Statistics for Engineers

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Part 2: Probability

Sample space, Events, Union and intersection of events, Probability of events, Conditional probabilities, Bayes theorem, Chebysev's inequality

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Sample space:

- Sample space: The set of all possible outcomes. It is usually denoted by S.

Example:

- 1. Roll a die: {1, 2, 3, 4, 5, 6}.
- 2. Flip a coin twice: {(H, H), (H, T), (T, H), (T, T)}.

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Finite sample space:

- If the sample space contains finite number of outcomes (elements), then it is known as finite sample space.

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Infinite sample space:

- If the sample space contains infinite number of outcomes (elements), then it is known as infinite sample space.
- Example: The experiment throwing a coin until a appears.

  S={H, TH, TTH, TTTH, ……}

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Discrete and continuous sample space:

- A sample space is said to be discrete if it contains only finitely or infinitely many points which can be arranged into a sample sequence. Otherwise, the sample space is continuous.

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Events:

Event: A subset of the sample space.

1. Roll a die: the outcome is even {2, 4, 6}.

2. Flip a coin twice and the two results are different:

{(H, T), (T, H)}.

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Events:

- Union of events: Let A and B be events in S and let E be the event "either A occurs or B occurs". Then E occurs if the outcome of the experiment is either in A, or in B, or in both A and B. Therefore, we conclude that

$$E = A \cup B$$

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Events:

- Intersection of events: Let A and B be events in S and let F be the event "both A occurs and B occurs". Then F occurs if the outcome of the experiment is in both A and B. Therefore, we conclude that

$$F = A \cap B$$

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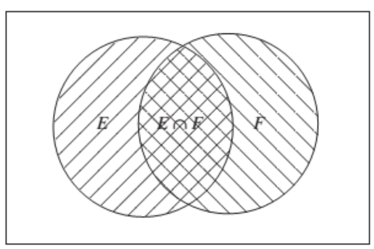
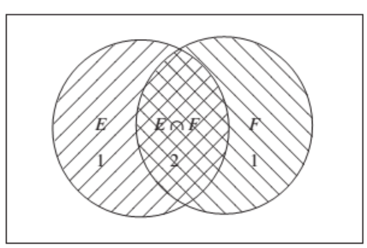
$$A \cup B = \{x | x \in A \ \underline{\text{or}} \ x \in B\} \quad \text{and} \quad A \cap B = \{x | x \in A \ \underline{\text{and}} \ x \in B\}.$$





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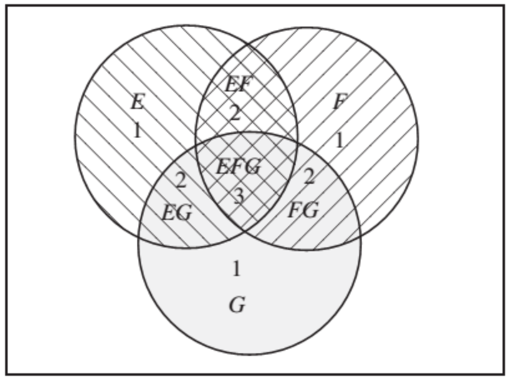

Example:

Roll a fair die. The sample space of equally likely simple events is:

$$S = \{1, 2, 3, 4, 5, 6\}.$$

Let A be the event "an odd number turns up" and let B be the event "the number that turns up is divisible by 3".

$$A = \{1, 3, 5\}, \quad B = \{3, 6\}\ , \ E = A \cup B = \{1, 3, 5, 6\}$$

$$F = A \cap B = \{3\}$$

- Mutually exclusive events: Two events E and F are said to be mutually exclusive if

$$EF = \phi$$

- Example:

Roll a die:
E_1: outcome is below 3: $\{1, 2\}$
E_2: outcome is above 4: $\{5, 6\}$
$E_1 \cap E_2 = \phi$.

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- Complement of and event:

E^c: outcome that is not in E.

