08-04-2014

Differential equations

$\frac{dy}{dx}$ → dependent, independent

* $\frac{d^2y}{dx^2}+4\frac{dy}{dx}+xy=5x$ (O.D.E)
* $\frac{du}{dx}+\frac{\delta u}{dy}=0$ (P.D.E)

* **Defn:** Equations which involve differential co-efficients w.r to a single independent variable is called ordinary differential equations.

Ex: $\left(\frac{dy}{dx}\right)^2+4\frac{dy}{dx}+xy=x^3y^3$

* **Defn:** Equations which involve differential co-efficients w.r to more than one independent variable are called partial differential equations.

Ex: $\frac{\delta z}{\delta x}+\frac{\delta z}{\delta y}=kz$

* Defn: The order of a differential equations in the highest order differential co-efficient involved ~~the~~ in the differential equations.

* Defn: The degree of a differential equation in the power of the highest differential co-efficient in the equation.

Ex: The order and degree of the differential equation $\frac{d^2y}{dx^2} + 7\left(\frac{dy}{dx}\right)^2 + 5\left(\frac{dy}{dx}\right) + 6xy = 0$ are 2 and 1 respectively.

Formation of differential equation from its Solution:

Ex: Form the differential equation of the family of curves $(y-k)^2 = 4(x-h)$ [h and k are parameter]

Sol^n: $(y-k)^2 = 4a(x-h)$

Differentiating w.r to x. ~~wh~~ we have

$$2(y-k)\frac{dy}{dx} = 4a$$

$$\Rightarrow (y-k)\frac{dy}{dx} = 2a$$

$$\Rightarrow (y-k)\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = 0$$

$$\Rightarrow \frac{2a}{\frac{dy}{dx}}\left(\frac{d^2y}{dx^2}\right) + \left(\frac{dy}{dx}\right)^2 = 0$$

$$\Rightarrow 2a\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^3 = 0$$

Problem: Find the differential equation of all circle of radius a $(x-h)^2+(y-k)^2=a^2$

[H.W]

Problem: Find the differential equation of all circles of radius a which have their ~~centre~~ center on x axis.

16-04-2018

DE of first order and first degree –

(i) variable sp anable

(ii) Exact

(iii) homogeneous

(iv) linear

Solve: $\frac{dy}{dx} = \frac{1+y^2}{1+x^2}$

we have,

$$\frac{dy}{dx} = \frac{1+y^2}{1+x^2}$$

$$\Rightarrow \frac{dy}{1+y^2} = \frac{dx}{1+x^2}$$

$$\Rightarrow \int \frac{dy}{1+y^2} = \int \frac{dx}{1+x^2}$$

$$\Rightarrow \tan^{-1} y = \tan^{-1} x + c$$

Problem:

Solve - $\sec^2 x \tan y \, dx + \sec^2 y \tan x \, dy = 0$

Soln: We have

$$\sec^2 x \tan y \, dx + \sec^2 y \tan x \, dy = 0$$

$$\Rightarrow \frac{\sec^2 x}{\tan x} \cdot dx + \frac{\sec^2 y}{\tan y} \, dy = 0$$

$$\therefore \int \frac{\sec^2 x}{\tan x} \, dx + \int \frac{\sec^2 y}{\tan y} \, dy = 0$$

$$\Rightarrow \ln(\tan x) + \ln(\tan y) = \ln c$$

$$\Rightarrow \ln(\tan x \cdot \tan y) = \ln c$$

$$\Rightarrow \tan x \tan y = c$$

Ans

Solve: $\frac{dy}{dx} = \sin(x+y) + \cos(x+y)$

Sol^n: put,

$$x+y = u$$

$$\Rightarrow 1 + \frac{dy}{dx} = \frac{du}{dx}$$ $\left[dxy = xdy + ydx\right]$

The given equation becomes

$$\frac{du}{dx} - 1 = \sin u + \cos u$$

$$\Rightarrow \frac{du}{1 + \sin u + \cos u} = dx$$

$$\Rightarrow \int \frac{du}{1 + \sin u + \cos u} = \int dx$$

$$\Rightarrow \int \frac{du}{2\cos^2 u/2 + 2\sin u/2 \cos u/2} = x$$

$$\Rightarrow \int \frac{du \sec^2 u/2}{2+2\tan u/2} = x$$

$$\Rightarrow \ln\left(2+2\tan u/2\right) = x+c$$

$$\Rightarrow \ln\left(2+\tan (x+y)/2\right) = x+c$$ **Ans**

Solve: $(x+y)^2 \frac{dy}{dx} = a^2$

$\boxed{\frac{Ex-3}{P-8}}$, $\boxed{\frac{Ex-7(i)}{P-16}}$

$$df(x,y) = \frac{\delta f}{\delta x} dx + \frac{\delta f}{\delta y} dy$$

$M(x,y)\,dx + N(x,y)\,dy$, exact

$\Rightarrow \exists$ a function $f(x,y)$

such that $M = \frac{\delta f}{\delta x}$ and $N = \frac{\delta f}{\delta y}$

$$\therefore \frac{\delta M}{\delta y} = \frac{\delta^2 f}{\delta x \delta y} \quad \text{and}$$

$$\frac{\delta N}{\delta x} = \frac{\delta^2 f}{\delta x \delta y}$$

$$\Rightarrow \frac{\delta M}{\delta y} = \frac{\delta N}{\delta x}$$

$(x^2+y^2)\,dx + 2xy\,dy = 0$ $\qquad df = \left(\frac{\delta f}{\delta x}dx + \frac{\delta f}{\delta y}dy\right)$

$$\Rightarrow d\left(x^3/3 + xy^2\right) = 0$$

$$\Rightarrow \boxed{x^3/3 + xy^2 = c}$$

Ans

17-04-2018

Solve:

$$(x^2-2xy+3y^2)\,dx+(4y^3+6xy-x^2)\,dy=0 \; \text{---} \; (1)$$

Soln:

Here,

$$M=x^2-2xy+3y^2$$

and $N=4y^3+6xy-x^2$

$\therefore \dfrac{\delta M}{\delta y}=-2x+6y$. and $\dfrac{\delta N}{\delta x}=6y-2x$

$\therefore \dfrac{\delta M}{\delta y}=\dfrac{\delta N}{\delta x}$. Hence the equation (1) is exact.

