Discrete-Time Filter Design

1. Filter Design Process & Specifications
2. FIR vs IIR
3. IIR Filter Design:
   - Analog Filters: Butterworth; Chebyshev I & II; Ellipical
   - IIR Filter Design by Impulse Invariance
   - IIR Filter Design by Bilinear Transformation
4. FIR Filter Design by Windowing
5. Kaiser Window FIR Design Method (not in the exam)
6. Optimum Approximation of FIR Filters (not in the exam)

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Figures, examples and some text in these course slides are taken from the following sources:

. Oppenheim, Schafer, **DSP, 3ed.**

•**Dr.** Zheng-Hua Tan**,** Digital Signal Processing **III**, 2009, http://kom.aau.dk/~zt/cources/DSP/

Filter design process

- Filter, in broader sense, covers any system
- Order of a filter M or N
- Filters: FIR or IIR

$$H(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 - \sum_{k=1}^{N} a_k z^{-k}}$$

- Three design steps

Performance constraints

System function

Solution

Problem → **Specifications** → **Approximations** → **Realization** →

Magnitude &
Phase responses
Complexity

IIR or FIR
Subtype

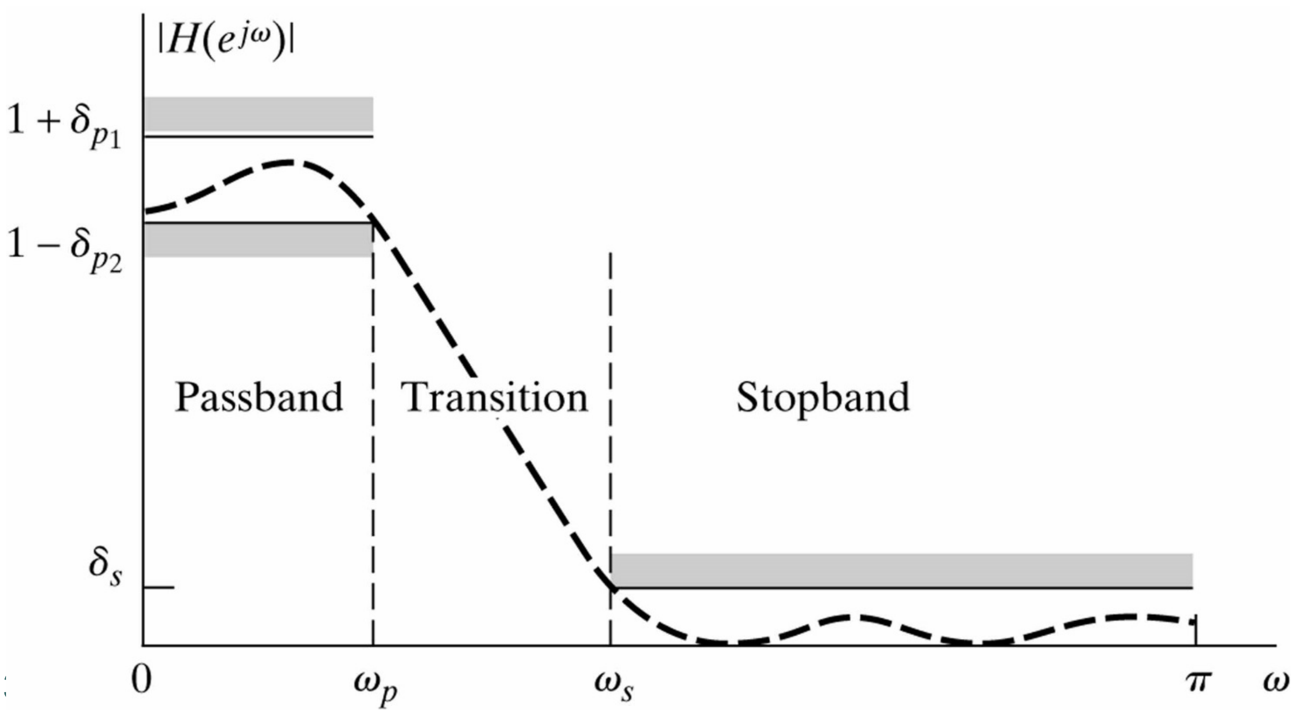
Structure

Specifications – an example

- Specifications for a discrete-time lowpass filter

$$1-0.01 \le |H(e^{j\omega})| \le 1+0.01, \quad 0 \le \omega \le \omega_p$$

$$|H(e^{j\omega})| \le 0.001, \quad \omega \ge \omega_s$$



$$\delta_1 = 0.01$$

$$\delta_2 = 0.001$$

Figure 7.1 Lowpass filter tolerance scheme.

Specifications of frequency response

- Typical lowpass filter specifications in terms of tolerable
  - Passband distortion, as smallest as possible
  - Stopband attenuation, as greatest as possible
  - Width of transition band: as narrowest as possible

- Improving one often worsens others
  → a trade-off

- Increasing filter order improves all

The filter design problem

$$H(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 - \sum_{k=1}^{N} a_k z^{-k}}$$

1. To find coefficients *ak* and *bk* of H(z) (IIR or FIR)
2. To meet the frequency response specifications
3. Specifications should be known or can be determined from an application
4. Specifications should be “reasonable” and have tolerance
5. Specifications have both amplitude and phase
   - Often only amplitude is of interest

6. How to apply specifications to design?

Design a filter

- Design goal:
  - find system function to make frequency response meet the specifications (tolerances)

- Infinite impulse response IIR filter
  - Poles inside unit circle due to causality and stability
  - Rational function approximation

- Finite impulse response FIR filter
  - Linear phase is often required
  - Polynomial approximation

Design a filter

- For rational system function

$$H(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 - \sum_{k=1}^{N} a_k z^{-k}}$$

- Find the system coefficients such that the corresponding frequency response

$$H(e^{j\omega}) = H(z)|_{z=e^{j\omega}}$$

provides a good approximation to a desired response

$$H(e^{j\omega}) \approx H_{desired}(e^{j\omega}) \longrightarrow$$

H(z)

•Rational system function

•Stable

•causal

DT filter for CT signals

- DT filter for the processing of CT signals
  - Bandlimited input signal
  - High enough sampling frequency
- Then, specifications (often given in frequency domain) conversion is straightforward

$$H_{eff}(j\Omega) = \begin{cases} H(e^{j\Omega T}), & |\Omega| < \pi / T \\ 0, & |\Omega| > \pi / T \end{cases}$$

•Signal is band-limited;
•T is small enough

$$H(e^{j\omega}) = H_{eff}(j\frac{\omega}{T}), \qquad |\omega| < \pi$$

$$\omega = \Omega T$$

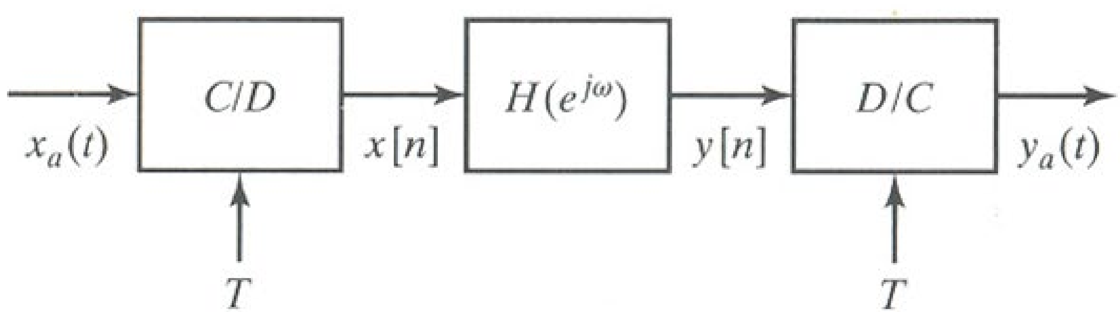


Figure 7.2 Basic system for discrete-time filtering of continuous-time signals.

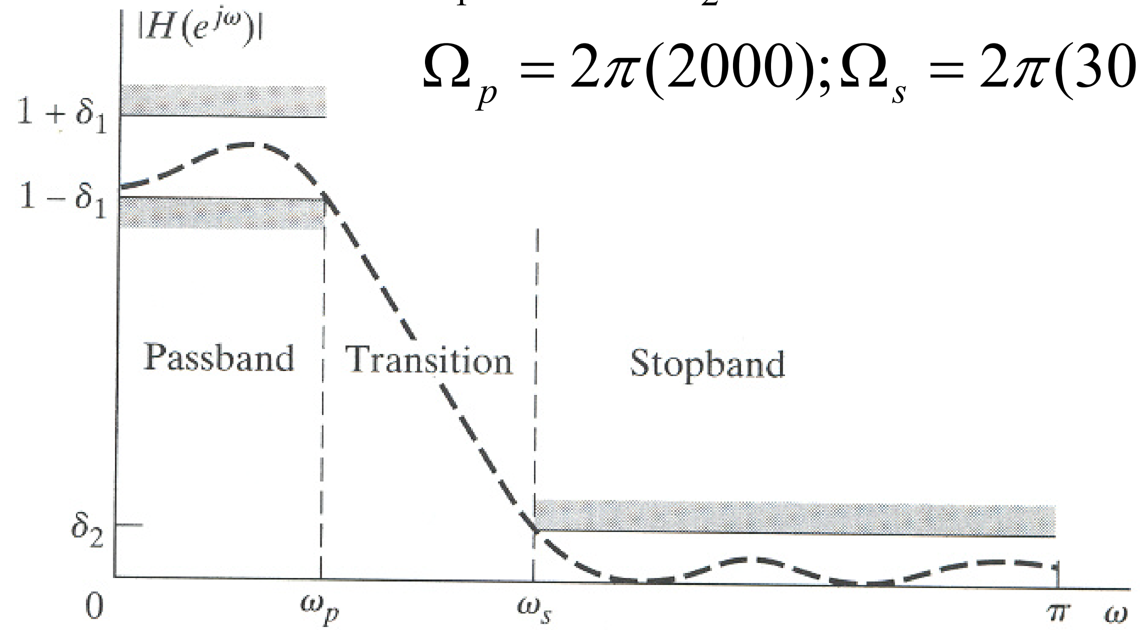
Specifications for a CT lowpass filter

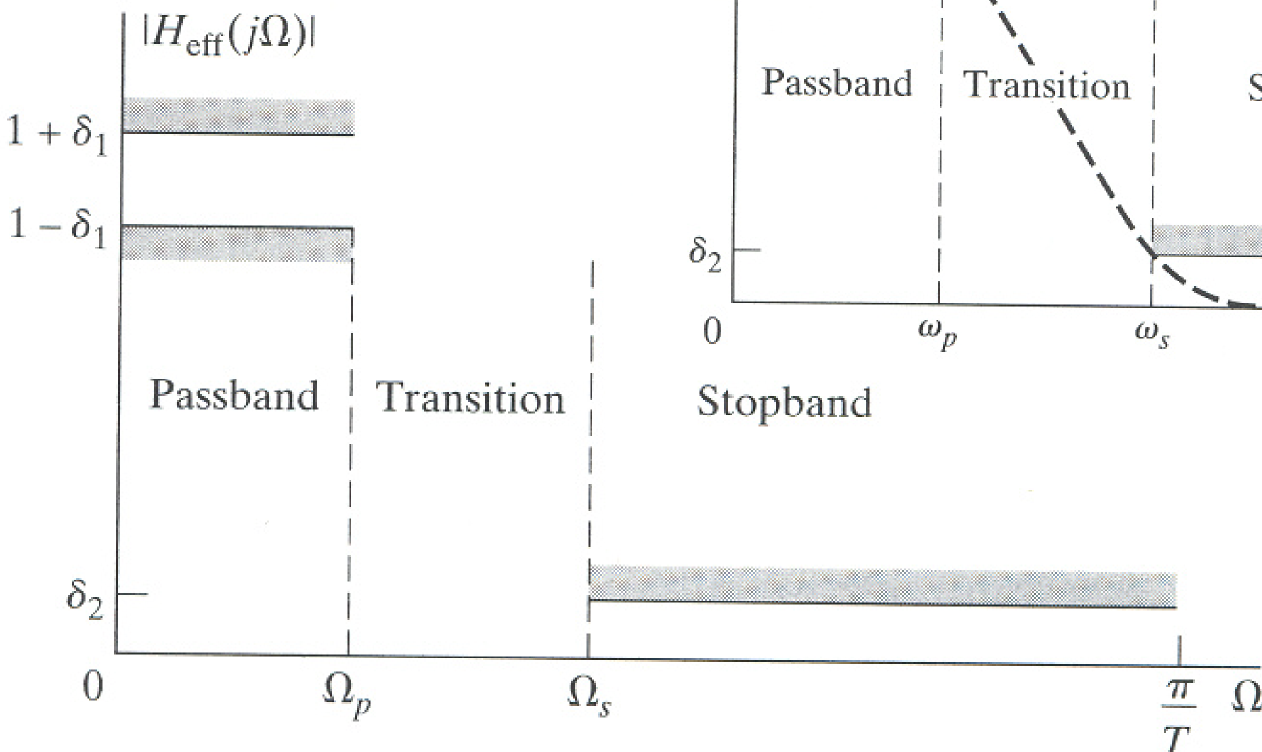
$$1-0.01 \le |H_{eff}(j\Omega)| \le 1+0.01, \quad 0 \le \Omega \le 2\pi(2000)$$

$$|H_{eff}(j\Omega)| \le 0.001, \quad \Omega \ge 2\pi(3000)$$

$$\delta_1 = 0.01; \delta_2 = 0.001$$

$$\Omega_p = 2\pi(2000); \Omega_s = 2\pi(3000)$$





$$\omega = \Omega T$$

Outline

- Filter Design Process & Specifications
- FIR vs IIR
- IIR Filter Design:
  - Analog Filters
  - IIR Filter Design by Impulse Invariance
  - IIR Filter Design by Bilinear Transformation
- FIR Filter Design
  - Design by Windowing
  - Generalized Linear Phase FIR (GLP-FIR)
- Kaiser Window FIR Design Method
- Optimum Approximation of FIR Filters

IIR (Infinite Impulse Response) Systems

- Linear constant-coefficient difference equation $\sum_{k=0}^{N} a_k y[n-k] = \sum_{m=0}^{M} b_m x[n-m]$
- **Rational system function**

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{m=0}^{M} b_m z^{-m}}{\sum_{k=0}^{N} a_k z^{-k}}$$

$$= \left(\frac{b_0}{a_0}\right) \frac{\prod_{m=1}^{M} (1 - c_m z^{-1})}{\prod_{k=1}^{N} (1 - d_k z^{-1})}$$

- If at least one nonzero pole does not cancel with a zero, in h[n] there will at least one term of the form $a^n u[n] \quad \text{OR} \quad -a^n u[-n-1]$

➔ h[n], the impulse response, will be <u>infinite length</u> ➔IIR system

FIR (Finite Impulse Response) Systems

- If H(z) does not have any poles except at z=0
  - In this case N=0

➔ The rational system function $$H(z) = \sum_{k=0}^{M} b_k z^{-k}$$

- The impulse response $$h[n] = \sum_{k=0}^{M} b_k \delta[n-k]$$

- Impulse response is of finite length ➔ FIR system

FIR and IIR

IIR

- Rational system function
- Poles + zeros
- Stable/unstable
- Hard to control phase
- Low order (4-20)
- Designed on the basis of analog filter
- z-domain design &
  s-domain design

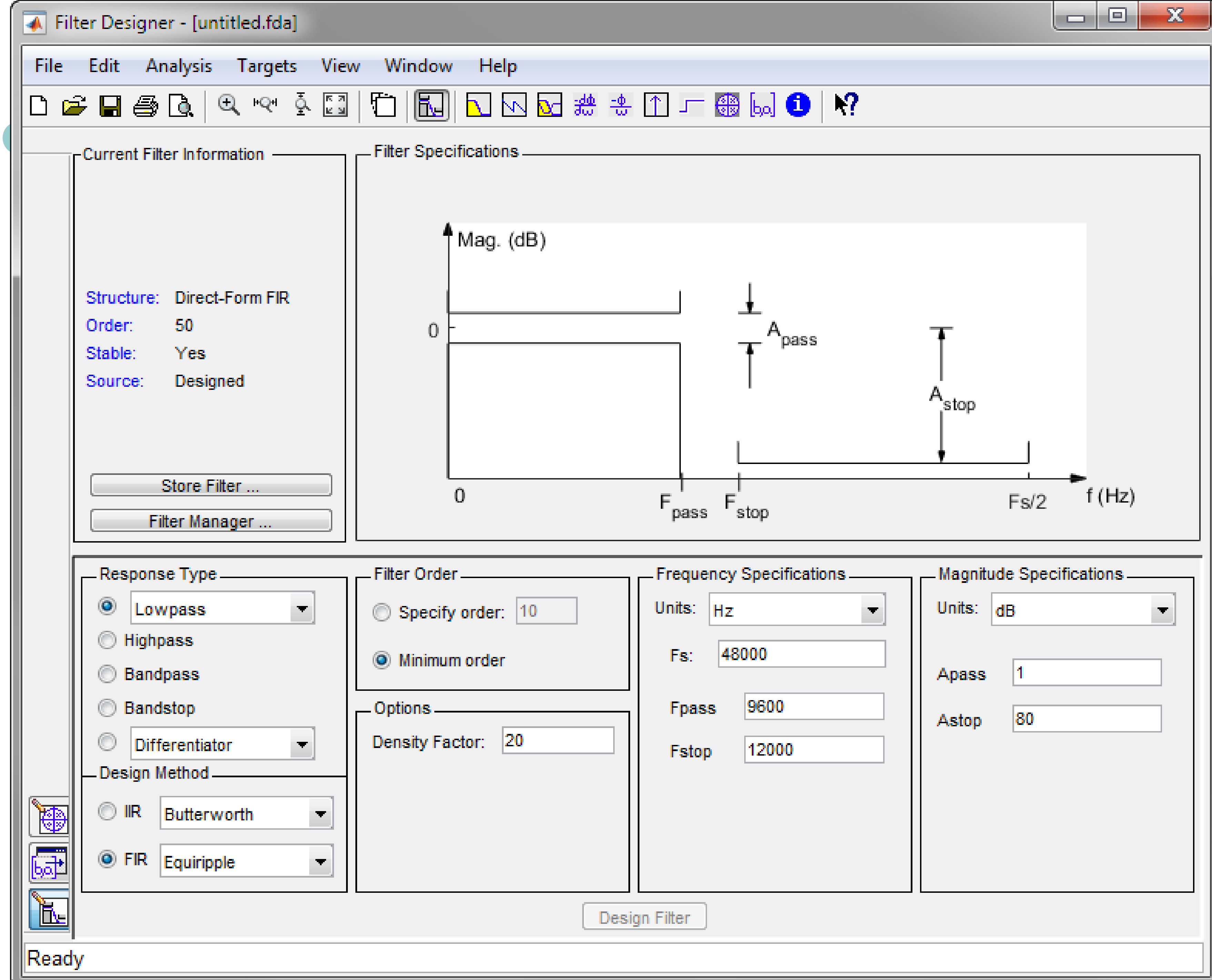
FIR

- Polynomial system function
- Zeros
- Stable
- Easy to get linear phase
- High order (20-2000)
- Unrelated to analog filter
- z-domain design only

FIR or IIR

- Whether FIR or IIR often depends on the phase requirements

- Design principle
  - If Generalized Linear Phase (GLP) is essential → FIR
  - If not → IIR preferable (can meet specifications with lower complexity)



Filter Designer - [untitled.fda]
File
Edit
Analysis
Targets
View
Window
Help
Current Filter Information
Structure: Direct-Form FIR
Order: 50
Stable: Yes
Source: Designed
Store Filter ...
Filter Manager ...
Filter Specifications
Mag. (dB)
0
0
Apass
Astop
Fpass
Fstop
Fs/2
f (Hz)
Response Type
Lowpass
Highpass
Bandpass
Bandstop
Differentiator
Design Method
IIR
Butterworth
FIR
Equiripple
Filter Order
Specify order: 10
Minimum order
Options
Density Factor: 20
Frequency Specifications
Units: Hz
Fs: 48000
Fpass 9600
Fstop 12000
Magnitude Specifications
Units: dB
Apass 1
Astop 80
Design Filter
Ready

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    - **Chebyshev I & II**
    - **Ellipical**
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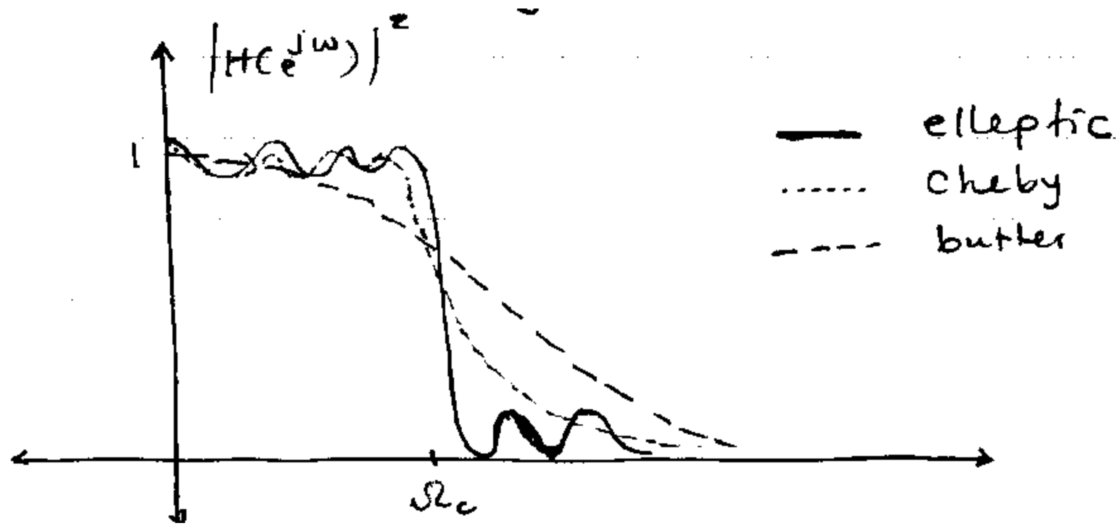
Analog Filters

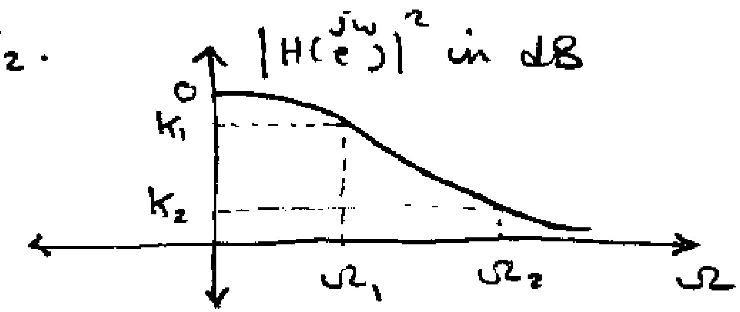
- Butterworth
- Chebyshev I & II
- Ellipical



- Analog filter design procedures begin with a specification of the desired frequency response.

- Approximations to the ideal filter (LP, BP, HP, etc.) are judged by the degree with which they approach the ideal response. This gives rise to a set of specs. for "closeness" of fit.

- The filter requirements are normally given in terms of a set of critical frequencies, say Ω_1 and Ω_2 and gains K_1 and K_2.