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Properties:

Commutativity:

$$E \cup F = F \cup E, \quad E \cap F = F \cap E.$$

Associativity:

$$(E \cup F) \cup G = E \cup (F \cup G), \quad (E \cap F) \cap G = E \cap (F \cap G).$$

Distributivity:

$$(E \cup F) \cap G = (E \cap G) \cup (F \cap G), \quad (E \cap F) \cup G = (E \cup G) \cap (F \cup G).$$

De Morgan's law:

$$(\cup_{i=1}^{n} E_i)^c = \cap_{i=1}^{n} E_i^c, \quad (\cap_{i=1}^{n} E_i)^c = \cup_{i=1}^{n} E_i^c,$$
$$(\cup_{i=1}^{\infty} E_i)^c = \cap_{i=1}^{\infty} E_i^c, \quad (\cap_{i=1}^{\infty} E_i)^c = \cup_{i=1}^{\infty} E_i^c.$$

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Probability:

"Given a sample description space, probability is a function which assigns a non-negative real number to every event A, denoted by P(A) and is called the probability of the event A."

The probability of event E is a number $P(E)$ assigned to E that satisfies the following conditions:

1. $0 \leq P(E) \leq 1$.
2. $P(S) = 1$.
3. For any sequence of events E_1, E_2, ..., which are mutually exclusive, that is $E_n E_m = \phi$ for any $n \neq m$, $P(\cup_{n=1}^{\infty} E_n) = \sum_{n=1}^{\infty} P(E_n)$.

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How to compute:

4·3·1. Mathematical or Classical or 'a priori' Probabality

Definition. If a trial results in n exhaustive, mutually exclusive and equally likely cases and m of them are favourable to the happening of an event E, then the probability 'p' of happening of E is given by

$$p = P(E) = \frac{\textit{Favourable number of cases}}{\textit{Exhaustive number of cases}} = \frac{m}{n} \qquad \text{....(4·1)}$$

Compute Probabilities:

1. Roll a fair die, 6 equally likely outcomes: $\{1, 2, 3, 4, 5, 6\}$

   $P(\{1\}) = 1/6, P(\{2\}) = 1/6, ..., P(\{6\}) = 1/6$.

2. E: the outcome is even. $P(E) = ?$:

   $P(E) = P(\{2, 4, 6\}) = P(\{2\}) + P(\{4\}) + P(\{6\}) = 1/2$.

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Roll a fair die. The sample space of equally likely simple events is:

$$S = \{1, 2, 3, 4, 5, 6\}.$$

Let A be the event "an odd number turns up" and let B be the event "the number that turns up is divisible by 3".

(a) Find the probability of the event E = "the number that turns up is odd <u>or</u> is divisible by 3".

(b) Find the probability of the event F = "the number that turns up is odd <u>and</u> is divisible by 3".

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Solutions:

(a) $A = \{1, 3, 5\}$, $B = \{3, 6\}$, $E = A \cup B = \{1, 3, 5, 6\}$

Since the simple events are equally likely,

$$P(E) = P(A \cup B) = \frac{n(A \cup B)}{n(S)} = \frac{4}{6} = \frac{2}{3},$$

(b) $F = A \cap B = \{3\}$ and

$$P(F) = P(A \cap B) = \frac{n(A \cap B)}{n(S)} = \frac{1}{6}.$$

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Theorem 4·2. *Probability of the impossible event is zero, i.e., $P(\phi) = 0$.*

Proof. Impossible event contains no sample point and hence the certain event S and the impossible event ϕ are mutually exclusive.

Hence $S \cup \phi = S$

$\therefore \quad P(S \cup \phi) = P(S)$

$\Rightarrow \quad P(S) + P(\phi) = P(S)$ [By Axiom 3]

$\Rightarrow \quad P(\phi) = 0$

Theorem 4·3. **Probability of the complementary event $\bar{A}$ of A is given by**

$$P(\bar{A}) = 1 - P(A)$$

Proof. A and $\bar{A}$ are disjoint events.

Moreover, $A \cup \bar{A} = S$

From axioms 2 and 3 of probability, we have

$$P(A \cup \bar{A}) = P(A) + P(\bar{A}) = P(S) = 1$$

$$\Rightarrow \quad P(\bar{A}) = 1 - P(A)$$

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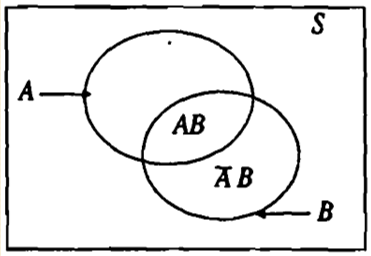
Theorem 4·4. *For any two events A and B,*

$$P(\bar{A} \cap B) = P(B) - P(A \cap B)$$

Proof.