The equation (1) can be written as,

$$d\left(\frac{x^3}{3}-x^2y+3xy^2+y^4\right)=0$$

Integrating. we have

$$x^3/3-x^2y+3xy^2+y^4=c$$

Ans

* $\frac{Ex-7}{P-34}$, $\frac{Ex-9}{P-40}$, $\frac{Ex-8}{P-40}$

Defn: If an equation $Mdx + Ndy = 0$ is not exact but if we can multiply the ~~qu~~ equation by a function so that the resultant equation is exact then such a function is called an integrating factor for the ~~q~~ equation.

Ex: $\frac{xdy - ydx}{y^2} = 0$ $\quad \frac{xdy}{y^2} - \frac{ydx}{y^2}$

$M = -\frac{1}{y^2}$

$N = \frac{x}{y^2}$

Solve: $ydx - xdy = 0$

Soln:

$\frac{1}{y}dx - \frac{x}{y^2}dy = 0$

$\Rightarrow d\left(\frac{x}{y}\right) = 0$

$\Rightarrow \boxed{\frac{x}{y} = c}$

$\frac{dx}{x} - \frac{dy}{y} = 0$

$\Rightarrow \ln x - \ln y = \ln c$

$\Rightarrow \boxed{\frac{x}{y} = c}$

* **Linear equation:** A differential equation of the form $\frac{dy}{dx} + Py = Q$ where P and Q are functions of x or constant is called a linear equation.

I.F (Integrating factor) for linear equation

~~$e^{\int P dx}$~~

$\frac{dy}{dx} + Py = Q$ is $e^{\int P dx}$

* Solve: $\frac{dy}{dx} + xy = e^{4x}$ ——— (1)

Soln: The given equation is a linear equation

$\therefore$ I.F $= e^{\int x dx} = e^{x^2/2}$

Multiplying (1) by $e^{x^2/2}$ we have

$$e^{x^2/2}\frac{dy}{dx} + e^{\frac{x^2}{2}} xy = e^{4x + \frac{x^2}{2}}$$

$$\Rightarrow \frac{d}{dx}\left(y e^{x^2/2}\right) = e^{x^2/2 + 4x}$$

By Integrating, we have

$$\boxed{y e^{x^2/2} = \int e^{x^2/2+4x}\,dx + c} \quad \text{Ans}$$

** Solve- $\frac{dy}{dx} + \sin x\, y = \sin x$ —— ①

Soln:

The given equation is a linear equation.

$\therefore$ I.F $= e^{\int \sin x\,dx} = e^{-\cos x}$

Multiplying ① ~~we have~~ by $e^{-\cos x}$ we have,

$$e^{-\cos x}\frac{dy}{dx} + e^{-\cos x}\sin x\, y = e^{-\cos x}\sin x$$

$$\Rightarrow \frac{d}{dx} y\left(e^{-\cos x}\right) = \sin x \cdot e^{-\cos x}$$

$$\Rightarrow y \cdot e^{-\cos x} = \int \sin x \cdot e^{-\cos x}\,dx + c$$

$$= \int e^{z}\,dz$$

$$= e^{z}$$

$$= e^{-\cos x} + c \quad \text{Ans}$$

$z = -\cos x$

$dz = \sin x\,dx$

22-04-2018

Ex-3
P-22

Solve $(1+y^2)\frac{dx}{dy}+(x-\tan^{-1}y)=0$

Soln: The equation can be written as

$$\frac{dx}{dy}+\frac{x}{1+y^2}=\frac{\tan^{-1}y}{1+y^2} \quad \text{---(1)}$$

$\therefore$ I.F $= e^{\int \frac{dy}{1+y^2}} = e^{\tan^{-1}y}$

I.F $= e^{\int P dx}$

$\frac{dy}{dx}+P(x)y$

$\frac{dx}{dy}+P(y)$

I.F $= e^{\int}$

Multiplying (1) by $e^{\tan^{-1}y}$ we have

$$e^{\tan^{-1}y}\frac{dx}{dy}+\frac{x}{1+y^2}e^{\tan^{-1}y}=\frac{\tan^{-1}y\, e^{\tan^{-1}y}}{1+y^2}$$

$$\Rightarrow \frac{d}{dy}\left(xe^{\tan^{-1}y}\right)=\frac{\tan^{-1}y}{1+y^2}e^{\tan^{-1}y}$$

Hence,

$$xe^{\tan^{-1}y} = \int \frac{\tan^{-1}y}{1+y^2}e^{\tan^{-1}y}\,dy$$

$\int ze^z dz$

$= ze^z - \int e$

$= ze^z - e$

$\Rightarrow x e^{\tan^{-1}y} = \tan^{-1}y \, e^{\tan^{-1}y} - e^{\tan^{-1}y} + c$ Ans

Bernoulli's equation:

A differential equation of the form,

$\frac{dy}{dx} + Py = Qy^n$ where P and Q are functions of x or constant is called a Bernoulli's equation.

How to solve Bernoulli's equation:

The equation, $\frac{dy}{dx} + Py = Qy^n$ can be written as

$$y^{-n}\frac{dy}{dx} + Py^{1-n} = Q \quad \text{--- (1)}$$

put, $v = y^{1-n}$, so that

$$\frac{dv}{dx} = (1-n)y^{-n}\frac{dy}{dx}$$

$\therefore$ ① becomes $\frac{1}{1-n}\frac{dv}{dx} + P.v = Q$

$\Rightarrow \frac{dv}{dx} + (1-n)Pv = (1-n)Q$ which is a linear equation.

** Ex-1 / P-23 Solve $\frac{dy}{dx} = x^3y^3 - xy$.

Soln: We have, $\frac{dy}{dx} + xy = x^3y^3$

$\Rightarrow y^{-3}\frac{dy}{dx} + xy^{-2} = x^3$ ---- ①

put, $v = y^{-2}$, so that $\frac{dv}{dx} = -2y^{-3}\frac{dy}{dx}$

Hence ① becomes

$-\frac{1}{2}\frac{dv}{dx} + xv = x^3$

$\Rightarrow \frac{dv}{dx} - 2xv = -2x^3$ ---- ② which is a linear equation.

