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State and prove De Moivre's theorem.

Ans: **Statement:** If n be a integer, positive or negative, the value of $(\cos\theta + i\sin\theta)^n$ is $\cos n\theta + i\sin n\theta$. If n be a fraction, positive or negative then one of the values of $(\cos\theta + i\sin\theta)^n$ is $\cos n\theta + i\sin n\theta$

Prove: Case-1: If n is a positive integer.

Now $(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)$

$$= \cos\theta_1\cos\theta_2 + \cos\theta_1 \cdot i\sin\theta_2 + i\sin\theta_1\cos\theta_2 + i^2\sin\theta_1\sin\theta_2$$

$$= (\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2)$$

$$= \cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)$$

Similarly $(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)(\cos\theta_3 + i\sin\theta_3)$

$$= \{\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)\}(\cos\theta_3 + i\sin\theta_3)$$

$$= \cos(\theta_1+\theta_2+\theta_3) + i\sin(\theta_1+\theta_2+\theta_3)$$

Proceeding in this way, we can write

$$(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)\cdots\cdots(\cos\theta_n + i\sin\theta_n) = \cos(\theta_1+\theta_2+\cdots+\theta_n) + i\sin(\theta_1+\theta_2+\cdots+\theta_n)$$

Putting $\theta_1 = \theta_2 = \theta_3 = \cdots\cdots = \theta_n = \theta$

then we can write, $\cos(\theta+\theta+\cdots+\theta) + i\sin(\theta+\theta+\cdots+\theta_n) = \cos n\theta + i\sin n\theta$

$\therefore (\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$.

Case-2: If n is a negative integer. Let $n = -m$

$$\text{Now } (\cos\theta + i\sin\theta)^n = (\cos\theta + i\sin\theta)^{-m} = \frac{1}{(\cos\theta + i\sin\theta)^m} = \frac{1}{\cos m\theta + i\sin m\theta}$$

$$= \frac{\cos m\theta - i\sin m\theta}{(\cos m\theta + i\sin m\theta)(\cos m\theta - i\sin m\theta)} = \frac{\cos m\theta - i\sin m\theta}{\cos^2 m\theta - i^2\sin^2 m\theta} = \frac{\cos m\theta - i\sin m\theta}{\cos^2 m\theta + \sin^2 m\theta}$$

$$= \cos m\theta - i\sin m\theta = \cos(-m)\theta + i\sin(-m)\theta = \cos n\theta + i\sin n\theta \quad [\because -m = n]$$

$\therefore (\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$.

Case-3: If n is a fraction, positive or negative.

Let $n = p/q$ where q is a positive integer and p is any integer positive or negative.

$$\text{Now } (\cos\theta + i\sin\theta)^n = (\cos\theta + i\sin\theta)^{p/q} = \cos\tfrac{p}{q}\theta + i\sin\tfrac{p}{q}\theta$$

$$= \cos n\theta + i\sin n\theta \quad [\because p/q = n]$$

$\therefore (\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$.

Hence for any value of n, we can write,

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta \quad \text{(Proved)}$$

Rules: Particular Cases:

i) $1 = \cos 0 + i\sin 0$; ii) $-1 = \cos\pi + i\sin\pi$

iii) $i = \cos\pi/2 + i\sin\pi/2$; iv) $-i = \cos 3\pi/2 + i\sin 3\pi/2 = \cos\pi/2 - i\sin\pi/2$

If $\sin\alpha + \sin\beta + \sin\gamma = \cos\alpha + \cos\beta + \cos\gamma = 0$ then prove that $\cos 3\alpha + \cos 3\beta + \cos 3\gamma = 3\cos(\alpha+\beta+\gamma)$ and $\sin 3\alpha + \sin 3\beta + \sin 3\gamma = 3\sin(\alpha+\beta+\gamma)$.

Solution: We know that, if $a+b+c=0$ then $a^3+b^3+c^3 = 3abc$

Let, $a = \cos\alpha + i\sin\alpha$
$b = \cos\beta + i\sin\beta$
$c = \cos\gamma + i\sin\gamma$

$\therefore a+b+c = 0 \Rightarrow (\cos\alpha + \cos\beta + \cos\gamma) + i(\sin\alpha + \sin\beta + \sin\gamma) = 0$

and $a^3+b^3+c^3 = 3abc \Rightarrow (\cos\alpha + i\sin\alpha)^3 + (\cos\beta + i\sin\beta)^3 + (\cos\gamma + i\sin\gamma)^3 = 3(\cos\alpha + i\sin\alpha)(\cos\beta + i\sin\beta)(\cos\gamma + i\sin\gamma)$

$\Rightarrow (\cos 3\alpha + i\sin 3\alpha) + (\cos 3\beta + i\sin 3\beta) + (\cos 3\gamma + i\sin 3\gamma) = 3\{\cos(\alpha+\beta+\gamma) + i\sin(\alpha+\beta+\gamma)\}$

