What is Digital Signal Processing (DSP)?

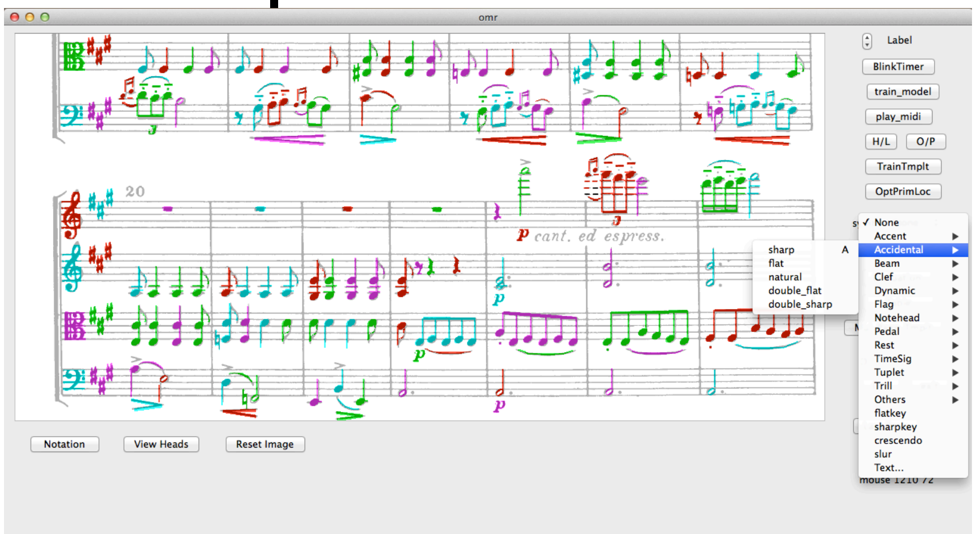
- **Signals convey information**

- DSP: modify signals with computers (and/or DSP circuits)

- DSP: represent,
  transform,
  manipulate, enhance, and analyze signals

- Digital ?
  x[n] = a sequence of numbers; *x[n]*={50,10,-50,…}
  x values from an interval

What is DSP ?



omr
Label
BlinkTimer
train_model
play_midi
H/L
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TrainTmplt
OptPrimLoc
None
Accent
Accidental
Beam
Clef
Dynamic
Flag
Notehead
Pedal
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TimeSig
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Others
flatkey
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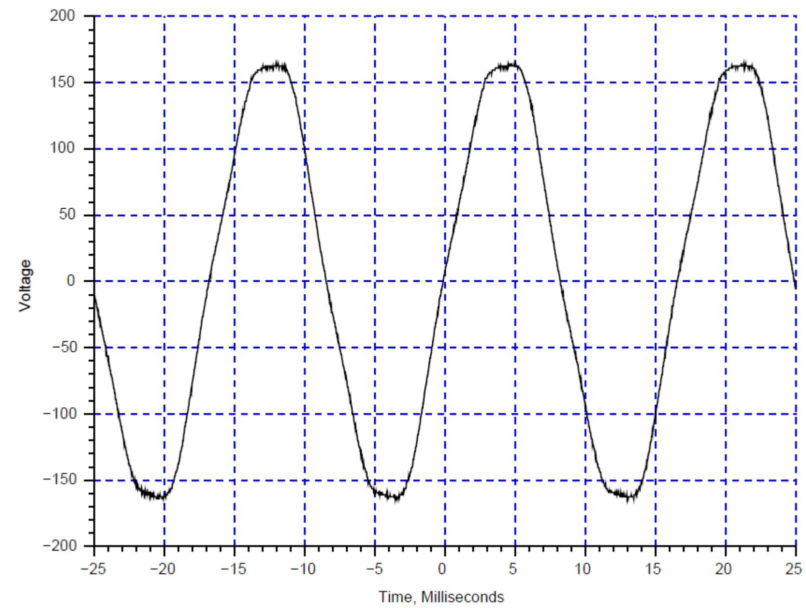
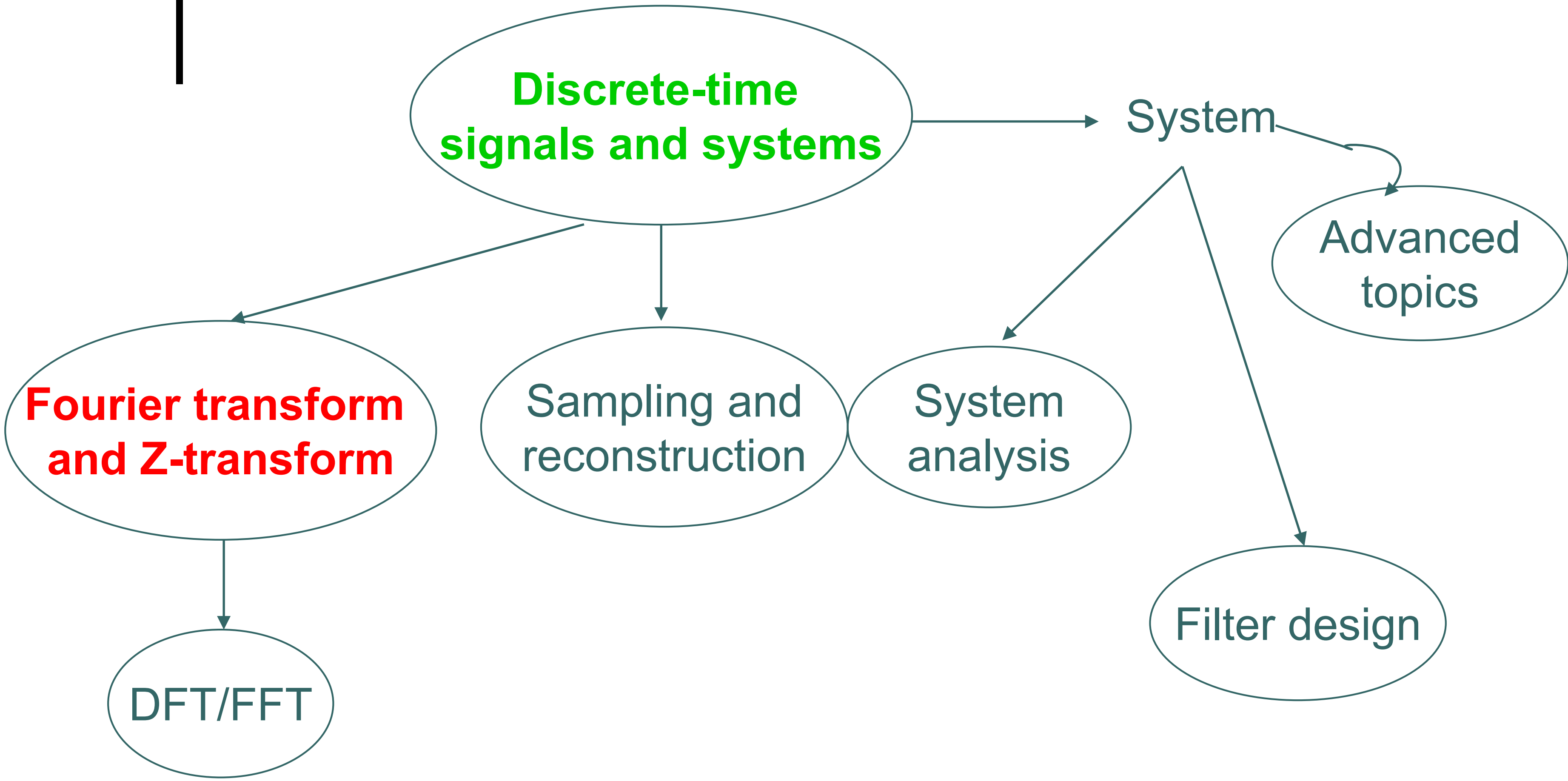


Voltage
Time, Milliseconds
200
150
100
50
0
−50
−100
−150
−200
−25
−20
−15
−10
−5
0
5
10
15
20
25

Applications of DSP

- Speech processing
  - Enhancement – noise filtering
  - Coding, synthesis, and recognition
- Image processing
  - Enhancement, coding, pattern recognition
- Multimedia processing
  - Media transmission, digital TV, video conferencing
- Telecommunications
- Biomedical engineering
- Navigation, radar, GPS
- Control, robotics, machine vision
- …

Course at a glance

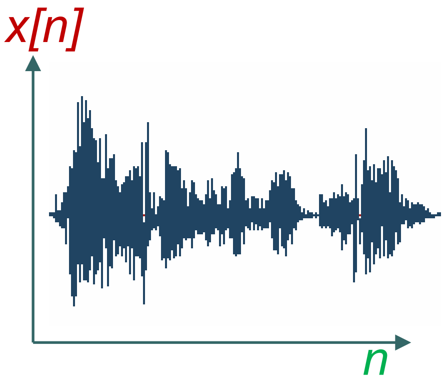


Outline

- **Introduction**
- DT signals
- Transformation of the independent variable
- DT LTI Systems & Convolution
- Frequency content of signals
- Fourier Transform
- Frequency response & difference equations

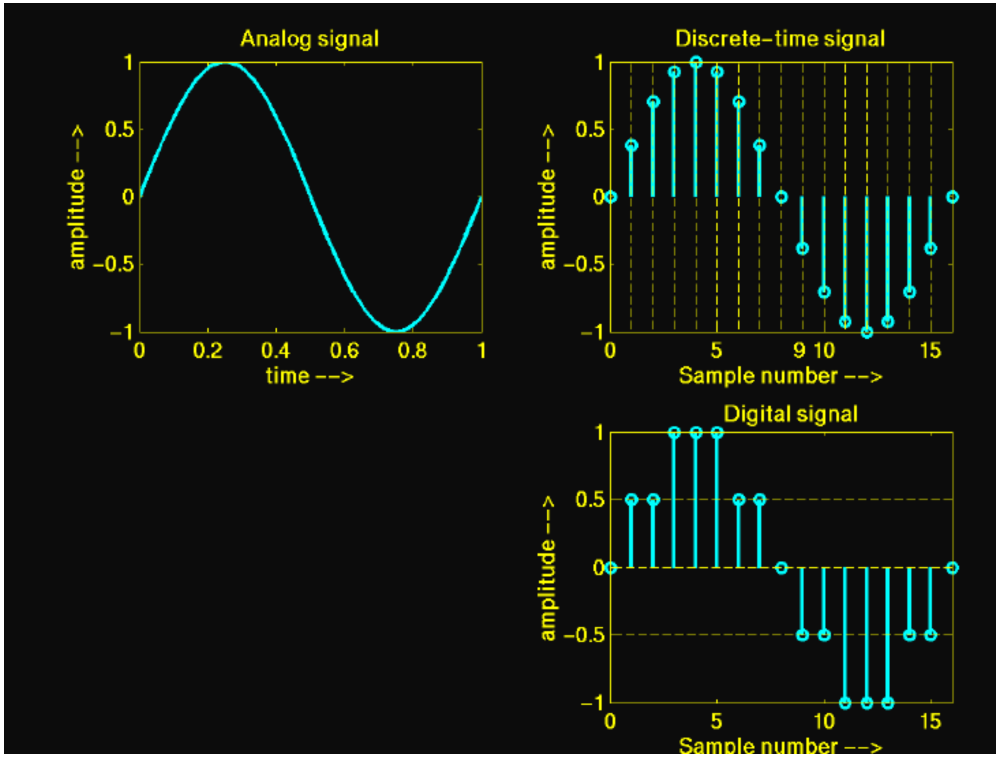
What is a signal?

- A flow of information
  - **Speech**: 1-Dimension signal as a function of time *x*(n)
  - **Image**: 2-dimension signal as a function of space *x*(i,j)
  - **Video**: 3-dimension signal as a function of space and time *x*(I,j,n)



- *x[n]* a function of the independent variable *n*
  - *n* represents the time of a speech signal
  - *x* the strength (amplitude) of the speech signal

Types of signals



- Discrete-time signals *x[n]*
  - *n* a discrete time

➔ *x(n)* defined at discrete n

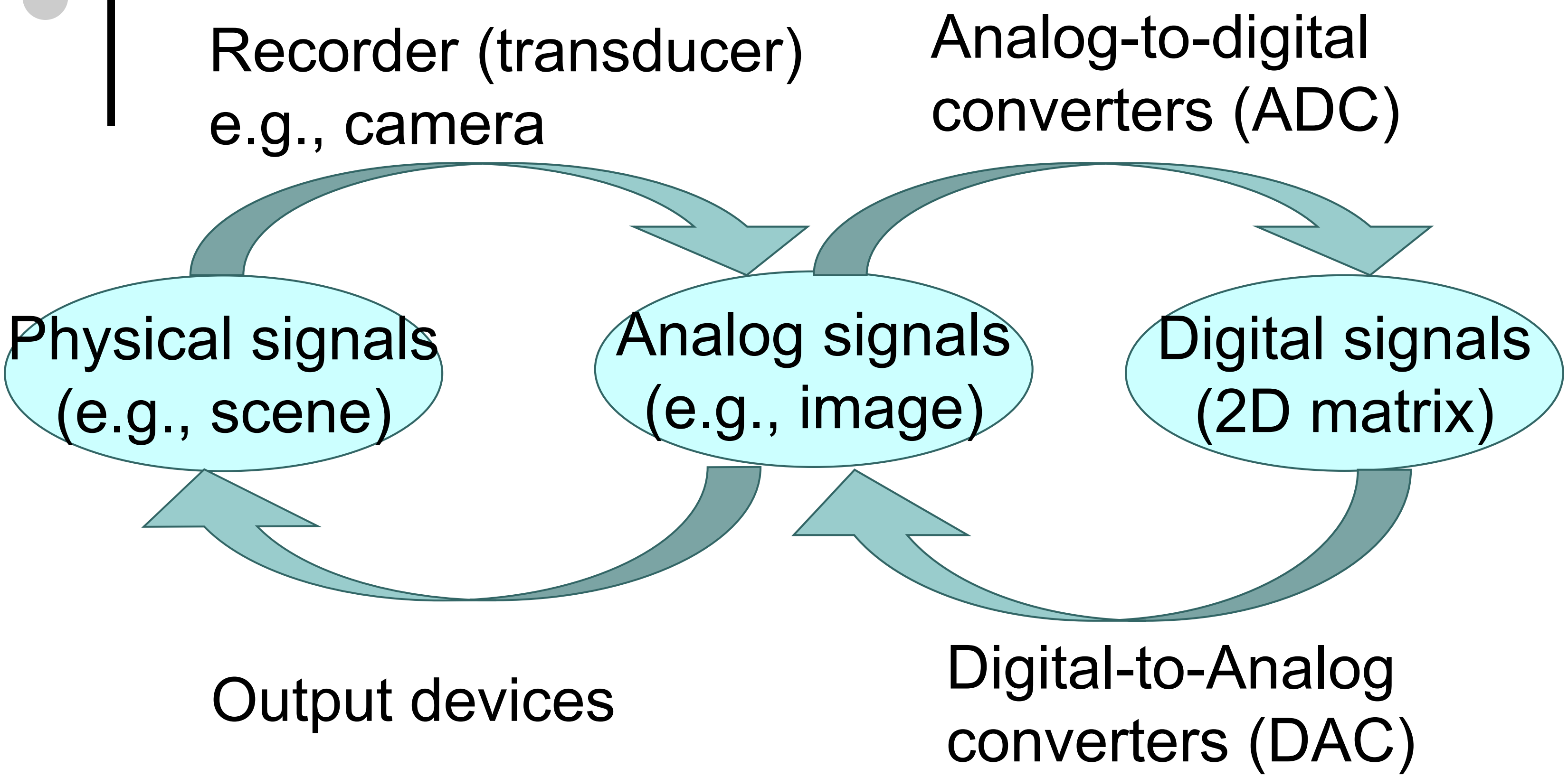
- Continuous-time signals x(t)
  - t continuous

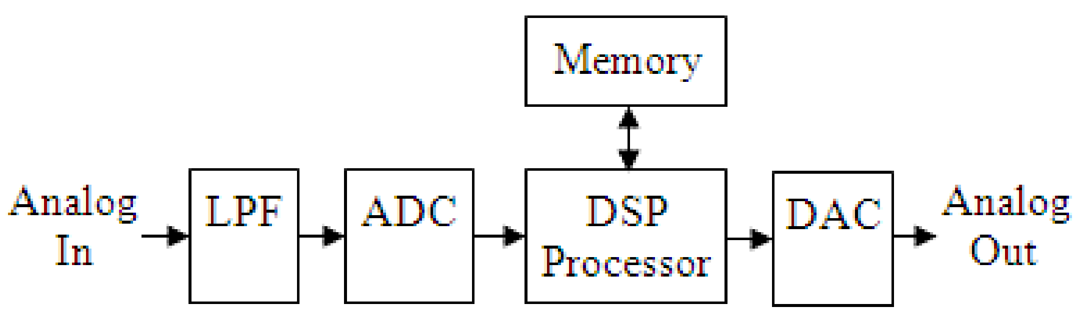
➔ x(t) defined at all times t

- *x(n)* continuous-amplitude
- *x(n)* discrete-amplitude

- Analog signals: both time and amplitude are continuous
- Digital signals: both time and amplitude are discrete

Typical DSP system components





Pros and cons of DSP

- Pros
  - Easy to duplicate
  - Stable and robust: not varying with temperature, storage without deterioration
  - Flexibility and upgrade: use a general computer or microprocessor

- Cons
  - Limitations of ADC and DAC
    ➔Lost of information
  - High power consumption due to complexity of a DSP implementation
    ➔ unsuitable for simple, low-power applications
  - Limited to signals with relatively low bandwidths (frequency)

Key History of DSP

- Prior to 1950's: analog signal processing using specialized electronic circuits or mechanical devices
- 1950's: computer simulation before analog implementation, thus cheap to try out
- 1965: Fast Fourier Transforms (FFTs) by Cooley and Tukey – make real time DSP possible
- 1980's: integrated circuit (IC) technology boosting DSP

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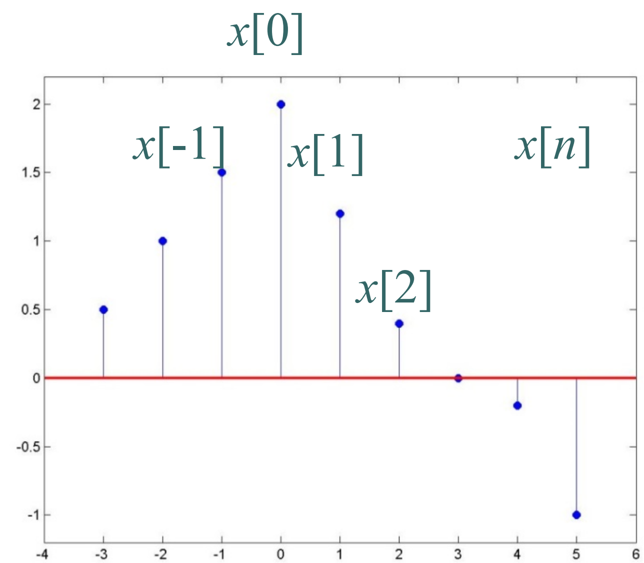
- Frequency response