$\therefore I.F = e^{-\int 2x dx} = e^{-x^2}$

Multiplying (2) by e^{-x^2} we have

$$e^{-x^2}\frac{dv}{dx} - 2xe^{-x^2}v = -2x^3e^{-x^2}$$

$$\Rightarrow \frac{d}{dx}\left(ve^{-x^2}\right) = -2x^3e^{-x^2}$$

Hence, $ve^{-x^2} = -\int 2x^3e^{-x^2}dx$

$z = -x^2$

$dz = -2x dx$

$$\Rightarrow ve^{-x^2} = -\int ze^{z}dz$$

$$\Rightarrow ve^{-x^2} = -(ze^{z} - e^{z}) + c$$

$$\Rightarrow y^{-2}e^{-x^2} = -\left(-x^2e^{-x^2} - e^{-x^2}\right) + c$$ Ans

** Ex-15 / P-29 [try yourself]

Rules for finding the integrating factor:

Rule-I: If $\dfrac{\dfrac{\delta M}{\delta y}-\dfrac{\delta N}{\delta x}}{N}=f(x)$

then $e^{\int f(x)\,dx}$ is an I.F.

Rule-II: If $\dfrac{\dfrac{\delta M}{\delta y}-\dfrac{\delta N}{\delta x}}{M}=g(y)$

then $e^{-\int g(y)\,dy}$ is an I.F.

Rule-III: If $M\,dx+N\,dy=0$ is homogeneous and $Mx+Ny\neq 0$ then

$\dfrac{1}{Mx+Ny}$ is an I.F.

** EX-5 / P-45

Solve $\left(y+\frac{1}{3}y^3+\frac{1}{2}x^2\right)dx+\frac{1}{4}\left(x+xy^2\right)dy=0$ ——— (1)

Soln: Here, $M=y+\frac{1}{3}y^3+\frac{1}{2}x^2$, $N=\frac{1}{4}\left(x+xy^2\right)$

$$\therefore \frac{\delta M}{\delta y}-\frac{\delta N}{\delta x}=(1+y^2)-\frac{1}{4}(1+y^2)$$

$$=\frac{3}{4}(1+y^2)$$

$$\therefore \frac{\frac{\delta M}{\delta y}-\frac{\delta N}{\delta x}}{N}=\frac{\frac{3}{\cancel{4}}\cancel{(1+y^2)}}{\frac{1}{\cancel{4}}x\cancel{(1+y^2)}}$$

$$=\frac{3}{x}$$

$$\therefore I.F=e^{\int\frac{3}{x}dx}=e^{\ln x^3}=x^3$$

Multiplying (1) by x^3, we have

$$x^3\left(y+\frac{1}{3}y^3+\frac{1}{2}x^2\right)dx+\frac{1}{4}\left(x^4+x^4y^2\right)dy=0$$

$$\Rightarrow d\left(\frac{x^4y}{4}+\frac{x^4y^3}{12}+\frac{x^6}{12}\right)=0$$

Hence, $\frac{x^4y}{4}+\frac{x^4y^3}{12}+\frac{x^6}{12}=c_1$

$$\boxed{3x^4y + x^4y^3 + x^6 = c}$$ Ans

** $\frac{\text{Ex-8}}{\text{P-46}}$ [try yourself]

$\frac{\text{Ex-2}}{\text{P-48}}$ [try yourself]

23-04-2018

Linear differential equations with constant coefficients:

A differential equation of the form

$$\frac{d^ny}{dx^n} + P_1 \frac{d^{n-1}y}{dx^{n-1}} + P_2 \frac{d^{n-2}y}{dx^{n-2}} + \cdots + P_n = X$$

Where $P_1, P_2, \cdots, P_n$ and X are function of x or constant is called a linear differential equation.

** Solve the differential equation

$$\frac{d^3y}{dx^3} - 13\frac{dy}{dx} - 12y = 0$$ [3 order 1st degree equation]

Soln:

Let, $y = e^{mx}$ be a trial solution then

$$m^3 e^{mx} - 13me^{mx} - 12e^{mx} = 0$$

$$\Rightarrow (m^3 - 13m - 12)e^{mx} = 0$$

$$\Rightarrow m^3 - 13m - 12 = 0 \quad [\because e^{mx} \neq 0]$$

$$\Rightarrow m^2(m+1) - m(m+1) - 12(m+1) = 0$$

$$\Rightarrow (m+1)(m^2 - m - 12) = 0$$

$$\Rightarrow (m+1)(m-4)(m+3) = 0$$

$$\therefore m = -1, 4, -3$$

Hence, $y = c_1 e^{-x} + c_2 e^{4x} + c_3 e^{-3x}$

is the general solution of the given equation.

** Ex-1
p-62

Solve: $\frac{d^4y}{dx^4} - \frac{d^3y}{dx^3} - 9\frac{d^2y}{dx^2} - 11\frac{dy}{dx} - 4y =$

Soln:

Let $y = e^{mx}$ be a trial soln of the given equation.

Then the auxiliary equation is

$$m^4 - m^3 - 9m^2 - 11m - 4 = 0$$

~~$\Rightarrow m = -1$~~

~~$\Rightarrow (m+1)(m+1)(m+1)$~~

$\Rightarrow m = -1, -1, -1, 4$

Hence the general solution of the given equation is

$$y = c_1 e^{-x} + c_2 x e^{-x} + c_3 x^2 e^{-x} + c_4 e^{4x}$$

Ans

Ex-1
P-64

Solve: $\frac{d^4y}{dx^4} - a^4y = 0$

Soln: Let, $y = e^{mx}$ be a trial solution of the given equation.

Then the A.E is $m^4 - a^4 = 0$

$\Rightarrow (m^2+a^2)(m^2-a^2) = 0$

$\Rightarrow m = a, -a, ai, -ai$

Here the general solution of the given equation

$y = c_1 e^{ax} + c_2 e^{-ax} + c_3 e^{aix} + c_4 e^{-aix}$

$\boxed{e^{i\theta} = \cos\theta + i\sin\theta \quad e^{-i\theta} = \cos\theta - i\sin\theta}$

$= c_1 e^{ax} + c_2 e^{-ax} + c_3 [\cos ax + i\sin ax]$

$+ c_4 [\cos ax - i\sin ax]$

$= c_1 e^{ax} + c_2 e^{-ax} + (c_3+c_4)\cos ax + (c_3-c_4) i\sin ax$

$= c_1 e^{ax} + c_2 e^{-ax} + A\cos ax + B\sin ax$

Hence the general solution of the given equation,

$y = c_1 e^{ax} + c_2 e^{-ax} + c_3 \cos ax + c_4 \sin ax$

Ans

if $m = 1 \pm i,\ 3 \pm 2i$ then,

$$\boxed{y = e^{x}(c_1 \cos x + c_2 \sin x) + e^{3x}(c_3 \cos 2x + c_4 \sin 2x)}$$

24-04-2018

* General solution of

$$(D^n + a_1 D^{n-1} + \cdots + a_n)y = X \quad \text{--- (1)}$$

$$D = \frac{d}{dx}, \quad D^n y = \frac{d^n y}{dx^n}$$

The general solution of

$$D^n + a_1 D^{n-1} + \cdots + a_n)y = 0 \quad (2)$$ is

called the complementary function of (1)

The particular integral of (1) is a Particular solution of (1).