$\Rightarrow (\cos 3\alpha + \cos 3\beta + \cos 3\gamma) + i(\sin 3\alpha + \sin 3\beta + \sin 3\gamma) = 3\cos(\alpha+\beta+\gamma) + i3\sin(\alpha+\beta+\gamma)$

Now equating real and imaginary part:-

$\cos 3\alpha + \cos 3\beta + \cos 3\gamma = 3\cos(\alpha+\beta+\gamma)$ and $\sin 3\alpha + \sin 3\beta + \sin 3\gamma = 3\sin(\alpha+\beta+\gamma)$ (proved)

If $x + \frac{1}{x} = 2\cos\theta$ then show that $x^n + \frac{1}{x^n} = 2\cos n\theta$

Solution: Given that, $x + \frac{1}{x} = 2\cos\theta \Rightarrow x^2 + 1 = 2x\cos\theta \Rightarrow x^2 - 2x\cos\theta + 1 = 0$

$$\therefore x = \frac{-b \pm \sqrt{b^2-4ac}}{2a} = \frac{-(-2\cos\theta) \pm \sqrt{(-2\cos\theta)^2 - 4\cdot 1\cdot 1}}{2\cdot 1}$$

$$= \frac{2\cos\theta \pm \sqrt{4\cos^2\theta - 4}}{2} = \frac{2\cos\theta \pm 2\sqrt{\cos^2\theta - 1}}{2} = \cos\theta \pm \sqrt{-\sin^2\theta}$$

$$= \cos\theta \pm \sqrt{-1}\cdot\sqrt{\sin^2\theta} = \cos\theta \pm i\sin\theta$$

Taking (+) sign, we get $x = \cos\theta + i\sin\theta$

$\therefore x^n = (\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$.

and $x^{-n} = (\cos\theta + i\sin\theta)^{-n} = \cos(-n)\theta + i\sin(-n)\theta$
$= \cos(-n)\theta - i\sin n\theta = \cos n\theta - i\sin n\theta$

$\therefore x^n + \frac{1}{x^n} = \cos n\theta + \cancel{i\sin n\theta} + \cos n\theta - \cancel{i\sin n\theta} = 2\cos n\theta$

$\therefore x^n + \frac{1}{x^n} = 2\cos n\theta$. (showed)

Ex If $x_r = \cos\frac{\pi}{2^r} + i\sin\frac{\pi}{2^r}$ then prove that $x_1 x_2 x_3 \cdots\cdots \alpha = -1$.

Solution: Given that $x_r = \cos\frac{\pi}{2^r} + i\sin\frac{\pi}{2^r}$ ——— (i)

Putting $r = 1, 2, 3 \cdots\cdots \alpha$ then in equation (i), we get

$x_1 = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2}$

$x_2 = \cos\frac{\pi}{2^2} + i\sin\frac{\pi}{2^2}$

$x_3 = \cos\frac{\pi}{2^3} + i\sin\frac{\pi}{2^3}$

$\cdots\cdots\cdots\cdots$

$x_\alpha = \cos\frac{\pi}{2^\alpha} + i\sin\frac{\pi}{2^\alpha}$

Now $x_1 x_2 x_3 \cdots \alpha = \left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)\left(\cos\frac{\pi}{2^2} + i\sin\frac{\pi}{2^2}\right)\left(\cos\frac{\pi}{2^3} + i\sin\frac{\pi}{2^3}\right) \cdots\cdots \left(\cos\frac{\pi}{2^\alpha} + i\sin\frac{\pi}{2^\alpha}\right)$

$= \cos\left(\frac{\pi}{2} + \frac{\pi}{2^2} + \frac{\pi}{2^3} + \cdots + \frac{\pi}{2^\alpha}\right) + i\sin\left(\frac{\pi}{2} + \frac{\pi}{2^2} + \frac{\pi}{2^3} + \cdots + \frac{\pi}{2^\alpha}\right)$

$= \cos\pi/2\left(1 + \frac{1}{2} + \frac{1}{2^2} + \cdots + \alpha\right) + i\sin\pi/2\left(1 + \frac{1}{2} + \frac{1}{2^2} + \cdots + \alpha\right)$

$= \cos\pi/2\left(\frac{1}{1-1/2}\right) + i\sin\pi/2\left(\frac{1}{1-1/2}\right)$

$= \cos\pi/2\,(2/1) + i\sin\pi/2\,(2/1)$

$= \cos\pi + i\sin\pi$

$= -1 + i\cdot 0$

$= -1$

$\therefore x_1 x_2 x_3 \cdots\cdots \alpha = -1$. (proved)

Ex If $(1+x)^n = P_0 + P_1 x + P_2 x^2 \cdots\cdots$ then prove that $P_0 - P_2 + P_4 - \cdots\cdots = 2^{n/2}\cos\frac{n\pi}{4}$ and $P_1 - P_3 + P_5 - \cdots\cdots = 2^{n/2}\sin\frac{n\pi}{4}$.