Discrete-time signals

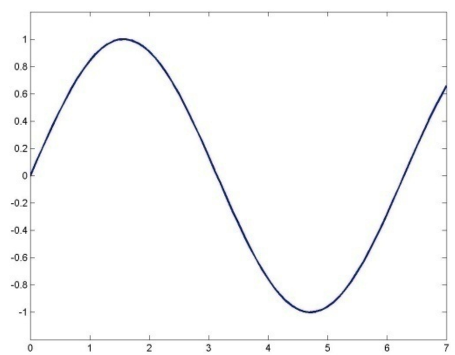
- Sequences of numbers

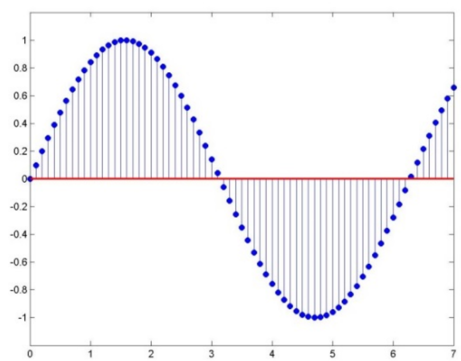
$$x = \{x[n]\}, \qquad -\infty < n < \infty;\ n \text{ integer}$$



- Sampling of an analog signal

$$x[n] = x_a(nT), -\infty < n < \infty; T \text{ the sampling period}$$





- Total energy $$E = \sum_{n=-\infty}^{\infty} |x[n]|^2$$
- Average power $$P = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x[n]|^2$$

Sequence operations

- **Product & sum** of two sequences x[n] and y[n]:
  ➔ sample-by-sample production and sum

- **Multiplication** of a sequence x[n] by a number α
  ➔multiplication of each sample value by α

- **Delay** or shift of a sequence x[n]

  $$y[n] = x[n - n_0]$$

  where n_0 is an integer

- …

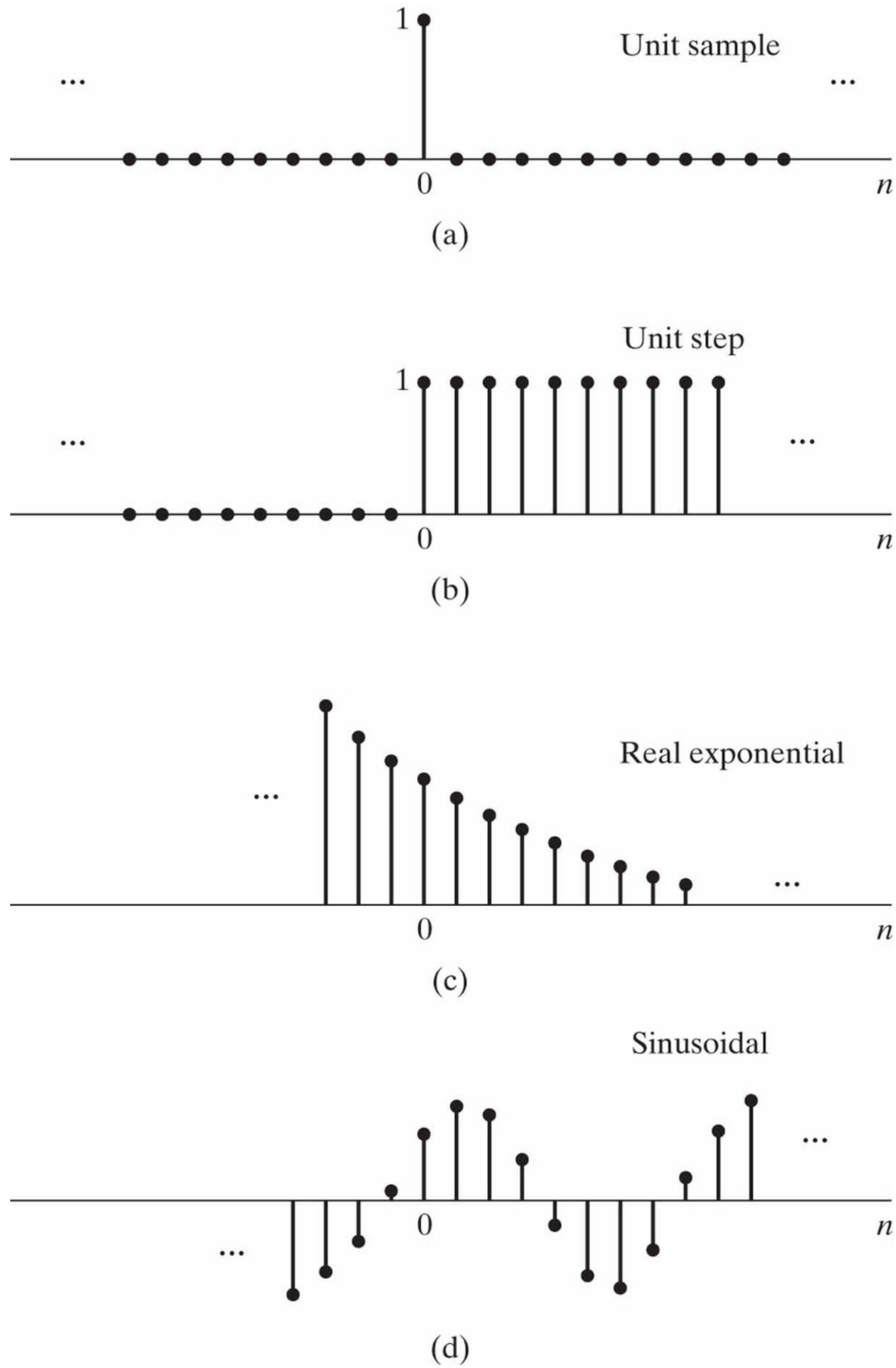
Some basic signals

Combining basic sequences:

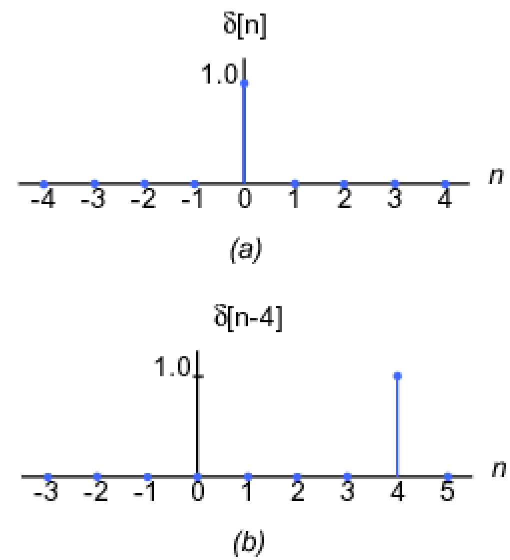
- Exponential and unit step:

an exponential that is zero for n<0

$$x[n] = A\alpha^n u[n] = \begin{cases} A\alpha^n, & n \geq 0 \\ 0, & n < 0 \end{cases}$$



Basic Signals: Impulse



- Unit sample sequence

$$\delta[n] = \begin{cases} 0, & n \neq 0, \\ 1, & n = 0, \end{cases}$$

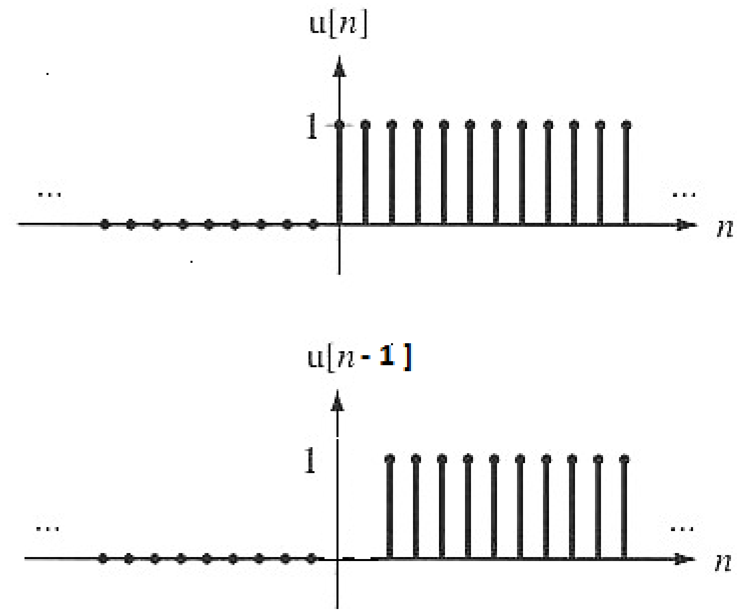
- Any sequence can be represented as a sum of scaled, delayed impulses

$$x[n] = a_{-3}\delta[n+3] + a_{-2}\delta[n+2] + \ldots + a_5\delta[n-5]$$

  - More generally $x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$

Basic Signals: Unit step

- $$u[n] = \begin{cases} 1, & n \geq 0, \\ 0, & n < 0, \end{cases}$$



- Related to the impulse by

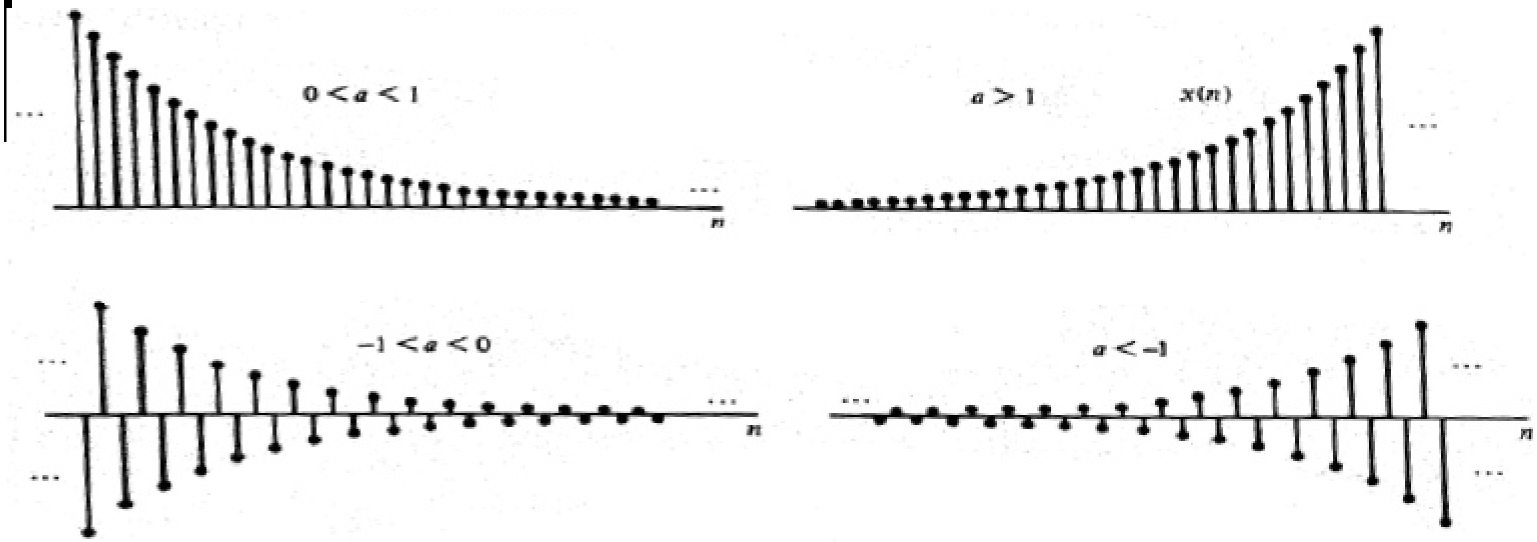
$$u[n] = \delta[n] + \delta[n-1] + \delta[n-2] + \ldots$$

or

$$u[n] = \sum_{k=-\infty}^{\infty} u[k]\delta[n-k] = \sum_{k=0}^{\infty} \delta[n-k]$$

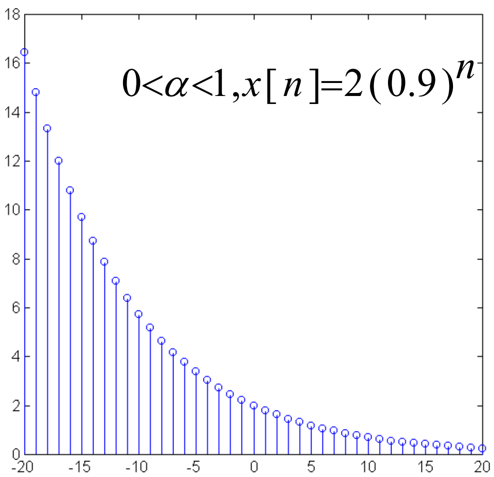
- Conversely, $$\delta[n] = u[n] - u[n-1]$$

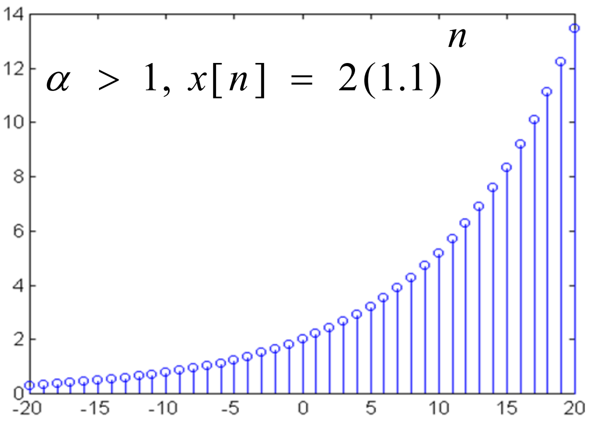
Basic Signals: Exponential $x[n] = A\alpha^n$

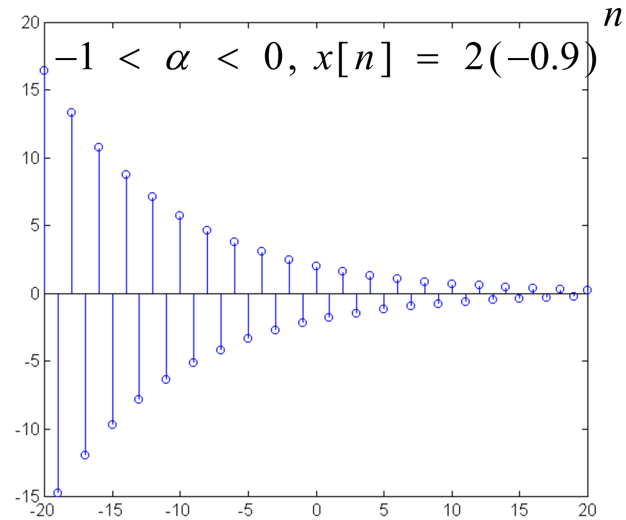


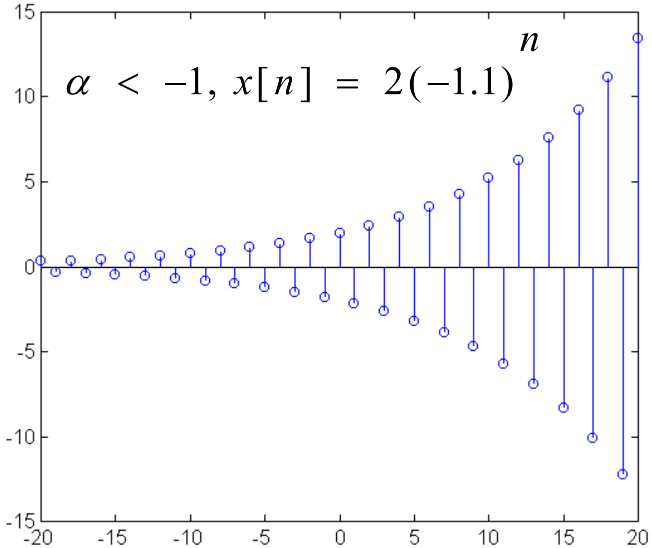
- If A and α are real numbers, the sequence is real
- If $0 < \alpha < 1$ and A is positive, the sequence values are positive and decrease with increasing n
- If $-1 < \alpha < 0$, the sequence values alternate in sign, but decrease in magnitude with increasing n
- If $|\alpha| > 1$, the sequence values increase with increasing n

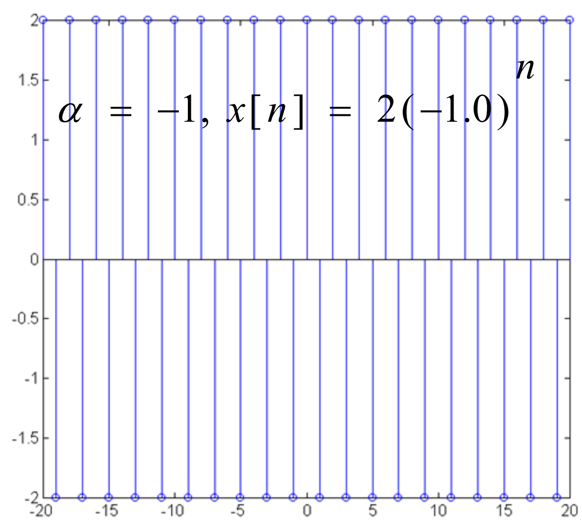
Real Exponential Signal $x[n] = A\alpha^n$











Basic Signals: Sinusoidal Signals

$$x[n] = A\cos(\omega_o n + \theta)$$

- Sinusoidal signals: important because they can be used to synthesize any signal
  - An arbitrary signal can be expressed as a sum of many sinusoidal signals with different frequencies, amplitudes and phases
  - Music notes are essentially sinusoids at different frequencies
- **Phase shift:** how much the max. of the sinusoidal signal is shifted away from t=0

$\omega_o = 2\pi f_o$

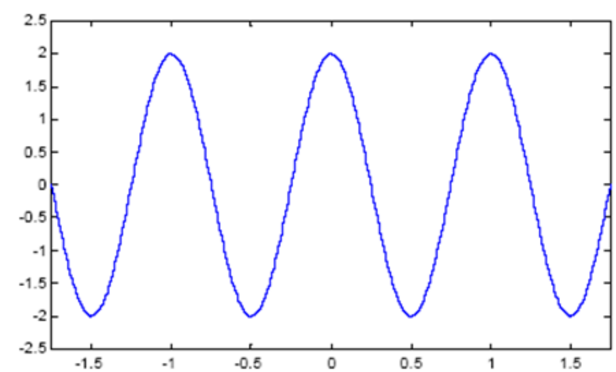
ω_o: angular frequency (radians)

f_o: frequency (cycles/second, Hz)

N: Period $N = 1/f = 2\pi/\omega_o$

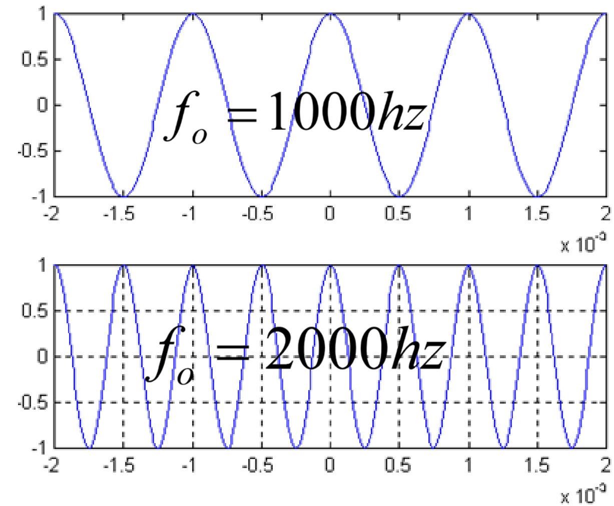
θ: Phase (shift)

A: Amplitude



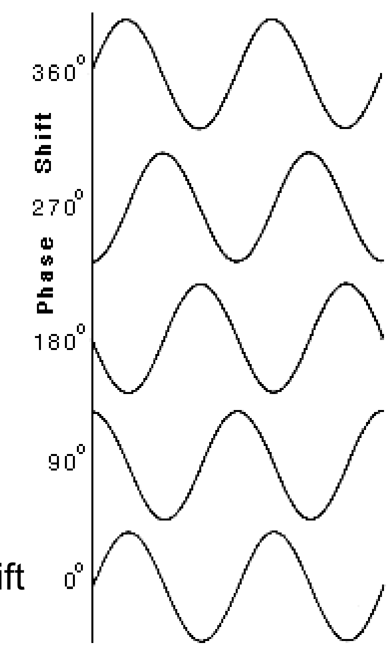
Sinusoidal Signals $x[n] = A\cos(\omega_o n + \theta)$

- Higher frequency : higher oscillation : more cycles/second (?)