- A common set of conditions for the lowpass filter response are:

$$0 \geq 20\log|H(j\Omega)| \geq K_1 \qquad \text{for } \forall\ \Omega \leq \Omega_1$$

$$20\log|H(j\Omega)| \leq K_2 \qquad \text{for } \forall\ \Omega \geq \Omega_2.$$

- These specs. are then used to solve for the order of the filter N and the cut-off frequency Ω_c.

$$\Rightarrow N = f_1(K_1, K_2, \Omega_1, \Omega_2)$$

$$\Omega_c = f_2(K_1, K_2, \Omega_1, \Omega_2)$$

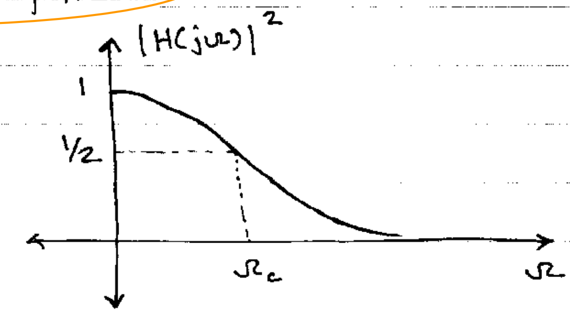
Butterworth LP Filter

- The butterworth filter of order N is described by its magnitude squared response:

All pole (N poles) filter.

$$|H(j\Omega)|^2 = \frac{1}{1+\left(\frac{j\Omega}{j\Omega_c}\right)^{2N}}$$

cut-off frequency



① $|H(0)|^2 = 1$

② $|H(j\Omega_c)|^2 = \frac{1}{2}$ or $20\log_{10}|H(j\Omega_c)| = -3.01\text{ dB}$

③ $|H(j\Omega)|^2$ is monotonically decreasing function of Ω.

④ As N gets larger, $|H(j\Omega)|^2$ approaches the ideal LP filter magnitude response.

⑤ $|H(j\Omega)|^2$ is maximally flat at $\Omega=0$, since the first $2N-1$ derivatives of $|H(j\Omega)|^2$ are zero at $\Omega=0$.

⑥ The filter rolls off at $\sim -20\text{dB/decade}$

Butterworth low pass filters

- The magnitude response
  - Maximally flat in the pass band
  - Monotonic in both pass band and stop band
- The squared magnitude response

$$|H_c(j\Omega)|^2 = \frac{1}{1+(\Omega/\Omega_c)^{2N}}$$

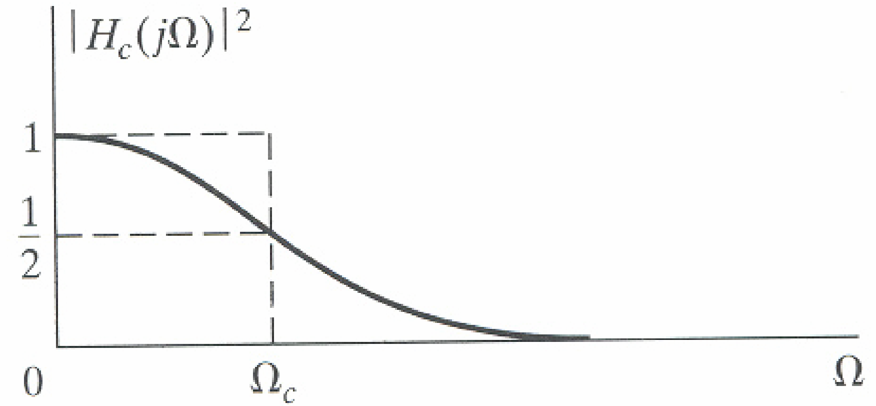


Figure B.1 Magnitude-squared function for continuous-time Butterworth filter.

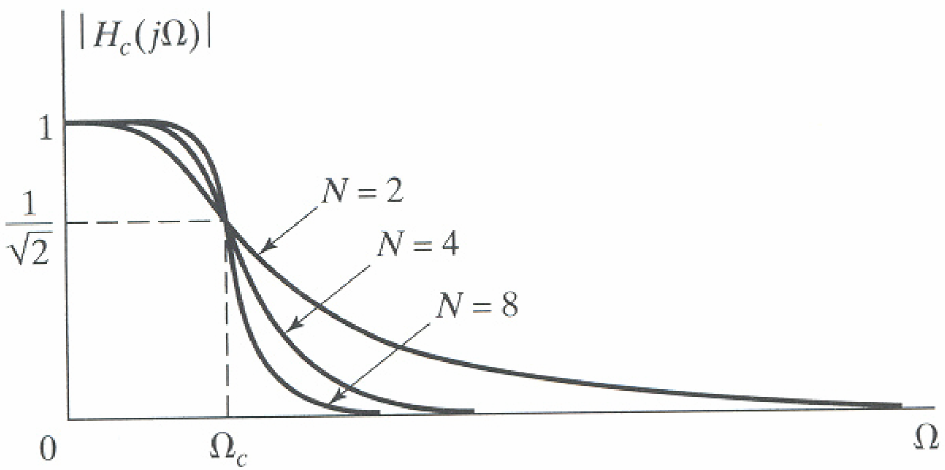
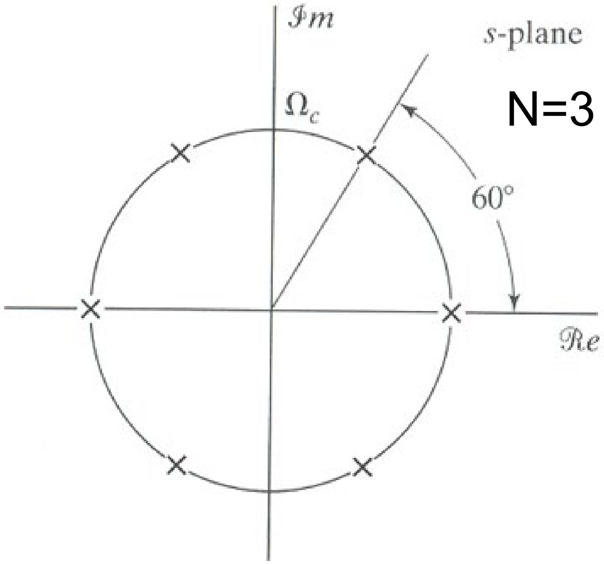


Figure B.2 Dependence of Butterworth magnitude characteristics on the order N.

For the magnitude-squared response defined as butterworth, we would like to find $H(s)$ where $j\Omega = s$. We then have

$$|H(j\Omega)|^2 = H(j\Omega)\,H(-j\Omega)$$
$$= H(s)\,H(-s)$$
$$= \frac{1}{1+\left(\frac{s}{j\Omega_c}\right)^{2N}}$$



the poles of $H(s)\,H(-s)$ are given by

$$1+\left(\frac{s}{j\Omega_c}\right)^{2N} = 0$$

or $$(s)^{2N} = -\left(j\Omega_c\right)^{2N}$$

➔A pole never falls on the j-axis

➔A pole occurs on real-axis if N odd only

➔Angular spacing between the poles is **π/N**

the poles are at :

$$s_n = \begin{cases} \Omega_c\, e^{j\frac{n\pi}{N}}, & N \text{ odd} \\ \Omega_c\, e^{j\left(\frac{n\pi}{N}+\frac{\pi}{2N}\right)}, & N \text{ even} \end{cases} \quad \text{for } n = 0, 1, \cdots, 2N-1$$

- Thus there are 2N poles equally spaced in angle around the circle of radius Ω_c in the s-plane.

- For N-odd, we have a pole @ s=1 and the others are uniformly spaced around the circle of radius Ω_c. For N-even, we have the 1st pole @ $s = \pi/2N$ and the rest are spaced as above.

- To determine the system function of the analog filter associated with the butterworth magnitude-squared response, we must perform the factorization $H(s)\,H(-s)$.

- For the butterworth filter, the poles of $H(s)H(-s)$ always occur in pairs: if there is a pole @ $s = s_n$ then there is also a pole @ $s = -s_n$.

- Consequently, to construct $H(s)$ we choose one pole from each pair. If we wish $H(s)$ to be stable, we select those poles on the LHP part of the s-plane.

Therefore we write:

$$H(s) = \frac{1}{\prod_n (s - s_n)}$$

s_n are LHP poles.

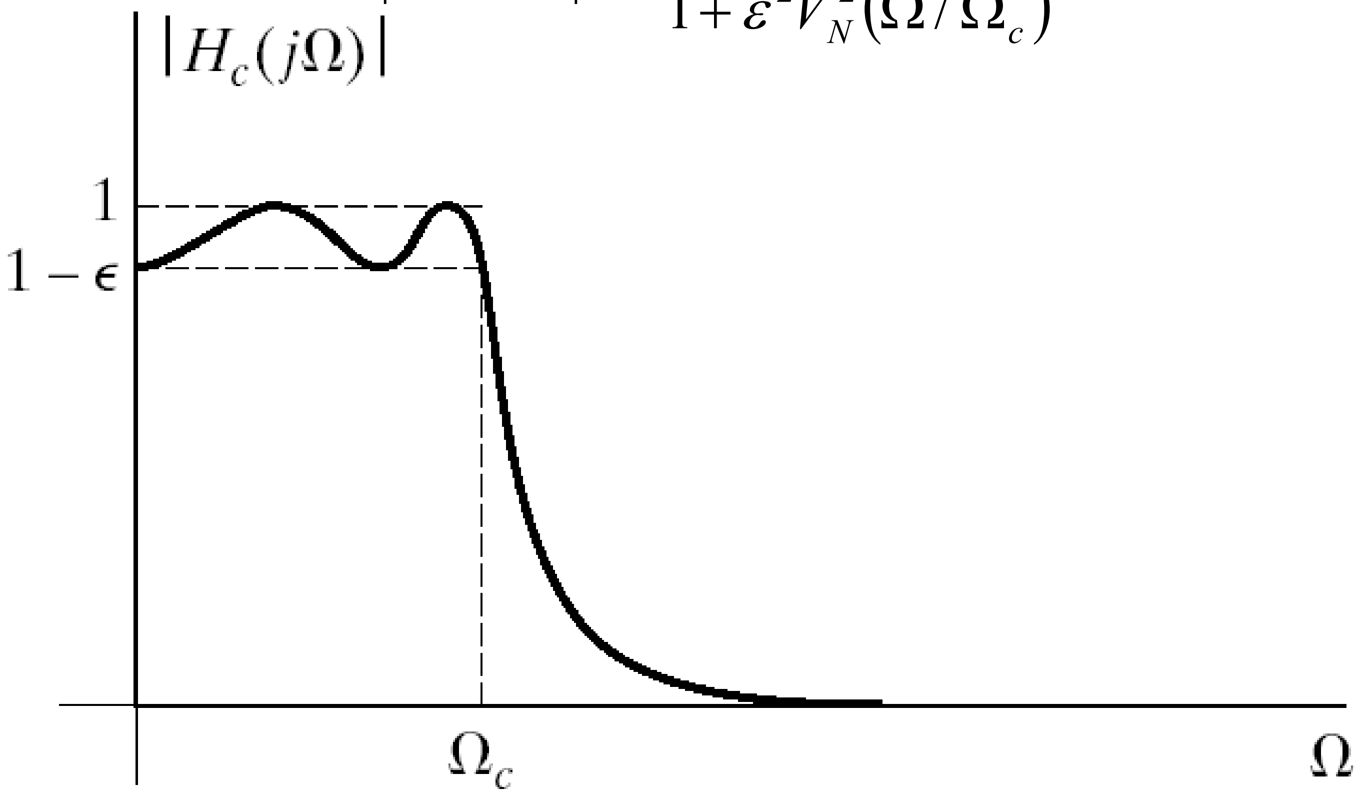
Chebyshev Filters

- Equiripple in the passband and monotonic in the stopband

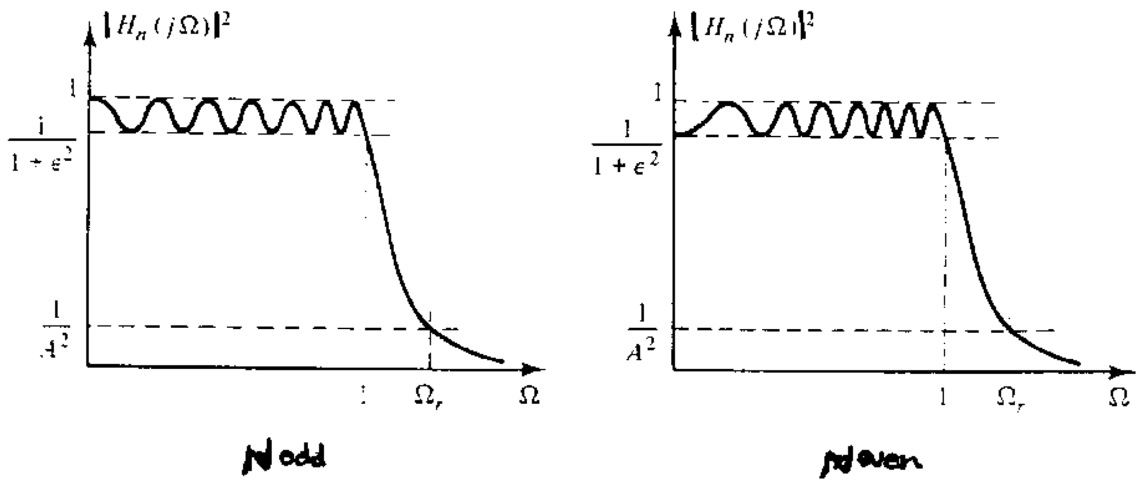
OR

- equiripple in the stopband and monotonic in the passband

$$|H_c(j\Omega)|^2 = \frac{1}{1+\varepsilon^2 V_N^2(\Omega/\Omega_c)} \qquad V_N(x) = \cos\left(N\cos^{-1}x\right)$$



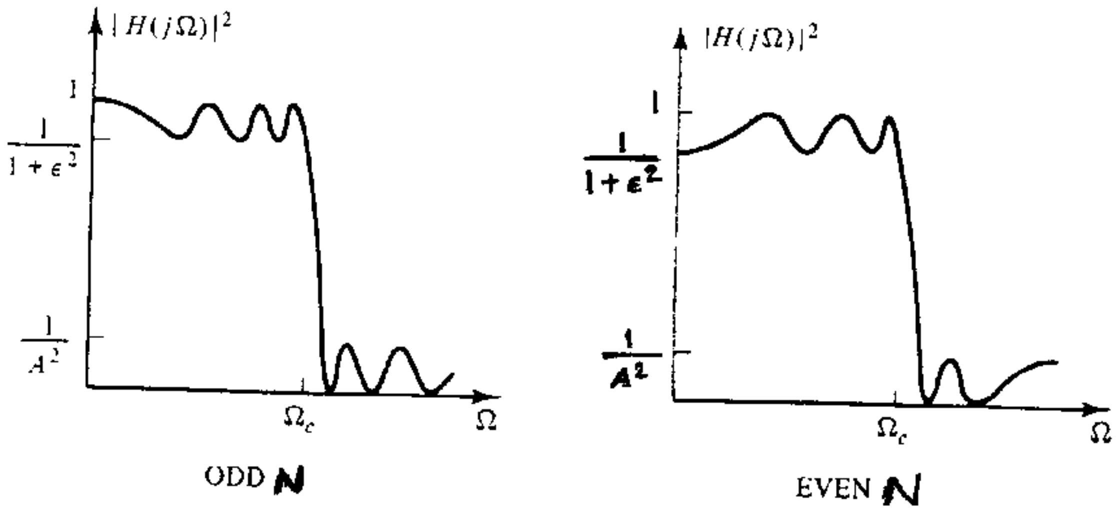
- What applies to butterworth filters in terms of filter order and cutoff frequency applies to Cheby filters. Where both N & Ω_c can be obtained from the given specs.



- By accepting passband (or stopband in the type II case) ripple, for a fixed order number, the chebychev filter can have sharper cutoff than the butterworth.

Ellipical Filter

- A typical frequency response of order N of an elleptic LP filter is shown below.



- By accepting a ripple in the passband & Stopband, a narrower transition width (Sharper) might be possible as compared to that of butterworth & Cheby of the same order.

Analog - Analog Transforms **LP ➔ HP,BP, BS, ..**

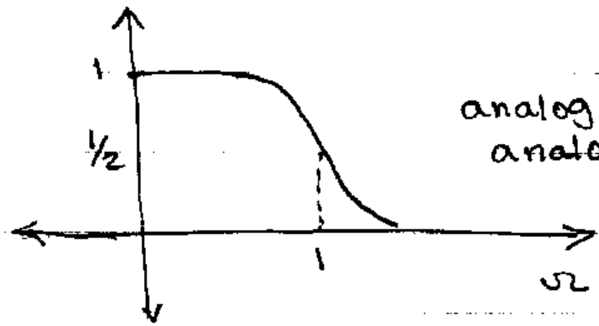
- Once a LP filter (butterworth) prototype has been designed, it can be used to design other types of filters (e.g., BP, HP, BS, etc.) with different specs. This is achieved through analog-to-analog transformation. (See below.)

- In the design of analog LP filters, we began by designing a prototype ($\Omega_c = 1$) analog LP filter, $H(s)$.

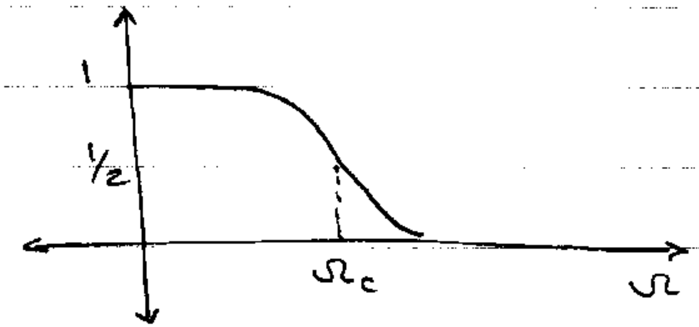
- We then applied LP-LP transform

$$s \longrightarrow \frac{s}{\Omega_c}$$

to obtain a LP filter with an arbitrary Ω_c.

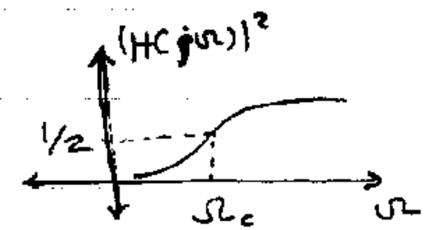


analog - analog

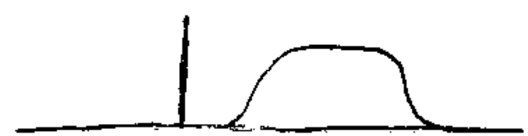


- To design other filter types (HP, BP, BS) of digital filters, we could first apply the corresponding analog transform to the prototype.

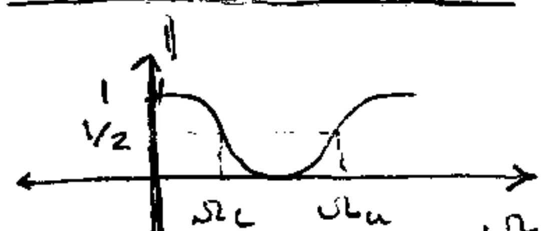
LP → HP $\qquad s \to \Omega_c / s$

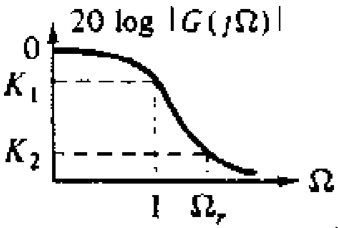
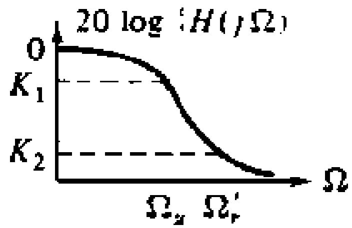
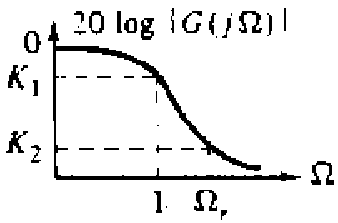
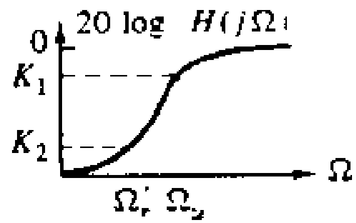
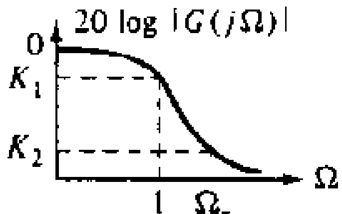
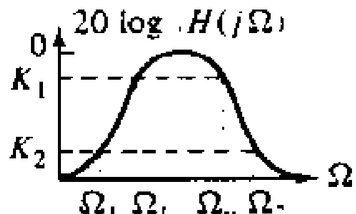
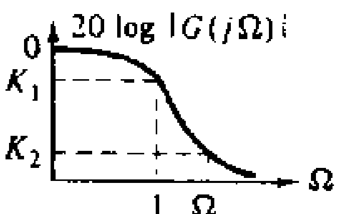
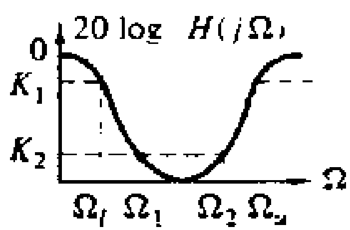


LP → BP $\qquad s \to \dfrac{(s^2 + \Omega_L \Omega_u)}{s(\Omega_u - \Omega_L)}$



LP → BS $\qquad s \to \dfrac{s(\Omega_u - \Omega_L)}{s^2 + \Omega_L \Omega_u}$



Prototype response	Transformed filter response	Design equations
 Low-pass $G(S)$	$S \to S/\Omega_u$  Low-pass $H(S)$	Forward: $\Omega_r' = \Omega_r\Omega_u$ Backward: $\Omega_r = \Omega_r'/\Omega_u$
 Low-pass $G(S)$	$S \to \Omega_u/S$  High-pass $H(S)$	Forward: $\Omega_r' = \Omega_u/\Omega_r$ Backward: $\Omega_r = \Omega_u/\Omega_r'$
 Low-pass $G(S)$	$S \to \dfrac{S^2 + \Omega_l\Omega_u}{S(\Omega_u - \Omega_l)}$  Bandpass $H(S)$	Forward: $\Omega_{av} = (\Omega_u - \Omega_l)/2$ $\Omega_1 = (\Omega_r^2\Omega_{av}^2 + \Omega_l\Omega_u)^{1/2} - \Omega_{av}\Omega_r$ $\Omega_2 = (\Omega_r^2\Omega_{av}^2 + \Omega_l\Omega_u)^{1/2} + \Omega_{av}\Omega_r$ Backward: $\Omega_r = \min\{\lvert A\rvert, \lvert B\rvert\}$ $A = (-\Omega_1^2 + \Omega_l\Omega_u)/[\Omega_1(\Omega_u - \Omega_l)]$ $B = (+\Omega_2^2 - \Omega_l\Omega_u)/[\Omega_2(\Omega_u - \Omega_l)]$
 Low-pass $G(S)$	$S \to \dfrac{S(\Omega_u - \Omega_l)}{S^2 + \Omega_l\Omega_u}$  Bandstop $H(S)$	Forward: $\Omega_{av} = (\Omega_u - \Omega_l)/2$ $\Omega_1 = [(\Omega_{av}/\Omega_r)^2 + \Omega_l\Omega_u]^{1/2} - \Omega_{av}/\Omega_r$ $\Omega_2 = [(\Omega_{av}/\Omega_r)^2 + \Omega_l\Omega_u]^{1/2} + \Omega_{av}/\Omega_r$ Backward: $\Omega_r = \min\{\lvert A\rvert, \lvert B\rvert\}$ $A = \Omega_1(\Omega_u - \Omega_l)/[-\Omega_1^2 + \Omega_l\Omega_u]$ $B = \Omega_2(\Omega_u - \Omega_l)/[-\Omega_2^2 + \Omega_l\Omega_u]$