Solution: Given that $(1+x)^n = P_0 + P_1 x + P_2 x^2 + \cdots\cdots$

Putting $x = i$, then we get, $(1+i)^n = P_0 + iP_1 + i^2P_2 + i^3P_3 + i^4P_4 + i^5P_5 + \cdots$

$\Rightarrow (1+i)^n = P_0 + iP_1 - P_2 - iP_3 + P_4 + iP_5 - \cdots$ ——— (i)

Let $1 + i = r(\cos\theta + i\sin\theta)$

Now equating real and imaginary part, we get

$r\cos\theta = 1$ —— (ii) and $r\sin\theta = 1$ ——— (iii)

(ii)^2 + (iii)^2 $\Rightarrow r^2(\cos^2\theta + \sin^2\theta) = 1^2 + 1^2 \Rightarrow r^2 = 2 \therefore r = \sqrt{2}$

and (iii) ÷ (ii) $\Rightarrow \tan\theta = 1 \Rightarrow \theta = \tan^{-1}1 = \pi/4$

P.T.O

$$r^n = (1+i)^n = r^n(\cos\theta + i\sin\theta)^n = r^n(\cos n\theta + i\sin n\theta)$$

$$= (\sqrt{2})^n \cdot \left(\cos\frac{n\pi}{4} + i\sin\frac{n\pi}{4}\right) = 2^{n/2}\left(\cos\frac{n\pi}{4} + i\sin\frac{n\pi}{4}\right)$$

From the equation (i), we can write

$$\therefore 2^{n/2}\left(\cos\frac{n\pi}{4} + i\sin\frac{n\pi}{4}\right) = P_0 + iP_1 - P_2 - iP_3 + P_4 + iP_5 - \cdots\cdots$$

$$\Rightarrow 2^{n/2}\left(\cos\frac{n\pi}{4} + i\sin\frac{n\pi}{4}\right) = \cancel{P_0 - P_2 + P_4) + i(P_1 - P_3 + P_5}$$

$$= (P_0 - P_2 + P_4 - \cdots\cdots) + i(P_1 - P_3 + P_5 - \cdots\cdots)$$

equating real and imaginary part, we get

$$\left.\begin{aligned} P_0 - P_2 + P_4 - \cdots\cdots &= 2^{n/2}\cos\frac{n\pi}{4} \text{ and} \\ P_1 - P_3 - P_5 - \cdots\cdots &= 2^{n/2}\sin\frac{n\pi}{4}. \end{aligned}\right\}\text{(Proved)}$$

(iii) If $(a_1+ib_1)(a_2+ib_2)\cdots\cdots(a_n+ib_n) = A+iB$ then proved that

i) $\tan^{-1}\frac{b_1}{a_1} + \tan^{-1}\frac{b_2}{a_2} + \cdots\cdots + \tan^{-1}\frac{b_n}{a_n} = \tan^{-1}B/A$ and

ii) $(a_1^2+b_1^2)(a_2^2+b_2^2)\cdots\cdots(a_n^2+b_n^2) = A^2+B^2$

Solution. Let $a_1+ib_1 = r_1(\cos\theta_1 + i\sin\theta_1)$

equating real and imaginary part, we get

$$a_1 = r_1\cos\theta_1 \text{ — (i) and } b_1 = r_1\sin\theta_1 \text{ — (ii)}$$

(i)^2 + (ii)^2 $\Rightarrow r_1^2(\cos^2\theta_1 + \sin^2\theta_1) = a_1^2 + b_1^2$ and (ii) ÷ (i) $\Rightarrow$

$$\Rightarrow r_1^2 = a_1^2 + b_1^2 \qquad \tan\theta_1 = \frac{b_1}{a_1}$$

$$\Rightarrow r_1 = \sqrt{a_1^2+b_1^2} \qquad \Rightarrow \theta_1 = \tan^{-1}\frac{b_1}{a_1}$$

Similarly $r_2 = \sqrt{a_2^2+b_2^2}$ and $\theta_2 = \tan^{-1}\frac{b_2}{a_2}$

$$\cdots\cdots\cdots\cdots$$

$$r_n = \sqrt{a_n^2+b_n^2} \text{ and } \theta_n = \tan^{-1}\frac{b_n}{a_n}$$

Now $a_1+ib_1 = r_1(\cos\theta_1 + i\sin\theta_1)$

$$a_2+ib_2 = r_2(\cos\theta_2 + i\sin\theta_2)$$

$$\cdots\cdots\cdots\cdots$$

$$a_n+ib_n = r_n(\cos\theta_n + i\sin\theta_n)$$

P.T.O

Now multiplication, we get

$(a_1+ib_1)(a_2+ib_2)\cdots\cdots(a_n+ib_n)=(r_1r_2\cdots\cdots r_n)\{\cos(\theta_1+\theta_2+\cdots\cdots+\theta_n)+i\sin(\theta_1+\theta_2+\cdots\cdots+\theta_n)\}$

$\Rightarrow A+iB=(r_1r_2\cdots r_n)\{\cos(\theta_1+\theta_2+\cdots+\theta_n)+i\sin(\theta_1+\theta_2+\cdots\cdots+\theta_n)\}$