ω_o

- The **phase θ** is the offset in the displacement from a specified reference point at time t = 0
- θ represents a "shift" from zero phase
- For infinitely long sinusoids, a change in θ is the same as a shift in time



Signals with different phase shift compared to the bottom signal

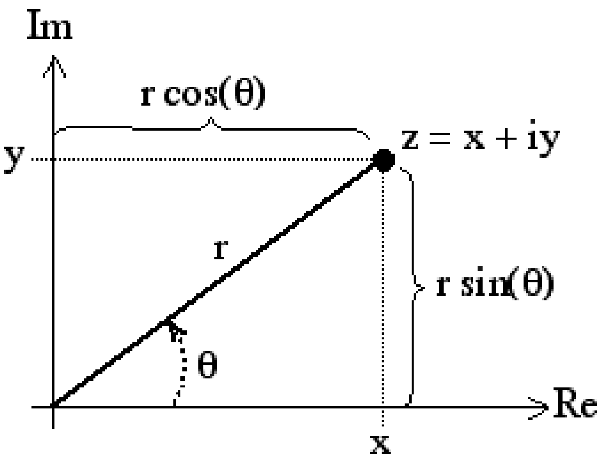
Basic signals: Complex Exponential Sequence $x[n] = Ae^{(j\omega_0 n+\theta)}$

- Complex number: Cartesian representation: $z = x + jy$

  $j = \sqrt{-1}$ and x,y are real numbers

  Magnitude of z is $|z| = \sqrt{x^2 + y^2}$

  Phase of z is $\theta = \angle z = \tan^{-1}\dfrac{y}{x}$



- Polar representation: $z = |z|e^{j\theta} = |z|\cos\theta + j|z|\sin\theta$
- Complex conjugate:

  $z^* = x - jy$ Note: $(z + z^*)$ *and* (zz^*) *are real*

- Euler forumla:

$$e^{j\theta} = \cos(\theta) + j\sin(\theta)$$

$$e^{\theta} + e^{-\theta} = 2\cos(\theta)$$

$$e^{\theta} - e^{-\theta} = j2\sin(\theta)$$

Complex Exponential Sequence

- $x[n] = Ae^{(j\omega_0 n+\theta)} = A\cos(\omega_0 n+\theta) + jA\sin(\omega_0 n+\theta)$

  θ is the phase shift (initial phase)

- $Ae^{(j\omega_0 n+\theta)}$ are closely related to sinusoidal $A\cos(\omega_o n+\theta)$

$$\text{Example}: A\cos(\omega_o n+\theta) = \frac{A}{2}e^{j\theta}e^{j\omega_o n} + \frac{A}{2}e^{-j\theta}e^{-j\omega_o n}$$

- n dimensionless $\Rightarrow$ both ω_o and θ have units of radians

- We note, for $\omega_{\mathrm{o}} = \pi$, odd multiple of π, $\mathrm{e}^{j\pi\,\mathrm{n}} = (e^{j\pi})^n = (-1)^n$,

the signal oscillates rapidly, changing sign at each point in time

Complex Exponentials: Periodicity

- CT : a sinusoid and a complex exponential signal are <u>both</u> periodic

  $x(t) = A\cos(\Omega_0 t + \theta)$ and $x(t) = e^{j\Omega_0 t}$

- A DT periodic sequence is defined as $x[n] = x[n+N], \quad \forall\ n$
  where the period *N* is <u>necessarily</u> an integer

For $e^{j\omega_0 n}$ to be periodic with period $N > 0$,

$e^{j\omega_0 (n+N)} = e^{j\omega_0 n}$, or equivalently $e^{j\omega_0 N} = 1 \;\therefore\; \omega_0 N$ must be a multiple of 2π

i.e. $\omega_0 N = 2\pi\, m$, or equvalently $\dfrac{\omega_0}{2\pi} = \dfrac{m}{N}$,

$\Rightarrow e^{j\omega_0 n}$ is periodic if $\omega_0 / 2\pi$ a rational number

is not periodic otherwise

Complex Exponentials: Frequency

- For a CT signal $x(t) = A\cos(\Omega_0 t + \theta)$

  as Ω_0 increases, $x(t)$ oscillates more and more rapidly

- For the DT signal
  - Consider $e^{j(\omega_o + 2\pi)n} = e^{j2\pi n} e^{j\omega_o n} = e^{j\omega_o n}$

  $\Rightarrow$ the exponential at frequency $\omega_o + 2\pi$ is the same as that at ω_o

  - Similarly at frequencies $\omega_o \pm 2\pi, \omega_o \pm 4\pi,$ and so on

  $\Rightarrow$ Because of this periodicity of 2π, we need only to consider frequency interval of 2π in the case of DT signals

  - Consider $x[n] = A\cos(\omega_0 n + \theta)$,

  as ω_0 increases from 0 to π, $x[n]$ oscillates more and more rapidly

  as ω_0 increases from π to 2π, the oscillations become slower

Complex Exponentials: Frequency

Because of implied periodicity, $e^{j\omega_o n}$ does not have a continually increasing rate of oscillation as ω_{o} is increased in magnitude

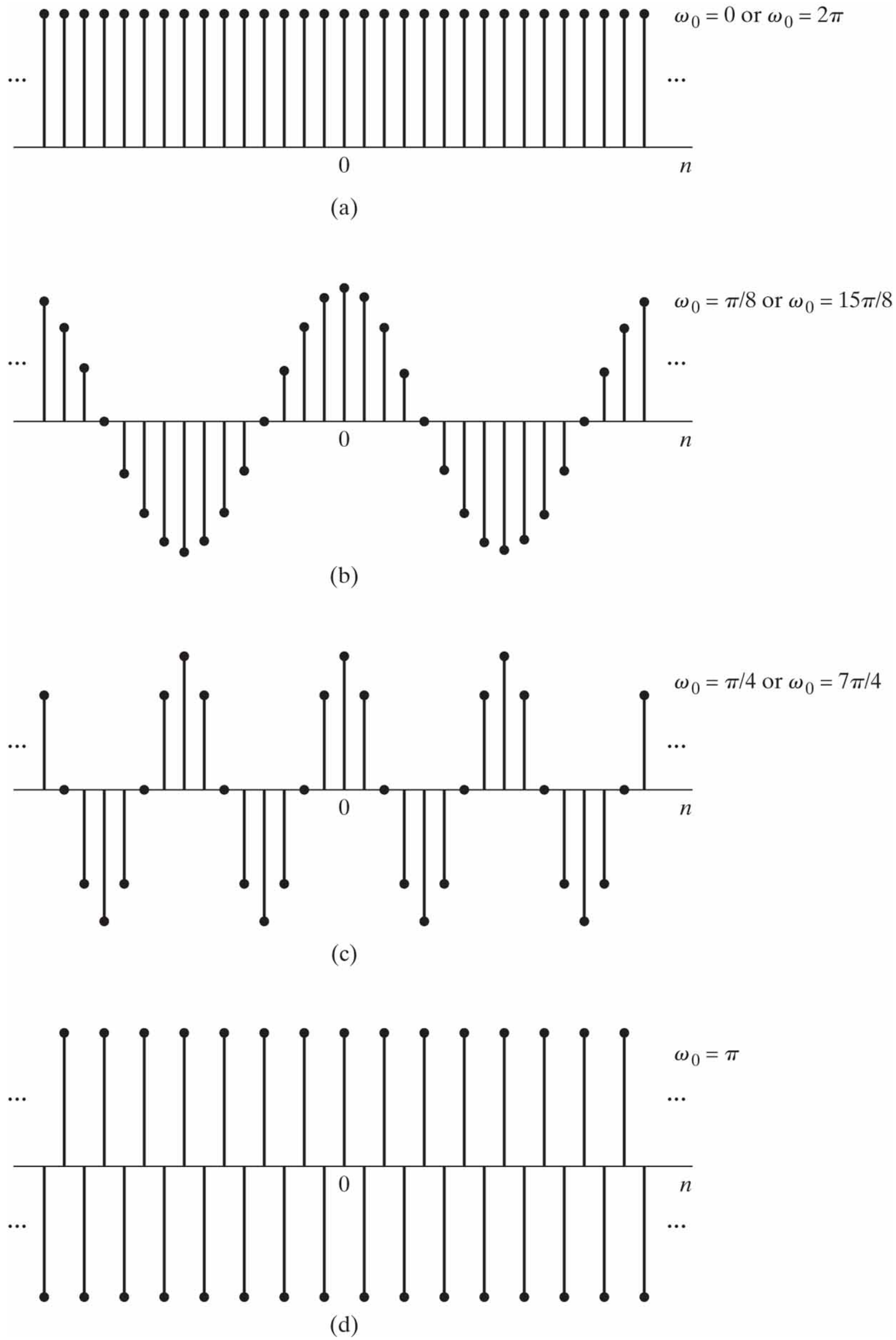
◦ Increasing ω_{o} from 0 (d.c., constant sequence, no oscillation) the oscillation increases until $\omega_{\text{o}} = \pi$, thereafter the oscillation will decrease to 0, i.e. a d.c. signal at $\omega_{\text{o}} = 2\pi$

$\Rightarrow$ low frequencies occurs at $\omega_{\text{o}} = 0, \quad \pm 2\pi, \pm$even multiple of π

$\Rightarrow$ High frequencies are at $\omega_{\text{o}} = \pm\pi, \pm 3\pi, \pm$odd multiple of π

cos $\omega_0 n$ for different ω_0

As ω_0 increases from zero toward π (parts a-d), the sequence oscillates more rapidly



CT versus DT complex exponentials

$e^{j\Omega_o t}$

Distinct signals for distinct values of Ω_o

The larger Ω_o , the higher the rate of oscillation

Periodic for any choice of Ω_o

Fundamental frequency Ω_o

Fundamental period T

$\Omega_o = 0$: T undefined

$$\Omega_o \neq 0 : \; T = \frac{2\pi}{\Omega_o}$$

$e^{j\omega_0 n}$

Identical signals for $\omega_o = 2\pi k$ (multiples of 2π)

Periodic only if $\omega_o = \dfrac{2\pi m}{N}$ (integer $N > 0$ and m)

Fundamental frequency $\dfrac{\omega_o}{m}$

Fundamental period N

$\omega_o = 0$: N undefined

$$\omega_o \neq 0 : \; N = m\frac{2\pi}{\omega_o} \; \left(\text{or } \frac{N}{m} = \frac{2\pi}{\omega_o} \text{ rational \#}\right)$$

General Complex Exponential Sequence

$x[n] = A\alpha^{n}$,

where A and α are complex numbers

- Case 1: Let $\alpha = e^{j\omega_o}$ and A=$e^{j\theta}$ $\therefore$ $x[n] = e^{j\theta}e^{j\omega_o n} = e^{j(\omega_o n+\theta)}$

$\Rightarrow x[n] = \cos(\omega_o n + \theta) + j\sin(\omega_o n + \theta)$

ω_o the frequency

θ the phase

And

$\operatorname{Re}\{x[n]\} = \cos(\omega_o n + \theta)$

$\operatorname{Im}\{x[n]\} = \sin(\omega_o n + \theta)$

General Complex Exponential Signals

- Case 2: For general complex A and α

$$\alpha = r_\alpha e^{j\omega_o} \text{ and } A = r_A e^{j\theta} \therefore \quad x[n] = A\alpha^n = r_A r_\alpha^n e^{j(\omega_o n+\theta)}$$

$$\Rightarrow x[n] = r_A r_\alpha^n \cos(\omega_o n + \theta) + j r_A r_\alpha^n \sin(\omega_o n + \theta)$$

$r_\alpha = 1 \Rightarrow$ real & imaginary parts are sinusoidal

$r_\alpha < 1 \Rightarrow$ sinusoidal decaying exponentially

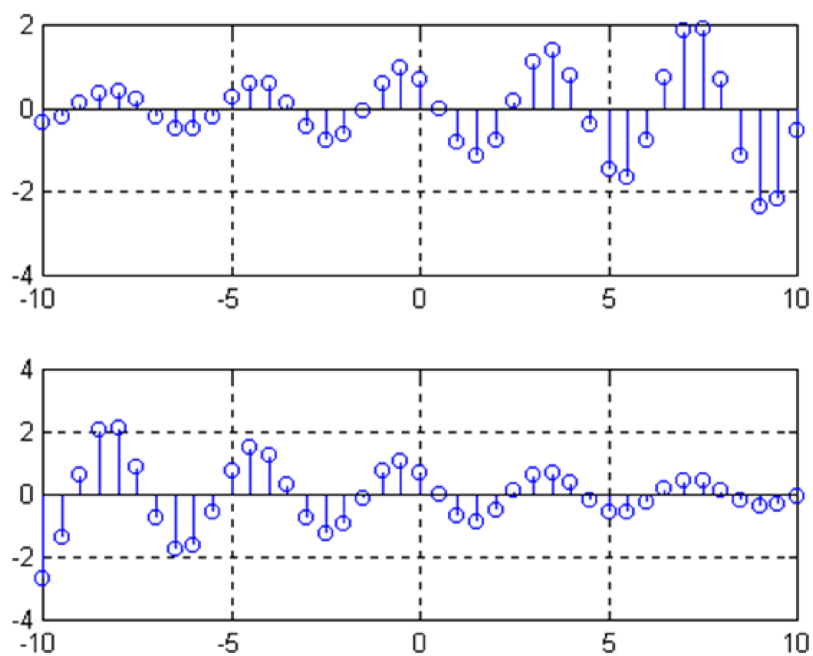
$r_\alpha > 1 \Rightarrow$ sinusoidal growing exponentially

Growing & Decaying exponential

Growing
- Population growth as function of generation
- Total return on investment as a function of day, month or quarter
- Credit card interest

Decaying:
- Response of RLC circuits
- Mechanical systems having both damping & restoring forces, e.g., automotive suspension system

```
pi=3.142;
t=-10:.1:10;
f=2000;
w=2*pi*f;
r=0.1;
x=zeros(size(t));
x=exp((r+w*i)*t);
theta=pi/4;
c=1*exp(i*theta);
y=c*x;
subplot(2,1,1);
plot(t,y);
grid;
```




```
r=-0.1;
x=zeros(size(t));
x=exp((r+w*i)*t);
theta=pi/4;
c=1*exp(i*theta);
y=c*x;
subplot(2,1,2);
plot(t,y);
grid;
```

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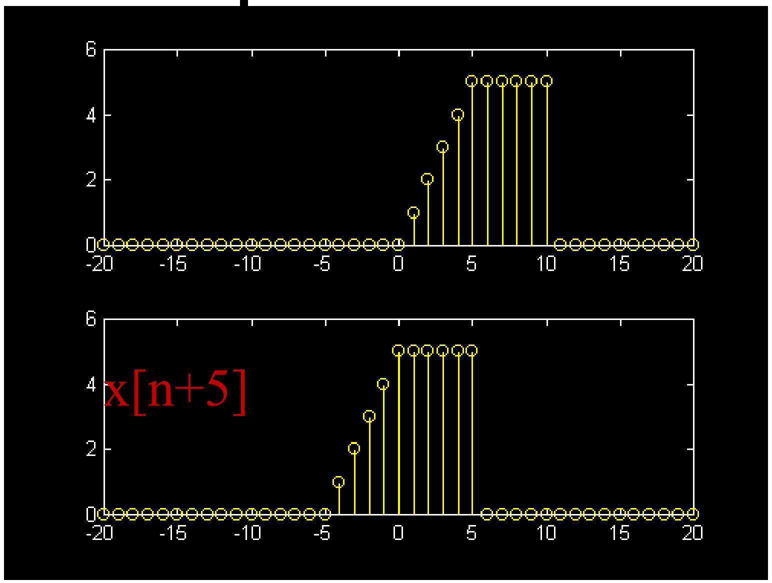
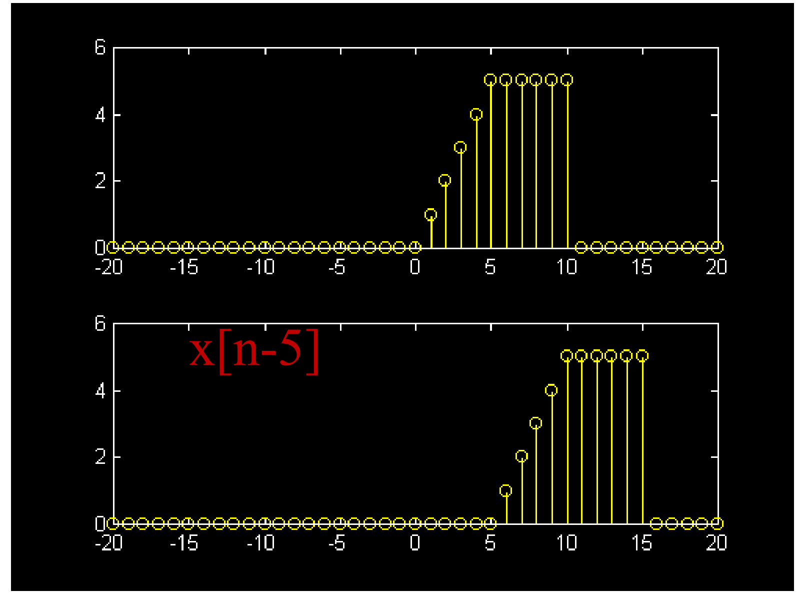
Transformation of Independent Variable (time axes)

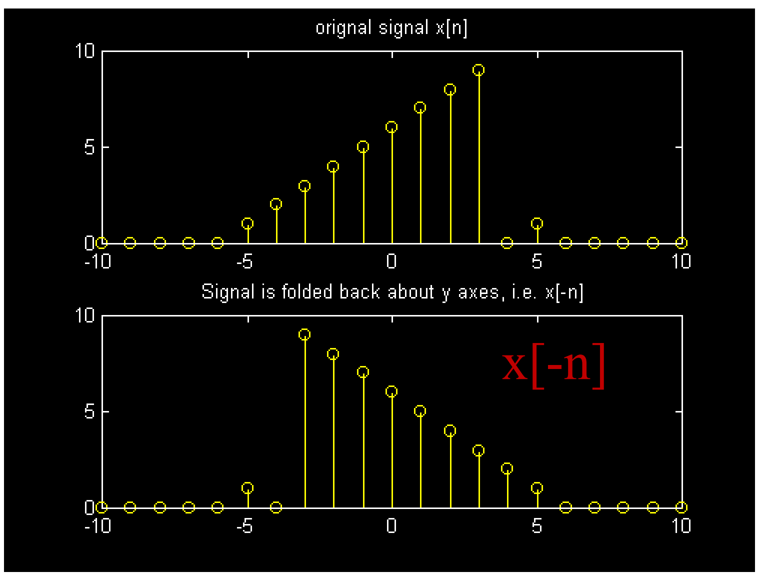
- Time shift *x[n-b]*
- Time reversal *x[-n]*
- Time scaling *x[2n]*
- Combinations of these: *x[an-b]*, a & b are constants