If $y = y_c(x)$ is the general solution of (2) and

$y = y_p(x)$ is a particular solution of (1) then

$y = y_c + y_p$ is the general solution of (1).

The particular integral of (1) will be denoted

by the notation $\dfrac{1}{D^n + a_1 D^{n-1} + \cdots a_n} x$

Evaluation of $\dfrac{1}{F(D)} e^{ax}$

** Solve: $(D^2+3D-4)y = e^{4x}$ —— (1)

Soln:

Let, $y = e^{mx}$ be a trial solution of

$$(D^2+3D-4)y = 0. \text{ Then}$$

A.E is $m^2+3m-4=0$

$\Rightarrow (m+4)(m-1)=0$

$\Rightarrow m = -4, 1$

$\therefore$ C.F. of (1) is $y_c = c_1 e^{-4x} + c_2 e^{x}$ Ans

A.E = Auxiliary Equation

C.F = Complementary Function

Particular integral of (1)

$$y_p = \frac{1}{D^2+3D-4} e^{4x}$$

$$= \frac{1}{24} e^{4x}$$

$$\boxed{\frac{1}{F(D)} e^{ax} = \frac{1}{F(a)} e^{ax} \text{ if } F(a) \neq 0}$$

Hence, the G.S. of (1) is

$$y = c_1 e^{-4x} + c_2 e^{x} + \frac{1}{24} e^{4x}$$ Ans

$$\frac{1}{D^2-3D+2} e^{2x}$$

$$= \frac{1}{(D-2)(D-1)} e^{2x}$$

$$= \left(\frac{1}{D-2}\right)\left(\frac{1}{D-1}\right)(e^{2x})$$

$$= \frac{1}{D-2} e^{2x}$$

$$= x e^{2x}$$

$\frac{1}{(D-a)} e^{ax}$

$$= x e^{ax}$$

$\frac{1}{(D-a)^2} e^{ax}$

$$= \frac{x^2}{2} e^{ax}$$

$\frac{1}{(D-a)^n} e^{ax}$

$$= \frac{x^n}{\lfloor n} e^{ax}$$

30-04-2018

Formula:

i) $\frac{1}{f(D^2)} \sin ax = \frac{1}{f(-a^2)} \sin ax$

if $f(-a^2) \neq 0$

ii) $\frac{1}{f(D^2)} \cos ax = \frac{1}{f(-a^2)} \cos ax$ if $f(-a^2) \neq 0$

Ex-1
P-69

Solve $\frac{d^2y}{dx^2} + \frac{dy}{dx} + y = \sin 2x$

Soln: Let $y = e^{mx}$ be a trial solution of

$\frac{d^2y}{dx^2} + \frac{dy}{dx} + y = 0$, then

A.E is $m^2 + m + 1 = 0$

$\Rightarrow m = -\frac{1}{2} \pm \frac{\sqrt{3}}{2} i$.

Hence C.E. $y_c = e^{-1/2x}\left(c_1 \cos \frac{\sqrt{3}}{2} x + c_2 \sin \frac{\sqrt{3}}{2} x\right)$ ——— (1)

Now P.I $= y_p = \frac{1}{D^2+D+1} \sin 2x$

$$= \frac{1}{(D^2+1)+D} \sin 2x$$

$$= \frac{1}{(-4+1)+D} \sin 2x$$

$$= \frac{1}{-3+D} \sin 2x$$

$$= \frac{D+3}{D^2-9} \sin 2x$$

$$= (D+3) \frac{\sin 2x}{-13}$$

$$= -\frac{1}{13} (D+3) \sin 2x$$

$$= -\frac{1}{13} (2\cos 2x + 3\sin 2x)$$

Hence, $GS = y = y_c + y_p$

$=$

** $\frac{1}{D^2+4} \sin 2x$

$e^{2ix} = \cos 2x + i \sin 2x$

$\sin 2x = I.P.$ of e^{2ix}

$\cos 2x = R.P.$ of e^{2ix}

= Imaginary Part

$$\left(\frac{1}{D^2+4} e^{2ix} \right)$$

$$= I.P\left(\frac{1}{(D+2i)(D-2i)} e^{2ix}\right)$$

$$= I.P\left(\frac{1}{4i} \frac{1}{D-2i} e^{2ix}\right)$$

$$= I.P.\left(\frac{1}{4i} x e^{2ix}\right)$$

$$= I.P.\left(\frac{-1}{4} x e^{2ix}\right)$$

$$= -\frac{x}{4} \cos 2x$$

* **Formula:**

$$\frac{1}{f(D)}(x^m) = (1 + a_1 D + a_2 D^2 + \cdots + a_m D^m + \cdots) x^m$$

\# **Solve** P-76 $(D^3 + 2D^2 + D)y = e^{2x} + x^2 + x.$

P.I $\quad y_p = \frac{1}{D^3+2D^2+D}(e^{2x} + x^2 + x)$

$$= \frac{1}{D^3+2D^2+D}(e^{2x}) + \frac{1}{D^3+2D^2+D}(x^2+x)$$

$$= \frac{1}{18} e^{2x} + \frac{1}{D(D^2+2D+1)} (x^2+x)$$

$$= \frac{1}{18} e^{2x} + \frac{1}{D} \left\{ 1 - (2D+D^2) + (2D+D^2)^2 - \cdots \right\}$$
$$(x^2+x)$$

$$= \frac{e^{2x}}{18} + \frac{1}{D} \left\{ 1 - 2D + 3D^2 + \cdots \right\} (x^2+x)$$

$$= \frac{e^{2x}}{18} + \frac{1}{D} (x^2 + x - 4x - 2 + 6)$$

$$= \frac{e^{2x}}{18} + \frac{x^3}{3} - \frac{3x^2}{2} + 4x$$

Ans

* Formula:

$$\frac{1}{f(D)}\left(e^{ax}V\right) = e^{ax}\frac{1}{f(D+a)}V\text{ , } V \text{ is a}$$

function of x.

