Solutions to Additional Problems

2.32. A discrete-time LTI system has the impulse response $h[n]$ depicted in Fig. P2.32 (a). Use linearity and time invariance to determine the system output $y[n]$ if the input $x[n]$ is
Use the fact that:

$$\begin{aligned}\delta[n-k] * h[n] &= h[n-k] \\ (ax_1[n] + bx_2[n]) * h[n] &= ax_1[n] * h[n] + bx_2[n] * h[n]\end{aligned}$$

(a) $x[n] = 3\delta[n] - 2\delta[n-1]$

$$\begin{aligned}y[n] &= 3h[n] - 2h[n-1] \\ &= 3\delta[n+1] + 7\delta[n] - 7\delta[n-2] + 5\delta[n-3] - 2\delta[n-4]\end{aligned}$$

(b) $x[n] = u[n+1] - u[n-3]$

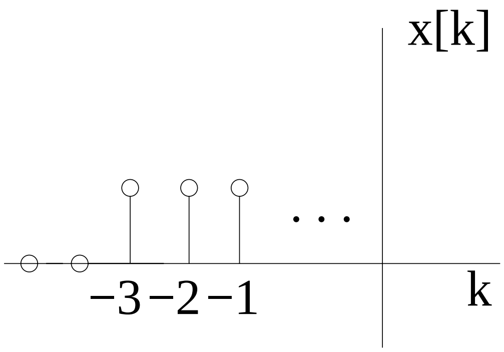
$$\begin{aligned}x[n] &= \delta[n] + \delta[n-1] + \delta[n-2] \\ y[n] &= h[n] + h[n-1] + h[n-2] \\ &= \delta[n+1] + 4\delta[n] + 6\delta[n-1] + 4\delta[n-2] + 2\delta[n-3] + \delta[n-5]\end{aligned}$$

(c) $x[n]$ as given in Fig. P2.32 (b)
.

$$\begin{aligned}x[n] &= 2\delta[n-3] + 2\delta[n] - \delta[n+2] \\ y[n] &= 2h[n-3] + 2h[n] - h[n+2] \\ &= -\delta[n+3] - 3\delta[n+2] + 7\delta[n] + 3\delta[n-1] + 8\delta[n-3] + 4\delta[n-4] - 2\delta[n-5] + 2\delta[n-6]\end{aligned}$$

2.33. Evaluate the discrete-time convolution sums given below.
(a) $y[n] = u[n+3] * u[n-3]$

Let $u[n+3] = x[n]$ and $u[n-3] = h[n]$



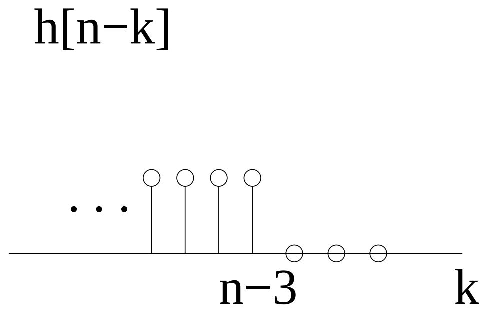
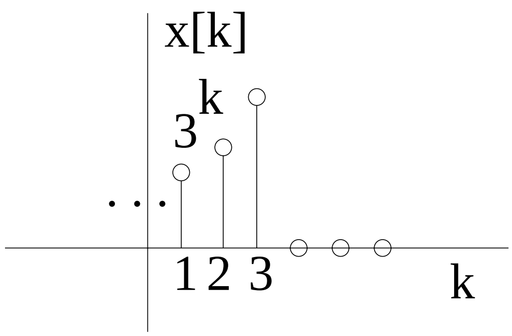


Figure P2.33. (a) Graph of $x[k]$ and $h[n-k]$

$$\text{for } n-3 < -3 \qquad n < 0$$
$$y[n] = 0$$
$$\text{for } n-3 \geq -3 \qquad n \geq 0$$
$$y[n] = \sum_{k=-3}^{n-3} 1 = n+1$$
$$y[n] = \begin{cases} n+1 & n \geq 0 \\ 0 & n < 0 \end{cases}$$

(b) $y[n] = 3^n u[-n+3] * u[n-2]$



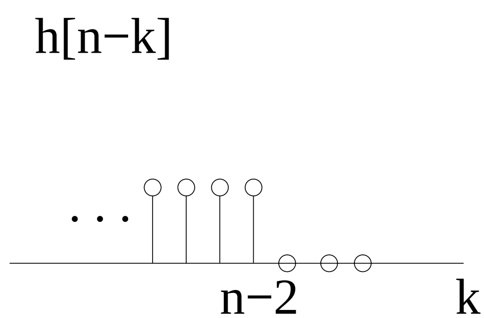


Figure P2.33. (b) Graph of $x[k]$ and $h[n-k]$

$$\text{for } n-2 \leq 3 \qquad n \leq 5$$
$$y[n] = \sum_{k=-\infty}^{n-2} 3^k$$
$$y[n] = \frac{1}{6}3^n$$
$$\text{for } n-2 \geq 4 \qquad n \geq 6$$
$$y[n] = \sum_{k=-\infty}^{3} 3^k$$

$$y[n] = \frac{81}{2}$$

$$y[n] = \begin{cases} \frac{1}{6}3^n & n \le 5 \\ \frac{81}{2} & n \ge 6 \end{cases}$$

(c) $y[n] = \left(\frac{1}{4}\right)^n u[n] * u[n+2]$

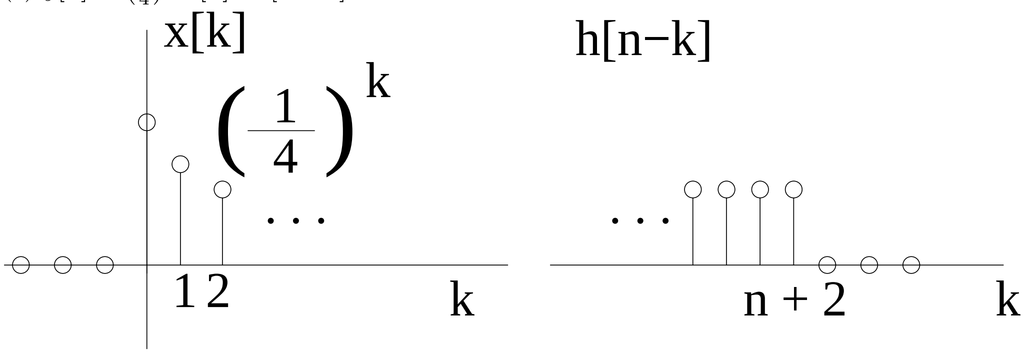


Figure P2.33. (c) Graph of $x[k]$ and $h[n-k]$

$$\begin{aligned}
&\text{for } n+2 < 0 && n < -2 \\
& && y[n] = 0 \\
&\text{for } n+2 \ge 0 && n \ge -2 \\
& && y[n] = \sum_{k=0}^{n+2} \left(\frac{1}{4}\right)^k \\
& && y[n] = \frac{4}{3} - \frac{1}{12}\left(\frac{1}{4}\right)^n
\end{aligned}$$

$$y[n] = \begin{cases} \frac{4}{3} - \frac{1}{12}\left(\frac{1}{4}\right)^n & n \ge -2 \\ 0 & n < -2 \end{cases}$$

(d) $y[n] = \cos(\frac{\pi}{2}n)u[n] * u[n-1]$

$$\begin{aligned}
&\text{for } n-1 < 0 && n < 1 \\
& && y[n] = 0 \\
&\text{for } n-1 \ge 0 && n \ge 1 \\
& && y[n] = \sum_{k=0}^{n-1} \cos\left(\frac{\pi}{2}k\right) \\
& && y[n] = \begin{cases} 1 & n = 4v+1, \quad 4v+2 \\ 0 & n = 4v, \quad 4v+3 \end{cases}
\end{aligned}$$

$$y[n] = u[n-1]f[n]$$

where

$$f[n] = \begin{cases} 1 & n = 4v+1, \quad 4v+2 \\ 0 & n = 4v, \quad 4v+3 \end{cases}$$

(e) $y[n] = (-1)^n * 2^n u[-n+2]$

$$\begin{aligned}
y[n] &= \sum_{k=n-2}^{\infty} (-1)^k 2^{n-k} \\
&= 2^n \sum_{k=n-2}^{\infty} \left(-\frac{1}{2}\right)^k \\
&= 2^n \frac{\left(-\frac{1}{2}\right)^{n-2}}{1-\left(-\frac{1}{2}\right)} \\
&= \frac{8}{3}(-1)^n
\end{aligned}$$

(f) $y[n] = \cos(\frac{\pi}{2}n) * \left(\frac{1}{2}\right)^n u[n-2]$

$$\begin{aligned}
y[n] &= \sum_{k=-\infty}^{n-2} \cos\left(\frac{\pi}{2}k\right)\left(\frac{1}{2}\right)^{n-k} \\
&\quad \text{substituting } p=-k \\
y[n] &= \sum_{p=-(n-2)}^{\infty} \cos\left(\frac{\pi}{2}p\right)\left(\frac{1}{2}\right)^{n+p} \\
y[n] &= \begin{cases} \sum_{p=-(n-2)}^{\infty} (-1)^{\frac{p}{2}} \left(\frac{1}{2}\right)^{n+p} & n \text{ even} \\ \sum_{p=-(n-3)}^{\infty} (-1)^{\frac{p}{2}} \left(\frac{1}{2}\right)^{n+p} & n \text{ odd} \end{cases} \\
y[n] &= \begin{cases} \frac{1}{5}(-1)^n & n \text{ even} \\ \frac{1}{10}(-1)^{n+1} & n \text{ odd} \end{cases}
\end{aligned}$$

(g) $y[n] = \beta^n u[n] * u[n-3], \quad |\beta| < 1$

$$\begin{aligned}
&\text{for } n-3<0 && n<3 \\
& && y[n]=0 \\
&\text{for } n-3\geq 0 && n\geq 3 \\
& && y[n]=\sum_{k=0}^{n-3}\beta^k \\
& && y[n]=\left(\frac{1-\beta^{n-2}}{1-\beta}\right)
\end{aligned}$$

$$y[n] = \begin{cases} \left(\frac{1-\beta^{n-2}}{1-\beta}\right) & n \geq 3 \\ 0 & n < 3 \end{cases}$$

(h) $y[n] = \beta^n u[n] * \alpha^n u[n-10], \quad |\beta| < 1, \ |\alpha| < 1$

$$\text{for } n-10<0 \qquad n<10$$

$$y[n] = 0$$

$$\text{for } n-10 \geq 0 \qquad n \geq 10$$

$$y[n] = \sum_{k=0}^{n-10} \left(\frac{\beta}{\alpha}\right)^k \alpha^n$$

$$y[n] = \begin{cases} \alpha^n \left(\frac{1-\left(\frac{\beta}{\alpha}\right)^{n-9}}{1-\frac{\beta}{\alpha}}\right) & \alpha \neq \beta \\ \alpha(n-9) & \alpha = \beta \end{cases}$$

(i) $y[n] = (u[n+10] - 2u[n] + u[n-4]) * u[n-2]$

$$\text{for } n-2 < -10 \qquad n < -8$$

$$y[n] = 0$$

$$\text{for } n-2 < 0 \qquad -8 \leq n < 2$$

$$y[n] = \sum_{k=-10}^{n-2} 1 = n+9$$

$$\text{for } n-2 \leq 3 \qquad 2 \leq n \leq 5$$

$$y[n] = \sum_{k=-10}^{-1} 1 - \sum_{k=0}^{n-2} 1 = 11-n$$

$$\text{for } n-2 \geq 4 \qquad n \geq 6$$

$$y[n] = \sum_{k=-10}^{-1} 1 - \sum_{k=0}^{3} 1 = 6$$

$$y[n] = \begin{cases} 0 & n < -8 \\ n+9 & -8 \leq n < 2 \\ 11-n & -2 \leq n \leq 5 \\ 6 & n > 5 \end{cases}$$

(j) $y[n] = (u[n+10] - 2u[n] + u[n-4]) * \beta^n u[n], \quad |\beta| < 1$

for $n < -10$

$$y[n] = 0$$

for $n < 0$

$$y[n] = \beta^n \sum_{k=-10}^{n} \left(\frac{1}{\beta}\right)^k$$

$$y[n] = \frac{\beta^{n+11} - 1}{\beta - 1}$$

for $n \leq 3$

$$y[n] = \beta^n \sum_{k=-10}^{-1} \left(\frac{1}{\beta}\right)^k - \beta^n \sum_{k=0}^{n} \left(\frac{1}{\beta}\right)^k$$

$$y[n] = \frac{\beta^{n+11} - \beta^{n+1}}{\beta - 1} - \frac{\beta^{n+1} - 1}{\beta - 1}$$

for $n > 3$

$$y[n] = \beta^n \sum_{k=-10}^{-1} \left(\frac{1}{\beta}\right)^k - \beta^n \sum_{k=0}^{3} \left(\frac{1}{\beta}\right)^k$$

$$y[n] = \frac{\beta^{n+11} - \beta^{n+1}}{\beta - 1} - \frac{\beta^{n+1} - \beta^{n-3}}{\beta - 1}$$

$$y[n] = \begin{cases} 0 & n < -10 \\ \frac{\beta^{n+11}-1}{\beta-1} & -10 \le n < 0 \\ \frac{\beta^{n+11}-\beta^{n+1}}{\beta-1} - \frac{\beta^{n+1}-1}{\beta-1} & 0 \le n \le 3 \\ \frac{\beta^{n+11}-\beta^{n+1}}{\beta-1} - \frac{\beta^{n+1}-\beta^{n-3}}{\beta-1} & n > 3 \end{cases}$$

(k) $y[n] = (u[n+10] - 2u[n+5] + u[n-6]) * \cos(\frac{\pi}{2}n)$
There are four different cases:

(i) $n = 4v$ v is any integer

$$y[n] = (1)[-1+0+1+0-1] + (-1)[0+1+0-1+0+1+0-1+0+1+0] = -2$$

(ii) $n = 4v + 2$

$$y[n] = (1)[1+0-1+0+1] + (-1)[0-1+0+1+0-1+0+1+0-1+0] = 2$$

(iii) $n = 4v + 3$

$$y[n] = (1)[0-1+0+1+0] + (-1)[-1+0+1+0-1+0+1+0-1+0+1] = 0$$

(iv) $n = 4v + 1$

$$y[n] = 0$$

$$y[n] = \begin{cases} -2 & n = 4v \\ 2 & n = 4v+2 \\ 0 & \text{otherwise} \end{cases}$$

(l) $y[n] = u[n] * \sum_{p=0}^{\infty} \delta[n-4p]$

for $n < 0$

$$y[n] = 0$$

for $n \ge 0$ $n = 0, 4, 8, ...$

$$y[n] = \frac{n}{4} + 1$$

for $n \ge 0$ $n \ne 0, 4, 8, ...$

$$y[n] = \left\lceil \frac{n}{4} \right\rceil$$

where $\lceil x \rceil$ is the smallest integer larger than x. Ex. $\lceil 3.2 \rceil = 4$

(m) $y[n] = \beta^n u[n] * \sum_{p=0}^{\infty} \delta[n-4p]$, $|\beta| < 1$

for $n < 0$

$$y[n] = 0$$

for $n \geq 0$ $\quad n = 0, 4, 8, ...$

$$y[n] = \sum_{k=0}^{\frac{n}{4}} \beta^{4k}$$

$$y[n] = \frac{1-\beta^{4\left(\frac{n}{4}+1\right)}}{1-\beta^4}$$

for $n \geq 0$ $\quad n = 1, 5, 9, ...$

$$y[n] = \sum_{k=0}^{\frac{n-1}{4}} \beta^{4k-1}$$

$$y[n] = \frac{1}{\beta}\left(\frac{1-\beta^{4\left(\frac{n-1}{4}+1\right)}}{1-\beta^4}\right)$$

for $n \geq 0$ $\quad n = 2, 6, 10, ...$

$$y[n] = \sum_{k=0}^{\frac{n-2}{4}} \beta^{4k-2}$$

$$y[n] = \frac{1}{\beta^2}\left(\frac{1-\beta^{4\left(\frac{n-2}{4}+1\right)}}{1-\beta^4}\right)$$

for $n \geq 0$ $\quad n = 3, 7, 11, ...$

$$y[n] = \sum_{k=0}^{\frac{n-3}{4}} \beta^{4k-3}$$

$$y[n] = \frac{1}{\beta^3}\left(\frac{1-\beta^{4\left(\frac{n-3}{4}+1\right)}}{1-\beta^4}\right)$$

(n) $y[n] = \left(\frac{1}{2}\right)^n u[n+2] * \gamma^{|n|}$

for $n+2 \leq 0$ $\quad n \leq -2$

$$y[n] = \sum_{k=-\infty}^{n+2} \left(\frac{1}{2}\right)^{n-k} \gamma^{-k}$$

$$y[n] = \left(\frac{1}{2}\right)^n \sum_{k=-\infty}^{n+2} \left(\frac{\gamma}{2}\right)^{-k}$$

let $l = -k$

$$y[n] = \left(\frac{1}{2}\right)^n \sum_{l=-(n+2)}^{\infty} \left(\frac{\gamma}{2}\right)^{l}$$

$$y[n] = \left(\frac{1}{2}\right)^n \frac{\left(\frac{\gamma}{2}\right)^{-(n+2)}}{1-\frac{\gamma}{2}}$$

$$y[n] = \left(\frac{2}{\gamma}\right)^2 \frac{\left(\frac{1}{\gamma}\right)^n}{1-\frac{\gamma}{2}}$$

for $n+2 \geq 0$ $\quad n > -2$

$$y[n] = \sum_{k=-\infty}^{0} \left(\frac{1}{2}\right)^{n-k} \gamma^{-k} + \sum_{k=1}^{n+2} \left(\frac{1}{2}\right)^{n-k} \gamma^{k}$$

$$y[n] = \left(\frac{1}{2}\right)^{n} \sum_{k=-\infty}^{0} \left(\frac{\gamma}{2}\right)^{-k} + \left(\frac{1}{2}\right)^{n} \sum_{k=1}^{n+2} (2\gamma)^{k}$$

$$y[n] = \left(\frac{1}{2}\right)^{n} \left[\frac{1}{1-\frac{\gamma}{2}} + \left(\frac{1-(2\gamma)^{n+3}}{1-2\gamma} - 1\right)\right]$$

2.34. Consider the discrete-time signals depicted in Fig. P2.34. Evaluate the convolution sums indicated below.

(a) $m[n] = x[n] * z[n]$

for $n+5 < 0$ $\quad n < -5$

$$m[n] = 0$$

for $n+5 < 4$ $\quad -5 \le n < -1$

$$m[n] = \sum_{k=0}^{n+5} 1 = n+6$$

for $n-1 < 1$ $\quad -1 \le n < 2$

$$m[n] = \sum_{k=0}^{3} 1 + 2\sum_{k=4}^{n+5} 1 = 2n+8$$

for $n+5 < 9$ $\quad 2 \le n < 4$

$$m[n] = \sum_{k=n-1}^{3} 1 + 2\sum_{k=4}^{n+5} 1 = 9+n$$

for $n-1 < 4$ $\quad 4 \le n < 5$

$$m[n] = \sum_{k=n-1}^{3} 1 + 2\sum_{k=4}^{8} 1 = 15-n$$

for $n-1 < 9$ $\quad 5 \le n < 10$

$$m[n] = 2\sum_{k=n-1}^{8} 1 = 20-2n$$

for $n-1 \ge 9$ $\quad n \ge 10$

$$m[n] = 0$$

$$m[n] = \begin{cases} 0 & n < -5 \\ n+6 & -5 \le n < -1 \\ 2n+8 & -1 \le n < 2 \\ 9+n & 2 \le n < 4 \\ 15-n & 4 \le n < 5 \\ 20-2n & 5 \le n < 10 \\ 0 & n \ge 10 \end{cases}$$

(b) $m[n] = x[n] * y[n]$

for $n+5 < -3$ $\quad n < -8$

$$
\begin{aligned}
& & m[n] &= 0 \\
&\text{for } n+5<1 & -8 &\le n < -4 \\
& & m[n] &= \sum_{k=-3}^{n+5} 1 = n+9 \\
&\text{for } n-1<-2 & -4 &\le n < -1 \\
& & m[n] &= \sum_{k=-3}^{0} 1 - \sum_{k=1}^{n+5} 1 = -n-1 \\
&\text{for } n+5<5 & -1 &\le n < 0 \\
& & m[n] &= \sum_{k=n-1}^{0} 1 - \sum_{k=1}^{n+5} 1 = -2n-4 \\
&\text{for } n-1<1 & 0 &\le n < 2 \\
& & m[n] &= \sum_{k=n-1}^{0} 1 - \sum_{k=1}^{4} 1 = -n-2 \\
&\text{for } n-1<5 & 2 &\le n < 6 \\
& & m[n] &= -\sum_{k=n-1}^{4} 1 = n-6 \\
&\text{for } n-1\ge 5 & n &\ge 6 \\
& & m[n] &= 0
\end{aligned}
$$

$$
m[n] = \begin{cases} 0 & n < -8 \\ n+3 & -8 \le n < -4 \\ -n-1 & -4 \le n < -1 \\ -2n-4 & -1 \le n < 0 \\ -n-2 & 0 \le n < 2 \\ n-6 & 2 \le n < 6 \\ 0 & n \ge 6 \end{cases}
$$

(c) $m[n] = x[n] * f[n]$

$$
\begin{aligned}
&\text{for } n+5<-5 & n &< -10 \\
& & m[n] &= 0 \\
&\text{for } n-1<-5 & -10 &\le n < -4 \\
& & m[n] &= \frac{1}{2}\sum_{k=-5}^{n+5} k = -5n-55+\frac{1}{2}(n+10)(n+11) \\
&\text{for } n+5<6 & -4 &\le n < 1 \\
& & m[n] &= \frac{1}{2}\sum_{k=n-1}^{n+5} k = \frac{7}{2}(n-1)+\frac{21}{2} \\
&\text{for } n-1<6 & 1 &\le n < 7 \\
& & m[n] &= \frac{1}{2}\sum_{k=n-1}^{5} k = \frac{1}{2}(7-n)\left[(n-1)+\frac{1}{2}(6-n)\right] \\
&\text{for } n-1\ge 6 & n &\ge 7
\end{aligned}
$$

$$m[n] = 0$$

$$m[n] = \begin{cases} 0 & n < -10 \\ -5n - 55 + \frac{1}{2}(n+10)(n+11) & -10 \le n < -4 \\ \frac{7}{2}(n-1) + \frac{21}{2} & -4 \le n < 1 \\ \frac{1}{2}(7-n)\left[(n-1) + \frac{1}{2}(6-n)\right] & 1 \le n < 7 \\ 0 & n \ge 7 \end{cases}$$

(d) $m[n] = x[n] * g[n]$

$$\begin{aligned}
\text{for } n+5 < -8 \qquad & n < -13 \\
& m[n] = 0 \\
\text{for } n-1 < -7 \qquad & -14 \le n < -6 \\
& m[n] = \sum_{k=-8}^{n+5} 1 = n + 14 \\
\text{for } n+5 < 4 \qquad & -6 \le n < -1 \\
& m[n] = \sum_{k=n-1}^{-2} 1 = -n \\
\text{for } n-1 < -1 \qquad & -1 \le n < 0 \\
& m[n] = \sum_{k=n-1}^{-2} 1 + \sum_{k=4}^{n+5} 1 = -2 \\
\text{for } n-1 < 4 \qquad & 0 \le n < 5 \\
& m[n] = \sum_{k=4}^{n+5} 1 = n + 2 \\
\text{for } n-1 < 11 \qquad & 5 \le n < 12 \\
& m[n] = \sum_{k=n-1}^{10} 1 = 12 - n \\
\text{for } n-1 \ge 11 \qquad & n \ge 12 \\
& m[n] = 0
\end{aligned}$$

$$m[n] = \begin{cases} 0 & n < -13 \\ n+14 & -13 \le n < -6 \\ -n & -6 \le n < -1 \\ -2 & -1 \le n < 0 \\ n+2 & 0 \le n < 5 \\ 12-n & 5 \le n < 12 \\ 0 & n \ge 12 \end{cases}$$

(e) $m[n] = y[n] * z[n]$

The remaining problems will not show all of the steps of convolution, instead figures and intervals will be given for the solution.