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IIR Filter design by impulse invariance $\omega = T\Omega$

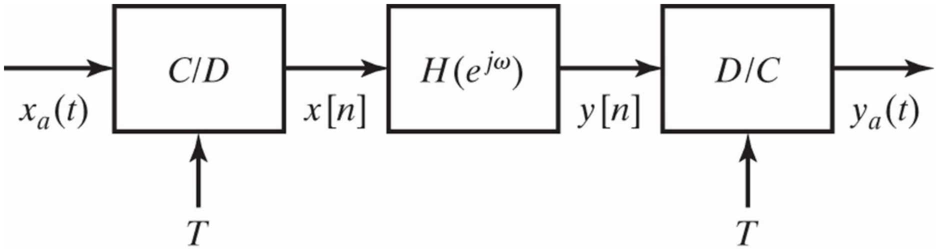
- Impulse invariance:

a method for obtaining a DT system whose $H(e^{j\omega})$ is determined by the of a CT system $H_c(j\Omega)$

$$h[n] = T_d h_c(nT_d)$$

$$T_d \ \text{- 'design' sampling interval}$$

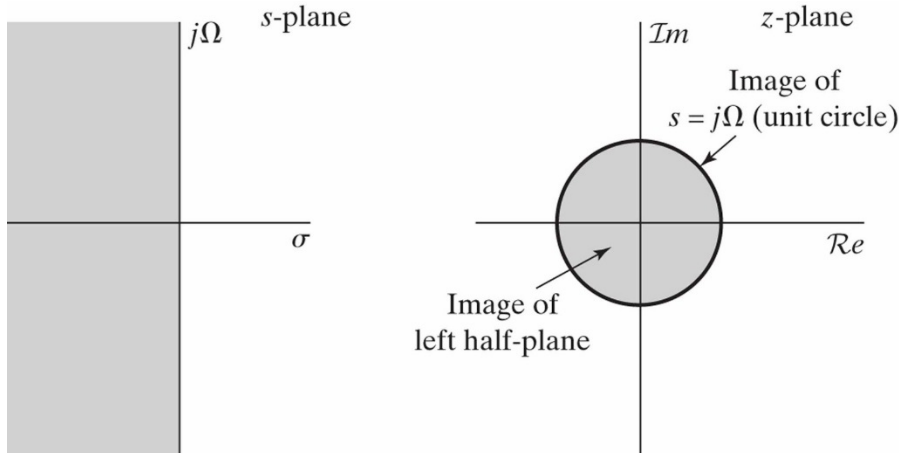
- In DT filter design:
  - the specifications are provided in the DT, so T_d has no role
  - T_d is included for discussion though
  - T_d also has nothing to do with C/D and D/C conversion in Fig. 7.2, i.e., T_d need **not** be the same as the sampling period T of the C/D and D/C conversion



Relationship between frequency responses

- h[n] is the impulse response of a DT LTI system
- $H(e^{i\omega})$ is the DT frequency response
- H(z) is the DT system function; $z=re^{j\omega}$, $|z|=r$, $\angle z=\omega$
  - Poles are at $z_p=re^{j\omega}$
  - H(z) reduces to the DTFT for $|z| = 1$

- H(s) is the CT transfer function $s = \sigma + j\omega$
  - Poles are at s_k



Relationship between frequency responses

- Impulse response

$$h[n] = T_d h_c(nT_d)$$

- Frequency response

$$H(e^{j\omega}) = \sum_{k=-\infty}^{\infty} H_c(j\frac{\omega}{T_d} + j\frac{2\pi}{T_d}k)$$

if the CT filter is band limited

$$H_c(j\Omega) = 0, \quad |\Omega| \geq \pi / T_d$$

then

$$H(e^{j\omega}) = H_c(j\frac{\omega}{T_d}), \quad |\omega| \leq \pi$$

sampling:

$$x[n] = x_c(nT)$$

$$X(e^{j\omega}) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c(j\frac{\omega}{T} - j\frac{2\pi k}{T})$$

$$H(e^{j\omega}) = H_{eff}(j\frac{\omega}{T}), \ |\omega| < \pi$$

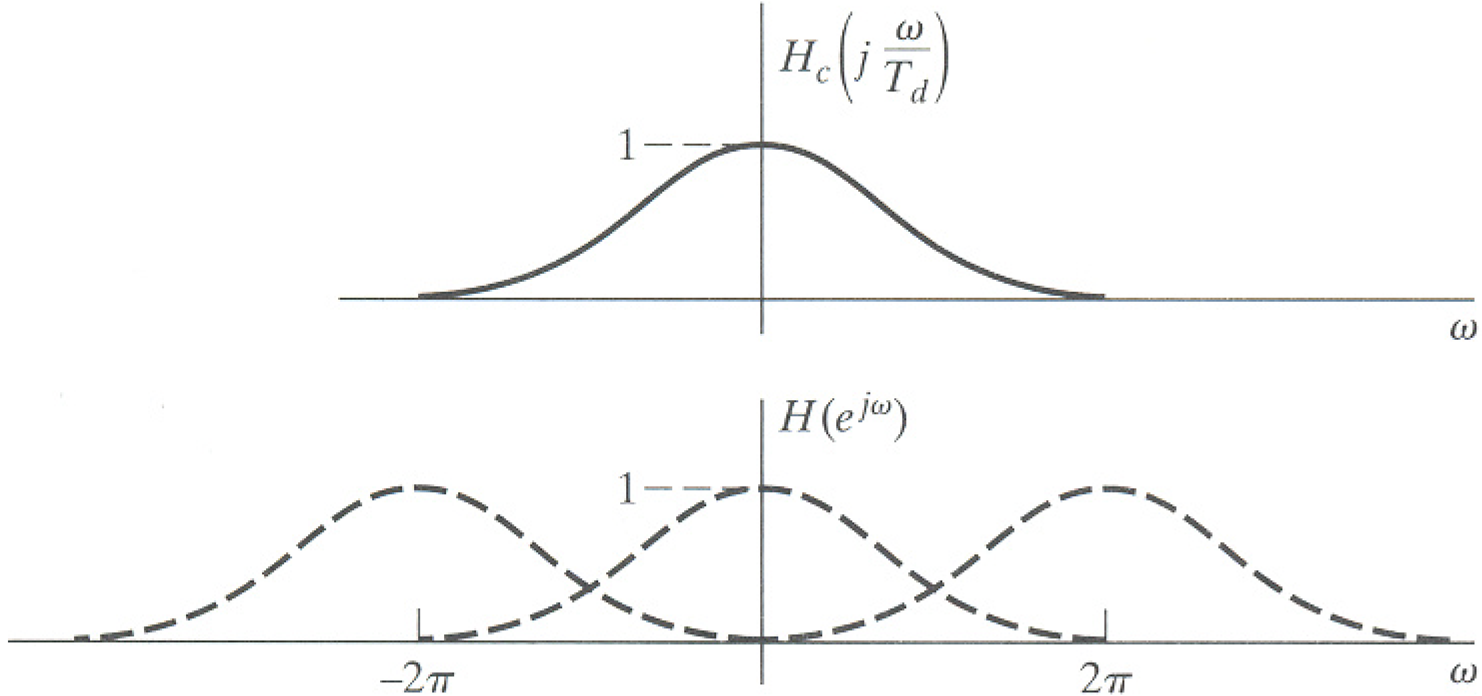
This is also the way to get CT filter specifications from

$H(e^{j\omega})$ by applying the relation $\Omega = \omega / T_d$

Aliasing in the impulse invariance design

$$h[n] = T_d h_c(nT_d)$$

$$H(e^{j\omega}) = \sum_{k=-\infty}^{\infty} H_c\left(j\frac{\omega}{T_d} + j\frac{2\pi}{T_d}k\right)$$



The continuous-time filter may be designed to exceed the specifications, particularly in the stopband

Design procedure:
Impulse Invariance of System Functions

- Develop impulse invariance relation between system functions

$$H_c(s)=\sum_{k=1}^{N}\frac{A_k}{s-s_k}$$

- Find transfer function using partial fraction expansion

Pole location

- Corresponding impulse response (Inverse Laplace transform)

$$h_c(t)=\begin{cases}\sum_{k=1}^{N}A_k e^{s_k t} & t\ge 0\\ 0 & t<0\end{cases}$$

Stable system
➔ROC includes the jw axis
➔*Re*(s_k) < 0

- Impulse response of discrete-time filter

Sampling hc

$$h[n]=T_d h_c(nT_d)=\sum_{k=1}^{N}T_d A_k e^{s_k n T_d}u[n]=\sum_{k=1}^{N}T_d A_k\left(e^{s_k T_d}\right)^n u[n]$$

➔System function
1) A pole s= s_k ➔z= $e^{s_k T_d}$
2) Stable➔|z|<1

$$H(z)=\sum_{k=1}^{N}\frac{T_d A_k}{1-e^{s_k T_d}z^{-1}};\qquad |z|>|e^{s_k T_d}|$$

Observations: Impulse Invariance of System Functions

- Pole $s=s_k$ in s-domain transform into z-pole at $e^{s_k T_d}$
- Stability of H(z) depends on locations of s_k:
  - If all s_k are located in the left-half of the s-plane, H(z) stable? Yes
- Does the impulse invariance yield frequency response invariance? Not exact !
  - The zero of H(z) are a function of the coefficients and the poles

$$H(z)=\sum_{k=1}^{N}\frac{T_d A_k}{1-e^{s_k T_d} z^{-1}}; \quad |z|>|e^{s_k T_d}|$$

Example 7.2

- **Impulse invariance applied to Butterworth**

$$0.89125 \le \left|H\left(e^{j\omega}\right)\right| \le 1 \qquad 0 \le |\omega| \le 0.2\pi$$
$$\left|H\left(e^{j\omega}\right)\right| \le 0.17783 \quad 0.3\pi \le |\omega| \le \pi$$

- For Convenience we choose T_d=1

$$\Big\downarrow \omega = T\Omega$$

- Map spec to continuous time

$$0.89125 \le \left|H\left(j\Omega\right)\right| \le 1 \qquad 0 \le |\Omega| \le 0.2\pi$$
$$\left|H\left(j\Omega\right)\right| \le 0.17783 \quad 0.3\pi \le |\Omega| \le \pi$$

- Butterworth filter is monotonic so spec will be satisfied if

$$\left|H_c\left(j0.2\pi\right)\right| \ge 0.89125 \ \text{ and } \ \left|H_c\left(j0.3\pi\right)\right| \le 0.17783$$

$$\left|H_c\left(j\Omega\right)\right|^2 = \frac{1}{1 + \left(j\Omega / j\Omega_c\right)^{2N}}$$

- **Determine N and Ω_c to satisfy these conditions**

- **Satisfy both constrains**

$$1+\left(\frac{0.2\pi}{\Omega_c}\right)^{2N}=\left(\frac{1}{0.89125}\right)^2 \quad \text{and} \quad 1+\left(\frac{0.3\pi}{\Omega_c}\right)^{2N}=\left(\frac{1}{0.17783}\right)^2$$

- **Solve these equations to get**

$$N=5.8858\cong 6 \quad \text{and} \quad \Omega_c=0.70474$$

N must be an integer >5.8 to meet the specs; with N=6→Ω=0.7032

- **Poles of transfer function**

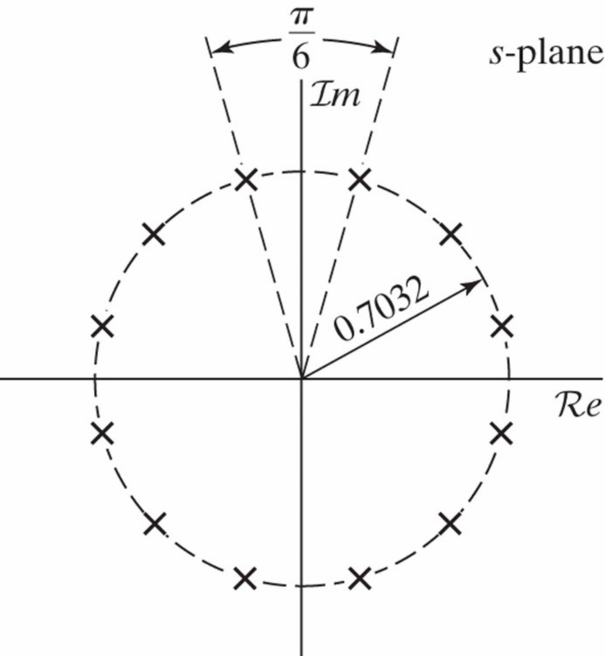
$$s_k=(-1)^{1/12}(j\Omega_c)=\Omega_c e^{(j\pi/12)(2k+11)} \quad \text{for } k=0,1,\ldots,11$$

- **The transfer function** $H(s)=\dfrac{1}{\prod_n (s-s_n)}$ with only the poles s_k in the LHP, i.e., for k=3,4,5,6,7,8

$$H(s)=\frac{0.12093}{(s^2+0.364s+0.4945)(s^2+0.9945s+0.4945)(s^2+1.3585s+0.4945)}$$

- **Mapping to z-domain**

$$H(z)=\frac{0.2871-0.4466z^{-1}}{1-1.2971z^{-1}+0.6949z^{-2}}+\frac{-2.1428+1.1455z^{-1}}{1-1.0691z^{-1}+0.3699z^{-2}}+\frac{1.8557-0.6303z^{-1}}{1-0.9972z^{-1}+0.257z^{-2}}$$



$$s_k = (-1)^{1/12}(j\Omega_c) = \Omega_c e^{(j\pi/12)(2k+11)} \quad \text{for } k = 0,1,\ldots,11$$

The only the poles s_k in the LHP are for k=3,4,5,6,7,8

Figure 7.4 *s*-plane locations for poles of $H_c(s)H_c(-s)$ for 6th-order Butterworth filter in Example 7.2

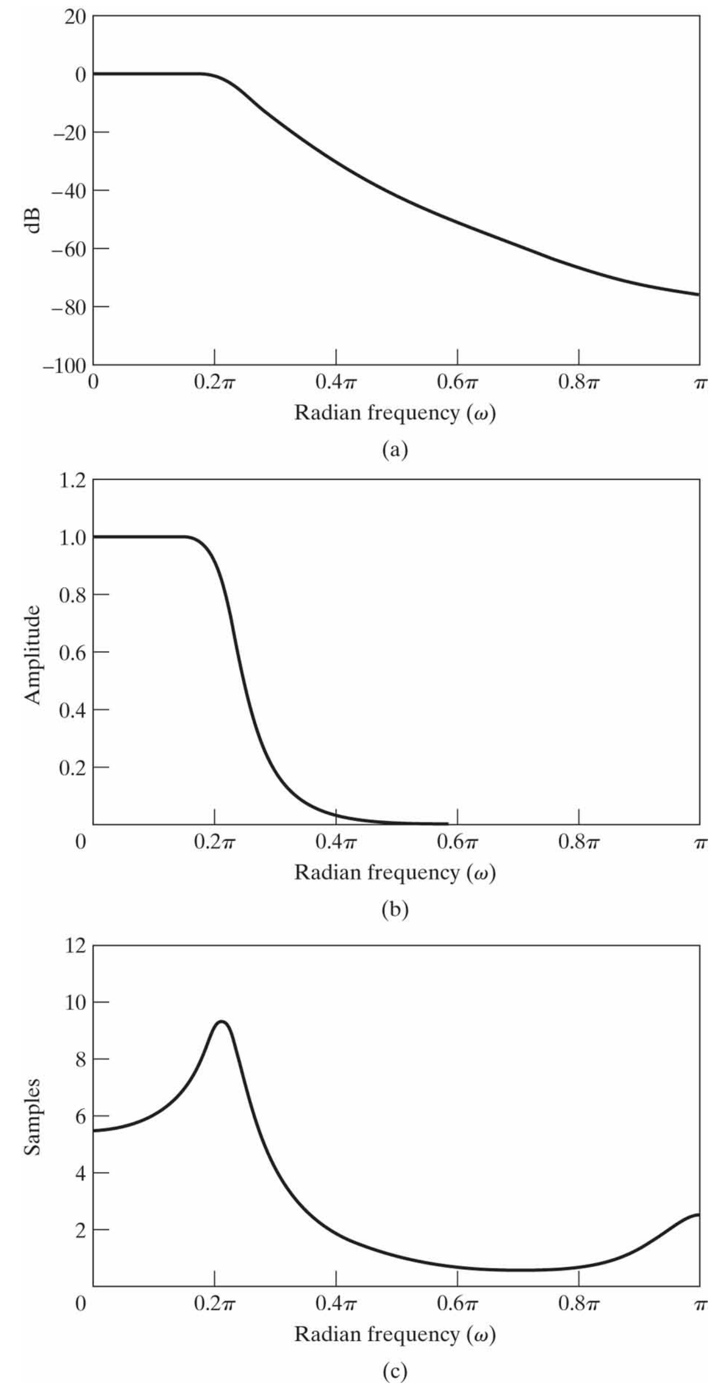


Figure 7.5 Frequency response of 6th-order Butterworth filter transformed by impulse invariance. (a) Log magnitude in dB. (b) Magnitude. (c) Group delay.

Outline

- Filter Design Process & Specifications
- FIR vs. IIR
- IIR Filter Design:
    - Analog Filters
    - IIR Filter Design by Impulse Invariance
    - IIR Filter Design by Bilinear Transformation
- FIR Filter Design
    - Design by Windowing
    - Generalized Linear Phase FIR (GLP-FIR)
- Kaiser Window FIR Design Method
- Optimum Approximation of FIR Filters

- The bilinear transformation is an algebraic transformation between the variables s & z that maps the entire $j\Omega$-axis in the s-plane to one revolution of the unit circle in the z-plane

- Since, $-\infty \leq \Omega \leq \infty$ maps onto $-\pi \leq \omega \leq \pi$, the transformation between the CT and DT variables in nonlinear. Therefore, the frequency axis will be "warped".

- Let $H_c(s)$ denote the CT system function and $H(z)$ the DT system function. The BLT Corresponds to replacing s by:

$$z = \frac{1 + \left(\frac{T_d}{2}\right)s}{1 - \left(\frac{T_d}{2}\right)s}$$

$$s = \frac{2}{T_d}\left(\frac{1 - z^{-1}}{1 + z^{-1}}\right) \qquad H(z) = H_c\left[\frac{2}{T_d}\left(\frac{1 - z^{-1}}{1 + z^{-1}}\right)\right]$$

Bilinear Transformation: Properties

1. Causal stable analog filter maps into causal stable DT filter
2. No aliasing introduced by frequency mapping

$$0 \le \Omega \le \infty \leftrightarrow 0 \le \omega \le \pi$$

$$-\infty \le \Omega \le 0 \leftrightarrow -\pi \le \omega \le 0$$

➔Non-linear compression of the frequency axis introduced

Property1

- Map the entire s-plane onto the unit-circle in the z-plane
  ➔Nonlinear transformation: Freq. response subject to warping

$$s = \frac{2}{T_d}\left(\frac{1 - z^{-1}}{1 + z^{-1}}\right) \qquad s = \sigma + j\Omega$$

- Transformed system function $H(z) = H_c\left[\frac{2}{T_d}\left(\frac{1 - z^{-1}}{1 + z^{-1}}\right)\right]$
- T_d cancels out so we can ignore it
- We can solve the transformation for z as

$$z = \frac{1 + (T_d/2)s}{1 - (T_d/2)s} = \frac{1 + \sigma T_d/2 + j\Omega T_d/2}{1 - \sigma T_d/2 - j\Omega T_d/2}$$

On the unit circle $(s = j\Omega)$:

$$z = \frac{1 + j\Omega T_d/2}{1 - j\Omega T_d/2} = e^{j\omega}$$

➔Property 1: Maps the left-half s-plane into the inside of the unit-circle in z

- Stable in one domain would stay in the other

Property #1: The interior of the LHP maps to the interior of the unit circle; the $j\Omega$ axis maps onto the unit circle

Substituting $s = \sigma + j\Omega$ we obtain:

$$z = \frac{1 + \sigma\left(\frac{T_d}{2}\right) + j\Omega\left(\frac{T_d}{2}\right)}{1 - \sigma\left(\frac{T_d}{2}\right) - j\Omega\left(\frac{T_d}{2}\right)}$$

With a bit of algebra, we can obtain the first part of Property #1 of the BLT, i.e.,

$$|z| < 1, \quad \sigma < 0 \quad (\text{LHP})$$

$$|z| > 1, \quad \sigma > 0 \quad (\text{RHP})$$

That's if a pole of $H_c(s)$ is in the LHP, its image in the z-plane will be inside the unit circle. Therefore, causal, stable CT filters map into causal, stable DT filters under the BLT.

To show that the $j\Omega$-axis of the s-plane maps onto the unit circle (the boundary condition) substitute $\sigma = 0$ and we obtain:

$$z = \frac{1 + j\Omega\left(\frac{T_d}{2}\right)}{1 - j\Omega\left(\frac{T_d}{2}\right)}$$

Clearly $|z| = 1$ (unit circle)

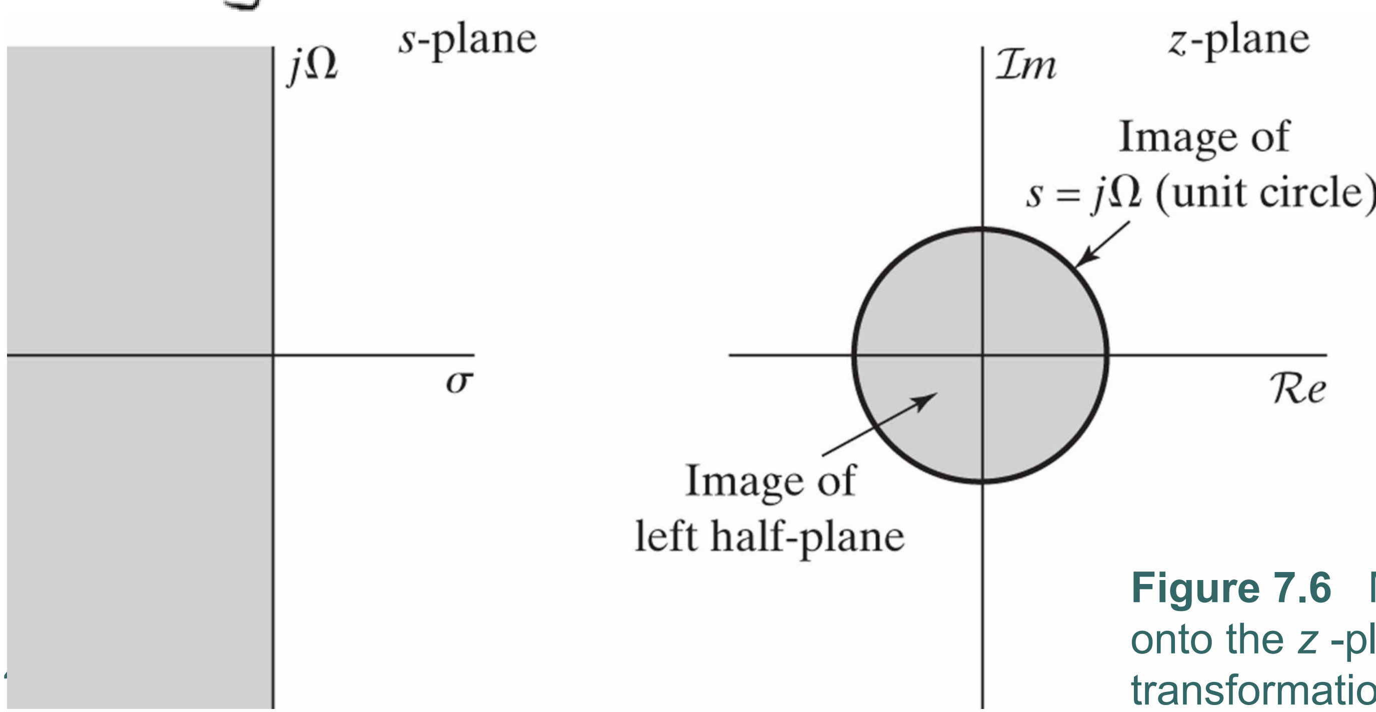


Figure 7.6 Mapping of the *s*-plane onto the *z* -plane using the bilinear transformation.