$\dfrac{\text{Ex-3}}{\text{P-78}}$ Solve $(D^3-7D-6)y = e^{2x}x^2$

$$y_p = \frac{1}{D^3-7D-6}\left(e^{2x}x^2\right)$$

$$= e^{2x}\frac{1}{(D+2)^3-7(D+2)-6}\left(x^2\right)$$

$$= e^{2x}\frac{1}{(D^3+6D^2+12D+8)-7D-14-6}\left(x^2\right)$$

$$= e^{2x}\frac{1}{D^3+6D^2+5D-12}\left(x^2\right)$$

$$= \frac{-e^{2x}}{12}\left[1+\frac{1}{12}\left(D^3+6D^2+5D\right)+\frac{1}{144}\left(D^3+6D^2+5D\right)^2+ \cdots\right](x^2)$$

$$= -\frac{e^{2x}}{12}\left[1+\frac{5}{12}D+\frac{1}{2}D^2+\frac{25}{144}D^2+\cdots\right](x^2)$$

$$= -\frac{e^{2x}}{12}\left(x^2+\frac{5}{6}x+\frac{97}{144}\times 2\right)$$

08-05-2018

Ex-4 p-78. Solve $\frac{d^3y}{dx^3} - 2\frac{dy}{dx} + 4y = e^x \cos x$

Soln:

A.E.

$m^3 - 2m + 4 = 0$

$\Rightarrow m^2(m+2) - 2m(m+2) + 2(m+2) = 0$

$\Rightarrow (m+2)(m^2 - 2m + 2) = 0$

$\therefore m = -2,\ 1 \pm i$

$y_c = c_1 e^{-2x} + e^x(c_2 \cos x + c_3 \sin x)$

$\because$ P.I $= \frac{1}{D^3 - 2D + 4}(e^x \cos x)$

$= e^x \frac{1}{(D+1)^3 - 2(D+1) + 4}(\cos x)$

$= e^x \frac{1}{D^3 + 3D^2 + D + 3}(\cos x)$

$= e^x \frac{1}{D(D^2+1) + 3(D^2+1)}(\cos x)$

$= e^x \frac{1}{(D^2+1)(D+3)}(\cos x)$

$= e^x \frac{(D-3)}{(D^2+1)(D^2-9)}(\cos x)$

$= \frac{-e^x}{10} \frac{1}{D^2+1}(-\sin x - 3\cos x)$

$$= \frac{e^x}{10} \frac{1}{D^2+1}(\sin x + 3\cos x)$$

$$= \frac{e^x}{10}\left[\frac{1}{D^2+1}(\sin x) + 3 \cdot \frac{1}{D^2+1} \cdot (\cos x)\right]$$

$$= \frac{e^x}{10}[Q+3P] \quad (\text{say}) \text{ --- } (1)$$

Now, $P+iQ = \frac{1}{D^2+1}(\cos x) + i\frac{1}{D^2+1}(\sin x)$

$$= \frac{1}{D^2+1}(\cos x + i\sin x)$$

$$= \frac{1}{D^2+1} e^{ix}$$

$$= \frac{1}{(D-i)(D+i)} e^{ix}$$

$$= \frac{1}{2i} \cdot \frac{1}{D-i} e^{ix}$$

$$= \frac{1}{2i} \cdot e^{ix} \cdot \frac{1}{D}(1)$$

$$= \frac{xe^{ix}}{2i} = \frac{-ix}{2}(\cos x + i\sin x)$$

$$= \frac{x}{2}\sin x - \frac{1}{2}x\cos x$$

$$P.I = \frac{e^x}{10}[Q+3P] \quad (\text{say}) \text{ --- } (1)$$

$\therefore P = \frac{x}{2}\sin x$

$Q = \frac{-x}{2}\cos x$

Hence, $y_p = \frac{e^x}{10}\left[\frac{-x}{2}\cos x + \frac{3x}{2}\sin x\right]$

$= \frac{e^x . x}{20}\left[3\sin x - \cos x\right]$ Ans

$\left(\frac{\text{NO}-1}{\text{page}-80}, \frac{\text{NO}-04}{\text{page}-82}\right)$

Homogeneous linear equation:

An equation of the form

$$x^n \frac{d^n y}{dx^n} + P_1 x^{n-1}\frac{d^{n-1}y}{dx^{n-1}} + \cdots + P_n y = X$$

where $P_1, P_2, \cdots, P_n$ are constants and X is a function of x is called a homogeneous linear equation.

** Ex-3, P-97 Solve $x^2\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} - 4y = x^4$ ∴

soln: put $x = e^z$ and $D \equiv \frac{d}{dz}$. The equation (1) becomes

$$\{D(D-1) - 2D - 4\}y = e^{4z}$$

$$\Rightarrow (D^2 - 3D - 4)y = e^{4z} \quad \text{(ii)}$$

$$\therefore C.F = c_1e^{-z} + c_2e^{4z} \quad \text{(iii)}$$

Particular integral

$$\boxed{P.I} = \frac{1}{D^2 - 3D - 4}(e^{4z})$$

$$= \frac{1}{(D-4)(D+1)}e^{4z}$$

$$= \frac{1}{5}ze^{4z} \quad \text{(iv)}$$

$$y = c_1e^{-z} + c_2e^{4z} + \frac{z}{5}e^{4z}$$

$$\Rightarrow y = \frac{c_1}{x} + c_2x^4 + \frac{x^4}{5}\ln x \quad \text{Ans}$$

$$\frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{dx}$$

$$= \frac{1}{x}\frac{dy}{dz}$$

$$\boxed{x\frac{dy}{dx} = \frac{dy}{dz}}$$

$$\frac{d^2y}{dz^2} = \frac{d}{dz}\left(x\frac{dy}{dx}\right)$$

$$= \frac{d}{dx}\left(x\frac{dy}{dx}\right)\frac{dx}{dz}$$

$$= x\left[\frac{dy}{dx} + \frac{d^2y}{dx^2}\right]$$

$$= x\frac{dy}{dx} + x^2\frac{d^2y}{dx^2}$$

$$= \frac{dy}{dz} + \frac{x^2d^2y}{dx^2}$$

$$\frac{x^2d^2y}{dx^2} = D(D-1)y, \; D=$$

$$\frac{x^3d^3y}{dx^3} = D(D-1)(D-2)y$$

Ex-9
p-109

09-05-2018

Ex-2
p-49

Solve $\frac{d^2y}{dx^2} - x^2\frac{dy}{dx} + xy = x$, Given that $y=x$ is a part of C.F.