Intervals

$n < -3$

$-3 \leq n < 1$

$1 \leq n < 5$

$5 \leq n < 6$

$6 \leq n < 9$

$9 \leq n < 13$

$n \geq 13$

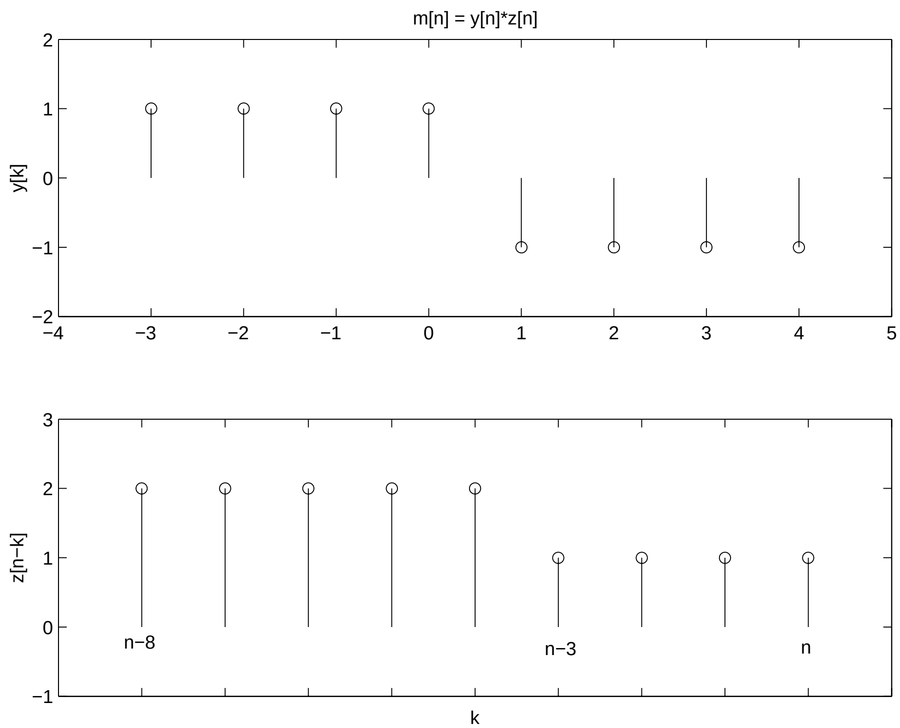


Figure P2.34. Figures of $y[n]$ and $z[n-k]$

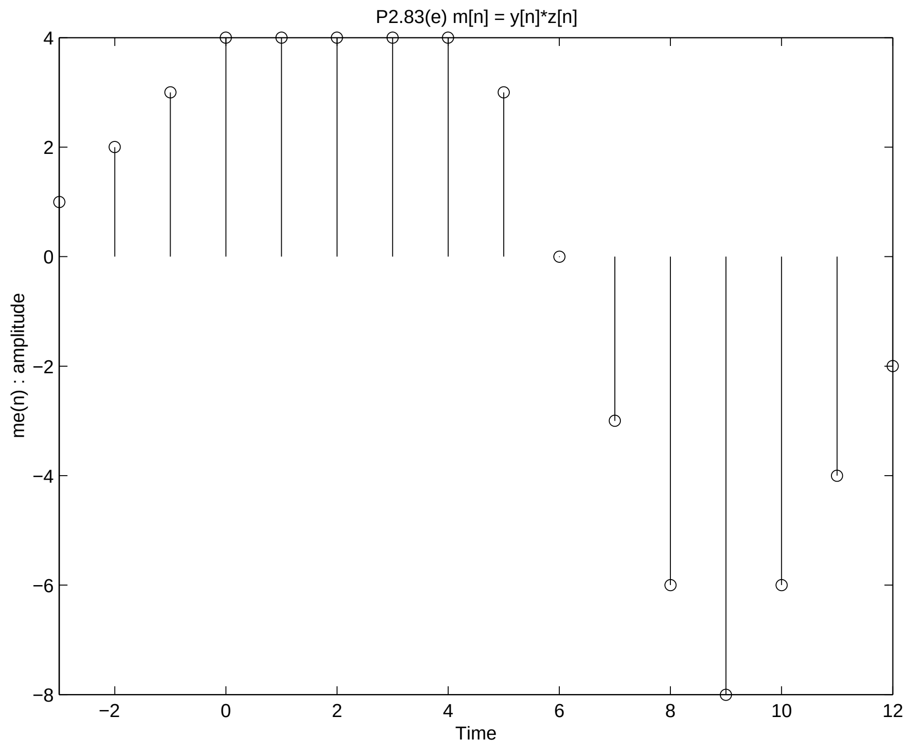


Figure P2.34. $m[n] = y[n] * z[n]$

(f) $m[n] = y[n] * g[n]$

Intervals

$n < -11$

$-11 \leq n < -7$

$-7 \leq n \leq -5$

$-4 \leq n < -3$

$-3 \leq n < -1$

$-1 \leq n < 1$

$1 \leq n < 3$

$3 \leq n < 5$

$5 \leq n < 7$

$7 \leq n < 9$

$9 \leq n < 11$

$11 \leq n < 15$

$n \geq 15$

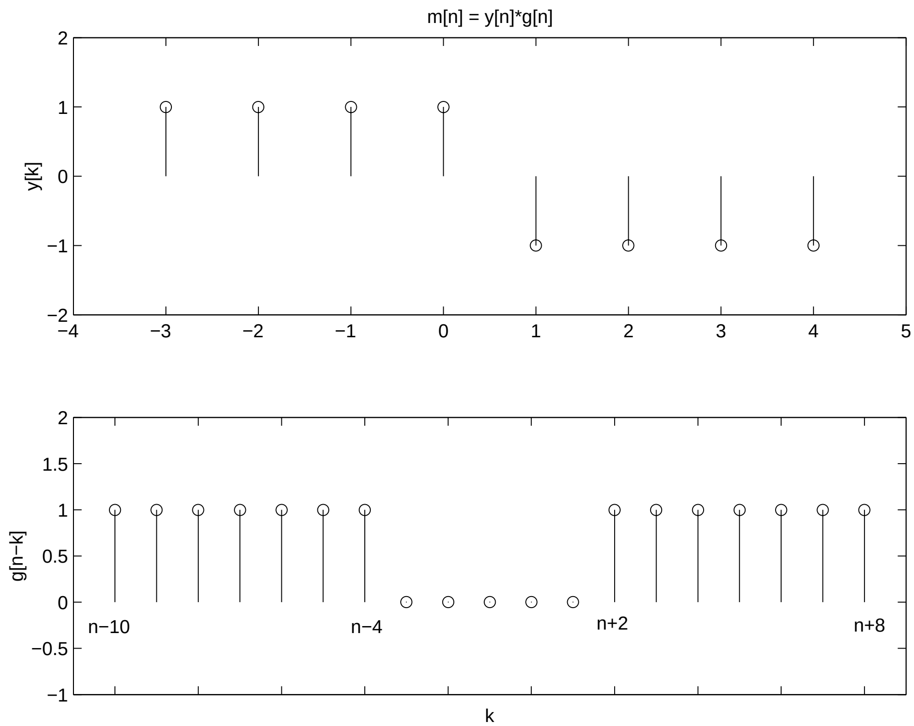


Figure P2.34. Figures of $y[n]$ and $g[n-k]$

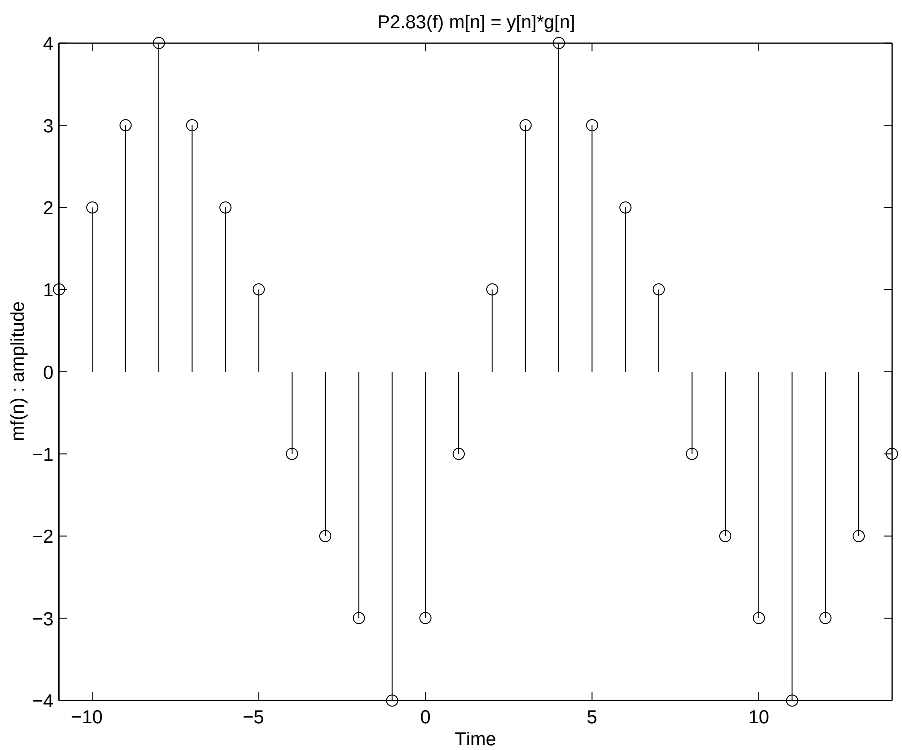


Figure P2.34. $m[n] = y[n] * g[n]$

(g) $m[n] = y[n] * w[n]$

Intervals

$n < -7$

$-7 \leq n < -3$

$-3 \leq n < -2$

$-2 \leq n < 1$

$1 \leq n < 2$

$2 \leq n < 5$

$5 \leq n < 9$

$n \geq 9$

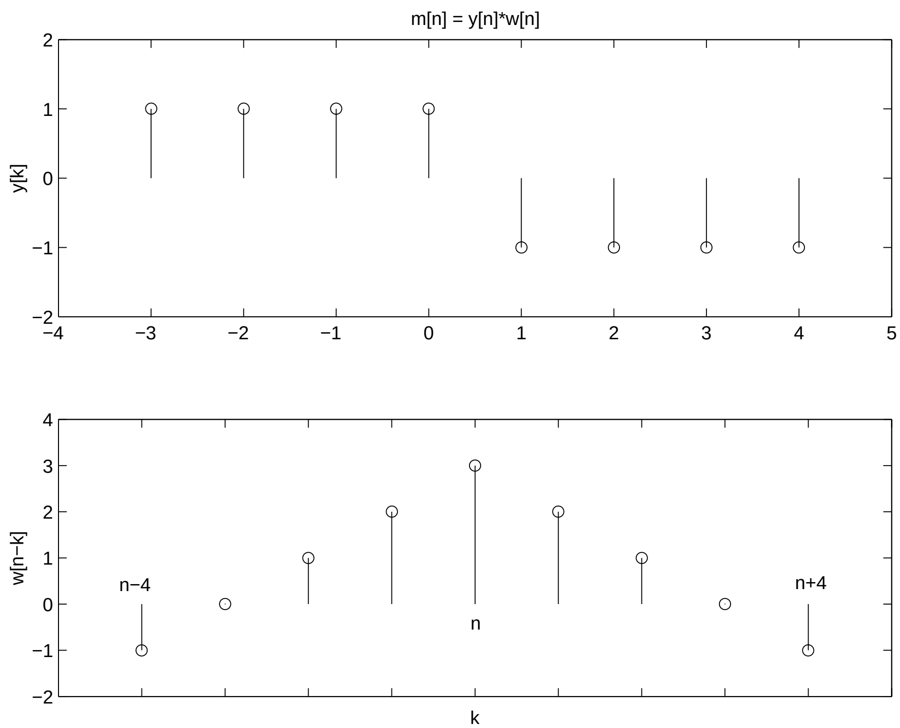


Figure P2.34. Figures of $y[n]$ and $w[n-k]$

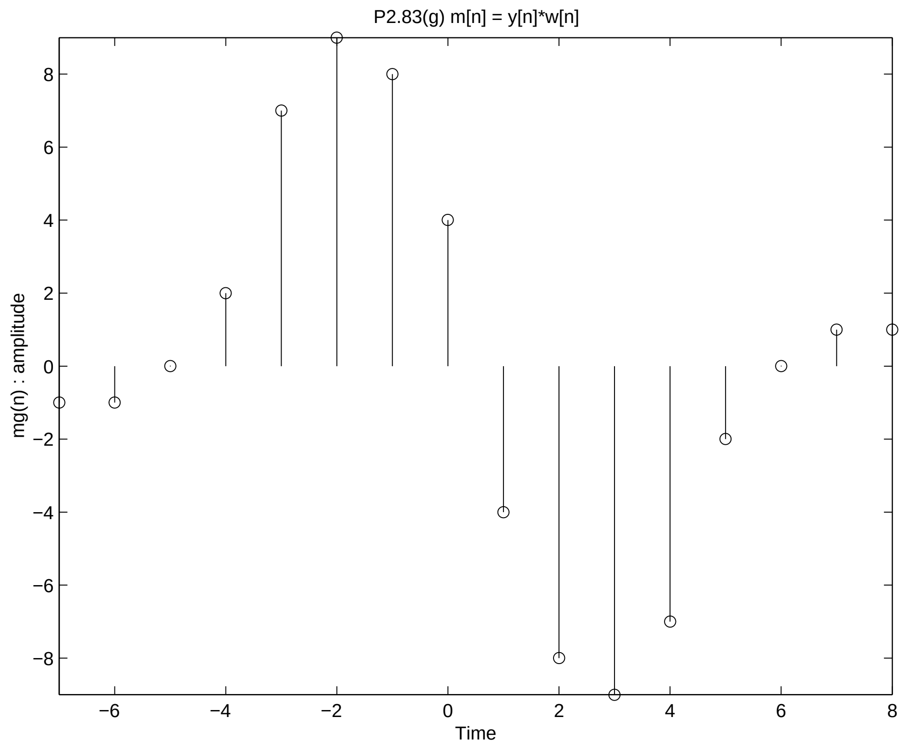


Figure P2.34. $m[n] = y[n] * w[n]$

(h) $m[n] = y[n] * f[n]$

Intervals

$n < -8$

$-8 \leq n < -4$

$-4 \leq n < 0$

$0 \leq n < 2$

$2 \leq n < 6$

$6 \leq n < 10$

$n \geq 10$

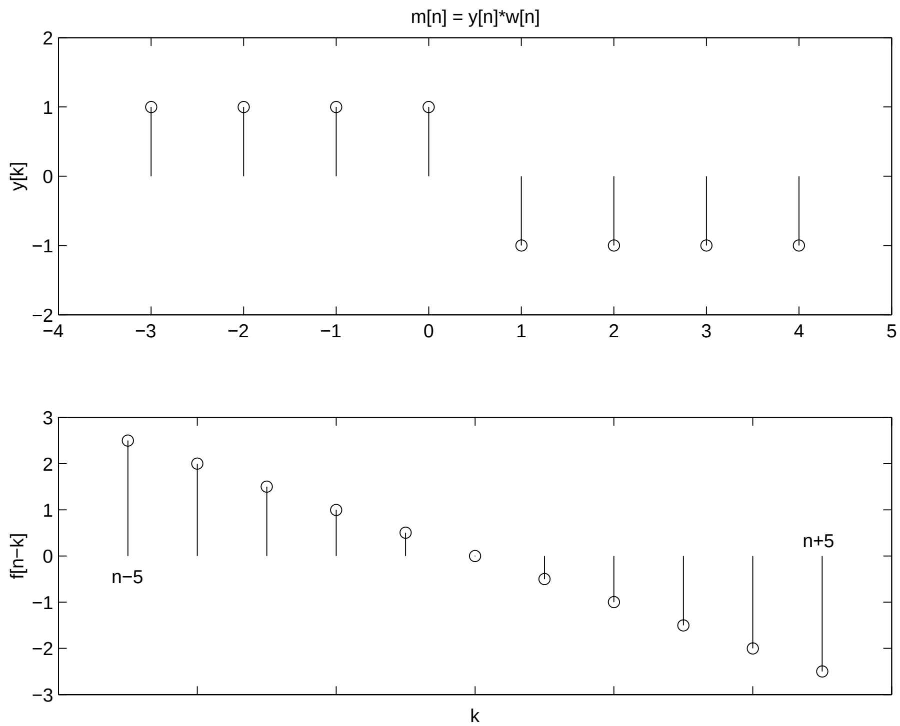


Figure P2.34. Figures of $y[n]$ and $f[n-k]$

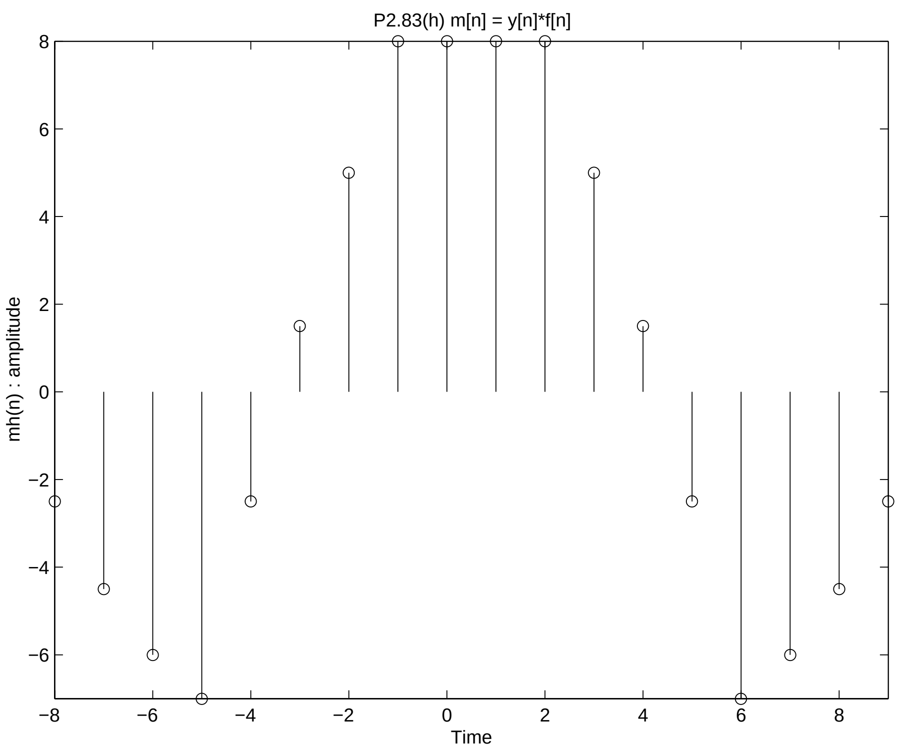


Figure P2.34. $m[n] = y[n] * f[n]$

(i) $m[n] = z[n] * g[n]$

Intervals

$n < -8$

$-8 \leq n < -4$

$-4 \leq n < -1$

$-1 \leq n < 1$

$1 \leq n < 2$

$2 \leq n < 4$

$4 \leq n < 7$

$7 \leq n < 8$

$8 \leq n < 11$

$11 \leq n < 13$

$13 \leq n < 14$

$14 \leq n < 19$

$n \geq 19$

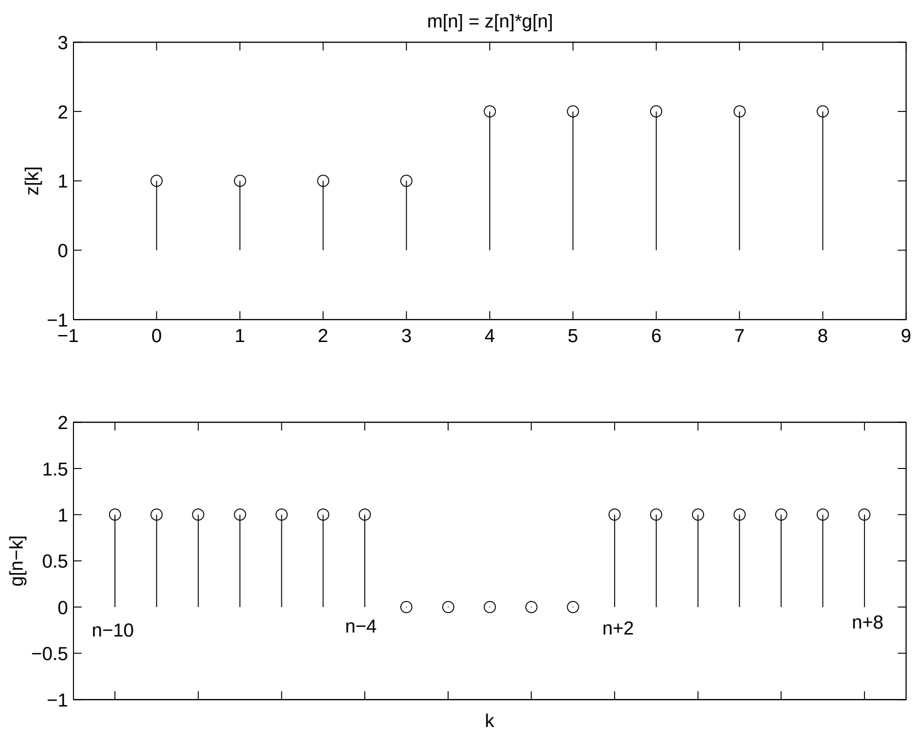


Figure P2.34. Figures of $z[n]$ and $g[n-k]$

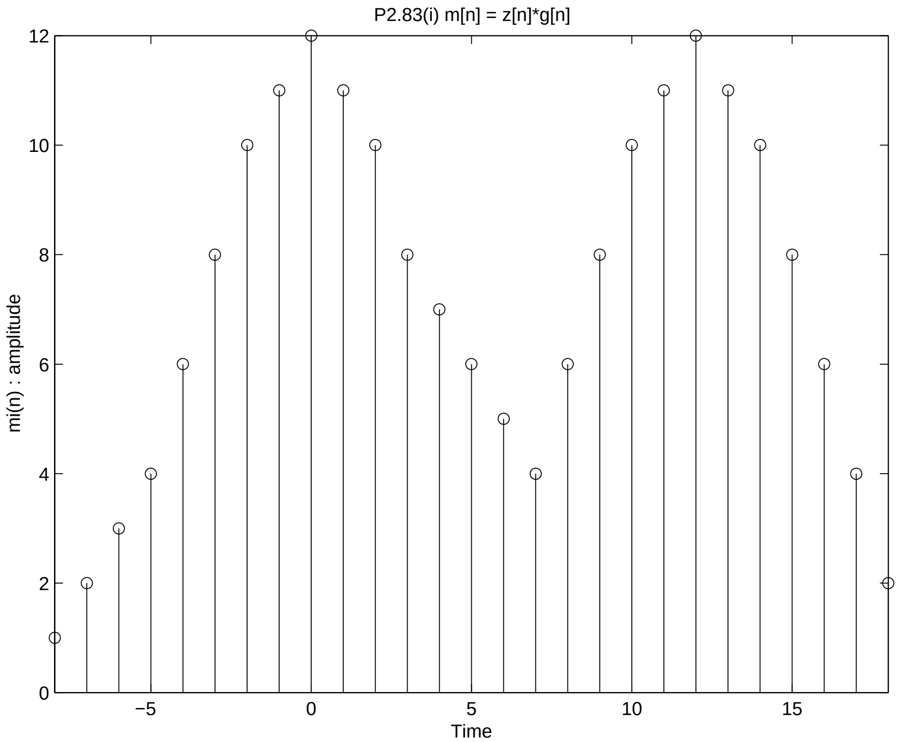


Figure P2.34. $m[n] = z[n] * g[n]$

(j) $m[n] = w[n] * g[n]$

Intervals

$n < -12$

$-12 \leq n < -7$

$-7 \leq n < -6$

$-6 \leq n < -3$

$-3 \leq n < -1$

$-1 \leq n < 0$

$0 \leq n < 3$

$3 \leq n < 5$

$5 \leq n < 7$

$7 \leq n < 9$

$9 \leq n < 11$

$11 \leq n < 15$

$n \geq 15$

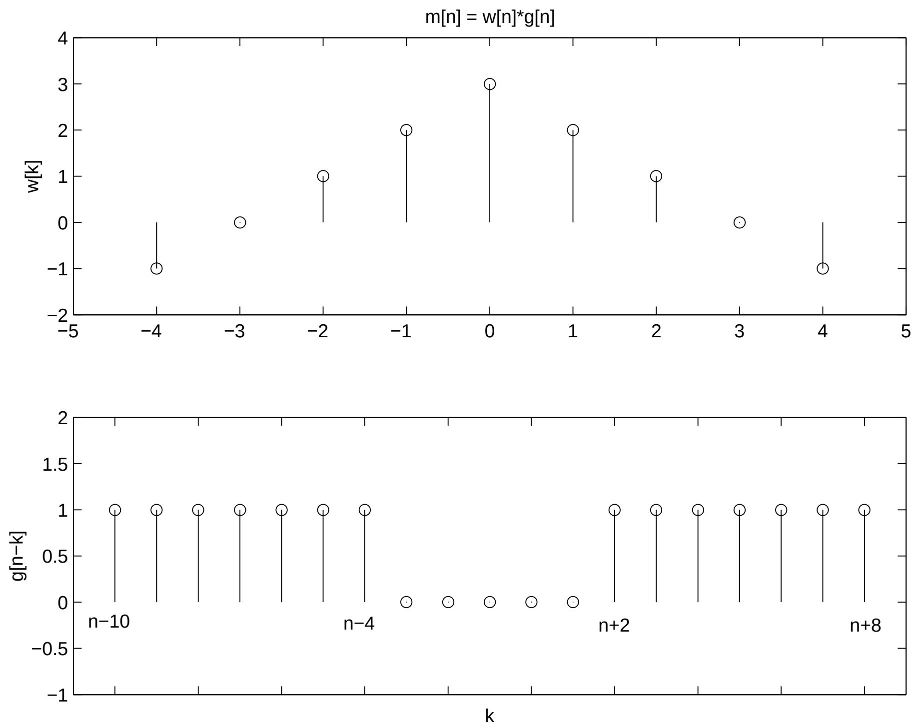


Figure P2.34. Figures of $w[n]$ and $g[n-k]$

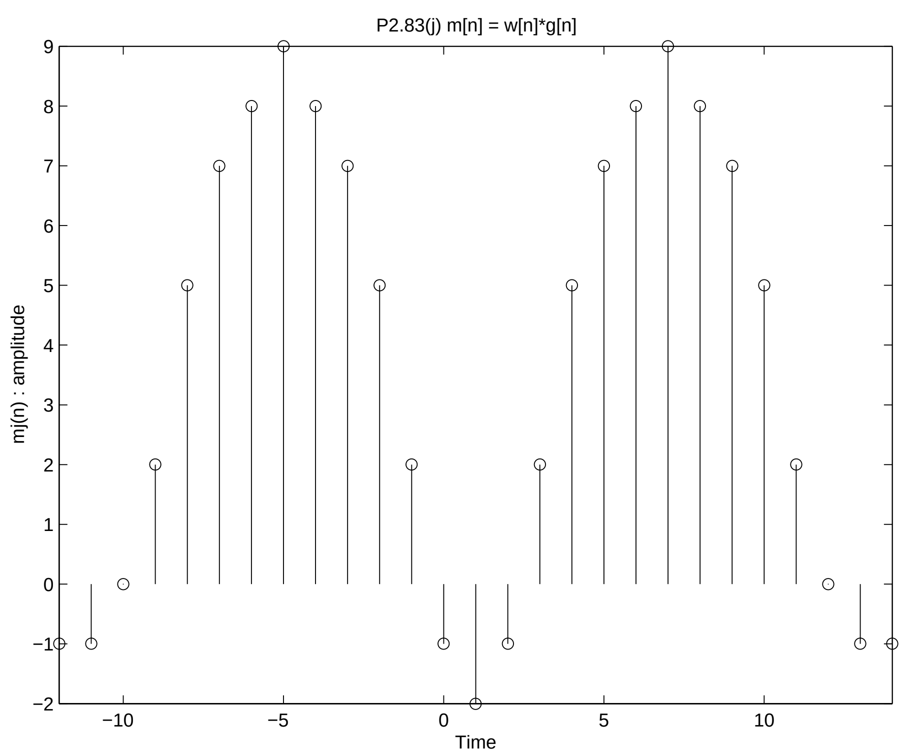


Figure P2.34. $m[n] = w[n] * g[n]$

(k) $m[n] = f[n] * g[n]$

Intervals

$n < -13$

$-13 \leq n < -7$

$-7 \leq n < -2$

$-2 \leq n < -1$

$-1 \leq n < 4$

$4 \leq n < 5$

$5 \leq n < 10$

$10 \leq n < 16$

$n \geq 16$

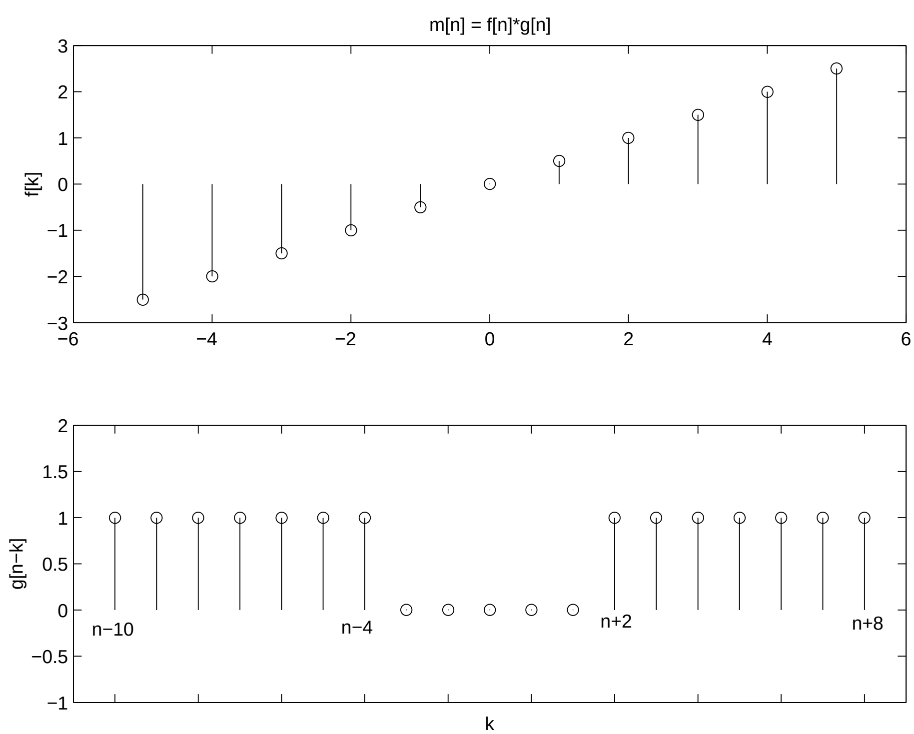


Figure P2.34. Figures of $f[n]$ and $g[n-k]$

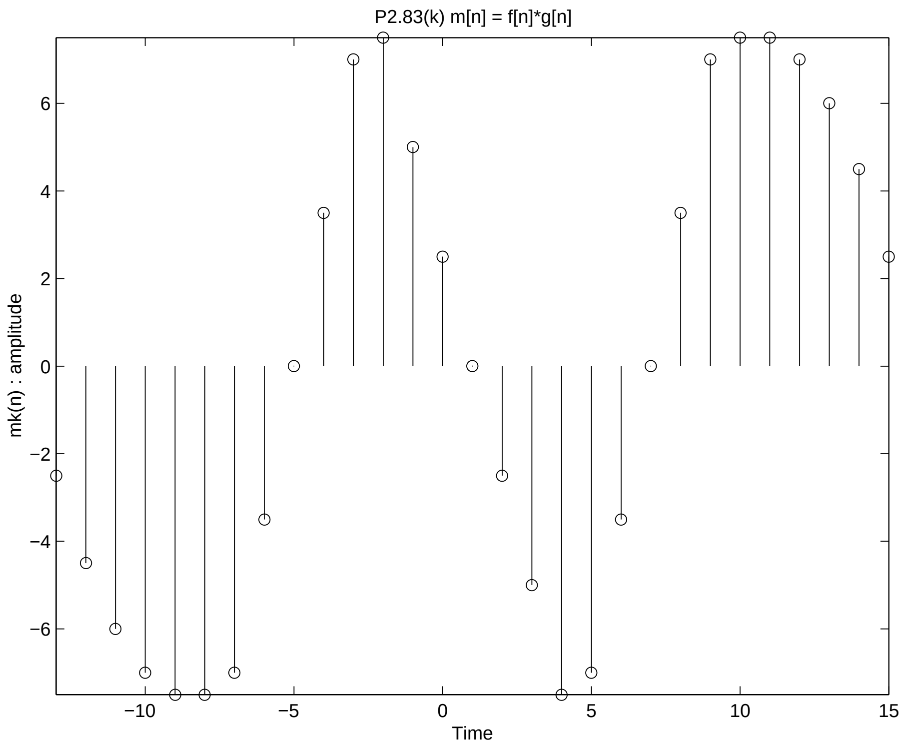


Figure P2.34. $m[n] = f[n] * g[n]$

2.35. At the start of the first year \$10,000 is deposited in a bank account earning 5% per year. At the start of each succeeding year \$1000 is deposited. Use convolution to determine the balance at the start of each year (after the deposit). Initially \$10000 is invested.

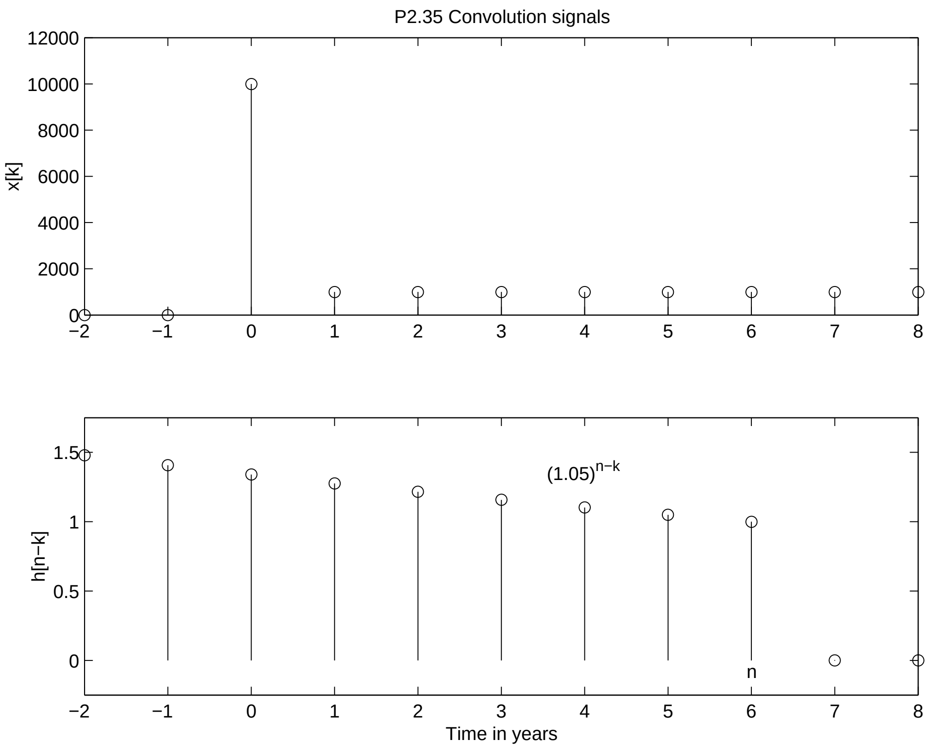


Figure P2.35. Graph of $x[k]$ and $h[n-k]$

for $n = -1$

$$y[-1] = \sum_{k=-1}^{-1} 10000(1.05)^{n-k} = 10000(1.05)^{n+1}$$

\$1000 is invested annually, similar to example 2.5

for $n \geq 0$

$$y[n] = 10000(1.05)^{n+1} + \sum_{k=0}^{n} 1000(1.05)^{n-k}$$

$$y[n] = 10000(1.05)^{n+1} + 1000(1.05)^{n} \sum_{k=0}^{n} (1.05)^{-k}$$

$$y[n] = 10000(1.05)^{n+1} + 1000(1.05)^{n} \frac{1 - \left(\frac{1}{1.05}\right)^{n+1}}{1 - \frac{1}{1.05}}$$

$$y[n] = 10000(1.05)^{n+1} + 20000\left[1.05^{n+1} - 1\right]$$

The following is a graph of the value of the account.

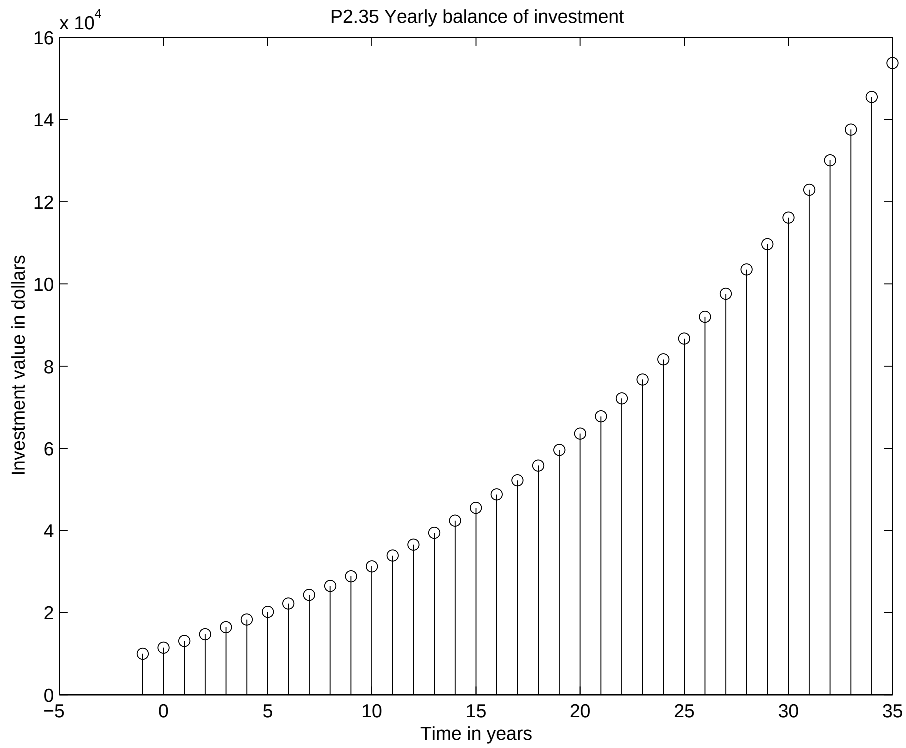


Figure P2.35. Yearly balance of the account

2.36. The initial balance of a loan is \$20,000 and the interest rate is 1% per month (12% per year). A monthly payment of \$200 is applied to the loan at the start of each month. Use convolution to calculate the loan balance after each monthly payment.

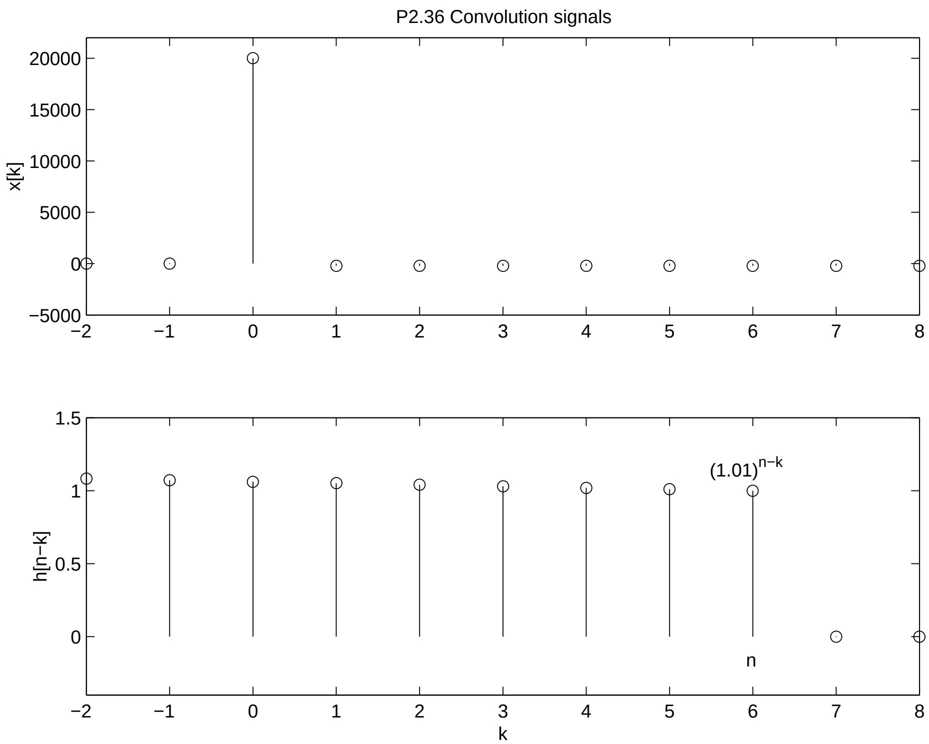


Figure P2.36. Plot of $x[k]$ and $h[n-k]$

for $n=-1$

$$y[n] = \sum_{k=-1}^{-1} 20000(1.01)^{n-k} = 20000(1.01)^{n+1}$$

for ≥ 0

$$y[n] = 20000(1.01)^{n+1} - \sum_{k=0}^{n} 200(1.01)^{n-k}$$

$$y[n] = 20000(1.01)^{n+1} - 200(1.01)^{n} \sum_{k=0}^{n} (1.01)^{-k}$$

$$y[n] = 20000(1.01)^{n+1} - 20000[(1.01)^{n+1} - 1]$$

The following is a plot of the monthly balance.

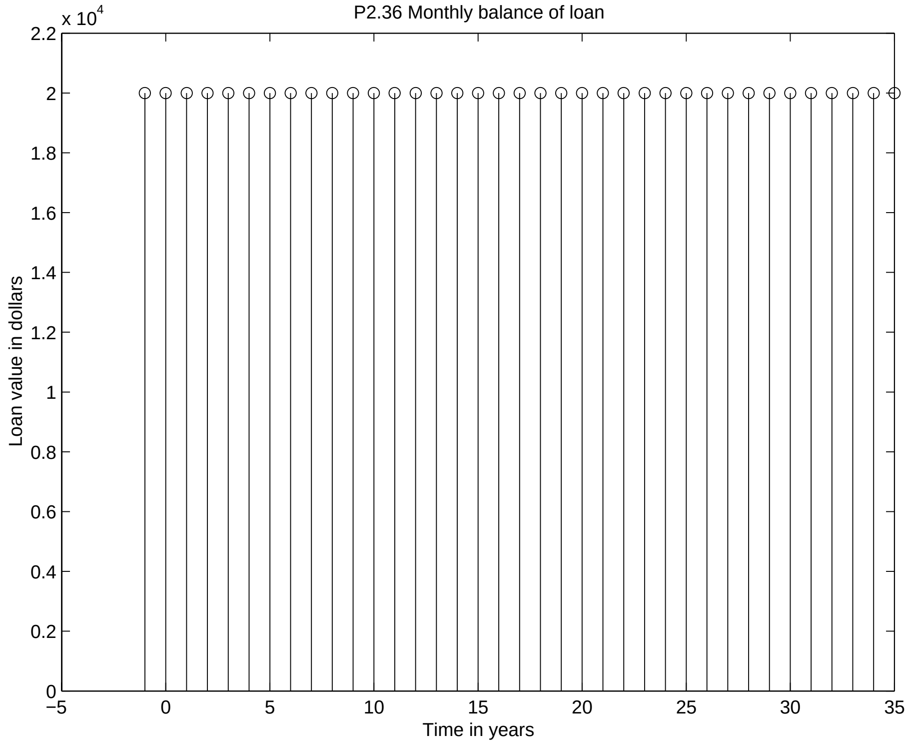


Figure P2.36. Monthly loan balance

Paying \$200 per month only takes care of the interest, and doesn't pay off any of the principle of the loan.

2.37. The convolution sum evaluation procedure actually corresponds to a formal statement of the well-known procedure for multiplication of polynomials. To see this, we interpret polynomials as signals by setting the value of a signal at time n equal to the polynomial coefficient associated with monomial z^n. For example, the polynomial $x(z) = 2+3z^2-z^3$ corresponds to the signal $x[n] = 2\delta[n]+3\delta[n-2]-\delta[n-3]$. The procedure for multiplying polynomials involves forming the product of all polynomial coefficients that result in an n-th order monomial and then summing them to obtain the polynomial coefficient of the n-th order monomial in the product. This corresponds to determining $w_n[k]$ and summing over k to obtain $y[n]$.

Evaluate the convolutions $y[n] = x[n] * h[n]$ using both the convolution sum evaluation procedure and as a product of polynomials.

(a) $x[n] = \delta[n] - 2\delta[n-1] + \delta[n-2], \quad h[n] = u[n] - u[n-3]$

$$\begin{aligned} x(z) &= 1 - 2z + z^2 \\ h(z) &= 1 + z + z^2 \\ y(z) &= x(z)h(z) \\ &= 1 - z - z^3 + z^4 \\ y[n] &= \delta[n] - \delta[n-1] - \delta[n-3] - \delta[n-4] \end{aligned}$$

$$\begin{aligned} y[n] &= x[n]*h[n] = h[n]-2h[n-1]+h[n-2] \\ &= \delta[n]-\delta[n-1]-\delta[n-3]-\delta[n-4] \end{aligned}$$

(b) $x[n]=u[n-1]-u[n-5], \;\; h[n]=u[n-1]-u[n-5]$

$$\begin{aligned} x(z) &= z+z^2+z^3+z^4 \\ h(z) &= z+z^2+z^3+z^4 \\ y(z) &= x(z)h(z) \\ &= z^2+2z^3+3z^4+4z^5+3z^6+2z^7+z^8 \\ y[n] &= \delta[n-2]+2\delta[n-3]+3\delta[n-4]+4\delta[n-5]+3\delta[n-6]+2\delta[n-7]+\delta[n-8] \end{aligned}$$

$$\begin{aligned} &\text{for } n-1 \le 0 && n \le 1 \\ & && y[n]=0 \\ &\text{for } n-1 \le 4 && n \le 5 \\ & && y[n]=\sum_{k=1}^{n-1} 1 = n-1 \\ &\text{for } n-4 \le 4 && n \le 8 \\ & && y[n]=\sum_{k=n-4}^{4} 1 = 9-n \\ &\text{for } n-4 \ge 5 && n \ge 9 \\ & && y[n]=0 \end{aligned}$$

$$y[n] = \delta[n-2]+2\delta[n-3]+3\delta[n-4]+4\delta[n-5]+3\delta[n-6]+2\delta[n-7]+\delta[n-8]$$

2.38. An LTI system has impulse response $h(t)$ depicted in Fig. P2.38. Use linearity and time invariance to determine themsystem output $y(t)$ if the input $x(t)$ is

(a) $x(t)=2\delta(t+2)+\delta(t-2)$

$$y(t) = 2h(t+2)+h(t-2)$$

(b) $x(t)=\delta(t-1)+\delta(t-2)+\delta(t-3)$

$$y(t) = h(t-1)+h(t-2)+h(t-3)$$

(c) $x(t)=\sum_{p=0}^{\infty}(-1)^p\delta(t-2p)$

$$y(t)=\sum_{p=0}^{\infty}(-1)^p h(t-2p)$$

2.39. Evaluate the continuous-time convolution integrals given below.

(a) $y(t) = (u(t) - u(t-2)) * u(t)$

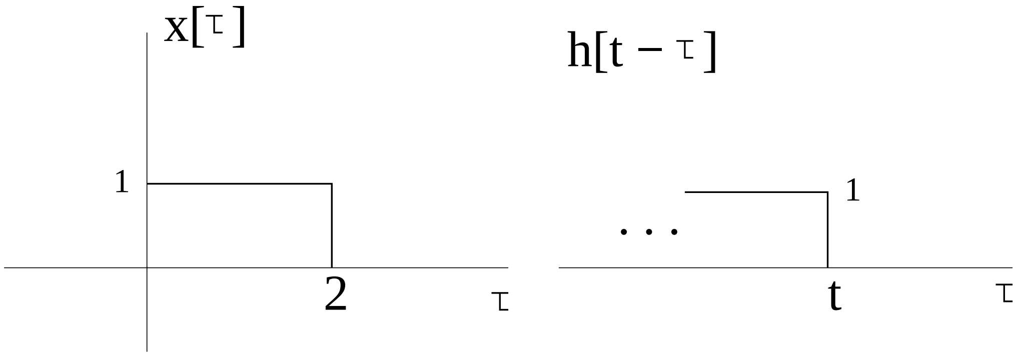


Figure P2.39. (a) Graph of $x[\tau]$ and $h[t-\tau]$

for $t < 0$

$$y(t) = 0$$

for $t < 2$

$$y(t) = \int_0^t d\tau = t$$

for $t \geq 2$

$$y(t) = \int_0^2 d\tau = 2$$

$$y(t) = \begin{cases} 0 & t < 0 \\ t & 0 \leq t < 2 \\ 2 & t \geq 2 \end{cases}$$

(b) $y(t) = e^{-3t}u(t) * u(t+3)$

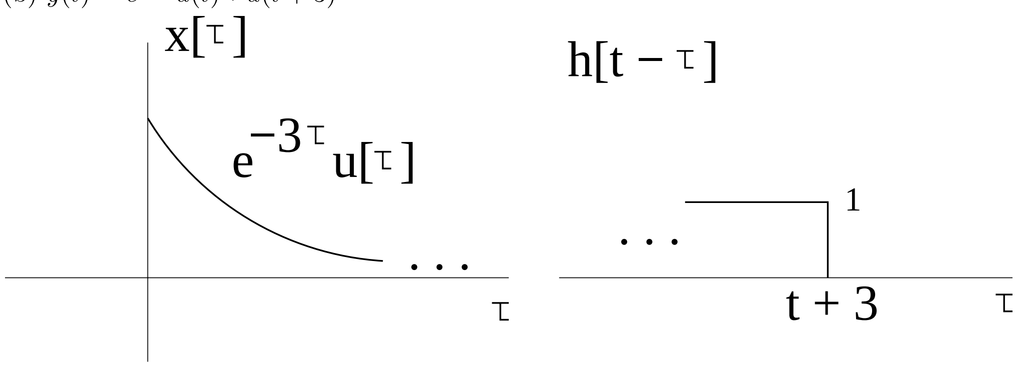


Figure P2.39. (b) Graph of $x[\tau]$ and $h[t-\tau]$

for $t + 3 < 0$ $\quad t < -3$

$$y(t) = 0$$

for $t \geq -3$

$$y(t) = \int_0^{t+3} e^{-3\tau} d\tau$$
$$y(t) = \frac{1}{3}\left[1 - e^{-3(t+3)}\right]$$
$$y(t) = \begin{cases} 0 & t < -3 \\ \frac{1}{3}\left[1 - e^{-3(t+3)}\right] & t \geq -3 \end{cases}$$

(c) $y(t) = \cos(\pi t)(u(t+1) - u(t-1)) * u(t)$

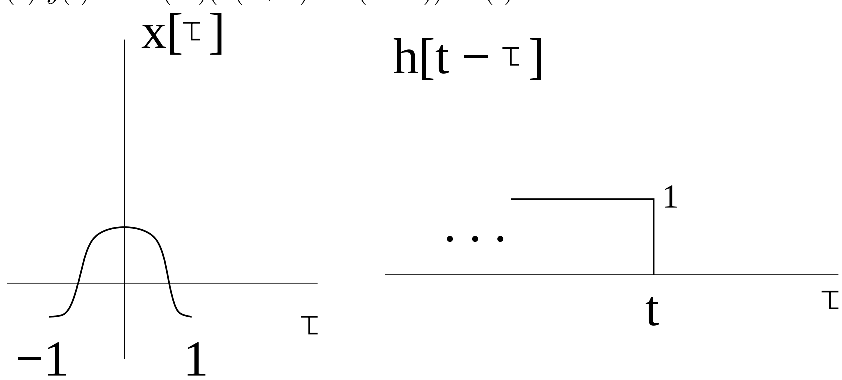


Figure P2.39. (c) Graph of $x[\tau]$ and $h[t-\tau]$

for $t < -1$

$$y(t) = 0$$

for $t < 1$

$$y(t) = \int_{-1}^{t} \cos(\pi t) d\tau$$
$$y(t) = \frac{1}{\pi}\sin(\pi t)$$

for $t > 1$

$$y(t) = \int_{-1}^{1} \cos(\pi t) d\tau$$
$$y(t) = 0$$
$$y(t) = \begin{cases} \frac{1}{\pi}\sin(\pi t) & -1 \leq t < 1 \\ 0 & \text{otherwise} \end{cases}$$

(d) $y(t) = (u(t+3) - u(t-1)) * u(-t+4)$

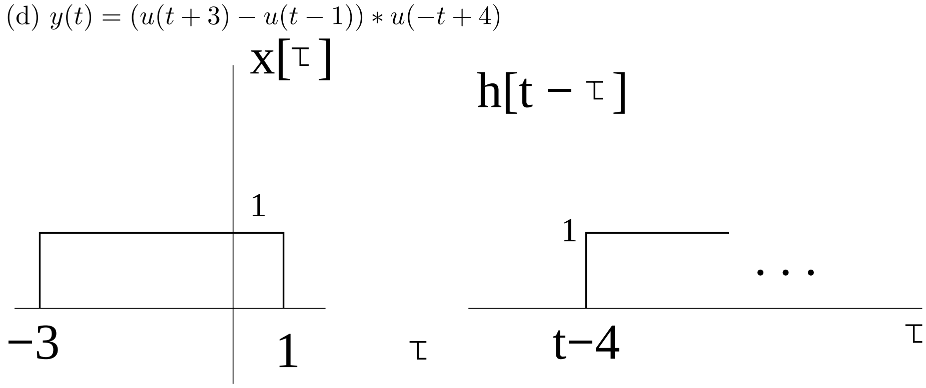


Figure P2.39. (d) Graph of $x[\tau]$ and $h[t-\tau]$

$$\begin{array}{rl}
\text{for } t-4<-3 & t<1 \\
& y(t)=\int_{-3}^{1} d\tau = 4 \\
\text{for } t-4<1 & t<5 \\
& y(t)=\int_{t-4}^{1} d\tau = 5-t \\
\text{for } t-4\geq 1 & t\geq 5 \\
& y(t)=0
\end{array}$$

$$y(t) = \begin{cases} 4 & t<1 \\ 5-t & 1\leq t<5 \\ 0 & t\geq 5 \end{cases}$$

(e) $y(t)=(tu(t)+(10-2t)u(t-5)-(10-t)u(t-10))*u(t)$

for $t<0$

$$y(t)=0$$

for $0\leq t<5$

$$y(t)=\int_0^t \tau d\tau = \frac{1}{2}t^2$$

for $5\leq t<10$

$$y(t)=\int_0^5 \tau d\tau + \int_5^t (10-\tau)d\tau = -\frac{1}{2}t^2+10t-25$$

for $t\geq 10$

$$y(t)=\int_0^5 \tau d\tau + \int_5^{10} (10-\tau)d\tau = 25$$

$$y(t) = \begin{cases} 0 & t<0 \\ \frac{1}{2}t^2 & 0\leq t<5 \\ -\frac{1}{2}t^2+10t-25 & 5\leq t<10 \\ 25 & t\geq 10 \end{cases}$$