Now equating real and imaginary part, we get

$A=(r_1r_2\cdots r_n)\cos(\theta_1+\theta_2+\cdots+\theta_n)$ —— (iii) and

$B=(r_1r_2\cdots r_n)\cdot\sin(\theta_1+\theta_2+\cdots+\theta_n)$ —— (iv)

(iv) $\div$ (iii) $\Rightarrow \tan(\theta_1+\theta_2+\cdots+\theta_n)=B/A$

$\Rightarrow \theta_1+\theta_2+\cdots\cdots+\theta_n=\tan^{-1}B/A$

$\Rightarrow \tan^{-1}\frac{b_1}{a_1}+\tan^{-1}\frac{b_2}{a_2}+\cdots\cdots+\tan^{-1}\frac{b_3}{a_3}=\tan^{-1}B/A$ (Proved)

Again (iii)^2 + (iv)^2 $\Rightarrow$

$A^2+B^2=(r_1^2r_2^2\cdots r_n^2)\{\cos^2(\theta_1+\theta_2+\cdots+\theta_n)+\sin^2(\theta_1+\theta_2+\cdots+\theta_n)\}$

$\Rightarrow (r_1^2r_2^2\cdots r_n^2)1=A^2+B^2$

$\Rightarrow r_1^2r_2^2\cdots\cdots r_n^2=A^2+B^2$

$\Rightarrow (a_1^2+b_1^2)(a_2^2+b_2^2)\cdots\cdots(a_n^2+b_n^2)=A^2+B^2$. (Proved)

If n be a positive integer then prove that $(1+i)^n+(1-i)^n=2^{n/2+1}\cdot\cos\frac{n\pi}{4}$

Solution: L.H.S $=(1+i)^n+(1-i)^n=\left(\frac{\sqrt2}{\sqrt2}+\frac{i\sqrt2}{\sqrt2}\right)^n+\left(\frac{\sqrt2}{\sqrt2}-\frac{i\sqrt2}{\sqrt2}\right)^n$

$=\left\{\sqrt2\left(\frac{1}{\sqrt2}+i\frac{1}{\sqrt2}\right)\right\}^n+\left\{\sqrt2\left(\frac{1}{\sqrt2}-i\frac{1}{\sqrt2}\right)\right\}^n$

$=\left\{\sqrt2\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)\right\}^n+\left\{\sqrt2\left(\cos\frac{\pi}{4}-i\sin\frac{\pi}{4}\right)\right\}^n$

$=(\sqrt2)^n\left(\cos\frac{n\pi}{4}+i\sin\frac{n\pi}{4}\right)+(\sqrt2)^n\left(\cos\frac{n\pi}{4}-i\sin\frac{n\pi}{4}\right)$

$=(\sqrt2)^n\left(\cos\frac{n\pi}{4}+\cancel{i\sin\frac{n\pi}{4}}+\cos\frac{n\pi}{4}-\cancel{i\sin\frac{n\pi}{4}}\right)$

$=2^{n/2}\cdot 2\cos\frac{n\pi}{4}$

$=2^{n/2+1}\cdot\cos\frac{n\pi}{4}$

$=$ R.H.S

$\therefore (1+i)^n+(1-i)^n=2^{n/2+1}\cdot\cos\frac{n\pi}{4}$. (Proved)

Ex. ~~Given that~~ If $(1+x)^n = a_0 + a_1 x + a_2 x^2 + \dots\dots$ (n being positive integer) then show/prove that $a_0 + a_4 + a_8 + \dots = 2^{n-2} + 2^{\frac{n}{2}-1}\cos\frac{n\pi}{4}$

Solution: Given that $(1+x)^n = a_0 + a_1 x + a_2 x^2 + \dots\dots$

Putting $x = 1$ and -1, then we get

$(1+1)^n = a_0 + a_1 \cdot 1 + a_2 \cdot 1^2 + a_3 \cdot 1^3 + a_4 \cdot 1^4 + a_5 1^5 + a_6 \cdot 1^6 + a_7 \cdot 1^7 + a_8 \cdot 1^8 + \dots\dots$

$\Rightarrow 2^n = a_0 + a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + a_7 + a_8 + \dots\dots$ — (i)

Again

$(1-1)^n = a_0 + a_1(-1) + a_2(-1)^2 + a_3(-1)^3 + a_4(-1)^4 + a_5(-1)^5 + a_6(-1)^6 + a_7(-1)^7 + a_8(-1)^8 + \dots\dots$ — (ii)

$\Rightarrow 0 = a_0 - a_1 + a_2 - a_3 + a_4 - a_5 + a_6 - a_7 + a_8 \dots\dots$ — (iii)

(i) + (ii) $\Rightarrow 2^n + 0 = 2(a_0 + a_2 + a_4 + a_6 + a_8 + \dots\dots)$ — (iii)

Again putting $x = i$ and $-i$, we get

$(1+i)^n = a_0 + a_1 \cdot i + a_2 i^2 + a_3 \cdot i^3 + a_4 \cdot i^4 + a_5 i^5 + a_6 i^6 + a_7 i^7 + a_8 i^8 + \dots$ — (iv)