Property 2

- To derive the relation between ω and Ω

$$s = \frac{2}{T_d}\left(\frac{1-e^{-j\omega}}{1+e^{-j\omega}}\right) = \sigma + j\Omega$$

$$= \frac{2}{T_d}\left[\frac{2e^{-j\omega/2}\, j\sin(\omega/2)}{2e^{-j\omega/2}\cos(\omega/2)}\right] = \frac{2j}{T_d}\tan\left(\frac{\omega}{2}\right)$$

- Which yields

$$\Omega = \frac{2}{T_d}\tan\left(\frac{\omega}{2}\right) \quad \text{or} \quad \omega = 2\arctan\left(\frac{\Omega T_d}{2}\right)$$

Property 2: Warping of the freq. axis

Property #2: The DT frequency axis is non-linearly warped (compressed).

To derive a relationship between ω (DT frequency) and Ω (CT frequency) variables, substitute $z = e^{j\omega}$ into the bilinear map:

$$s = \frac{2}{T_d} \frac{(1 - e^{-j\omega})}{(1 + e^{-j\omega})}$$

which with some algebra becomes:

$$\sigma + j\Omega = s$$

$$= \frac{2}{T_d} \frac{e^{-j\omega/2}\left(e^{j\omega/2} - e^{-j\omega/2}\right)}{e^{-j\omega/2}\left(e^{j\omega/2} + e^{-j\omega/2}\right)}$$

$$= \frac{2}{T_d}\left[\frac{j2\sin(\omega/2)}{2\cos(\omega/2)}\right]$$

$$= \frac{j2}{T_d} \tan(\omega/2)$$

Therefore

$$\Omega = \frac{2}{T_d} \tan(\omega/2)$$

or

$$\omega = 2 \arctan\left(\frac{\Omega T_d}{2}\right)$$

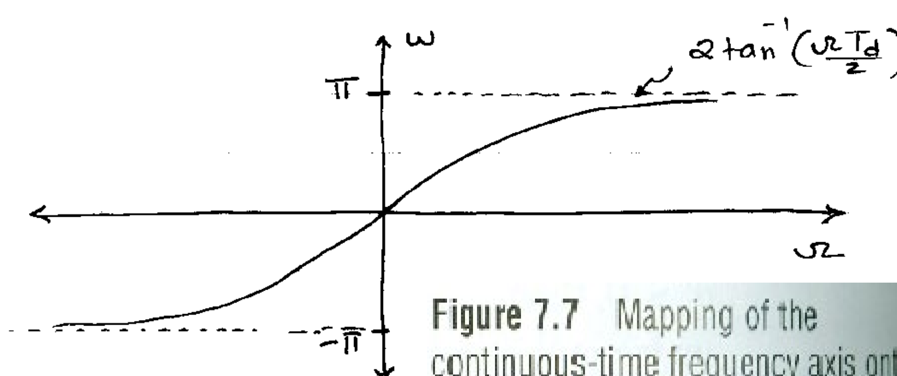


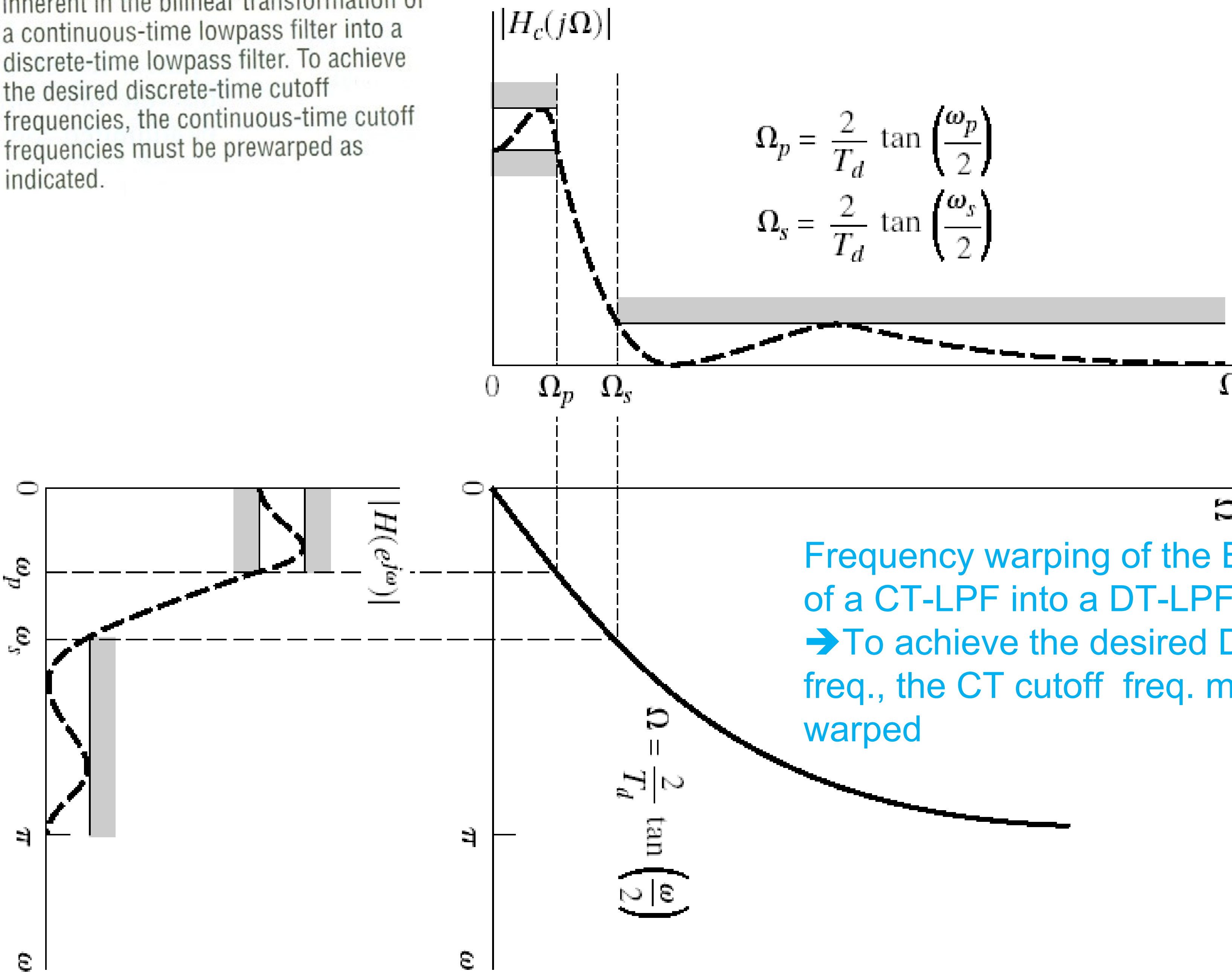
Figure 7.7 Mapping of the continuous-time frequency axis onto the discrete-time frequency axis by bilinear transformation.

- Due to the frequency warping, critical frequencies (cutoff frequency, passband and stopband edge frequencies, etc.) for the analog filter will not necessarily by the same value for the digital filter. For this problem, though we can "prewarp" the critical frequencies $(0, \Omega_p, \Omega_s)$ according to:

$$\Omega = \frac{2}{T_d} \tan(\omega/2)$$

so that when the CT filter is transformed to the DT filter it will meet the desired specifications.

Figure 7.8 Frequency warping inherent in the bilinear transformation of a continuous-time lowpass filter into a discrete-time lowpass filter. To achieve the desired discrete-time cutoff frequencies, the continuous-time cutoff frequencies must be prewarped as indicated.



Frequency warping of the BL trans. of a CT-LPF into a DT-LPF
➔To achieve the desired DT cutoff freq., the CT cutoff freq. must be pre-warped

Procedure for Designing a Digital Filter using the Bilinear Transformation

- Suppose that we are given the specs of the digital filter that we need to design.

1. Prewarp the digital filter specs.
2. Design an analog filter to meet the prewarped specs.
3. Apply bilinear transformation.

Example 7.3:
Bilinear transform **applied to Butterworth**

- Assume DT filter specs

$$0.89125 \le \left|H\left(e^{j\omega}\right)\right| \le 1 \qquad 0 \le |\omega| \le 0.2\pi$$

$$\left|H\left(e^{j\omega}\right)\right| \le 0.17783 \qquad 0.3\pi \le |\omega| \le \pi$$

- Prewarp DT filter critical freq. (pass/stop bands) to obtain CT freq.:

$$0.89125 \le \left|H\left(j\Omega\right)\right| \le 1 \qquad 0 \le |\Omega| \le \frac{2}{T_d}\tan\left(\frac{0.2\pi}{2}\right)$$

$$\Omega = \frac{2}{T_d}\tan(\omega/2)$$

$$\left|H\left(j\Omega\right)\right| \le 0.17783 \qquad \frac{2}{T_d}\tan\left(\frac{0.3\pi}{2}\right) \le |\Omega| < \infty$$

- Assume T_d=1 and apply the specs to $\left|H_c\left(j\Omega\right)\right|^2 = \dfrac{1}{1+\left(\Omega/\Omega_c\right)^{2N}}$

- Taking equalities (=0.89125 & =0.17783) at passband and stopband frequencies, we get

$$1+\left(\frac{2\tan 0.1\pi}{\Omega_c}\right)^{2N} = \left(\frac{1}{0.89125}\right)^2 \quad \text{and} \quad 1+\left(\frac{2\tan 0.15\pi}{\Omega_c}\right)^{2N} = \left(\frac{1}{0.17783}\right)^2$$

Example Cont'd

- Solve to N and Ω_c

$$\Omega_c = 0.766$$

$$N = \frac{\log\left[\left(\left(\frac{1}{0.17783}\right)^2 - 1\right) \Big/ \left(\left(\frac{1}{0.89125}\right)^2 - 1\right)\right]}{2\log\left[\tan(0.15\pi)/\tan(0.1\pi)\right]} = 5.305 \cong 6$$

- The resulting H(s) has the poles

$$s_k = (-1)^{1/12}(j\Omega_c) = \Omega_c e^{(j\pi/12)(2k+11)} \quad \text{for } k = 0,1,\ldots,11$$

➔12 poles equally spaced on a circle of radius Ω_c

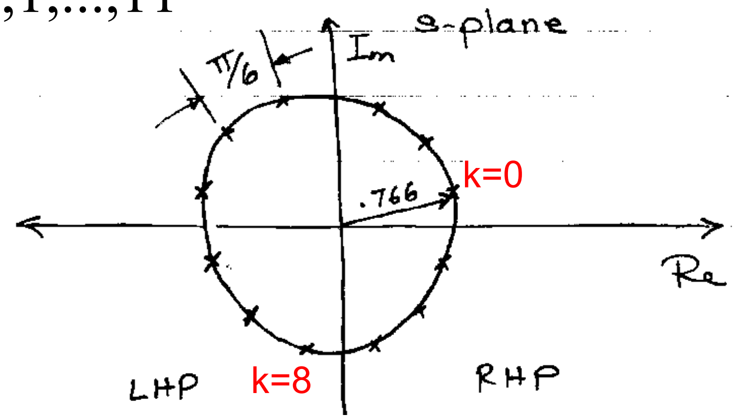
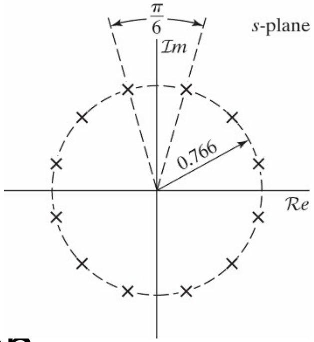


Figure 7.10 *s*-plane locations for poles of $H_c(s)H_c(-s)$ for 6th-order Butterworth filter

Example Cont'd



- Constructing stable H(s): 6 poles equally spaced in the LHP on a circle of radius Ω_c

$$H(s) = \frac{1}{\prod_n (s - s_n)}$$

with only the poles s_k in the LHP, i.e., for k=3,4,5,6,7,8

$$H_c(s) = \frac{(0.76622)^6}{(s^2 + 0.3996s + 0.5871)(s^2 + 1.0836s + 0.5871)(s^2 + 1.4802s + 0.5871)}$$

- To obtain H(z), we apply the bilinear transform

$$H(z) = H_c\left[\frac{2}{T_d}\left(\frac{1 - z^{-1}}{1 + z^{-1}}\right)\right]$$

$$H(z) = \frac{0.0007378(1 + z^{-1})^6}{(1 - 1.2686z^{-1} + 0.7051z^{-2})(1 - 1.0106z^{-1} + 0.3583z^{-2})(1 - 0.9044z^{-1} + 0.2155z^{-2})}$$

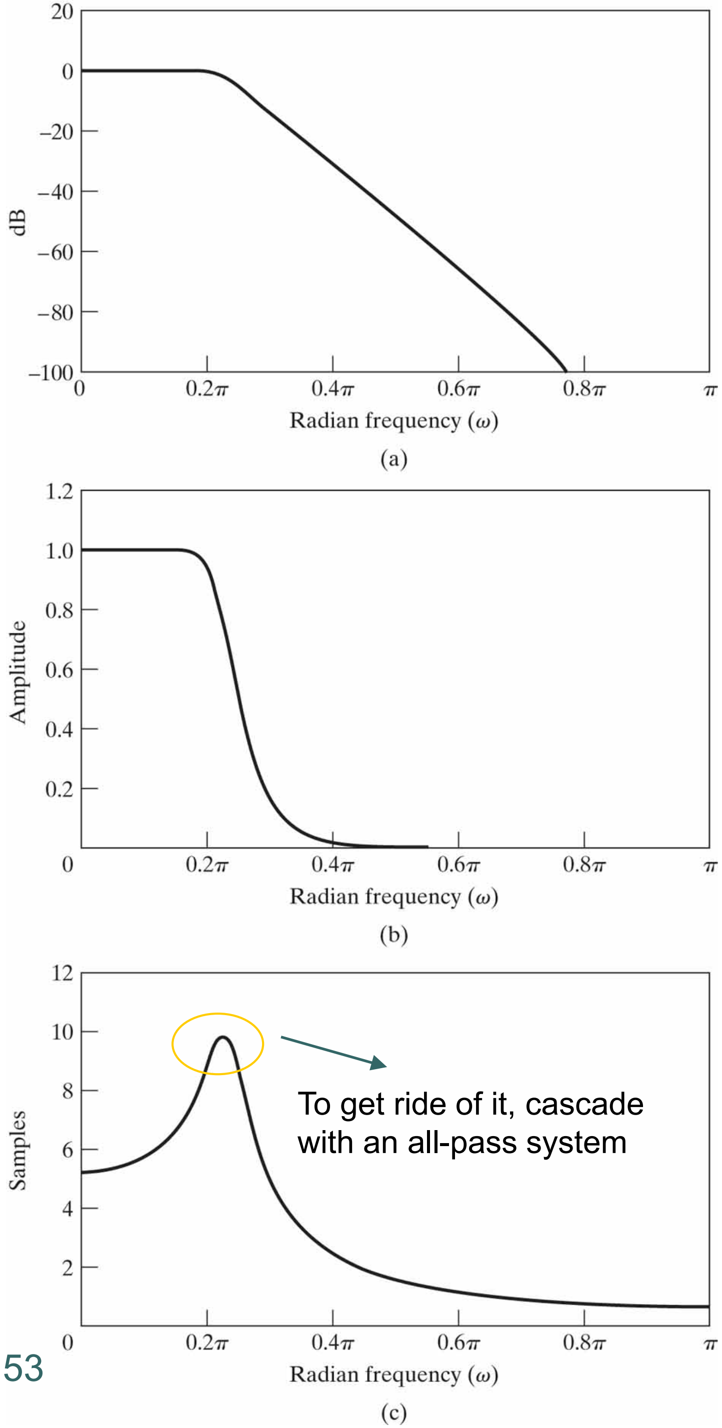


Figure 11.Frequency response of 6th-order Butterworth filter transformed by bilinear transform. (a) Log magnitude in dB. (b) Magnitude. (c) Group delay.

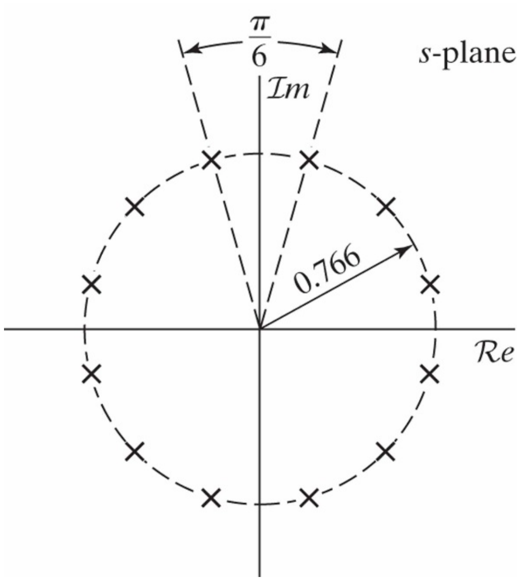


Figure 7.10 *s*-plane locations for poles of $H_c(s)H_c(-s)$ for 6th-order Butterworth filter in Example 7.3.

Ex7.3 Design a filter to meet

$$0.89125 \le |H(e^{j\omega})| \le 1 \qquad 0 \le \omega \le 0.2\pi$$

$$|H(e^{j\omega})| \le 0.17783 \qquad 0.3\pi \le \omega \le \pi$$

1°. Calculate Ω_p, Ω_s according to ω_p, ω_s

2° Design an analog filter to meet

$$0.89125 \le |H_c(j\Omega)| \le 1 \qquad 0 \le \Omega \le \Omega_p$$

$$|H_c(j\Omega)| \le 0.17783 \qquad \Omega_s \le \Omega \le \infty$$

a) choose approximation type (Butterworth, Chebyshev, elliptic)

b). Determine the analog filter order

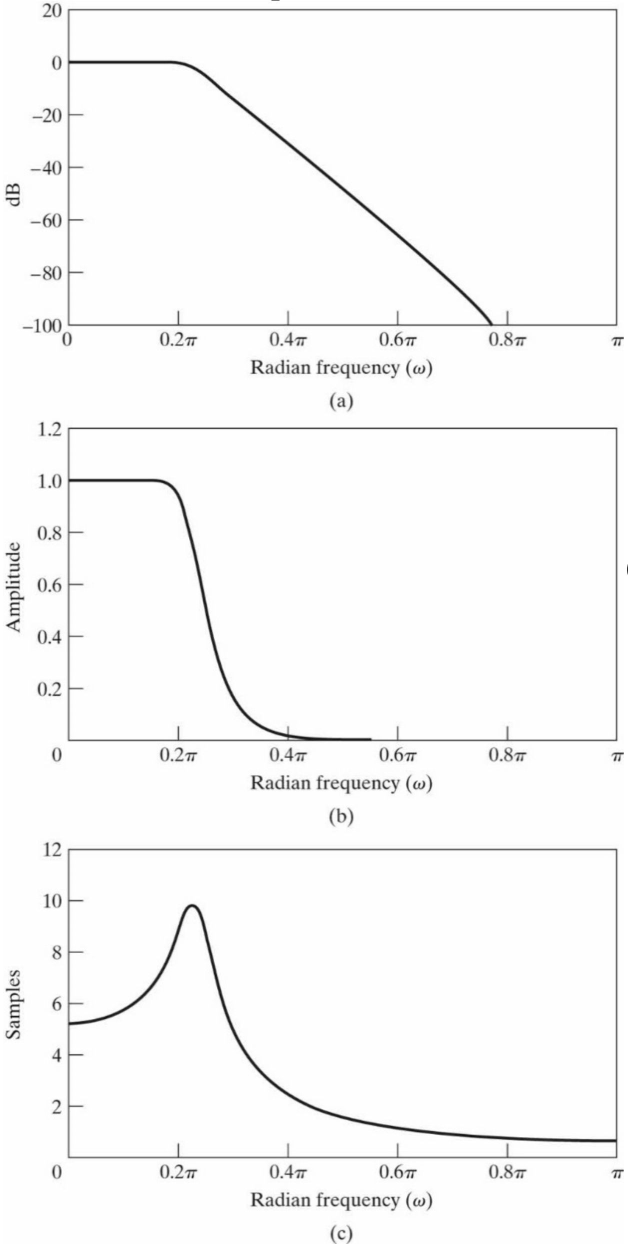
c). calculate the pole locations

d). Determine s function $H_c(s)$

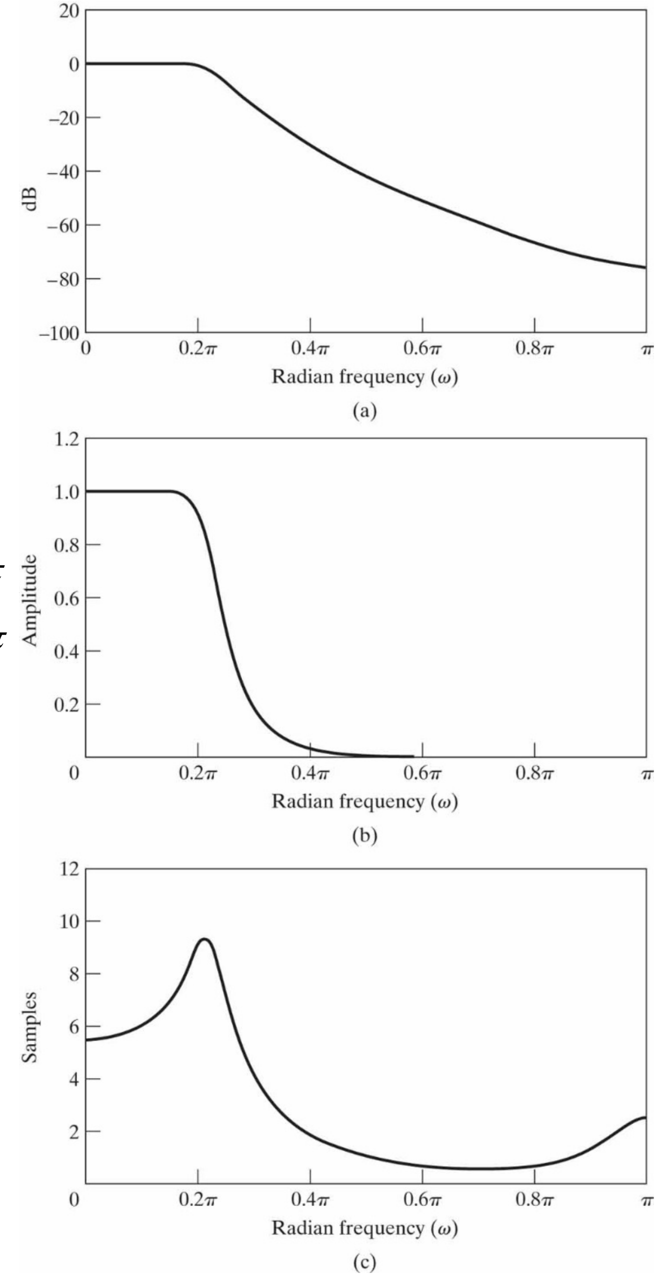
3° Apply the Bilinear Transformation to $H_c(s)$

4° Plot amplitude response of $H_c(z)$

by bilinear transform



by impulse invariance



Specs

$$0.89125 \le \left|H\left(e^{j\omega}\right)\right| \le 1 \qquad 0 \le |\omega| \le 0.2\pi$$

$$\left|H\left(e^{j\omega}\right)\right| \le 0.17783 \qquad 0.3\pi \le |\omega| \le \pi$$

Frequency response of 6th-order Butterworth filter
(a) Log magnitude in dB. (b) Magnitude. (c) Group delay.