(f) $y(t)=2t^2(u(t+1)-u(t-1))*2u(t+2)$

$$\begin{array}{rl}
\text{for } t+2<-1 & t<-3 \\
& y(t)=0 \\
\text{for } t+2<1 & -3\leq t<-1 \\
& y(t)=2\int_{-1}^{t+2} 2\tau^2 d\tau = \frac{4}{3}\left[(t+2)^3+1\right] \\
\text{for } t+2\geq -1 & t\geq -1 \\
& y(t)=2\int_{-1}^{1} 2\tau^2 d\tau = \frac{8}{3}
\end{array}$$

$$y(t) = \begin{cases} 0 & t < -3 \\ \frac{4}{3}\left[(t+2)^3+1\right] & -3 \leq t < -1 \\ \frac{8}{3} & t \geq -1 \end{cases}$$

(g) $y(t) = \cos(\pi t)(u(t+1) - u(t-1)) * (u(t+1) - u(t-1))$

$$\begin{aligned}
\text{for } t+1 < -1 \quad & t < -2 \\
& y(t) = 0 \\
\text{for } t+1 < 1 \quad & -2 \leq t < 0 \\
& y(t) = \int_{-1}^{t+1} \cos(\pi t)d\tau = \frac{1}{\pi}\sin(\pi(t+1)) \\
\text{for } t-1 < 1 \quad & 0 \leq t < 2 \\
& y(t) = \int_{t-1}^{1} \cos(\pi t)d\tau = -\frac{1}{\pi}\sin(\pi(t-1)) \\
\text{for } t-1 \geq 1 \quad & t \geq 2 \\
& y(t) = 0
\end{aligned}$$

$$y(t) = \begin{cases} 0 & t < -2 \\ \frac{1}{\pi}\sin(\pi(t+1)) & -2 \leq t < 0 \\ -\frac{1}{\pi}\sin(\pi(t-1)) & 0 \leq t < 2 \\ 0 & t \geq 2 \end{cases}$$

(h) $y(t) = \cos(2\pi t)(u(t+1) - u(t-1)) * e^{-t}u(t)$

$$\begin{aligned}
\text{for } t < -1 \quad & \\
& y(t) = 0 \\
\text{for } t < 1 \quad & -1 \leq t < 1 \\
& y(t) = \int_{-1}^{t} e^{-(t-\tau)}\cos(2\pi t)d\tau \\
& y(t) = e^{-t}\left[\frac{e^{\tau}}{1+4\pi^2}\left(\cos(2\pi\tau) + 2\pi\sin(2\pi\tau)\right)\Big|_{-1}^{t}\right] \\
& y(t) = \frac{\cos(2\pi t) + 2\pi\sin(2\pi t) - e^{-(t+1)}}{1+4\pi^2} \\
\text{for } t \geq 1 \quad & t \geq 1 \\
& y(t) = \int_{-1}^{1} e^{-(t-\tau)}\cos(2\pi t)d\tau \\
& y(t) = e^{-t}\left[\frac{e^{\tau}}{1+4\pi^2}\left(\cos(2\pi\tau) + 2\pi\sin(2\pi\tau)\right)\Big|_{-1}^{1}\right] \\
& y(t) = \frac{e^{-(t-1)} - e^{-(t+1)}}{1+4\pi^2}
\end{aligned}$$

$$y(t) = \begin{cases} 0 & t < -1 \\ \frac{\cos(2\pi t)+2\pi\sin(2\pi t)-e^{-(t+1)}}{1+4\pi^2} & -1 \leq t < -1 \\ \frac{e^{-(t-1)}-e^{-(t+1)}}{1+4\pi^2} & t \geq 1 \end{cases}$$

(i) $y(t) = (2\delta(t+1) + \delta(t-5)) * u(t-1)$

$$
\begin{aligned}
\text{for } t-1 < -1 \quad & t < 0 \\
& y(t) = 0 \\
\text{for } t-1 < 5 \quad & 0 \le t < 6 \\
& \text{By the sifting property.} \\
& y(t) = \int_{-\infty}^{t-1} 2\delta(t+1)d\tau = 2 \\
\text{for } t-1 \ge 5 \quad & t \ge 6 \\
& y(t) = \int_{-\infty}^{t-1} \left(2\delta(t+1) + \delta(t-5)\right) d\tau = 3
\end{aligned}
$$

$$
y(t) = \begin{cases} 0 & t < 0 \\ 2 & 0 \le t < 6 \\ 3 & t \ge 6 \end{cases}
$$

(j) $y(t) = (\delta(t+2) + \delta(t-2)) * (tu(t) + (10-2t)u(t-5) - (10-t)u(t-10))$

$$
\begin{aligned}
\text{for } t < -2 \quad & \\
& y(t) = 0 \\
\text{for } t < 2 \quad & -2 \le t < 2 \\
& y(t) = \int_{t-10}^{t} (t-\tau)\delta(\tau+2)d\tau = t+2 \\
\text{for } t-5 < -2 \quad & 2 \le t < 3 \\
& y(t) = \int_{t-10}^{t} (t-\tau)\delta(\tau+2)d\tau + \int_{t-10}^{t} (t-\tau)\delta(\tau-2)d\tau = 2t \\
\text{for } t-5 < 2 \quad & 3 \le t < 7 \\
& y(t) = \int_{t-10}^{t} [10-(t-\tau)]\delta(\tau+2)d\tau + \int_{t-10}^{t} (t-\tau)\delta(\tau-2)d\tau = 6 \\
\text{for } t-10 < -2 \quad & 7 \le t < 8 \\
& y(t) = \int_{t-10}^{t} [10-(t-\tau)]\delta(\tau+2)d\tau + \int_{t-10}^{t} [10-(t-\tau)]\delta(\tau-2)d\tau = 20-2t \\
\text{for } t-10 < 2 \quad & 8 \le t < 12 \\
& y(t) = \int_{t-10}^{t} [10-(t-\tau)]\delta(\tau-2)d\tau = 12-t \\
\text{for } t-10 \ge 2 \quad & t \ge 12 \\
& y(t) = 0
\end{aligned}
$$

$$
y(t) = \begin{cases} 0 & t < -2 \\ t+2 & -2 \le t < 2 \\ 2t & 2 \le t < 3 \\ 6 & 3 \le t < 7 \\ 20-2t & 7 \le t < 8 \\ 12-t & 8 \le t < 12 \\ 0 & t \ge 12 \end{cases}
$$

(k) $y(t) = e^{-\gamma t}u(t) * (u(t+2) - u(t))$

for $t < -2$

$$y(t) = 0$$

for $t < 0$ $\quad -2 \le t < 0$

$$y(t) = \int_{-2}^{t} e^{-\gamma(t-\tau)} d\tau$$

$$y(t) = \frac{1}{\gamma}\left[1 - e^{-\gamma(t+2)}\right]$$

for $t \ge 0$

$$y(t) = \int_{-2}^{0} e^{-\gamma(t-\tau)} d\tau$$

$$y(t) = \frac{1}{\gamma}\left[e^{-\gamma t} - e^{-\gamma(t+2)}\right]$$

$$y(t) = \begin{cases} 0 & t < -2 \\ \frac{1}{\gamma}\left[1 - e^{-\gamma(t+2)}\right] & -2 \le t < 0 \\ \frac{1}{\gamma}\left[e^{-\gamma t} - e^{-\gamma(t+2)}\right] & t \ge 0 \end{cases}$$

(l) $y(t) = e^{-\gamma t}u(t) * \sum_{p=0}^{\infty} \left(\frac{1}{4}\right)^p \delta(t - 2p)$

for $t < 0$

$$y(t) = 0$$

for $t \ge 0$

$$y(t) = \int_{0}^{t} e^{-\gamma\tau} \sum_{p=0}^{\infty} \left(\frac{1}{4}\right)^p \delta(t - 2p - \tau) d\tau$$

$$y(t) = \sum_{p=0}^{\infty} \left(\frac{1}{4}\right)^p \int_{0}^{t} e^{-\gamma\tau} \delta(t - 2p - \tau) d\tau$$

Using the sifting property yields

$$y(t) = \sum_{p=0}^{\infty} \left(\frac{1}{4}\right)^p e^{-\gamma(t-2p)} u(t - 2p)$$

for $0 \le t < 2$

$$y(t) = e^{-\gamma t}$$

for $2 \le t < 4$

$$y(t) = e^{-\gamma t} + \frac{1}{4} e^{-\gamma(t-2)}$$

for $4 \le t < 6$

$$y(t) = e^{-\gamma t} + \frac{1}{4} e^{-\gamma(t-2)} + \frac{1}{16} e^{-\gamma(t-4)}$$

for $2l \le t < 2l + 2$

$$y(t) = \left(\sum_{p=0}^{l} \left(\frac{1}{4}\right)^p e^{2p\gamma}\right) e^{-\gamma t}$$

(m) $y(t) = (2\delta(t) + \delta(t-2)) * \sum_{p=0}^{\infty} \left(\frac{1}{2}\right)^p \delta(t - p)$

$$\text{let} \qquad x_1(t) = \sum_{p=0}^{\infty} \left(\frac{1}{2}\right)^p \delta(t-p)$$

for $t < 0$

$$y(t) = 0$$

for $t < 2$

$$y(t) = 2\delta(t) * x_1(t) = 2x_1(t)$$

for $t \geq 2$

$$y(t) = 2\delta(t) * x_1(t) + \delta(t-2) * x_1(t) = 2x_1(t) + x_1(t-2)$$

$$y(t) = \begin{cases} 0 & t < 0 \\ 2\sum_{p=0}^{\infty} \left(\frac{1}{2}\right)^p \delta(t-p) & 0 \leq t < 2 \\ 2\sum_{p=0}^{\infty} \left(\frac{1}{2}\right)^p \delta(t-p) + \sum_{p=0}^{\infty} \left(\frac{1}{2}\right)^p \delta(t-p-2) & t \geq 0 \end{cases}$$

(n) $y(t) = e^{-\gamma t}u(t) * e^{\beta t}u(-t)$

for $t < 0$

$$y(t) = \int_0^{\infty} e^{\beta t} e^{-(\beta+\gamma)\tau} d\tau$$

$$y(t) = \frac{e^{\beta t}}{\beta + \gamma}$$

for $t \geq 0$

$$y(t) = \int_t^{\infty} e^{\beta t} e^{-(\beta+\gamma)\tau} d\tau$$

$$y(t) = \frac{e^{\beta t}}{\beta + \gamma} e^{-(\beta+\gamma)t}$$

$$y(t) = \frac{e^{-\gamma t}}{\beta + \gamma}$$

$$y(t) = \begin{cases} \frac{e^{\beta t}}{\beta+\gamma} & t < 0 \\ \frac{e^{-\gamma t}}{\beta+\gamma} & t \geq 0 \end{cases}$$

(o) $y(t) = u(t) * h(t)$ where $h(t) = \begin{cases} e^{2t} & t < 0 \\ e^{-3t} & t \geq 0 \end{cases}$

for $t < 0$

$$y(t) = \int_{-\infty}^{t} e^{2\tau} d\tau$$

$$y(t) = \frac{1}{2} e^{2t}$$

for $t \geq 0$

$$y(t) = \int_{-\infty}^{0} e^{2\tau} d\tau + \int_0^t e^{-3\tau} d\tau$$

$$y(t) = \frac{1}{2} + \frac{1}{3}\left[1 - e^{-3t}\right]$$

$$y(t) = \begin{cases} \frac{1}{2}e^{2t} & t < 0 \\ \frac{1}{2} + \frac{1}{3}\left[1 - e^{-3t}\right] & t \geq 0 \end{cases}$$

2.40. Consider the continuous-time signals depicted in Fig. P2.40. Evaluate the following convolution integrals:.
(a)$m(t) = x(t) * y(t)$

$$
\begin{aligned}
\text{for } t+1<0 \qquad & t<-1 \\
& m(t)=0 \\
\text{for } t+1<2 \qquad & -1 \le t < 1 \\
& m(t)=\int_0^{t+1} d\tau = t+1 \\
\text{for } t+1<4 \qquad & 1 \le t < 3 \\
& m(t)=\int_{t-1}^{2} d\tau + \int_2^{t+1} 2d\tau = t+1 \\
\text{for } t-1<4 \qquad & 3 \le t < 5 \\
& m(t)=\int_{t-1}^{4} 2d\tau = 10-2t \\
\text{for } t-1\ge 4 \qquad & t \ge 5 \\
& m(t)=0
\end{aligned}
$$

$$
m(t) = \begin{cases} 0 & t<-1 \\ t+1 & -1 \le t < 1 \\ t+1 & 1 \le t < 3 \\ 10-2t & 3 \le t < 5 \\ 0 & t \ge 5 \end{cases}
$$

(b) $m(t) = x(t) * z(t)$

$$
\begin{aligned}
\text{for } t+1<-1 \qquad & t<-2 \\
& m(t)=0 \\
\text{for } t+1<0 \qquad & -2 \le t < -1 \\
& m(t)=-\int_{-1}^{t+1} d\tau = -t-2 \\
\text{for } t+1<1 \qquad & -1 \le t < 0 \\
& m(t)=-\int_{-1}^{0} d\tau + \int_0^{t+1} d\tau = t \\
\text{for } t-1<0 \qquad & 0 \le t < 1 \\
& m(t)=-\int_{t-1}^{0} d\tau + \int_0^{1} d\tau = t \\
\text{for } t-1<1 \qquad & 1 \le t < 2 \\
& m(t)=\int_{t-1}^{1} d\tau = 2-t \\
\text{for } t-1\ge 1 \qquad & t \ge 2 \\
& m(t)=0
\end{aligned}
$$

$$m(t) = \begin{cases} 0 & t < -2 \\ -t-2 & -2 \le t < -1 \\ t & -1 \le t < 0 \\ t & 0 \le t < 1 \\ 2-t & 1 \le t < 2 \\ 0 & t \ge 2 \end{cases}$$

(c) $m(t) = x(t) * f(t)$

for $t < -1$

$$m(t) = 0$$

for $t < 0$ $\quad -1 \le t < 0$

$$m(t) = \int_{-1}^{t} e^{-(t-\tau)} d\tau = 1 - e^{-(t+1)}$$

for $t < 1$ $\quad 0 \le t < 1$

$$m(t) = \int_{t-1}^{t} e^{-(t-\tau)} d\tau = 1 - e^{-1}$$

for $t < 2$ $\quad 1 \le t < 2$

$$m(t) = \int_{t-1}^{1} e^{-(t-\tau)} d\tau = e^{1-t} - e^{-1}$$

for $t \ge 2$

$$m(t) = 0$$

$$m(t) = \begin{cases} 0 & t < -1 \\ 1 - e^{-(t+1)} & -1 \le t < 0 \\ 1 - e^{-1} & 0 \le t < 1 \\ e^{1-t} - e^{-1} & 1 \le t < 2 \\ 0 & t \ge 2 \end{cases}$$

(d) $m(t) = x(t) * a(t)$
By inspection, since $x(\tau)$ has a width of 2 and $a(t-\tau)$ has the period 2 and duty cycle $\frac{1}{2}$, the area under the overlapping signals is always 1, thus $m(t) = 1$ for all t.

(e) $m(t) = y(t) * z(t)$

for $t < -1$

$$m(t) = 0$$

for $t < 0$ $\quad -1 \le t < 0$

$$m(t) = -\int_{-1}^{t} d\tau = -t - 1$$

for $t < 1$ $\quad 0 \le t < 1$

$$m(t) = -\int_{-1}^{0} d\tau + \int_{0}^{t} d\tau = -1 + t$$

for $t < 2$ $\quad 1 \le t < 2$

$$m(t) = -2\int_{-1}^{t-2} d\tau - \int_{t-2}^{0} d\tau + \int_{0}^{1} d\tau = -t+1$$

for $t-2<1$ $\quad 2 \le t < 3$

$$m(t) = -2\int_{-1}^{0} d\tau + 2\int_{0}^{t-2} d\tau + \int_{t-2}^{1} d\tau = t-3$$

for $t-4<0$ $\quad 3 \le t < 4$

$$m(t) = -2\int_{t-4}^{0} d\tau + 2\int_{0}^{1} d\tau = 2t-6$$

for $t-4<1$ $\quad 4 \le t < 5$

$$m(t) = 2\int_{t-4}^{1} d\tau = 10-2t$$

for $t \ge 5$

$$m(t) = 0$$

$$m(t) = \begin{cases} 0 & t < -1 \\ -t-1 & -1 \le t < 0 \\ -1+t & 0 \le t < 1 \\ -t+1 & 1 \le t < 2 \\ t-3 & 2 \le t < 3 \\ 2t-6 & 3 \le t < 4 \\ 10-2t & 4 \le t < 5 \\ 0 & t \ge 5 \end{cases}$$

(f) $m(t) = y(t) * w(t)$

for $t < 0$

$$m(t) = 0$$

for $t < 1$ $\quad 0 \le t < 1$

$$m(t) = \int_{0}^{t} d\tau = t$$

for $t < 2$ $\quad 1 \le t < 2$

$$m(t) = \int_{0}^{1} d\tau - \int_{1}^{t} d\tau = 2-t$$

for $t < 3$ $\quad 2 \le t < 3$

$$m(t) = 2\int_{0}^{t-2} d\tau + \int_{t-2}^{1} d\tau - \int_{1}^{t} d\tau = 0$$

for $t-4<0$ $\quad 3 \le t < 4$

$$m(t) = 2\int_{0}^{1} d\tau - 2\int_{1}^{t-2} d\tau - \int_{t-2}^{3} d\tau = 3-t$$

for $t-4<1$ $\quad 4 \le t < 5$

$$m(t) = 2\int_{t-4}^{1} d\tau - 2\int_{1}^{t-2} d\tau - \int_{t-2}^{3} d\tau = 11-3t$$

for $t-4<3$ $\quad 5 \le t < 7$

$$m(t) = -2\int_{t-4}^{3} d\tau = 2t-14$$

for $t \ge 7$

$$m(t) = 0$$

$$m(t) = \begin{cases} 0 & t < 0 \\ t & 0 \le t < 1 \\ 2 - t & 1 \le t < 2 \\ 0 & 2 \le t < 3 \\ 3 - t & 3 \le t < 4 \\ 11 - 3t & 4 \le t < 5 \\ 2t - 14 & 5 \le t < 7 \\ 0 & t \ge 7 \end{cases}$$

(g) $m(t) = y(t) * g(t)$

for $t < -1$

$$m(t) = 0$$

for $t < 1$ $\quad -1 \le t < 1$

$$m(t) = \int_{-1}^{t} \tau d\tau = 0.5[t^2 - 1]$$

for $t - 2 < 1$ $\quad 1 \le t < 3$

$$m(t) = \int_{-1}^{t-2} 2\tau d\tau + \int_{t-2}^{1} \tau d\tau = 0.5t^2 + 0.5(t-2)^2 - 1$$

for $t - 4 < 1$ $\quad 3 \le t < 5$

$$m(t) = \int_{t-4}^{1} 2\tau d\tau = 1 - (t-4)^2$$

for $t \ge 5$

$$m(t) = 0$$

$$m(t) = \begin{cases} 0 & t < -1 \\ 0.5[t^2 - 1] & -1 \le t < 1 \\ 0.5t^2 + 0.5(t-2)^2 - 1 & 1 \le t < 3 \\ 1 - (t-4)^2 & 3 \le t < 5 \\ 0 & t \ge 5 \end{cases}$$

(h) $m(t) = y(t) * c(t)$

for $t + 2 < 0$ $\quad t < -2$

$$m(t) = 0$$

for $t + 2 < 2$ $\quad -2 \le t < 0$

$$m(t) = 1$$

for $t - 1 < 0$ $\quad 0 \le t < 1$

$$m(t) = \int_{0}^{t} d\tau + 2 = t + 2$$

for $t < 2$ $\quad 1 \le t < 2$

$$m(t) = \int_{t-1}^{t} d\tau + 2 = 3$$

$$\begin{aligned}
\text{for } t < 3 \qquad & 2 \le t < 3 \\
& m(t) = -1 + \int_{t-1}^{2} d\tau + 2\int_{2}^{t} d\tau = t - 2 \\
\text{for } t < 4 \qquad & 3 \le t < 4 \\
& m(t) = -1 + 2\int_{t-2}^{t} d\tau = 1 \\
\text{for } t < 5 \qquad & 4 \le t < 5 \\
& m(t) = -2 + 2\int_{t-1}^{4} d\tau = 8 - 2t \\
\text{for } t - 2 < 4 \qquad & 5 \le t < 6 \\
& m(t) = -2 \\
\text{for } t \ge 6 \qquad & \\
& m(t) = 0
\end{aligned}$$

$$m(t) = \begin{cases} 0 & t < -2 \\ 1 & -2 \le t < 0 \\ t+2 & 0 \le t < 1 \\ 3 & 1 \le t < 2 \\ t-2 & 2 \le t < 3 \\ 1 & 3 \le t < 4 \\ 8-2t & 5 \le t < 5 \\ -2 & 5 \le t < 6 \\ 0 & t \ge 6 \end{cases}$$

(i) $m(t) = z(t) * f(t)$

$$\begin{aligned}
\text{for } t + 1 < 0 \qquad & t < -1 \\
& m(t) = 0 \\
\text{for } t + 1 < 1 \qquad & -1 \le t < 0 \\
& m(t) = -\int_{0}^{t+1} e^{-\tau} d\tau = e^{-(t+1)} - 1 \\
\text{for } t < 1 \qquad & 0 \le t < 1 \\
& m(t) = \int_{0}^{t} e^{-\tau} d\tau - \int_{t}^{1} e^{-\tau} d\tau = 1 + e^{-1} - 2e^{-t} \\
\text{for } t - 1 < 1 \qquad & 1 \le t < 2 \\
& m(t) = \int_{t-1}^{1} e^{-\tau} d\tau = -e^{-1} + e^{-(t-1)} \\
\text{for } t \ge 2 \qquad & \\
& m(t) = 0
\end{aligned}$$

$$m(t) = \begin{cases} 0 & t < -1 \\ e^{-(t+1)} - 1 & -1 \le t < 0 \\ 1 + e^{-1} - 2e^{-t} & 0 \le t < 1 \\ -e^{-1} + e^{-(t-1)} & 1 \le t < 2 \\ 0 & t \ge 2 \end{cases}$$

(j) $m(t) = z(t) * g(t)$

$$
\begin{aligned}
\text{for } t+1 < -1 \qquad & t < -2 \\
& m(t) = 0 \\
\text{for } t+1 < 0 \qquad & -2 \leq t < -1 \\
& m(t) = -\int_{-1}^{t+1} \tau d\tau = -0.5[(t+1)^2 - 1] \\
\text{for } t < 0 \qquad & -1 \leq t < 0 \\
& m(t) = -\int_{t}^{t+1} \tau d\tau + \int_{-1}^{t} \tau d\tau = -\frac{1}{2}\left((t+1)^2 - t^2\right) + \frac{1}{2}\left(t^2 - 1\right) \\
\text{for } t-1 < 0 \qquad & 0 \leq t < 1 \\
& m(t) = \int_{t-1}^{t} \tau d\tau - \int_{t}^{1} \tau d\tau = 0.5[t^2 - (t-1)^2] - 0.5[1 - t^2] \\
\text{for } t-1 < 1 \qquad & 1 \leq t < 2 \\
& m(t) = \int_{t-1}^{1} \tau d\tau = 0.5[1 - (t-1)^2] \\
\text{for } t \geq 2 \qquad & \\
& m(t) = 0
\end{aligned}
$$

$$
m(t) = \begin{cases} 0 & t < -2 \\ -0.5[(t+1)^2 - 1] & -2 \leq t < -1 \\ -\frac{1}{2}\left((t+1)^2 - t^2\right) + \frac{1}{2}\left(t^2 - 1\right) & -1 \leq t < 0 \\ 0.5[t^2 - (t-1)^2] - 0.5[1 - t^2] & 0 \leq t < 1 \\ 0.5[1 - (t-1)^2] & 1 \leq t < 2 \\ 0 & t \geq 2 \end{cases}
$$

(k) $m(t) = z(t) * b(t)$

$$
\begin{aligned}
\text{for } t+1 < -3 \qquad & t < -4 \\
& m(t) = 0 \\
\text{for } t+1 < -2 \qquad & -4 \leq t < -3 \\
& m(t) = -\int_{-3}^{t+1} (\tau+3) d\tau = -0.5(t+1)^2 + \frac{9}{2} - 3(t+1) - 9 \\
\text{for } t < -2 \qquad & -3 \leq t < -2 \\
& m(t) = \int_{-3}^{t} (\tau+3) d\tau - \int_{t}^{-2} (\tau+3) d\tau - \int_{-2}^{t} d\tau = t^2 + 5t + \frac{11}{2} \\
\text{for } t-1 < -2 \qquad & -2 \leq t < -1 \\
& m(t) = \int_{t-1}^{2} (\tau+3) d\tau + \int_{-2}^{t} d\tau - \int_{t}^{t+1} d\tau = 12 - \frac{1}{2}(t-1)^2 - 2t \\
\text{for } t-1 \geq -2 \qquad & t \geq -1 \\
& m(t) = 0
\end{aligned}
$$

$$m(t) = \begin{cases} 0 & t < -4 \\ -0.5(t+1)^2 - 3(t+1) - \frac{9}{2} & -4 \le t < -3 \\ t^2 + 5t + \frac{11}{2} & -3 \le t < -2 \\ 12 - \frac{1}{2}(t-1)^2 - 2t & -2 \le t < -1 \\ 0 & t \ge 2 \end{cases}$$

(l) $m(t) = w(t) * g(t)$

for $t < -1$

$$m(t) = 0$$

for $t < 0$ $\quad -1 \le t < 0$

$$m(t) = -\int_{-1}^{t} \tau d\tau = -0.5t^2 + 0.5$$

for $t < 1$ $\quad 0 \le t < 1$

$$m(t) = \int_{t-1}^{t} \tau d\tau - \int_{-1}^{t-1} \tau d\tau = 0.5t^2 - (t-1)^2 + 0.5$$

for $t - 1 < 1$ $\quad 1 \le t < 2$

$$m(t) = -\int_{-1}^{t-1} \tau d\tau + \int_{t-1}^{1} \tau d\tau = 1 - (t-1)^2$$

for $t - 3 < 1$ $\quad 2 \le t < 4$

$$m(t) = -\int_{t-3}^{1} \tau d\tau = 0.5(t-3)^2 - 0.5$$

for $t - 3 \ge 1$ $\quad t \ge 4$

$$m(t) = 0$$

$$m(t) = \begin{cases} 0 & t < -1 \\ -0.5t^2 + 0.5 & -1 \le t < 0 \\ 0.5t^2 - (t-1)^2 + 0.5 & 0 \le t < 1 \\ 1 - (t-1)^2 & 1 \le t < 2 \\ 0.5(t-3)^2 - 0.5 & 2 \le t < 4 \\ 0 & t \ge 4 \end{cases}$$

(m) $m(t) = w(t) * a(t)$

let $\quad a'(t) = \begin{cases} 1 & 0 \le t \le 1 \\ 0 & \text{otherwise} \end{cases}$

then $\quad a(t) = \sum_{k=-\infty}^{\infty} a'(t - 2k)$

consider $\quad m'(t) = w(t) * a'(t)$

for $t < 0$

$$m'(t) = 0$$

for $t < 1$ $\quad 0 \le t < 1$

$$m'(t) = \int_{0}^{t} d\tau = t$$

$$
\begin{array}{ll}
\text{for } t-1<1 & 1\le t<2 \\
& m'(t) = -\int_0^{t-1} d\tau + \int_{t-1}^{1} d\tau = 3-2t \\
\text{for } t-3<0 & 2\le t<3 \\
& m'(t) = -\int_0^{1} d\tau = -1 \\
\text{for } t-3<1 & 3\le t<4 \\
& m'(t) = -\int_{t-3}^{1} d\tau = t-4 \\
\text{for } t-3\ge 1 & t\ge 4 \\
& m'(t) = 0
\end{array}
$$

$$
m'(t) = \begin{cases} 0 & t<0 \\ t & 0\le t<1 \\ 3-2t & 1\le t<2 \\ -1 & 2\le t<3 \\ t-4 & 3\le t<4 \\ 0 & t\ge 4 \end{cases}
$$

$$
m(t) = \sum_{k=-\infty}^{\infty} m'(t-2k)
$$

(n) $m(t) = f(t) * g(t)$

$$
\begin{array}{rl}
\text{for } t<-1 & \\
& m(t) = 0 \\
\text{for } t<0 & -1\le t<0 \\
& m(t) = -\int_{-1}^{t} \tau e^{-(t-\tau)} d\tau = t-1+2e^{-(t+1)} \\
\text{for } t<1 & 0\le t<1 \\
& m(t) = \int_{t-1}^{t} \tau e^{-(t-\tau)} d\tau = t-1-(t-2)e^{-1} \\
\text{for } t-1<1 & 1\le t<2 \\
& m(t) = \int_{t-1}^{1} \tau e^{-(t-\tau)} d\tau = -e^{-1}(t-2) \\
\text{for } t-2\ge 1 & \\
& m(t) = 0
\end{array}
$$

$$
m(t) = \begin{cases} 0 & t<-1 \\ t-1+2e^{-(t+1)} & -1\le t<0 \\ t-1-(t-2)e^{-1} & 0\le t<1 \\ -e^{-1}(t-2) & 1\le t<2 \\ 0 & t\ge 1 \end{cases}
$$

(o) $m(t) = f(t) * d(t)$

$$
\begin{aligned}
d(t) &= \sum_{k=-\infty}^{\infty} f(t-k) \\
m'(t) &= f(t) * f(t) \\
m(t) &= \sum_{k=-\infty}^{\infty} m'(t-k)
\end{aligned}
$$

for $t < 0$

$$m'(t) = 0$$

for $t < 1$ $\quad 0 \le t < 1$

$$m'(t) = \int_0^t e^{-(t-\tau)} e^{-\tau} d\tau = te^{-t}$$

for $t < 2$ $\quad 1 \le t < 2$

$$m'(t) = e^{-t} \int_{t-1}^1 e^{\tau} e^{-\tau} d\tau = (2-t)e^{-t}$$

for $t \ge 2$

$$m'(t) = 0$$

$$
\begin{aligned}
m'(t) &= \begin{cases} 0 & t < 0 \\ te^{-t} & 0 \le t < 1 \\ (2-t)e^{-t} & 1 \le t < 2 \\ 0 & t \ge 2 \end{cases} \\
m(t) &= \sum_{k=-\infty}^{\infty} m'(t-k)
\end{aligned}
$$

(p) $m(t) = z(t) * d(t)$

let $\quad d'(t) = \begin{cases} e^{-t} & 0 \le t \le 1 \\ 0 & \text{otherwise} \end{cases}$

then $\quad d(t) = \sum_{k=-\infty}^{\infty} d'(t-k)$

consider $\quad m'(t) = z(t) * d'(t)$

for $t < -1$

$$m'(t) = 0$$

for $t < 0$ $\quad -1 \le t < 0$

$$m'(t) = \int_{-1}^t -e^{-(t-\tau)} d\tau = e^{-(t+1)} - 1$$

for $t < 1$ $\quad 0 \le t < 1$

$$m'(t) = -\int_{t-1}^0 -e^{-(t-\tau)} d\tau + \int_0^t e^{-(t-\tau)} d\tau = 1 + e^{-1} - 2e^{-t}$$

for $t - 1 < 1$ $\quad 1 \le t < 2$

$$m'(t) = \int_{t-1}^1 e^{-(t-\tau)} d\tau = e^{-(t-1)} - e^{-1}$$

$$\text{for } t-1 \geq 1 \qquad t \geq 2$$
$$m'(t) = 0$$

$$m'(t) = \begin{cases} 0 & t < -1 \\ e^{-(t+1)} - 1 & -1 \leq t < 0 \\ 1 + e^{-1} - 2e^{-t} & 0 \leq t < 1 \\ e^{-(t-1)} - e^{-1} & 1 \leq t < 2 \\ 0 & t \geq 2 \end{cases}$$

$$m(t) = \sum_{k=-\infty}^{\infty} m'(t-k)$$

2.41. Suppose we model the effect of imperfections in a communication channel as the RC circuit depicted in Fig. P2.41(a). Here the input $x(t)$ is the transmitted signal and the output $y(t)$ is the received signal. Assume the message is represented in binary format and that a "1" is communicated in an interval of length T by transmitting the symbol $p(t)$ depicted in Fig. P2.41 (b) in the pertinent interval and that a "0" is communicated by transmitting $-p(t)$ in the pertinent interval. Figure P2.41 (c) illustrates the transmitted waveform for communicating the sequence "1101001". Assume that $T = 1/(RC)$.
(a) Use convolution to calculate the received signal due to transmission of a single "1" at time $t = 0$. Note that the received waveform extends beyond time T and into the interval allocated for the next bit, $T < t < 2T$. This contamination is called intersymbol interference (ISI), since the received waveform at any time is interfered with by previous symbols.

The impulse response of an RC circuit is $h(t) = \frac{1}{RC} e^{-\frac{t}{RC}} u(t)$. The output of the system is the convolution of the input, $x(t)$ with the impulse response, $h(t)$.

$$y_p(t) = h(t) * p(t)$$

for $t < 0$

$$y_p(t) = 0$$

$$\text{for } t < T \qquad 0 \leq t < T$$
$$y_p(t) = \int_0^t \frac{1}{RC} e^{-\frac{(t-\tau)}{RC}} d\tau$$
$$y_p(t) = 1 - e^{-\frac{t}{RC}}$$

for $t \geq T$

$$y_p(t) = \int_0^T \frac{1}{RC} e^{-\frac{(t-\tau)}{RC}} d\tau$$
$$y_p(t) = e^{-\frac{(t-T)}{RC}} - e^{-\frac{t}{RC}}$$

$$y_p(t) = \begin{cases} 0 & t < 0 \\ 1 - e^{-\frac{t}{RC}} & 0 \leq t < T \\ e^{-\frac{(t-T)}{RC}} - e^{-\frac{t}{RC}} & t \geq T \end{cases}$$

(b) Use convolution to calculate the received signal due to transmission of the sequences "1110" and "1000". Compare the received waveforms to the output of an ideal channel ($h(t) = \delta(t)$) to evaluate the

effect of ISI for the following choices of RC:
(i) $RC = 1/T$
(ii) $RC = 5/T$
(iii) $RC = 1/(5T)$
Assuming $T = 1$

$$(1) \qquad \begin{aligned} x(t) &= p(t) + p(t-1) + p(t-2) - p(t-3) \\ y(t) &= y_p(t) + y_p(t-1) + y_p(t-2) + y_p(t-3) \end{aligned}$$

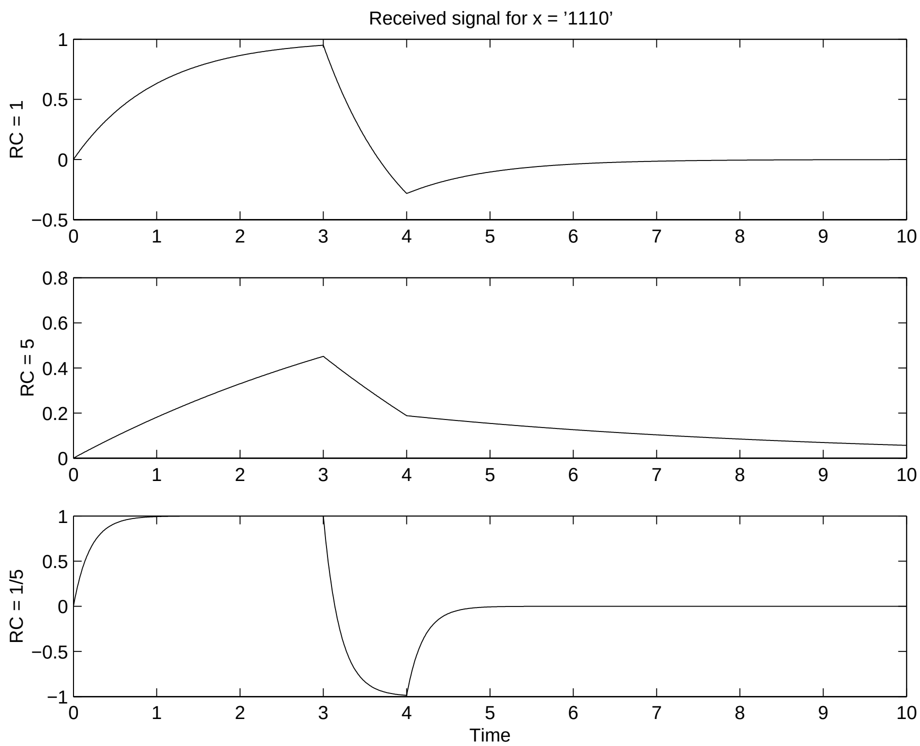


Figure P2.41. x = “1110”

$$(2) \qquad \begin{aligned} x(t) &= p(t) - p(t-1) - p(t-2) - p(t-3) \\ y(t) &= y_p(t) - y_p(t-1) - y_p(t-2) + y_p(t-3) \end{aligned}$$

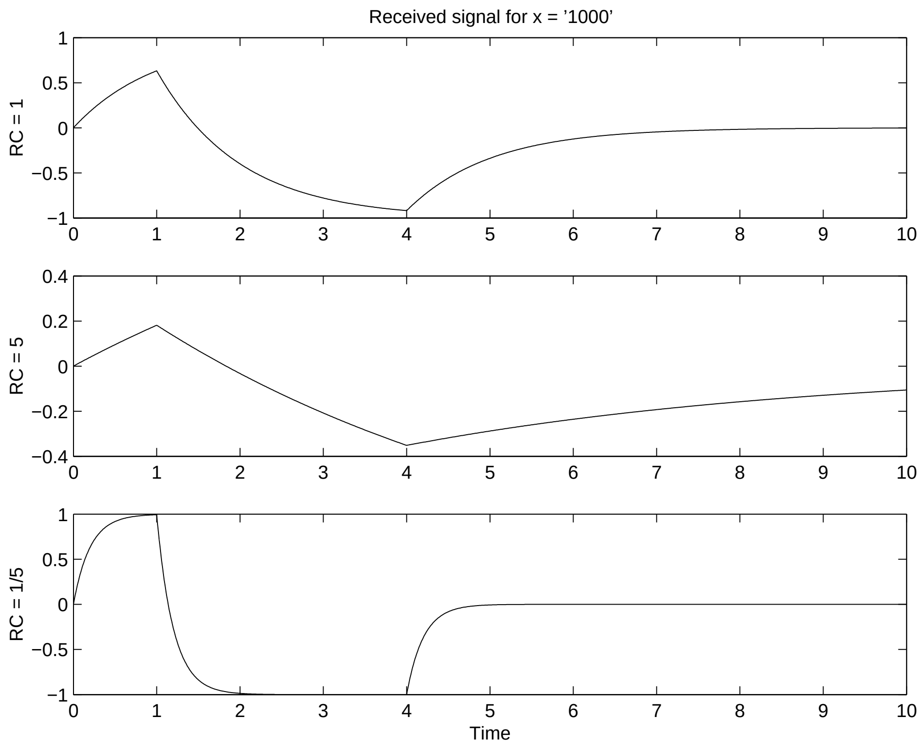


Figure P2.41. x = "1000"

From the two graphs it becomes apparent that as T becomes smaller, ISI becomes a larger problem for the communication system. The bits blur together and it becomes increasingly difficult to determine if a '1' or '0' is transmitted.