$= a_0 + ia_1 - a_2 - ia_3 + a_4 + ia_5 - a_6 - ia_7 + a_8 + \dots$

and $(1-i)^n = a_0 + a_1(-i) + a_2(-i)^2 + a_3(-i)^3 + a_4(-i)^4 + a_5(-i)^5 + a_6(-i)^6 + a_7(-i)^7 + a_8(-i)^8 + \dots$ — (v)

$= a_0 - ia_1 - a_2 + ia_3 + a_4 - ia_5 - a_6 + ia_7 + a_8 + \dots\dots$ — (vi)

(iv) + (v) $\Rightarrow (1+i)^n + (1-i)^n = 2(a_0 - a_2 + a_4 - a_6 + a_8 + \dots\dots)$

Now (iii) + (vi) $\Rightarrow$

$2^n + (1+i)^n + (1-i)^n = 2^2(a_0 + a_4 + a_8 + \dots\dots)$

$\Rightarrow a_0 + a_4 + a_8 + \dots\dots = \dfrac{2^n + (1+i)^n + (1-i)^n}{2^2}$ — (vii)

Let $1+i = r(\cos\theta + i\sin\theta)$

Now equating real and imaginary part, we get

$1 = r\cos\theta$ ~~and~~ — (viii) and $1 = r\sin\theta$ — (ix)

Now (viii)^2 + (ix)^2 $\Rightarrow r^2 = 2$ and (ix) ÷ (viii) $\Rightarrow$

$\Rightarrow r = \sqrt{2}$

$\tan\theta = 1$

$\Rightarrow \theta = \tan^{-1} 1 = \pi/4$

P.T.O

$$\therefore (1+i)^n = r^n(\cos\theta + i\sin\theta)^n = (\sqrt{2})^n\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)^n$$
$$= (\sqrt{2})^n\left(\cos\frac{n\pi}{4} + i\sin\frac{n\pi}{4}\right) = 2^{n/2}\left(\cos\frac{n\pi}{4} + i\sin\frac{n\pi}{4}\right)$$

and
$$(1-i)^n = r^n(\cos\theta - i\sin\theta)^n = (\sqrt{2})^n\left(\cos\frac{\pi}{4} - i\sin\frac{\pi}{4}\right)^n$$
$$= (\sqrt{2})^n\left(\cos\frac{n\pi}{4} - i\sin\frac{n\pi}{4}\right) = 2^{n/2}\left(\cos\frac{n\pi}{4} - i\sin\frac{n\pi}{4}\right)$$

From equation (vii) we get

$$a_0 + a_4 + a_8 + \cdots\cdots = \frac{2^n + 2^{n/2}\left(\cos\frac{n\pi}{4} + i\sin\frac{n\pi}{4}\right) + 2^{n/2}\left(\cos\frac{n\pi}{4} - i\sin\frac{n\pi}{4}\right)}{2^2}$$
$$= \frac{2^n + 2^{n/2}\left(\cos\frac{n\pi}{4} + \cancel{i\sin\frac{n\pi}{4}} + \cos\frac{n\pi}{4} - \cancel{i\sin\frac{n\pi}{4}}\right)}{2^2}$$
$$= \frac{2^n + 2^{n/2}\cdot 2\cos\frac{n\pi}{4}}{2^2}$$
$$= \frac{2^n + 2^{n/2+1}\cdot\cos\frac{n\pi}{4}}{2^2}$$
$$= 2^n\cdot 2^{-2} + 2^{n/2+1}\cdot 2^{-2}\cos\frac{n\pi}{4}$$
$$= 2^{n-2} + 2^{n/2+1-2}\cdot\cos\frac{n\pi}{4}$$
$$= 2^{n-2} + 2^{n/2-1}\cdot\cos\frac{n\pi}{4}$$

$$\therefore a_0 + a_4 + a_8 = 2^{n-2} + 2^{n/2-1}\cdot\cos\frac{n\pi}{4} \quad (\text{Proved})$$

⑤ If $a = \cos\theta_1 + i\sin\theta_1$, $b = \cos\theta_2 + i\sin\theta_2$, $c = \cos\theta_3 + i\sin\theta_3$ ~~and~~ ~~abc~~ and $a+b+c = abc$ then prove that $\cos(\theta_1-\theta_2) + \cos(\theta_2-\theta_3) + \cos(\theta_3-\theta_1) + 1 = 0$

__Solution:__ Given that $a+b+c = abc$

$$\Rightarrow (\cos\theta_1 + i\sin\theta_1) + (\cos\theta_2 + i\sin\theta_2) + (\cos\theta_3 + i\sin\theta_3) = (\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)(\cos\theta_3 + i\sin\theta_3)$$
$$\Rightarrow (\cos\theta_1 + \cos\theta_2 + \cos\theta_3) + i(\sin\theta_1 + \sin\theta_2 + \sin\theta_3)$$
$$= \cos(\theta_1+\theta_2+\theta_3) + i\sin(\theta_1+\theta_2+\theta_3)$$
$$\Rightarrow \Sigma\cos\theta_1 + i\,\Sigma\sin\theta_1 = \cos(\theta_1+\theta_2+\theta_3) + i\sin(\theta_1+\theta_2+\theta_3)$$

Now equating real and imaginary parts, we get
$$\Sigma\cos\theta_1 = \cos(\theta_1+\theta_2+\theta_3) \text{ and}$$
$$\Sigma\sin\theta_1 = \sin(\theta_1+\theta_2+\theta_3)$$

P.T.O.