IIR filter design: Concluding

① Filters can be designed using closed-form design formulas. Once the problem has been specified in terms appropriate for a given approximation (Butterworth, Chebychev, ..) the filter order meeting these specs can be computed and the coefficients can be obtained by straight-forward substitution into the design equations.

$\Longrightarrow$ Leads to non-iterative computer program for the design.

② Closed form design equations exist for LP, HP, BS, & BP IIR filters.

IIR filter design: Concluding comments

(3) Nonlinear phase response in all IIR filters.

(4) Most magnitude specifications can be met most efficiently with IIR designs.

(5) Can be made causal.

Outline

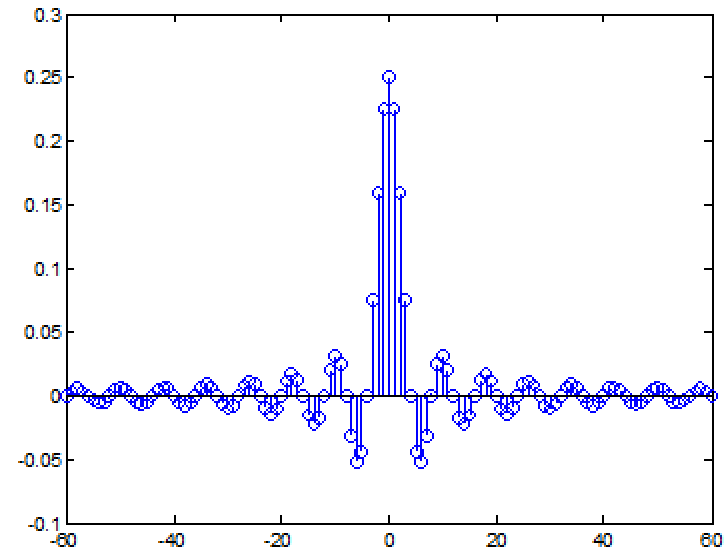
- Filter Design Process & Specifications
- FIR vs. IIR
- IIR Filter Design using CT filters:
  - Analog Filters
  - IIR Filter Design by Bilinear Transformation
  - IIR Filter Design by Impulse Invariance
- FIR Filter Design in DT
  - Design by Windowing
  - Generalized Linear Phase FIR (GLP-FIR)
- Kaiser Window FIR Design Method
- Optimum Approximation of FIR Filters

FIR filter design in DT
Low pass filter – as an example

- FIR filters are designed directly in DT
- Ideal low pass filter

$$H_{lp}(e^{j\omega}) = \begin{cases} 1, & |\omega| < \omega_c \\ 0, & \omega_c < \omega \le \pi \end{cases}$$

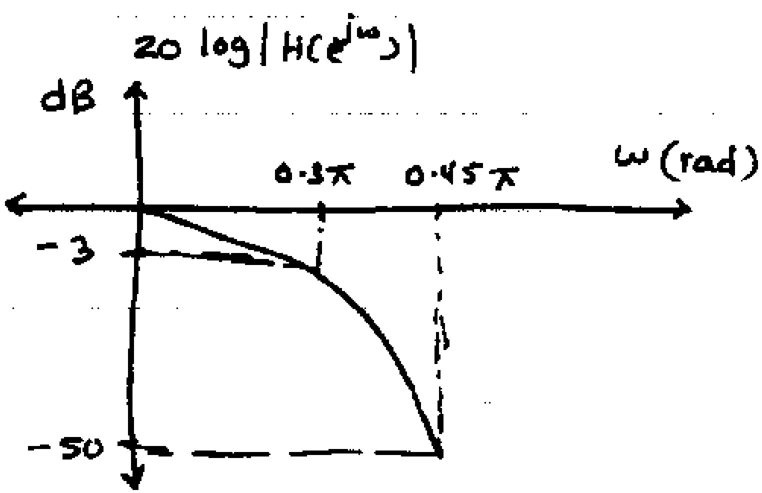
$$h[n] = \frac{\sin \omega_c n}{\pi n}, \quad -\infty < n < \infty$$

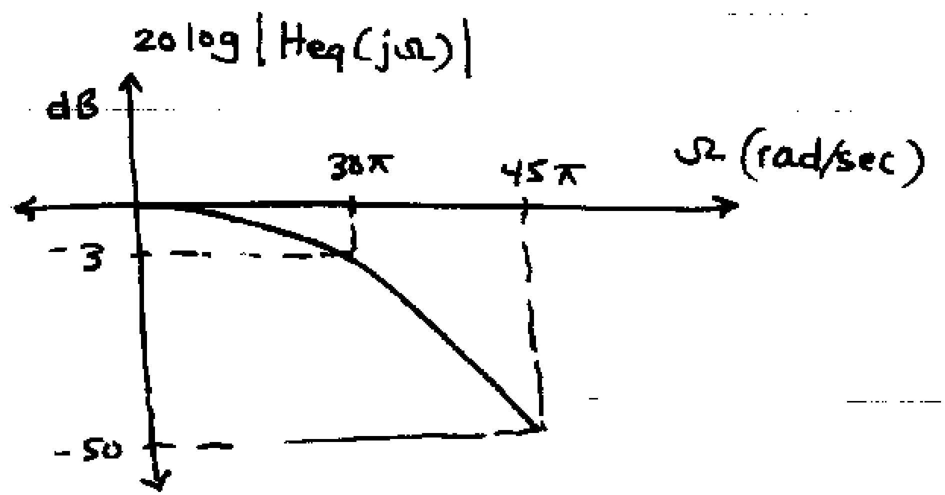


- FIR filter: if h[n] ideal then
  non-causal, infinite!

FIR filter design in A/D➔H(z)➔D/A systems

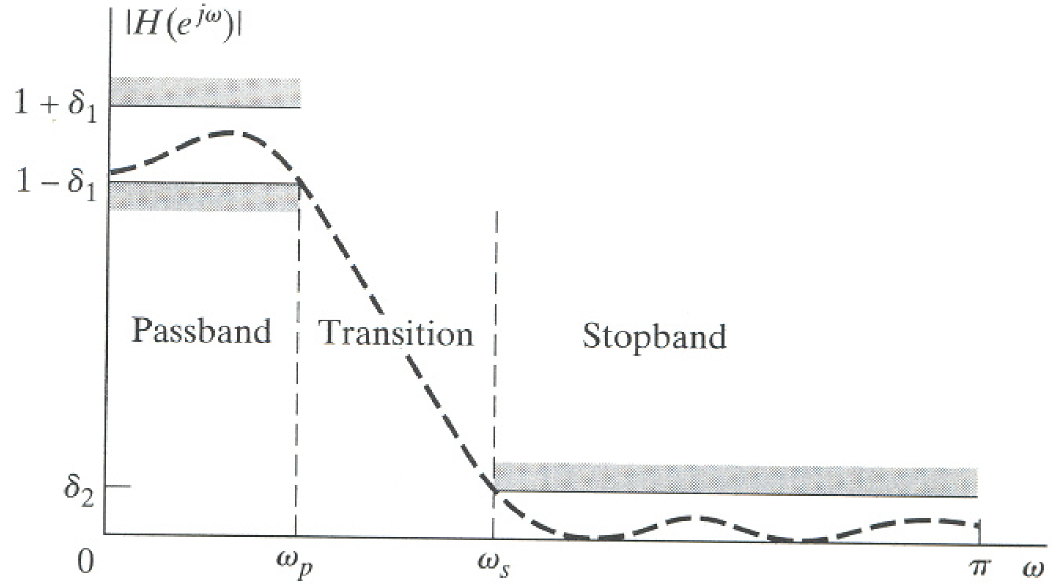
- If the FIR filter is to be used in an A/D➔H(z)➔D/A structure, the set of continuous specs must be first converted to digital specs before the design
- For analog critical frequencies, the corresponding digital specs using a sampling rate of T samples/sec are given by $\omega_i = T\Omega_i$





Desired lowpass filter specifications

- Passband distortion, as smallest as possible
- Stopband attenuation, as greatest as possible
- Width of transition band: as narrowest as possible



FIR filter design

- Design problem: the FIR system function

$$H(z) = \sum_{k=0}^{M} b_k z^{-k} \qquad h[n] = \begin{cases} b_n, & 0 \le n \le M \\ 0, & \text{otherwise} \end{cases}$$

- Start from impulse response directly

$$H(z) = h[0] + h[1]z^{-1} + \ldots + h[M]z^{-M}$$

Find
- the degree M and
- the filter coefficients $h[k]$

to approximate a desired frequency response

➔Objective: design FIR filter with H(z) close to ideal H(z), while maintaining a reasonable length/degree

Design by windowing

- Desired ideal frequency responses are often piecewise-constant with discontinuities at the boundaries between bands, resulting in non-causal and infinite impulse response extending from $-\infty$ to ∞, but

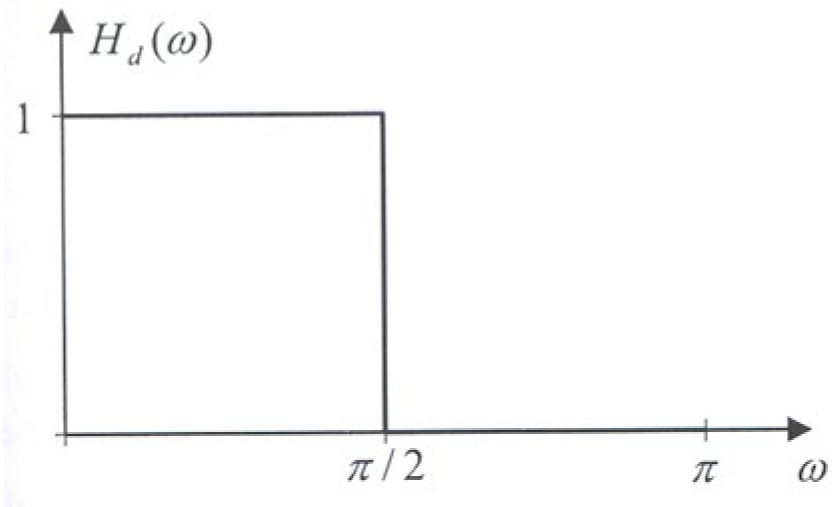
$$n \to \pm\infty, \qquad h_d[n] \to 0$$

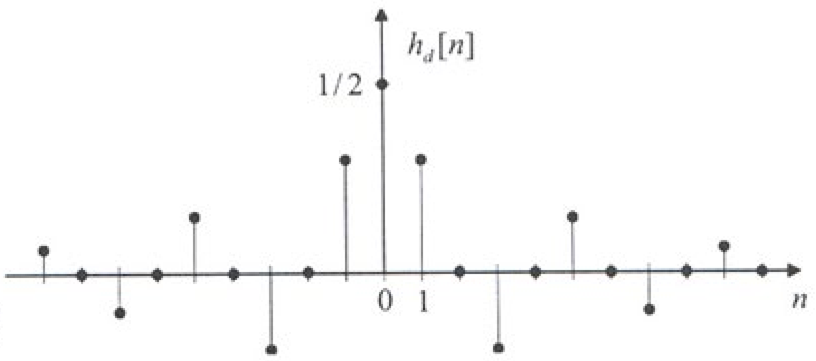
- So, the most straightforward method is to truncate the ideal impulse response by windowing and do time-shifting:

$$g[n] = \begin{cases} h_d[n], & |n| \le M \\ 0, & \text{otherwise} \end{cases}$$

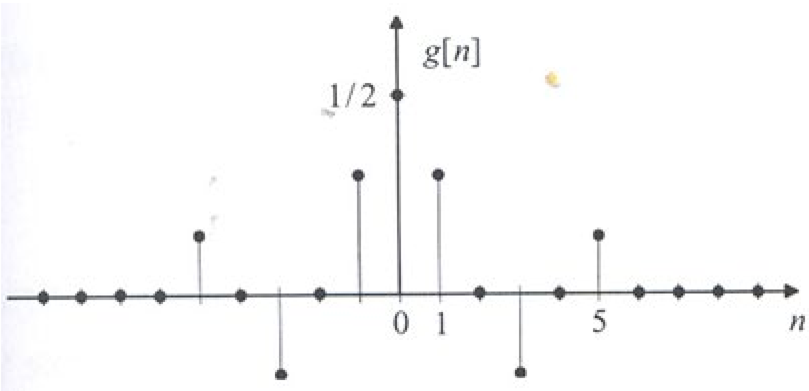
$$h[n] = g[n-M]$$

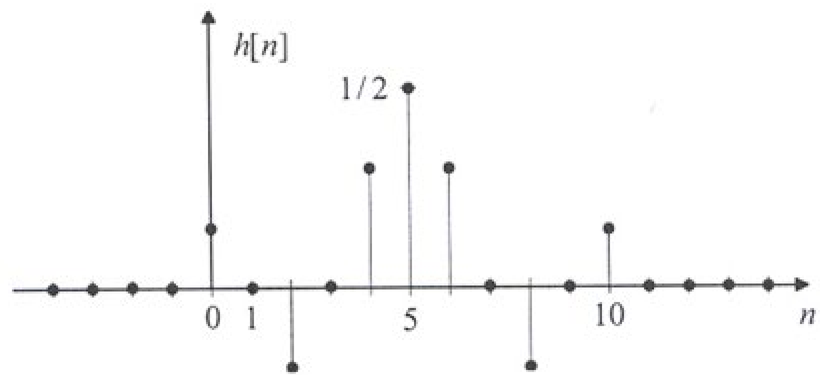
Design by windowing





Desired (s) frequency response and (b) impulse response





(a) Windowed impulse response g[n] and (b) shifted windowed impulse response h[n]

FIR Design by Windowing

- Simplest way of designing FIR filters
- Method is all discrete-time no continuous-time involved
- Start with ideal frequency response

$$H_d\left(e^{j\omega}\right) = \sum_{n=-\infty}^{\infty} h_d[n] e^{-j\omega n} \qquad h_d[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d\left(e^{j\omega}\right) e^{j\omega n} d\omega$$

- Choose ideal frequency response as desired response
- Most ideal impulse responses are of <u>infinite length</u>

- The easiest way to obtain a <u>causal</u> FIR filter from ideal is

$$h[n] = \begin{cases} h_d[n] & 0 \le n \le M \\ 0 & \text{else} \end{cases}$$

- More generally $\quad h[n] = h_d[n]w[n] \quad$ where $\quad w[n] = \begin{cases} 1 & 0 \le n \le M \\ 0 & \text{else} \end{cases}$

Design by rectangular window

$$h[n] = h_d[n]w[n]$$

- In general,

$$w[n] = \begin{cases} 1, & 0 \le n \le M \\ 0, & \text{otherwise} \end{cases}$$

- For simple truncation, the window is the rectangular window

- Convolution property:

$$H(e^{j\omega}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(e^{j\theta})H(e^{j(\omega-\theta)})d\theta$$

➔ Ideal (square) H will be ‘smeared’ as a result of convolution (Gibbs phenomenon)

Convolution process by truncation

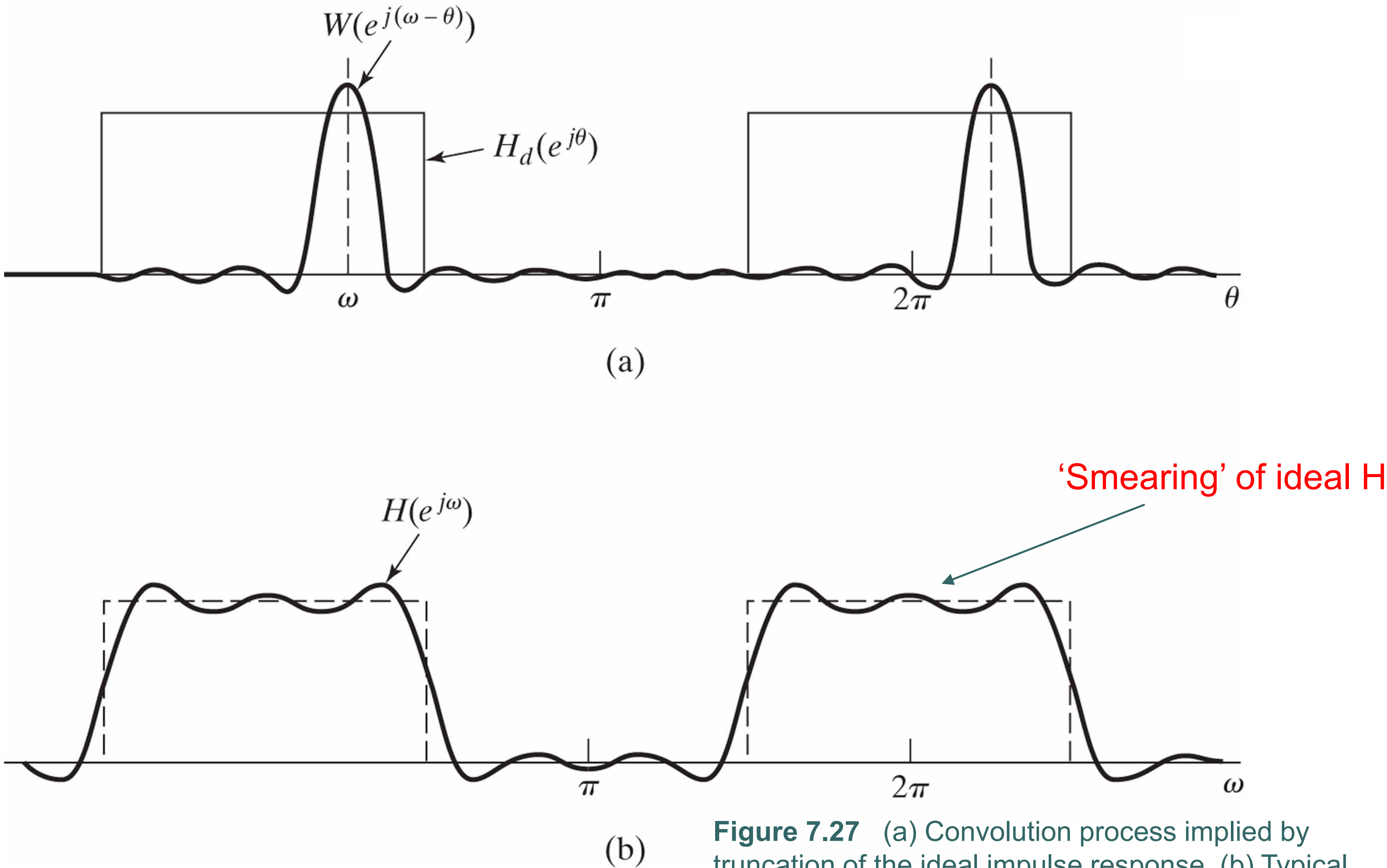


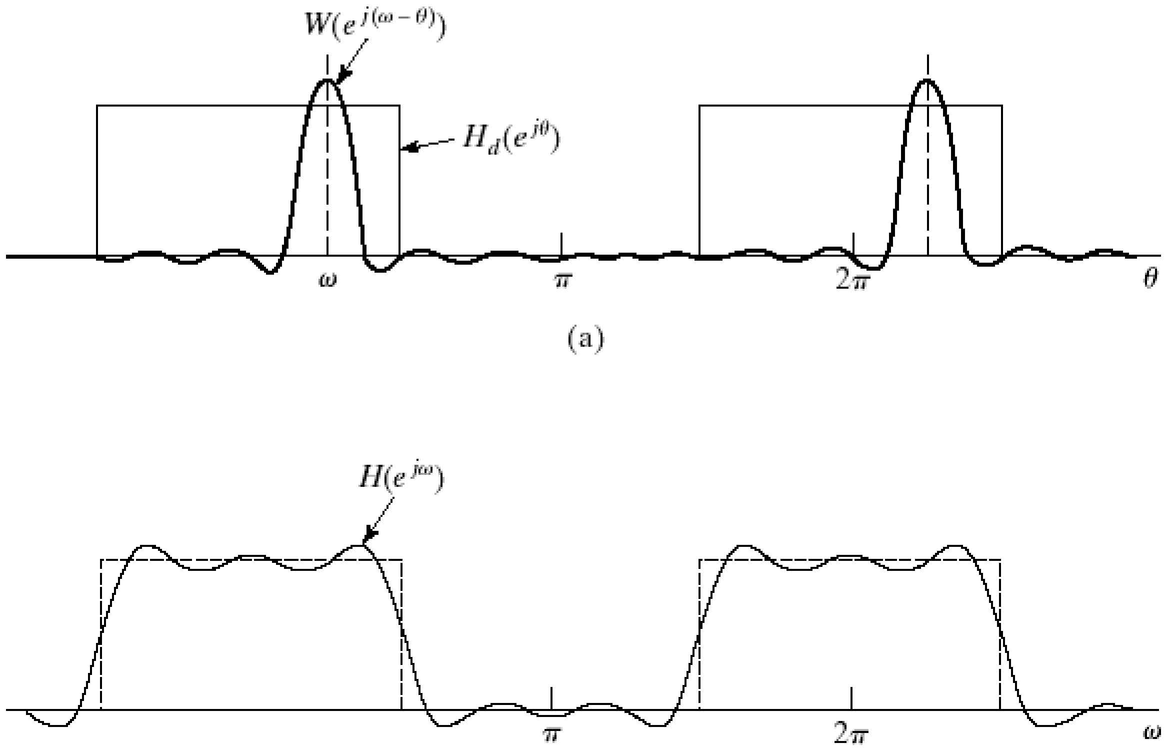
Figure 7.27 (a) Convolution process implied by truncation of the ideal impulse response. (b) Typical approximation resulting from windowing the ideal impulse response.