2.42. Use the definition of the convolution sum to prove the following properties

(a) Distributive: $x[n] * (h[n] + g[n]) = x[n] * h[n] + x[n] * g[n]$

$$\begin{aligned}
\text{LHS} &= x[n] * (h[n] + g[n]) \\
&= \sum_{k=-\infty}^{\infty} x[k]\left(h[n-k] + g[n-k]\right) : \text{ The definition of convolution.} \\
&= \sum_{k=-\infty}^{\infty} \left(x[k]h[n-k] + x[k]g[n-k]\right) : \text{ the dist. property of mult.} \\
&= \sum_{k=-\infty}^{\infty} x[k]h[n-k] + \sum_{k=-\infty}^{\infty} x[k]g[n-k] \\
&= x[n] * h[n] + x[n] * g[n] \\
&= \text{RHS}
\end{aligned}$$

(b) Associative: $x[n] * (h[n] * g[n]) = (x[n] * h[n]) * g[n]$

$$\text{LHS} = x[n] * (h[n] * g[n])$$

$$
\begin{aligned}
&= x[n] * \left(\sum_{k=-\infty}^{\infty} h[k]g[n-k] \right) \\
&= \sum_{l=-\infty}^{\infty} x[l] \left(\sum_{k=-\infty}^{\infty} h[k]g[n-k-l] \right) \\
&= \sum_{l=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} (x[l]h[k]g[n-k-l]) \\
&\quad \text{Use } v = k + l \text{ and exchange the order of summation} \\
&= \sum_{v=-\infty}^{\infty} \left(\sum_{l=-\infty}^{\infty} x[l]h[v-l] \right) g[n-v] \\
&= \sum_{v=-\infty}^{\infty} (x[v] * h[v])\, g[n-v] \\
&= (x[n] * h[n]) * g[n] \\
&= \text{RHS}
\end{aligned}
$$

(c) Commutative: $x[n] * h[n] = h[n] * x[n]$

$$
\begin{aligned}
\text{LHS} &= x[n] * h[n] \\
&= \sum_{k=-\infty}^{\infty} x[k]h[n-k] \\
&\quad \text{Use } k = n - l \\
&= \sum_{l=-\infty}^{\infty} x[n-l]h[l] \\
&= \sum_{l=-\infty}^{\infty} h[l]x[n-l] \\
&= \text{RHS}
\end{aligned}
$$

2.43. An LTI system has the impulse response depicted in Fig. P2.43.
(a) Express the system output $y(t)$ as a function of the input $x(t)$.

$$
\begin{aligned}
y(t) &= x(t) * h(t) \\
&= x(t) * \left(\frac{1}{\Delta}\delta(t) - \frac{1}{\Delta}\delta(t-\Delta) \right) \\
&= \frac{1}{\Delta} (x(t) - x(t-\Delta))
\end{aligned}
$$

(b) Identify the mathematical operation performed by this system in the limit as $\Delta \to 0$.

When $\Delta \to 0$

$$\lim_{\Delta\to 0} y(t) = \lim_{\Delta\to 0} \frac{x(t)-x(t-\Delta)}{\Delta}$$

is nothing but $\frac{d}{dt}x(t)$, the first derivative of $x(t)$ with respect to time t.

(c) Let $g(t) = \lim_{\Delta\to 0} h(t)$. Use the results of (b) to express the output of an LTI system with impulse response $h^n(t) = \underbrace{g(t) * g(t) * ... * g(t)}_{n \text{ times}}$ as a function of the input $x(t)$.

$$\begin{aligned}
g(t) &= \lim_{\Delta\to 0} h(t) \\
h^n(t) &= \underbrace{g(t) * g(t) * ... * g(t)}_{n \text{ times}} \\
y^n(t) &= x(t) * h^n(t) \\
&= (x(t) * g(t)) * \underbrace{g(t) * g(t) * ... * g(t)}_{(n-1) \text{ times}} \\
\text{let } x^{(1)}(t) &= x(t) * g(t) = x(t) * \lim_{\Delta\to 0} h(t) \\
&= \lim_{\Delta\to 0} (x(t) * h(t)) \\
&= \frac{d}{dt}x(t) \text{ from (b)} \\
\text{then } y^n(t) &= \left(x^{(1)}(t) * g(t)\right) * \underbrace{g(t) * g(t) * ... * g(t)}_{(n-2) \text{ times}} \\
\text{Similarly} & \\
x^{(1)}(t) * g(t) &= \frac{d^2}{dt^2}x(t) \\
& \text{Doing this repeatedly, we find that} \\
y^n(t) &= x^{(n-1)}(t) * g(t) \\
&= \frac{d^{n-1}}{dt^{n-1}}x(t) * g(t) \\
\text{Therefore} & \\
y^n(t) &= \frac{d^n}{dt^n}x(t)
\end{aligned}$$

2.44. If $y(t) = x(t) * h(t)$ is the output of an LTI system with input $x(t)$ and impulse response $h(t)$, then show that

$$\frac{d}{dt}y(t) = x(t) * \left(\frac{d}{dt}h(t)\right)$$

and

$$\frac{d}{dt}y(t) = \left(\frac{d}{dt}x(t)\right) * h(t)$$

$$\begin{aligned}
\frac{d}{dt}y(t) &= \frac{d}{dt}x(t) * h(t) \\
&= \frac{d}{dt}\int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau
\end{aligned}$$

$$
\begin{aligned}
& \text{Assuming that the functions are sufficiently smooth,} \\
& \text{the derivative can be pulled throught the integral} \\
\frac{d}{dt}y(t) &= \int_{-\infty}^{\infty} x(\tau)\frac{d}{dt}h(t-\tau)d\tau \\
& \text{Since } x(\tau) \text{ is independent of } t \\
\frac{d}{dt}y(t) &= x(t) * \left(\frac{d}{dt}h(t)\right) \\
& \text{The convolution integral can also be written as} \\
\frac{d}{dt}y(t) &= \frac{d}{dt}\int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau \\
&= \int_{-\infty}^{\infty} h(\tau)\frac{d}{dt}x(t-\tau)d\tau \\
&= \left(\frac{d}{dt}x(t)\right) * h(t)
\end{aligned}
$$

2.45. If $h(t) = H\{\delta(t)\}$ is the impulse response of an LTI system, express $H\{\delta^{(2)}(t)\}$ in terms of $h(t)$.

$$
\begin{aligned}
H\{\delta^{(2)}(t)\} &= h(t) * \delta^{(2)}(t) \\
&= \int_{-\infty}^{\infty} h(\tau)\delta^{(2)}(t-\tau)d\tau \\
& \text{By the doublet sifting property} \\
&= \frac{d}{dt}h(t)
\end{aligned}
$$

2.46. Find the expression for the impulse response relating the input $x[n]$ or $x(t)$ to the output $y[n]$ or $y(t)$ in terms of the impulse response of each subsystem for the LTI systems depicted in
(a) Fig. P2.46 (a)

$$
y(t) = x(t) * \{h_1(t) - h_4(t) * [h_2(t) + h_3(t)]\} * h_5(t)
$$

(b) Fig. P2.46 (b)

$$
y[n] = x[n] * \{-h_1[n] * h_2[n] * h_4[n] + h_1[n] * h_3[n] * h_5[n]\} * h_6[n]
$$

(c) Fig. P2.46 (c)

$$
y(t) = x(t) * \{[-h_1(t) + h_2(t)] * h_3(t) * h_4(t) + h_2(t)\}
$$

2.47. Let $h_1(t), h_2(t), h_3(t)$, and $h_4(t)$ be impulse responses of LTI systems. Construct a system with impulse response $h(t)$ using $h_1(t), h_2(t), h_3(t)$, and $h_4(t)$ as subsystems. Draw the interconnection of systems required to obtain

(a) $h(t) = \{h_1(t) + h_2(t)\} * h_3(t) * h_4(t)$

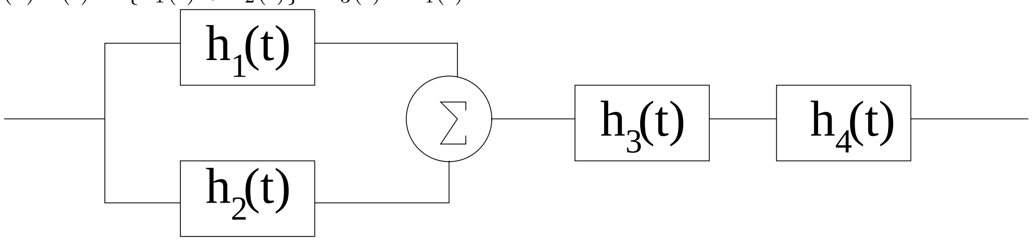


Figure P2.47. (a) Interconnections between systems

(b) $h(t) = h_1(t) * h_2(t) + h_3(t) * h_4(t)$

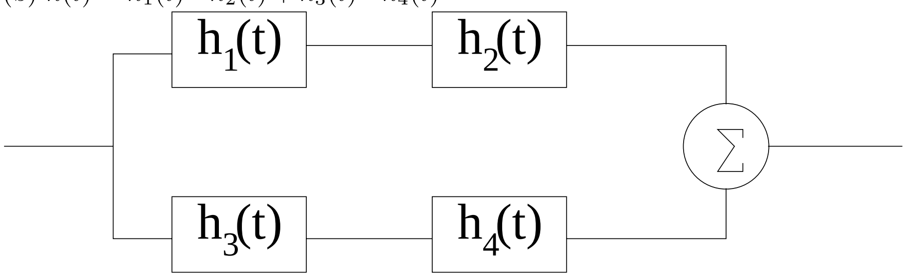


Figure P2.47. (b) Interconnections between systems

(c) $h(t) = h_1(t) * \{h_2(t) + h_3(t) * h_4(t)\}$

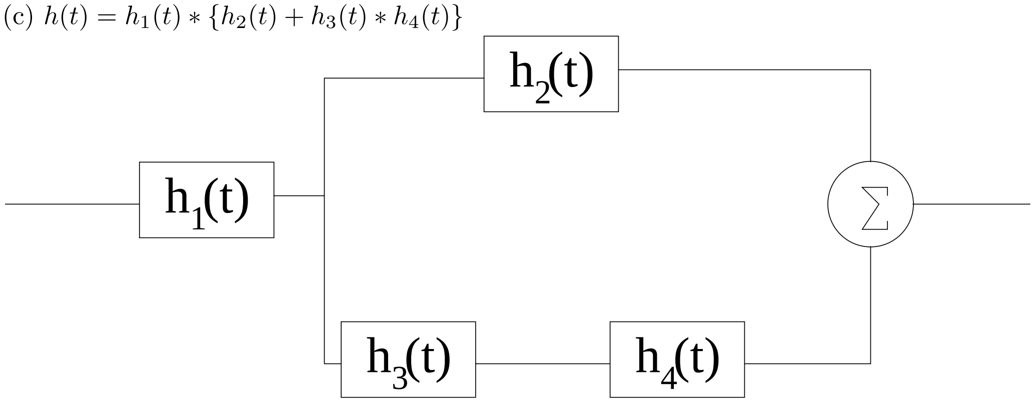


Figure P2.47. (c) Interconnections between systems

2.48. For the interconnection of LTI systems depicted in Fig. P2.46 (c) the impulse responses are $h_1(t) = \delta(t-1), h_2(t) = e^{-2t}u(t), h_3(t) = \delta(t-1)$ and $h_4(t) = e^{-3(t+2)}u(t+2)$. Evaluate $h(t)$, the impulse response of the overall system from $x(t)$ to $y(t)$.

$$\begin{aligned}
h(t) &= \left[-\delta(t-1) + e^{-2t}u(t)\right] * \delta(t-1) * e^{-3(t+2)}u(t+2) + \delta(t-1) \\
&= \left[-\delta(t-2) + e^{-2(t-1)}u(t-1)\right] * e^{-3(t+2)}u(t+2) + \delta(t-1) \\
&= -e^{-3t}u(t) + e^{-2(t-1)}u(t-1) * e^{-3(t+2)}u(t+2) + \delta(t-1)
\end{aligned}$$

$$= \quad -e^{-3t}u(t) + \left(e^{-2(t-1)} - e^{-3(t+3)}\right)u(t+1) + \delta(t-1)$$

2.49. For each impulse response listed below, determine whether the corresponding system is (i) memoryless, (ii) causal, and (iii) stable.

(i) Memoryless if and only if $h(t) = c\delta(t)$ or $h[n] = c\delta[k]$
(ii) Causal if and only if $h(t) = 0$ for $t < 0$ or $h[n] = 0$ for $n < 0$
(iii) Stable if and only if $\int_{-\infty}^{\infty} |h(t)|dt < \infty$ or $\sum_{k=-\infty}^{\infty} |h[k]| < \infty$

(a) $h(t) = \cos(\pi t)$
(i) has memory
(ii) not causal
(iii) not stable
(b) $h(t) = e^{-2t}u(t-1)$
(i) has memory
(ii) causal
(iii) stable
(c) $h(t) = u(t+1)$
(i) has memory
(ii) not causal
(iii) not stable
(d) $h(t) = 3\delta(t)$
(i) memoryless
(ii) causal
(iii) stable
(e) $h(t) = \cos(\pi t)u(t)$
(i) has memory
(ii) causal
(iii) not stable
(f) $h[n] = (-1)^n u[-n]$
(i) has memory
(ii) not causal
(iii) not stable
(g) $h[n] = (1/2)^{|n|}$
(i) has memory
(ii) not causal
(iii) stable
(h) $h[n] = \cos(\frac{\pi}{8}n)\{u[n] - u[n-10]\}$
(i) has memory
(ii) causal
(iii) stable
(i) $h[n] = 2u[n] - 2u[n-5]$
(i) has memory

(ii) causal
(iii) stable
(j) $h[n] = \sin(\frac{\pi}{2}n)$
(i) has memory
(ii) not causal
(iii) not stable
(k) $h[n] = \sum_{p=-1}^{\infty} \delta[n-2p]$
(i) has memory
(ii) not causal
(iii) not stable

2.50. Evaluate the step response for the LTI systems represented by the following impulse responses:
(a) $h[n] = (-1/2)^n u[n]$

for $n < 0$
$$s[n] = 0$$
for $n \geq 0$
$$s[n] = \sum_{k=0}^{n} (-\frac{1}{2})^k$$
$$s[n] = \frac{1}{3}\left(2 + \left(-\frac{1}{2}\right)^n\right)$$
$$s[n] = \begin{cases} \frac{1}{3}\left(2 + \left(-\frac{1}{2}\right)^n\right) & n \geq 0 \\ 0 & n < 0 \end{cases}$$

(b) $h[n] = \delta[n] - \delta[n-2]$

for $n < 0$
$$s[n] = 0$$
for $n = 0$
$$s[n] = 1$$
for $n \geq 1$
$$s[n] = 0$$
$$s[n] = \begin{cases} 1 & n = 0 \\ 0 & n \neq 0 \end{cases}$$

(c) $h[n] = (-1)^n\{u[n+2] - u[n-3]\}$

for $n < -2$
$$s[n] = 0$$
for $-2 \leq n \leq 2$

$$s[n] = \begin{cases} 1 & n = \pm 2,\ 0 \\ 0 & n = \pm 1 \end{cases}$$

for $n \geq 3$

$$s[n] = 1$$

(d) $h[n] = nu[n]$

for $n < 0$

$$s[n] = 0$$

for $n \geq 0$

$$s[n] = \sum_{k=0}^{n} k$$

$$s[n] \quad = \quad \begin{cases} \frac{n(n+1)}{2} & n \geq 0 \\ 0 & n < 0 \end{cases}$$

(e) $h(t) = e^{-|t|}$

for $t < 0$

$$s(t) = \int_{-\infty}^{t} e^{\tau} d\tau = e^{t}$$

for $t \geq 0$

$$s(t) = \int_{-\infty}^{0} e^{\tau} d\tau + \int_{0}^{t} e^{-\tau} d\tau = 2 - e^{-t}$$

$$s(t) \quad = \quad \begin{cases} e^{t} & t < 0 \\ 2 - e^{-t} & t \geq 0 \end{cases}$$

(f) $h(t) = \delta^{(2)}(t)$

for $t < 0$

$$s(t) = 0$$

for $t \geq 0$

$$s(t) = \int_{-\infty}^{t} \delta^{(2)}(t) d\tau = \delta(t)$$

$$s(t) \quad = \quad \delta(t)$$

(g) $h(t) = (1/4)(u(t) - u(t-4))$

for $t < 0$

$$s(t) = 0$$

for $t < 4$

$$s(t) = \frac{1}{4}\int_0^t d\tau = \frac{1}{4}t$$

for $t \geq 4$

$$s(t) = \frac{1}{4}\int_0^4 d\tau = 1$$

$$s(t) = \begin{cases} 0 & t < 0 \\ \frac{1}{4}t & 0 \leq t < 4 \\ 1 & t \geq 4 \end{cases}$$

(h) $h(t) = u(t)$

for $t < 0$

$$s(t) = 0$$

for $t \geq 0$

$$s(t) = \int_0^t d\tau = t$$

$$s(t) = \begin{cases} 0 & t < 0 \\ t & t \geq 0 \end{cases}$$

2.51. Suppose the multipath propagation model is generalized to a k-step delay between the direct and reflected paths as shown by the input-output equation

$$y[n] = x[n] + ax[n-k]$$

Find the impulse response of the inverse system.

$$\begin{aligned} & \quad \text{The inverse system must satisfy} \\ h^{inv}[n] + ah^{inv}[n-k] &= \delta[n] \\ h^{inv}[0] + ah^{inv}[-k] &= 1 \\ & \quad \text{Implies} \\ h^{inv}[0] &= 1 \\ & \quad \text{For the system to be causal} \\ h^{inv}[n] &= -ah^{inv}[n-k] \\ & \quad \text{Which means } h^{inv}[n] \text{ is nonzero} \\ & \quad \text{ only for positive multiples of } k \\ h^{inv}[n] &= \sum_{p=0}^{\infty}(-a)^p\delta[n-pk] \end{aligned}$$

2.52. Write a differential equation description relating the output to the input of the following electrical circuits.

(a) Fig. P2.52 (a)

Writing node equations, assuming node A is the node the resistor, inductor and capacitor share.

$$\begin{aligned}
(1)\;\; y(t) &+ \frac{v_A(t)-x(t)}{R}+C\frac{d}{dt}v_A(t)=0 \\
\text{For the inductor L:} & \\
(2)\;\; v_A(t) &= L\frac{d}{dt}y(t) \\
\text{Combining (1) and (2) yields} & \\
\frac{1}{RLC}x(t) &= \frac{d^2}{dt^2}y(t)+\frac{1}{RC}\frac{d}{dt}y(t)+\frac{1}{LC}y(t)
\end{aligned}$$

(b) Fig. P2.52 (b)
Writing node equations, assuming node A is the node the two resistors and C_2 share. $i(t)$ is the current through R_2.

$$\begin{aligned}
(1)\quad 0 &= C_2\frac{d}{dt}y(t)+\frac{y(t)-x(t)}{R_1}+i(t) \\
\text{Implies} & \\
i(t) &= -C_2\frac{d}{dt}y(t)-\frac{y(t)-x(t)}{R_1} \\
\text{For capacitor } C_1 & \\
\frac{d}{dt}V_{C1}(t) &= \frac{i(t)}{C_1} \\
y(t) &= i(t)R_2+V_{C1}(t) \\
(2)\qquad \frac{d}{dt}y(t) &= R_2\frac{d}{dt}i(t)+\frac{i(t)}{C_1} \\
\text{Combining (1) and (2)} &
\end{aligned}$$

$$\frac{d^2}{dt^2}y(t)+\left(\frac{1}{C_2R_2}+\frac{1}{C_2R_1}+\frac{1}{C_1R_2}\right)\frac{d}{dt}y(t)+\frac{1}{C_1C_2R_1R_2}y(t)=\frac{1}{C_1C_2R_1R_2}x(t)+\frac{1}{C_2R_1}\frac{d}{dt}x(t)$$

2.53. Determine the homogeneous solution for the systems described by the following differential equations:
(a) $5\frac{d}{dt}y(t)+10y(t)=2x(t)$

$$\begin{aligned}
5r+10 &= 0 \\
r &= -2 \\
y^{(h)}(t) &= c_1e^{-2t}
\end{aligned}$$

(b) $\frac{d^2}{dt^2}y(t)+6\frac{d}{dt}y(t)+8y(t)=\frac{d}{dt}x(t)$

$$\begin{aligned}
r^2+6r+8 &= 0 \\
r &= -4,-2 \\
y^{(h)}(t) &= c_1e^{-4t}+c_2e^{-2t}
\end{aligned}$$

(c) $\frac{d^2}{dt^2}y(t) + 4y(t) = 3\frac{d}{dt}x(t)$

$$\begin{aligned} r^2 + 4 &= 0 \\ r &= \pm j2 \\ y^{(h)}(t) &= c_1 e^{j2t} + c_2 e^{-j2t} \end{aligned}$$

(d) $\frac{d^2}{dt^2}y(t) + 2\frac{d}{dt}y(t) + 2y(t) = x(t)$

$$\begin{aligned} r^2 + 2r + 2 &= 0 \\ r &= -1 \pm j \\ y^{(h)}(t) &= c_1 e^{(-1+j)t} + c_2 e^{-(1+j)t} \end{aligned}$$

(e) $\frac{d^2}{dt^2}y(t) + 2\frac{d}{dt}y(t) + y(t) = \frac{d}{dt}x(t)$

$$\begin{aligned} r^2 + 2r + 1 &= 0 \\ r &= -1, -1 \\ y^{(h)}(t) &= c_1 e^{-t} + c_2 t e^{-t} \end{aligned}$$

2.54. Determine the homogeneous solution for the systems described by the following difference equations:
(a) $y[n] - \alpha y[n-1] = 2x[n]$

$$\begin{aligned} r - \alpha &= 0 \\ y^{(h)}[n] &= c_1 \alpha^n \end{aligned}$$

(b) $y[n] - \frac{1}{4}y[n-1] - \frac{1}{8}y[n-2] = x[n] + x[n-1]$

$$\begin{aligned} r^2 - \frac{1}{4}r - \frac{1}{8} &= 0 \\ r &= \frac{1}{2}, -\frac{1}{4} \\ y^{(h)}[n] &= c_1 \left(\frac{1}{2}\right)^n + c_2 \left(-\frac{1}{4}\right)^n \end{aligned}$$

(c) $y[n] + \frac{9}{16}y[n-2] = x[n-1]$

$$r^2 + \frac{9}{16} = 0$$

$$\begin{aligned} r &= \pm j\frac{3}{4} \\ y^{(h)}[n] &= c_1\left(j\frac{3}{4}\right)^n + c_2\left(-j\frac{3}{4}\right)^n \end{aligned}$$

(d) $y[n] + y[n-1] + \frac{1}{4}y[n-2] = x[n] + 2x[n-1]$

$$\begin{aligned} r^2 + r + \frac{1}{4} &= 0 \\ r &= -\frac{1}{2},\ -\frac{1}{2} \\ y^{(h)}[n] &= c_1\left(-\frac{1}{2}\right)^n + c_2 n\left(-\frac{1}{2}\right)^n \end{aligned}$$

2.55. Determine a particular solution for the systems described by the following differential equations for the given inputs:

(a) $5\frac{d}{dt}y(t) + 10y(t) = 2x(t)$

(i) $x(t) = 2$

$$\begin{aligned} y^{(p)}(t) &= k \\ 10k &= 2(2) \\ k &= \frac{2}{5} \\ y^{(p)}(t) &= \frac{2}{5} \end{aligned}$$

(ii) $x(t) = e^{-t}$

$$\begin{aligned} y^{(p)}(t) &= ke^{-t} \\ -5ke^{-t} + 10ke^{-t} &= 2e^{-t} \\ k &= \frac{2}{5} \\ y^{(p)}(t) &= \frac{2}{5}e^{-t} \end{aligned}$$

(iii) $x(t) = \cos(3t)$

$$\begin{aligned} y^{(p)}(t) &= A\cos(3t) + B\sin(3t) \\ \frac{d}{dt}y^{(p)}(t) &= -3A\sin(3t) + 3B\cos(3t) \\ 5\left(-3A\sin(3t) + 3B\cos(3t)\right) + 10A\cos(3t) + 10B\sin(3t) &= 2\cos(3t) \\ -15A + 10B &= 0 \\ 10A + 15B &= 2 \end{aligned}$$

$$
\begin{aligned}
A &= \frac{4}{65} \\
B &= \frac{6}{65} \\
y^{(p)}(t) &= \frac{4}{65}\cos(3t) + \frac{6}{65}\sin(3t)
\end{aligned}
$$

(b) $\frac{d^2}{dt^2}y(t) + 4y(t) = 3\frac{d}{dt}x(t)$
(i) $x(t) = t$

$$
\begin{aligned}
y^{(p)}(t) &= k_1 t + k_2 \\
4k_1 t + 4k_2 &= 3 \\
k_1 &= 0 \\
k_2 &= \frac{3}{4} \\
y^{(p)}(t) &= \frac{3}{4}
\end{aligned}
$$

(ii) $x(t) = e^{-t}$

$$
\begin{aligned}
y^{(p)}(t) &= ke^{-t} \\
ke^{-t} + 4ke^{-t} &= -3e^{-t} \\
k &= -\frac{3}{5} \\
y^{(p)}(t) &= -\frac{3}{5}e^{-t}
\end{aligned}
$$

(iii) $x(t) = (\cos(t) + \sin(t))$

$$
\begin{aligned}
y^{(p)}(t) &= A\cos(t) + B\sin(t) \\
\frac{d}{dt}y^{(p)}(t) &= -A\sin(t) + B\cos(t) \\
\frac{d^2}{dt^2}y^{(p)}(t) &= -A\cos(t) - B\sin(t) \\
-A\cos(t) - B\sin(t) + 4A\cos(t) + 4B\sin(t) &= -3\sin(t) + 3\cos(t) \\
-A + 4A &= 3 \\
-B + 4B &= -3 \\
A &= 1 \\
B &= -1 \\
y^{(p)}(t) &= \cos(t) - \sin(t)
\end{aligned}
$$

(c) $\frac{d^2}{dt^2}y(t) + 2\frac{d}{dt}y(t) + y(t) = \frac{d}{dt}x(t)$
(i) $x(t) = e^{-3t}$

$$
\begin{aligned}
y^{(p)}(t) &= ke^{-3t} \\
9ke^{-3t} - 6ke^{-3t} + ke^{-3t} &= -3e^{-3t} \\
k &= -\frac{3}{4} \\
y^{(p)}(t) &= -\frac{3}{4}e^{-3t}
\end{aligned}
$$

(ii) $x(t) = 2e^{-t}$

Since e^{-t} and te^{-t} are in the natural response, the particular soluction takes the form of

$$
\begin{aligned}
y^{(p)}(t) &= kt^2e^{-t} \\
\frac{d}{dt}y^{(p)}(t) &= 2kte^{-t} - kt^2e^{-t} \\
\frac{d^2}{dt^2}y^{(p)}(t) &= 2ke^{-t} - 4kte^{-t} + kt^2e^{-t} \\
-2e^{-t} &= 2ke^{-t} - 4kte^{-t} + kt^2e^{-t} + 2(2kte^{-t} - kt^2e^{-t}) + kt^2e^{-t} \\
k &= -1 \\
y^{(p)}(t) &= -t^2e^{-t}
\end{aligned}
$$

(iii) $x(t) = 2\sin(t)$

$$
\begin{aligned}
y^{(p)}(t) &= A\cos(t) + B\sin(t) \\
\frac{d}{dt}y^{(p)}(t) &= -A\sin(t) + B\cos(t) \\
\frac{d^2}{dt^2}y^{(p)}(t) &= -A\cos(t) - B\sin(t) \\
-A\cos(t) - B\sin(t) - 2A\sin(t) + 2B\cos(t) + A\cos(t) + B\sin(t) &= 2\cos(t) \\
-A - 2B + A &= 2 \\
-B - 2A + B &= 0 \\
A &= 0 \\
B &= -1 \\
y^{(p)}(t) &= -\sin(t)
\end{aligned}
$$

2.56. Determine a particular solution for the systems described by the following difference equations for the given inputs:

(a) $y[n] - \frac{2}{5}y[n-1] = 2x[n]$

(i) $x[n] = 2u[n]$

$$
\begin{aligned}
y^{(p)}[n] &= ku[n] \\
k - \frac{2}{5}k &= 4
\end{aligned}
$$

$$\begin{aligned} k &= \frac{20}{3} \\ y^{(p)}[n] &= \frac{20}{3}u[n] \end{aligned}$$

(ii) $x[n] = -(\frac{1}{2})^n u[n]$

$$\begin{aligned} y^{(p)}[n] &= k\left(\frac{1}{2}\right)^n u[n] \\ k\left(\frac{1}{2}\right)^n - \frac{2}{5}\left(\frac{1}{2}\right)^{n-1} k &= -2\left(\frac{1}{2}\right)^n \\ k &= -10 \\ y^{(p)}[n] &= -10\left(\frac{1}{2}\right)^n u[n] \end{aligned}$$

(iii) $x[n] = \cos(\frac{\pi}{5}n)$

$$\begin{aligned} y^{(p)}[n] &= A\cos(\frac{\pi}{5}n) + B\sin(\frac{\pi}{5}n) \\ 2\cos(\frac{\pi}{5}n) &= A\cos(\frac{\pi}{5}n) + B\sin(\frac{\pi}{5}n) - \frac{2}{5}\left[A\cos(\frac{\pi}{5}(n-1)) + B\sin(\frac{\pi}{5}(n-1))\right) \\ & \text{Using the trig identities} \\ \sin(\theta \pm \phi) &= \sin\theta\cos\phi \pm \cos\theta\sin\phi \\ \cos(\theta \pm \phi) &= \cos\theta\cos\phi \mp \sin\theta\sin\phi \\ y^{(p)}[n] &= 2.6381\cos(\frac{\pi}{5}n) + 0.9170\sin(\frac{\pi}{5}n) \end{aligned}$$

(b) $y[n] - \frac{1}{4}y[n-1] - \frac{1}{8}y[n-2] = x[n] + x[n-1]$
(i) $x[n] = nu[n]$

$$\begin{aligned} y^{(p)}[n] &= k_1 n u[n] + k_2 u[n] \\ k_1 n + k_2 - \frac{1}{4}\left[k_1(n-1) + k_2\right] - \frac{1}{8}\left[k_1(n-2) + k_2\right] &= n + n - 1 \\ k_1 &= \frac{16}{5} \\ k_2 &= -\frac{104}{5} \\ y^{(p)}[n] &= \frac{16}{5}nu[n] - \frac{104}{5}u[n] \end{aligned}$$

(ii) $x[n] = (\frac{1}{8})^n u[n]$

$$y^{(p)}[n] = k\left(\frac{1}{8}\right)^n u[n]$$

$$\begin{aligned}
k\left(\frac{1}{8}\right)^n - \frac{1}{4}\left(\frac{1}{8}\right)^{n-1}k - \frac{1}{8}\left(\frac{1}{8}\right)^{n-2}k &= \left(\frac{1}{8}\right)^n + \left(\frac{1}{8}\right)^{n-1} \\
k &= -1 \\
y^{(p)}[n] &= -1\left(\frac{1}{8}\right)^n u[n]
\end{aligned}$$

(iii) $x[n] = e^{j\frac{\pi}{4}n}u[n]$

$$\begin{aligned}
y^{(p)}[n] &= ke^{j\frac{\pi}{4}n}u[n] \\
ke^{j\frac{\pi}{4}n} - \frac{1}{4}ke^{j\frac{\pi}{4}(n-1)} - \frac{1}{8}ke^{j\frac{\pi}{4}(n-2)} &= e^{j\frac{\pi}{4}n} + e^{j\frac{\pi}{4}(n-1)} \\
k &= \frac{1 + e^{-j\frac{\pi}{4}}}{1 - \frac{1}{4}e^{-j\frac{\pi}{4}} - \frac{1}{8}ke^{-j\frac{\pi}{2}}}
\end{aligned}$$

(iv) $x[n] = (\frac{1}{2})^n u[n]$

Since $\left(\frac{1}{2}\right)^n u[n]$ is in the natural response, the particular solution takes the form of:

$$\begin{aligned}
y^{(p)}[n] &= kn\left(\frac{1}{2}\right)^n u[n] \\
kn\left(\frac{1}{2}\right)^n - k\frac{1}{4}(n-1)\left(\frac{1}{2}\right)^{n-1} - k\frac{1}{8}(n-2)\left(\frac{1}{2}\right)^{n-2} &= \left(\frac{1}{2}\right)^n + \left(\frac{1}{2}\right)^{n-1} \\
k &= 2 \\
y^{(p)}[n][n] &= 2n\left(\frac{1}{2}\right)^n u[n]
\end{aligned}$$

(c) $y[n] + y[n-1] + \frac{1}{2}y[n-2] = x[n] + 2x[n-1]$

(i) $x[n] = u[n]$

$$\begin{aligned}
y^{(p)}[n][n] &= ku[n] \\
k + k + \frac{1}{2}k &= 2 + 2 \\
k &= \frac{8}{5} \\
y^{(p)}[n] &= \frac{8}{5}u[n]
\end{aligned}$$

(ii) $x[n] = (\frac{-1}{2})^n u[n]$

$$\begin{aligned}
y^{(p)}[n] &= k\left(-\frac{1}{2}\right)^n u[n] \\
k\left(-\frac{1}{2}\right)^n + k\left(-\frac{1}{2}\right)^{n-1} + \frac{1}{2}\left(-\frac{1}{2}\right)^{n-2}k &= \left(-\frac{1}{2}\right)^n + 2\left(-\frac{1}{2}\right)^{n-1} \\
k &= -3 \\
y^{(p)}[n] &= -3\left(-\frac{1}{2}\right)^n u[n]
\end{aligned}$$

2.57. Determine the output of the systems described by the following differential equations with input andinitial conditions as specified:

(a) $\frac{d}{dt}y(t) + 10y(t) = 2x(t), \quad y(0^-) = 1, x(t) = u(t)$

$$
\begin{aligned}
t \geq 0 & \qquad \text{natural: characteristic equation} \\
r + 10 &= 0 \\
r &= -10 \\
y^{(n)}(t) &= ce^{-10t} \\
& \qquad \text{particular} \\
y^{(p)}(t) &= ku(t) = \frac{1}{5}u(t) \\
y(t) &= \frac{1}{5} + ce^{-10t} \\
y(0^-) = 1 &= \frac{1}{5} + c \\
c &= \frac{4}{5} \\
y(t) &= \frac{1}{5}\left[1 + 4e^{-10t}\right]u(t)
\end{aligned}
$$

(b) $\frac{d^2}{dt^2}y(t) + 5\frac{d}{dt}y(t) + 4y(t) = \frac{d}{dt}x(t), \quad y(0^-) = 0, \frac{d}{dt}y(t)\Big|_{t=0^-} = 1, x(t) = \sin(t)u(t)$

$$
\begin{aligned}
t \geq 0 & \qquad \text{natural: characteristic equation} \\
r^2 + 5r + 4 &= 0 \\
r &= -4, \ -1 \\
y^{(n)}(t) &= c_1e^{-4t} + c_2e^{-t} \\
& \qquad \text{particular} \\
y^{(p)}(t) &= A\sin(t) + B\cos(t) \\
&= \frac{5}{34}\sin(t) + \frac{3}{34}\cos(t) \\
y(t) &= \frac{5}{34}\sin(t) + \frac{3}{34}\cos(t) + c_1e^{-4t} + c_2e^{-t} \\
y(0^-) = 0 &= \frac{3}{34} + c_1 + c_2 \\
\frac{d}{dt}y(0)\Big|_{t=0^-} = 1 &= \frac{5}{34} - 4c_1 - c_2 \\
c_1 &= -\frac{13}{51} \\
c_2 &= \frac{1}{6} \\
y(t) &= \frac{5}{34}\sin(t) + \frac{3}{34}\cos(t) - \frac{13}{51}e^{-4t} + \frac{1}{6}e^{-t}
\end{aligned}
$$

(c) $\frac{d^2}{dt^2}y(t) + 6\frac{d}{dt}y(t) + 8y(t) = 2x(t), \quad y(0^-) = -1, \frac{d}{dt}y(t)\Big|_{t=0^-} = 1, x(t) = e^{-t}u(t)$

$$
\begin{aligned}
t \geq 0 & \qquad \text{natural: characteristic equation} \\
r^2 + 6r + 8 &= 0 \\
r &= -4, \ -2 \\
y^{(n)}(t) &= c_1 e^{-2t} + c_2 e^{-4t} \\
& \qquad \text{particular} \\
y^{(p)}(t) &= k e^{-t} u(t) \\
&= \frac{2}{3} e^{-t} u(t) \\
y(t) &= \frac{2}{3} e^{-t} u(t) + c_1 e^{-2t} + c_2 e^{-4t} \\
y(0^-) = -1 &= \frac{2}{3} + c_1 + c_2 \\
\frac{d}{dt} y(0)\Big|_{t=0^-} = 1 &= -\frac{2}{3} - 2c_1 - 4c_2 \\
c_1 &= -\frac{5}{2} \\
c_2 &= \frac{5}{6} \\
y(t) &= \frac{2}{3} e^{-t} u(t) - \frac{5}{2} e^{-2t} + \frac{5}{6} e^{-4t}
\end{aligned}
$$

(d) $\frac{d^2}{dt^2} y(t) + y(t) = 3 \frac{d}{dt} x(t), \quad y(0^-) = -1,\ \frac{d}{dt} y(t)\Big|_{t=0^-} = 1, x(t) = 2te^{-t}u(t)$

$$
\begin{aligned}
t \geq 0 & \qquad \text{natural: characteristic equation} \\
r^2 + 1 &= 0 \\
r &= \pm j \\
y^{(n)}(t) &= A\cos(t) + B\sin(t) \\
& \qquad \text{particular} \\
y^{(p)}(t) &= kte^{-t} u(t) \\
\frac{d^2}{dt^2} y^{(p)}(t) &= -2ke^{-t} + kte^{-t} \\
-2ke^{-t} + kte^{-t} + kte^{-t} &= 3[2e^{-t} - 2te^{-t}] \\
k &= -3 \\
y^{(p)}(t) &= -3te^{-t} u(t) \\
y(t) &= -3te^{-t} u(t) + A\cos(t) + B\sin(t) \\
y(0^-) &= -1 = 0 + A + 0 \\
\frac{d}{dt} y(t)\Big|_{t=0^-} &= 1 = -3 + 0 + B \\
y(t) &= -3te^{-t} u(t) - \cos(t) + 4\sin(t)
\end{aligned}
$$

2.58. Identify the natural and forced response for the systems in Problem 2.57.
See problem 2.57 for the derivations of the natural and forced response.