Now $\frac{1}{a}+\frac{1}{b}+\frac{1}{c} = (\cos\theta_1 + i\sin\theta_1)^{-1} + (\cos\theta_2 + i\sin\theta_2)^{-1} + (\cos\theta_3 + i\sin\theta_3)^{-1}$

$= (\cos\theta_1 - i\sin\theta_1) + (\cos\theta_2 - i\sin\theta_2) + (\cos\theta_3 - i\sin\theta_3)$

$= (\cos\theta_1 + \cos\theta_2 + \cos\theta_3) - i(\sin\theta_1 + \sin\theta_2 + \sin\theta_3)$

$= \sum\cos\theta_1 - i\sum\sin\theta_1$ —— (i)

$\therefore \frac{1}{a}+\frac{1}{b}+\frac{1}{c} = \cos(\theta_1+\theta_2+\theta_3) - i\sin(\theta_1+\theta_2+\theta_3)$ —— (ii)

and $\frac{1}{abc} = (abc)^{-1} = a^{-1}.b^{-1}.c^{-1}$

$= (\cos\theta_1 + i\sin\theta_1)^{-1}.(\cos\theta_2 + i\sin\theta_2)^{-1}.(\cos\theta_3 + i\sin\theta_3)^{-1}$

$= (\cos\theta_1 - i\sin\theta_1)(\cos\theta_2 - i\sin\theta_2)(\cos\theta_3 - i\sin\theta_3)$

$= \cos(\theta_1+\theta_2+\theta_3) - i\sin(\theta_1+\theta_2+\theta_3)$ —— (iii)

From the equation (ii) and (iii) we get

$\frac{1}{a}+\frac{1}{b}+\frac{1}{c} = \frac{1}{abc} \Rightarrow abc\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) = 1$

$\Rightarrow (a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) = 1$

$\Rightarrow 1+1+1+\frac{a}{b}+\frac{b}{a}+\frac{b}{c}+\frac{c}{b}+\frac{c}{a}+\frac{a}{c} = 1$

$\Rightarrow 2+\left(\frac{a}{b}+\frac{b}{a}\right)+\left(\frac{b}{c}+\frac{c}{b}\right)+\left(\frac{c}{a}+\frac{a}{c}\right) = 0$ —— (iv)

Here $\frac{a}{b}+\frac{b}{a} = \cos(\theta_1-\theta_2) + \cancel{i\sin(\theta_1-\theta_2)} + \cos(\theta_1-\theta_2) - \cancel{i\sin(\theta_1-\theta_2)}$

$= 2\cos(\theta_1-\theta_2)$

Similarly $\frac{b}{c}+\frac{c}{b} = 2\cos(\theta_2-\theta_3)$ and $\frac{c}{a}+\frac{a}{c} = 2\cos(\theta_3-\theta_1)$

From the equation (iv), we get

$2 + 2\cos(\theta_1-\theta_2) + 2\cos(\theta_2-\theta_3) + 2\cos(\theta_3-\theta_1) = 0$

$\Rightarrow \cos(\theta_1-\theta_2) + \cos(\theta_2-\theta_3) + \cos(\theta_3-\theta_1) + 1 = 0.$

(Proved)

If $x = \frac{2}{\underline{|1}} - \frac{4}{\underline{|3}} + \frac{6}{\underline{|5}} - \frac{8}{\underline{|7}} + \cdots$ then show that $x^2 = y$

$y = 1 + \frac{2}{\underline{|1}} - \frac{2^3}{\underline{|3}} + \frac{2^5}{\underline{|5}} - \cdots$

Solution: Given that,

$$x = \frac{2}{\underline{|1}} - \frac{4}{\underline{|3}} + \frac{6}{\underline{|5}} - \frac{8}{\underline{|7}} + \cdots\cdots$$

$$= \frac{1+1}{\underline{|1}} - \frac{1+3}{\underline{|3}} + \frac{1+5}{\underline{|5}} - \frac{1+7}{\underline{|7}} + \cdots\cdots$$

$$= \left(\frac{1}{\underline{|1}} - \frac{1}{\underline{|3}} + \frac{1}{\underline{|5}} - \frac{1}{\underline{|7}} + \cdots\right) + \left(\frac{1}{\underline{|1}} - \frac{3}{\underline{|3}} + \frac{5}{\underline{|5}} - \frac{7}{\underline{|7}} + \cdots\right)$$

