Sets

What is a set?

- A set is a unordered collection of “objects”
  - People in a class: {Alice, Bob, Chris }
  - States in the US: {Alabama, Alaska, Virginia, … }
  - Sets can contain non-related elements: {3, a, Virginia}
  - All positive numbers less than or equal to 5: {1, 2, 3, 4, 5}

- Properties
  - Order does not matter
    - {1, 2, 3, 4, 5} is equivalent to {3, 5, 2, 4, 1}
  - Sets do not have duplicate elements
    - Consider the list of students in this class
      - It does not make sense to list somebody twice

Specifying a set

- A set "contains" the various "members" or "elements" that make up the set
  - If an element *a* is a member of (or an element of) a set S, we use then notation $a \in S$
    - $4 \in \{1, 2, 3, 4\}$
  - If not, we use the notation $a \notin S$
    - $7 \notin \{1, 2, 3, 4\}$

Often used sets

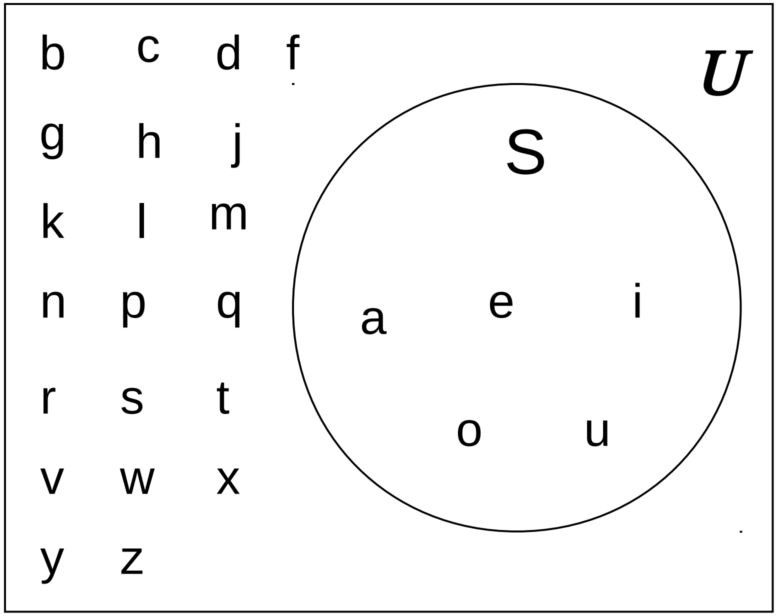
- $\mathbf{N} = \{0, 1, 2, 3, \ldots\}$ is the set of natural numbers
- $\mathbf{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$ is the set of integers
- $\mathbf{Z}^+ = \{1, 2, 3, \ldots\}$ is the set of positive integers (a.k.a whole numbers)
  - Note that people disagree on the exact definitions of whole numbers and natural numbers
- $\mathbf{Q} = \{p/q \mid p \in \mathbf{Z}, q \in \mathbf{Z}, q \neq 0\}$ is the set of rational numbers
  - Any number that can be expressed as a fraction of two integers (where the bottom one is not zero)
- $\mathbf{R}$ is the set of real numbers

The universal set

- U is the universal set – the set of all of elements (or the “universe”) from which given any set is drawn
  - For the set {-2, 0.4, 2}, U would be the real numbers
  - For the set {0, 1, 2}, U could be the **N**, **Z**, **Q**, **R** depending on the context
  - For the set of the vowels of the alphabet, U would be all the letters of the alphabet

Venn diagrams

- Represents sets graphically
  - The box represents the universal set
  - Circles represent the set(s)
- Consider set S, which is the set of all vowels in the alphabet
- The individual elements are usually not written in a Venn diagram



Sets of sets

- Sets can contain other sets
  - S = { {1}, {2}, {3} }
  - T = { {1}, {{2}}, {{{3}}} }
  - V = { {{1}, {{2}}}, {{{3}}}, { {1}, {{2}}, {{{3}}} } }
    - V has only 3 elements!
- Note that 1 ≠ {1} ≠ {{1}} ≠ {{{1}}}
  - They are all different

The Empty Set

- If a set has zero elements, it is called the empty (or null) set
  - Written using the symbol $\varnothing$
  - Thus, $\varnothing = \{ \}$ ← **VERY IMPORTANT**

- It can be a element of other sets
  - $\{ \varnothing, 1, 2, 3, x \}$ is a valid set

- $\varnothing \neq \{ \varnothing \}$
  - The first is a set of zero elements
  - The second is a set of 1 element

- Replace $\varnothing$ by $\{ \}$, and you get: $\{ \} \neq \{\{ \}\}$
  - It's easier to see that they are not equal that way

Set Equality, Subsets

- Two sets are equal if they have the same elements
  - $\{1, 2, 3, 4, 5\} = \{5, 4, 3, 2, 1\}$
  - $\{1, 2, 3, 2, 4, 3, 2, 1\} = \{4, 3, 2, 1\}$
  - Two sets are not equal if they do not have the same elements
    - $\{1, 2, 3, 4, 5\} \neq \{1, 2, 3, 4\}$

- If all the elements of a set S are also elements of a set T, then S is a subset of T
  - If $S = \{2, 4, 6\}$, $T = \{1, 2, 3, 4, 5, 6, 7\}$, S is a subset of T
  - This is specified by $S \subseteq T$ meaning that $\forall x\ (x \in S \rightarrow x \in T)$
  - For any set S, $S \subseteq S$ ($\forall S\ \ S \subseteq S$)
  - For any set S, $\varnothing \subseteq S$ ($\forall S\ \ \varnothing \subseteq S$)