(a)

(i) Natural Response

$$\begin{aligned} r+10 &= 0 \\ r &= -10 \\ y^{(n)}(t) &= c_1 e^{-10t} \\ y(0^-)=1 &= c_1 \\ y^{(n)}(t) &= e^{-10t} \end{aligned}$$

(ii) Forced Response

$$\begin{aligned} y^{(f)}(t) &= \frac{1}{5} + ke^{-10t} \\ y(0)=0 &= \frac{1}{5} + k \\ y^{(f)}(t) &= \frac{1}{5} - \frac{1}{5}e^{-10t} \end{aligned}$$

(b)

(i) Natural Response

$$\begin{aligned} y^{(n)}(t) &= c_1 e^{-4t} + c_2 e^{-t} \\ y(0^-)=0 &= c_1 + c_2 \\ \left.\frac{d}{dt}y(t)\right|_{t=0^-} = 1 &= -4c_1 - c_2 \\ y^{(n)}(t) &= -\frac{1}{3}e^{-4t} + \frac{1}{3}e^{-t} \end{aligned}$$

(ii) Forced Response

$$\begin{aligned} y^{(f)}(t) &= \frac{5}{34}\sin(t) + \frac{3}{34}\cos(t) + c_1 e^{-4t} + c_2 e^{-t} \\ y(0)=0 &= \frac{3}{34} + c_1 + c_2 \\ \left.\frac{d}{dt}y(t)\right|_{t=0^-} = 0 &= \frac{5}{34} - 4c_1 - c_2 \\ y^{(f)}(t) &= \frac{5}{34}\sin(t) + \frac{3}{34}\cos(t) + \frac{4}{51}e^{-4t} - \frac{1}{6}e^{-t} \end{aligned}$$

(c)

(i) Natural Response

$$\begin{aligned}
y^{(n)}(t) &= c_1 e^{-4t} + c_2 e^{-2t} \\
y(0^-) = -1 &= c_1 + c_2 \\
\left.\frac{d}{dt}y(t)\right|_{t=0^-} = 1 &= -4c_1 - 2c_2 \\
y^{(n)}(t) &= \frac{1}{2}e^{-4t} - \frac{3}{2}e^{-2t}
\end{aligned}$$

(ii) Forced Response

$$\begin{aligned}
y^{(f)}(t) &= \frac{2}{3}e^{-t}u(t) + c_1 e^{-2t}u(t) + c_2 e^{-4t}u(t) \\
y(0) = 0 &= \frac{2}{3} + c_1 + c_2 \\
\left.\frac{d}{dt}y(t)\right|_{t=0^-} = 0 &= -\frac{2}{3} - 2c_1 - 4c_2 \\
y^{(f)}(t) &= \frac{2}{3}e^{-t}u(t) - e^{-2t}u(t) + \frac{1}{3}e^{-4t}u(t)
\end{aligned}$$

(d)

(i) Natural Response

$$\begin{aligned}
y^{(n)}(t) &= c_1 \cos(t) + c_2 \sin(t) \\
y(0^-) = -1 &= c_1 \\
\left.\frac{d}{dt}y(t)\right|_{t=0^-} = 1 &= c_2 \\
y^{(n)}(t) &= -\cos(t) + \sin(t)
\end{aligned}$$

(ii) Forced Response

$$\begin{aligned}
y^{(f)}(t) &= -3te^{-t}u(t) + c_1 \cos(t)u(t) + c_2 \sin(t)u(t) \\
y(0) = 0 &= c_1 + \\
\left.\frac{d}{dt}y(t)\right|_{t=0^-} = 0 &= -3 + c_2 \\
y^{(f)}(t) &= -3te^{-t}u(t) + 3\sin(t)u(t)
\end{aligned}$$

2.59. Determine the output of the systems described by the following difference equations with input and initial conditions as specified:

(a) $y[n] - \frac{1}{2}y[n-1] = 2x[n], \quad y[-1] = 3, x[n] = (\frac{-1}{2})^n u[n]$

$$\begin{aligned}
n \geq 0 & \qquad \text{natural: characteristic equation} \\
r - \frac{1}{2} &= 0 \\
y^{(n)}[n] &= c\left(\frac{1}{2}\right)^n \\
& \qquad \text{particular} \\
y^{(p)}[n] &= k\left(-\frac{1}{2}\right)^n u[n] \\
k\left(-\frac{1}{2}\right)^n - \frac{1}{2}k\left(-\frac{1}{2}\right)^{n-1} &= 2\left(-\frac{1}{2}\right)^n \\
k &= 1 \\
y^{(p)}[n] &= \left(-\frac{1}{2}\right)^n u[n] \\
& \qquad \text{Translate initial conditions} \\
y[n] &= \frac{1}{2}y[n-1] + 2x[n] \\
y[0] &= \frac{1}{2}3 + 2 = \frac{7}{2} \\
y[n] &= \left(-\frac{1}{2}\right)^n u[n] + c\left(\frac{1}{2}\right)^n u[n] \\
\frac{7}{2} &= 1 + c \\
c &= \frac{5}{2} \\
y[n] &= \left(-\frac{1}{2}\right)^n u[n] + \frac{5}{2}\left(\frac{1}{2}\right)^n u[n]
\end{aligned}$$

(b) $y[n] - \frac{1}{9}y[n-2] = x[n-1], y[-1] = 1, y[-2] = 0, x[n] = u[n]$

$$\begin{aligned}
n \geq 0 & \qquad \text{natural: characteristic equation} \\
r^2 - \frac{1}{9} &= 0 \\
r &= \pm\frac{1}{3} \\
y^{(n)}[n] &= c_1\left(\frac{1}{3}\right)^n + c_2\left(-\frac{1}{3}\right)^n \\
& \qquad \text{particular} \\
y^{(p)}[n] &= ku[n] \\
k - \frac{1}{9}k &= 1 \\
k &= \frac{9}{8} \\
y^{(p)}[n] &= \frac{9}{8}u[n]
\end{aligned}$$

$$
\begin{aligned}
y[n] &= \frac{9}{8}u[n] + c_1\left(\frac{1}{3}\right)^n + c_2\left(-\frac{1}{3}\right)^n \\
& \quad \text{Translate initial conditions} \\
y[n] &= \frac{1}{9}y[n-2] + x[n-1] \\
y[0] &= \frac{1}{9}0 + 0 = 0 \\
y[1] &= \frac{1}{9}1 + 1 = \frac{10}{9} \\
0 &= \frac{9}{8} + c_1 + c_2 \\
\frac{10}{9} &= \frac{9}{8} + \frac{1}{3}c_1 - \frac{1}{3}c_2 \\
y[n] &= \frac{9}{8}u[n] - \frac{7}{12}\left(\frac{1}{3}\right)^n - \frac{13}{24}\left(-\frac{1}{3}\right)^n
\end{aligned}
$$

(c) $y[n] + \frac{1}{4}y[n-1] - \frac{1}{8}y[n-2] = x[n] + x[n-1], \quad y[-1] = 4, y[-2] = -2, x[n] = (-1)^n u[n]$

$$
\begin{aligned}
n \geq 0 & \quad \text{natural: characteristic equation} \\
r^2 + \frac{1}{4}r - \frac{1}{8} &= 0 \\
r &= -\frac{1}{4},\ \frac{1}{2} \\
y^{(n)}[n] &= c_1\left(\frac{1}{2}\right)^n + c_2\left(-\frac{1}{4}\right)^n \\
& \quad \text{particular} \\
y^{(p)}[n] &= k(-1)^n u[n] \\
\text{for } n \geq 1 & \\
k(-1)^n + k\frac{1}{4}(-1)^{n-1} - k\frac{1}{8}(-1)^{n-2} &= (-1)^n + (-1)^{n-1} = 0 \\
k &= 0 \\
y^{(p)}[n] &= 0 \\
y[n] &= c_1\left(\frac{1}{2}\right)^n + c_2\left(-\frac{1}{4}\right)^n \\
& \quad \text{Translate initial conditions} \\
y[n] &= -\frac{1}{4}y[n-1] + \frac{1}{8}y[n-2] + x[n] + x[n-1] \\
y[0] &= \frac{1}{4}4 + \frac{1}{8}(-2) + 1 + 0 = -\frac{1}{4} \\
y[1] &= -\frac{1}{4}\left(-\frac{1}{4}\right) + \frac{1}{8}4 + -1 + 1 = \frac{9}{16} \\
-\frac{1}{4} &= c_1 + c_2 \\
\frac{9}{16} &= \frac{1}{2}c_1 - \frac{1}{4}c_2 \\
y[n] &= \frac{2}{3}\left(\frac{1}{2}\right)^n - \frac{11}{12}\left(-\frac{1}{4}\right)^n
\end{aligned}
$$

(d) $y[n]-\frac{3}{4}y[n-1]+\frac{1}{8}y[n-2]=2x[n], \quad y[-1]=1, y[-2]=-1, x[n]=2u[n]$

$$\begin{aligned}
n \geq 0 & \quad \text{natural: characteristic equation} \\
r^2-\frac{3}{4}r+\frac{1}{8} &= 0 \\
r &= \frac{1}{4}, \frac{1}{2} \\
y^{(n)}[n] &= c_1\left(\frac{1}{2}\right)^n + c_2\left(\frac{1}{4}\right)^n \\
& \quad \text{particular} \\
y^{(p)}[n] &= ku[n] \\
k-k\frac{3}{4}+k\frac{1}{8} &= 4 \\
k &= \frac{32}{3} \\
y^{(p)}[n] &= \frac{32}{3}u[n] \\
y[n] &= \frac{32}{3}u[n]+c_1\left(\frac{1}{2}\right)^n + c_2\left(\frac{1}{4}\right)^n \\
& \quad \text{Translate initial conditions} \\
y[n] &= \frac{3}{4}y[n-1]-\frac{1}{8}y[n-2]+2x[n] \\
y[0] &= \frac{3}{4}1-\frac{1}{8}(-1)+2(2)=\frac{39}{8} \\
y[1] &= \frac{3}{4}\left(\frac{39}{8}\right)-\frac{1}{8}1+2(2)=\frac{241}{32} \\
\frac{39}{8} &= \frac{32}{3}+c_1+c_2 \\
\frac{241}{32} &= \frac{32}{3}+\frac{1}{2}c_1+\frac{1}{4}c_2 \\
y[n] &= \frac{32}{3}u[n]-\frac{27}{4}\left(\frac{1}{2}\right)^n+\frac{23}{24}\left(\frac{1}{4}\right)^n
\end{aligned}$$

2.60. Identify the natural and forced response for the systems in Problem 2.59. (a)

(i) Natural Response

$$\begin{aligned}
y^{(n)}[n] &= c\left(\frac{1}{2}\right)^n \\
y[-1]=3 &= c\left(\frac{1}{2}\right)^{-1} \\
c &= \frac{3}{2} \\
y^{(n)}[n] &= \frac{3}{2}\left(\frac{1}{2}\right)^n
\end{aligned}$$

(ii) Forced Response

$$
\begin{aligned}
y^{(f)}[n] &= k\left(\frac{1}{2}\right)^n + \left(-\frac{1}{2}\right)^n \\
&\quad \text{Translate initial conditions} \\
y[n] &= \frac{1}{2}y[n-1] + 2x[n] \\
y[0] &= \frac{1}{2}(0) + 2 = 2 \\
y[0] = 2 &= k + 1 \\
k &= 1 \\
y^{(f)}[n] &= \left(\frac{1}{2}\right)^n + \left(-\frac{1}{2}\right)^n
\end{aligned}
$$

(b)

(i) Natural Response

$$
\begin{aligned}
y^{(n)}[n] &= c_1\left(\frac{1}{3}\right)^n u[n] + c_2\left(-\frac{1}{3}\right)^n u[n] \\
y[-2] = 0 &= c_1\left(\frac{1}{3}\right)^{-2} + c_2\left(-\frac{1}{3}\right)^{-2} \\
y[-1] = 1 &= c_1\left(\frac{1}{3}\right)^{-1} + c_2\left(-\frac{1}{3}\right)^{-1} \\
c_1 &= \frac{1}{6} \\
c_2 &= -\frac{1}{6} \\
y^{(n)}[n] &= \frac{1}{6}\left(\frac{1}{3}\right)^n u[n] - \frac{1}{6}\left(-\frac{1}{3}\right)^n u[n]
\end{aligned}
$$

(ii) Forced Response

$$
\begin{aligned}
y^{(f)}[n] &= \frac{9}{8}u[n] + c_1\left(\frac{1}{3}\right)^n u[n] + c_2\left(-\frac{1}{3}\right)^n u[n] \\
&\quad \text{Translate initial conditions} \\
y[n] &= \frac{1}{9}y[n-2] + x[n-1] \\
y[0] &= \frac{1}{9}0 + 0 = 0 \\
y[1] &= \frac{1}{9}0 + 1 = 1 \\
y[0] = 0 &= \frac{9}{8} + c_1 + c_2 \\
y[1] = 1 &= \frac{9}{8} + \frac{1}{3}c_1 - \frac{1}{3}c_2 \\
y^{(f)}[n] &= \frac{9}{8}u[n] - \frac{3}{4}\left(\frac{1}{3}\right)^n u[n] - \frac{3}{8}\left(-\frac{1}{3}\right)^n u[n]
\end{aligned}
$$

(c)

(i) Natural Response

$$
\begin{aligned}
y^{(n)}[n] &= c_1\left(-\frac{1}{2}\right)^n u[n] + c_2\left(\frac{1}{4}\right)^n u[n] \\
y[-2] = -2 &= c_1\left(-\frac{1}{2}\right)^{-2} + c_2\left(\frac{1}{4}\right)^{-2} \\
y[-1] = 4 &= c_1\left(-\frac{1}{2}\right)^{-1} + c_2\left(\frac{1}{4}\right)^{-1} \\
c_1 &= -\frac{3}{2} \\
c_2 &= \frac{1}{4} \\
y^{(n)}[n] &= -\frac{3}{2}\left(-\frac{1}{2}\right)^n u[n] - \frac{3}{8}\left(\frac{1}{4}\right)^n u[n]
\end{aligned}
$$

(ii) Forced Response

$$
\begin{aligned}
y^{(f)}[n] &= \frac{16}{5}(-1)^n u[n] + c_1\left(-\frac{1}{2}\right)^n u[n] + c_2\left(\frac{1}{4}\right)^n u[n] \\
& \text{Translate initial conditions} \\
y[n] &= -\frac{1}{4}y[n-1] + \frac{1}{8}y[n-2] + x[n] + x[n-1] \\
y[0] &= -\frac{1}{4}0 + \frac{1}{8}0 + 1 + 0 = 1 \\
y[1] &= -\frac{1}{4}1 + \frac{1}{8}0 - 1 + 1 = -\frac{1}{4} \\
y[0] = 1 &= \frac{16}{5} + c_1 + c_2 \\
y[1] = -\frac{1}{4} &= -\frac{16}{5} - \frac{1}{2}c_1 + \frac{1}{4}c_2 \\
y^{(f)}[n] &= \frac{16}{5}(-1)^n u[n] - \frac{14}{3}\left(-\frac{1}{2}\right)^n u[n] + \frac{37}{15}\left(\frac{1}{4}\right)^n u[n]
\end{aligned}
$$

(d)

(i) Natural Response

$$
\begin{aligned}
y^{(n)}[n] &= c_1\left(\frac{1}{2}\right)^n u[n] + c_2\left(\frac{1}{4}\right)^n u[n] \\
y[-2] = -1 &= c_1\left(\frac{1}{2}\right)^{-2} + c_2\left(\frac{1}{4}\right)^{-2} \\
y[-1] = 1 &= c_1\left(\frac{1}{2}\right)^{-1} + c_2\left(\frac{1}{4}\right)^{-1} \\
y^{(n)}[n] &= \frac{5}{4}\left(\frac{1}{2}\right)^n u[n] - \frac{3}{8}\left(\frac{1}{4}\right)^n u[n]
\end{aligned}
$$

(ii) Forced Response

$$
\begin{aligned}
y^{(f)}[n] &= \frac{32}{3}u[n] + c_1\left(\frac{1}{2}\right)^n u[n] + c_2\left(\frac{1}{4}\right)^n u[n] \\
& \quad \text{Translate initial conditions} \\
y[n] &= \frac{3}{4}y[n-1] - \frac{1}{8}y[n-2] + 2x[n] \\
y[0] &= \frac{3}{4}0 - \frac{1}{8}0 + 2(2) = 4 \\
y[1] &= \frac{3}{4}4 - \frac{1}{8}0 + 2(2) = 7 \\
y[0] = 4 &= \frac{32}{3} + c_1 + c_2 \\
y[1] = 7 &= \frac{32}{3} + \frac{1}{2}c_1 + \frac{1}{4}c_2 \\
y^{(f)}[n] &= \frac{32}{3}u[n] - 8\left(\frac{1}{2}\right)^n u[n] + \frac{4}{3}\left(\frac{1}{4}\right)^n u[n]
\end{aligned}
$$

2.61. Write a differential equation relating the output $y(t)$ to the circuit in Fig. P2.61 and find the step response by applying an input $x(t) = u(t)$. Next use the step response to obtain the impulse response. *Hint:* Use principles of circuit analysis to translate the $t = 0^-$ initial conditions to $t = 0^+$ before solving for the undetermined coefficients in the homogeneous component of the complete solution.

$$
\begin{aligned}
x(t) &= i_R(t) + i_L(t) + i_c(t) \\
y(t) &= i_R(t) \\
y(t) &= L\frac{d}{dt}i_L(t) \\
i_L(t) &= \frac{1}{L}\int_0^t y(\tau)d\tau + i_L(0^-) \\
i_c(t) &= C\frac{d}{dt}y(t) \\
x(t) &= y(t) + \frac{1}{L}\int_0^t y(\tau)d\tau + i_L(0^-) + C\frac{d}{dt}y(t) \\
\frac{d}{dt}x(t) &= C\frac{d^2}{dt^2}y(t) + \frac{d}{dt}y(t) + \frac{1}{L}y(t)
\end{aligned}
$$

Given $y(0^-) = 0$, $\frac{d}{dt}y(t)\big|_{t=0^-} = 0$, find $y(0^+)$, $\frac{d}{dt}y(t)\big|_{t=0^+}$.
Since current cannot change instantaneously through an inductor and voltage cannot change instantaneously across a capacitor,

$$
\begin{aligned}
i_L(0^+) &= 0 \\
y(0^+) &= y(0^-) \\
& \text{Implies} \\
i_R(0^+) &= 0
\end{aligned}
$$

$$
\begin{aligned}
i_C(0^+) &= 1 \quad \text{since} \\
i_C(t) &= C\frac{d}{dt}y(t) \\
\left.\frac{d}{dt}y(t)\right|_{t=0^+} &= \frac{i_C(0^+)}{C} = \frac{1}{C}
\end{aligned}
$$

Solving for the step response,

$$
\begin{aligned}
5\frac{d}{dt}x(t) &= \frac{d^2}{dt^2}y(t) + 5\frac{d}{dt}y(t) + 20y(t) \\
& \text{Finding the natural response} \\
r^2 + 5r + 20 &= 0 \\
r &= \frac{-5 \pm j\sqrt{55}}{2} \\
y^{(n)}(t) &= c_1 e^{-\frac{5}{2}t}\cos(\omega_o t) + c_2 e^{-\frac{5}{2}t}\sin(\omega_o t) \\
\omega_o &= \frac{\sqrt{55}}{2} \\
& \text{Particular} \\
y^{(p)}(t) &= k \\
x(t) &= u(t) \\
0 &= 0 + 0 + 20k \\
k &= 0 \\
& \text{Step Response} \\
y(t) &= y^{(n)}(t) \\
& \text{Using the initial conditions to solve for the constants} \\
y(0^+) = 0 &= c_1 \\
\left.\frac{d}{dt}y(t)\right|_{t=0^+} = 5 &= c_2\omega_o \\
y(t) &= \frac{5}{\omega_o}e^{-\frac{5}{2}t}\sin(\omega_o t) \\
&= \frac{10}{\sqrt{55}}e^{-\frac{5}{2}t}\sin(\frac{\sqrt{55}}{2}t)
\end{aligned}
$$

2.62. Use the first-order difference equation to calculate the monthly balance of a \$100,000 loan at 1% per month interest assuming monthly payments of \$1200. Identify the natural and forced response. In this case the natural response represents the loan balance assuming no payments are made. How many payments are required to pay off the loan?

$$
\begin{aligned}
y[n] - 1.01y[n-1] &= x[n] \\
x[n] &= -1200
\end{aligned}
$$

Natural

$$
r - 1.01 = 0
$$

$$
\begin{aligned}
y^{(n)}[n] &= c_n(1.01)^n \\
y[-1] = 100000 &= c_n(1.01)^{-1} \\
y^{(n)}[n] &= 101000(1.01)^n \\
\text{Particular} & \\
y^{(p)}[n] &= c_p \\
c_p - 1.01c_p &= -1200 \\
y^{(p)}[n] &= 120000 \\
\text{Forced} & \\
& \quad \text{Translate initial conditions} \\
y[0] &= 1.01y[-1] + x[0] \\
&= 1.01(100000) - 1200 = -1200 \\
y^{(f)}[n] &= 120000 + c_f(1.01)^n \\
y^{(f)}[0] = -1200 &= 120000 + c_f \\
y^{(f)}[n] &= 120000 - 121200(1.01)^n
\end{aligned}
$$

To solve for the required payments to pay off the loan, add the natural and forced response to obtain the complete solution, and solve for the number of payments to where the complete response equals zero.

$$
\begin{aligned}
y^{(c)}[n] &= 120000 - 20200(1.01)^n \\
y^{(c)}[\tilde{n}] = 0 &= 120000 - 20200(1.01)^{\tilde{n}} \\
n &\cong 179
\end{aligned}
$$

Since the first payment is made at $n = 0$, 180 payments are required to pay off the loan.

2.63. Determine the monthly payments required to pay off the loan in Problem 2.62 in 30 years (360 payments) and 15 years (180 payments).

$$
\begin{aligned}
p &= 1.01 \\
y[-1] &= 100000 \\
x[n] &= b \\
& \quad \text{The homogeneous solution is} \\
y^{(h)}[n] &= c_h(1.01)^n \\
& \quad \text{The particular solution is} \\
y^{(p)}[n] &= c_p \\
y[n] - 1.01y[n-1] &= x[n] \\
& \quad \text{Solving for } c_p \\
c_p - 1.01c_p &= b
\end{aligned}
$$

$$
\begin{aligned}
c_p &= -100b \\
&\quad \text{The complete solution is of the form} \\
y[n] &= c_h(1.01)^n - 100b \\
&\quad \text{Translating the initial conditions} \\
y[0] &= 1.01y[-1] + x[0] \\
&= 101000 + b \\
101000 + b &= c_h - 100b \\
c_h &= 101000 + 101b \\
y[n] &= (101000 + 101b)1.01^n - 100b \\
&\quad \text{Now solving for } b \text{ by setting } y[359] = 0 \\
b &= \frac{-101000(1.01)^{359}}{(101)1.01^{359} - 100} = -1028.61
\end{aligned}
$$

Hence a monthly payment of \$1028.61 will pay off the loan in 30 years.

$$
\begin{aligned}
&\quad \text{Now solving for } b \text{ by setting } y[179] = 0 \\
b &= \frac{-101000(1.01)^{179}}{(101)1.01^{179} - 100} = -1200.17
\end{aligned}
$$

A monthly payment of \$1200.17 will pay off the loan in 15 years.

2.64. The portion of a loan payment attributed to interest is given by multiplying the balance after the previous payment was credited times $\frac{r}{100}$ where r is the rate per period expressed in percent. Thus if $y[n]$ is the loan balance after the n^{th} payment, the portion of the n^{th} payment required to cover the interest cost is $y[n-1](r/100)$. The cumulative interest paid over payments for period n_1 through n_2 is thus

$$
I = (r/100) \sum_{n=n_1}^{n_2} y[n-1]
$$

Calculate the total interest paid over the life of the 30-year and 15-year loans described in Problem 2.63.

For the 30-year loan:

$$
I = (1/100) \sum_{n=0}^{359} y[n-1] = \$270308.77
$$

For the 15-year loan:

$$
I = (1/100) \sum_{n=0}^{179} y[n-1] = \$116029.62
$$

2.65. Find difference-equation descriptions for the three systems depicted in Fig. P2.65. (a)

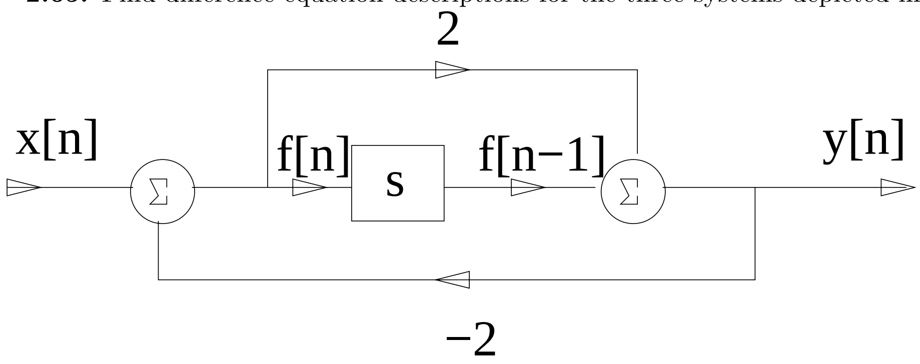


Figure P2.65. (a) Block diagram

$$
\begin{aligned}
f[n] &= -2y[n] + x[n] \\
y[n] &= f[n-1] + 2f[n] \\
&= -2y[n-1] + x[n-1] - 4y[n] + 2x[n] \\
5y[n] + 2y[n-1] &= x[n-1] + 2x[n]
\end{aligned}
$$

(b)

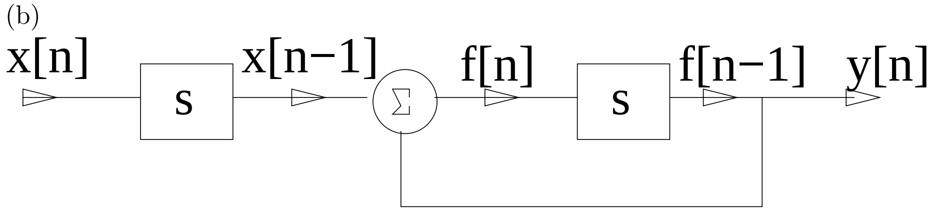


Figure P2.65. (b) Block diagram

$$
\begin{aligned}
f[n] &= y[n] + x[n-1] \\
y[n] &= f[n-1] \\
&= y[n-1] + x[n-2]
\end{aligned}
$$

(c)

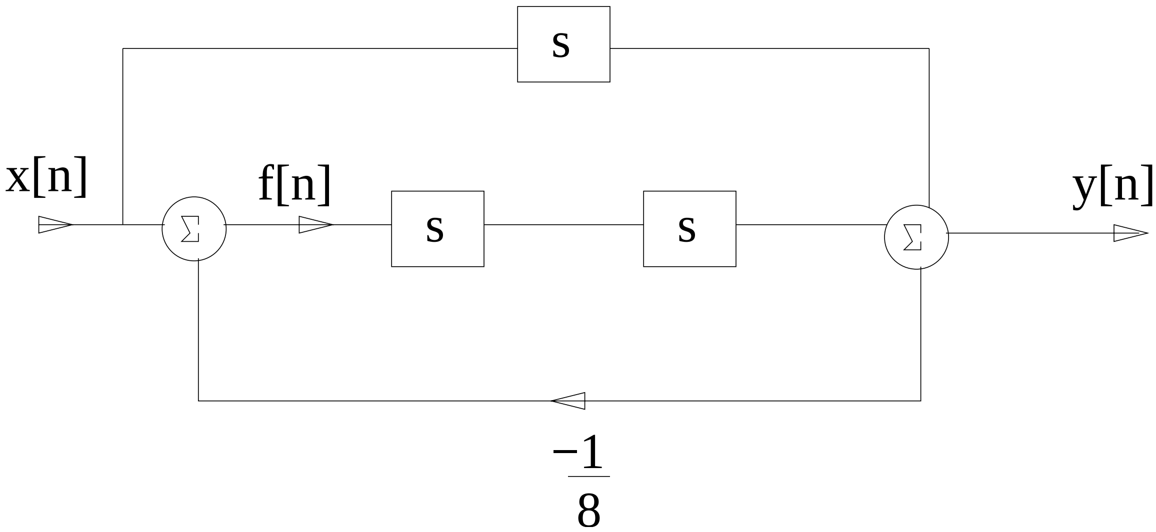


Figure P2.65. (c) Block diagram

$$\begin{aligned} f[n] &= x[n] - \frac{1}{8}y[n] \\ y[n] &= x[n-1] + f[n-2] \\ y[n] + \frac{1}{8}y[n-2] &= x[n-1] + x[n-2] \end{aligned}$$

2.66. Draw direct form I and direct form II implementations for the following difference equations.
(a) $y[n] - \frac{1}{4}y[n-1] = 6x[n]$
(i) Direct Form I

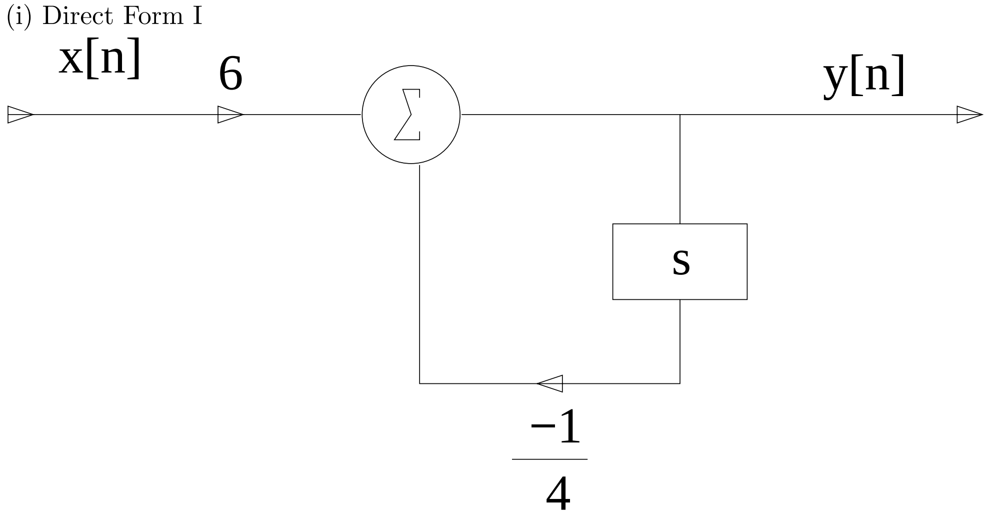
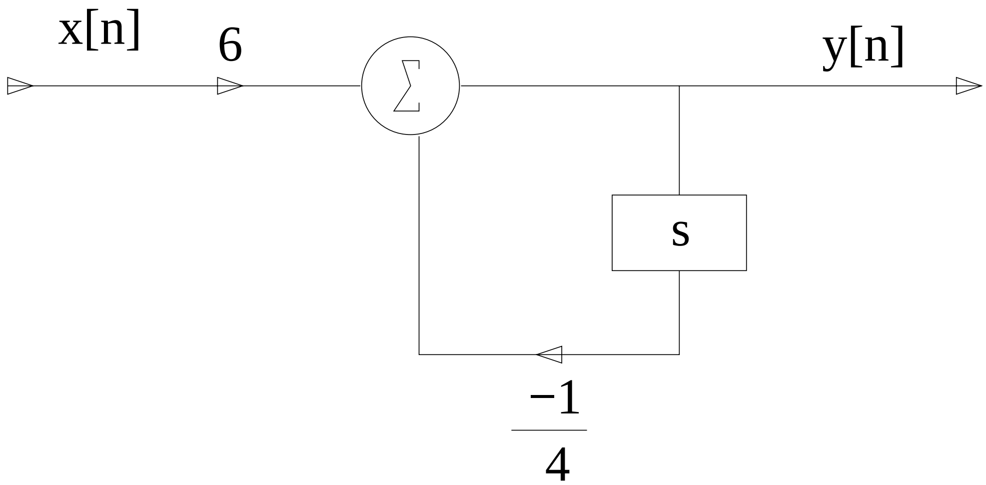


Figure P2.66. (a) Direct form I



(ii) Direct form II

Figure P2.66. (a) Direct form II

(b) $y[n] + \frac{1}{2}y[n-1] - \frac{1}{8}y[n-2] = x[n] + 2x[n-1]$

(i) Direct Form I

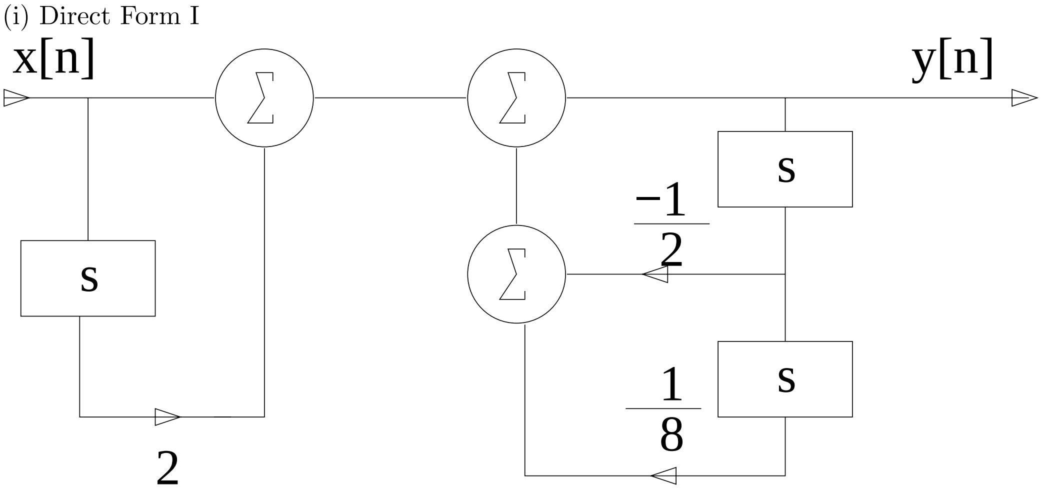
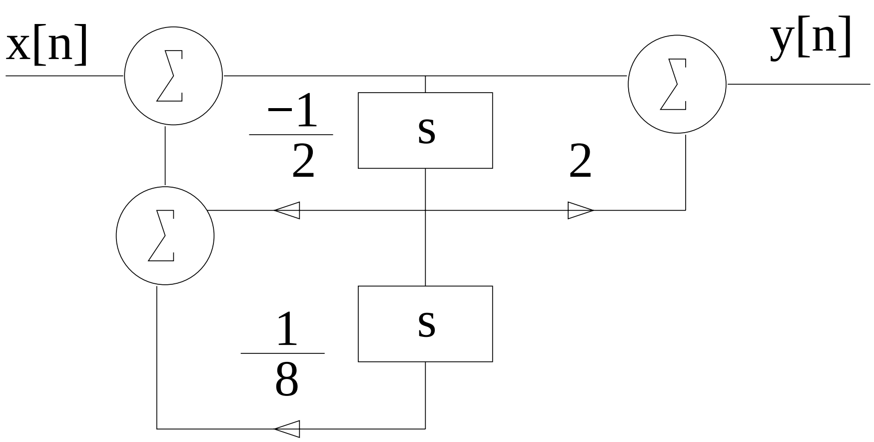


Figure P2.66. (b) Direct form I



(ii) Direct form II

Figure P2.66. (b) Direct form II

(c) $y[n] - \frac{1}{9}y[n-2] = x[n-1]$

(i) Direct Form I

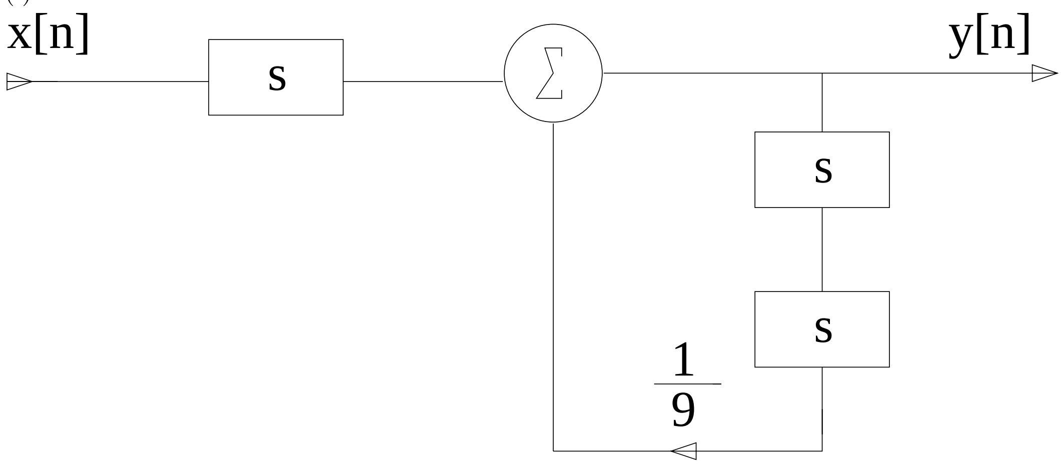
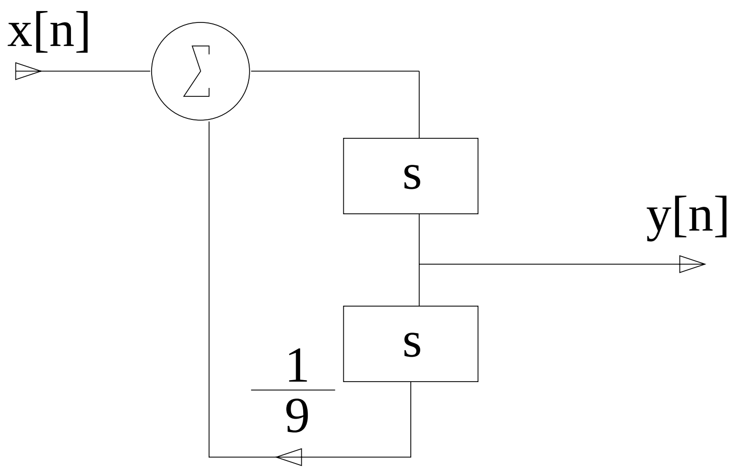


Figure P2.66. (c) Direct form I



(ii) Direct form II

Figure P2.66. (c) Direct form II

(d) $y[n] + \frac{1}{2}y[n-1] - y[n-3] = 3x[n-1] + 2x[n-2]$

(i) Direct Form I

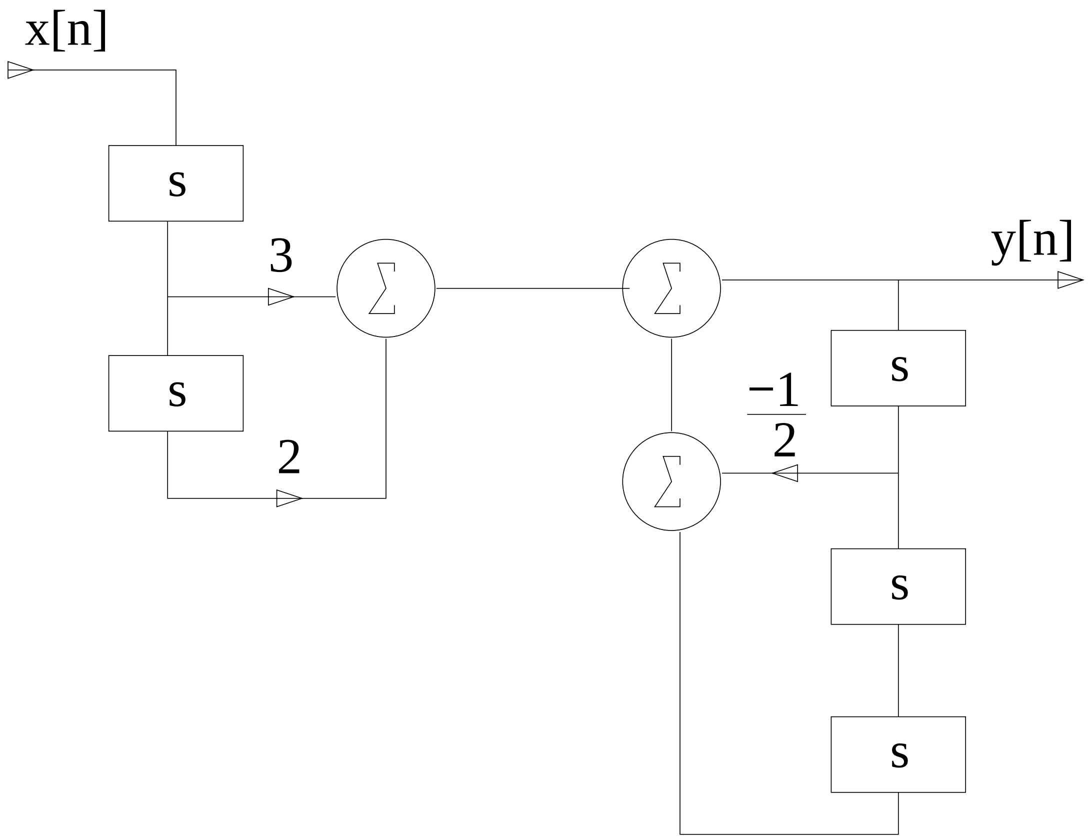
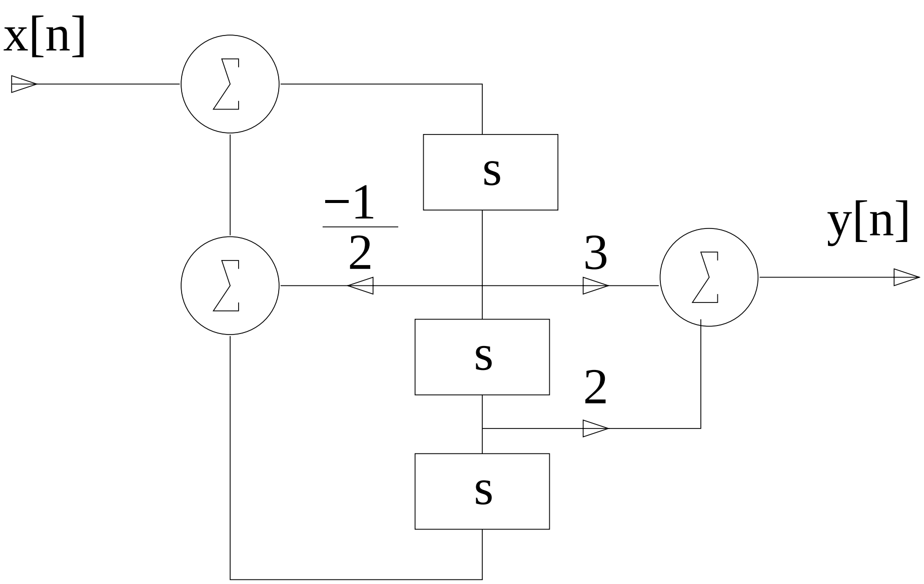


Figure P2.66. (d) Direct form I



(ii) Direct form II

Figure P2.66. (d) Direct form II

2.67. Convert the following differential equations to integral equations and draw direct form I and direct form II implementations of the corresponding systems.