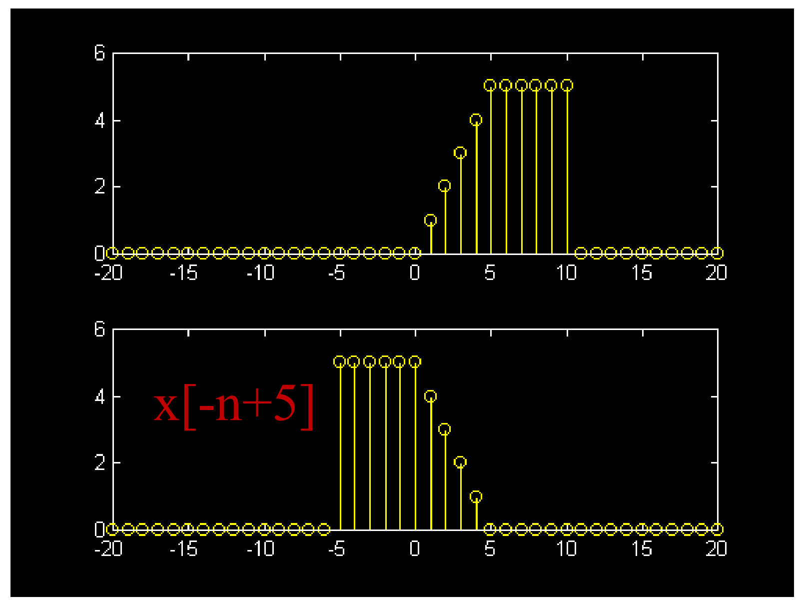
- An application: Aircraft control system:
  - Input = pilot actions
  - These action are transformed by electrical & mechanical system of the aircraft to changes to aircraft trust or position control surfaces such as the rudder & ailerons
  - Finally these changes affect the dynamics & kinematics such as the aircraft velocity and heading

Examples

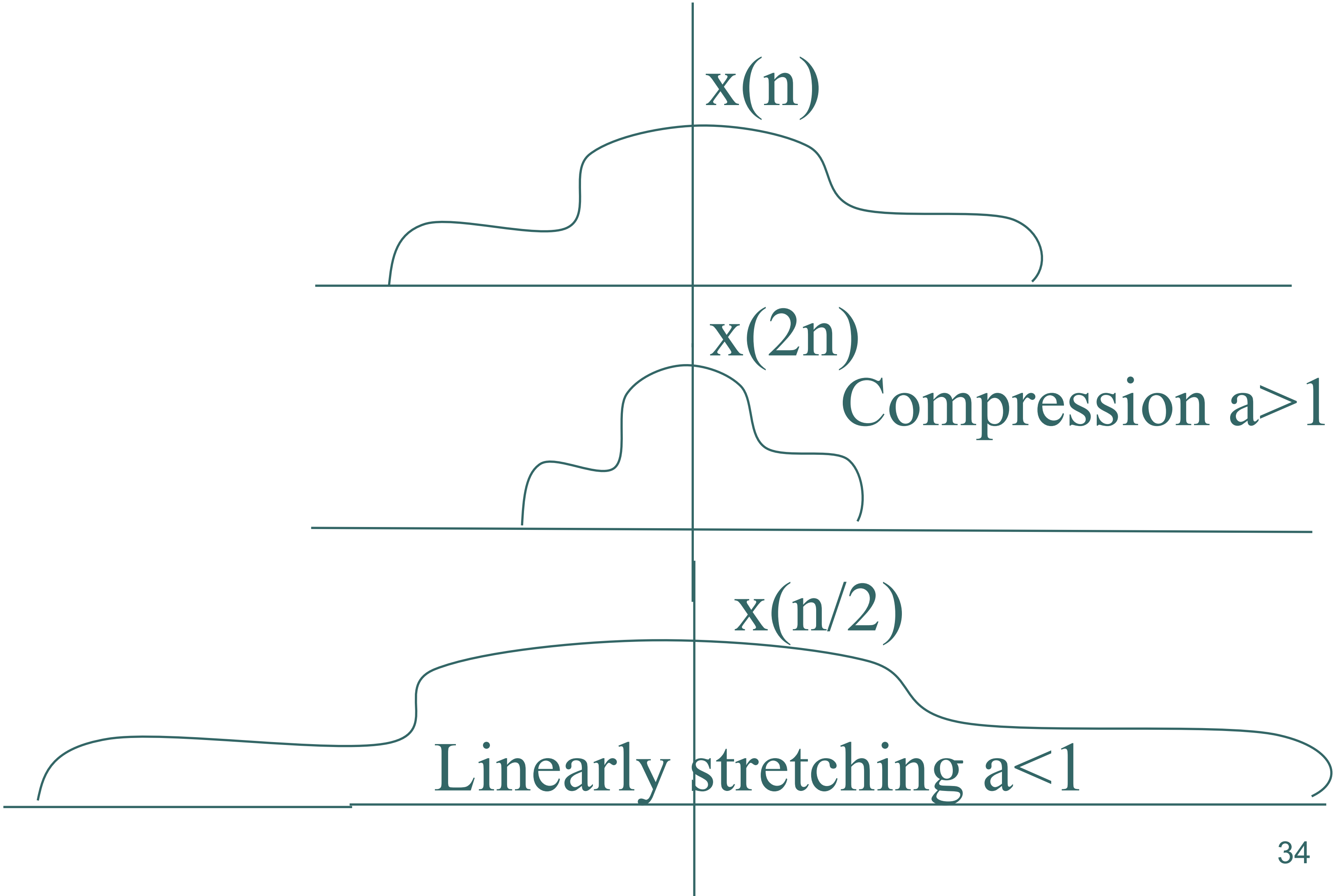

x[n+5]


x[n-5]


orignal signal x[n]
Signal is folded back about y axes, i.e. x[-n]
x[-n]


x[-n+5]

Examples
Time scaling $y(n)=x(an)$



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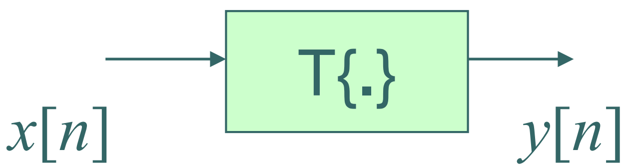
Discrete-time systems

$$y[n] = T\{x[n]\}$$

- An operator (transformation) that maps input into output



- Examples:
  - Delay system $\quad y[n] = x[n - n_d], \qquad -\infty < n < \infty$
  - Memoryless system $\quad y[n] = (x[n])^2, \qquad -\infty < n < \infty$
  - Accumulator: $\quad y[n] = \sum_{k=-\infty}^{n} x[k]$

LTI systems
Impulse response

- What if the input to the system is an impulse δ(n) ?

- The output is called the response to an impulse (impulse response) *h(n)*

- *But what properties should such system has?*
  1. *Linear &*
  2. *Time Invariant*

  ➔ *LTI*

Linear systems (Linearity)

- A system is linear if and only if

additivity property

$$T\{x_1[n]+x_2[n]\}=T\{x_1[n]\}+T\{x_2[n]\}=y_1[n]+y_2[n]$$

and

$$T\{ax[n]\}=aT\{x[n]\}=ay[n]$$

scaling property

where a is an arbitrary constant

- Combined into superposition

$$T\{ax_1[n]+bx_2[n]\}=aT\{x_1[n]\}+aT\{x_2[n]\}=ay_1[n]+ay_2[n]$$

Examples

- Accumulator system – a linear system

$$y[n]=\sum_{k=-\infty}^{n} x[k]$$

$$y_1[n]=\sum_{k=-\infty}^{n} x_1[k], \qquad y_2[n]=\sum_{k=-\infty}^{n} x_2[k]$$

$$y_3[n]=\sum_{k=-\infty}^{n} (ax_1[k]+bx_2[k]) = ay_1[n]+by_2[n]$$

- A nonlinear system

$$y[n]=\log_{10}(|\,x[n]\,|)$$

Consider $x_1[n]=1$ and $x_2[n]=10$

Time-invariant systems

$$x_1[n] = x[n-n_0] \Rightarrow y_1[n] = y[n-n_0]$$

- For which a time shift (or delay) of the input sequence causes a corresponding shift in the output sequence

- Example: Accumulator system

$$y[n] = \sum_{k=-\infty}^{n} x[k] \quad ; \quad y[n-n_0] = \sum_{k=-\infty}^{n-n_0} x[k]$$

$$y_1[n] = \sum_{k=-\infty}^{n} x_1[k] = \sum_{k=-\infty}^{n} x[k-n_0] = \sum_{k_1=-\infty}^{n-n_0} x[k_1] = y[n-n_0]$$

LTI systems and Convolution
Proof

- A linear system is completely characterised by its impulse response

$$\text{We know: } x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$$

$$h[n] = T\{\delta[n]\} \qquad \text{(response of } T\{\}\text{ to an impulse)}$$

$$h[n-k] = T\{\delta[n-k]\} \qquad \text{(time-invariant)}$$

$$\Rightarrow y[n] = T\{x[n]\} = T\{\sum_{k=-\infty}^{\infty} x[k]\delta[n-k]\} = \sum_{k=-\infty}^{\infty} x[k]T\{\delta[n-k]\} \quad \text{(linear)}$$

- LTI $y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] = \sum_{k=-\infty}^{\infty} x[n-k]h[n]$

$$= x[n] * h[n] = h[n] * x[n]$$

←Convolution sum

Computation of the convolution sum

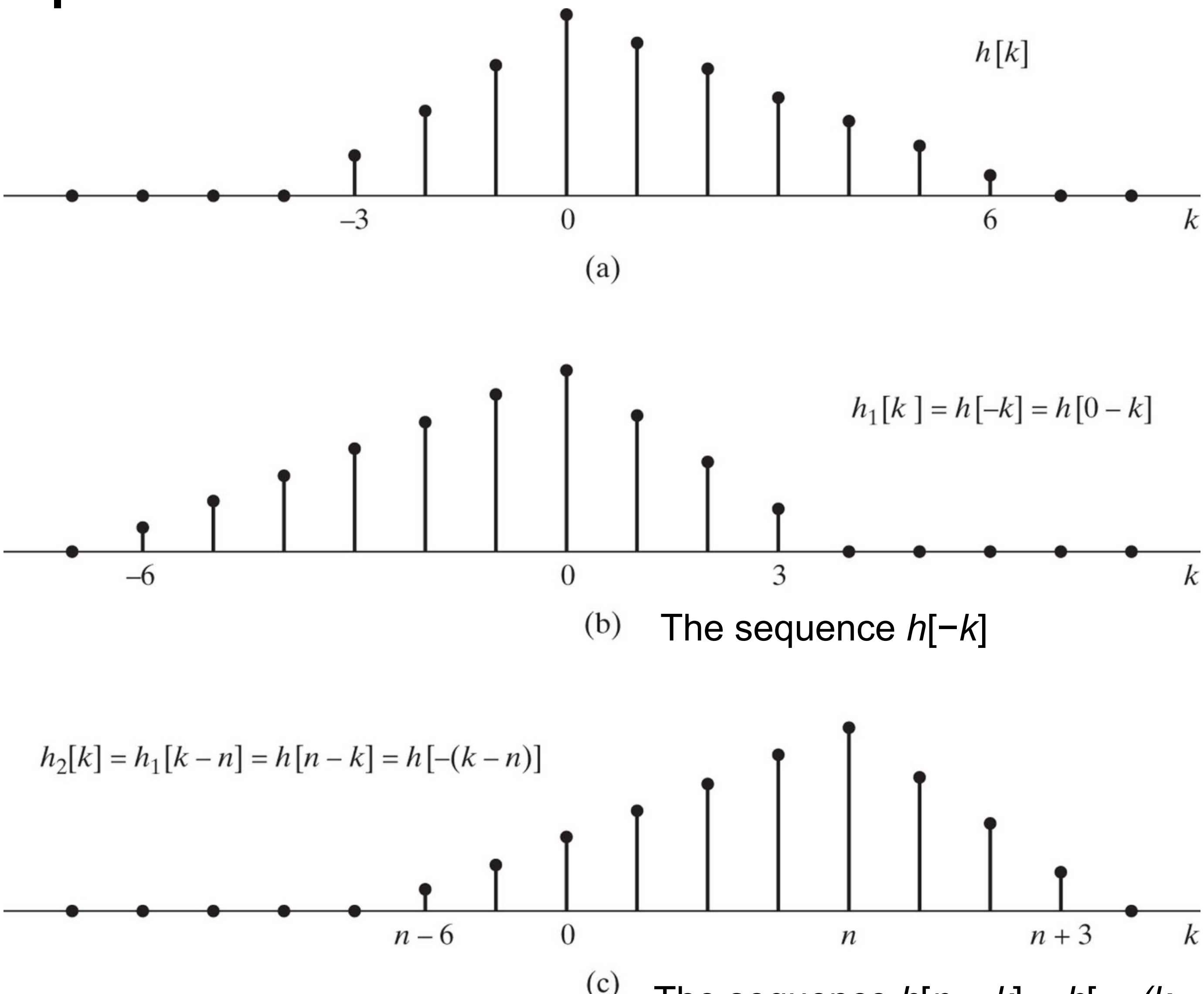
$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

- Obtain the sequence h[n-k]
  - Reflecting h[k] about the origin to get h[-k]
  - Shifting the origin of the reflected sequence to k=n
- Multiply x[k] and h[n-k] for
- Sum the products to compute the output sample y[n]

- Interpretation of convolution operation
  - replacing each pixel by a weighted sum of its neighbors

Example
  - Low-pass: the weights sum = weighted average
  - High-pass: the weighted sum = left neighbors –right neighbors

Forming the sequence $h[n\text{-}k]$



The sequence $h[-k]$

The sequence $h[n-k] = h[-(k-n)]$ for $n = 4$

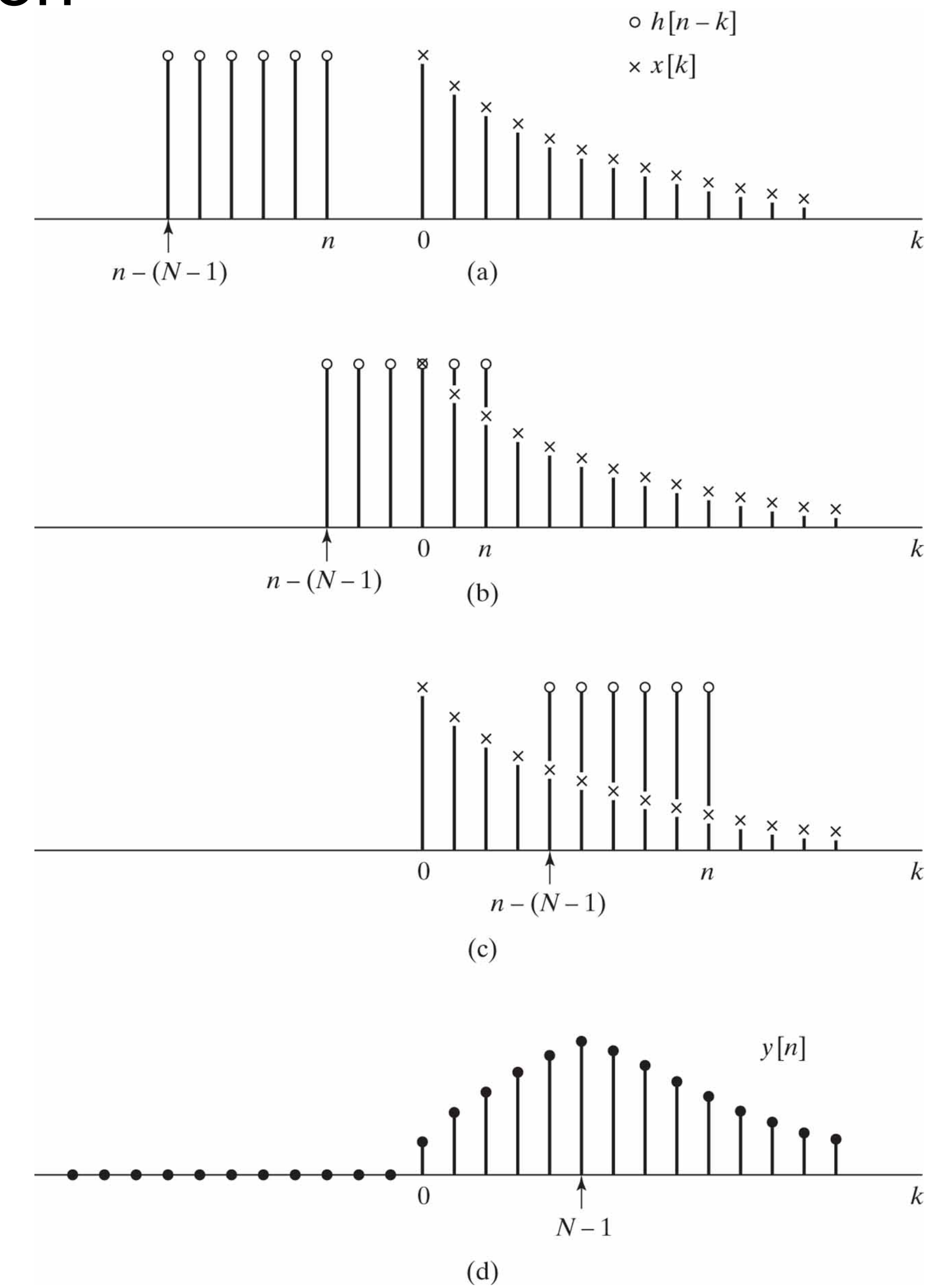
Discrete convolution Example

- Impulse response

$$h[n] = u[n] - u[n-N]$$
$$= \begin{cases} 1, & 0 \le n \le N-1, \\ 0, & \text{otherwise.} \end{cases}$$

- Input $x[n] = a^n u[n]$

$$y[n] = \begin{cases} 0, & n < 0, \\ \dfrac{1-a^{n+1}}{1-a}, & 0 \le n \le N-1, \\ a^{n-N+1}\left(\dfrac{1-a^N}{1-a}\right), & N-1 < n. \end{cases}$$



Convolution Example

$$h[n] = 2\delta[n] - 2\delta[n-1]$$
$$x[n] = u[n+1] - u[n-1] + 2\delta[n-2]$$
$$\text{Find } y[n] = x[n] * h[n]$$

- The sequence h[n] consists of two samples.
  - Therefore, convolving x[n] and h[n] can be simplified by convolving x[n] with h[n] one sample at a time.
  - For example, $h[n] = 4\delta[n] - 2\delta[n-1]$ can be convolved by convolving x[n] first with $h_1[n] = 4\delta[n]$ and then convolving x[n] with $h_2[n] = -2\delta[n-1]$
- Finally, the convolution sum (y[n]) can be then obtained by adding the two sequences (adding sample by corresponding sample).
- In doing this, the output y[n] is

$$y[n] = 3\delta[n+1] + \delta[n] - \delta[n-1] + 6\delta[n-2] - 4\delta[n-3]$$

- The same can be represented graphically which is just as good.