Soln: put $y = vx$, where v is a function of x is the given equation.

We have $\frac{d}{dx}\left(v + x\frac{dv}{dx}\right) - x^2\left(v + x\frac{dv}{dx}\right) + vx^2 = x$

$$\Rightarrow 2\frac{dv}{dx} + x\frac{d^2v}{dx^2} - vx^2 - x^3\frac{dv}{dx} + vx^2 = x$$

$$\Rightarrow x\frac{d^2v}{dx^2} + (2-x^3)\frac{dv}{dx} = x$$

putting $\frac{dv}{dx} = p$, we have

$$x\frac{dp}{dx} + (2-x^3)p = x$$

$$\Rightarrow \frac{dp}{dx} + \frac{2-x^3}{x}p = 1 \text{ ---- } (1)$$

This is a 1st order linear equation.

$$\therefore I.F = e^{\int \frac{2-x^3}{x} dx} = e^{2\ln x - \frac{x^3}{3}}$$

$$= x^2 . e^{-\frac{x^3}{3}}$$

Hence the solution of ①

$$p x^2 e^{-x^3/3} = \int x^2 e^{-x^3/3} dx$$

$$\Rightarrow p x^2 e^{-x^3/3} = -e^{-x^3/3} + c_1$$

$$\Rightarrow p = -\frac{1}{x^2} + \frac{4}{x^2} e^{x^3/3}$$

$$\Rightarrow \frac{dv}{dx} = -\frac{1}{x^2} + \frac{c_1}{x^2} e^{x^3/3}$$

$$\therefore v = \frac{1}{x} + c_1 \int \frac{e^{x^3/3}}{x^2} dx + c_2$$

Hence the G.S of the ~~linear~~ given equa

is $y = \left[\frac{1}{x} + c_1 \int \frac{e^{x^3/3}}{x^2} dx + c_2\right] x$

Ans

** Ex-7
p-52

Solve $x\frac{d^2y}{dx^2}-(2x-1)\frac{dy}{dx}+(x-1)y=0$, given that $y=e^x$ is a solⁿ.

Method of variation of parameters:

Working rule: Given the equation $\frac{d^2y}{dx^2}+P\frac{dy}{dx}+Qy=X$

find the solⁿ. of $\frac{d^2y}{dx^2}+P\frac{dy}{dx}+Qy=0$ ---- (2)

If $y=c_1u+c_2v$ is a G.S of (2) then we assume $y=Au+Bv$ is solⁿ of (1), where A and B are functions of x.

To determine A and B, use the following two equations:

(i) $y=Au+Bv$ is a solⁿ of (1) implies

$A_1u_1+B_1v_1=X$ ---- (3)

(ii) $A_1u+B_1v=0$ ---- (4)

$$\boxed{\begin{aligned}A_1&=\frac{dA}{dx}\\ B_1&=\frac{dB}{dx}\\ u_1&=\frac{du}{dx}\\ v_1&=\frac{dv}{dx}\end{aligned}}$$

Solving (3) and (4), find A_1 and B_1, and then integrate to get A and B.

** Ex-1(a)
P-86

Apply the method of variation of parameters to solve $\frac{d^2y}{dx^2} + n^2y = \sec nx$

Soln: The given equation is

$$\frac{d^2y}{dx^2} + n^2y = \sec nx \quad \text{---} (1)$$

A.E
m^2+n^2
$m=$

The C.F of (1) is $y = C_1 \cos nx + C_2 \sin nx$

Let, $y = A\cos nx + B\sin nx$ (2) be a soln of (1) whe

A and B are function of x.

From (2) $\frac{dy}{dx} = -An\sin nx + A_1\cos nx + Bn\cos$
$+ B_1 \sin nx$

$= (Bn\cos nx - An\sin nx) + (A_1\cos nx + B_1$

put, $A_1\cos nx + B_1\sin nx = 0$ ---- (3)

Then $\frac{dy}{dx} = Bn\cos nx - An\sin nx$

$\therefore \frac{d^2y}{dx^2} = -n^2(A\cos nx + B\sin nx) - A_1 n\sin nx$
$+ B_1 n\cos nx$

$$= -n^2 y - A_1 n \sin nx + B_1 n \cos nx$$

$$\Rightarrow \frac{d^2 y}{dx^2} + n^2 y = -A_1 n \sin nx + B_1 n \cos nx$$

$$\Rightarrow \sec nx = -A_1 n \sin nx + B_1 n \cos nx$$

$$\Rightarrow A_1 n \sin nx - B_1 n \cos nx = -\sec nx \quad \text{--- (4)}$$

Solving (3) and (4), we have $B_1 = \frac{1}{n}$ and

$\Rightarrow$ next part home work.

10-05-2018

Method of operational factors

Example-1

Solve $x\frac{d^2y}{dx^2}+(1-x)\frac{dy}{dx}-y=e^x$ using the method of operational factors.

Soln: Writting D for $\frac{d}{dx}$, the given equation bec

$$\{xD^2+(1-x)D-1\}y=e^x$$

$$\Rightarrow (xD^2-xD+D-1)y=e^x$$

$$\Rightarrow (xD+1)(D-1)y=e^x$$

Now, Let $(D-1)y=v$, then

$$(xD+1)v=e^x$$

$$\Rightarrow xDv+v=e^x$$

$\Rightarrow \frac{dv}{dx}+\frac{1}{x}v=\frac{e^x}{x}$. This is a 1st orde linear equation.

$I.F = e^{\int \frac{1}{x} dx} = x$. Multiplying by I.F,

we have

$$x \, dv + v = e^x$$

$$\Rightarrow \frac{d}{dx}(xv) = e^x$$

$$\Rightarrow xv = e^x + c_1$$

$$\Rightarrow v = \frac{e^x}{x} + \frac{c_1}{x}$$

$$\Rightarrow (D-1)y = \frac{e^x}{x} + \frac{c_1}{x}$$

$\Rightarrow \frac{dy}{dx} - y = \frac{e^x}{x} + \frac{c_1}{x}$ which is again a first order linear equation.