Windowing in Frequency Domain

- Windowed frequency response

$$H\left(e^{j\omega}\right) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d\left(e^{j\omega}\right)W\left(e^{j(\omega-\theta)}\right)d\theta$$

- The windowed version is smeared version of desired response



- If w[n]=1 for all n, then $W(e^{j\omega})$ is pulse train with 2π period

Requirements on the window

- Consider infinitely long window

$$w[n]=1,\quad -\infty<n<\infty \Leftrightarrow W(e^{j\omega})=\sum_{r=-\infty}^{\infty}2\pi\delta(\omega+2\pi r)$$

$$\Rightarrow H(e^{j\omega})=H_d(e^{j\omega})$$

➔ If *w[n]* is chosen so that *W(e^jw)* is concentrated in a narrow band around w=0, then *H(e^jw)* will ‘look like’ *Hd(e^jw),* except at abrupt changes

➔ However, doing so increases the length of *w[n]*

- These are conflicting requirements:
  - *W(e^jw)* approximates an impulse to faithfully reproduce the desired frequency response
  - $w[n]$ as short as possible in duration (the order of the filter) to minimize computation in the implementation of the filter

Properties of Windows

- Prefer windows that concentrate around DC in frequency
  - Less smearing, closer approximation
- Prefer window that has minimal span in time
  - Less coefficient in designed filter, computationally efficient

➔ So we want concentration in time and in frequency
  - Contradictory requirements
- Example:

Rectangular window

$$W\left(e^{j\omega}\right)=\sum_{n=0}^{M} e^{-j\omega n}=\frac{1-e^{-j\omega(M+1)}}{1-e^{-j\omega}}$$

$$=e^{-j\omega M/2}\frac{\sin[\omega(M+1)/2]}{\sin[\omega/2]}$$

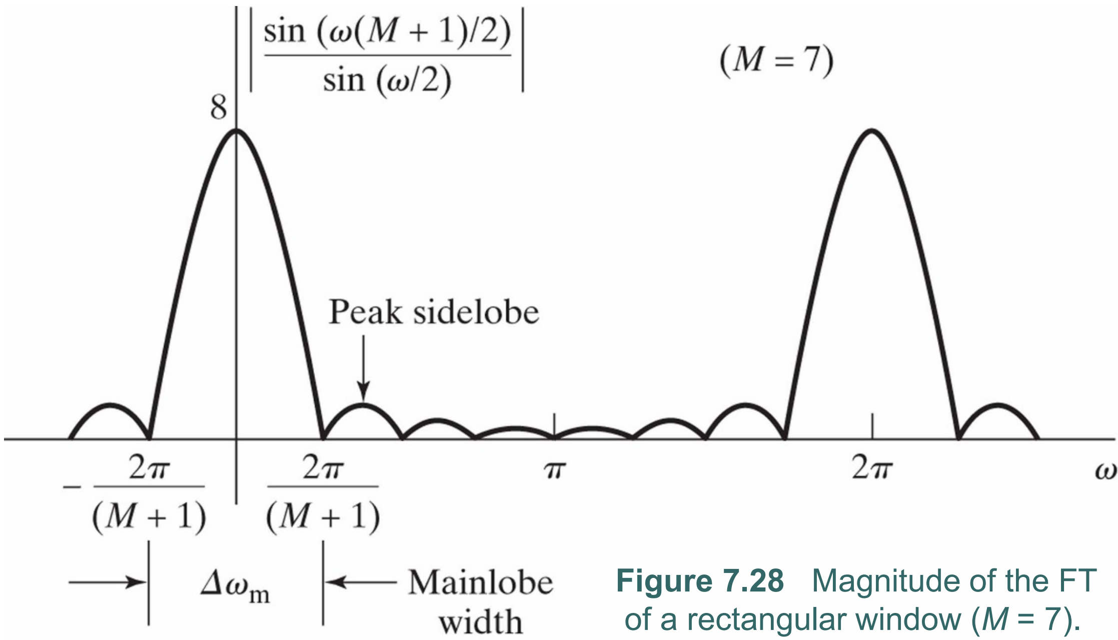
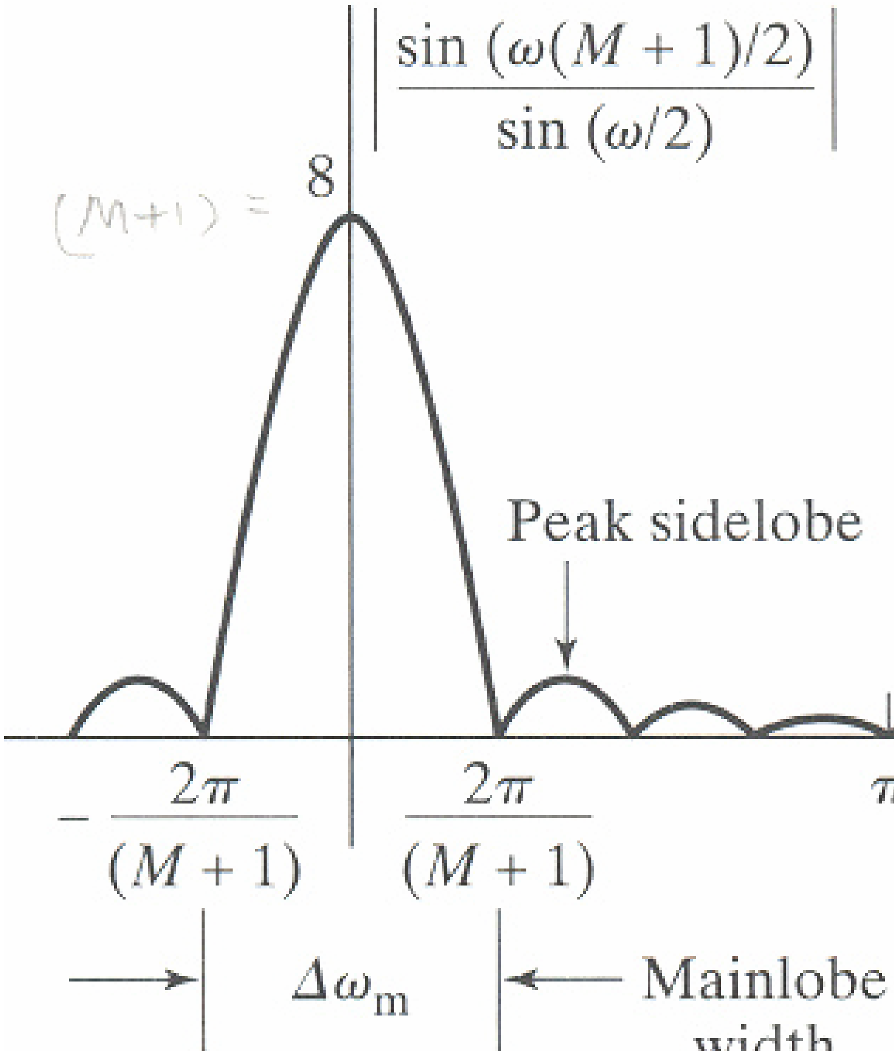
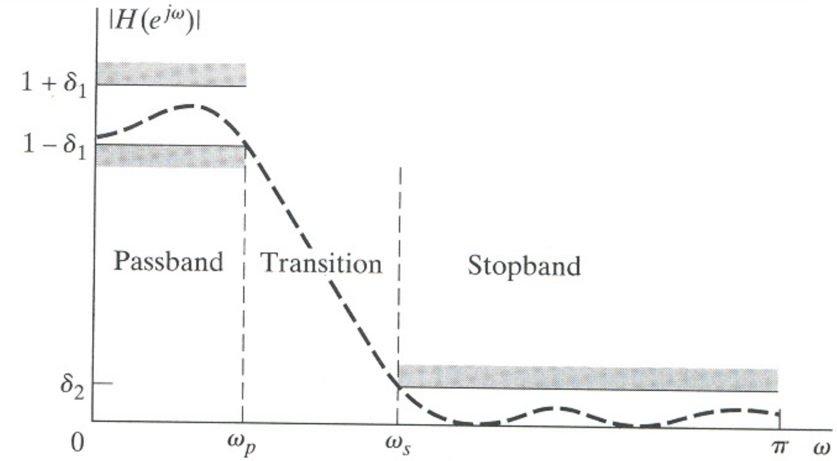


Figure 7.28 Magnitude of the FT of a rectangular window (M = 7).



(M = 7)



constant $\Delta\omega_m = 4\pi / M + 1$

$\Rightarrow M\uparrow$, widthes of main & sidelobe $\downarrow$, but their peak amplitudes $\uparrow$

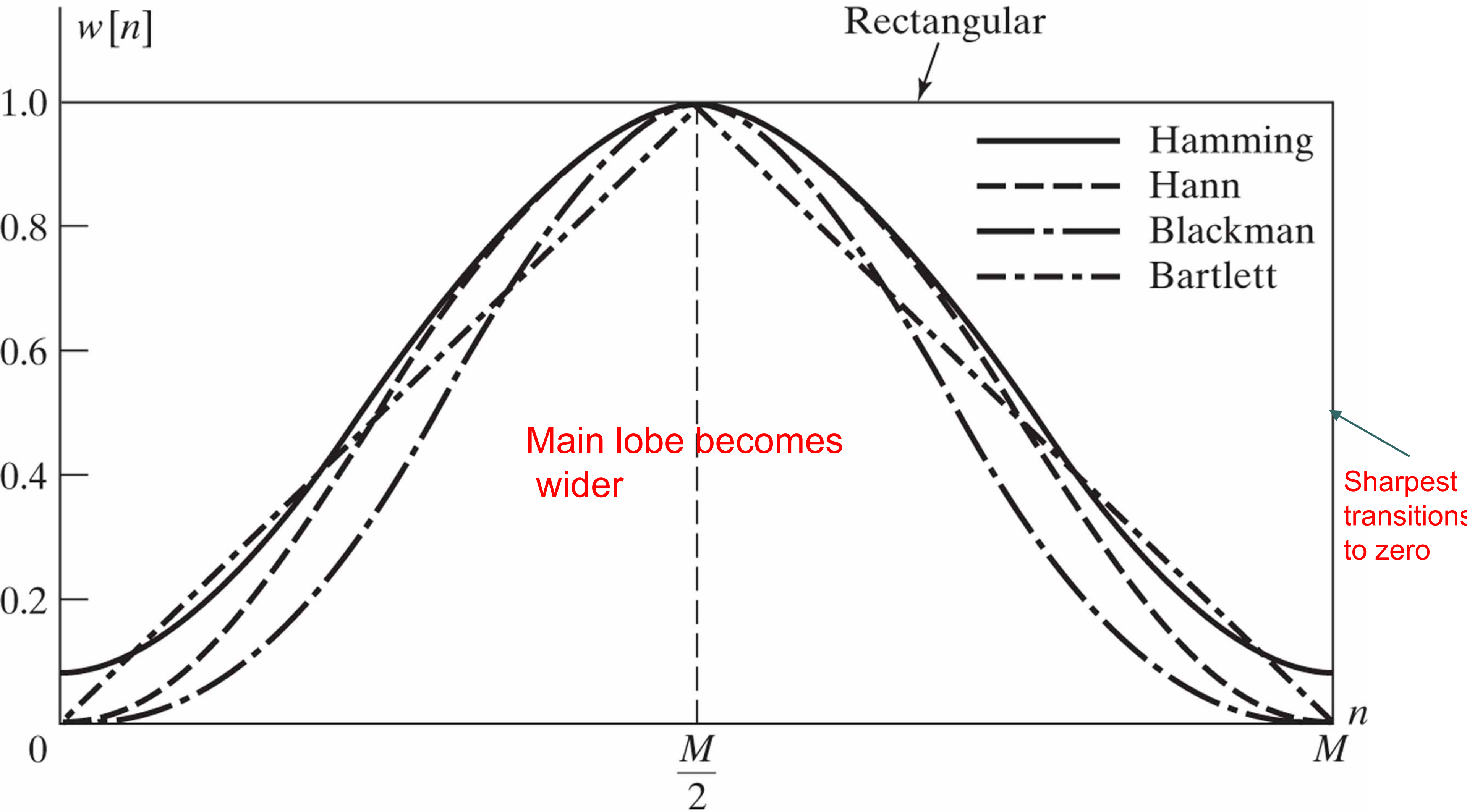
$\Rightarrow M\uparrow$, the ratio of the height of main & side lobes remains constant

Properties of Design by Windowing

- Windows **length** and approximation accuracy
  - Larger length increase, lower transition width
  - Stopband gain is insensitive to length
  - Transition width = main lobe
- Window **shape** and Gibbs phenomenon
  - Rectangular window has the sharpest transition to zero
  - The large the side lobe, the low the stopband attenuation !
  - The side lobe height can be reduced by tapering the window slower to zero

    ➔This leads to a family of windows

- Linear phase considerations
  - Symmetric h[n]
- Redesign needed



To meet the requirement of a FIR filter, choose

- Shape of the window
- Duration of the window

➔ A 'trial and error' method !

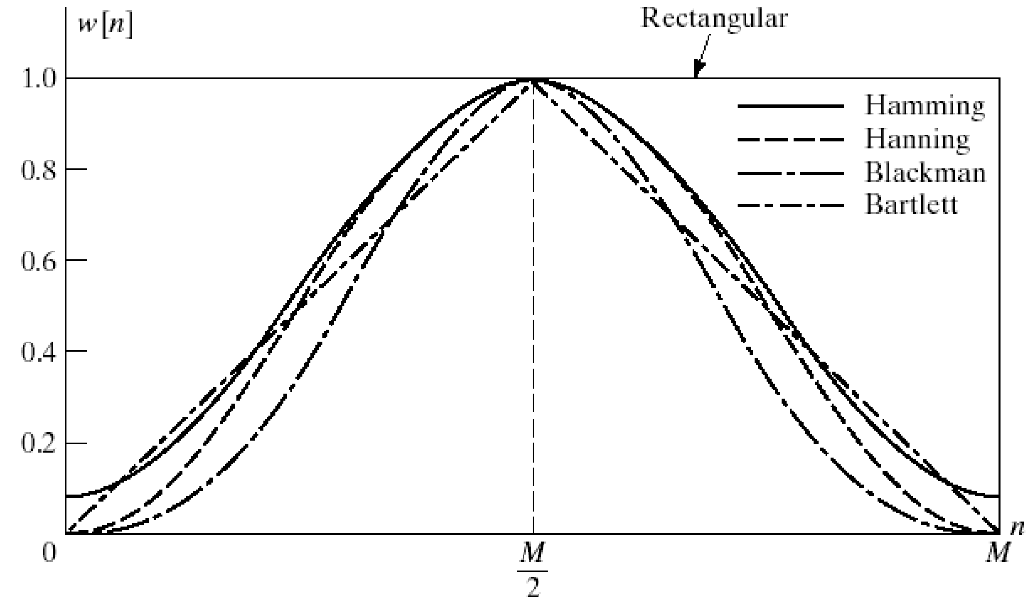
Not optimal but simple and useful

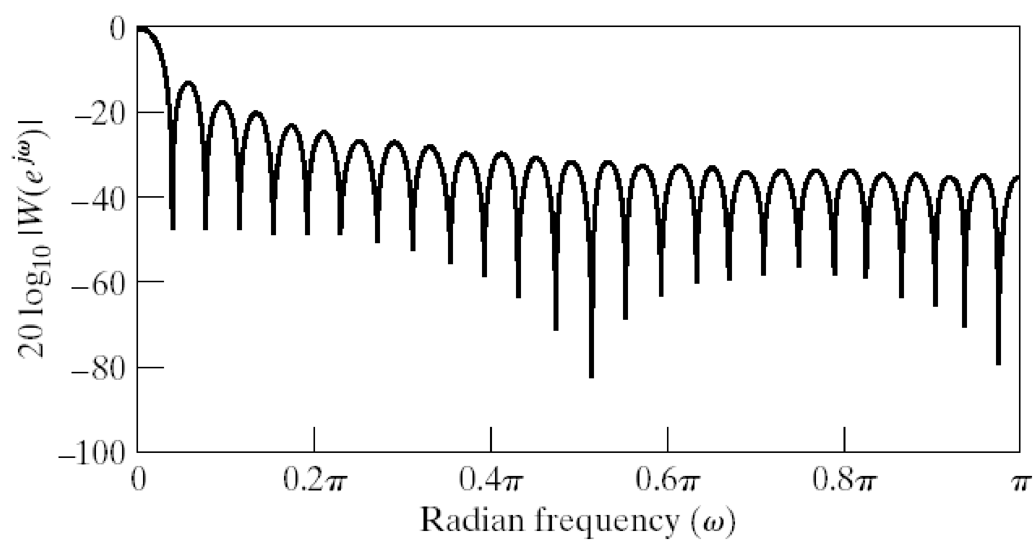
Window	Formula	Mainlobe Width	Peak Sidelobe Amplitude
Rectangular	$w(n)=\begin{cases}1, & 0\le n\le M\\ 0, & o.w.\end{cases}$	$\dfrac{4\pi}{M+1}$	–13dB
Bartlett (triangular)	$w(n)=\begin{cases}\dfrac{2n}{M}, & 0\le n\le \dfrac{M}{2}\\ 2-\dfrac{2n}{M}, & \dfrac{M}{2}\le n\le M\\ 0, & o.w.\end{cases}$	$\dfrac{8\pi}{M}$	–25dB
Hanning	$w(n)=\begin{cases}\left[1-\cos\left(\dfrac{2\pi n}{M}\right)\right]/2, & 0\le n\le M\\ 0, & o.w.\end{cases}$	$\dfrac{8\pi}{M}$	–31dB
Hamming	$w(n)=\begin{cases}0.54-0.46\cos\left(\dfrac{2\pi n}{M}\right), & 0\le n\le M\\ 0, & o.w.\end{cases}$	$\dfrac{8\pi}{M}$	–41dB
Blackman	$w(n)=\begin{cases}0.42-0.5\cos\left(\dfrac{2\pi n}{M}\right)+ 0.08\cos\left(\dfrac{4\pi n}{M}\right), & 0\le n\le M\\ 0, & o.w.\end{cases}$	$\dfrac{12\pi}{M}$	–57dB

Rectangular Window

- Narrowest main lob
  - $4\pi/(M+1)$
  - Sharpest transitions at discontinuities in frequency
- Large side lobs
  - -13 dB below main peak
  - ➔Large oscillation around discontinuities
  - ➔Stopband attenuation not high
- Simplest window possible

$$w[n] = \begin{cases} 1 & 0 \le n \le M \\ 0 & \text{else} \end{cases}$$

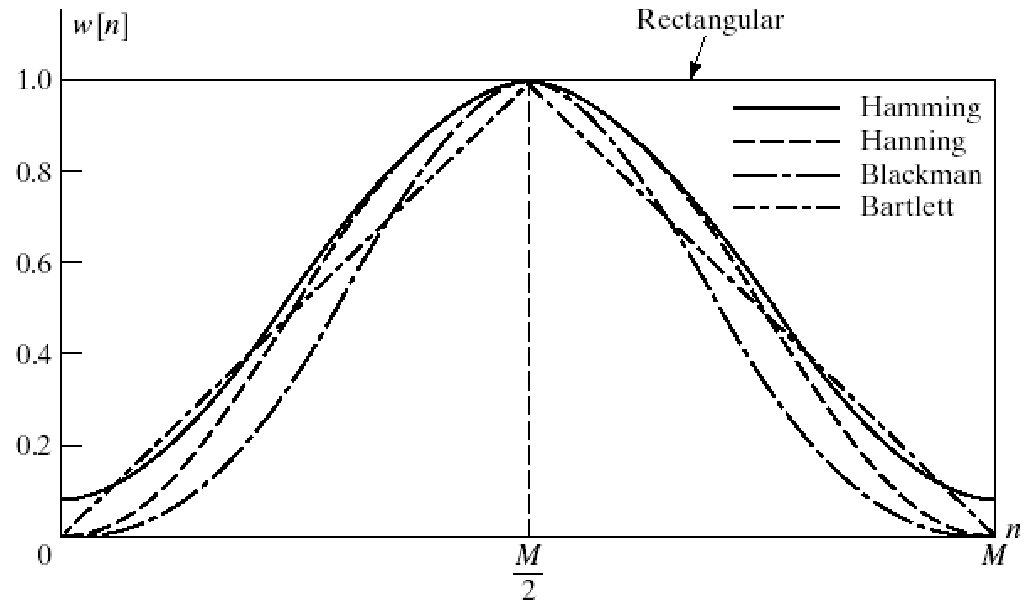


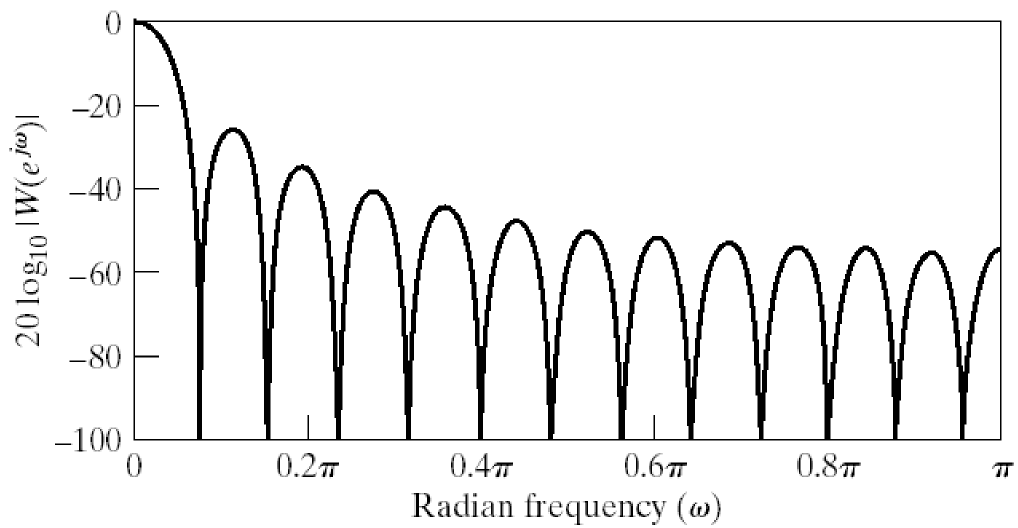


Bartlett (Triangular) Window

- Medium main lob
  - $8\pi/M$
- Side lobs
  - -25 dB
- Hamming window performs better
- Simple equation

$$w[n] = \begin{cases} 2n/M & 0 \le n \le M/2 \\ 2 - 2n/M & M/2 \le n \le M \\ 0 & \text{else} \end{cases}$$





Hanning Window

- Medium main lob
  - $8\pi/M$
- Side lobs
  - -31 dB
- Hamming window performs better
- Same complexity as Hamming

$$w[n] = \begin{cases} \frac{1}{2}\left[1 - \cos\left(\frac{2\pi n}{M}\right)\right] & 0 \le n \le M \\ 0 & \text{else} \end{cases}$$

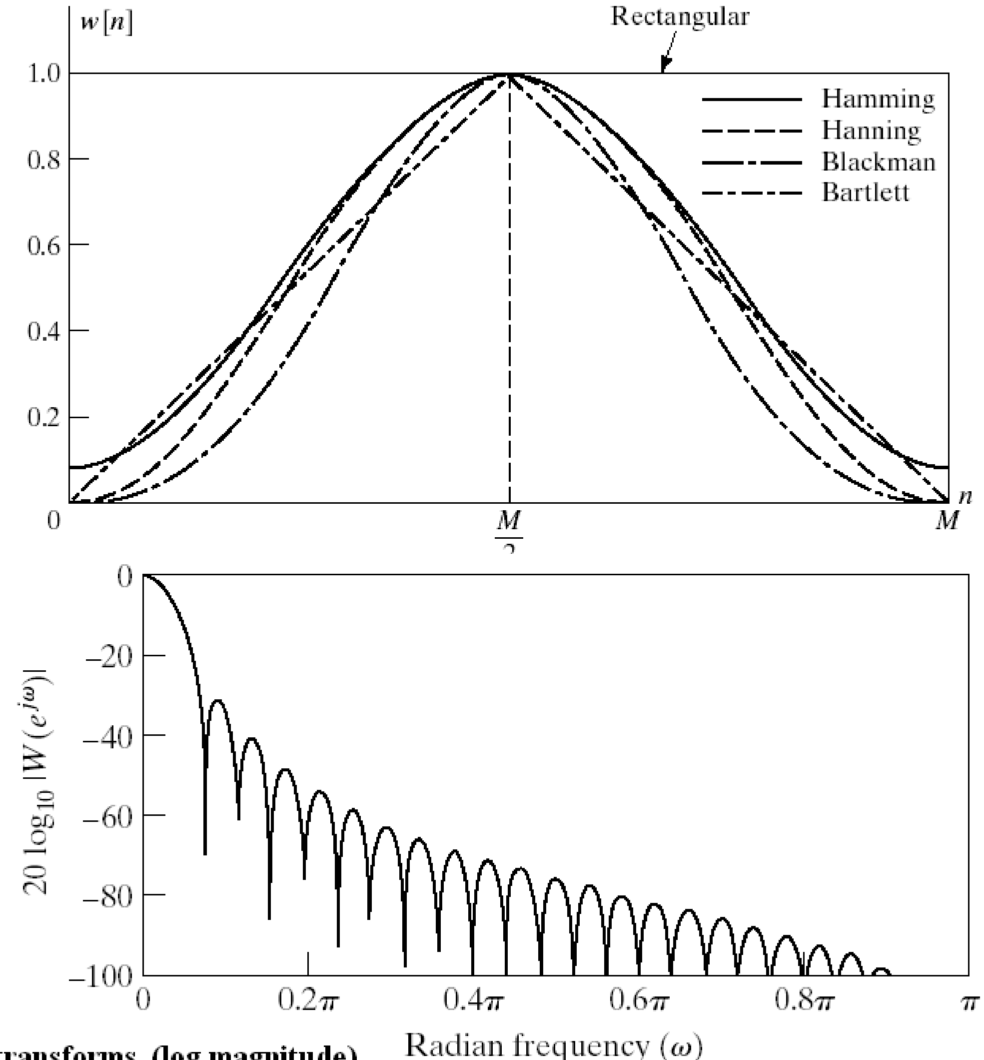
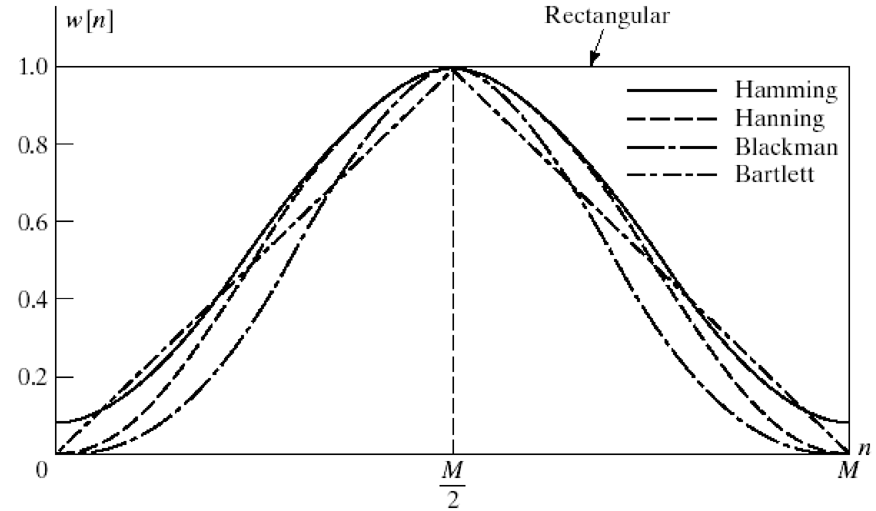


Figure 7.30 Fourier transforms (log magnitude) of windows of figure 7.29 with M=50. (a)Rectangular. (b) Barlett.