Properties of LTI systems Defined by convolution and *h[n]*

- Causality: the output sequence value at the index $n=n_0$ depends only on the input sequence values for $n<=n$
- Example $y[n] = x[n-n_d], \qquad -\infty < n < \infty$
  - Causal for $n_d>=0$
  - Noncausal for $n_d<0$
- Stability: A system is stable in the (bounded input bounded output) BIBO sense if and only if every bounded input sequence produces a bounded output sequence
- Example $y[n] = (x[n])^2, \qquad -\infty < n < \infty$

Properties of LTI systems Defined by convolution and *h[n]*

- Commutative
$$x[n]*h[n]=h[n]*x[n]$$
- Linear
$$x[n]*(h_1[n]+h_2[n])=x[n]*h_1[n]+x[n]*h_2[n]$$
- Cascade connection
$$h[n]=h_1[n]*h_2[n]$$
- Parallel connection
$$h[n]=h_1[n]+h_2[n]$$
- Stable
$$\sum_{k=-\infty}^{\infty}|h[k]|<\infty$$
- Causality
$$h[n]=0, \qquad n<0$$

Outline

- Introduction
- DT signals
- Transformation of the independent variable
- LTI Systems & Convolution
- **Frequency content of signals**
- Fourier transform
- Relation between Fourier representations
- Frequency response

Signal Representation

- Time-domain representation $x(n)$
  - Waveform based
  - Periodic / non-periodic signals
  - Sound amplitude, temperature reading, stock price

- Frequency-domain representation $X(e^{j\omega})$
  - Periodic signals
  - Sinusoidal signals
  - Frequency analysis for periodic signals
  - Concepts of frequency, bandwidth, filtering

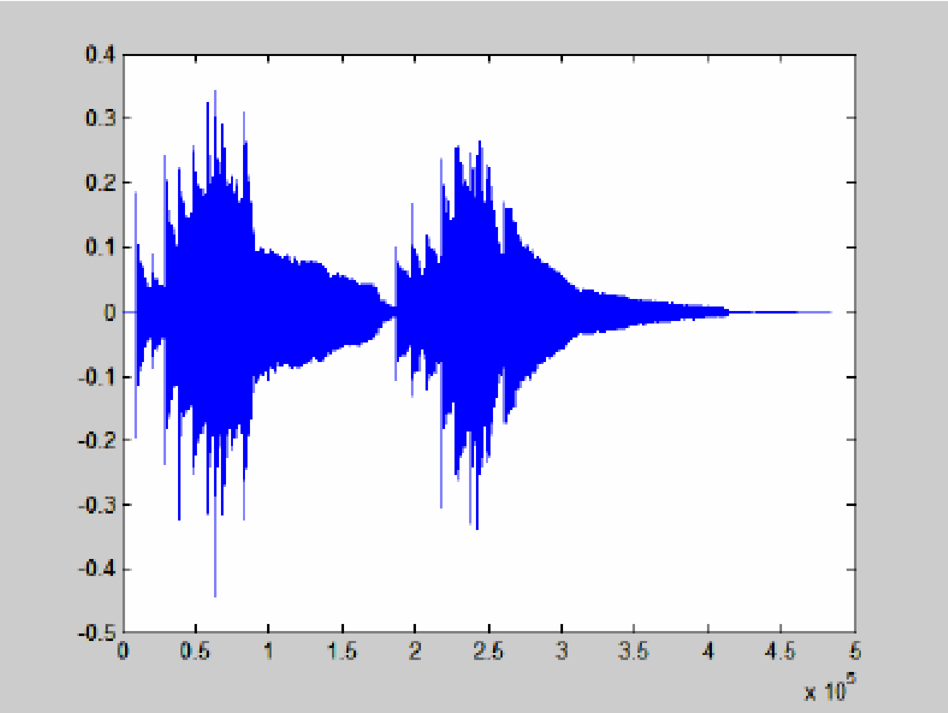
Frequency content in signals

- A **constant** : only zero frequency component (DC component)
- **Slowly** varying : contain low frequency only
- **Fast** varying : contain very high frequency
- **Sharp** transition : contain from low to high frequency

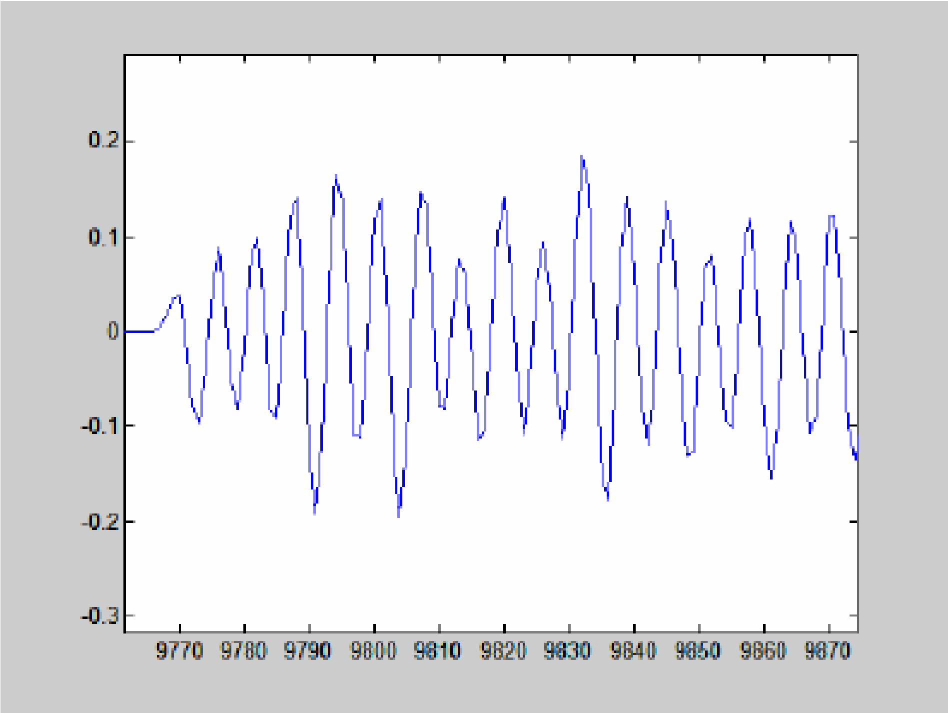
- A **sinusoid** : Contain only a single frequency component
- **Periodic** signals : Contain the fundamental frequency and harmonics : Line spectrum

- **Real-world** signal: contain different frequencies = Σ sinusoid
- Music: :
  - contain both slowly varying and fast varying components, wide bandwidth

Sample Music Waveform



Entire waveform



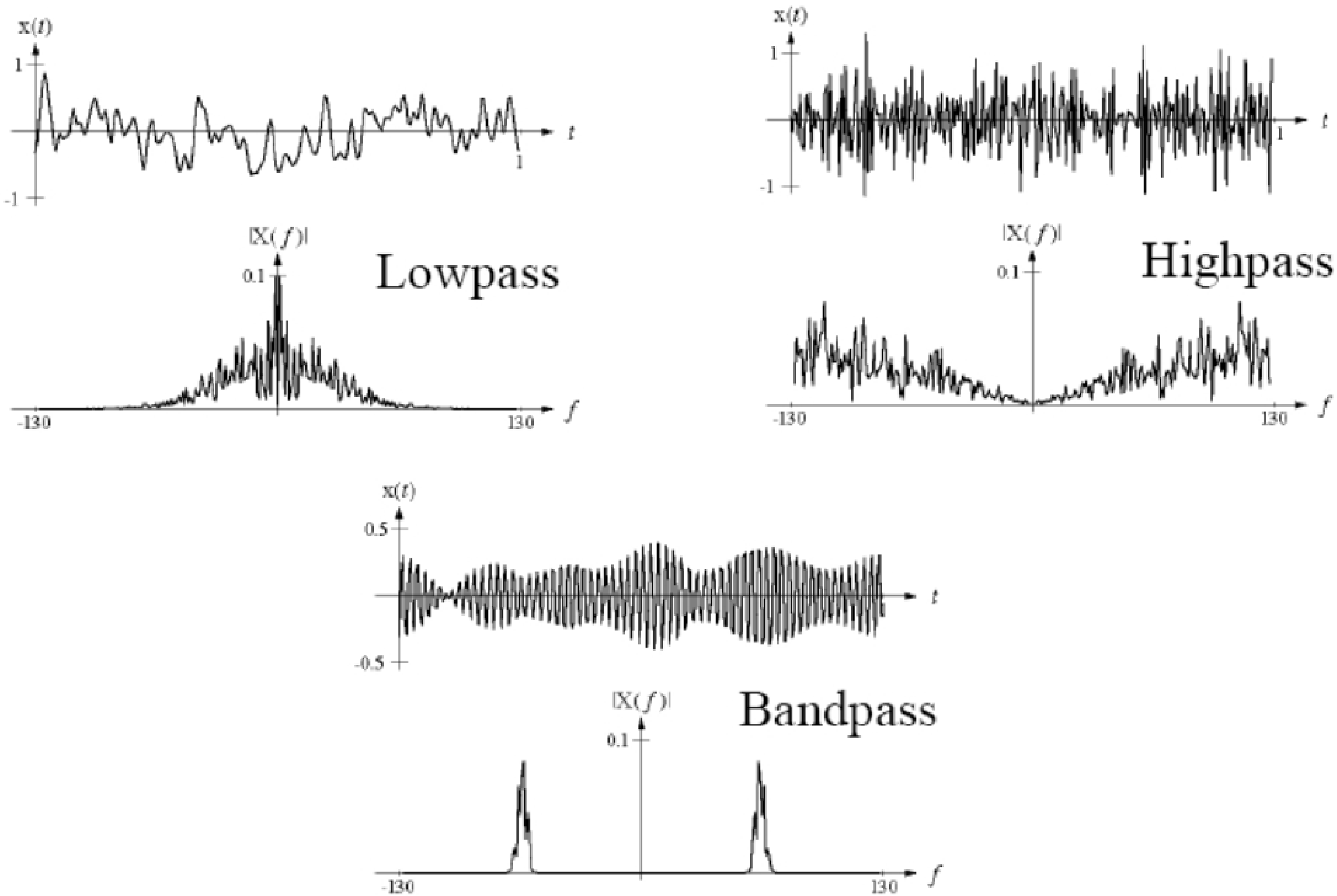
Blown-up of a section

» [y,fs]=wavread('sc01_L.wav');
» sound(y,fs);
» figure; plot(y);

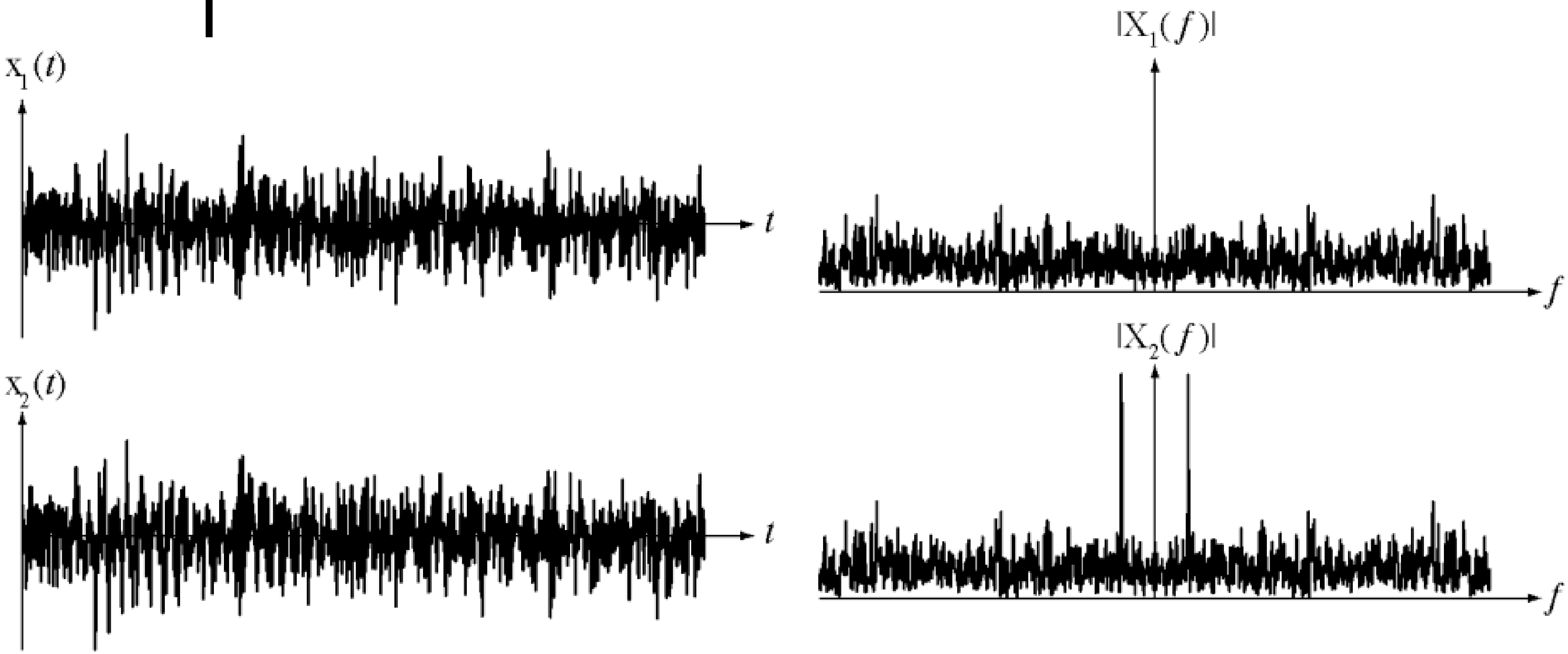
» v=axis;
» axis([1.1e4,1.2e4,-.2,.2])

Music typically has more periodic structure than speech
Structure depends on the note being played

Frequency content in signals



Spectrum analyzer



- x1(t) & x2(t) look similar
- A spectrum analyzer reveals the difference:
  - ➔ x2(t) contains a sinusoid that causes the two large “spikes”

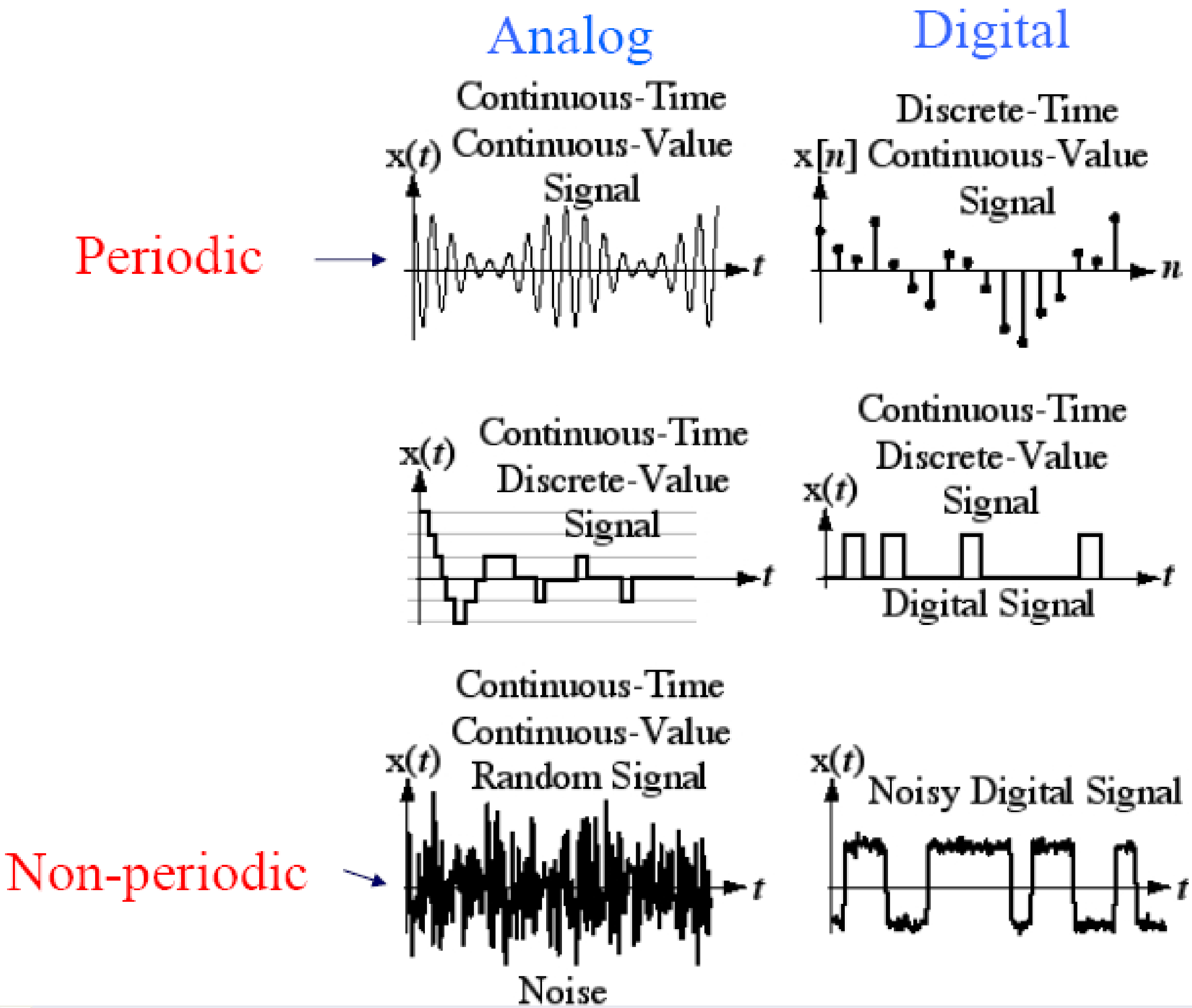
What is the frequency of an arbitrary signal?