(a) $\frac{d}{dt}y(t) + 10y(t) = 2x(t)$

$$\begin{aligned} y(t) + 10y^{(1)}(t) &= 2x^{(1)}(t) \\ y(t) &= 2x^{(1)}(t) - 10y^{(1)}(t) \end{aligned}$$

Direct Form I

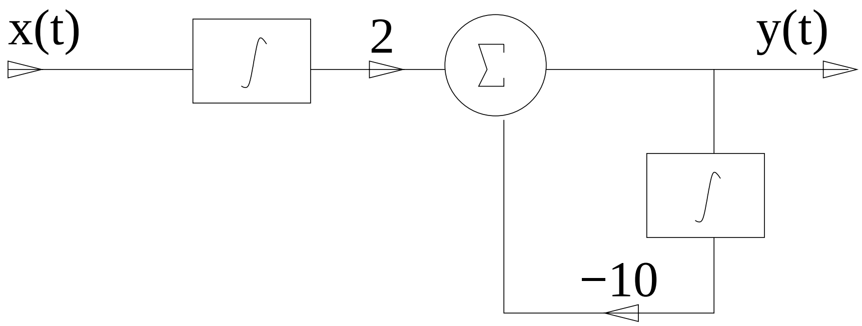


Figure P2.67. (a) Direct Form I

Direct Form II

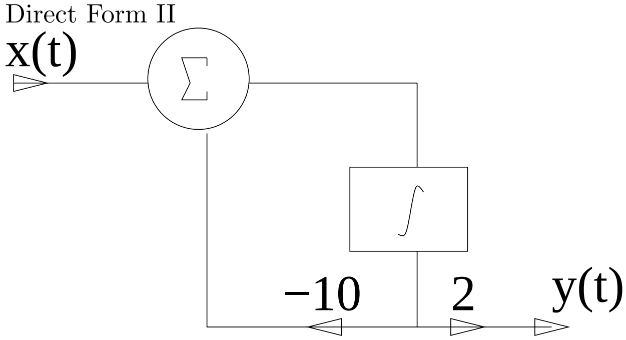


Figure P2.67. (a) Direct Form II

(b) $\frac{d^2}{dt^2}y(t) + 5\frac{d}{dt}y(t) + 4y(t) = \frac{d}{dt}x(t)$

$$\begin{aligned} y(t) + 5y^{(1)}(t) + 4y^{(2)}(t) &= x^{(1)}(t) \\ y(t) &= x^{(1)}(t) - 5y^{(1)}(t) - 4y^{(2)}(t) \end{aligned}$$

Direct Form I

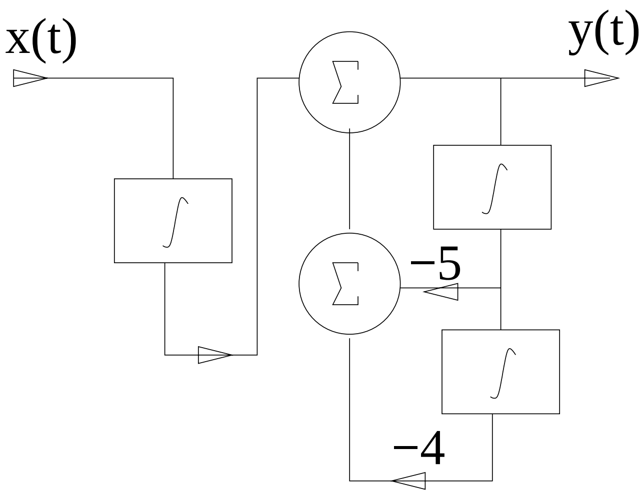


Figure P2.67. (b) Direct Form I

Direct Form II

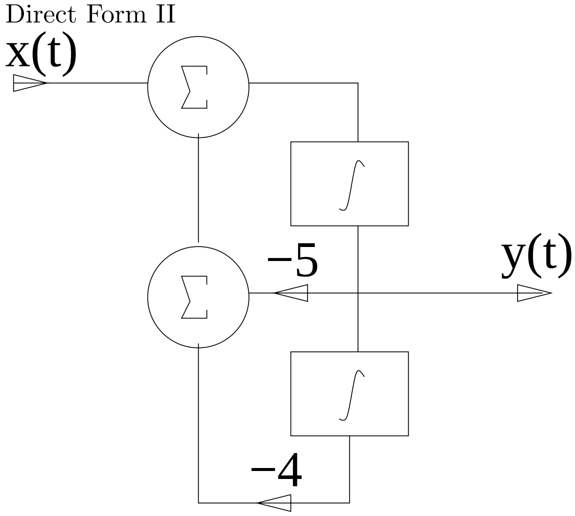


Figure P2.67. (b) Direct Form II

(c) $\frac{d^2}{dt^2}y(t) + y(t) = 3\frac{d}{dt}x(t)$

$$\begin{aligned} y(t) + y^{(2)}(t) &= 3x^{(1)}(t) \\ y(t) &= 3x^{(1)}(t) - y^{(2)}(t) \end{aligned}$$

Direct Form I

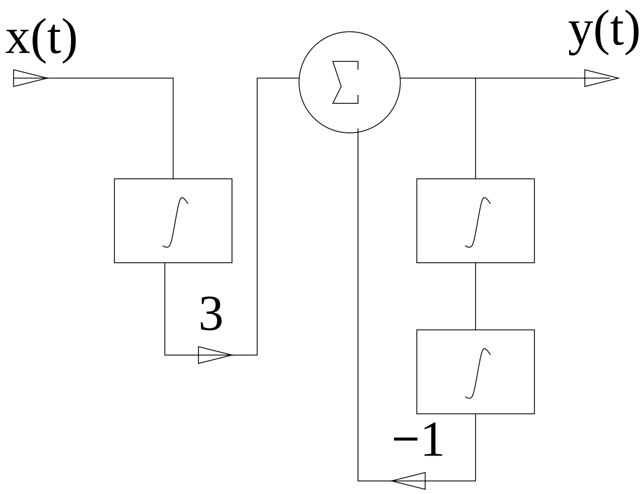


Figure P2.67. (c) Direct Form I

Direct Form II

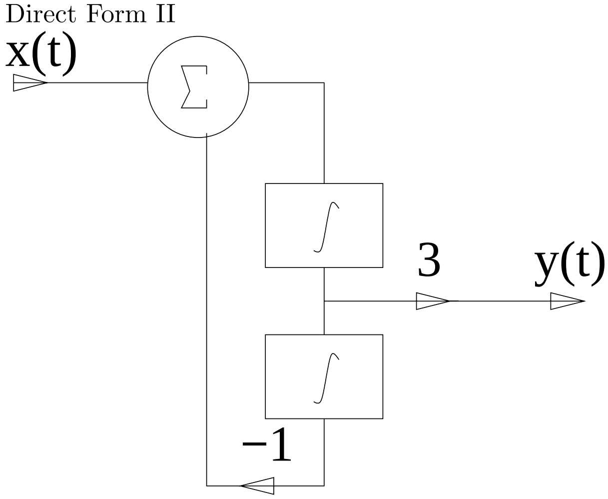


Figure P2.67. (c) Direct Form II

(d) $\frac{d^3}{dt^3}y(t) + 2\frac{d}{dt}y(t) + 3y(t) = x(t) + 3\frac{d}{dt}x(t)$

$$\begin{aligned} y(t) + 2y^{(2)}(t) + 3y^{(3)}(t) &= x^{(3)}(t) + 3x^{(2)}(t) \\ y(t) &= x^{(3)}(t) + 3x^{(2)}(t) - 2y^{(2)}(t) - 3y^{(3)}(t) \end{aligned}$$

Direct Form I

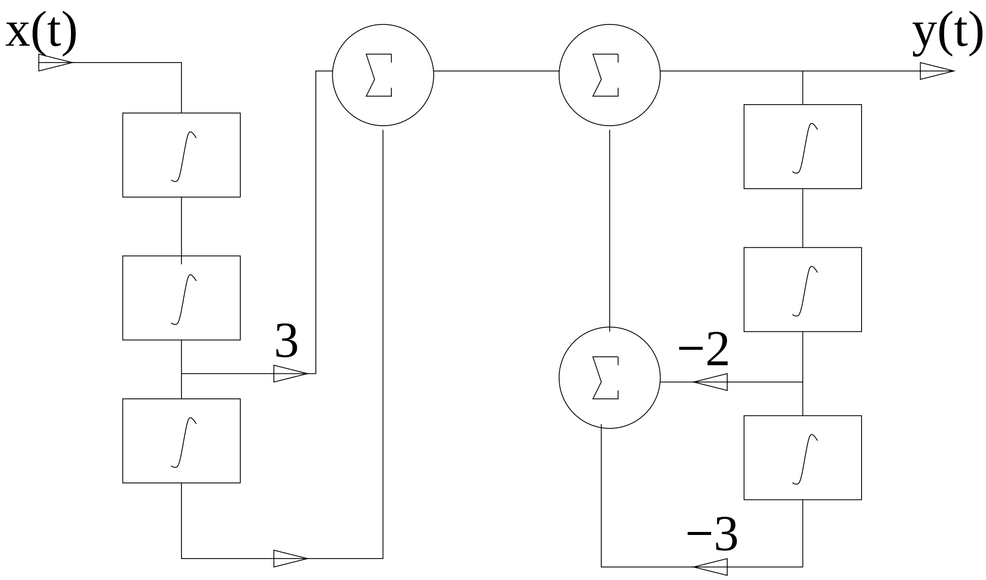


Figure P2.67. (d) Direct Form I

Direct Form II

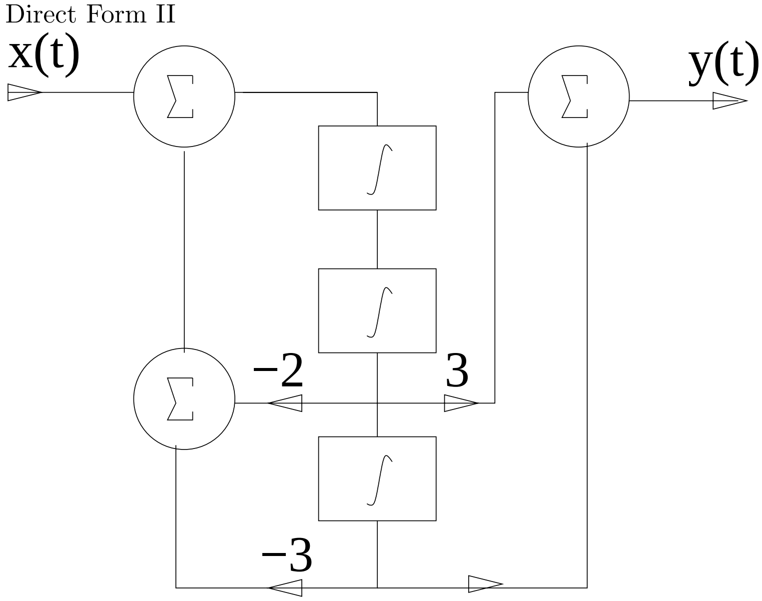


Figure P2.67. (d) Direct Form II

2.68. Find differential-equation descriptions for the two systems depicted in Fig. P2.68.
(a)

$$\begin{aligned} y(t) &= x^{(1)}(t) + 2y^{(1)}(t) \\ \frac{d}{dt}y(t) - 2y(t) &= x(t) \end{aligned}$$

(b)

$$
\begin{aligned}
y(t) &= x^{(1)}(t) + 2y^{(1)}(t) - y^{(2)}(t) \\
\frac{d^2}{dt^2}y(t) - 2\frac{d}{dt}y(t) + y(t) &= \frac{d}{dt}x(t)
\end{aligned}
$$

2.69. Determine a state variable description for the four discrete-time systems depicted in Fig. P2.69.
(a)

$$
\begin{aligned}
q[n+1] &= -2q[n] + x[n] \\
y[n] &= 3x[n] + q[n]
\end{aligned}
$$

$\mathbf{A} = [-2]\,, \quad \mathbf{b} = [1] \quad \mathbf{c} = [1]\,, \quad D = [3]$
(b)

$$
\begin{aligned}
q_1[n+1] &= -q_2[n] + 2x[n] \\
q_2[n+1] &= \frac{1}{4}q_1[n] + \frac{1}{2}q_2[n] - x[n] \\
y[n] &= -2q_2[n]
\end{aligned}
$$

$\mathbf{A} = \begin{bmatrix} 0 & -1 \\ \frac{1}{4} & \frac{1}{2} \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}, \quad \mathbf{c} = \begin{bmatrix} -2 & 0 \end{bmatrix}, \quad D = [0]$
(c)

$$
\begin{aligned}
q_1[n+1] &= -\frac{1}{8}q_3[n] + x[n] \\
q_2[n+1] &= q_1[n] + \frac{1}{4}q_3[n] + 2x[n] \\
q_3[n+1] &= q_2[n] - \frac{1}{2}q_3[n] + 3x[n] \\
y[n] &= q_3[n]
\end{aligned}
$$

$\mathbf{A} = \begin{bmatrix} 0 & 0 & -\frac{1}{8} \\ 1 & 0 & \frac{1}{4} \\ 0 & 1 & -\frac{1}{2} \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad \mathbf{c} = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}, \quad D = [0]$
(d)

$$
\begin{aligned}
q_1[n+1] &= -\frac{1}{4}q_1[n] + \frac{1}{6}q_2[n] + x[n] \\
q_2[n+1] &= q_1[n] + q_2[n] + 2x[n] \\
y[n] &= q_2[n] - x[n]
\end{aligned}
$$

$$\mathbf{A} = \begin{bmatrix} -\frac{1}{4} & \frac{1}{6} \\ 1 & 1 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad \mathbf{c} = \begin{bmatrix} 0 & 1 \end{bmatrix}, \quad D = [-1]$$

2.70. Draw block-diagram representations corresponding to the discrete-time state-variable descriptions of the following LTI systems:

(a) $\mathbf{A} = \begin{bmatrix} 1 & -\frac{1}{2} \\ \frac{1}{3} & 0 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad \mathbf{c} = \begin{bmatrix} 1 & 1 \end{bmatrix}, \quad D = [0]$

$$\begin{aligned} q_1[n+1] &= q_1[n] - \frac{1}{2}q_2[n] + x[n] \\ q_2[n+1] &= \frac{1}{3}q_1[n] + 2x[n] \\ y[n] &= q_1[n] + q_2[n] \end{aligned}$$

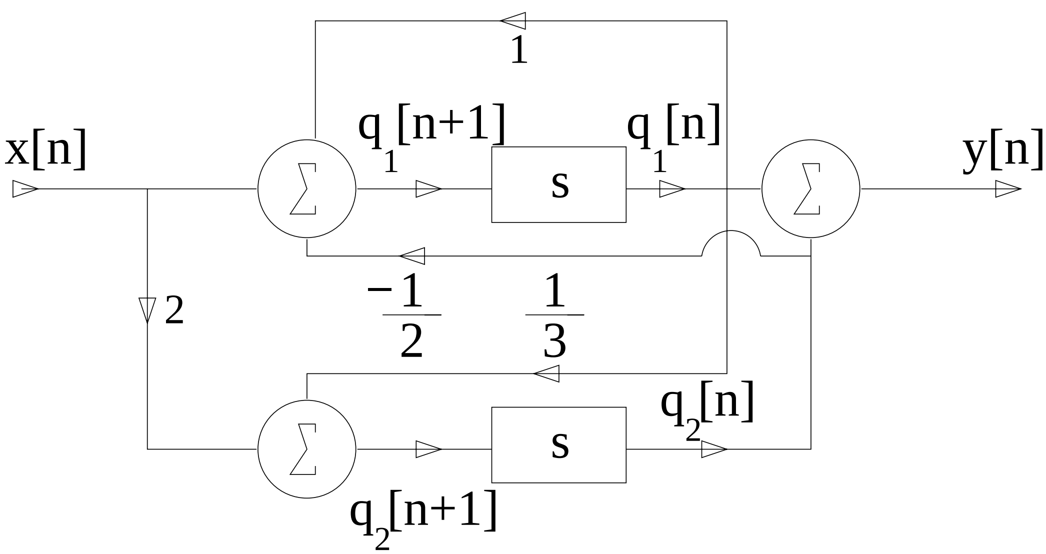


Figure P2.70. (a) Block Diagram

(b) $\mathbf{A} = \begin{bmatrix} 1 & -\frac{1}{2} \\ \frac{1}{3} & 0 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad \mathbf{c} = \begin{bmatrix} 1 & -1 \end{bmatrix}, \quad D = [0]$

$$\begin{aligned} q_1[n+1] &= q_1[n] - \frac{1}{2}q_2[n] + x[n] \\ q_2[n+1] &= \frac{1}{3}q_1[n] + 2x[n] \\ y[n] &= q_1[n] - q_2[n] \end{aligned}$$

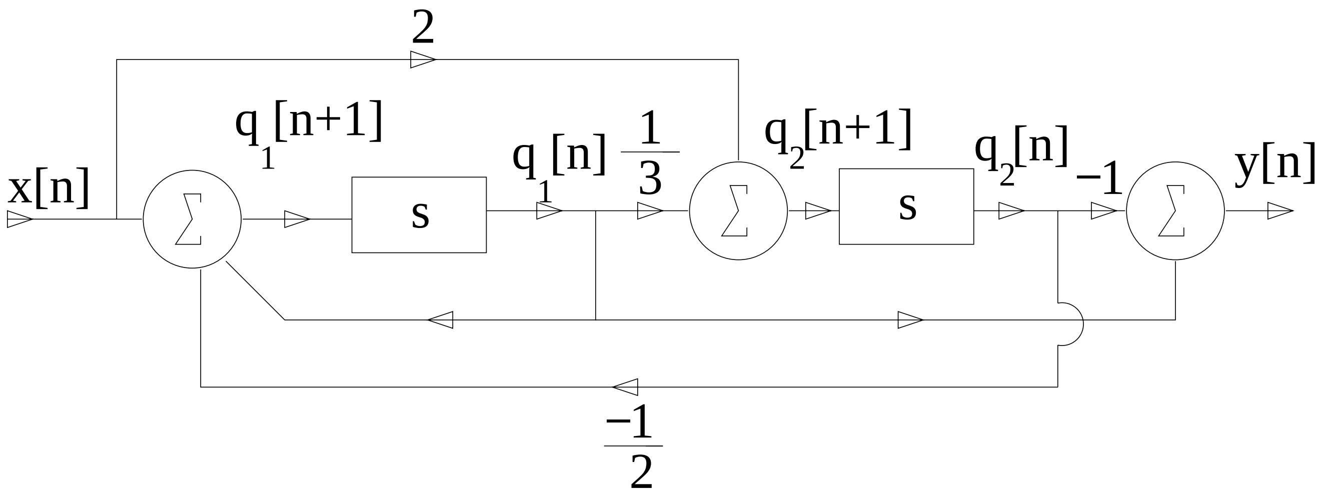


Figure P2.70. (b) Block Diagram

(c) $\mathbf{A} = \begin{bmatrix} 0 & -\frac{1}{2} \\ \frac{1}{3} & -1 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $\mathbf{c} = \begin{bmatrix} 1 & 0 \end{bmatrix}$, $D = [1]$

$$\begin{aligned} q_1[n+1] &= -\frac{1}{2}q_2[n] \\ q_2[n+1] &= \frac{1}{3}q_1[n] - q_2[n] + x[n] \\ y[n] &= q_1[n] + x[n] \end{aligned}$$

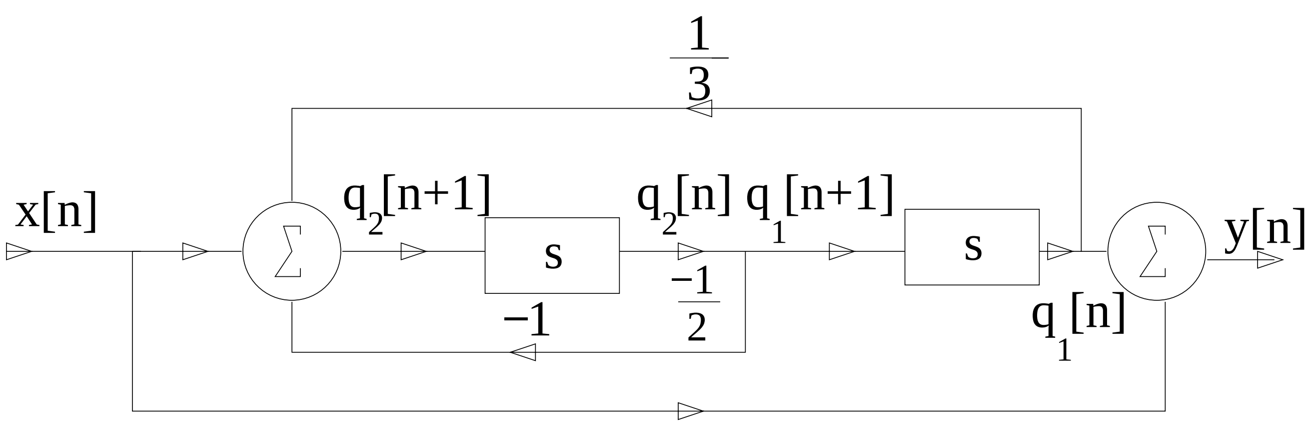


Figure P2.70. (c) Block Diagram

(d) $\mathbf{A} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$, $\mathbf{c} = \begin{bmatrix} 1 & -1 \end{bmatrix}$, $D = [0]$

$$\begin{aligned} q_1[n+1] &= 2x[n] \\ q_2[n+1] &= q_2[n] + 3x[n] \\ y[n] &= q_1[n] - q_2[n] \end{aligned}$$

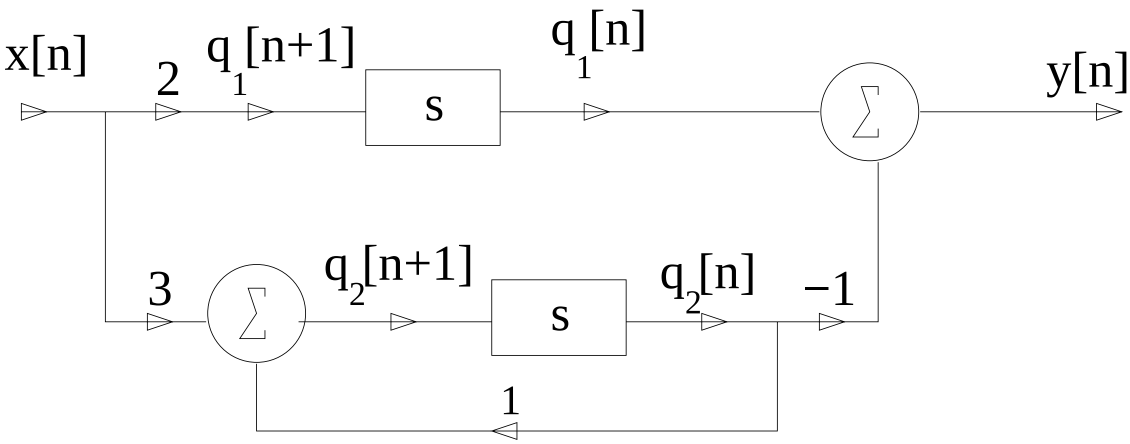


Figure P2.70. (d) Block Diagram

2.71. Determine a state-variable description for the five continuous-time LTI systems depicted in Fig. P2.71.

(a)

$$\begin{aligned} \frac{d}{dt}q(t) &= -q(t) + x(t) \\ y(t) &= 2q(t) + 6x(t) \end{aligned}$$

$\mathbf{A} = [-1]\,, \quad \mathbf{b} = [1]\,, \quad \mathbf{c} = [2]\,, \quad D = [6]$

(b)

$$\begin{aligned} \frac{d}{dt}q_1(t) &= q_2(t) + 2x(t) \\ \frac{d}{dt}q_2(t) &= q_1(t) + q_2(t) \\ y(t) &= q_1(t) \end{aligned}$$

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 1 & -2 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \quad \mathbf{c} = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad D = [0]$$

(c)

$$\begin{aligned} \frac{d}{dt}q_1(t) &= -8q_2(t) - 3q_3(t) + x(t) \\ \frac{d}{dt}q_2(t) &= q_1(t) + 4q_2(t) + 3x(t) \\ \frac{d}{dt}q_3(t) &= 2q_1(t) + q_2(t) - q_3(t) \\ y(t) &= q_3(t) \end{aligned}$$

$$\mathbf{A} = \begin{bmatrix} 0 & -8 & -3 \\ 1 & 4 & 0 \\ 2 & 1 & -1 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}, \quad \mathbf{c} = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}, \quad D = [0]$$

(d)

$$\begin{aligned}
x(t) &= q_1(t)R + L\frac{d}{dt}q_1(t) + q_2(t) \\
(1) \quad \frac{d}{dt}q_1(t) &= -\frac{R}{L}q_1(t) - \frac{1}{L}q_2(t) + \frac{1}{L}x(t) \\
q_1(t) &= C\frac{d}{dt}q_2(t) \\
(2) \quad \frac{d}{dt}q_2(t) &= \frac{1}{C}q_1(t) \\
x(t) &= q_1(t)R + y(t) + q_2(t) \\
(3) \quad y(t) &= -Rq_1(t) - q_2(t) + x(t) \\
&\text{Combining } (1), (2), (3)
\end{aligned}$$

$$\mathbf{A} = \begin{bmatrix} -\frac{R}{L} & -\frac{1}{L} \\ \frac{1}{C} & 0 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} \frac{1}{L} \\ 0 \end{bmatrix}, \quad \mathbf{c} = \begin{bmatrix} -R & -1 \end{bmatrix}, \quad D = [1]$$

(e)

$$\begin{aligned}
x(t) &= y(t)R + q_1(t) \\
(3) \quad y(t) &= -\frac{1}{R}q_1(t) + \frac{1}{R}x(t) \\
y(t) &= C\frac{d}{dt}q_1(t) + q_2(t) \\
-\frac{1}{R}q_1(t) + \frac{1}{R}x(t) &= C\frac{d}{dt}q_1(t) + q_2(t) \\
(1) \quad \frac{d}{dt}q_1(t) &= -\frac{1}{RC}q_1(t) - \frac{1}{C}q_2(t) + \frac{1}{RC}x(t) \\
q_1(t) &= L\frac{d}{dt}q_2(t) \\
(2) \quad \frac{d}{dt}q_2(t) &= \frac{1}{L}q_1(t) \\
&\text{Combining } (1), (2), (3)
\end{aligned}$$

$$\mathbf{A} = \begin{bmatrix} -\frac{1}{RC} & -\frac{1}{C} \\ \frac{1}{L} & 0 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} \frac{1}{RC} \\ 0 \end{bmatrix}, \quad \mathbf{c} = \begin{bmatrix} -\frac{1}{R} & 0 \end{bmatrix}, \quad D = [\tfrac{1}{R}]$$

2.72. Draw block-diagram representations corresponding to the continuous-time state variable descriptions of the following LTI systems:

(a) $\mathbf{A} = \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & -\frac{1}{2} \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}, \quad \mathbf{c} = \begin{bmatrix} 1 & 1 \end{bmatrix}, \quad D = [0]$

$$\frac{d}{dt}q_1(t) = \frac{1}{3}q_1(t) - x(t)$$

$$\begin{aligned}\frac{d}{dt}q_2(t) &= -\frac{1}{2}q_2(t) + 2x(t)\\ y(t) &= q_1(t) + q_2(t)\end{aligned}$$

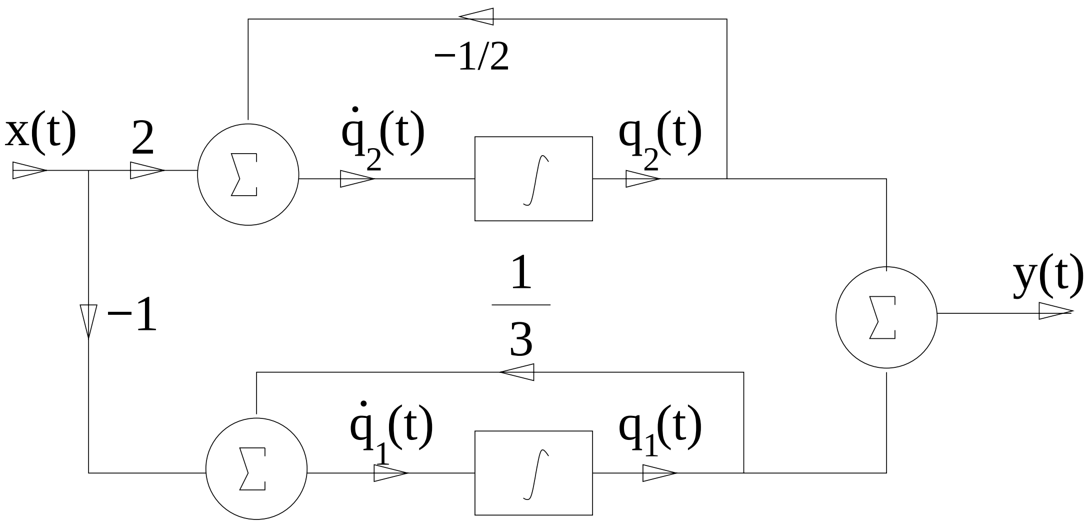


Figure P2.72. (a) Block Diagram

(b) $\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$, $\mathbf{c} = \begin{bmatrix} 0 & -1 \end{bmatrix}$, $D = [0]$

$$\begin{aligned}\frac{d}{dt}q_1(t) &= q_1(t) + q_2(t) - x(t)\\ \frac{d}{dt}q_2(t) &= q_1(t) + 2x(t)\\ y(t) &= -q_2(t)\end{aligned}$$

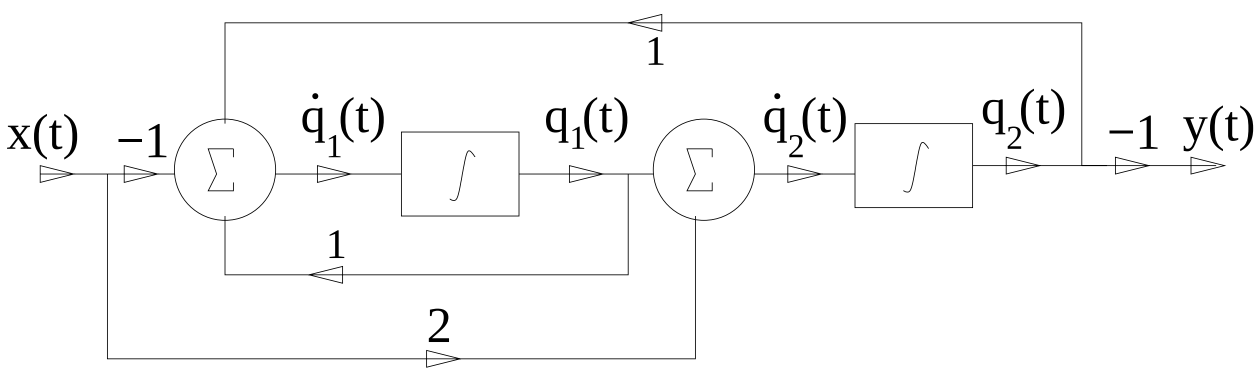


Figure P2.72. (b) Block Diagram

(c) $\mathbf{A} = \begin{bmatrix} 1 & -1 \\ 0 & -1 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 0 \\ 5 \end{bmatrix}$, $\mathbf{c} = \begin{bmatrix} 1 & 0 \end{bmatrix}$, $D = [0]$

$$\frac{d}{dt}q_1(t) = q_1(t) - q_2(t)$$

$$
\begin{aligned}
\frac{d}{dt}q_2(t) &= -q_2(t) + 5x(t) \\
y(t) &= q_1(t)
\end{aligned}
$$

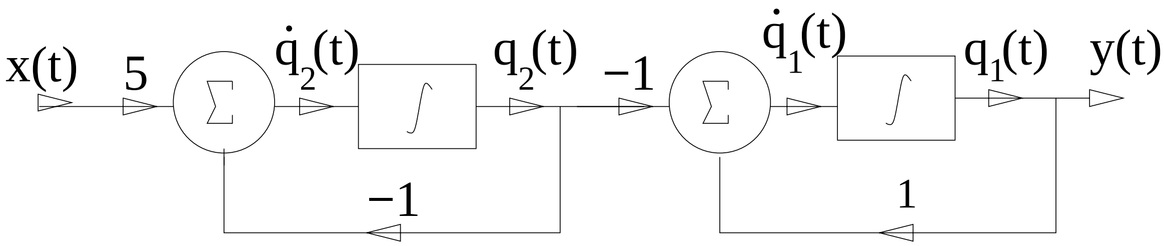


Figure P2.72. (c) Block Diagram

(d) $\mathbf{A} = \begin{bmatrix} 1 & -2 \\ 1 & 1 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \quad \mathbf{c} = \begin{bmatrix} 1 & 1 \end{bmatrix}, \quad D = [0]$

$$
\begin{aligned}
\frac{d}{dt}q_1(t) &= q_1(t) - 2q_2(t) + 2x(t) \\
\frac{d}{dt}q_2(t) &= q_1(t) + q_2(t) + 3x(t) \\
y(t) &= q_1(t) + q_2(t)
\end{aligned}
$$

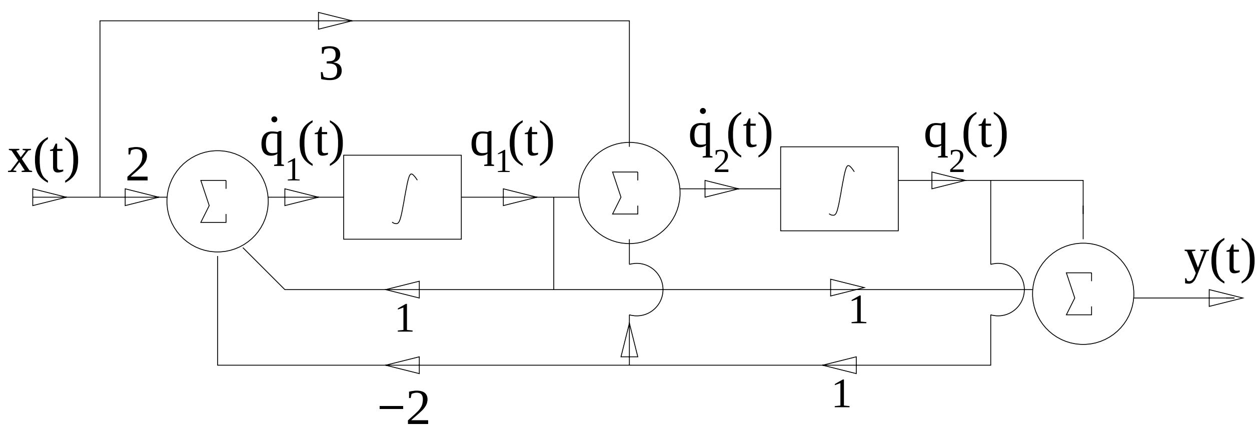


Figure P2.72. (d) Block Diagram

2.73. Let a discrete-time system have the state-variable description

$$
\mathbf{A} = \begin{bmatrix} 1 & -\frac{1}{2} \\ \frac{1}{3} & 0 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad \mathbf{c} = \begin{bmatrix} 1 & -1 \end{bmatrix}, \quad D = [0]
$$

(a) Define new states $q_1'[n] = 2q_1[n], q_2'[n] = 3q_2[n]$. Find the new state-variable description $\mathbf{A}', \mathbf{b}', \mathbf{c}', D'$.