Hamming Window

- Medium main lob
  - $8\pi/M$
- Good side lobs
  - -41 dB
  - ➔ High stopband attenuation
  - Smallest transition band (small M)
- Simpler than Blackman

$$w[n] = \begin{cases} 0.54 - 0.46\cos\left(\frac{2\pi n}{M}\right) & 0 \le n \le M \\ 0 & \text{else} \end{cases}$$



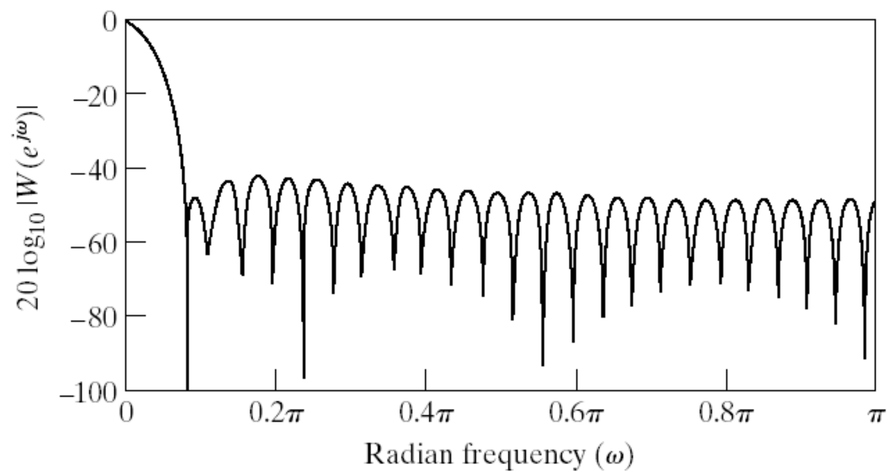
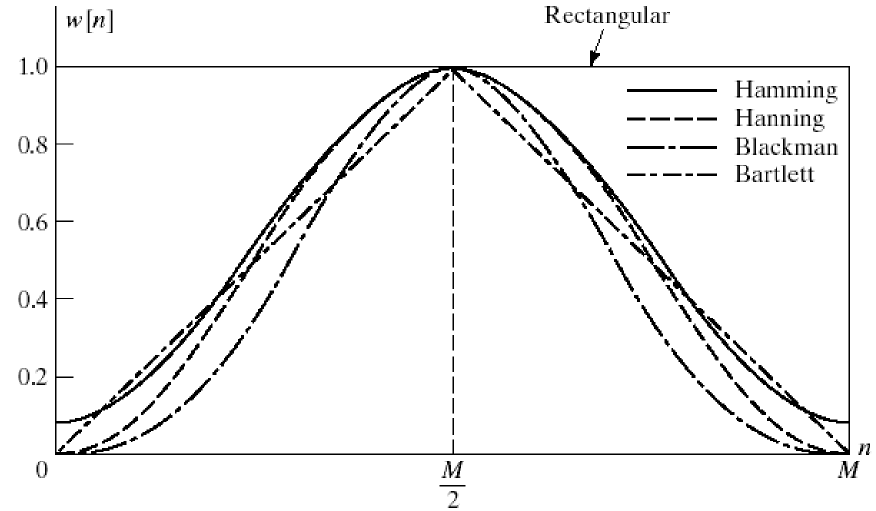


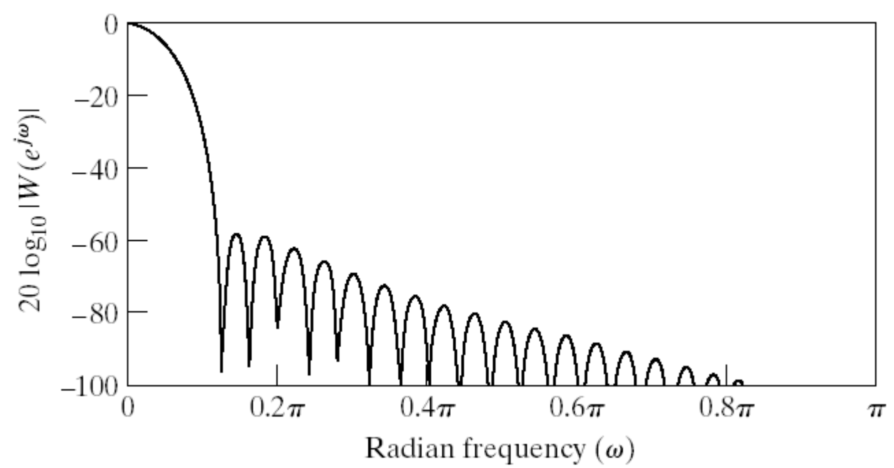
Figure 7.30 Fourier transforms (log magnitude) of windows of figure 7.29 with M=50. (a)Rectangular. (b) Barlett.

Blackman Window



- Large main lob
  - $12\pi/M$
- Very good side lobs
  - -57 dB
- Complex equation

$$w[n] = \begin{cases} 0.42 - 0.5\cos\left(\frac{2\pi n}{M}\right) + 0.08\cos\left(\frac{4\pi n}{M}\right) & 0 \le n \le M \\ 0 & \text{else} \end{cases}$$



Outline

- Filter Design Process & Specifications
- FIR vs. IIR
- IIR Filter Design:
  - Analog Filters
  - IIR Filter Design by Bilinear Transformation
  - IIR Filter Design by Impulse Invariance
- FIR Filter Design
  - Design by Windowing
  - Generalized Linear Phase FIR (GLP-FIR)
- Kaiser Window FIR Design Method
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Generalized Linear Phase FIR (GLP-FIR)

- Linear phase filters

$$H(e^{j\omega}) = |H(e^{j\omega})| e^{-j\omega\alpha}$$

- Generalized linear phase filters

$$H(e^{j\omega}) = A(e^{j\omega}) e^{-j\omega\alpha + j\beta}$$

$A(e^{j\omega})$ is a real function of ω,

α and β are real constants

Generalized Linear Phase FIR (GLP-FIR)

- If the system is causal and generalized linear-phase

$$h[M-n]=\pm h[n]$$

- Since h[n]=0 for n<0 we get

$$h[n]=0 \qquad n<0 \quad \text{and} \quad n>M$$

- **An FIR impulse response of length M+1 is generalized linear phase if they are symmetric (**M is an even integer)

➔Causal FIR system with generalized linear phase are symmetric:

$$\text{Symmetry:} \qquad h[M-n]=h[n] \qquad \text{n}=0,1,\ldots,\text{M} \quad (\text{type I or III})$$

$$\text{Antisymmetry:}\; h[M-n]=-h[n] \qquad \text{n}=0,1,\ldots,\text{M} \quad (\text{type II or IV})$$

Incorporation of Generalized Linear Phase

- Windows are designed with linear phase in mind
  - Symmetric around M/2

$$w[n] = \begin{cases} w[M-n] & 0 \le n \le M \\ 0 & \text{else} \end{cases}$$

- So their Fourier transform are of the form

$$W\left(e^{j\omega}\right) = W_e\left(e^{j\omega}\right)e^{-j\omega M/2} \qquad \text{where } W_e\left(e^{j\omega}\right) \text{ is a real and even}$$

- Will keep symmetry properties of the desired impulse response
- Assume symmetric desired response

$$H_d\left(e^{j\omega}\right) = H_e\left(e^{j\omega}\right)e^{-j\omega M/2}$$

- With symmetric window

$$A_e\left(e^{j\omega}\right) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_e\left(e^{j\theta}\right)W\left(e^{j(\omega-\theta)}\right)d\theta$$

  - Periodic convolution of real functions

Linear-Phase Low pass filter

- Desired frequency response

$$H_{lp}\left(e^{j\omega}\right) = \begin{cases} e^{-j\omega M/2} & |\omega| < \omega_c \\ 0 & \omega_c < |\omega| \leq \pi \end{cases}$$

- Corresponding impulse response

$$h_{lp}[n] = \frac{\sin[\omega_c(n - M/2)]}{\pi(n - M/2)}$$

- Desired response is even symmetric, use symmetric window

$$h[n] = \frac{\sin[\omega_c(n - M/2)]}{\pi(n - M/2)} w[n]$$

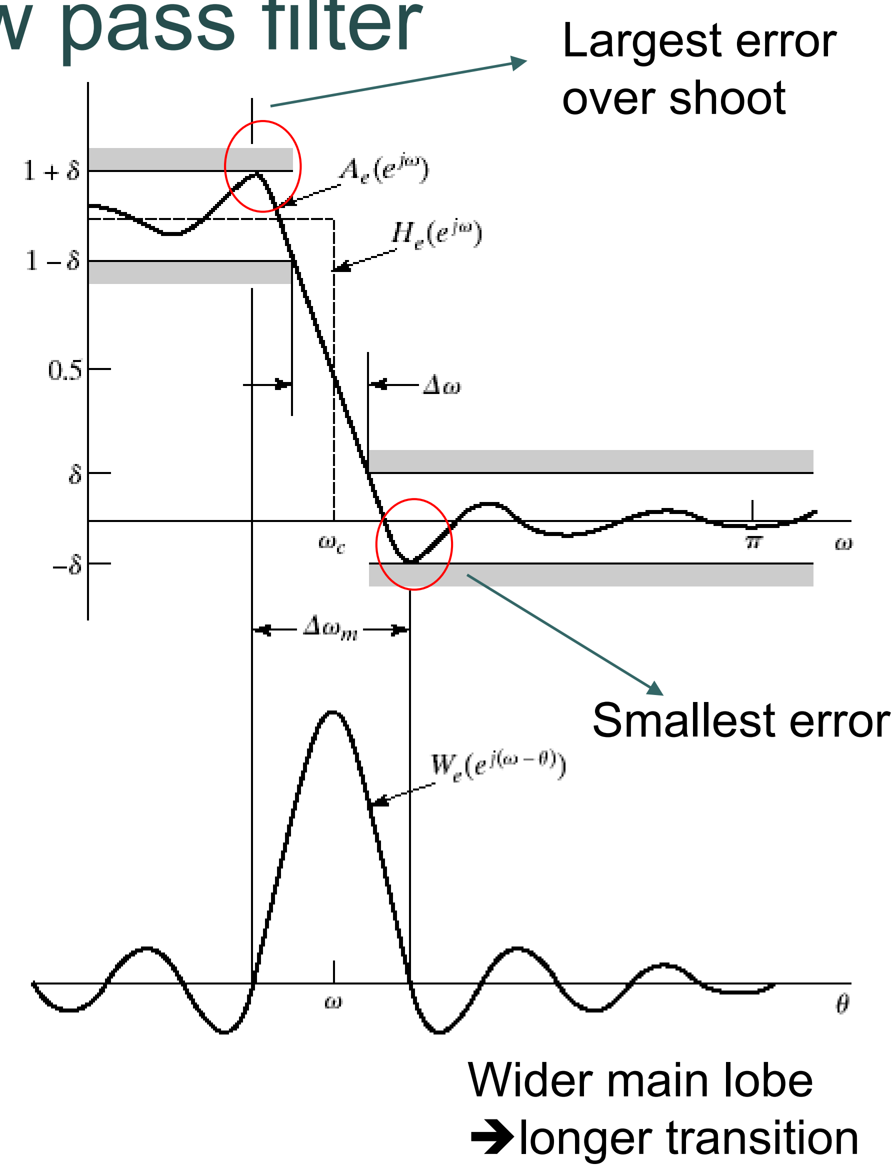

Figure 7.31 Illustration of type of approximation obtained at a discontinuity of the ideal frequency response.

FIR filter design: Concluding comments

1. FIR filters are always stable
2. Linear phase easily achievable
3. we can shift to make causal
4. Closed form equations do not exist for FIR filters. Though it can be easily designed using windows (sometimes have to iterate to achieve specs.)
5. An optimal design can be obtained (using optimal approximation techniques such as Minimax designs). These techniques are complex as compared to using windows.

$$\min_{\{h_e[n]:0\le n\le L\}}\left(\max_{\omega\in F}|E(\omega)|\right)$$

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Key parameters in FIR filter design

- To meet the requirement of FIR filter, choose
  - Shape of the window
  - Duration of the window
- Trail and error is not a satisfactory method to design filters

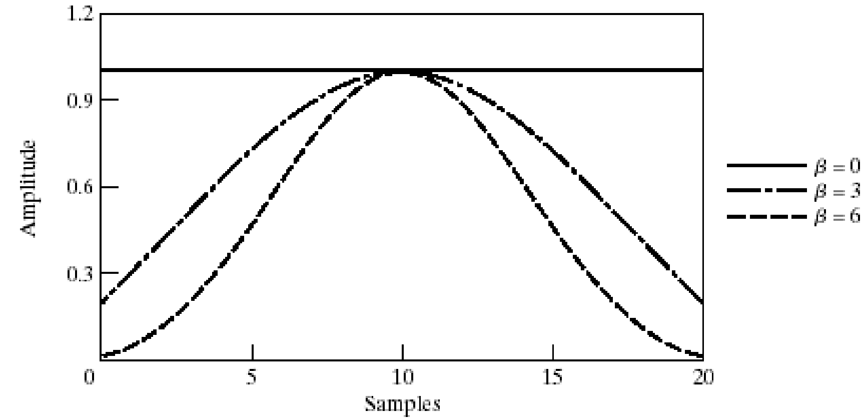
→ a simple formalization of the window method by Kaiser

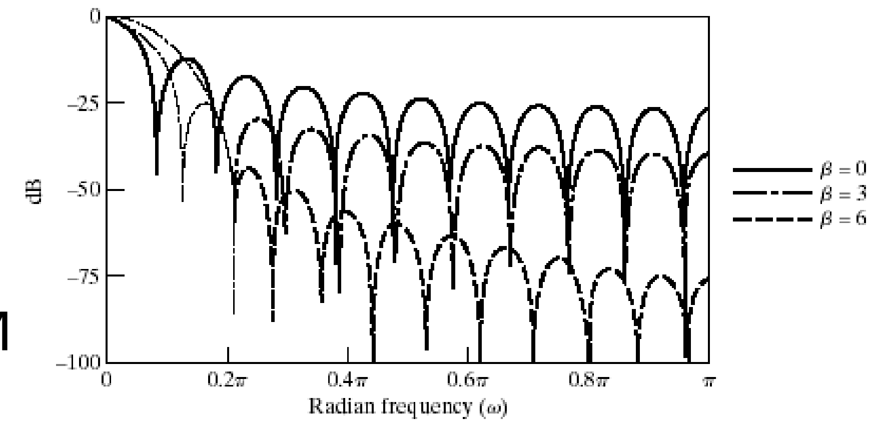
Kaiser Window Filter Design Method

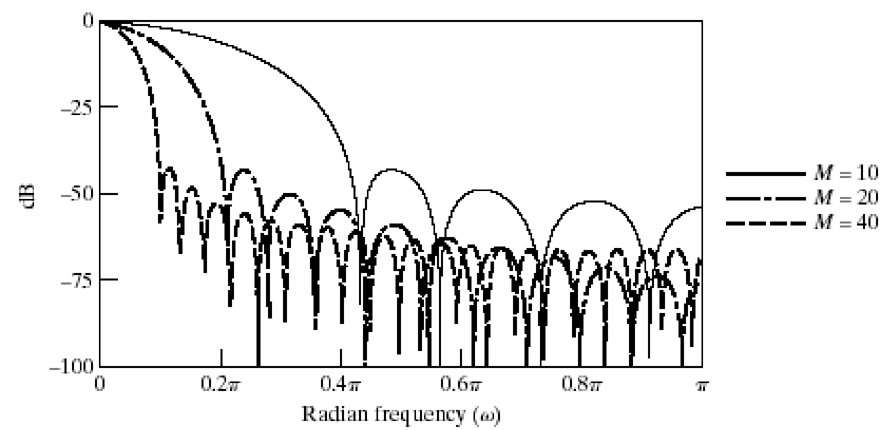
- Parameterized equation forming a set of windows
  - Parameter to change main-lob width and side-lob area trade-off

$$w[n] = \begin{cases} \dfrac{I_0\left[\beta\sqrt{1-\left(\dfrac{n-M/2}{M/2}\right)^2}\right]}{I_0(\beta)} & 0 \le n \le M \\ 0 & \text{else} \end{cases}$$

  - $I_0(.)$ represents zeroth-order modified Bessel function of 1st kind







Determining Kaiser Window Parameters

- Given filter specifications Kaiser developed empirical equations
  - Given the peak approximation error δ or in dB as $A=-20\log_{10}\delta$
  - and transition band width $\Delta\omega = \omega_s - \omega_p$
- The shape parameter β should be

$$\beta = \begin{cases} 0.1102(A-8.7) & A > 50 \\ 0.5842(A-21)^{0.4} + 0.07886(A-21) & 21 \le A \le 50 \\ 0 & A < 21 \end{cases}$$

- The filter order M is determined approximately by

$$M = \frac{A-8}{2.285\Delta\omega}$$

Example 7.6: Kaiser Window Design of a Low pass Filter

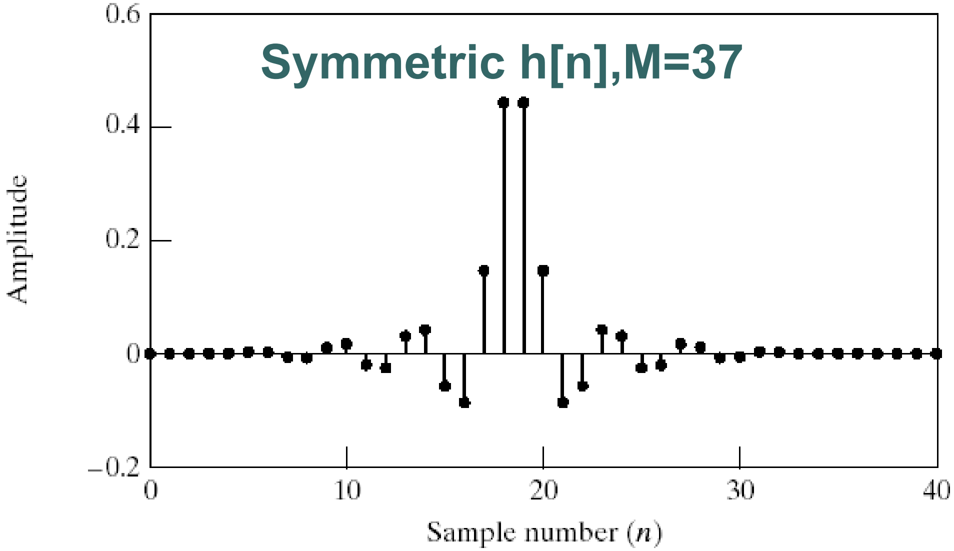
- Specifications $\omega_p = 0.4\pi, \omega_p = 0.6\pi, \delta_1 = 0.01, \delta_2 = 0.001$
- Window design methods assume $\delta_1 = \delta_2 = 0.001$
- Determine cut-off frequency
  - Due to the symmetry we can choose it to be $\omega_c = 0.5\pi$
- Compute

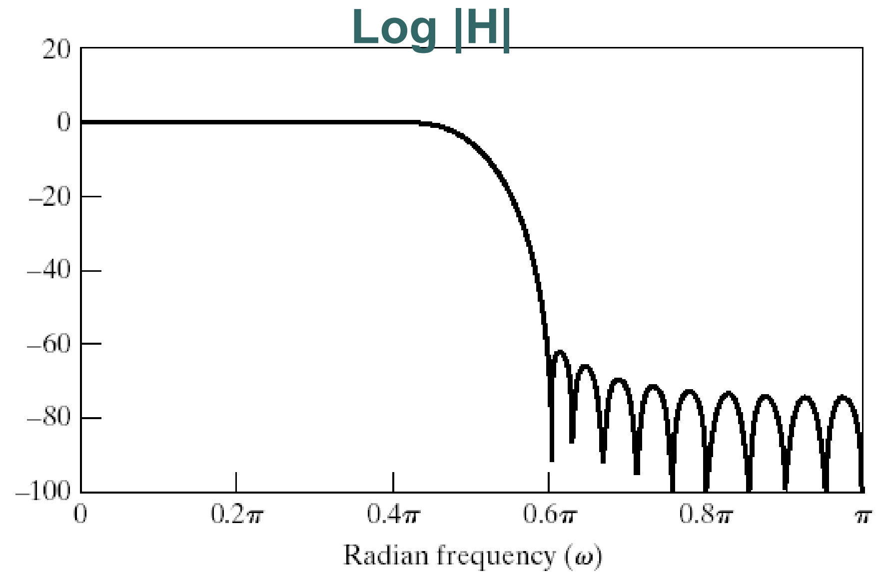
  $$\Delta\omega = \omega_s - \omega_p = 0.2\pi \qquad A = -20\log_{10}\delta = 60$$
- And Kaiser window parameters