- Sinusoidal signals have a distinct (unique) frequency
- An arbitrary signal x(t) does not have a unique frequency
  - x(t) can be decomposed into many sinusoidal signals with **different** frequencies, each with **different** magnitude and phase
- ***Spectrum*** of *x(t)*:
  - the plot of the magnitudes and phases of different frequency components of *x(t)*
- ***Fourier analysis:*** find spectrum for signals
- ***Bandwidth*** of x(t): the spread of the frequency components with significant energy existing in a signal
  - Difference between min and max frequency

Outline

- Introduction
- DT signals
- Transformation of the independent variable
- DT LTI Systems & Convolution
- Impulse response and diff. equations
- Frequency content of signals
- **Fourier Transform**
- Frequency response

Fourier Analysis: Types of signals



Overview of Fourier Analysis Methods

	Periodic in Time Discrete in Frequency	Aperiodic in Time Continuous in Frequency
Continuous in Time Aperiodic in Frequency	$\otimes$ CT Fourier Series : $\mathrm{CT\text{-}P_T} \Rightarrow \mathrm{DT}$ $a_k = \frac{1}{T}\int_0^T x(t)e^{-jk\omega_0 t}dt$ $\otimes$ CT Inverse Fourier Series : $\mathrm{DT} \Rightarrow \mathrm{CT\text{-}P_T}$ $x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$	$\otimes$ CT Fourier Transform : $\mathrm{CT} \Rightarrow \mathrm{CT}$ $X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$ $\otimes$ Inverse CT Fourier Transform : $\mathrm{CT} \Rightarrow \mathrm{CT}$ $x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega$
Discrete in Time Periodic in Frequency	$\otimes$ DT Fourier Series $\mathrm{DT\text{-}P_N} \Rightarrow \mathrm{DT\text{-}P_N}$ $X[k] = \sum_{n=0}^{N-1} x[n]e^{-j\omega_0 kn}$ $\otimes$ Inverse DT Fourier Series $\mathrm{DT\text{-}P_N} \Rightarrow \mathrm{DT\text{-}P_N}$ $x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]e^{j\omega_0 kn}$	$\otimes$ DT Fourier Transform : $\mathrm{DT} \Rightarrow \mathrm{CT} + \mathrm{P}_{2\pi}$ $X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$ $\otimes$ Inverse DT Fourier Transform : $\mathrm{CT} + \mathrm{P}_{2\pi} \Rightarrow \mathrm{DT}$ $x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})e^{j\omega n}d\omega$

Fourier Analysis Methods

Continuous-time analog signal
x(t) C

Sampling in **time**
Sampling period = T

Discrete-time sequence
x [n] D

Laplace Transform
X(s) C
$s = \sigma + j\Omega$
$$\int_{-\infty}^{\infty} x(t)\, e^{-st} dt$$

Continuous-time Fourier Transform
$X(j\Omega)$ C
$$\int_{-\infty}^{\infty} x(t)\, e^{-j\Omega t}\, dt$$

z-Transform
X(z) C
$z = re^{j\omega}$
$$\sum_{n=-\infty}^{\infty} x[n]\, z^{-n}$$

Discrete-Time Fourier Transform
$X(e^{j\omega})$ C
$$\sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n}$$
$-\pi \le \omega \le \pi$

Discrete Fourier Transform
X(k)
$$\sum_{n=0}^{N-1} x[n]\, e^{-j\frac{2\pi n k}{N}}$$
D
k integer

$s = j\Omega$
$\Omega = 2\pi F$

$z = e^{j\omega}$
$\omega = 2\pi f$

ω frequency of DT signal
Ω frequency of CT signal
$\Omega = \omega / T; \omega = \Omega T$
→Scaling of DT freq. by T

Sampling in **frequency**
N = Length of *x[n]*

Overview of Fourier symbols

	Variable	Period	Continuous Frequency	Discrete Frequency
DT *x[n]*	*n*	*N*	ω	*k* $\omega_k = 2\pi k / N$
CT *x(t)*	*t*	*T*	Ω	*k* $\omega_k = 2\pi k / T$

- **DT-FT: Discrete in time; Aperiodic in time; Continuous in Frequency; Periodic in Frequency**
- **CT-FT: Continuous in time; Aperiodic in time; Continues in Frequency; Aperiodic in Frequenc**
- **DT-FS: Discrete in time; Periodic in time; Discrete in Frequency; Periodic in Frequency**
- **CT-FS: Continuous in time; Periodic in time; Discrete in Frequency; Aperiodic in Frequency**

DT Fourier Transform

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}, \qquad x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}d\omega$$

➔DTFT represents a DT aperiodic signal

- as a sum of infinitely many complex exponentials
- with the frequency varying continuously in (-π, π)

- DTFT is periodic ➔ only need to determine it for (-π, π)
- Often FT is a complex function (magnitude+phase)

$$X(e^{j\omega}) = X_{\mathrm{Re}}(e^{j\omega}) + jX_{\mathrm{Im}}(e^{j\omega}) \qquad = |X(e^{j\omega})|e^{j\angle X(e^{j\omega})}$$

- A FT is unique, i.e., different signals have different FT

➔ FT is the frequency domain representation of the original function

➔ FT describes which frequencies are present in the original function

➔ The original signal can be recovered from its FT, and vice versa

Limitation of Fourier transform

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}, \qquad x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}d\omega$$

- Condition for the convergence of the infinite sum

$$|X(e^{j\omega})| = |\sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}| \le \sum_{n=-\infty}^{\infty} |x[n]||e^{-j\omega n}| \le \sum_{n=-\infty}^{\infty} |x[n]| < \infty$$

- If x[n] is **absolutely** summable, its FT (sufficient condition)

- Example: Exponential + unit step

$$x[n] = a^n u[n] \qquad |a| < 1: \; X(e^{j\omega}) = \frac{1}{1 - ae^{-j\omega}}$$

$$a = 1: \; X(e^{j\omega}) = \frac{1}{1 - e^{-j\omega}} + \sum_{k=-\infty}^{\infty} \pi\delta(\omega + 2\pi k)$$

$$|a| > 1: \quad \text{No}$$

Example: a constant sequence

- Constant sequence $x[n] = 1$

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}d\omega \quad \text{where} \quad X(e^{j\omega}) = \sum_{-\infty}^{\infty} x[n]e^{-j\omega n}$$

➔Its FT is defined as the periodic impulse train

$$X(e^{j\omega}) = \sum_{r=-\infty}^{\infty} 2\pi\delta(\omega + 2\pi r)$$

Example: impulse

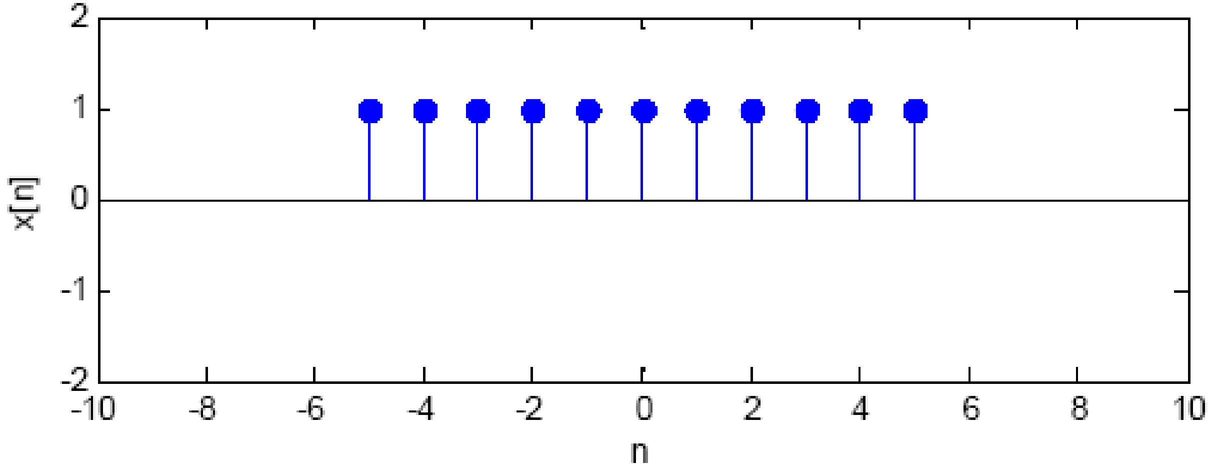
1. $x(n) = \delta(n)$

$$X(e^{j\omega}) = \sum_n \delta(n)\, e^{-j\omega n}$$

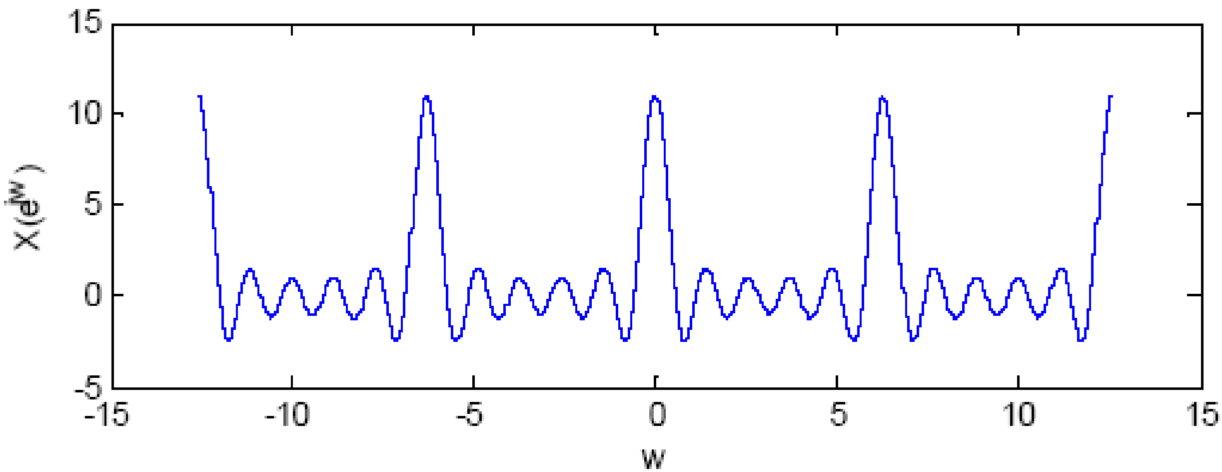
$$= 1 \quad \text{(by sifting property)}$$

Example: rectangle

$$x[n] = \begin{cases} 1, & |n| \le 5 \\ 0, & \text{e.w.} \end{cases} \longrightarrow \quad X(e^{jw}) = \sum_{n=-5}^{5} e^{-jwn} = \frac{\sin[w(11/2)]}{\sin[w/2]}$$



Aperiodic



Periodic

Example: exp / sinusoid

$$3.\quad e^{j\omega_0 n} \overset{\text{DTFT}}{\longleftrightarrow} \text{rep}_{2\pi}[2\pi\ \delta(\omega-\omega_0)]$$

(by modulation property)

$$4.\quad \cos(\omega_0 n) \overset{\text{DTFT}}{\longleftrightarrow} \text{rep}_{2\pi}[\pi\ \delta(\omega-\omega_0)+\pi\ \delta(\omega+\omega_0)]$$

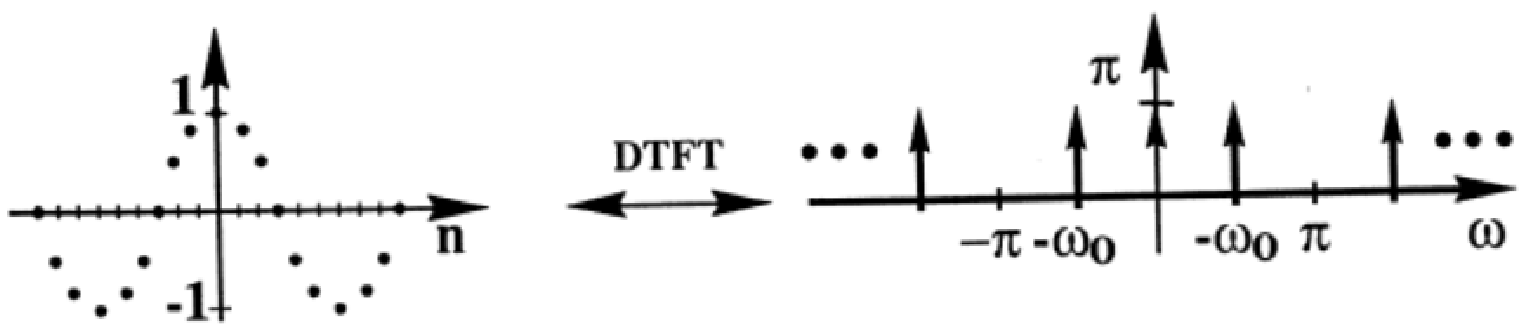


TABLE 2.2 FOURIER TRANSFORM THEOREMS

Sequence	Fourier Transform
$x[n]$	$X(e^{j\omega})$
$y[n]$	$Y(e^{j\omega})$
1. $ax[n] + by[n]$	$aX(e^{j\omega}) + bY(e^{j\omega})$
2. $x[n - n_d]$ (n_d an integer)	$e^{-j\omega n_d} X(e^{j\omega})$
3. $e^{j\omega_0 n} x[n]$	$X(e^{j(\omega-\omega_0)})$
4. $x[-n]$	$X(e^{-j\omega})$ $X^*(e^{j\omega})$ if $x[n]$ real.
5. $nx[n]$	$j\dfrac{dX(e^{j\omega})}{d\omega}$
6. $x[n] * y[n]$	$X(e^{j\omega})Y(e^{j\omega})$
7. $x[n]y[n]$	$\dfrac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\theta})Y(e^{j(\omega-\theta)})d\theta$

Parseval's theorem:

8. $$\sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |X(e^{j\omega})|^2 d\omega$$

9. $$\sum_{n=-\infty}^{\infty} x[n]y^*[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})Y^*(e^{j\omega})d\omega$$

TABLE 2.3 FOURIER TRANSFORM PAIRS

Sequence	Fourier Transform
1. $\delta[n]$	1
2. $\delta[n-n_0]$	$e^{-j\omega n_0}$
3. $1 \quad (-\infty < n < \infty)$	$\sum_{k=-\infty}^{\infty} 2\pi\delta(\omega + 2\pi k)$
4. $a^n u[n] \quad (\lvert a\rvert < 1)$	$\frac{1}{1 - ae^{-j\omega}}$
5. $u[n]$	$\frac{1}{1 - e^{-j\omega}} + \sum_{k=-\infty}^{\infty} \pi\delta(\omega + 2\pi k)$
6. $(n+1)a^n u[n] \quad (\lvert a\rvert < 1)$	$\frac{1}{(1 - ae^{-j\omega})^2}$
7. $\frac{r^n \sin\omega_p(n+1)}{\sin\omega_p} u[n] \quad (\lvert r\rvert < 1)$	$\frac{1}{1 - 2r\cos\omega_p e^{-j\omega} + r^2 e^{-j2\omega}}$
8. $\frac{\sin\omega_c n}{\pi n}$	$X(e^{j\omega}) = \begin{cases} 1, & \lvert\omega\rvert < \omega_c, \\ 0, & \omega_c < \lvert\omega\rvert \le \pi \end{cases}$
9. $x[n] = \begin{cases} 1, & 0 \le n \le M \\ 0, & \text{otherwise} \end{cases}$	$\frac{\sin[\omega(M+1)/2]}{\sin(\omega/2)} e^{-j\omega M/2}$
10. $e^{j\omega_0 n}$	$\sum_{k=-\infty}^{\infty} 2\pi\delta(\omega - \omega_0 + 2\pi k)$
11. $\cos(\omega_0 n + \phi)$	$\sum_{k=-\infty}^{\infty} [\pi e^{j\phi}\delta(\omega - \omega_0 + 2\pi k) + \pi e^{-j\phi}\delta(\omega + \omega_0 + 2\pi k)]$

Examples: Determining a FT using FT theorems

$$x[n] = a^n u[n-5]$$

$$x_1[n] = a^n u[n] \stackrel{F}{\longleftrightarrow} X_1(e^{j\omega}) = \frac{1}{1-ae^{-j\omega}}$$

$$x_2[n] = x_1[n-5] \qquad (\text{i.e.} = a^{n-5}u[n-5])$$

$$X_2(e^{j\omega}) = e^{-j5\omega} X_1(e^{j\omega}) = \frac{e^{-j5\omega}}{1-ae^{-j\omega}}$$

$$x[n] = a^5 x_2[n] \qquad (\text{i.e.} = a^n u[n-5])$$

$$X(e^{j\omega}) = \frac{a^5 e^{-j5\omega}}{1-ae^{-j\omega}}$$

Example: Convolution

- Let $h[n]=\left(\frac{1}{2}\right)^n u[n]$ and $x[n]=\left(\frac{1}{7}\right)^n u[n]$
- Find the output y[n]

- **Solution 1:** solve using convolution of h[n] and x[n] !!!
- **Solution 2:** Use properties of FT:
  - Convolution in time → multiplication in frequency

$$y[n]=h[n]*x[n]\Rightarrow Y(e^{j\omega})=H(e^{j\omega})X(e^{j\omega})$$

  - We know:

$$h[n]=a^n u[n]\Rightarrow H(e^{j\omega})=\frac{1}{1-ae^{-j\omega}}, \qquad a<1$$

$$\Rightarrow H(e^{j\omega})=\frac{1}{1-\frac{1}{2}e^{-j\omega}} \quad \text{and } X(e^{j\omega})=\frac{1}{1-\frac{1}{7}e^{-j\omega}}$$

Example: Convolution

- Thus $$Y(e^{j\omega}) = H(e^{j\omega})X(e^{j\omega}) = \left(\frac{1}{1-\frac{1}{7}e^{-j\omega}}\right)\left(\frac{1}{1-\frac{1}{2}e^{-j\omega}}\right)$$
- What is *y[n]*? It is the inverse FT

➔ Use <u>partial fraction expansion</u> method $$Y(e^{j\omega}) = \frac{-2/5}{1-\frac{1}{7}e^{-j\omega}} + \frac{7/5}{1-\frac{1}{2}e^{-j\omega}}$$

- We know

$$x[n] = a^n u[n] \Leftrightarrow X(e^{j\omega}) = \frac{1}{1-ae^{-j\omega}} \quad ; \quad a<1$$

- Recall: a FT is unique (different signals in time give different FT)

$$\frac{-2/5}{1-\frac{1}{7}e^{-j\omega}} \Leftrightarrow -\frac{2}{5}\left(\frac{1}{7}\right)^n u[n] \text{ and } \frac{7/5}{1-\frac{1}{2}e^{-j\omega}} \Leftrightarrow \frac{7}{5}\left(\frac{1}{2}\right)^n u[n]$$

- The inverse FT $$y[n] = -\frac{2}{5}\left(\frac{1}{7}\right)^n u[n] + \frac{7}{5}\left(\frac{1}{2}\right)^n u[n]$$

Symmetry properties of the FT

x[n] is even (or symmetric) when x[n]= x[-n]
x[n] is odd (or antisymmetric) when x[n]=-x[-n]

x[n]=$x_e[n]$+ $x_o[n]$

conjugate-symmetric sequence: $x_e[n] = \frac{1}{2}(x[n] + x^*[-n]) = x^*{}_e[-n]$

conjugate-antisymmetric sequence: $x_o[n] = \frac{1}{2}(x[n] - x^*[-n]) = -x^*{}_o[-n]$

even sequence is conjugate-symmetric : $x_e[n] = x_e[-n]$

odd sequence is conjugate-antisymmetric : $x_o[n] = -x_o[-n]$

According to Table 2.1(property 5&6)

$$x_e[n] \quad \overset{\mathcal{F}}{\rightarrow} \quad X_R(e^{jw}) = \mathcal{R}e\{X(e^{jw})\}$$

$$x_o[n] \quad \overset{\mathcal{F}}{\rightarrow} \quad jX_I(e^{jw}) = jIm\{X(e^{jw})\}$$

$$X(e^{j\omega}) = X_{\mathrm{Re}}(e^{j\omega}) + jX_{\mathrm{Im}}(e^{j\omega})$$
$$= |X(e^{j\omega})|\, e^{j\angle X(e^{j\omega})}$$

Duality property of Fourier transform

$X(e^{jw}) = X_e(e^{jw}) + X_o(e^{jw})$

conjugate-symmetric FT: $X_e(e^{jw}) = \frac{1}{2}(X(e^{jw}) + X^*(e^{-jw}))$

conjugate-antisymmetric FT: $X_o(e^{jw}) = \frac{1}{2}(X(e^{jw}) - X^*(e^{-jw}))$

conjugate-symmetric Function : $X_e(e^{jw}) = X^*{}_e(e^{-jw})$

conjugate-antisymmetric Function : $X_o(e^{jw}) = -X^*{}_o(e^{-jw})$

According to Table 2.1(property 3&4)