$\bar{A} \cap B$ and $A \cap B$ are disjoint events and

$(A \cap B) \cup (\bar{A} \cap B) = B$



Hence by axiom 3, we get

$P(B) = P(A \cap B) + P(\bar{A} \cap B)$

$\Rightarrow P(\bar{A} \cap B) = P(B) - P(A \cap B)$

Remark. Similarly, we shall get

$P(A \cap \bar{B}) = P(A) - P(A \cap B)$

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Theorem 4·6. *If $B \subset A$, then*

(i) $P(A \cap \bar{B}) = P(A) - P(B)$,

(ii) $P(B) \le P(A)$

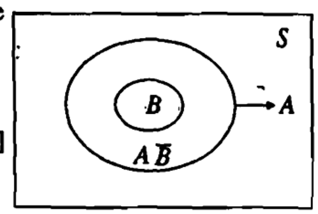
Proof. *(i)* When $B \subset A$, B and $A \cap \bar{B}$ are mutually exclusive events and their union is A

Therefore

$P(A) = P[B \cup (A \cap \bar{B})]$

$= P(B) + P(A \cap \bar{B})$ [By axiom 3]

$\Rightarrow P(A \cap \bar{B}) = P(A) - P(B)$



(ii) Using axiom 1,

$P(A \cap \bar{B}) \ge 0 \Rightarrow P(A) - P(B) \ge 0$

Hence $P(B) \le P(A)$

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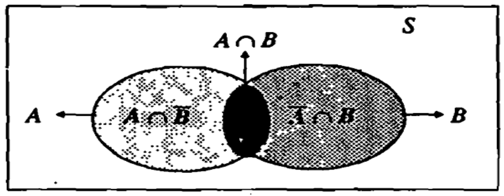
4·6·2. Law of Addition of Probabilities

Statement. *If A and B are any two events [subsets of sample space S] and are not disjoint, then*

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) \qquad ...(4\cdot5)$$

Proof.



We have

$$A \cup B = A \cup (\bar{A} \cap B)$$

Since A and $(\bar{A} \cap B)$ are disjoint,

$$\begin{aligned} P(A \cup B) &= P(A) + P(\bar{A} \cap B) \\ &= P(A) + [P(\bar{A} \cap B) + P(A \cap B)] - P(A \cap B) \\ &= P(A) + P[(\bar{A} \cap B) \cup (A \cap B)] - P(A \cap B) \end{aligned}$$

[$\because (\bar{A} \cap B)$ and $(A \cap B)$ *are disjoint*]

$$\Rightarrow \quad P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

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4·6·3. Extention of General Law of Addition of Probabilities. *For n events* $A_1, A_2, \ldots, A_n$, *we have*

$$P\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} P(A_i) - \sum\sum_{1 \le i < j \le n} P(A_i \cap A_j) + \sum\sum\sum_{1 \le i < j < k \le n} P(A_i \cap A_j \cap A_k) - \ldots + (-1)^{n-1} P(A_1 \cap A_2 \cap \ldots \cap A_n) \qquad ...(4\cdot6)$$

Theorem 4·7. *(Boole's inequality). For n events* $A_1, A_2, \ldots, A_n$, *we have*

(a) $$P\left(\bigcap_{i=1}^{n} A_i\right) \ge \sum_{i=1}^{n} P(A_i) - (n-1) \qquad ...(4\cdot7)$$

(b) $$P\left(\bigcup_{i=1}^{n} A_i\right) \le \sum_{i=1}^{n} P(A_i) \qquad ...(4\cdot7a)$$

Theorem 4·8. *For n events* $A_1, A_2, \ldots, A_n$,

$$P\left[\bigcup_{i=1}^{n} A_i\right] \ge \sum_{i=1}^{n} P(A_i) - \sum_{1 \le i < j \le n} P(A_i \cap A_j)$$

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4·7. Multiplication Law of Probability and Conditional Probability

Theorem 4·8. *For two events A and B*

$$\left.\begin{aligned} P(A \cap B) &= P(A) \, . \, P(B \mid A), \; P(A) > 0 \\ &= P(B) \, . \, P(A \mid B), \; P(B) > 0 \end{aligned}\right\} \quad ...(4{\cdot}8)$$

3. Multiplication Law of Probability for Independent Events. If A and B are independent then

$$P(A \mid B) = P(A) \quad \text{and} \quad P(B \mid A) = P(B)$$

Hence (4·8) gives :

$$P(A \cap B) = P(A) \, P(B) \quad ...(4{\cdot}8a)$$

provided A and B are independent.