$$= \left(1 - \frac{1}{\underline{|3}} + \frac{1}{\underline{|5}} - \frac{1}{\underline{|7}} + \cdots\right) + \left(1 - \frac{3}{3\underline{|2}} + \frac{5}{5\underline{|4}} - \frac{7}{7\underline{|6}} + \cdots\right)$$

$$= \left(1 - \frac{1}{\underline{|3}} + \frac{1}{\underline{|5}} - \frac{1}{\underline{|7}} + \cdots\right) + \left(1 - \frac{1}{\underline{|2}} + \frac{1}{\underline{|4}} - \frac{1}{\underline{|6}} + \cdots\right)$$

$$= \sin 1 + \cos 1$$

Now $\because x = \sin 1 + \cos 1 \Rightarrow x^2 = (\sin 1 + \cos 1)^2 = \sin^2 1 + \cos^2 1 + 2\sin 1\cos 1$

$$\Rightarrow x^2 = 1 + \sin 2$$

$$= 1 + \left(\frac{2}{\underline{|1}} - \frac{2^3}{\underline{|3}} + \frac{2^5}{\underline{|5}} - \cdots\cdots\right)$$

$$= 1 + \left(2 - \frac{2^3}{\underline{|3}} + \frac{2^5}{\underline{|5}} - \cdots\cdots\right)$$

$$= y$$

Hence $x^2 = y$. (Proved)

Show that $\tan^{-1} x_1 + \tan^{-1} x_2 + \cdots\cdots + \tan^{-1} x_n = \tan^{-1}\left(\frac{P_1 - P_3 + P_5 - \cdots\cdots}{1 - P_2 + P_4 - \cdots\cdots}\right)$

where P denotes the sum of the products of $x_1 x_2 \cdots\cdots x_n$ taken at a time.

Solution: Let $P_1 = x_1 + x_2 + x_3 + \cdots\cdots + x_n$

$$= \Sigma x_1$$

Similarly, $P_2 = \Sigma x_1 x_2$

$P_3 = \Sigma x_1 x_2 x_3$

$\cdots\cdots\cdots\cdots$

$P_n = \Sigma x_1 x_2 \cdots\cdots x_n$

$\cdots\cdots \quad \cdots\cdots$

P.T.O

$$LHS = \tan^{-1}x_1 + \tan^{-1}x_2 + \tan^{-1}x_3 + \cdots + \tan^{-1}x_n$$

$$= \tan^{-1}\frac{x_1+x_2}{1-x_1x_2} + \tan^{-1}x_3 + \cdots + \tan^{-1}x_n$$

$$= \tan^{-1}\frac{\frac{x_1+x_2}{1-x_1x_2}+x_3}{1-\frac{x_1+x_2}{1-x_1x_2}\cdot x_3} + \tan^{-1}x_4 + \cdots + \tan^{-1}x_n$$

$$= \tan^{-1}\frac{x_1+x_2+x_3-x_1x_2x_3}{1-x_1x_2-x_2x_3-x_3x_1} + \tan^{-1}x_4 + \cdots + \tan^{-1}x_n$$

$$= \tan^{-1}\frac{\sum x_1 - \sum x_1x_2x_3 + \sum x_1x_2x_3x_4x_5 - \cdots}{1-\sum x_1x_2 + \sum x_1x_2x_3x_4 - \cdots}$$

$$= \tan^{-1}\frac{P_1 - P_3 + P_5 - \cdots}{1 - P_2 + P_4 - \cdots}$$

$= R.H.S$

$\therefore L.H.S = R.H.S$ (Proved)

E. Express $\sin 5\theta$ in terms of $\sin\theta$ and $\cos 5\theta$ in terms of $\cos\theta$.

Solution: We know that

$$\cos 5\theta + i\sin 5\theta = (\cos\theta + i\sin\theta)^5$$

Now, applying binomial theorem, we get

$$(\cos\theta + i\sin\theta)^5 = \cos^5\theta + {}^5C_1\cos^4\theta\cdot i\sin\theta + {}^5C_2\cos^3\theta\cdot i^2\sin^2\theta + {}^5C_3\cos^2\theta\cdot i^3\sin^3\theta + {}^5C_4\cos\theta\cdot i^4\sin^4\theta + {}^5C_5\cos^0\theta\cdot i^5\sin^5\theta$$

$$= \cos^5\theta + 5i\cos^4\theta\sin\theta - 10\cos^3\theta\sin^2\theta - i\,10\cos^2\theta\sin^3\theta + 5\cos\theta\sin^4\theta + i\sin^5\theta$$

$$= (\cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta) + i(\sin^5\theta - 10\cos^2\theta\sin^3\theta + 5\cos^4\theta\sin\theta)$$