$\mathcal{R}e\{x[n]\} \quad \xrightarrow{\mathcal{F}} \quad X_e(e^{jw})$

$jIm\{x[n]\} \quad \xrightarrow{\mathcal{F}} \quad X_o(e^{jw})$

TABLE 2.1 SYMMETRY PROPERTIES OF THE FOURIER TRANSFORM

Sequence $x[n]$	Fourier Transform $X(e^{j\omega})$
1. $x^*[n]$	$X^*(e^{-j\omega})$
2. $x^*[-n]$	$X^*(e^{j\omega})$
3. $\mathcal{R}e\{x[n]\}$	$X_e(e^{j\omega})$ (conjugate-symmetric part of $X(e^{j\omega})$)
4. $j\mathcal{I}m\{x[n]\}$	$X_o(e^{j\omega})$ (conjugate-antisymmetric part of $X(e^{j\omega})$)
5. $x_e[n]$ (conjugate-symmetric part of $x[n]$)	$X_R(e^{j\omega}) = \mathcal{R}e\{X(e^{j\omega})\}$
6. $x_o[n]$ (conjugate-antisymmetric part of $x[n]$)	$jX_I(e^{j\omega}) = j\mathcal{I}m\{X(e^{j\omega})\}$
The following properties apply only when $x[n]$ is real:	
7. Any real $x[n]$	$X(e^{j\omega}) = X^*(e^{-j\omega})$ (Fourier transform is conjugate symmetric)
8. Any real $x[n]$	$X_R(e^{j\omega}) = X_R(e^{-j\omega})$ (real part is even)
9. Any real $x[n]$	$X_I(e^{j\omega}) = -X_I(e^{-j\omega})$ (imaginary part is odd)
10. Any real $x[n]$	$\|X(e^{j\omega})\| = \|X(e^{-j\omega})\|$ (magnitude is even)
11. Any real $x[n]$	$\angle X(e^{j\omega}) = -\angle X(e^{-j\omega})$ (phase is odd)
12. $x_e[n]$ (even part of $x[n]$)	$X_R(e^{j\omega})$
13. $x_o[n]$ (odd part of $x[n]$)	$jX_I(e^{j\omega})$

Example: Problem 2.55/c

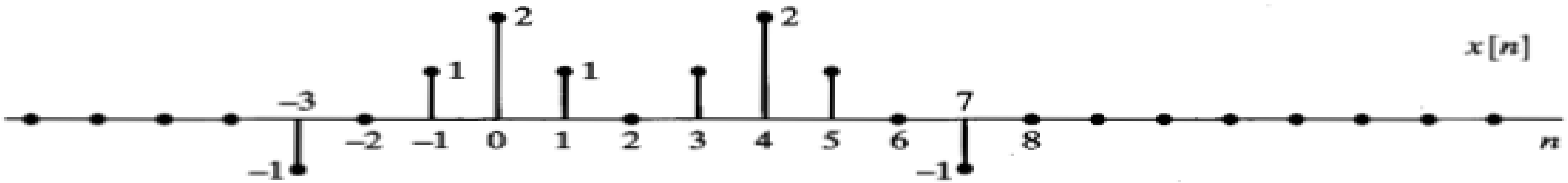


Figure P2.44-1

x[n] is a real even function shifted by 2 to right.

$x_e[n]$ is equal to x[n] before shifting → $x_e[n] = x_e^*[-n] \xrightarrow{\mathcal{F}\ by\ property5/Table.2.1} \mathcal{Re}\{X(e^{jw})\}$

Thus, It has a real value in FT or $\sphericalangle\, \mathcal{F}\,(x_e[n]) = 0$

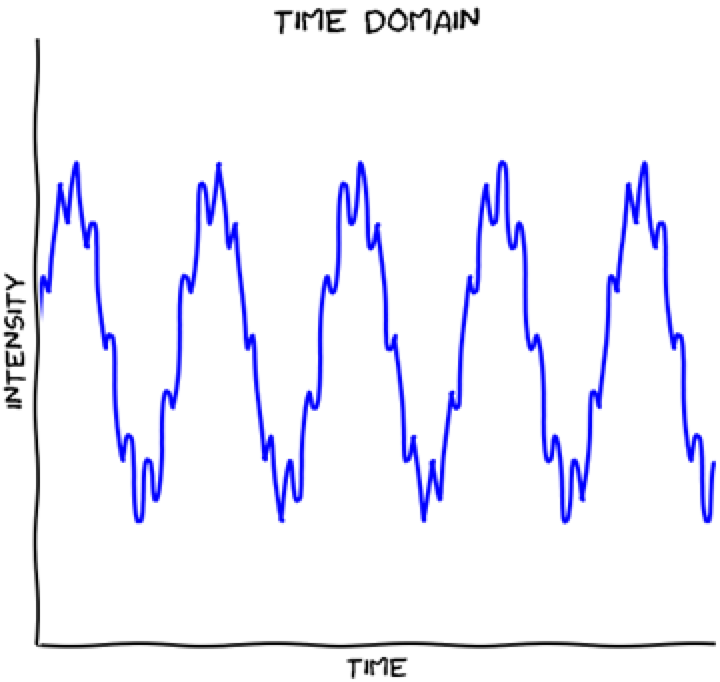
According to shifting property by 2 to right:

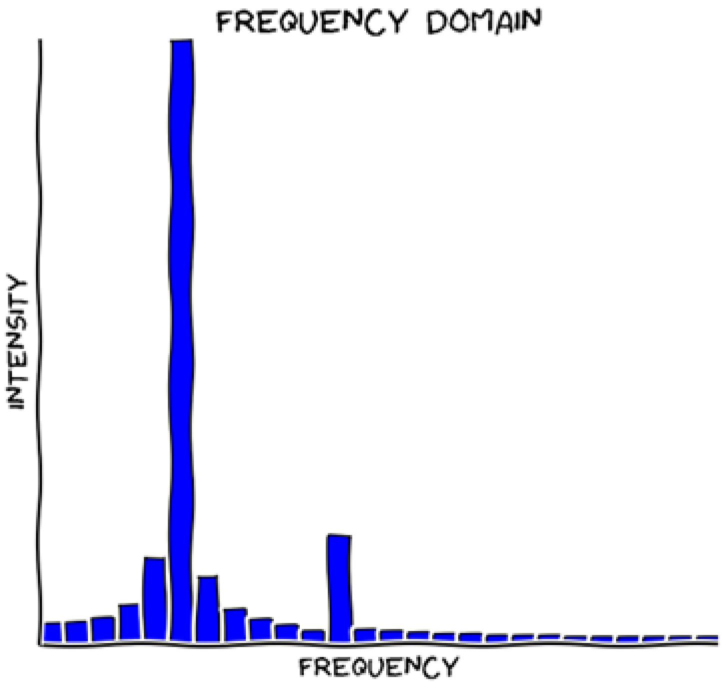
$$X(e^{jw}) = \mathcal{F}\,(x_e[n])\, e^{-j2w}$$

$\mathcal{F}\,(x_e[n])$ is a zero phase real function of w.

Application of FT: $Y(e^{j\omega}) = DSP\{\ X(e^{j\omega})\ \}$

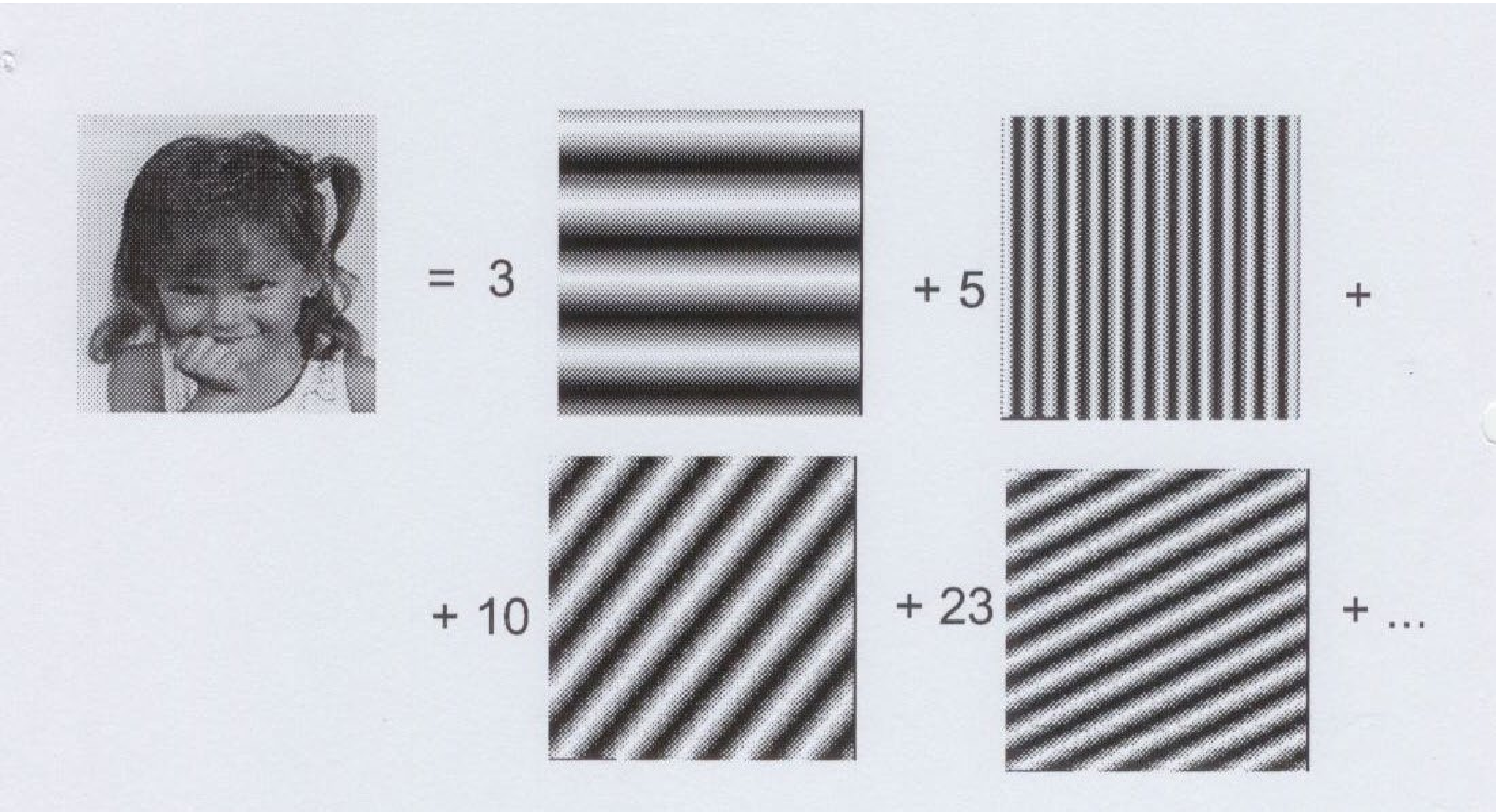
$$x[n] \Leftrightarrow X(e^{j\omega}) \quad \forall\, \omega$$





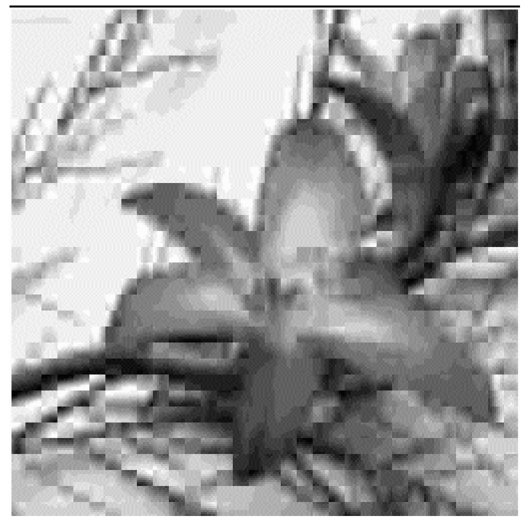
$$Y(e^{j\omega}) = DSP\{\ X(e^{j\omega})\ \} \Leftrightarrow y[n] \neq x[n]$$

Application of FT: Filtering (remove noise)



Application of FT: Signal Compression

Original



With 4/64 Coefficients ω

With 8/64 Coefficients ω

With 16/64 Coefficients ω

Outline

- Introduction
- DT signals
- Transformation of the independent variable
- DT LTI Systems & Convolution
- Frequency content of signals
- Fourier Transform
- **Frequency response & filters**
- Difference equation & systems

System response

- A DT signal is a sum of scaled, delayed impulses $x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$
- System response to an impulse ➔ impulse response of LTI systems

$$x[n]=\delta[n]\rightarrow y[n]=h[n]$$

- System response to any *x[n]*

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] = h[n] * x[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k]$$

Convolution sum !

- System response to exp $x[n]=e^{j\omega n}$

$$\boxed{y[n] = T\{e^{j\omega n}\} = h[n] * e^{j\omega n} = \sum_{k=-\infty}^{\infty} h[k]e^{j\omega(n-k)} \left(\sum_{k=-\infty}^{\infty} h[k]e^{-j\omega k} \right) e^{j\omega n} = \underline{H(e^{j\omega n})e^{j\omega n}}}$$

$$H(e^{j\omega}) = \sum_{k=-\infty}^{\infty} h[k]e^{-j\omega k}$$

LTI System input and output

Input | Output

$$x[n]=\delta[n] \quad \Rightarrow \quad y[n]=h[n] \Leftrightarrow H(e^{j\omega})=\sum_{k=-\infty}^{\infty} h[k]e^{-j\omega k}$$

$$x[n] \quad \Rightarrow \quad y[n]=h[n]*x[n]$$

$$x[n]=\sum_{k=-\infty}^{\infty} x[k]\delta[n-k] \quad \Rightarrow \quad y[n]=\sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

$$x[n]=e^{j\omega n} \quad \Rightarrow \quad y[n]=H(e^{j\omega})\cdot e^{j\omega n}$$

$$x[n] \Leftrightarrow X(e^{j\omega}) \quad \Rightarrow \quad \mathrm{Y}(e^{j\omega})=H(e^{j\omega})X(e^{j\omega})$$

$$x[n]=\frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}d\omega \quad \Rightarrow \quad y[n]=\frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})\cdot H(e^{j\omega})\cdot e^{j\omega n}d\omega$$

Frequency response

- The frequency response is always a periodic function of the frequency variable ω with period 2π

$$H(e^{j(\omega+2\pi)}) = \sum_{n=-\infty}^{\infty} h[n]e^{-j(\omega+2\pi)n} = \sum_{n=-\infty}^{\infty} h[n]e^{-j\omega n}e^{-j2\pi n} = H(e^{j\omega})$$

  - Only specify over the interval $-\pi < \omega \leq \pi$
  - The 'low frequencies' are close to 0
  - The 'high frequencies' are close to $\pm\pi$

- The frequency response is generally complex

$$H(e^{j\omega}) = H_R(e^{j\omega}) + jH_I(e^{j\omega})$$

$$= |H(e^{j\omega})|\, e^{j\angle H(e^{j\omega})}$$

← describes changes in magnitude and phase

Example: The frequency response of the ideal delay system

$$y[n]=x[n-n_d] \quad \rightarrow \quad h[n]=\delta[n-n_d]$$

$$h[n] \Leftrightarrow H(e^{j\omega})=\sum_{n=-\infty}^{\infty}\delta[n-n_d]e^{-j\omega n}=e^{-j\omega n_d}$$

$$H_R(e^{j\omega})=\cos(\omega n_d), \qquad H_I(e^{j\omega})=-\sin(\omega n_d)$$

$$|H(e^{j\omega})|=1, \qquad \angle H(e^{j\omega})=-\omega n_d$$

Filters & frequency response

A filter remove unwanted signal components and/or enhance wanted ones

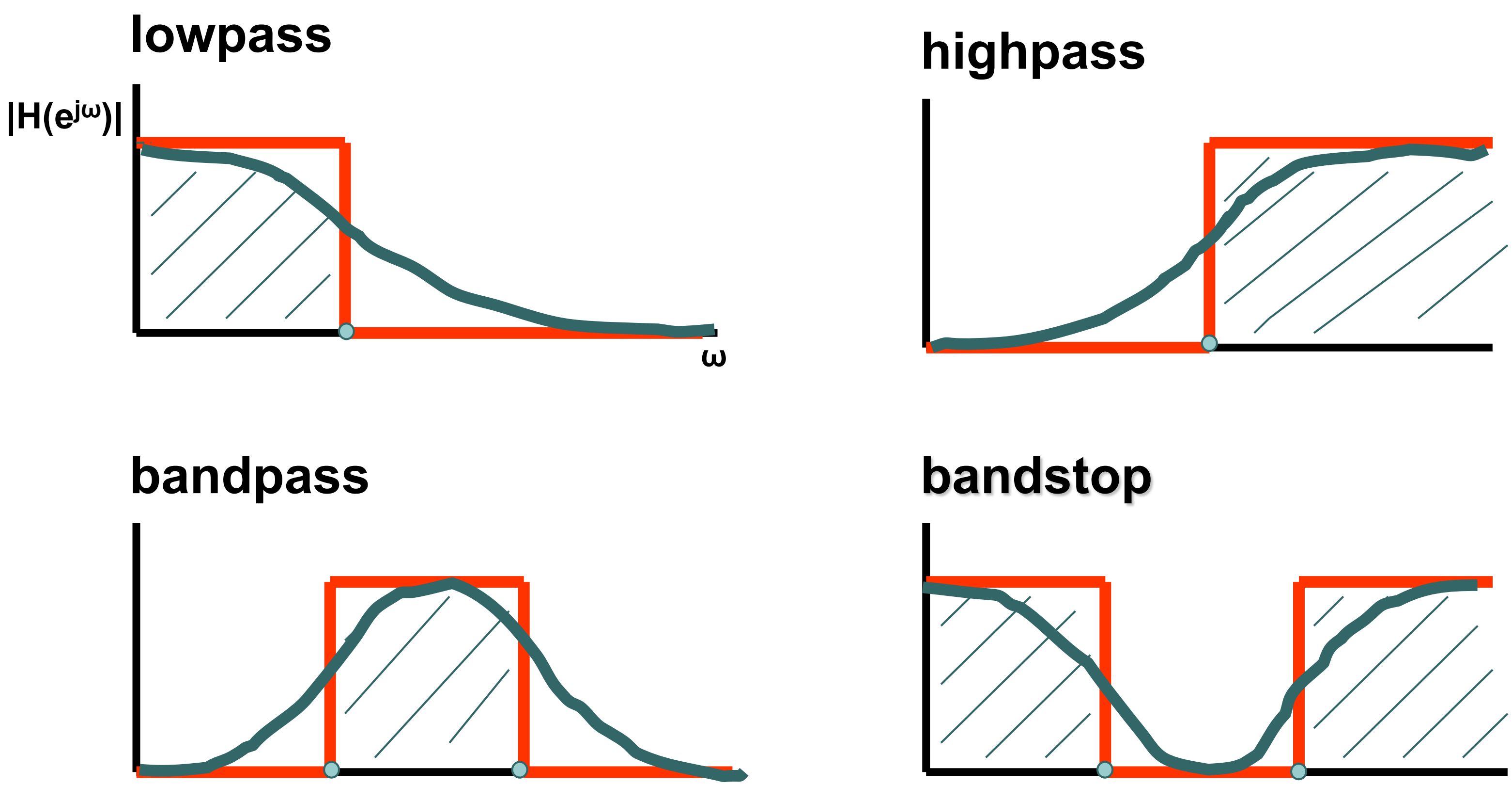
Four Main Filter Types:

- Low-pass: most common
  - Passes low frequencies, attenuates highs
- High-pass:
  - Passes high frequencies, attenuates lows
  - Used to brighten a signal
  - Careful: can also increase noise
- Band-pass:
  - Passes band of frequencies, attenuates those above and below band
  - Most common in implementations of DFF to separate out harmonics
- Band-stop (band-reject):
  - Stops band of frequencies, passes those above and below band

Filter specification through frequency response (magnitude and phase)

Types of Filters

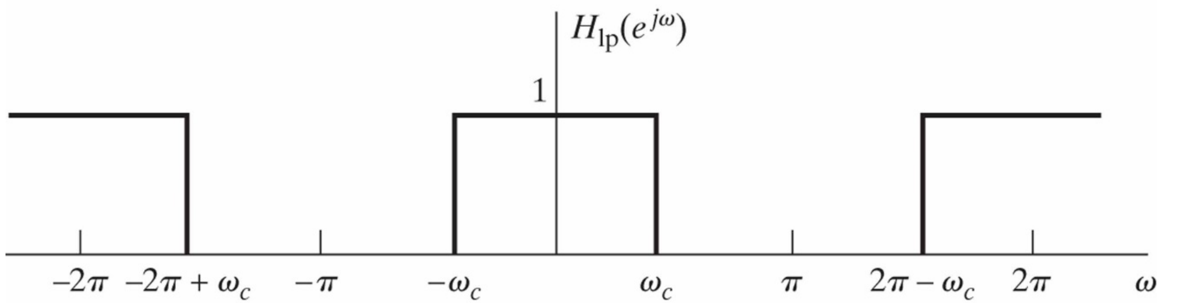
Ideal filters **Real filters**



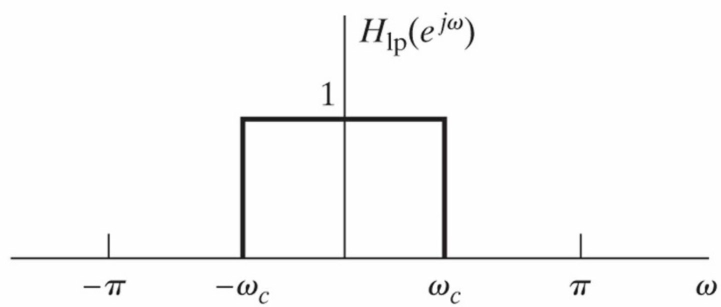
Ideal frequency-selective filter has **unity** frequency response over a certain range of frequencies, and is **zero** at the remaining frequencies

Example: Lowpass filter

- Low-pass filter: passes only low frequencies and rejects high frequencies of an input signal x[n]



(a) periodicity of the frequency response

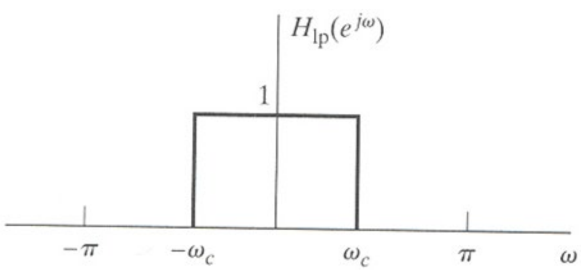


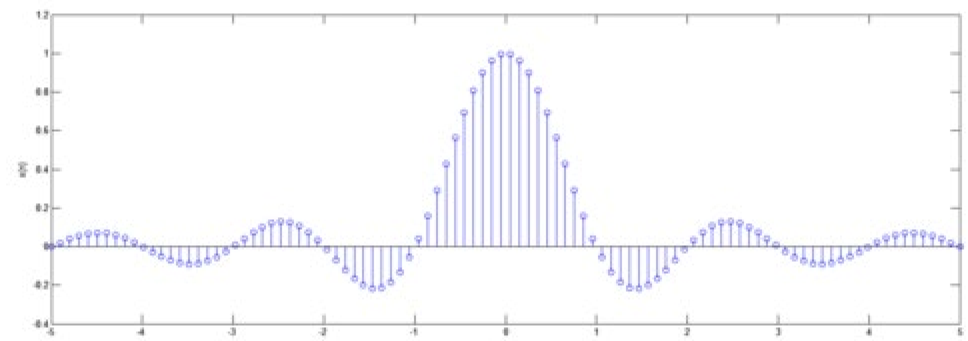
(b) One period of the periodic frequency response

Example : ideal lowpass filter

- Frequency response

$$H_{lp}(e^{j\omega}) = \begin{cases} 1, & |\,\omega\,| < \omega_c \\ 0, & \omega_c < |\,\omega\,| \leq \pi \end{cases} \qquad \leftrightarrow h_{lp}[n] = \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} e^{j\omega n}\, d\omega = \frac{\sin \omega_c n}{\pi n}, \qquad -\infty < n < \infty$$





➔ *Not stable?* *h[n]* is not absolutely summable $\sum_{k=-\infty}^{\infty} |\, h[k]\,| < \infty$

➔ *Non-caussal ?* $h[n] = 0, \qquad n < 0$

Example : ideal lowpass filter

$$H_{lp}(e^{j\omega}) = \begin{cases} 1, & |\omega| < \omega_c \\ 0, & \omega_c < |\omega| \le \pi \end{cases} \quad \leftrightarrow h_{lp}[n] = \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} e^{j\omega n} d\omega = \frac{\sin \omega_c n}{\pi n}, \quad -\infty < n < \infty$$

➔We cannot implement the ideal lowpass filter in practice because h[n] is *infinitely long* in time

➔it is *noncausal* (we cannot shift it to make causal h[n] extends all the way to time)

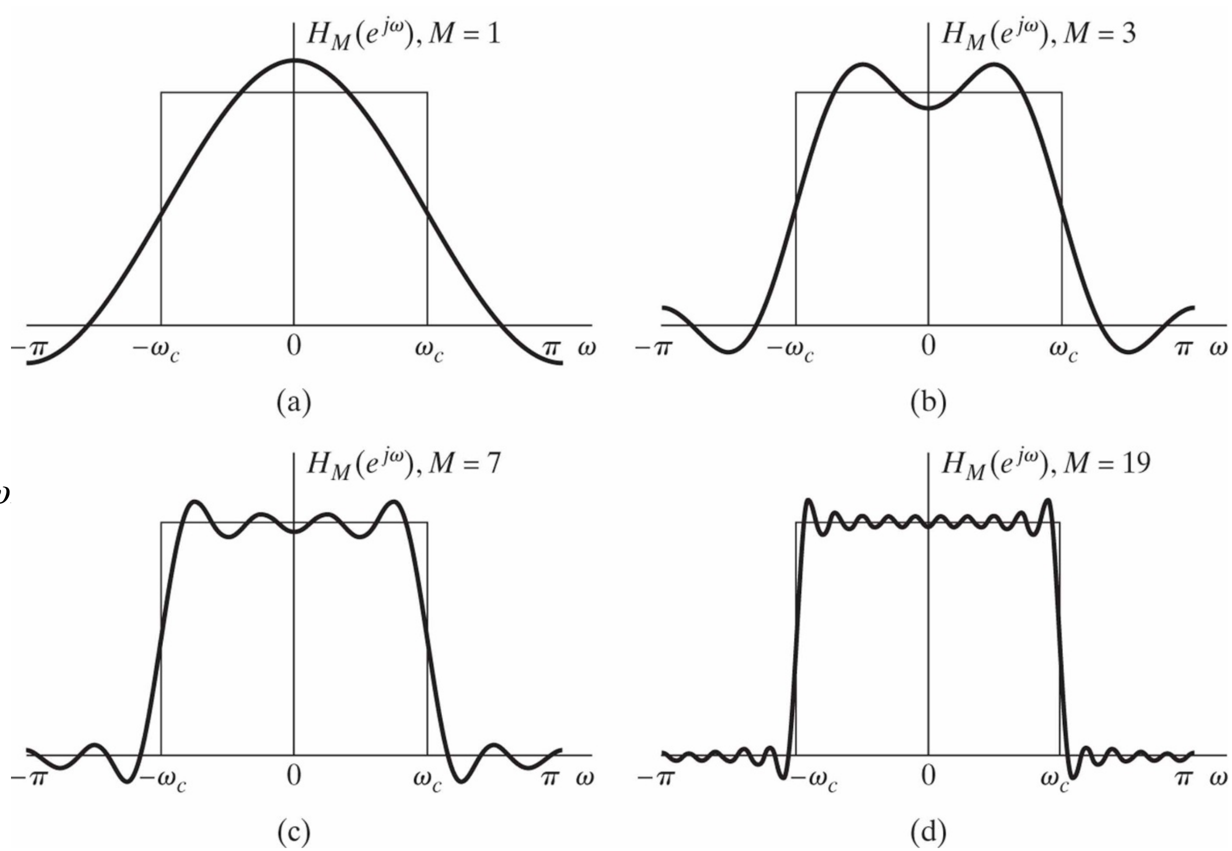
➔We will have to accept some sort of compromise in the design of any practical lowpass filter

Example : real lowpass filter

- *Filter design?*

$$H_{lp}(e^{j\omega}) = \sum_{n=-\infty}^{\infty} \frac{\sin \omega_c n}{\pi n} e^{-j\omega n}$$

$$H_M(e^{j\omega}) = \sum_{n=-M}^{M} \frac{\sin \omega_c n}{\pi n} e^{-j\omega n}$$

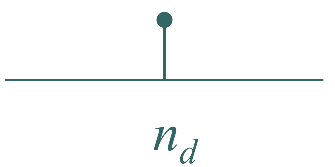


Convergence of the $H_M(e^{j\omega})$
(The oscillatory behavior at $\omega = \omega_c$ is called the Gibbs phenomenon)

FIR Systems via $h[n]$

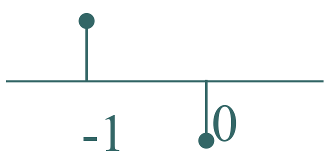
- Finite-duration impulse response (FIR) filters
  - The impulse response has only a finite number of nonzero samples
- Examples:
  - Ideal delay

$$y[n] = x[n-n_d], \quad -\infty < n < \infty$$
$$h[n] = \delta[n-n_d], \quad n_d \text{ a positive integer}$$



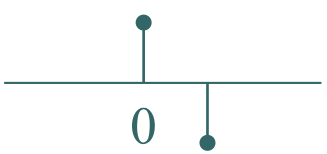
  - Forward difference

$$y[n] = x[n+1] - x[n]$$
$$h[n] = \delta[n+1] - \delta[n]$$



  - Backward difference

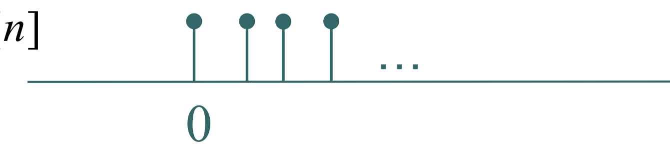
$$y[n] = x[n] - x[n-1]$$
$$h[n] = \delta[n] - \delta[n-1]$$



IIR systems via $h[n]$

- Infinite-duration impulse response (IIR) system
  - The impulse response is infinitive in duration
- Example: Accumulator $y[n] = \sum_{k=-\infty}^{n} x[k]$

$$h[n] = \sum_{k=-\infty}^{n} \delta[k] = u[n]$$



- Stability? $\sum_{n=-\infty}^{\infty} |h[n]| \overset{?}{<} \infty$
  - FIR systems are always stable, if each of h[n] values is finite in magnitude
  - IIR systems can be stable, e.g. $h[n] = a^n u[n]$ with $|a| < 1$ $\Rightarrow \sum_0^{\infty} |a|^n = 1/(1-|a|) < \infty$

Outline

- Introduction to DSP
- DT signals
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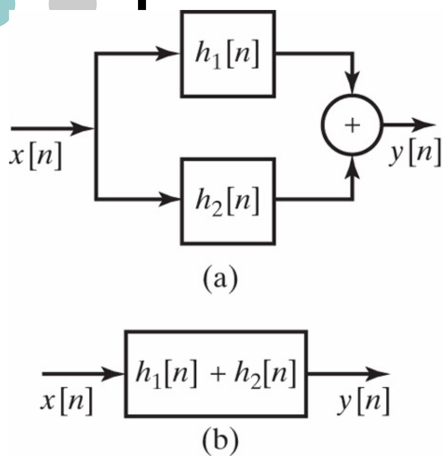
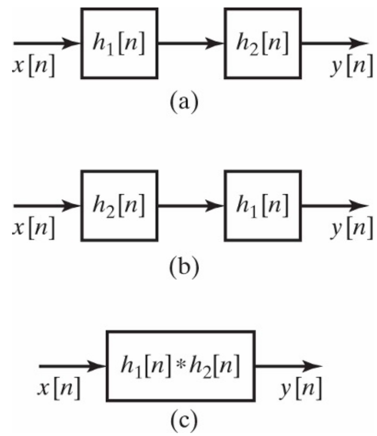
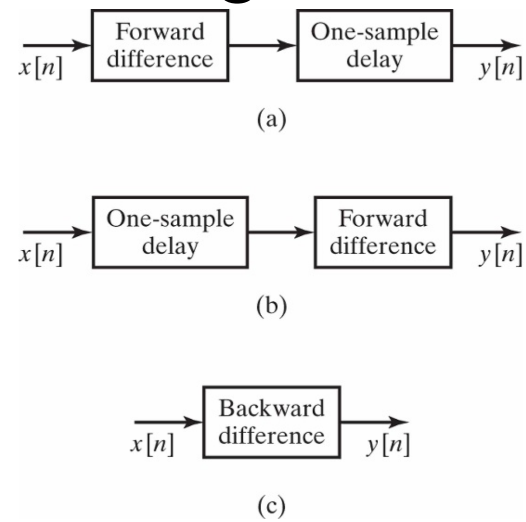
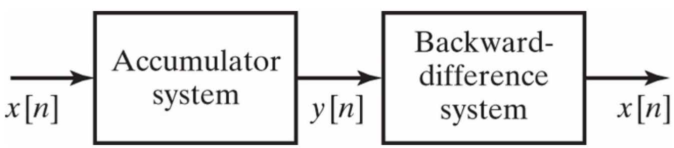
- **Difference equations & systems**

Combining systems; block diagrams

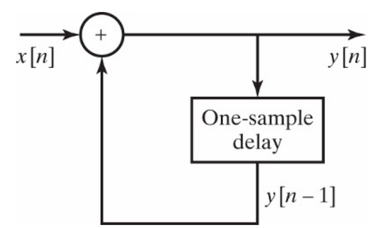






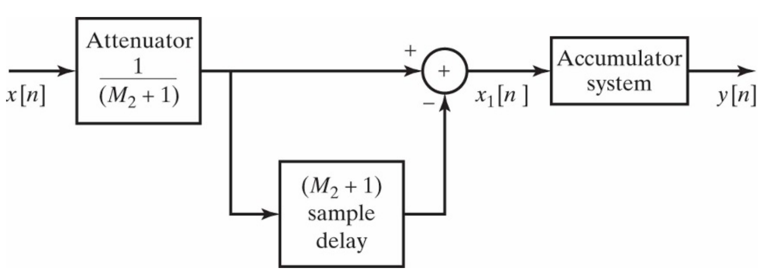


Identity system: accumulator cascaded with backward difference (backward difference is the inverse system for the accumulator)



A recursive system representing an accumulator

$$y[n] = \sum_{k=-\infty}^{n} x[k] = x[n] + y[n-1]$$



A recursive form of a moving-average system

$$y[n] = \frac{1}{M+1} \sum_{k=0}^{M} x[n-k]$$

LTI systems characterized by Difference Equations

- remember linear differential equations? $\frac{d}{dt}y(t) - y(t) = x(t)$

- A **difference equation** is the discrete-time analogue of a *differentail equation*. We simply use differences ($x[n] - x[n-1]$) rather than derivatives ($\frac{d}{dt}x(t)$)

- An important *subclass* of linear systems consists of those whose input $x[n]$ and output $x[n]$ obey an N-th order Linear Constact Coefficient Difference Equation

$$\sum_{k=0}^{N} (a_k\, y[n-k]) = \sum_{k=0}^{M} (b_k\, x[n-k])$$

Example : Moving average system $y[n] = \frac{1}{M_1 + M_2 + 1} \sum_{k=M_1}^{M_2} (x[n-k])$

Example : Recursive System $y[n] = \sum_{k=1}^{N} (\alpha_k\, y[n-k]) + x[n]$

- y [n − k] represents delayed outputs & x [n − k] represents delayed inputs
- N represents the order (delay/memory) of the difference equation
- Because this equation relies on past values of the output, to compute a numerical solution, certain past outputs (called “initial conditions”) must be known

LTI systems characterized by Difference Equations

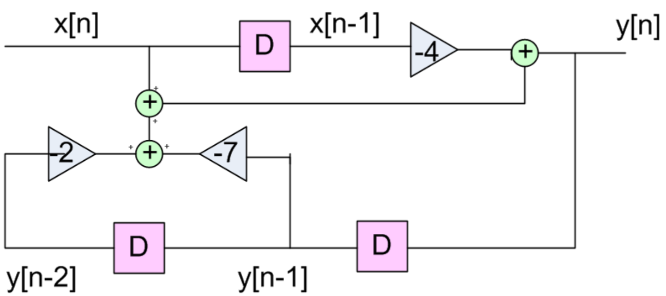
- A linear constant-coefficient difference equation (LCCDE) shows the relationship between consecutive values of a sequence and the difference among them

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{m=0}^{M} b_m x[n-m]$$

- LCCDE is often a recursive formula: a system output can be computed from current and past values

$$y[n] + 7y[n-1] + 2y[n-2] = x[n] - 4x[n-1]$$

$$\Rightarrow y[n] = x[n] - 4x[n-1] - 7y[n-1] - 2y[n-2]$$

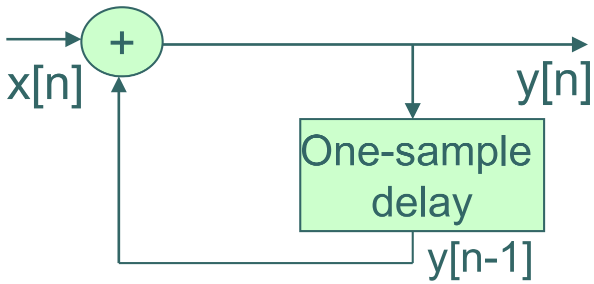


- Accumulator

$$y[n] = \sum_{k=-\infty}^{n} x[k]; y[n-1] = \sum_{k=-\infty}^{n-1} x[k]$$

$$y[n] = x[n] + \sum_{k=-\infty}^{n-1} x[k] = x[n] + y[n-1]$$

$$y[n] - y[n-1] = x[n]$$



LTI systems characterized by Difference Equations or Frequency response

- Consider an LTI system with the LCCDE

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

- Re-write to easily express a recursive output:

$$y[n] = -\left(\sum_{k=1}^{N} (a_k y[n-k])\right) + \sum_{k=0}^{M} (b_k x[n-k])$$

- In FT domain:

$$Y(e^{j\omega})\sum_{k=-0}^{N} a_k e^{j\omega k} = X(e^{j\omega})\sum_{k=-0}^{M} b_k e^{j\omega k} \Rightarrow H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{\sum_{k=-0}^{M} b_k e^{j\omega k}}{\sum_{k=-0}^{N} a_k e^{j\omega k}}$$

- Given LCCD; we can find the frequency response

- Given the frequency response; we can find LCCD