4·7·1. Extension of Multiplication Law of Probability. *For n events* $A_1, A_2, ..., A_n$, *we have*

$$P(A_1 \cap A_2 \cap \ldots \cap A_n) = P(A_1) \, P(A_2 \mid A_1) \, P(A_3 \mid A_1 \cap A_2) \ldots \times P(A_n \mid A_1 \cap A_2 \cap \ldots \cap A_{n-1}) \quad ...(4{\cdot}8b)$$

where $P(A_i \mid A_j \cap A_k \cap \ldots \cap A_l)$ *represents the conditional probability of the event* A_i *given that the events* $A_j, A_k, ..., A_l$ *have already happened.*

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Theorem 4·9. *For any three events A, B and C*

$$P(A \cup B \mid C) = P(A \mid C) + P(B \mid C) - P(A \cap B \mid C)$$

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Example 4·12. *Two dice, one green and the other red, are thrown. Let A be the event that the sum of the points on the faces shown is odd, and B be the event of at least one ace (number '1').*

(a) Describe the (i) complete sample space, (ii) events A, B, $\overline{B}$, $A \cap B$, $A \cup B$, and $A \cap \overline{B}$ and find their probabilities assuming that all the 36 sample points have equal probabilities.

(b) Find the probabilities of the events :

(i) $(\overline{A} \cup \overline{B})$ *(ii)* $(\overline{A} \cap \overline{B})$ *(iii)* $(A \cap \overline{B})$ *(iv)* $(\overline{A} \cap B)$ *(v)* $(\overline{A \cap B})$ *(vi)* $(\overline{A} \cup B)$ *(vii)* $(\overline{A \cup B})$ *(viii)* $\overline{A} \cap (A \cup B)$ *(ix)* $A \cup (\overline{A} \cap B)$ *(x)* $(A \mid B)$ and $(B \mid A)$, and *(xi)* $(\overline{A} \mid \overline{B})$ *and* $(\overline{B} \mid \overline{A})$. .

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Solution.,*(a)* The sample space consists of the 36 elementary events .

$$\begin{array}{cccccc} (1,1) ; & (1,2) ; & (1,3) ; & (1,4) ; & (1,5) ; & (1,6) \\ (2,1) ; & (2,2) ; & (2,3) ; & (2,4) ; & (2,5) ; & (2,6) \\ (3,1) ; & (3,2) ; & (3,3) ; & (3,4) ; & (3,5) ; & (3,6) \\ (4,1) ; & (4,2) ; & (4,3) ; & (4,4) ; & (4,5) ; & (4,6) \\ (5,1) ; & (5,2) ; & (5,3) ; & (5,4) ; & (5,5) ; & (5,6) \\ (6,1) ; & (6,2) ; & (6,3) ; & (6,4) ; & (6,5) ; & (6,6) \end{array}$$

where, for example, the ordered pair (4, 5) refers to the elementary event that th green die shows 4 and and the red die shows 5.

A = The event that the sum of the numbers shown by the two dice is odd.

= {(1,2);(2,1);(1,4);(2,3);(3,2);(4,1);(1,6);(2,5) (3,4);(4,3);(5,2);(6,1);(3,6); (4,5); (5,4); (6,3) (5,6);(6,5)} and therefore

$$P(A) = \frac{n(A)}{n(S)} = \frac{18}{36}$$

B = The event that at least one face is 1,

= {(1,1) ; (1,2) ; (1,3) ; (1,4) ; (1,5) ; (1,6) (2,1) ; (3,1) ; (4,1) ; (5,1) ; (6,1)} and therefore

$$P(B) = \frac{n(B)}{n(S)} = \frac{11}{36}$$

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$\overline{B}$ = The event that each of the face obtained is not an ace.