$\therefore I.F = e^{\int - dx} = e^{-x}$, Multiplying by I.F,

we have, $e^{-x}\frac{dy}{dx} - ye^{-x} = \frac{1}{x} + \frac{c_1}{x} e^{-x}$

$$\Rightarrow \frac{d}{dx}(ye^{-x}) = \frac{1}{x} + \frac{c_1}{x} e^{-x}$$

$$\therefore ye^{-x} = \int \left(\frac{1}{x} + \frac{c_1}{x} e^{-x}\right) dx$$

$$\Rightarrow ye^{-x} = \ln x + c_1 \int \frac{e^{-x}}{x} dx + c_2$$

$$\therefore y = e^x \ln x + c_1 e^x \int \frac{e^{-x}}{x} dx + c_2 e^x \quad \text{Ans}$$

Example - 2

Page - 97

Series Solution

Example - 1

Page - 146

Obtain a series solution of the differential equation

$$\frac{x d^2y}{dx^2} + \frac{dy}{dx} + xy = 0$$

Solution:

Let. $y = x^k \left[C_0 + C_1 x + C_2 x^2 + \cdots\right]$ be a sol^n

of the given equation.

Then $\frac{dy}{dx} = C_0 k x^{k-1} + C_1 (k+1) x^k + C_2 (k+2) x^{k+1} + \cdots$

and $\frac{d^2y}{dx^2} = C_0 k (k-1) x^{k-2} + C_1 (k+1) k x^{k-1} + C_2 (k+2)$

$+ \cdots$

putting the value of $y, \frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$

is the given equation, we have

$x \{ C_0 k (k-1) x^{k-2} + C_1 (k+1) k x^{k-1} + C_2 (k+2)(k+1) x^k + \cdots$

$+\{c_0 k x^{k-1} + c_1(k+1)x^k + c_2(k+2)x^{k+1} + \cdots\}$
$+ x\{c_0 x^k + c_1 x^{k+1} + c_2 x^{k+2} + \cdots\} = 0$

$\Rightarrow \{c_0(k-1)x^{k-1} + c_1(k+1)k x^k + c_2(k+2)(k+1)x^{k+1} + \cdots\}$
$+ \{c_0 k x^k + c_1(k+1)x^{k+1} + c_2(k+2)x^{k+2} + \cdots\} + \{c_0 x^{k+1} +$
$c_1 x^{k+2} + c_2 x^{k+2} + \cdots\} = 0$

$\cancel{\Rightarrow \{c_0(k-1)x^{k-1} + c_1(k+1)kx^k + c_2(k+2)}$

$\Rightarrow \{c_0(k-1)k + c_0 k\}x^{k-1} + \{c_1(k+1)k + c_1(k+1)\}x^k +$
$\{c_2(k+2)(k+1) + c_2(k-2) + c_0\}x^{k+1} + \cdots = 0$

Equating to zero co-efficient of various power of x, we get

Series Solution:

$c_0\{k(k-1)+k\} = 0 \Rightarrow k^2 = 0 \Rightarrow k = 0 \quad [\because c_0 \neq 0]$

$c_1\{(k+1)k + (k+1)\} = 0 \Rightarrow c_1 = 0$

$c_2(k+2)(k+1) + c_2(k+2) + c_0 = 0 \Rightarrow c_2 = \frac{-c_0}{4}$

$c_3(k+3)(k+2) + c_3(k+3) + c_1 = 0 \Rightarrow -\frac{c_1}{9} = 0$

$c_4(k+4)(k+3) + c_4(k+4) + c_2 = 0 \Rightarrow -\frac{c_2}{16} = -\frac{c_0}{4} \times \frac{1}{16}$

$$= \frac{-c_0}{64}$$

$$\therefore y = \left(c_0 - \frac{c_0}{4}x^2 + \frac{c_0}{64}x^4 - \cdots\cdots\right)$$

$k=0$

Other coefficient are given by

$c_1(k+1)^2 = 0 \Rightarrow c_1 = 0$

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Hence our soln for $k=0$ may be obtained putting $k=0$ in

$$y = c_0 x^k \left(1 - \frac{1}{(k+2)^2}x^2 + \frac{1}{(k+4)^2(k+2)^2}x^4 - \cdots\right) \quad (1)$$

Differentiating ① w.r.to k

$$\frac{dy}{dk} = C_0 x^k \ln x \left\{1 - \frac{1}{(k+2)^2}x^2 + \cdots\right\} + C_0 x^k \left\{\frac{2}{(k+2)^3}x^2 - \left(\frac{2}{(k+2)^2(k+3)^3} + \frac{2}{(k+4)^3(k+2)^2}\right)x^4 \cdots\right\}$$

$$= y\ln x + C_0 x^k \left\{\frac{2x^2}{(k+2)^3} - \left(\frac{2}{(k+2)^2(k+4)^3} + \frac{2}{(k+4)^3(k+2)^2}\right\}x^4 - \cdots\right\} \text{ --- } ②$$

and $\frac{dy}{dk} = c' u \ln x + c' \left[\frac{x^2}{2} - \frac{1}{2^2 4^2}\left(1+\frac{1}{2}\right)x^4 + \cdots\right]$

$= c' v$ (say)

Hence the G.S is

$$y = cu + c'v$$

14-05-2016

Partial Differential Equation:

Lagrane's Method: Consider the 1st order line

p.d.e

$$P(x,y,z)\cdot p + Q(x,y,z)\, q = R(x,y,z)$$

Then the Lagrane's auxiliary equation are

Notation.
$\frac{\delta z}{\delta x} = p$
$\frac{\delta z}{\delta y} = q$
x, y independ variable
z dependent variable

$$\frac{dx}{P(x,y,z)} = \frac{dy}{Q(x,y,z)} = \frac{dz}{R(x,y,z)} \quad \text{--- (1)}$$

which give the independent solution

$u(x,y,z) = c_1$ and

$v(x,y,z) = c_2$

Where c_1 and c_2 are arbitary constant.

The general solution of the equation is

$$\phi\left(u(x,y,z)\; v(x,y,z)\right) = 0$$

where ϕ is arbitary function.