$\mathbf{q}_1' = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \mathbf{q}_1$

Thus the transformation matrix $\mathbf{T}$ is

$$
\begin{aligned}
\mathbf{T} &= \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \\
\mathbf{T}^{-1} &= \frac{1}{6}\begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} \\
&= \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{3} \end{bmatrix} \\
\mathbf{A}' = \mathbf{T}\mathbf{A}\mathbf{T}^{-1} &= \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 1 & -\frac{1}{2} \\ \frac{1}{3} & 0 \end{bmatrix}\begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{3} \end{bmatrix} \\
&= \begin{bmatrix} 1 & -\frac{1}{3} \\ \frac{1}{2} & 0 \end{bmatrix} \\
\mathbf{b}' = \mathbf{T}\mathbf{b} &= \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} \\
&= \begin{bmatrix} 2 \\ 6 \end{bmatrix} \\
\mathbf{c}' = \mathbf{c}\mathbf{T}^{-1} &= \begin{bmatrix} 1 & -1 \end{bmatrix}\begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{3} \end{bmatrix} \\
&= \begin{bmatrix} \frac{1}{2} & -\frac{1}{3} \end{bmatrix} \\
D' = D &= 0
\end{aligned}
$$

(b) Define new states $q_1'[n] = 3q_2[n], q_2'[n] = 2q_1[n]$. Find the new state-variable description $\mathbf{A}', \mathbf{b}', \mathbf{c}', D'$.
$\mathbf{q}_1' = \begin{bmatrix} 0 & 3 \\ 2 & 0 \end{bmatrix} \mathbf{q}_1$
Thus the transformation matrix $\mathbf{T}$ is

$$
\begin{aligned}
\mathbf{T} &= \begin{bmatrix} 0 & 3 \\ 2 & 0 \end{bmatrix} \\
\mathbf{T}^{-1} &= -\frac{1}{6}\begin{bmatrix} 0 & -3 \\ -2 & 0 \end{bmatrix} \\
&= \begin{bmatrix} 0 & \frac{1}{2} \\ \frac{1}{3} & 0 \end{bmatrix} \\
\mathbf{A}' = \mathbf{T}\mathbf{A}\mathbf{T}^{-1} &= \begin{bmatrix} 0 & \frac{1}{2} \\ -\frac{1}{3} & 1 \end{bmatrix} \\
\mathbf{b}' = \mathbf{T}\mathbf{b} &= \begin{bmatrix} 6 \\ 2 \end{bmatrix} \\
\mathbf{c}' = \mathbf{c}\mathbf{T}^{-1} &= \begin{bmatrix} -\frac{1}{3} & \frac{1}{2} \end{bmatrix} \\
D' = D &= 0
\end{aligned}
$$

(c) Define new states $q_1'[n] = q_1[n] + q_2[n], q_2'[n] = q_1[n] - q_2[n]$. Find the new state-variable description $\mathbf{A}', \mathbf{b}', \mathbf{c}', D'$.
The transformation matrix $\mathbf{T}$ is

$$\begin{aligned}
\mathbf{T} &= \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \\
\mathbf{T}^{-1} &= -\frac{1}{2}\begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix} \\
&= \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix} \\
\mathbf{A}' = \mathbf{TAT}^{-1} &= \begin{bmatrix} \frac{5}{12} & \frac{11}{12} \\ \frac{1}{12} & \frac{7}{12} \end{bmatrix} \\
\mathbf{b}' = \mathbf{Tb} &= \begin{bmatrix} 3 \\ -1 \end{bmatrix} \\
\mathbf{c}' = \mathbf{cT}^{-1} &= \begin{bmatrix} 0 & 1 \end{bmatrix} \\
D' = D &= 0
\end{aligned}$$

2.74. Consider the continuous-time system depicted in Fig. P2.74.
(a) Find the state-variable description for this system assuming the states $q_1(t)$ and $q_2(t)$ are as labeled.

$$\begin{aligned}
\frac{d}{dt}q_1(t) &= \alpha_1 q_1(t) + b_1 x(t) \\
\frac{d}{dt}q_2(t) &= \alpha_2 q_2(t) + b_2 x(t) \\
y(t) &= c_1 q_1(t) + c_2 q_2(t)
\end{aligned}$$

$$\mathbf{A} = \begin{bmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}, \quad \mathbf{c} = \begin{bmatrix} c_1 & c_2 \end{bmatrix}, \quad D = [0]$$

(b) Define new states $q_1'(t) = q_1(t) - q_2(t), q_2'(t) = 2q_1(t)$. Find the new state-variable description $\mathbf{A}', \mathbf{b}', \mathbf{c}', D'$.

$$\begin{aligned}
\mathbf{T} &= \begin{bmatrix} 1 & -1 \\ 2 & 0 \end{bmatrix} \\
\mathbf{T}^{-1} &= \frac{1}{2}\begin{bmatrix} 0 & 1 \\ -2 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 0 & \frac{1}{2} \\ -1 & \frac{1}{2} \end{bmatrix}
\end{aligned}$$

$$
\begin{aligned}
\mathbf{A}' = \mathbf{TAT}^{-1} &= \begin{bmatrix} \alpha_2 & \frac{1}{2}(\alpha_1 - \alpha_2) \\ 0 & \alpha_1 \end{bmatrix} \\
\mathbf{b}' = \mathbf{Tb} &= \begin{bmatrix} b_1 - b_2 \\ 2b_1 \end{bmatrix} \\
\mathbf{c}' = \mathbf{cT}^{-1} &= \begin{bmatrix} -c_2 & \frac{1}{2}(c_1 + c_2) \end{bmatrix} \\
D = D' &= 0
\end{aligned}
$$

(c) Draw a block diagram corresponding the new state-variable description in (b).
The corresponding differential equations are:

$$
\begin{aligned}
\frac{d}{dt}q_1(t) &= \alpha_2 q_1(t) + \frac{1}{2}(\alpha_1 - \alpha_2)q_2(t) + (b_1 - b_2)x(t) \\
\frac{d}{dt}q_2(t) &= \alpha_1 q_2(t) + 2b_1 x(t) \\
y(t) &= -c_1 q_1(t) + \frac{1}{2}c_1 c_2 q_2(t)
\end{aligned}
$$

'a' will replace α in the following figures.

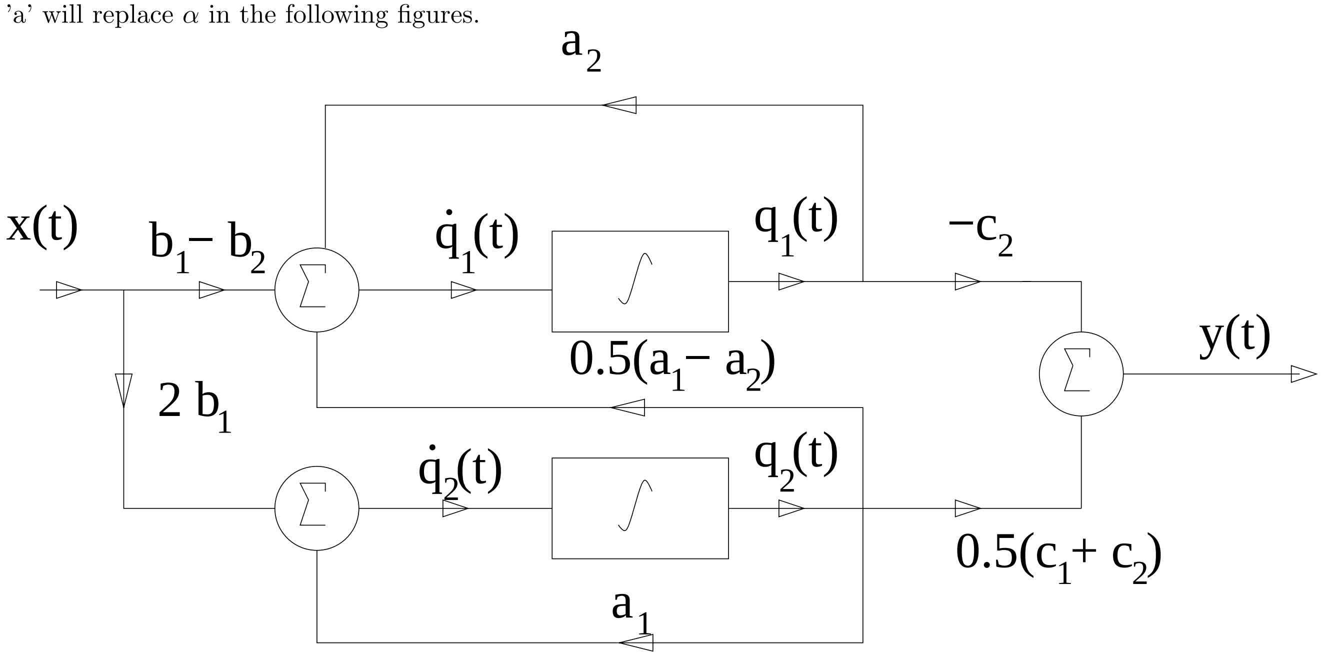


Figure P2.74. (c) Block Diagram

(d) Define new states $q_1'(t) = \frac{1}{b_1}q_1(t), q_2'(t) = b_2 q_1(t) - b_1 q_2(t)$. Find the new state-variable description $\mathbf{A}', \mathbf{b}', \mathbf{c}', D'$.

$$
\mathbf{T} = \begin{bmatrix} \frac{1}{b_1} & 0 \\ b_2 & -b_1 \end{bmatrix}
$$

$$
\begin{aligned}
\mathbf{T}^{-1} &= \begin{bmatrix} b_1 & 0 \\ b_2 & -\frac{1}{b_1} \end{bmatrix} \\
\mathbf{A}' = \mathbf{TAT}^{-1} &= \begin{bmatrix} \alpha_1 & 0 \\ b_1 b_2(\alpha_1 - \alpha_2) & \alpha_2 \end{bmatrix} \\
\mathbf{b}' = \mathbf{Tb} &= \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\
\mathbf{c}' = \mathbf{cT}^{-1} &= \begin{bmatrix} c_1 b_1 + c_2 b_2 & -\frac{c_2}{b_1} \end{bmatrix} \\
D = D' &= 0
\end{aligned}
$$

(e) Draw a block diagram corresponding the new state-variable description in (d).
The corresponding differential equations are:

$$
\begin{aligned}
\frac{d}{dt} q_1(t) &= \alpha_1 q_1(t) + x(t) \\
\frac{d}{dt} q_2(t) &= b_1 b2_2(\alpha_1 - \alpha_2) q_1(t) + \alpha_2 q_2(t) \\
y(t) &= c_1(b_1 + b_2) q_1(t) - \frac{c_2}{b_1} q_2(t)
\end{aligned}
$$

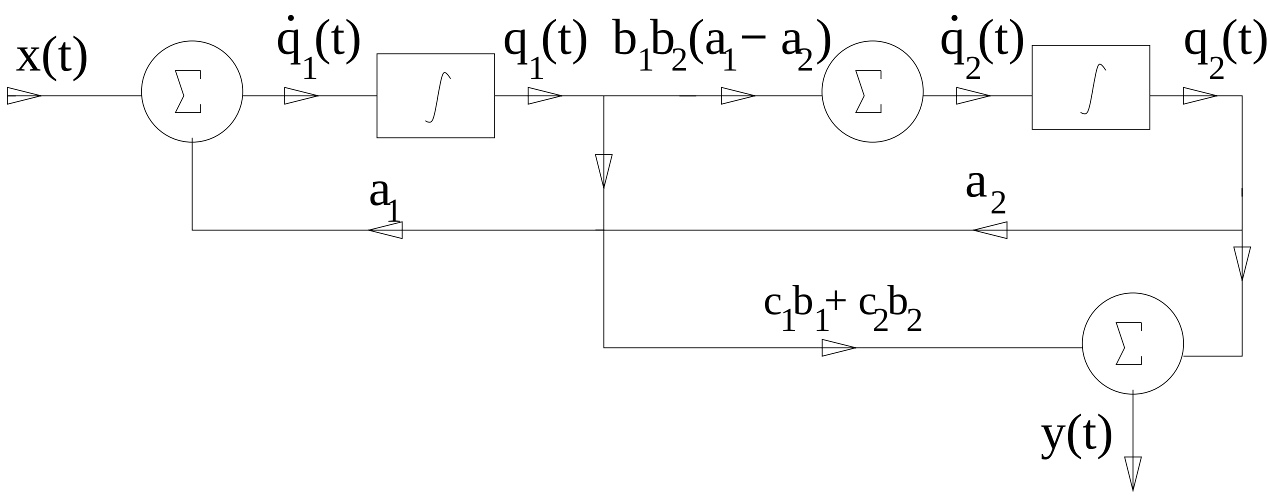


Figure P2.74. (e) Block Diagram

Solutions to Advanced Problems

2.75. We may develop the convolution integral using linearity, time invariance, and the limiting form of a stairstep approximation to the input signal. Define $g_\Delta(t)$ as the unit area rectangular pulse depicted in Fig. P2.75 (a).
(a) A stairstep approximation to a signal $x(t)$ is depicted in Fig. P2.75 (b). Express $\tilde{x}(t)$ as a weighted

sum of shifted pulses $g_\Delta(t)$. Does the approximation quality improve as Δ decreases?

$$\tilde{x}(t) \quad = \quad \sum_{k=-\infty}^{\infty} x(k\Delta) g_\Delta(t - k\Delta)\Delta$$

As Δ decreases and approaches zero, the approximation quality improves.

(b) Let the response of an LTI system to an input $g_\Delta(t)$ be $h_\Delta(t)$. If the input to this system is $\tilde{x}(t)$, find an expression for the the output of this system in terms of $h_\Delta(t)$.

Let the system be represented by $H\{.\}$ such that $H\{g_\Delta(t)\} = h_\Delta(t)$.

$$\begin{aligned}
H\{\tilde{x}(t)\} &= H\left\{\sum_{k=-\infty}^{\infty} x(k\Delta) g_\Delta(t - k\Delta)\Delta\right\} \\
&\quad \text{By the linearity of } H \\
&= \sum_{k=-\infty}^{\infty} (H\{x(k\Delta) g_\Delta(t - k\Delta)\Delta) \\
&= \sum_{k=-\infty}^{\infty} x(k\Delta) H\{g_\Delta(t - k\Delta)\}\Delta \\
&\quad \text{By the time-invariance of } H \\
H\{\tilde{x}(t)\} &= \sum_{k=-\infty}^{\infty} x(k\Delta) h_\Delta(t - k\Delta)\Delta
\end{aligned}$$

(c) In the limit as Δ goes to zero, $g_\Delta(t)$ satisfies the properties of an impulse and we may interpret $h(t) = \lim_{\Delta\to 0} h_\Delta(t)$ as the impulse response of the system. Show that the expression for the system output derived in (b) reduces to $x(t) * h(t)$ in the limit as Δ goes to zero.

$$\begin{aligned}
&\quad \text{When } \Delta \to 0 \\
h(t) &= \lim_{\Delta\to 0} h_\Delta(t) \\
\lim_{\Delta\to 0} H\{x^2(t)\} &= \lim_{\Delta\to 0} \sum_{k=-\infty}^{\infty} x(k\Delta) h_\Delta(t - k\Delta)\Delta
\end{aligned}$$

As $\Delta \to 0$, the limit is a Riemann sum, which represents an integral.

$$\begin{aligned}
y(t) &\cong \int_{-\infty}^{\infty} x(\tau) h_\Delta(t - \tau) d\tau \\
&\quad \text{Using the fact that } h(t) = \lim_{\Delta\to 0} h_\Delta(t) \\
y(t) &\cong \int_{-\infty}^{\infty} x(\tau) h(t - \tau) d\tau \\
y(t) &\cong x(t) * h(t)
\end{aligned}$$

2.76. Convolution of finite-duration discrete-time signals may be expressed as the product of a matrix and a vector. Let the input $x[n]$ be zero outside of $n = 0, 1, \ldots L-1$ and the impulse response $h[n]$ zero outside $n = 0, 1, \ldots M-1$. The output $y[n]$ is then zero outside $n = 0, 1, \ldots, L+M-1$. Define column vectors $\mathbf{x} = [x[0], x[1], \cdots x[L-1]]^T$ and $\mathbf{y} = [y[0], y[1], \cdots y[L+M-1]]^T$. Use the definition of the convolution sum to find a matrix $\mathbf{H}$ such that $\mathbf{y} = \mathbf{H}\mathbf{x}$.

$$
\begin{aligned}
y[n] &= \sum_{k=0}^{\infty} x[k]h[n-k] \\
&\quad \text{Applying the appropriate range for } x[n] \text{ and } h[n]\text{, starting with } n = 0 \\
y[0] &= x[0]h[0] \\
&\quad \text{Since all other values of } x[n] \text{ and } h[n] \text{ are 0, similarly} \\
y[1] &= x[1]h[0] + x[0]h[1] \\
y[2] &= x[2]h[0] + x[1]h[1] + x[0]h[2] \\
&\vdots \\
&\quad \text{Which can be written in matrix form as}
\end{aligned}
$$

$$
\begin{bmatrix} y[0] \\ y[1] \\ y[2] \\ \vdots \\ y[M-1] \\ \vdots \\ y[L+M-1] \end{bmatrix}
=
\begin{bmatrix}
h[0] & 0 & 0 & \ldots & & & 0 \\
h[1] & h[0] & 0 & \ldots & & & 0 \\
 & & \vdots & & & & \\
h[M-1] & h[M-2] & \ldots & h[0] & 0 & \ldots & 0 \\
 & & & \vdots & & & \\
0 & \ldots & & 0 & h[M-1] & \ldots & h[0]
\end{bmatrix}
\begin{bmatrix} x[0] \\ x[1] \\ x[2] \\ \vdots \\ x[L-1] \end{bmatrix}
$$

$$
\begin{aligned}
&\quad \text{Yields a solution of the form} \\
\mathbf{y} &= \mathbf{H}\mathbf{x}
\end{aligned}
$$

Where $\mathbf{y}$ is an $(L+M-1)$ by 1 matrix, $\mathbf{H}$ is an $(L+M-1)$ by L matrix, and $\mathbf{x}$ is L by 1.

2.77. Assume the impulse response of a continous-time system is zero outside the interval $0 < t < T_o$. Use a Riemann sum approximation to the convolution integral to convert the convolution integral to a convolution sum that relates uniformly spaced samples of the output signal to uniformly spaced samples of the input signal.

$$
\begin{aligned}
y(t) &= x(t) * h(t) \\
&= \int_0^{T_o} h(\tau)x(t-\tau)d\tau
\end{aligned}
$$

Since $h(t)$ is zero outside of that interval.

The Riemann sum approximation is

$$\begin{aligned}
\int_0^{T_o} f(\tau)d\tau &\approx \sum_{k=0}^{N-1} f(k\Delta)\Delta \\
\text{Where } \Delta = \frac{T_o}{N} & \\
\text{Using this approximation} & \\
y(t) &= \sum_{k=0}^{N-1} h(k\Delta)x(t-k\Delta)\Delta \\
& \text{Evaluate at } t = n\Delta \\
y(n\Delta) &= \sum_{k=0}^{N-1} h(k\Delta)x(n\Delta - k\Delta)\Delta \\
\text{Setting} & \\
y[n] &= y(n\Delta) \\
h[k] &= h(k\Delta) \\
x[k] &= x(k\Delta) \\
\text{Implies} & \\
y(n\Delta) = y[n] &= \sum_{k=0}^{N-1} h[k]x[n-k] \\
\text{Which is the discrete time convolution sum.} &
\end{aligned}$$

2.78. The cross-correlation between two real signals $x(t)$ and $y(t)$ is defined as

$$r_{xy}(t) = \int_{-\infty}^{\infty} x(\tau)y(\tau - t)d\tau$$

This is the area under the product of $x(t)$ and a shifted version of $y(t)$. Note that the independent variable $\tau - t$ is the negative of that found in the definition of convolution. The autocorrelation, $r_{xx}(t)$, of a signal $x(t)$ is obtained by replacing $y(t)$ with $x(t)$.

(a) Show that $r_{xy}(t) = x(t) * y(-t)$

$$\begin{aligned}
r_{xy}(t) &= \int_{-\infty}^{\infty} x(\tau)y(\tau - t)d\tau \\
&= \int_{-\infty}^{\infty} x(\tau)y(-(t-\tau))d\tau \qquad (1)
\end{aligned}$$

First assume $r_{xy}(t)$ can be expressed in terms of a convolution integral, i.e.,

$$\begin{aligned}
r_{xy}(t) &= \int_{-\infty}^{\infty} f_1(\tau)f_2(t-\tau)d\tau \\
&= f_1(t) * f_2(t) \qquad (2)
\end{aligned}$$

By (1), we can see that (1) and (2) are equivalent if:

$$f_1(v_1) = x(v_1), \text{ and}$$

$$f_2(v_2) = y(-v_2)$$

where v_1, and v_2 are arguments, for this case $v_1 = \tau$ and $v_2 = t - \tau$, then

$$\begin{aligned} r_{xy}(t) &= f_1(t) * f_2(t) \\ &= x(t) * y(-t) \end{aligned}$$

(b) Derive a step-by-step procedure for evaluating the cross-correlation analogous to the one for evaluating convolution integral given in Section 2.2.

1. Graph both $x(\tau)$ and $y(\tau - t)$ as a function of τ. To obtain $y(\tau - t)$, shift $y(\tau)$ by t.
2. Shift t to $-\infty$.
3. Write a mathematical representation for $x(\tau)y(\tau - t)$.
4. Increase the shift until the mathematical representation for $x(\tau)y(\tau - t)$ changes. The value t at which the change occurs defines the end of the current set and begins a new one.
5. Let t be in the new set. Repeat (3) and (4) until all sets of the shifts by t and the corresponding representations for $x(\tau)$ and $y(\tau - t)$ are identified, i.e., shift t until it reaches ∞.
6. For each set of shifts for t, integrate $x(\tau)$ and $y(\tau - t)$ from $\tau = -\infty$ to $\tau = \infty$ to obtain $r_{xy}(t)$ on each set.

(c) Evaluate the cross-correlation between the following signals:

(i) $x(t) = e^{-t}u(t), y(t) = e^{-3t}u(t)$

$$\begin{aligned} &\text{for } t < 0 \\ r_{xy}(t) &= \int_0^\infty e^{3t}e^{-4\tau}\,d\tau \\ &= \frac{1}{4}e^{3t} \\ &\text{for } t \geq 0 \\ r_{xy}(t) &= \int_t^\infty e^{3t}e^{-4\tau}\,d\tau \\ &= \frac{1}{4}e^{-t} \\ r_{xy}(t) &= \begin{cases} \frac{1}{4}e^{3t} & t < 0 \\ \frac{1}{4}e^{-t} & t \geq 0 \end{cases} \end{aligned}$$

(ii) $x(t) = \cos(\pi t)[u(t+2) - u(t-2)], y(t) = \cos(2\pi t)[u(t+2) - u(t-2)]$

for $t < -4$

$$r_{xy}(t) = 0$$

for $-4 \leq t < 0$

$$\begin{aligned}
r_{xy}(t) &= \int_{-2}^{t+2} \cos(\pi\tau)\cos(2\pi\tau - 2\pi t)d\tau \\
&= \int_{-2}^{t+2} \cos(\pi\tau)\left[\cos(2\pi\tau)\cos(2\pi t) + \sin(2\pi\tau)\sin(2\pi t)\right]d\tau \\
&= \frac{1}{2}\cos(2\pi t)\int_{-2}^{t+2}\left(\cos(\pi\tau) + \cos(3\pi\tau)\right)d\tau + \frac{1}{2}\sin(2\pi t)\int_{-2}^{t+2}\left(\sin(\pi\tau) + \sin(3\pi\tau)\right)d\tau \\
&= \frac{1}{2\pi}\cos(2\pi t)\left[\sin(\pi t) + \frac{1}{3}\sin(3\pi t)\right] - \frac{1}{2\pi}\sin(2\pi t)\left[\cos(\pi t) + \frac{1}{3}\cos(3\pi t) - \frac{4}{3}\right]
\end{aligned}$$

for $0 \le t < 4$

$$\begin{aligned}
r_{xy}(t) &= \int_{t-2}^{2} \cos(\pi\tau)\cos(2\pi\tau - 2\pi t)d\tau \\
&= \int_{t-2}^{2} \cos(\pi\tau)\left[\cos(2\pi\tau)\cos(2\pi t) + \sin(2\pi\tau)\sin(2\pi t)\right]d\tau \\
&= \frac{1}{2}\cos(2\pi t)\int_{t-2}^{2}\left(\cos(\pi\tau) + \cos(3\pi\tau)\right)d\tau + \frac{1}{2}\sin(2\pi t)\int_{t-2}^{t}\left(\sin(\pi\tau) + \sin(3\pi\tau)\right)d\tau \\
&= -\frac{1}{2\pi}\cos(2\pi t)\left[\sin(\pi t) + \frac{1}{3}\sin(3\pi t)\right] + \frac{1}{2\pi}\sin(2\pi t)\left[\cos(\pi t) + \frac{1}{3}\cos(3\pi t) - \frac{4}{3}\right]
\end{aligned}$$

for $t \ge 4$

$$\begin{aligned}
r_{xy}(t) &= 0 \\
r_{xy}(t) &= \begin{cases} 0 & t < -4 \\ \frac{1}{2\pi}\cos(2\pi t)\left[\sin(\pi t) + \frac{1}{3}\sin(3\pi t)\right] - \frac{1}{2\pi}\sin(2\pi t)\left[\cos(\pi t) + \frac{1}{3}\cos(3\pi t) - \frac{4}{3}\right] & -4 \le t < 0 \\ -\frac{1}{2\pi}\cos(2\pi t)\left[\sin(\pi t) + \frac{1}{3}\sin(3\pi t)\right] + \frac{1}{2\pi}\sin(2\pi t)\left[\cos(\pi t) + \frac{1}{3}\cos(3\pi t) - \frac{4}{3}\right] & 0 \le t < 4 \\ 0 & t \ge 4 \end{cases}
\end{aligned}$$

(iii) $x(t) = u(t) - 2u(t-1) + u(t-2), y(t) = u(t+1) - u(t)$

for $t < 0$

$$r_{xy}(t) = 0$$

for $0 \le t < 1$

$$r_{xy}(t) = t$$

for $1 \le t < 2$

$$r_{xy}(t) = 3 - 2t$$

for $2 \le t < 3$

$$r_{xy}(t) = t - 3$$

for $t \ge 3$

$$r_{xy}(t) = 0$$

$$r_{xy}(t) = \begin{cases} 0 & t < 0 \\ t & 0 \le t < 1 \\ 3 - 2t & 1 \le t < 2 \\ t - 3 & 2 \le t < 3 \\ 0 & t \ge 3 \end{cases}$$

(iv) $x(t) = u(t-a) - u(t-a-1), y(t) = u(t) - u(t-1)$

$$
\begin{aligned}
&\text{for } t < a-1 \\
r_{xy}(t) &= 0 \\
&\text{for } a-1 \le t < a \\
r_{xy}(t) &= t+1-a \\
&\text{for } a \le t < a+1 \\
r_{xy}(t) &= a+1-t \\
&\text{for } t \ge a+1 \\
r_{xy}(t) &= 0 \\
r_{xy}(t) &= \begin{cases} 0 & t < a-1 \\ t+1-a & a-1 \le t < a \\ a+1-t & a \le t < a+1 \\ 0 & t \ge a+1 \end{cases}
\end{aligned}
$$

(d) Evaluate the autocorrelation of the following signals:

(i) $x(t) = e^{-t}u(t)$

$$
\begin{aligned}
&\text{for } t < 0 \\
r_{xx}(t) &= \int_0^\infty e^t e^{-2\tau} d\tau \\
&= \frac{1}{2}e^t \\
&\text{for } t \ge 0 \\
r_{xx}(t) &= \int_t^\infty e^t e^{-2\tau} d\tau \\
&= \frac{1}{2}e^{-t} \\
r_{xx}(t) &= \begin{cases} \frac{1}{2}e^t & t < 0 \\ \frac{1}{2}e^{-t} & t \ge 0 \end{cases}
\end{aligned}
$$

(ii) $x(t) = \cos(\pi t)[u(t+2) - u(t-2)]$

$$
\begin{aligned}
&\text{for } t < -4 \\
r_{xx}(t) &= 0 \\
&\text{for } -4 \le t < 0 \\
r_{xx}(t) &= \int_{-2}^{t+2} \cos(\pi\tau)\cos(\pi\tau - \pi t) d\tau \\
&= \int_{-2}^{t+2} \cos(\pi\tau)\left(\cos(\pi\tau)\cos(\pi t) + \sin(\pi\tau)\sin(\pi t)\right) d\tau \\
&= \frac{1}{2}\cos(\pi t)\int_{-2}^{t+2} \left(1 + \cos(2\pi\tau)\right) d\tau + \frac{1}{2}\sin(\pi t)\int_{-2}^{t+2} \sin(2\pi\tau) d\tau \\
&= \frac{1}{2}\cos(\pi t)\left[t + 4 + \frac{1}{2\pi}\sin(2\pi t)\right] - \frac{1}{4\pi}\sin(\pi t)\left[\cos(2\pi t) + 2\right]
\end{aligned}
$$

for $0 \le t < 4$

$$\begin{aligned}
r_{xx}(t) &= \int_{t-2}^{2} \cos^2(\pi\tau)d\tau \\
&= \int_{t-2}^{2} \cos(\pi\tau)\left(\cos(\pi\tau)\cos(\pi t) + \sin(\pi\tau)\sin(\pi t)\right)d\tau \\
&= \frac{1}{2}\cos(\pi t)\int_{t-2}^{2}\left(1+\cos(2\pi\tau)\right)d\tau + \frac{1}{4\pi}\sin(\pi t)\int_{t-2}^{2}\sin(2\pi\tau)d\tau \\
&= \frac{1}{2}\cos(\pi t)\left[4-t-\frac{1}{2\pi}\sin(2\pi t)\right] + \frac{1}{4\pi}\sin(\pi t)\left[2+\cos(2\pi t)\right]
\end{aligned}$$

for $t \ge 4$

$$r_{xx}(t) = 0$$

$$r_{xx}(t) = \begin{cases} 0 & t < -4 \\ \frac{1}{2}\cos(\pi t)\left[t+4+\frac{1}{2\pi}\sin(2\pi t)\right] - \frac{1}{4\pi}\sin(\pi t)\left[\cos(2\pi t)+2\right] & -4 \le t < 0 \\ \frac{1}{2}\cos(\pi t)\left[4-t-\frac{1}{2\pi}\sin(2\pi t)\right] + \frac{1}{4\pi}\sin(\pi t)\left[2+\cos(2\pi t)\right] & 0 \le t < 4 \\ 0 & t \ge 4 \end{cases}$$

(iii) $x(t) = u(t) - 2u(t-1) + u(t-2)$

for $t < -2$

$$r_{xx}(t) = 0$$

for $-2 \le t < -1$

$$r_{xx}(t) = -t-2$$

for $-1 \le t < 0$

$$r_{xx}(t) = 3t+2$$

for $0 \le t < 1$

$$r_{xx}(t) = 2-3t$$

for $1 \le t < 2$

$$r_{xx}(t) = t-2$$

for $t \ge 2$

$$r_{xx}(t) = 0$$

$$r_{xx}(t) = \begin{cases} 0 & t < -2 \\ -t-2 & -2 \le t < -1 \\ 3t+2 & -1 \le t < 0 \\ 2-3t & 0 \le t < 1 \\ t-2 & 1 \le t < 2 \\ 0 & t \ge 3 \end{cases}$$

(iv) $x(t) = u(t-a) - u(t-a-1)$

for $t < -1$

$$r_{xx}(t) = 0$$

$$
\begin{aligned}
\text{for } -1 \le t < 0 & \\
r_{xx}(t) &= t+1 \\
\text{for } 0 \le t < 1 & \\
r_{xx}(t) &= 1-t \\
\text{for } t \ge 1 & \\
r_{xx}(t) &= 0 \\
r_{xx}(t) &= \begin{cases} 0 & t < -1 \\ t+1 & -1 \le t < 0 \\ 1-t & 0 \le t < 1 \\ 0 & t \ge 1 \end{cases}
\end{aligned}
$$

(e) Show that $r_{xy}(t) = r_{yx}(-t)$

$$
\begin{aligned}
r_{xy}(t) &= \int_{-\infty}^{\infty} x(\tau)y(\tau - t)d\tau \\
& \text{Let } u = \tau - t \\
r_{xy}(t) &= \int_{-\infty}^{\infty} y(u)x(u+t)du \\
r_{xy}(t) &= \int_{-\infty}^{\infty} y(\tau)x(\tau+t)d\tau \\
r_{xy}(t) &= r_{yx}(-t)
\end{aligned}
$$

(f) Show that $r_{xx}(t) = r_{xx}(-t)$

$$
\begin{aligned}
r_{xx}(t) &= \int_{-\infty}^{\infty} x(\tau)x(\tau - t)d\tau \\
& \text{Let } u = \tau - t \\
r_{xx}(t) &= \int_{-\infty}^{\infty} x(u)x(u+t)du \\
r_{xx}(t) &= \int_{-\infty}^{\infty} x(\tau)x(\tau+t)d\tau \\
r_{xx}(t) &= r_{xx}(-t)
\end{aligned}
$$

2.79. Prove that absolute summability of the impulse response is a necessary condition for stability of a discrete-time system. *Hint:* find a bounded input $x[n]$ such that the output at some time n_o satisfies $|y[n_o]| = \sum_{k=-\infty}^{\infty} |h[k]|$.

$$
\begin{aligned}
\text{let } n_o = 0 & \\
y[0] &= \sum_{k=-\infty}^{\infty} h[k]x[-k] \\
\text{let} \quad & x[-k] = \text{sign}\{h[k]\} \\
\text{then} \quad & h[k]x[-k] = |h[k]|
\end{aligned}
$$

$$\begin{aligned}
y[0] &= \sum_{k=-\infty}^{\infty} |h[k]| \\
|y[0]| &= \left| \sum_{k=-\infty}^{\infty} |h[k]| \right| \\
&= \sum_{k=-\infty}^{\infty} |h[k]|
\end{aligned}$$

Hence there exists an input for which $|y[n_o]| = \sum_{k=-\infty}^{\infty} |h[k]|$, and $\sum_{k=-\infty}^{\infty} |h[k]| < \infty$ is a necessary condition for stability.

2.80. Using the Fresnel approximation, light with a complex amplitude $f(x, y)$ in the xy-plane propagating over a distance d along the z-axis in free space generates a complex amplitude $g(x, y)$ given by

$$g(x, y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x', y') h(x - x', y - y') dx' dy'$$

where

$$h(x, y) = h_0 e^{-jk(x^2+y^2)/2d}$$

Here $k = 2\pi/\lambda$ is the wavenumber, λ the wavelength, and $h_0 = j/(\lambda d) e^{-jkd}$.
(a) Determine whether free-space propagation represents a linear system.

Suppose $f(x', y') = af_1(x', y') + bf_2(x', y')$. The system is linear if $g(x, y) = ag_1(x, y) + bg_2(x, y)$

$$\begin{aligned}
g(x, y) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x', y') h(x - x', y - y') dx' dy' \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left[af_1(x', y') + bf_2(x', y')\right] h(x - x', y - y') dx' dy' \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} af_1(x', y') h(x - x', y - y') dx' dy' + \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} bf_2(x', y') h(x - x', y - y') dx' dy' \\
&= ag_1(x, y) + bg_2(x, y)
\end{aligned}$$

(b) Is this system space invariant? That is, does a spatial shift of the input, $f(x - x_0, y - y_0)$ lead to the identical spatial shift in the output?

$$\begin{aligned}
g'(x, y) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x' - x_o, y' - y_o) h(x - x', y - y') dx' dy' \\
& \text{let } z = x' - x_o, \ w = y' - y_o \\
g'(x, y) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(z, w) h(x - x_o - z, y - y_o - w) dw dz \\
&= g(x - x_o, y - y_o)
\end{aligned}$$

This shows that the system is space invariant since a shift in the input produces a similar shift in the output.

(c) Evaluate the result of a point source located at (x_1, y_1) propagating a distance d. In this case $f(x,y) = \delta(x - x_1, y - y_1)$ where $\delta(x,y)$ is the two-dimensional version of the impulse. Find the "impulse response" of this system.

$$\begin{aligned}
g(x,y) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x',y')h(x-x',y-y')dx'dy' \\
g(x,y) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \delta(x-x_1,y-y_1)h(x-x',y-y')dx'dy' \\
&\quad \text{Using the sifting property} \\
g(x,y) &= \int_{-\infty}^{\infty} \delta(y-y_1)h(x-x_1,y-y')dy' \\
&= h(x-x_1,y-y_1)
\end{aligned}$$

(d) Evaluate the result of two point sources located at (x_1, y_1) and (x_2, y_2) propagating a distance d. To account for the two point sources, $f(x,y)$ is of the form $f(x,y) = f_1(x_1,y_1) + f_2(x_2,y_2) = \delta(x-x_1,y-y_1) + \delta(x-x_2,y-y_2)$

$$\begin{aligned}
g(x,y) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x',y')h(x-x',y-y')dx'dy' \\
g(x,y) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} [\delta(x-x_1,y-y_1)+\delta(x-x_2,y-y_2)]h(x-x',y-y')dx'dy' \\
&\quad \text{Using linearity and the sifting property yields} \\
g(x,y) &= h(x-x_1,y-y_1)+h(x-x_2,y-y_2)
\end{aligned}$$

2.81. The motion of a vibrating string depicted in Fig. P2.81 may be described by the partial differential equation

$$\frac{\partial^2}{\partial l^2}y(l,t) = \frac{1}{c^2}\frac{\partial^2}{\partial t^2}y(l,t)$$

where $y(l,t)$ is the displacement expressed as a function of position l and time t, and c is a constant determined by the material properties of the string. The intial conditions may be specified as follows:

$$\begin{aligned}
y(0,t) &= 0, \quad y(a,t)=0, \quad t>0 \\
y(l,0) &= x(l), \quad 0<l<a \\
\left.\frac{\partial}{\partial t}y(l,t)\right|_{t=0} &= g(l), 0<l<a
\end{aligned}$$

Here $x(l)$ is the displacement of the string at $t = 0$ while $g(l)$ describes the velocity at $t = 0$. One approach to solving this equation is to assume the solution is separable, that is, $y(l,t) = \phi(l)f(t)$, in which case the partial differential equation becomes

$$f(t)\frac{d^2}{dl^2}\phi(l) = \phi(l)\frac{1}{c^2}\frac{d^2}{dt^2}f(t)$$

This implies

$$\frac{\frac{d^2}{dl^2}\phi(l)}{\phi(l)} = \frac{\frac{d^2}{dt^2}f(t)}{c^2 f(t)}, \quad 0 < l < a, \ \ 0 < t$$

For this equality to hold, both sides of the equation must be constant. Let the constant be $-\omega^2$ and separate the partial differential equation into two ordinary second-order differential equations linked by the common parameter ω^2.

$$\begin{aligned} \frac{d^2}{dt^2}f(t) + \omega^2 c^2 f(t) &= 0, \quad 0 < t \\ \frac{d^2}{dl^2}\phi(l) + \omega^2 \phi(l) &= 0 \quad 0 < l < a \end{aligned}$$

(a) Find the form of the solution for $f(t)$ and $\phi(l)$.

The homogeneous solution for $f(t)$ is of the form:

$$\begin{aligned} r^2 + \omega^2 c^2 &= 0 \\ r &= \pm jc\omega \\ f(t) &= a_1 e^{j\omega ct} + a_2 e^{-j\omega ct} \end{aligned}$$

The homogeneous solution for $\phi(l)$ is of the form:

$$\begin{aligned} r^2 + \omega^2 &= 0 \\ r &= \pm j\omega \\ \phi(l) &= b_1 e^{j\omega l} + b_2 e^{-j\omega l} \\ \phi(l) &= d_1 \cos(\omega l) + d_2 \sin(\omega l) \end{aligned}$$

(b) The boundary conditions at the end points of the string are

$$\phi(0)f(t) = 0 \quad \phi(a)f(t) = 0$$

and, since $f(t) = 0$ gives a trivial solution for $y(l,t)$, we must have $\phi(0) = 0$ and $\phi(a) = 0$. Determine how these constraints restrict the permissable values for ω and the form of the solution for $\phi(l)$.

$$\begin{aligned} \phi(0) = 0 &= d_1 \cos(\omega 0) + d_2 \sin(\omega 0) \\ &\quad \text{implies} \\ d_1 &= 0 \\ \phi(a) = 0 &= d_2 \sin(\omega l) \\ &\quad \text{implies} \\ \omega &= \frac{k\pi}{a} \quad \text{where } k \text{ is any integer.} \end{aligned}$$

(c) Use the boundary conditions in (b) to show that constant $(-\omega^2)$ used to separate the partial differential equation into two ordinary second-order differential equations must be negative.

If $(-\omega^2)$ is positive:

$$
\begin{aligned}
r^2 - \omega^2 &= 0 \\
r &= \pm\omega \\
\phi(l) &= b_1 e^{\omega t} + b_2 e^{-\omega t} \\
\phi(0) = 0 &= b_1 + b_2 \\
\phi(a) = 0 &= b_1 e^{j\omega a} + b_2 e^{-\omega a} \\
& \quad \text{implies} \\
b_1 = b_2 &= 0
\end{aligned}
$$

$(-\omega^2)$ being positive is the trivial solution, hence $(-\omega^2)$ must be negative.