$= \{(2,2);\ (2,3);\ (2,4);\ (2,5);\ (2,6);\ (3,2);\ (3,3);\ (3,4);\ (3,5);\ (3,6);\ (4,2);\ (4,3);\ (4,4);\ (4,5);\ (4,6);\ (5,2);\ (5,3);\ (5,4);\ (5,5);\ (5,6);\ (6,2);\ (6,3);\ (6,4);\ (6,5);\ (6,6)\}$ and therefore

$$P(\overline{B}) = \frac{n(\overline{B})}{n(S)} = \frac{25}{36}$$

$A \cap B$ = The event that sum is odd and at least one face is an ace.

$= \{(1,2);(2,1);(1,4);(4,1);(1,6);(6,1)\}$

$$\therefore\ P(A \cap B) = \frac{n(A \cap B)}{n(S)} = \frac{6}{36} = \frac{1}{6}$$

$A \cup B = \{(1,2);(2,1);(1,4);(2,3);(3,2);(4,1);(1,6);(2,5)\ (3,4);(4,3);(5,2);(6,1);\ (3,6);\ (4,5);\ (5,4);(6,3)\ (5,6);(6,5);(1,1);(1,3);(1,5);\ (3,1);(5,1)\}$

$$\therefore\ P(A \cup B) = \frac{n(A \cup B)}{n(S)} = \frac{23}{36}$$

$A \cap \overline{B} = \{(2,3);(3,2);(2,5);(3,4);(3,6);(4,3);(4,5);(5,2)\ (5,4);(5,6);\ (6,3);(6,5)\}$

$$P(A \cap \overline{B}) = \frac{n(A \cap \overline{B})}{n(S)} = \frac{12}{36} = \frac{1}{3}$$

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(b) (i) $P(\overline{A} \cup \overline{B}) = P(\overline{A \cap B}) = 1 - P(A \cap B) = 1 - \frac{1}{6} = \frac{5}{6}$

(ii) $P(\overline{A} \cap \overline{B}) = P(\overline{A \cup B}) = 1 - P(A \cup B) = 1 - \frac{23}{36} = \frac{13}{36}$

(iii) $P(A \cap \overline{B}) = P(A) - P(A \cap B) = \frac{18}{36} - \frac{6}{36} = \frac{12}{36} = \frac{1}{3}$

(iv) $P(\overline{A} \cap B) = P(B) - P(A \cap B) = \frac{11}{36} - \frac{6}{36} = \frac{5}{36}$

(v) $P(\overline{A \cap B}) = 1 - P(A \cap B) = 1 - \frac{1}{6} = \frac{5}{6}$

(vi) $P(\overline{A} \cup B) = P(\overline{A}) + P(B) - P(\overline{A} \cap B)$

$= \left(1 - \frac{18}{36}\right) + \frac{11}{36} - \frac{5}{36} = \frac{2}{3}$

(vii) $P(\overline{A \cup B}) = 1 - P(A \cup B) = 1 - \frac{23}{36} = \frac{13}{36}$

(viii) $P[\overline{A} \cap (A \cup B)] = P[(A \cap \overline{A}) \cup (\overline{A} \cap B)]$

$= P(\overline{A} \cap B) = \frac{5}{36}$ $\quad [\because A \cap \overline{A} = \phi]$

(ix) $P[A \cup (\overline{A} \cap B)] = P(A) + P(\overline{A} \cap B) - P(A \cap \overline{A} \cap B)$

$= P(A) + P(\overline{A} \cap B) = \frac{18}{36} + \frac{5}{36} = \frac{23}{36}$

(x) $P(A \mid B) = \frac{P(A \cap B)}{P(B)} = \frac{6/36}{11/36} = \frac{6}{11}$

$P(B \mid A) = \frac{P(A \cap B)}{P(A)} = \frac{6/36}{18/36} = \frac{6}{18} = \frac{1}{3}$

(xi) $P(\overline{A} \mid \overline{B}) = \frac{P(\overline{A} \cap \overline{B})}{P(\overline{B})} = \frac{13/36}{25/36} = \frac{13}{25}$

$P(\overline{B} \mid \overline{A}) = \frac{P(\overline{A} \cap \overline{B})}{P(\overline{A})} = \frac{13/36}{18/36} = \frac{13}{18}$

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Example 4·13. *If two dice are thrown, what is the probability that the sum is (a) greater than 8, and (b) neither 7 nor 11?*

(b) Let A denote the event of getting the sum of 7 and B denote the event of getting the sum of 11 with a pair of dice.