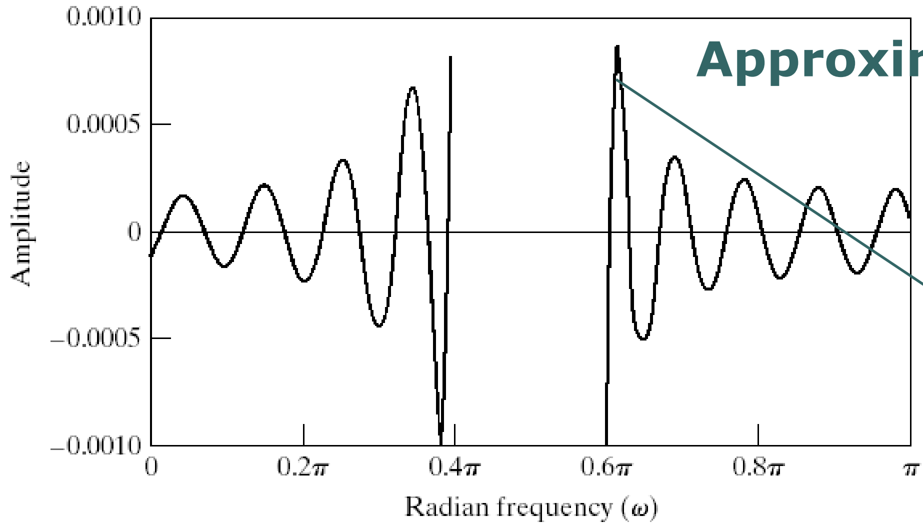
  $$\beta = 5.653 \qquad M = 37$$
- Then the impulse response is given as

$$h[n] = \begin{cases} \dfrac{\sin[0.5\pi(n-18.5)]}{\pi(n-18.5)} \dfrac{I_0\left[5.653\sqrt{1-\left(\dfrac{n-18.5}{18.5}\right)^2}\right]}{I_0(5.653)} & 0 \le n \le M \\ 0 & \text{else} \end{cases}$$

Example Cont'd







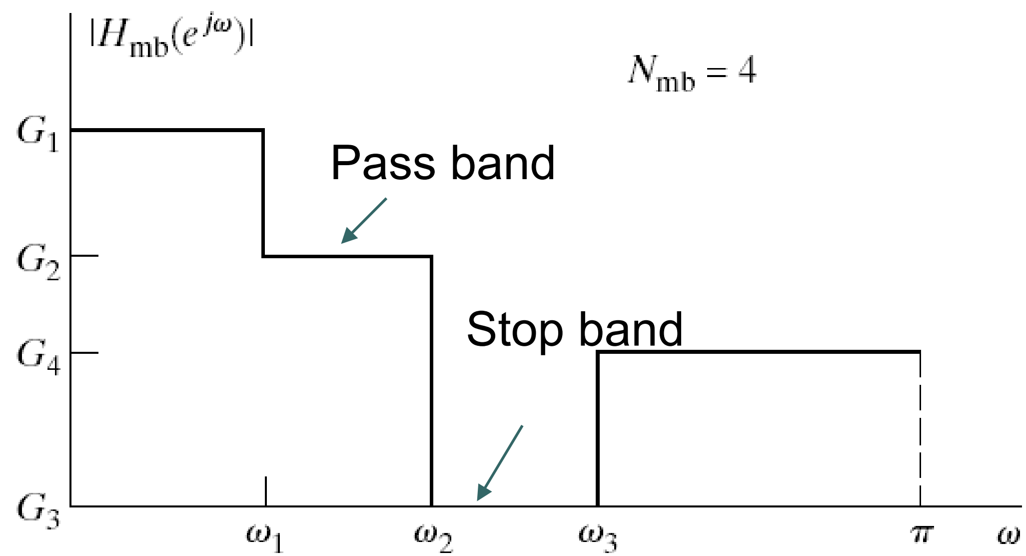
Large error when close to discontinuity

General Frequency Selective Filters

- A general multiband impulse response can be written as

$$h_{mb}[n] = \sum_{k=1}^{N_{mb}} (G_k - G_{k+1}) \frac{\sin \omega_k (n - M/2)}{\pi (n - M/2)}$$

- Window methods can be applied to multiband filters
- Example multiband frequency response
  - Special cases of
    - Bandpass
    - Highpass
    - Bandstop



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Optimum Filter Design

- Filter design by windows is simple but not optimal
  - Like to design filters with minimal length
- Optimality Criterion
  - Window design with rectangular filter is optimal in one sense
    - Minimizes the mean-squared approximation error to desired response
    - But causes large error around discontinuities

$$h[n] = \begin{cases} h_d[n] & 0 \le n \le M \\ 0 & \text{else} \end{cases} \qquad \varepsilon^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| H_d\left(e^{j\omega}\right) - H\left(e^{j\omega}\right) \right|^2 d\omega$$

  - Alternative criteria can give better results
    - Minimax: Minimize maximum error
    - Frequency-weighted error
- Most popular method: Parks-McClellan Algorithm
  - Reformulates filter design problem as function approximation

Function Approximation

- Consider the design of type I FIR filter
  - Assume zero-phase for simplicity
  - Can delay by sufficient amount to make causal

$$h_e[n] = h_e[-n] \longrightarrow A_e\left(e^{j\omega}\right) = \sum_{n=-L}^{L} h_e[n] e^{-j\omega n}$$

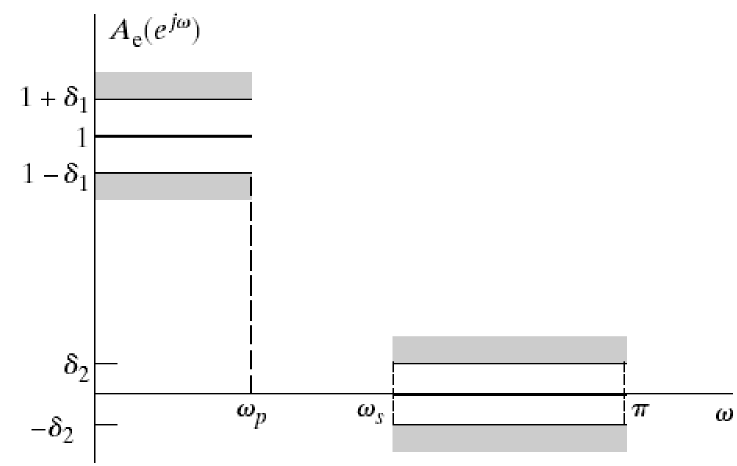
  - Assume L=M/2 an integer

$$A_e\left(e^{j\omega}\right) = h_e[0] + \sum_{n=1}^{L} 2h_e[n]\cos(\omega n)$$

- After delaying the resulting impulse response

$$h[n] = h_e[n - M/2] = h[M - n] \longrightarrow H\left(e^{j\omega}\right) = A_e\left(e^{j\omega}\right)e^{-j\omega M/2}$$

- Goal is to approximate a desired response with $A_e\left(e^{j\omega}\right)$
- Example approximation mask
  - Low-pass filter



Polynomial Approximation

- Using Chebyshev polynomials

$$\cos(\omega n) = T_n(\cos\omega) \quad \text{where} \quad T_n(x) = \cos(n\cos^{-1}x)$$

- Express the following as a sum of powers

$$A_e(e^{j\omega}) = h_e[0] + \sum_{n=1}^{L} 2h_e[n]\cos(\omega n) = \sum_{k=0}^{L} a_k(\cos\omega)^k$$

- Can also be represented as

$$A_e(e^{j\omega}) = P(x)\big|_{x=\cos\omega} \quad \text{where} \quad P(x) = \sum_{k=0}^{L} a_k x^k$$

- Parks and McClellan fix ω_p, ω_s, and L
  - Convert filter design to an approximation problem
- The approximation error is defined as

$$E(\omega) = W(\omega)\left[H_d(e^{j\omega}) - A_e(e^{j\omega})\right]$$

  - $W(\omega)$ is the weighting function
  - $H_d(e^{j\omega})$ is the desired frequency response
  - Both defined only over the passpand and stopband
  - Transition bands are unconstrained

Lowpass Filter Approximation

- The weighting function for lowpass filter is

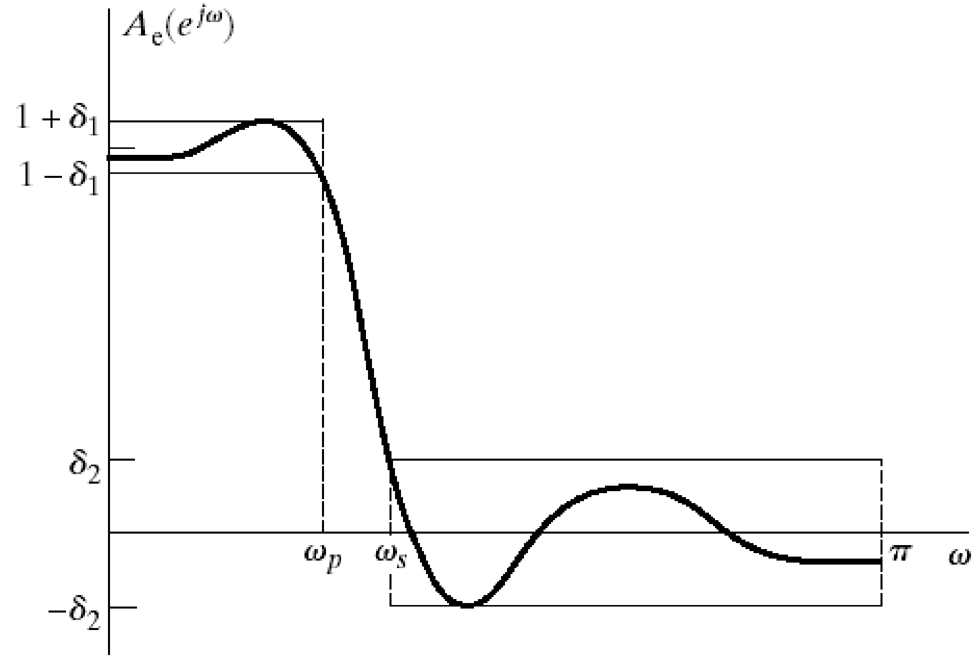
$$W(\omega) = \begin{cases} \dfrac{\delta_2}{\delta_1} & 0 \le \omega \le \omega_p \\ 1 & \omega_s \le \omega \le \pi \end{cases}$$

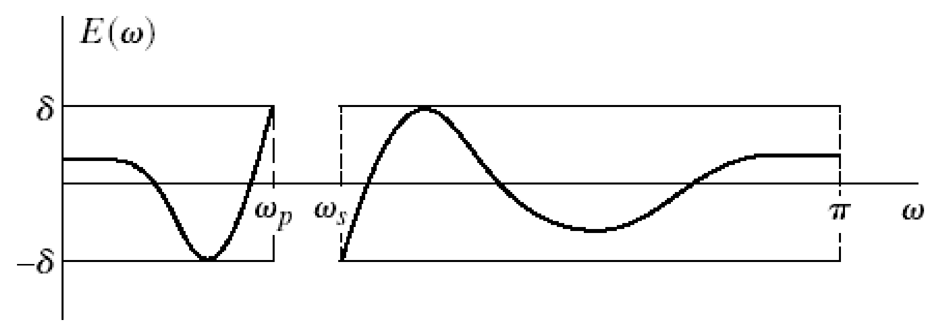
- This choice will force the error to $\delta = \delta_2$ in both bands

- Criterion used is minmax

$$\min_{\{h_e[n]:0\le n\le L\}} \left(\max_{\omega\in F} |E(\omega)| \right)$$

- F is the set of frequencies the approximations is made over





Alternation Theorem

- F_p denote the closed subset
  - consisting of the disjoint union of closed subsets of the real axis x
- The following is an r^{th} order polynomial

$$P(x) = \sum_{k=0}^{r} a_k x^k$$

- $D_p(x)$ denotes given desired function that is continuous on F_p
- $W_p(x)$ is a positive function that is continuous on F_p
- The weighted error is given as

$$E_p(x) = W_p(x)\left[D_p(x) - P(x)\right]$$

- The maximum error is defined as

$$\|E\| = \max_{x \in F_p} \left|E_p(x)\right|$$

- A necessary and sufficient condition that P(x) be the unique r^{th} order polynomial that minimizes $\|E\|$ is that $E_p(x)$ exhibit at least (r+2) alternations
- There must be at least (r+2) values x_i in F_p such that $x_1 < x_2 < \ldots < x_{r+2}$

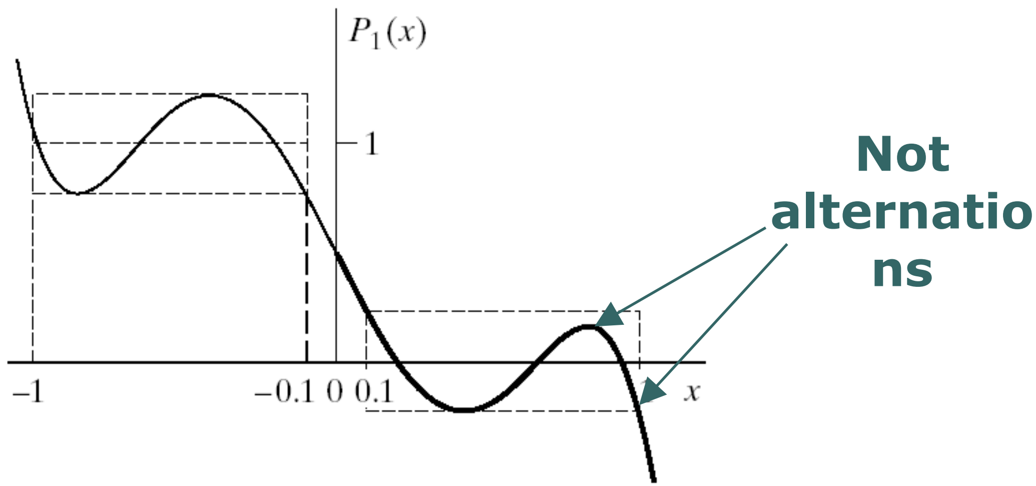
$$E_p(x_i) = -E_p(x_{i+1}) = \mp\|E\| \quad \text{for} \quad i = 1,2,\ldots,(r+2)$$

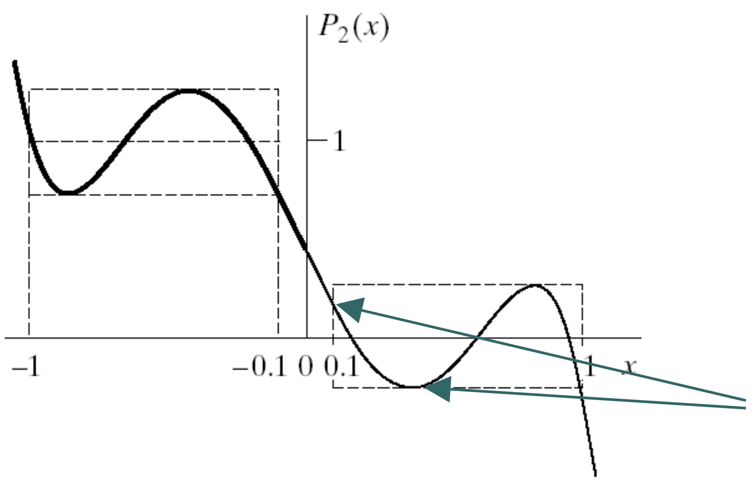
Example

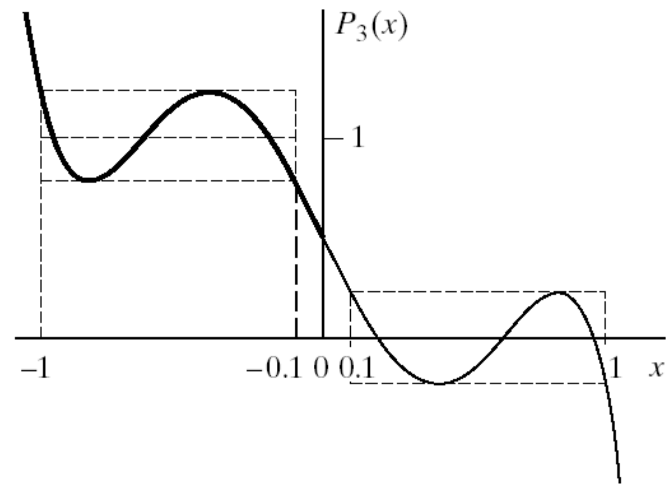
- Examine polynomials P(x) that approximate

  1 for $-1 \le x \le -0.1$

  0 for $0.1 \le x \le 1$
- Fifth order polynomials shown
- Which satisfy the theorem?







Optimal Type I Low pass Filters

- In this case the P(x) polynomial is the cosine polynomial

$$P(\cos\omega) = \sum_{k=0}^{L} a_k (\cos\omega)^k$$

- The desired lowpass filter frequency response (x=cosω)

$$D_p(\cos\omega) = \begin{cases} 1 & \cos\omega_p \le \omega \le 1 \\ 0 & -1 \le \omega \le \cos\omega_s \end{cases}$$

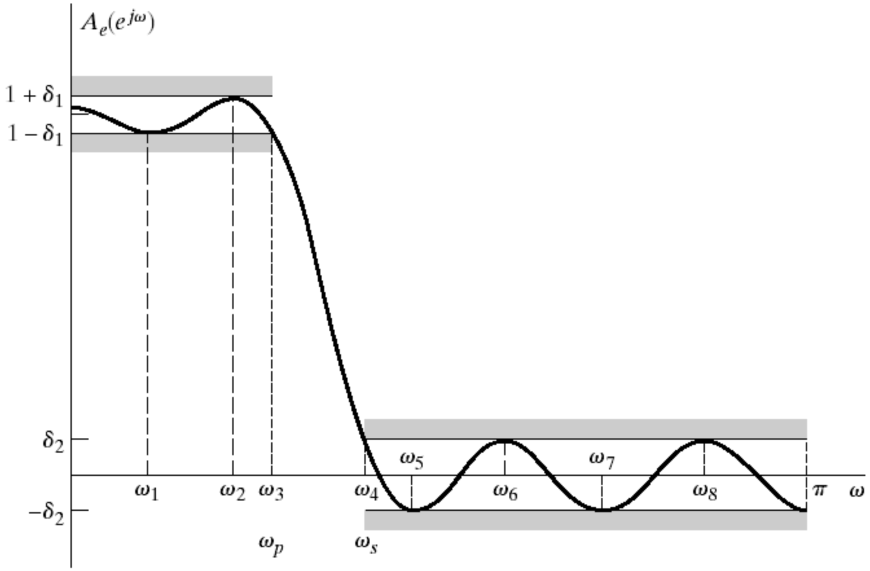
- The weighting function is given as

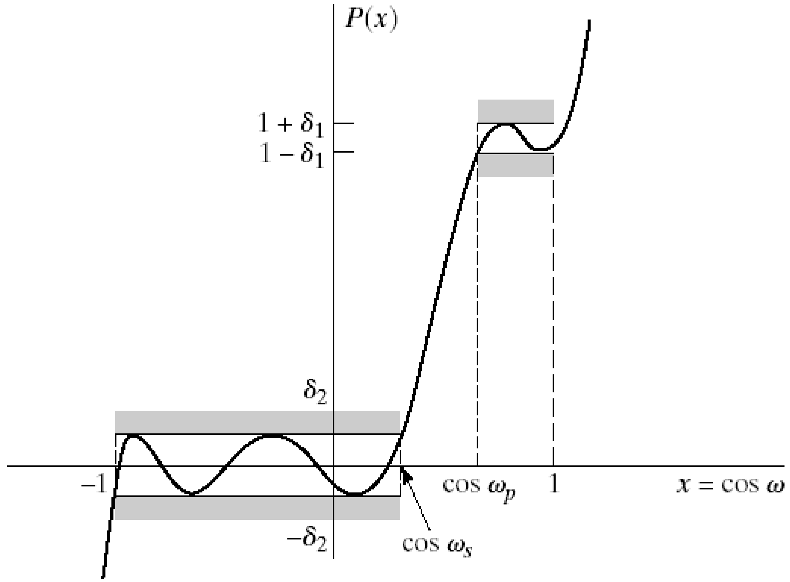
$$W_p(\cos\omega) = \begin{cases} 1/K & \cos\omega_p \le \omega \le 1 \\ 1 & -1 \le \omega \le \cos\omega_s \end{cases}$$

- The approximation error is given as

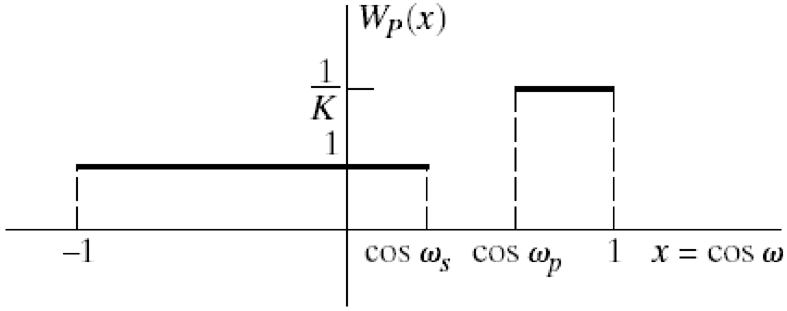
$$E_p(\cos\omega) = W_p(\cos\omega)\left[D_p(\cos\omega) - P(\cos\omega)\right]$$

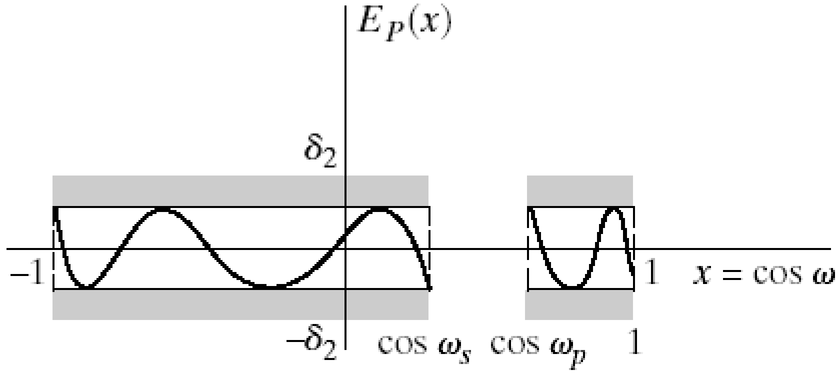
Typical Example Lowpass Filter Approximation





- 7^{th} order approximation





Properties of Type I Lowpass Filters

- Maximum possible number of alternations of the error is L+3
- Alternations will always occur at ω_p and ω_s
- All points with zero slope inside the passpand and all points with zero slope inside the stopband will correspond to alternations
  - The filter will be equiripple except possibly at 0 and π

Flowchart of Parks-McClellan Algorithm