(d) Assume the initial position of the string is $y(l,0) = x(l) = \sin(\pi l/a)$ and that the initial velocity is $g(l) = 0$. Find $y(l,t)$.

$$
\begin{aligned}
y(l,t) &= \phi(l) f(t) \\
&= d_2 \sin\left(\frac{k\pi l}{a}\right)\left[e_1 \sin\left(\frac{k\pi ct}{a}\right) + e_2 \cos\left(\frac{k\pi ct}{a}\right)\right] \\
y(l,0) = \sin\left(\frac{\pi l}{a}\right) &= d_2 e_2 \sin\left(\frac{k\pi l}{a}\right) \\
\text{Implies} & \\
k &= 1 \\
d_2 e_2 &= 1 \\
\left.\frac{d}{dt} y(l,t)\right|_{t=0} = 0 &= d_2 \sin\left(\frac{k\pi l}{a}\right) e_1 \frac{\pi c}{a} \cos\left(\frac{k\pi ct}{a}\right) \\
\text{Implies} & \\
e_1 &= 0 \\
y(l,t) &= \sin\left(\frac{\pi l}{a}\right) \cos\left(\frac{\pi ct}{a}\right)
\end{aligned}
$$

2.82. Suppose the N-by-N matrix $\mathbf{A}$ in a state-variable description has N linearly independent eigenvectors $\mathbf{e}_i, i = 1, 2, \ldots, N$ and corresponding distinct eigenvalues λ_i. Thus $\mathbf{A}\mathbf{e}_i = \lambda_i \mathbf{e}_i, i = 1, 2, \ldots, N$.
(a) Show that we may decompose $\mathbf{A}$ as $\mathbf{A} = \mathbf{E}\,\mathbf{\Lambda}\mathbf{E}^{-1}$ where $\mathbf{\Lambda}$ is a diagonal matrix with i-th diagonal element λ_i.

$$
\begin{aligned}
\mathbf{A}\mathbf{e}_1 &= \lambda_1 \mathbf{e}_1 \\
\mathbf{A}\mathbf{e}_2 &= \lambda_2 \mathbf{e}_2
\end{aligned}
$$

This can be written as

$$
\begin{aligned}
\mathbf{A}[\mathbf{e}_1 \;\; \mathbf{e}_2] &= [\mathbf{e}_1 \;\; \mathbf{e}_2] \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \\
\text{Define} & \\
\mathbf{E} &= [\mathbf{e}_1 \;\; \mathbf{e}_2] \\
\mathbf{\Lambda} &= \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \\
\mathbf{AE} &= \mathbf{E\Lambda} \\
\text{Which can be rewritten as} & \\
\mathbf{AEE}^{-1} &= \mathbf{E\Lambda E}^{-1} \\
\mathbf{A} &= \mathbf{E\Lambda E}^{-1}
\end{aligned}
$$

(b) Find a transformation of the state that will diagonalize $\mathbf{A}$.
The transformation $\mathbf{A}' = \mathbf{TAT}^{-1}$ will diagonalize $\mathbf{A}$. Setting $\mathbf{T} = \mathbf{E}^{-1}$.

$$
\begin{aligned}
\mathbf{A}' &= \mathbf{E}^{-1}\mathbf{AE} \\
\mathbf{A}' &= \mathbf{E}^{-1}\left[\mathbf{E\Lambda E}^{-1}\right]\mathbf{E} \\
\mathbf{A}' &= \mathbf{\Lambda}
\end{aligned}
$$

(c) Assume $\mathbf{A} = \begin{bmatrix} 0 & -1 \\ 2 & -3 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$, $\mathbf{c} = \begin{bmatrix} 1 & 0 \end{bmatrix}$, $D = [0]$.
Find a transformation that converts this system to diagonal form.
The eigenvalues and eigenvectors are $\lambda_{1,2} = -1, -2$ and $\mathbf{e}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\mathbf{e}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ respectively. The transformation is

$$
\begin{aligned}
\mathbf{A}' = \mathbf{E}^{-1}\mathbf{AE} &= \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} \\
\mathbf{b}' = \mathbf{E}^{-1}b &= \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\
\mathbf{c}' = c\mathbf{E} &= \begin{bmatrix} 1 & 1 \end{bmatrix} \\
D' = D &= 0
\end{aligned}
$$

(d) Sketch the block-diagram representation for a discrete-time system corresponding to part (c).

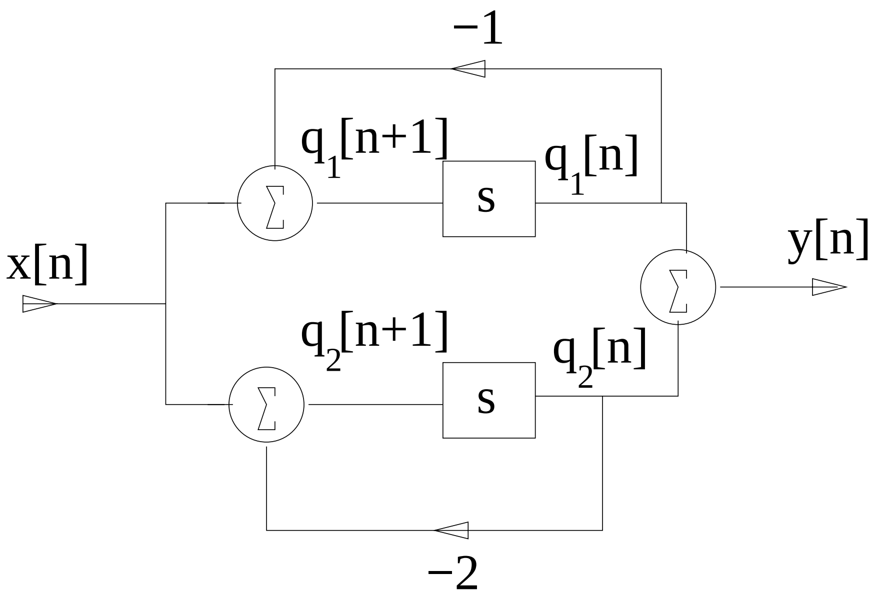


Figure P2.82. Block diagram of diagonal form

Solutions to Computer Experiments

2.83. Repeat Problem 2.34 using MATLABs `conv` command.

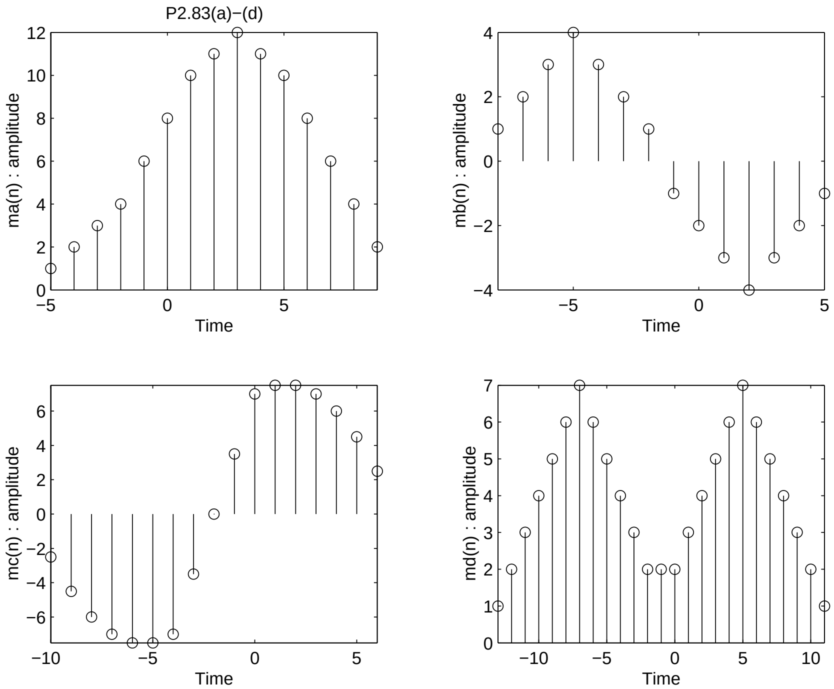


Figure P2.83. Convolution of (a)-(d) using MATLABs `conv` command

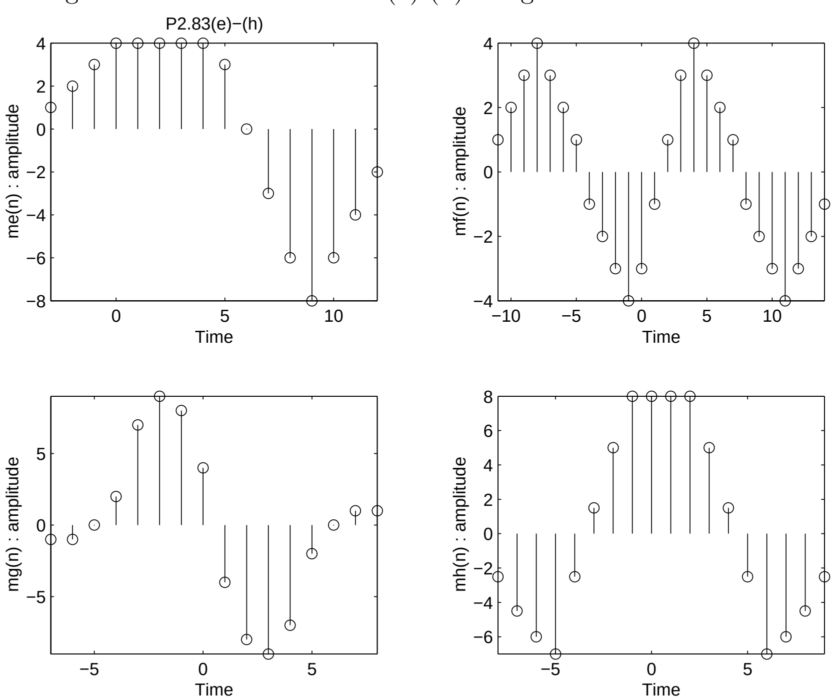


Figure P2.83. Convolution of (e)-(h) using MATLABs `conv` command

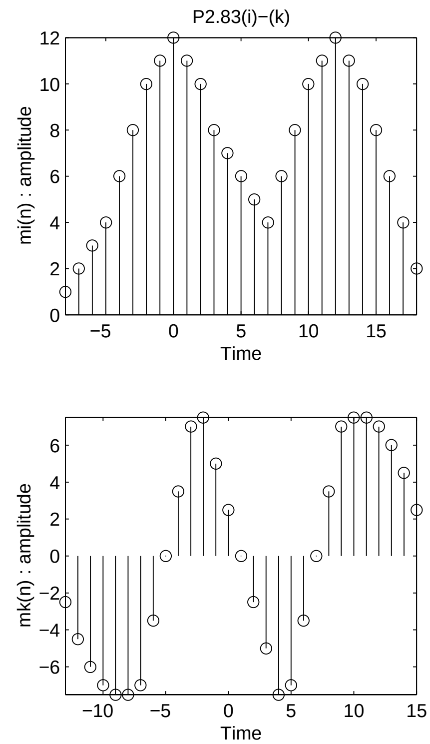


Figure P2.83. Convolution of (i)-(k) using MATLABs `conv` command

2.84. Use MATLAB to repeat Example 2.5.

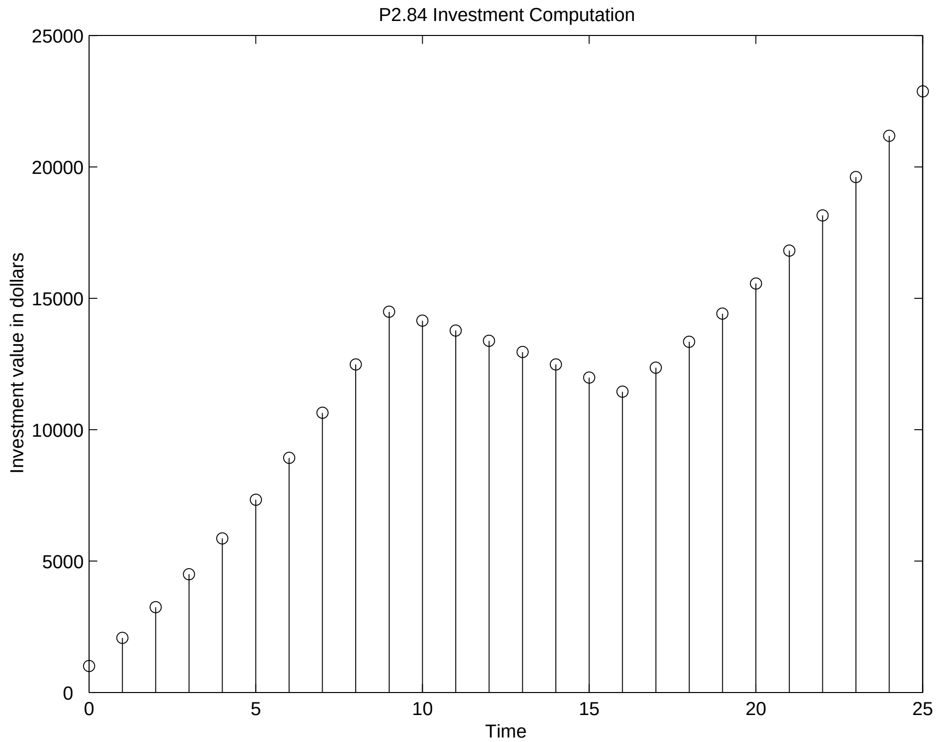


Figure P2.84. Investment computation

2.85. Use MATLAB to evaluate the first twenty values of the step response for Problem 2.50 (a) - (d).

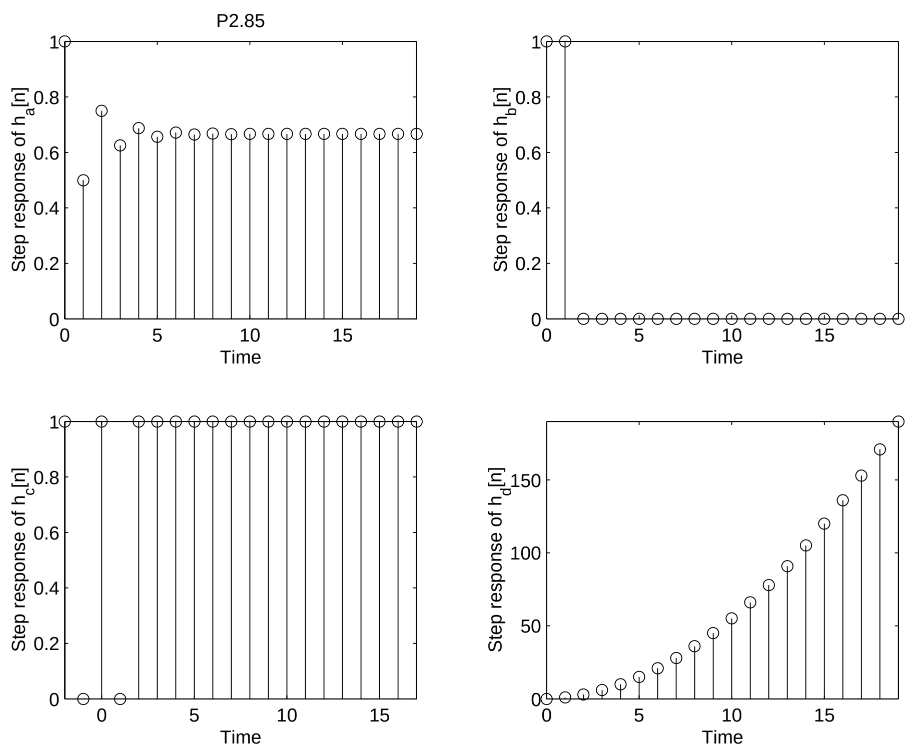


Figure P2.85. Step response for P2.50 (a)-(d)

2.86. Consider the two systems having impulse responses

$$h_1[n] = \begin{cases} \frac{1}{4}, & 0 \le n \le 3 \\ 0, & \text{otherwise} \end{cases}$$

$$h_2[n] = \begin{cases} \frac{1}{4}, & n = 0, 2 \\ -\frac{1}{4}, & n = 1, 3 \\ 0, & \text{otherwise} \end{cases}$$

Use the MATLAB command `conv` to plot the first 20 values of the step response.

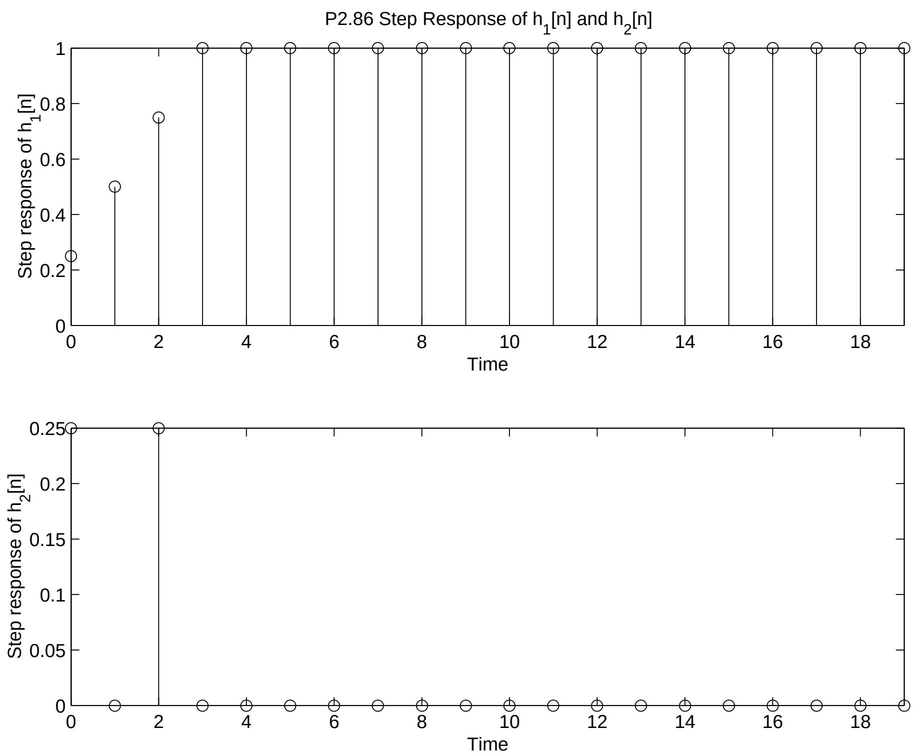


Figure P2.86. Step Response of the two systems

2.87. Use the MATLAB commands filter and filtic to repeat Example 2.16.

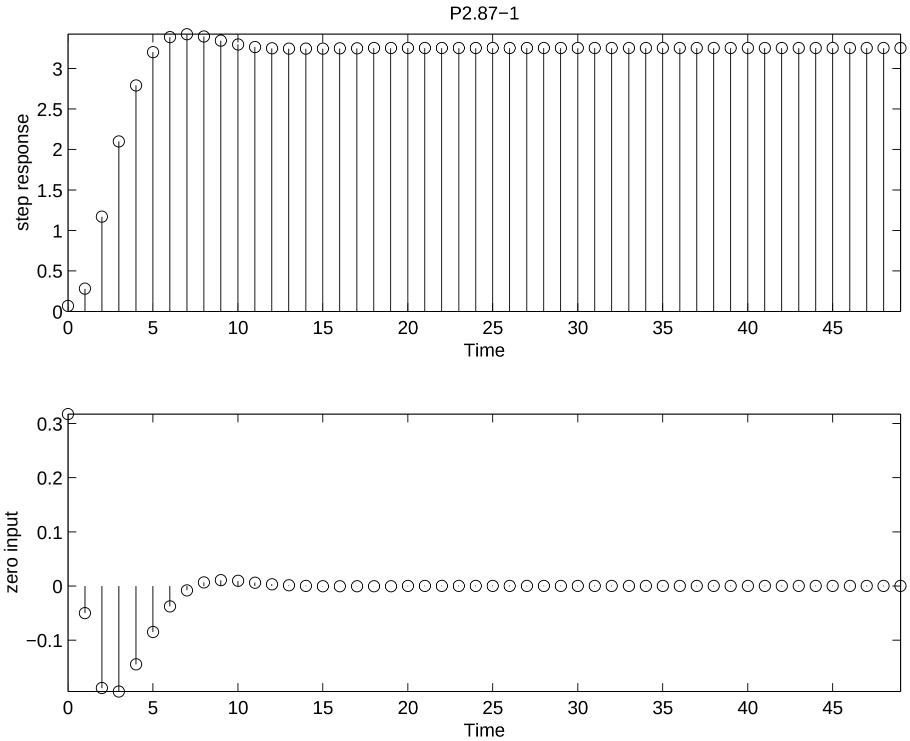


Figure P2.87. Step response and output due to no initial conditions.

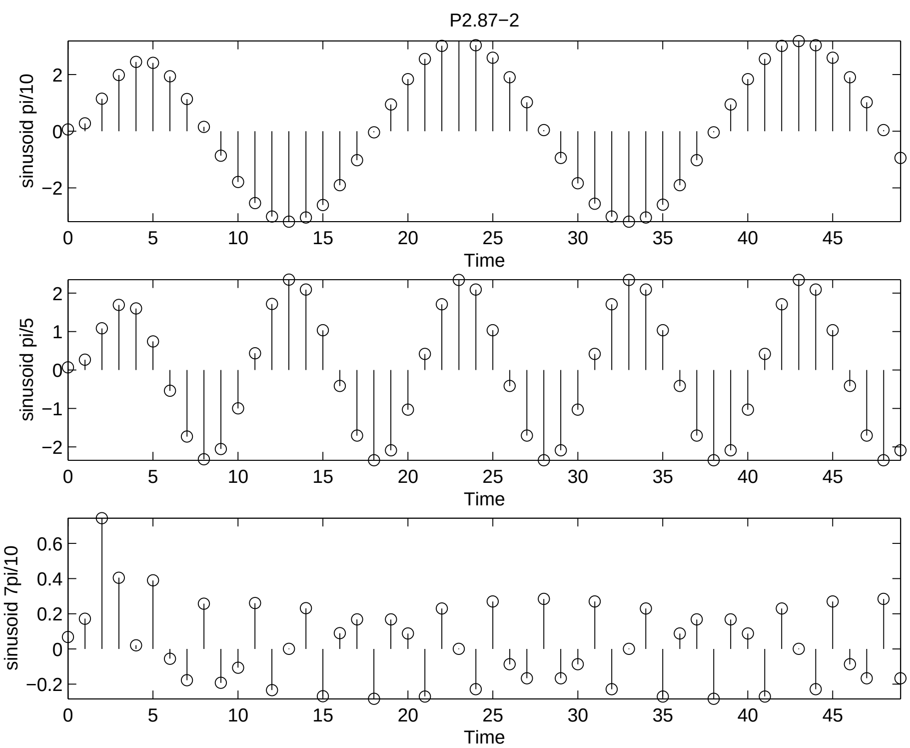


Figure P2.87. Output due to sinusoids of different frequencies.

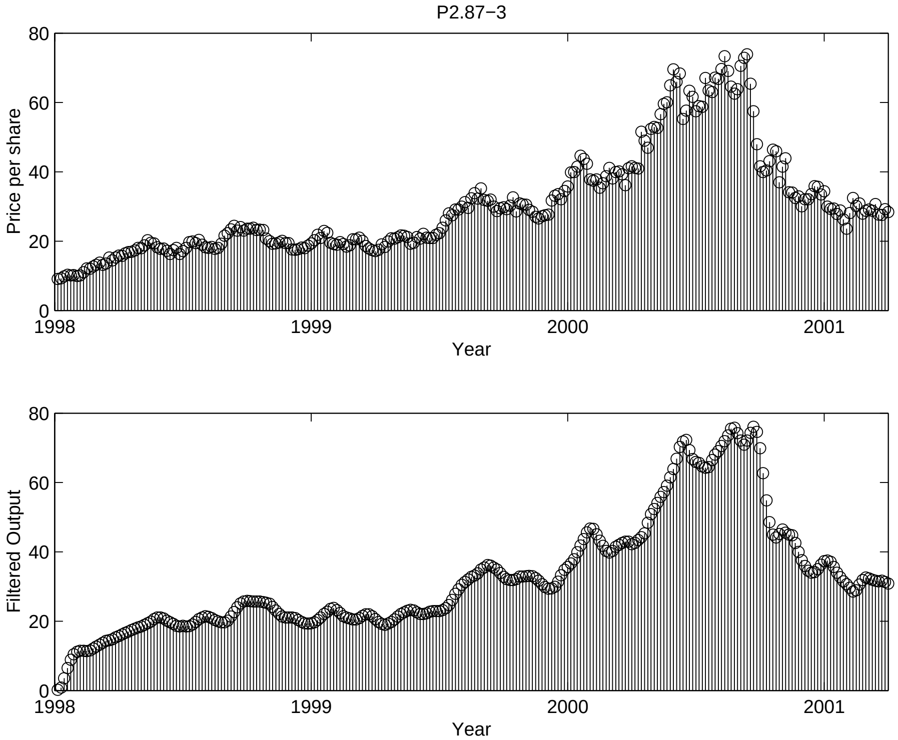


Figure P2.87. Input and filtered signals for Intel Stock Price.

2.88. Use the MATLAB commands filter and filtic to verify the loan balance in Example 2.23

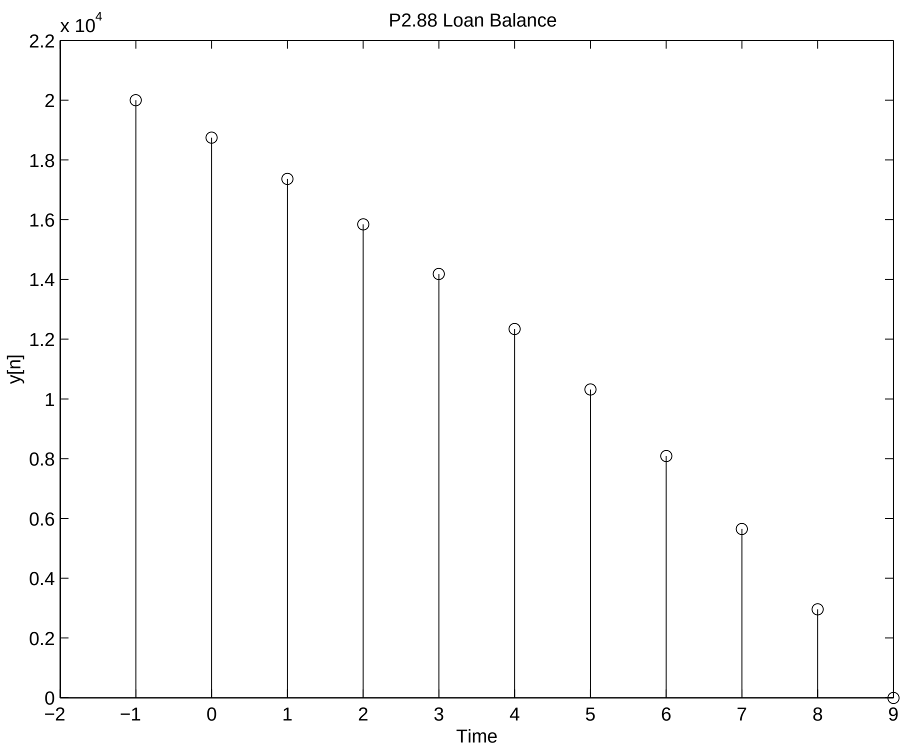


Figure P2.88. Loan Balance for Ex. 2.23

2.89. Use the MATLAB commands filter and filtic to determine the first fifty output values in Problem 2.59.

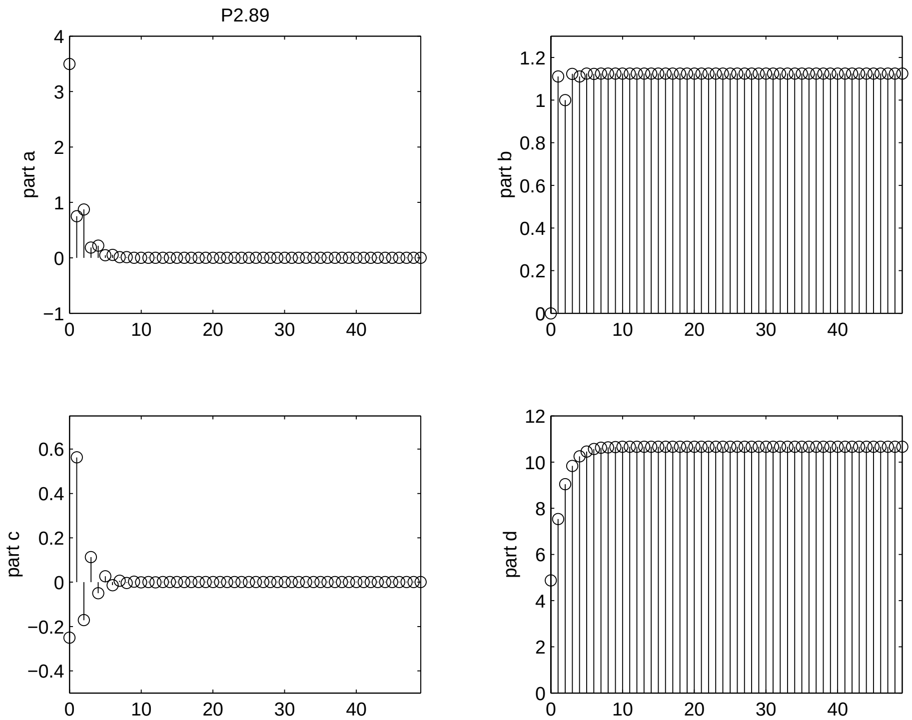


Figure P2.89. Output of first 50 values for Problem 2.59

2.90. Use the MATLAB command impz to determine the first 30 values of the impulse response for the systems described in Problem 2.59.

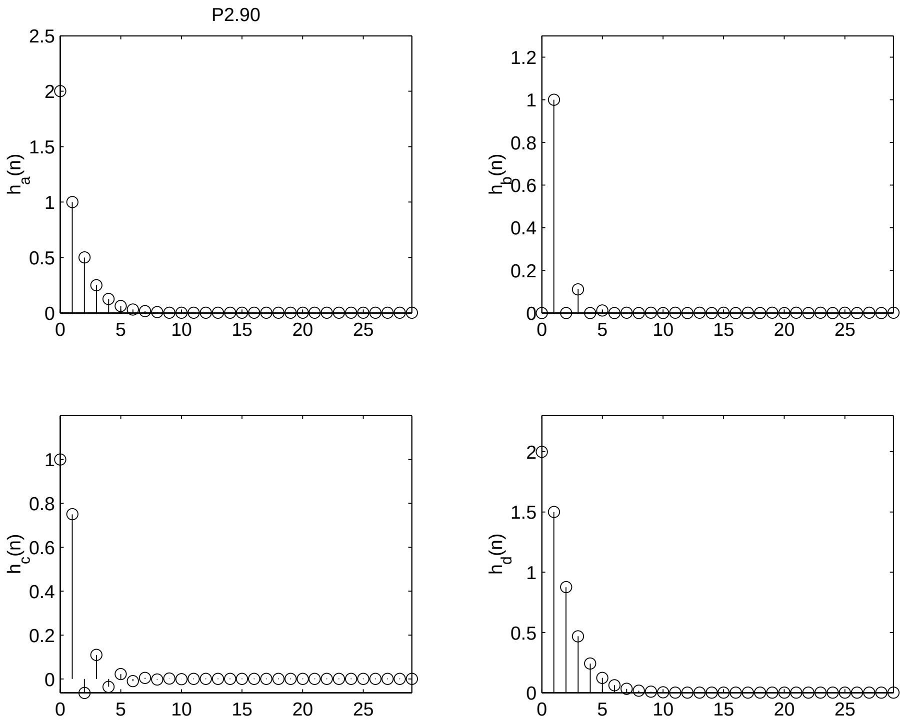


Figure P2.90. The first 30 values of the impulse response.

2.91. Use MATLAB to solve Problem 2.62.

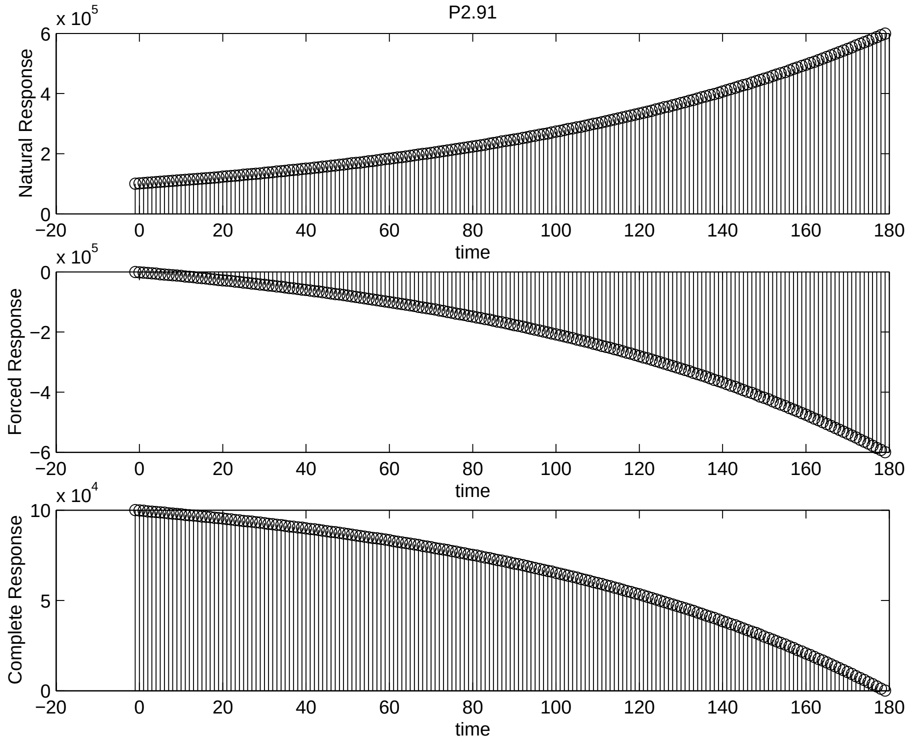


Figure P2.91. Using Matlab to solve Problem 2.62

2.92. Use MATLAB to solve Problem 2.63.

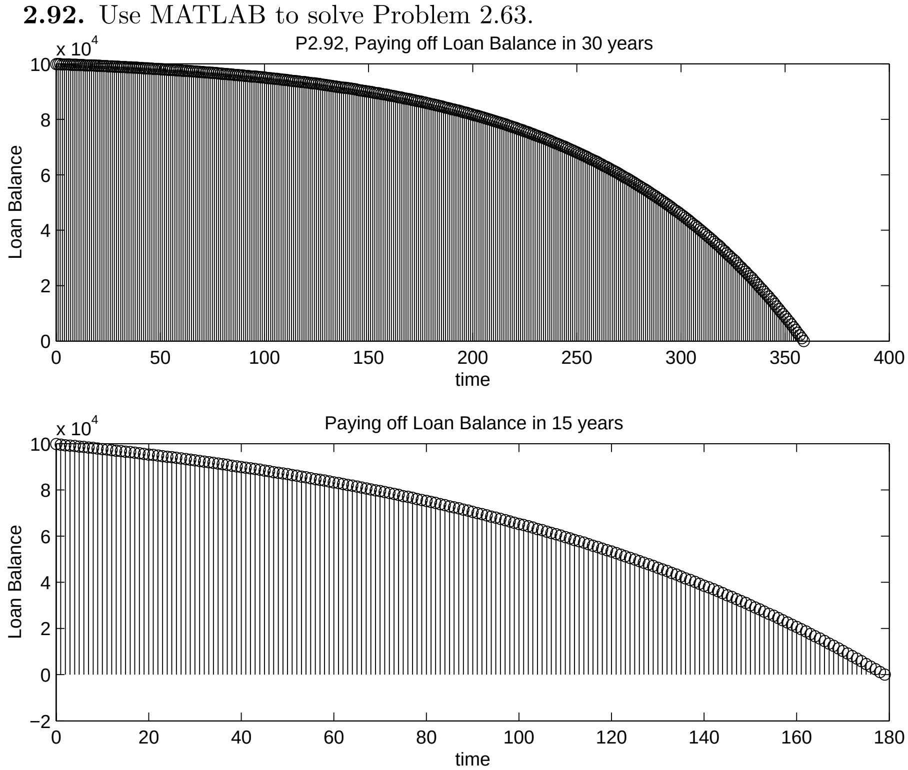


Figure P2.92. Plots of different loan payments, 30 vs. 15 years

2.93. Use the MATLAB command `ss2ss` to solve Problem 2.73.

P2.93 :
=======
Part (a) :
==========

```
a =
     x1 x2
x1 1 -0.3333
x2 0.5 0


b =
     u1
x1 2
x2 6


c =
     x1 x2
y1 0.5 -0.3333


d =
     u1
y1 0

Sampling time: unspecified
Discrete-time model.
```

Part (b) :
==========

```
a =
     x1 x2
x1 0 0.5
x2 -0.3333 1


b =
     u1
x1 6
x2 2
```

c =
x1 x2
y1 -0.3333 0.5

d =
u1
y1 0

Sampling time: unspecified
Discrete-time model.
Part (c) :
==========

a =
x1 x2
x1 0.4167 0.9167
x2 0.08333 0.5833

b =
u1
x1 3
x2 -1

c =
x1 x2
y1 0 1

d =
u1
y1 0

Sampling time: unspecified
Discrete-time model.

2.94. A system has the state-variable description

$$\mathbf{A} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{3} & 0 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad \mathbf{c} = \begin{bmatrix} 1 & -1 \end{bmatrix}, \quad D = [0]$$

(a) Use the MATLAB commands lsim and impulse to determine the first 30 values of the step and impulse responses of this system.

(b) Define new states $q_1[n] = q_1[n] + q_2[n]$ and $q_2[n] = 2q_1[n] - q_2[n]$. Repeat part (a) for the transformed system.

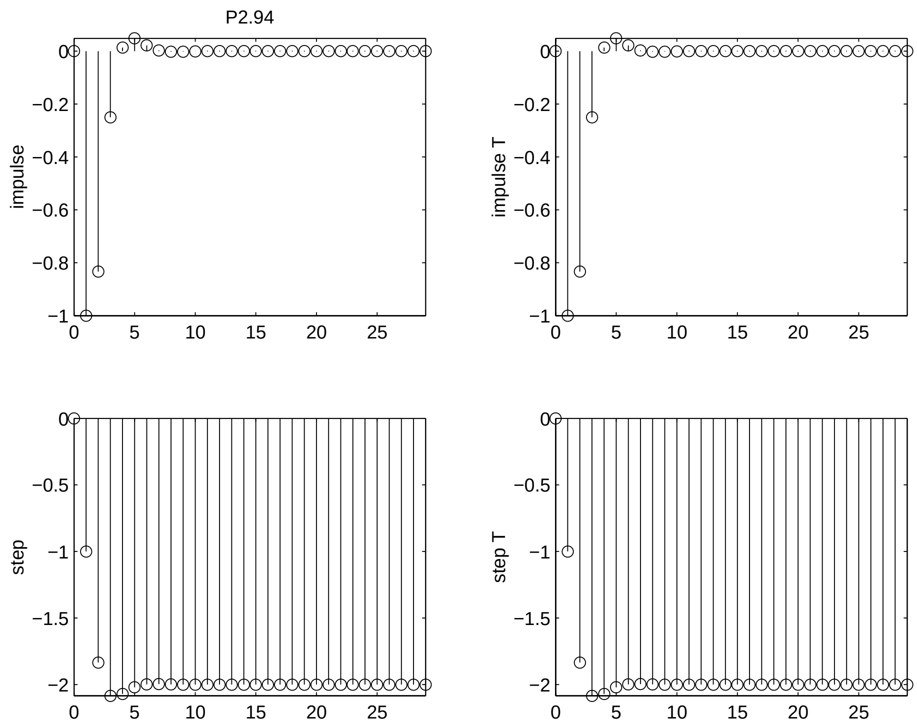


Figure P2.94. System step and impulse response