Example 4·15. *A card is drawn from a well-shuffled pack of playing cards. What is the probability that it is either a spade or an ace ?*

Example 4·16. *A box contains 6 red, 4 white and 5 black balls. A person draws 4 balls from the box at random. Find the probability that among the balls drawn there is at least one ball of each colour.* **(Nagpur Univ. B.Sc., 1992)**

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4·9. Bayes Theorem. *If $E_1, E_2, ..., E_n$ are mutually disjoint events with $P(E_i) \neq 0, (i = 1, 2, \ldots, n)$ then for any orbitrary event A which is a subset of $\bigcup_{i=1}^{n} E_i$ such that $P(A) > 0$, we have*

$$P(E_i \mid A) = \frac{P(E_i)\,P(A \mid E_i)}{\sum_{i=1}^{n} P(E_i)\,P(A \mid E_i)}, \quad i = 1, 2, ..., n. \qquad ...(4\cdot12)$$

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Proof. Since $A \subset \bigcup_{i=1} E_i$, we have

$$A = A \cap \left(\bigcup_{i=1}^{n} E_i\right) = \bigcup_{i=1}^{n} (A \cap E_i) \qquad \text{[By distributive law]}$$

Since $(A \cap E_i) \subset E_i$, $(i = 1, 2, ..., n)$ are mutually disjoint events, we have by addition theorem of probability (or Axiom 3 of probability)

$$P(A) = P\left[\bigcup_{i=1}^{n} (A \cap E_i)\right] = \sum_{i=1}^{n} P(A \cap E_i) = \sum_{i=1}^{n} P(E_i)\, P(A \mid E_i), \qquad ...(*)$$

by compound theorem of probability.

Also we have

$$P(A \cap E_i) = P(A)\, P(E_i \mid A)$$

$$P(E_i \mid A) = \frac{P(A \cap E_i)}{P(A)} = \frac{P(E_i)\, P(A \mid E_i)}{\sum_{i=1}^{n} P(E_i)\, P(A \mid E_i)} \qquad \text{[From (*)]}$$

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Example. 4·31. *The contents of urns I, II and III are as follows:*

1 white, 2 black and 3 red balls,

2 white, 1 black and 1 red balls, and

4 white, 5 black and 3 red balls.

One urn is chosen at random and two balls drawn. They happen to be white and red. What is the probability that they come from urns I, II or III ?

Example. 4·32. *In answering a question on a multiple choice test a student either knows the answer or he guesses. Let p be the probability that he knows the answer and 1−p the probability that he guesses. Assume that a student who guesses at the answer will be correct with probability 1/5, where 5 is the number of multiple-choice alternatives. What is the conditional probability that a student knew the answer to a question given that he answered it correctly?*

Example 4·33. *In a bolt factory machines A, B and C manufacture respectively 25%, 35% and 40% of the total. Of their output 5, 4, 2 per cent are defective bolts. A bolt is drawn at random from the product and is found to be defective. What are the probabilities that it was manufactured by machines A, B and C?*

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- Excercises:
- 3, 4, 5

Chebysev's inequality

Theorem 6·31. *If X is a random variable with mean μ and variance σ^2, then for any positive number k, we have*

$$P\{|X-\mu| \geq k\sigma\} \leq 1/k^2 \qquad \ldots(6\cdot73)$$

or $$P\{|X-\mu| < k\sigma\} \geq 1-(1/k^2) \qquad \ldots(6\cdot73\,a)$$

Proof. Case (i). *X is a continuous r.v. By def.,*

$$\sigma^2 = \sigma_X^2 = E[X-E(X)]^2 = E[X-\mu]^2$$

$$= \int_{-\infty}^{\infty} (x-\mu)^2 f(x)\,dx, \text{ where } f(x) \text{ is p.d.f. of } X.$$

$$= \int_{-\infty}^{\mu-k\sigma} (x-\mu)^2 f(x)dx + \int_{\mu-k\sigma}^{\mu+k\sigma} (x-\mu)^2 f(x)\,dx + \int_{\mu+k\sigma}^{\infty} (x-\mu)^2 f(x)\,dx$$

$$\geq \int_{-\infty}^{\mu-k\sigma} (x-\mu)^2 f(x)\,dx + \int_{\mu+k\sigma}^{\infty} (x-\mu)^2 f(x)\,dx \qquad \ldots(*)$$

