{1}+1, \frac{60}{1}+1$$

Hence −16 is the lower limit of −ve roots of $f(x)=0$

By Newton's Method

$f(-x)=x^4-4x^3-6x^2-24x-60$

$f_1(-x)=4x^3-12x^2-12x-24$, $f_2(-x)=12x^2-24x-12$; $f_3(-x)=24x-24$, $f_4(-x)=24$.

All the functions are +ve for x = 7 i. e. −7 is the lower limit of −ve roots $f(x)=0$

Hence limit of roots are 2 and −7

Ex. 22. Show that 5 is a negative superior limit and 0 is the positive limit of the roots of $x^4+10x^3+35x^2+50x+24=0$

By grouping : $x^2(x^2+10x+35)+50x+24=0$ ... (1)

Now $x^2+10x+35=0$ or, $x=-5\pm\sqrt{-10}$ which is imaginary.

Hence 0 is the upper limit of +ve roots.

Again $f(-x)=x^4-10x^3+35x^2-50x+24=0$ .... (2)

$x^3(x-10)+x(35x-50)+24=0$

This will be + ve if $x=11$. i. e., −11 is the upper limit of −ve roots.

Synthetic Division

By Synthetic Division Method. From (1)

write down the co-efficients in reverse order. The probable factors of 24 are ±1, ±2, ±3, ±4 etc.

−1	24	50	35	10	1	
		−24	−26	−9	−1	
		26	9	1	0	− 1 is a root.

−2	24	26	9	1	
		−12	−7	−1	
		14	2	0	−2 is a root.
−3	12	7	1		
		−4	−1		
		3	0		−3 is a root.

Also $4+x=0$ or, $x=-4$ is a root.

$\therefore\ S_n + P_1S_{n-1} + P_2S_{n-2} + \ldots\ldots + P_{n-1}S_1 + nP_n = 0$ .... (11)

$S_{n+1} + P_1S_n + P_2S_{n-1} + \ldots\ldots + P_{n-1}S_2 + P_nS_1 = 0$ .... (12)

$S_{n+2} + P_1S_{n+1} + P_2S_n + \ldots\ldots + P_{n-1}S_3 + P_nS_2 = 0$ .... (13)

Again if we are to find the values of $\Sigma a^{-1}, \Sigma a^{-2}, \Sigma a^{-3}$. i.e. of S_{-1}, S_{-2}, S_{-3} ....

then we should find S_1, S_2, S_3 ..... of the equation in which x has been replaced by $1/x$

Ex. 2. If α, β, γ be the roots of the equation $x^3+3x+9=0$, find the values of s_3, and s_9.

In the above equation we have

$p_1=0,\ p_2=3, p_3=9.$

$s_1+p_1=0$ or, $s_1=-p_1=0$, $s_2+p_1s_1+2p_2=0$ or, $s_2=-6$

$s_3+p_1s_2+p_2s_1+3p_3=0$ or, $s_3=-27$

Again $x^3=-3(x+3)$ or, $x^9=-27\ (x^3, +9x^2+27x+27)$ ... ... (1)

Put $x=\alpha, \beta, \gamma$ in (1) and

$\Sigma a^9=-27(\Sigma a^3+9\Sigma a^2+27\Sigma a+27)$

or, $S_9 \quad = -27\ (s_3+9s_2+27s_1+27s_o) = -27\ (-27-9.6+27.0+27.3) = 0$ and $s_o=1+1+1=3$

$\therefore\ s_9 = 0$

Ex. 2. Find the value of s_{-2} and S_5 for the equation $x^4-x^3-19x^2+49x-30=0$ ... ... (1)

$p_1=-1,\ p_2=-19,\ p_3=49,\ p_2=-30$

$\therefore\ s_1+p_1=0$ or, $s_1=1$, $s_2+p_1s_1+2p_2=0$ or, $s_2=39$.

Similarly, $S_3=-89$, $S_4=723$.

$\therefore\ s_5+p_1s_4+p_2s_3+p_3s_2+p_4s_1+5p_5=0$, where $p_5=0$ or; $S5=-2849$

Again replace x by $\frac{1}{x}$ in eq. (1) and the equation becomes after dividing by 30 throughout

$$\frac{1}{30x^4}-\frac{1}{30x^3}-\frac{19}{30x^2}+\frac{49}{30x}-1=0$$

$$\text{or, } x^4-\frac{49x^3}{30}+\frac{19x^2}{30}+\frac{x}{30}-\frac{1}{30}=0$$

Here $p_1=\frac{49}{30}, p_2=\frac{19}{30}, p_3=\frac{1}{30}, p_4=-\frac{1}{30}\ \therefore\ s_{-1}+p_1=0$ or, $s_{-1}=\frac{49}{30}$

$s_{-2}+p_1s_{-1}+2sp_2=0$ or, $s_{-2}=1261/900$

Ex. 3. If a, b, c, be the root, of the equation $x^n+\frac{x^{n-1}}{L1}+\frac{x^{n-1}}{L2}+\ldots\ldots\frac{1}{Ln}=0$

Show that $\Sigma a^r=0$ for $r=1, 2, 3, \ldots. n$

Here $s_1+p_1=0$ or, $s_1+1=0$ or, $s_1=-1$ i. e. Σa does not vanish.