P.T.O

Now equating real and imaginary part, we get

$$\cos 5\theta = \cos^5\theta - 10\cos^3\theta \cdot \sin^2\theta + 5\cos\theta \sin^4\theta$$

$$= \cos^5\theta - 10\cos^3\theta \sin^2\theta + 5\cos\theta (\sin^2\theta)^2$$

$$= \cos^5\theta - 10\cos^3\theta (1-\cos^2\theta) + 5\cos\theta (1-\cos^2\theta)^2$$

$$= \cos^5\theta - 10\cos^3\theta + 10\cos^5\theta + 5\cos\theta(1 - 2\cos^2\theta + \cos^4\theta)$$

$$= \cos^5\theta - 10\cos^3\theta + 10\cos^5\theta + 5\cos\theta - 10\cos^3\theta + 5\cos^5\theta$$

$$= 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta. \quad \text{Ans.}$$

$$\text{OR } \cos 5\theta = \cos^5\theta - 10\cos^3\theta \cdot \sin^2\theta + 5\cos\theta \sin^4\theta$$

$$= \cos^5\theta - 10\cos^3\theta (1-\cos^2\theta) + 5\cos\theta (1-\cos^4\theta)$$

$$= \cos^5\theta - 10\cos^3\theta + 10\cos^5\theta + 5\cos\theta - 5\cos^5\theta$$

$$= \cos^5\theta - 10\cos^3\theta - 5\cos\theta \quad \text{Ans.}$$

$$= 6\cos^5\theta - 10\cos^3\theta$$

$$\text{And } \sin 5\theta = \sin^5\theta - 10\cos^2\theta \cdot \sin^3\theta + 5\cos^4\theta \sin\theta$$

$$= \sin^5\theta - 10(1-\sin^2\theta)\sin^3\theta + 5(1-\sin^4\theta)\sin\theta$$

$$= \sin^5\theta - 10\sin^3\theta + 10\sin^5\theta + 5\sin\theta - 5\sin^5\theta$$

$$= \sin^5\theta - 10\sin^3\theta + 5\sin\theta. \quad \text{Ans}$$

$$= 6\sin^5\theta - 10\sin^3\theta$$

$$\text{OR } \sin 5\theta = \sin^5\theta - 10\cos^2\theta \cdot \sin^3\theta + 5\cos^4\theta \sin\theta$$

$$= \sin^5\theta - 10\cos^2\theta \cdot \sin^3\theta + 5(\cos^2\theta)^2 \sin\theta$$

$$= \sin^5\theta - 10(1-\sin^2\theta)\sin^3\theta + 5(1-\sin^2\theta)^2 \sin\theta$$

$$= \sin^5\theta - 10(1-\sin^2\theta)\sin^3\theta + 5(1 - 2\sin^2\theta + \sin^4\theta)\sin\theta$$

$$= \sin^5\theta - 10\sin^3\theta + 10\sin^5\theta + 5\sin\theta - 10\sin^3\theta + 5\sin^5\theta$$

$$= 16\sin^5\theta - 20\sin^3\theta + 5\sin\theta. \quad \text{Ans.}$$

Prove that $\frac{\sin^3\theta}{3!} = \frac{\theta^3}{3!} - (1+3^2)\frac{\theta^5}{5!} + (1+3^2+3^4)\frac{\theta^7}{7!} - \cdots$

Solution: We know

$$\sin 3\theta = 3\sin\theta - 4\sin^3\theta$$

$$\Rightarrow \sin^3\theta = \frac{1}{4}\left[3\sin\theta - \sin 3\theta\right]$$

$$= \frac{1}{4}\left[3\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots\right) - \left(3\theta - \frac{(3\theta)^3}{3!} + \frac{(3\theta)^5}{5!} - \cdots\right)\right]$$

$$= \frac{1}{4}\left[\left(3\theta - \frac{3\theta^3}{3!} + \frac{3\theta^5}{5!} - \cdots\right) - \left(3\theta - \frac{27\theta^3}{3!} + \frac{243\theta^5}{5!} - \cdots\right)\right]$$

$$= \frac{1}{4}\left[(3\theta - 3\theta) + \left(\frac{27\theta^3}{3!} - \frac{3\theta^3}{3!}\right) + \left(\frac{3\theta^5}{5!} - \frac{243\theta^5}{5!}\right) + \cdots\right]$$

$$= \frac{1}{4}\left[0 + \frac{24\theta^3}{3!} - \frac{240\theta^5}{5!} + \cdots\right]$$

$$\Rightarrow \sin^3\theta = \frac{1}{4}\left[\frac{24\theta^3}{3!} - \frac{240\theta^5}{5!} + \cdots\right]$$

$$\Rightarrow \frac{\sin^3\theta}{6} = \frac{1}{24}\left[\frac{24\theta^3}{3!} - \frac{240\theta^5}{5!} + \cdots\right]$$

$$\Rightarrow \frac{\sin^3\theta}{3!} = \left(\frac{\theta^3}{3!} - \frac{10\theta^5}{5!} + \cdots\right)$$

$$\Rightarrow \frac{\sin^3\theta}{3!} = \frac{\theta^3}{3!} - (1+3^2)\frac{\theta^5}{5!} + \cdots$$

$$\therefore \frac{\sin^3\theta}{3!} = \frac{\theta^3}{3!} - (1+3^2)\frac{\theta^5}{5!} + \cdots$$