We know that :

$$x \le \mu - k\sigma \text{ and } x \ge \mu + k\sigma \quad \Leftrightarrow \quad |x - \mu| \ge k\sigma \qquad \ldots(**)$$

Substituting in (*), we get

$$\therefore \qquad \sigma^2 \ge k^2\sigma^2 \left[\int_{-\infty}^{\mu - k\sigma} f(x)\,dx + \int_{\mu + k\sigma}^{\infty} f(x)\,dx \right]$$

$$= k^2\sigma^2 \left[P(X \le \mu - k\sigma) + P(X \ge \mu + k\sigma)\right] \qquad \text{[From (**)]}$$

$$= k^2\sigma^2 .\; P(|X - \mu| \ge k\sigma) \qquad \text{[From (**)]}$$

$$\Rightarrow \quad P(|X - \mu| \ge k\sigma) \le 1/k^2, \qquad \ldots(***)$$

which establishes (6·73)

Also since

$P\{|X - \mu| \ge k\sigma\} + P\{|X - \mu| < k\sigma\} = 1$, we get

$$P\{|X - \mu| < k\sigma\} = 1 - P\{|X - \mu| \ge k\sigma\} \ge 1 - \{1/k^2\} \qquad \text{[From (***)]}$$

Remark. In particular, if we take $k\sigma = c > 0$, then (*) and (**) give respectively

$$P\left\{|X - \mu| \ge c\right\} \le \frac{\sigma^2}{c^2} \text{ and } P\left\{|X - \mu| < c\right\} \ge 1 - \frac{\sigma^2}{c^2}$$

$$\left.\begin{aligned} \Rightarrow \quad & P\left\{|X - E(X)| \ge c\right\} \le \frac{\text{Var}(X)}{c^2} \\ \text{and} \quad & P\left\{|X - E(X)| < c\right\} \ge 1 - \frac{\text{Var}(X)}{c^2} \end{aligned}\right\} . \qquad \ldots(6{\cdot}73\,b)$$

Example 6·48. *A symmetric die is thrown 600 times. Find the lower bound for the probability of getting 80 to 120 sixes.*

Solution. Let S be total number of successes.

Then

$$E(S) = np = 600 \times \frac{1}{6} = 100$$

$$V(S) = npq = 600 \times \frac{1}{6} \times \frac{5}{6} = \frac{500}{6}$$

Using Chebychev's inequality, we get

$$P[\,|S - E(S)| < k\sigma\,] \geq 1 - \frac{1}{k^2}$$

$$\Rightarrow \qquad P\{\,|S - 100| < k\sqrt{500/6}\,\} \geq 1 - \frac{1}{k^2}$$

$$\Rightarrow \qquad P\{\,100 - k\sqrt{500/6} < S < 100 + k\sqrt{500/6}\,\} \geq 1 - \frac{1}{k^2}$$

Taking $k = \dfrac{20}{\sqrt{500/6}}$, we get

$$P(80 \leq S \leq 120) \geq 1 - \frac{1}{400 \times (6/500)} = \frac{19}{24}$$

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Example 6·53. *Two unbiased dice are thrown. If X is the sum of the numbers showing up, prove that*

$$P(|X - 7| \geq 3) \leq \frac{35}{54}.$$

Solution. The probability distribution of the r.v. X (the sum of the numbrs on the two dice) is as given below :

X	Favourable cases (distinct)	Probability (p)
2	(1, 1)	1/36
3	(1, 2), (2, 1)	2/36
4	(1, 3), (3, 1), (2, 2)	3/36
5	(1, 4), (4, 1), (2, 3), (3, 2)	4/36
6	(1, 5), (5, 1), (2, 4), (4, 2), (3, 3)	5/36
7	(1, 6), (6, 1), (2, 5), (5, 2), (3, 4), (4, 3)	6/36
8	(2, 6), (6, 2), (3, 5), (5, 3), (4, 4)	5/36
9	(3, 6), (6, 3), (4, 5), (5, 4)	4/36
10	(4, 6), (6, 4), (5, 5)	3/36
11	(5, 6), (6, 5)	2/36
12	(6, 6)	1/36