$s_2+p_1s_1+2p_2=0$ or, $s_2+1.(-1)+2\frac{1}{2!}=0$ $0r$, $s_2=0$

$s_3=s_4=s_5=\ldots\ldots=s_n=0$

Again $s_{n+1}+p_1s_n+p_2s_{n-1}+\ldots\ldots+p_ns_1=0$

$\frac{n(n-1)}{\lfloor 1 \lfloor 1} p_0 h^2 + (n-1)p_1 h + p_2 = 0$ and so on.

Ex. 30. Transform the equation $x^3+6x^2-7x-4=0$ into one in which the term with x^2 is absent. (D.U.)

The roots have to be diminished by

$h = -p_1/(np_0) = -6/(3.1) = -2$

Hence the co-efficients in the transformed equation can be obtained by dividing x^3+6x^2-7x-4 repeatedly by $x+2$. The co-efficients are obtained as follow

1	6	−7	−4	(−2
	−2	−8	30	
	4	−15	26	
	−2	−4		
	2	−19		
	−2			
	0			

we see that the transformed equation is $x^3-19x-26=0$

Ex. 31. Transform the equation $x^4-24x^2-13x+35=0$ into one in which the term x is absent.

Change x to $x+h$ the transformed equation is

$(x+h)^4-24(x+h)^2-13(x+h)+35=0$

or, $x^4+4hx^3+(6h^2-24)x^2+ \ldots\ldots =0$

The term with x^2 is absent if $6h^2-24=0$ or, $h=\pm 2$

(i) $h=2$ i. e. to diminish the roots of equation by 2 or divide equation by $x-2$

1	0	−24	−13	35	(2
	0	4	−40	−106	
	2	−20	−53	−71	
	2	8	−24		
	4	−12	−77		
	2	12			
	6	0			
	2				
	8				

Hence the transformed equation is $x^4+8x^3-77x-71=0$

(ii) When $h=-2$ i. e. it increased the roots of the equation by 2. Divide the equation repeatedly by $x+2$

1	0	−24	−13	35	(−2
	−2	4	+40	−54	
	−2	−20	27	−19	
	−2	8	24		
	−4	−12	51		
	−2	12			

4

THEORY OF EQUATIONS

Art. 50. The most general rational integral expression of the nth degree in x may be written as

$$a_0x^n+a_1x^{n-1}+a_2x^{n-2}+\dots\dots+a^n$$

Any algebraical expression which contains x is called a function of x and is denoted by $f(x)$, $\phi(x)$, or by some similar symbols

Let $f(x) \equiv a_0x^n+a_1x^{n-1}+a_2x^{n-2}+\dots\dots+a^n$

be a rational equation of nth degree. If $a_0 \neq 0$, after division by a_0 the equation can be written in the form

$$x^n+p_1x^{n-1}+p_2x^{n-2}+\dots\dots+p_{n-1}x+p_n=0 \quad \dots\dots\dots (1)$$

where P_1, p_2, p_3 do not contain x, unless otherwise stated, these co-efficients are supposed to be rational.

Any value of x which makes $f(x)$ vanish is called a root of the equation $f(x)=0$

We shall assume that every equation of the form $f(x)=0$ has a root real or imaginary which is called thre Fundamental Theorem of Algebra. Different proofs of this fundamental proposition have been given by Cauchy, Cliford and others. The proofs are however long and difficult. Interested students may consult "Theory of equation" by H. W. Turnbull page No. 56.

6 **Art. 51. Every equation of the nth degree has exactly n roots :**

Let $f(x)=x^n+p_1x^{n-1}+p_2x^{n-2}+\dots\dots+p_{n-1}x+p_n \dots\dots\dots (2)$

Let a_1 be a root of $f(x)=0$. Then we have $f(a_1)=0$

Therefore $f(x)$ must be divisible by $x-a_1$; so we write

$$f(x) \equiv (x-a_1)(x^{n-1}+q_1x^{n-2}+q_2x^{n-3}+\dots\dots+q_{n-2}x+q_{n-1}) \equiv (x-a_1)\,\phi(x)$$

Similarly, since the equation $\phi(x)=0$ has a root say a_2, we have as before

$$\phi(x) \equiv (x-a_2)(x^{n-2}+r_1x^{n-3}+r_2x^{n-1}+\dots\dots+r_{n-3}x+r_{n-2}) \equiv (x-a_2)\psi(x)$$

Hence $f(x) \equiv (x-a_1)(x-a_2)\psi(x)$

Proceding in this way, we can show that

$$f(x)=(x-a_1)(x-a_2)(x-a_3)\dots\dots(x-a_n)$$

It is now clear that $a_1, a_2 \dots\dots a_n$ are root of $f(x)=0$ and no other value of x will stisfy $f(x)=0$, so the equation $f(x)=0$ has only n roots.