Sheet - 12 no

Solve the P.d.e $x(y^2+z)p - y(x^2+z)q = z(x^2-y^2)$

Soln: Lagrange's auxiliary equations are

$$\frac{dx}{x(y^2+z)} = \frac{dy}{-y(x^2+z)} = \frac{dz}{z(x^2-y^2)} \quad \text{--- (1)}$$

(1) can be written as

$$\frac{dx/x}{y^2+z} = \frac{dy/y}{-(x^2+z)} = \frac{dz/z}{x^2-y^2} = \frac{dx/x + dy/y + dz/z}{y^2+z-x^2-z+x^2-y^2}$$

$$\Rightarrow \frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z} = 0$$

$$\Rightarrow \ln(xyz) = \ln c_1$$

$$\Rightarrow \boxed{xyz = c_1}$$

Again (1) can be written as

$$\frac{xdx}{x^2(y^2+z)} = \frac{ydy}{-y^2(x^2+z)} = \frac{dz}{(x^2-y^2)} = \frac{xdx+ydy-dz}{x^2(y^2+z)-y^2(x^2+z)-z(x^2-y^2)}$$

$$\Rightarrow xdx + ydy - dz = 0$$

$$\Rightarrow \frac{x^2}{2} + \frac{y^2}{2} - z = c_1'$$

$$\Rightarrow \boxed{x^2+y^2-2z = c_2}$$

Therefore the general solution of the given equation is

$\phi(xyz, x^2+y^2-2z) = 0$ where ϕ is an arbitory function

Problem: Find the integral surface of the P.d.e $x(y^2+z)p - y(x^2+z)q = z(x^2-y^2)$ which contains the stateline $x+y=0,\ z=1$

Two independent solutions are $xyz = c_1$ —— (2)

and $x^2+y^2-2z = c_2$ —— (3)

with $z=1$, the equations (2) and (3) becomes

$$xy = c_1 \text{ and } x^2+y^2-2 = c_2$$

with $x+y=0$, we have $(x+y)^2=0$

$$\Rightarrow x^2+y^2+2xy=0$$

$$\Rightarrow c_2+2+2c_1=0$$

$$\Rightarrow 2c_1+c_2+2=0$$

$$\Rightarrow 2(xyz)+(x^2+y^2-2z)+2=0$$

$$\Rightarrow x^2+y^2+2xyz-2z+2=0$$ wh

is the required integral surface.

15-06-2018

Charpit's method:

Consider a first order non-linear p.d.e

$$F(x, y, z, p, q) = 0 \quad \text{---} \ (1)$$

Charpit's auxiliary equations of (1) are given by

$$\frac{dp}{\frac{\delta F}{\delta x} + p\frac{\delta F}{\delta z}} = \frac{dq}{\frac{\delta F}{\delta y} + q\frac{\delta F}{\delta z}} = \frac{dz}{-p\frac{\delta F}{\delta p} - q\frac{\delta F}{\delta q}}$$

$$= \frac{\delta x}{-\frac{\delta F}{\delta p}} = \frac{\delta y}{-\frac{\delta F}{\delta q}}$$

Ex(1) sheet

Solve the p.d.e $px+qy=pq$ by charpit's method.

Soln: The given equation can be written as

$px+qy-pq=0$. Here $F(x, y, z, p, q)$

$= px+qy-pq$

Charpit's auxiliary equations are

$$\frac{dp}{p} = \frac{dq}{q} = \frac{dz}{-p(x-q)-q(y-p)} = \frac{dx}{-(x-q)} = \frac{dy}{-(y-p)}$$

So, $\frac{dp}{p} = \frac{dq}{q}$.

$\Rightarrow p = aq$ — (ii) where a is an ~~auxi'l~~ arbitary constant.

From (i) and (ii) we have

$$aqx + qy - aq^2 = 0$$

$$\Rightarrow q(ax+y) = aq^2$$

$$\therefore q = \frac{ax+y}{a} \quad \text{— (iii)}$$

Hence, from (ii) $p = ax+y$ — (iv)

Now, $dz = \frac{\delta z}{\delta x} dx + \frac{\delta z}{\delta y} dy$

$$adz = apdx + aqdy$$

$$\Rightarrow adz = a(ax+y)\,dx + (ax+y)\,dy$$

$$\Rightarrow adz = (ax+y)\,d\,(ax+y)$$

$$\Rightarrow az = \frac{(ax+y)^2}{2} + b,$$ where b is an arbitary constant.

Hence the complete solution of the given equation is

$$z = \frac{(ax+y)^2}{2a} + \frac{b}{a};$$

Ex-2 Solve $(p^2+q^2)x = pz$

Soln:

$$F(x,y,z,p,q) = (p^2+q^2)x - pz \quad \text{--- (1)}$$

Charpit's auxiliary equations are

$$\frac{dp}{\frac{\delta F}{\delta x} + p\frac{\delta F}{\delta z}} = \frac{dq}{\frac{\delta F}{\delta y} + q\frac{\delta F}{\delta z}} = \frac{dz}{-p\frac{\delta F}{\delta p} - \frac{q\delta F}{\delta q}}$$

$$= \frac{\delta x}{-\frac{\delta F}{\delta p}} = \frac{\delta y}{-\frac{\delta F}{\delta q}}$$

$$\Rightarrow \frac{dp}{p^2+q^2-p^2} = \frac{dq}{-pq} = \frac{dz}{-p(2px-z)-q(2qx)}$$

$$= \frac{dx}{-(2px-z)} = \frac{dy}{-2qx}$$

So, $\dfrac{dp}{q^2} = \dfrac{dq}{-pq}$

$\Rightarrow pdp = -qdq$

$\therefore p^2+q^2 = a^2$ where a is a arbitary constant —— (ii)

From (i) and (ii), we have

$$a^2x - pz = 0$$

$$\Rightarrow p = \frac{a^2x}{z} \quad \text{—— (iii)}$$

Hence, from (ii)

$$q^2 = a^2 - \frac{a^4x^2}{z^2}$$

$$\Rightarrow q = \frac{a}{z}\sqrt{z^2-a^2x^2} \quad \text{—— (iv)}$$

Now, since, $dz = \frac{\delta z}{\delta x}dx + \frac{\delta z}{\delta y}dy$

$$\Rightarrow dz = pdx + qdy$$

$$\Rightarrow dz = \frac{a^2x}{z}dx + \frac{a}{z}\sqrt{z^2-a^2x^2}$$

$$\Rightarrow zdz - a^2x\,dx = a\sqrt{z^2-a^2x^2}\,dy$$

$$\Rightarrow d(z^2-a^2x^2) = 2a\sqrt{z^2-a^2x^2}\,dy$$

$$\Rightarrow \frac{d(z^2-a^2x^2)}{\sqrt{z^2-a^2x^2}} = 2a\,dy$$

$\therefore \sqrt{z^2 - a^2x^2} = ay + b$, where b is an arbitary constant.

Hence the complete solution of the given equation is

$$z^2 = a^2x + (ay+b)^2 \quad \underline{\underline{\text{Ans}}}$$