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$$E(X) = \sum_x p \cdot x$$
$$= \frac{1}{36}(2 + 6 + 12 + 20 + 30 + 42 + 40 + 36 + 30 + 22 + 12)$$
$$= \frac{1}{36}(252) = 7$$
$$E(X^2) = \sum_x p \cdot x^2$$
$$= \frac{1}{36}[4 + 18 + 48 + 100 + 180 + 294 + 320 + 324 + 300 + 242 + 144]$$
$$= \frac{1}{36}(1974) = \frac{1974}{36} = \frac{329}{6}$$
$$\text{Var}(X) = E(X^2) - [E(X)]^2 = \frac{329}{6} - (7)^2 = \frac{35}{6}$$

By Chebychev's inequality, for $k > 0$, we have

$$P(|X - \mu| \geq k) \leq \frac{\text{Var}\, X}{k^2}$$
$$\Rightarrow \quad P(|X - 7| \geq 3) \leq \frac{35/6}{9} = \frac{35}{54} \qquad \text{(Taking } k = 3\text{)}$$

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Example 6·54. *If X is the number scored in a throw of a fair die, show that the Chebychev's inequality gives*

$$P[|X - \mu| > 2{\cdot}5] < 0{\cdot}47,$$

Solution. Here X is a random variable which takes the values 1,2,...,6, each with probability 1/6. Hence

$$E(X) = \tfrac{1}{6}(1 + 2 + \ldots + 6) = \tfrac{1}{6} \cdot \frac{6 \times 7}{2} = \frac{7}{2}$$
$$E(X^2) = \tfrac{1}{6}(1^2 + 2^2 + \ldots + 6^2) = \tfrac{1}{6}\,\frac{6 \times 7 \times 13}{6} = \frac{91}{6}$$
$$\therefore \quad \text{Var}(X) = E(X^2) - [E(X)]^2 = \frac{91}{6} - \frac{49}{4} = \frac{35}{12} = 2{\cdot}9167$$

For $k > 0$, Chebychev's inequality gives

$$P[|X - E(X)| > k] < \frac{\text{Var}\, X}{k^2}$$

Choosing $k = 2{\cdot}5$, we get

$$P[|X - \mu| > 2{\cdot}5] < \frac{2{\cdot}9167}{6{\cdot}25} = 0{\cdot}47$$

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Example 6·57. *How large a sample must be taken in order that the probability will be at least 0·95 that $\overline{X}_n$ will be lie within 0·5 of μ. μ is unknown and* $\sigma = 1$. **[Delhi Univ. B.Sc. (Maths Hons.) 1988]**

Solution. We have: $E(\overline{X}_n) = \mu$ and $\text{Var}(\overline{X}_n) = \sigma^2/n$

[c.f. § 6·15, = n (6·80)]

Applying Chebychev's inequality to the r.v. $\overline{X}_n$ we get, for any $c > 0$

$$P[|\overline{X}_n - E(\overline{X}_n)| < c] \geq 1 - \frac{\text{Var}(\overline{X}_n)}{c^2}$$

$$\Rightarrow \quad P[|\overline{X}_n - \mu| < c] \geq 1 - \frac{\sigma^2}{nc^2} \quad \text{...(*)}$$

We want n so that

$$P[|\overline{X}_n - \mu| < 0{\cdot}5] \geq 0{\cdot}95 \quad \text{...(**)}$$

Comparing (*) and (**) we get :

$$c = 0{\cdot}5 = 1/2 \quad \text{and} \quad 1 - \frac{\sigma^2}{nc^2} = 0{\cdot}95 \quad \text{and} \quad \sigma = 1 \quad \text{(Given)}$$

$$\therefore \quad 1 - \frac{4}{n} = 0{\cdot}95 \quad \Rightarrow \quad \frac{4}{n} = 0{\cdot}05 = \frac{1}{20} \quad \Rightarrow \quad n = 80.$$

Hence $n \geq 80$.

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- Excercises: 3 and 5

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