The number $a_1, a_2 \dots\dots a_n$ need not be all different from one another. Some of them may repeat.

Let $f(x)=(x-a_1)^p\,(x-a_2)^q\,(x-a_3)^r \dots\dots$

where $p+q+r \dots = n$,

The equation $f(x)=0$ has in this case p roots each equal to a_1, q roots each equal to a_2, r roots each equal to a_3, and so on but their sum i. e. total number of roots cannot exceed n; i.e.

$$p+q+r \dots\dots = n$$

Hence $f(x)$, $\phi(x)$, $\psi(x)$ etc. are integral function of x.

Taking logarithm of both sides, we have

$\log f(x) = \log (x-a_1) + \log (x-a_2) + \log (x-a_3) \ldots + \log (x-a_n)$. Differentiating, we have

$$\frac{f'(x)}{f(x)} = \frac{1}{x-a_1} + \frac{1}{x-a_2} + \frac{1}{x-a_3} + \ldots\ldots + \frac{1}{x-a_n} \quad \ldots\ldots\ldots \quad (i)$$

$$f'(x) = \frac{f(x)}{x-a_1} + \frac{f(x)}{x-a_2} \ldots\ldots + \frac{f(x)}{x-a_n} \quad \ldots\ldots\ldots \quad (ii)$$

Hence the theorem.

Art. 56. Sums of powers of the roots (মূলের শক্তির যোগফল)

From the equation (i) of Art. 55 we have

$$\frac{f'(x)}{f(x)} = \frac{1}{x-a_1} + \frac{1}{x-a_2} + \frac{1}{x-a_3} + \ldots\ldots + \frac{1}{x-a_n}$$

$$= \frac{1}{x}\left(1 - \frac{a_1}{x}\right)^{-1} + \frac{1}{x}\left(1 - \frac{a_2}{x}\right)^{-1} + \ldots\ldots\ldots + \frac{1}{x}\left(1 - \frac{a_n}{x}\right)^{-1}$$

$$= \frac{n}{x} + \frac{1}{x^2}\Sigma a_1 + \frac{1}{x^3}\Sigma a_1^2 + \frac{1}{x^4}\Sigma a_1^3 + \ldots\ldots + \frac{1}{x^{n+1}}\Sigma a_1^n \ldots\ldots$$

$$= \frac{n}{x} + \frac{s_1}{x^2} + \frac{s_2}{x^3} + \ldots\ldots\ldots + \frac{s_n}{x^{n+1}} + \ldots\ldots$$

Conclusion (সিদ্ধান্ত)

The sum of the nth powers of the roots, i.e. $S_n = \Sigma a_1^n$, is equal to the co-efficient of $x^{-(n+1)}$ in the expansion of $\frac{f'(x)}{f(x)}$ in power of x^{-1}.

Note : The following results are very useful.

If $a_1, a_2, a_3 \ldots\ldots a_n \ldots$ be the roots of the equation $f(x)=0$ i. e.

$x^n + p_1 x^{n-1} + p_2 x^{n-2} + \ldots\ldots + p_{n-1}x + p_n = 0$, then

(i) $\Sigma a_1^2 = p_1^2 - 2p_2$, $\Sigma a_1 = -p_1$, $\Sigma a_1 a_2 = p_2$

(ii) $\Sigma a_1^2 a_2 = 3p_3 - p_1 p_2$

(iii) $\Sigma a_1^3 = -p_1^3 + 3p_1 p_2 - 3p_3$

(iv) $\Sigma a_1^2 a_2 a_3 = p_1 p_3 - 4p_4$

(v) $\Sigma a_1^2 a_2^2 = p_2^2 - 2p_1 p_3 + 2p_4$

(vi) $\Sigma a_1^3 a_2 = p_1 p_2^2 - 2p_2^2 - p_1 p_3 + 4p_4$

(vii) $\Sigma a_1^4 = p_1^4 - 4p_1^2 p_2 + 4p_1 p_3 + 2p_2^2 - 4p_4$

Students are advised to establish the above results independently.

Ex. 4. Find the sum of the fourth powers and second powers of the roots of $f(x) \equiv x^3 - 2x^2 + x - 1 = 0$

Let a_1, a_2, a_3 be the roots of the equation

so that $f(x) = (x-a_1)(x-a_2)(x-a_3)$

$$\therefore \frac{f'(x)}{f(x)} = \frac{1}{x-a_1} + \frac{1}{x-a_2} + \frac{1}{x-a_3} = \frac{3}{x} + \frac{s_1}{x^2} + \frac{s_2}{x^3} + \frac{s_3}{x^4} + \frac{s_4}{x^5} + \ldots\ldots$$