Proper Subsets

- If S is a subset of T, and S is not equal to T, then S is a proper subset of T
  - Can be written as: R $\subseteq$ T and R $\not\subset$ T
  - Let T = {0, 1, 2, 3, 4, 5}
  - If S = {1, 2, 3}, S is not equal to T, and S is a subset of T
  - A proper subset is written as S $\subset$ T
  - Let Q = {4, 5, 6}. Q is neither a subset or T nor a proper subset of T

Set cardinality

- The cardinality of a set is the number of elements in a set, written as $|A|$

- Examples
  - Let $R = \{1, 2, 3, 4, 5\}$. Then $|R| = 5$
  - $|\emptyset| = 0$
  - Let $S = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$. Then $|S| = 4$

Power Sets

- Given S = {0, 1}. All the possible subsets of S?
  - ∅ (as it is a subset of all sets), {0}, {1}, and {0, 1}
  - The power set of S (written as P(S)) is the set of all the subsets of S
  - P(S) = { ∅, {0}, {1}, {0,1} }
    - Note that |S| = 2 and |P(S)| = 4
- Let T = {0, 1, 2}. The P(T) = { ∅, {0}, {1}, {2}, {0,1}, {0,2}, {1,2}, {0,1,2} }
    - Note that |T| = 3 and |P(T)| = 8
- P(∅) = { ∅ }
  - Note that |∅| = 0 and |P(∅)| = 1
- If a set has *n* elements, then the power set will have 2^n elements

Set Operations

Set operations: Union

- Formal definition for the union of two sets:
  A U B = { x | $x \in$ A **or** $x \in$ B }

- Further examples
  - {1, 2, 3} U {3, 4, 5} = {1, 2, 3, 4, 5}
  - {a, b} U {3, 4} = {a, b, 3, 4}
  - {1, 2} U ∅ = {1, 2}

- Properties of the union operation

A U ∅ = A	Identity law
A U U = U	Domination law
A U A = A	Idempotent law
A U B = B U A	Commutative law
A U (B U C) = (A U B) U C	Associative law

Set operations: Intersection

- Formal definition for the intersection of two sets:

  $A \cap B = \{ x \mid x \in A \textbf{ and } x \in B \}$

- Examples
  - $\{1, 2, 3\} \cap \{3, 4, 5\} = \{3\}$
  - $\{a, b\} \cap \{3, 4\} = \varnothing$
  - $\{1, 2\} \cap \varnothing = \varnothing$

- Properties of the intersection operation
  - $A \cap \mathcal{U} = A$ — Identity law
  - $A \cap \varnothing = \varnothing$ — Domination law
  - $A \cap A = A$ — Idempotent law
  - $A \cap B = B \cap A$ — Commutative law
  - $A \cap (B \cap C) = (A \cap B) \cap C$ — Associative law

Disjoint sets

- Formal definition for disjoint sets: two sets are disjoint if their intersection is the empty set
- Further examples
  - {1, 2, 3} and {3, 4, 5} are not disjoint
  - {a, b} and {3, 4} are disjoint
  - {1, 2} and ∅ are disjoint
    - Their intersection is the empty set
  - ∅ and ∅ are disjoint!
    - Their intersection is the empty set

Set operations: Difference

- Formal definition for the difference of two sets:
  $A - B = \{ x \mid x \in A \textbf{ and } x \notin B \}$
- Further examples
  - {1, 2, 3} - {3, 4, 5} = {1, 2}
  - {a, b} - {3, 4} = {a, b}
  - {1, 2} - $\emptyset$ = {1, 2}
    - The difference of any set S with the empty set will be the set S

Complement sets

- Formal definition for the complement of a set: $A = \{ x \mid x \notin A \} = A^c$
  - Or U – A, where U is the universal set

- Further examples (assuming $U = \mathbf{Z}$)
  - $\{1, 2, 3\}^c = \{ \ldots, -2, -1, 0, 4, 5, 6, \ldots \}$
  - $\{a, b\}^c = \mathbf{Z}$

- Properties of complement sets

$(A^c)^c = A$	Complementation law
$A \cup A^c = U$	Complement law
$A \cap A^c = \emptyset$	Complement law

Set identities

$A \cup \emptyset = A$ $A \cap U = A$	Identity Law	$A \cup U = U$ $A \cap \emptyset = \emptyset$	Domination law
$A \cup A = A$ $A \cap A = A$	Idempotent Law	$(A^c)^c = A$	Complement Law
$A \cup B = B \cup A$ $A \cap B = B \cap A$	Commutative Law	$(A \cup B)^c = A^c \cap B^c$ $(A \cap B)^c = A^c \cup B^c$	De Morgan's Law
$A \cup (B \cup C)$ $= (A \cup B) \cup C$ $A \cap (B \cap C)$ $= (A \cap B) \cap C$	Associative Law	$A \cap (B \cup C) =$ $(A \cap B) \cup (A \cap C)$ $A \cup (B \cap C) =$ $(A \cup B) \cap (A \cup C)$	Distributive Law
$A \cup (A \cap B) = A$ $A \cap (A \cup B) = A$	Absorption Law	$A \cup A^c = U$ $A \cap A^c = \emptyset$	Complement Law

How to prove a set identity

- For example: $A \cap B = B - (B - A)$
- Four methods:
  - Use the basic set identities
  - Use membership tables
  - Prove each set is a subset of each other
  - Use set builder notation and logical equivalences

Proof by Set Identities

- $A \cap B = A - (A - B)$

Proof) $A - (A - B) = A - (A \cap B^c)$

$= A \cap (A \cap B^c)^c$

$= A \cap (A^c \cup B)$

$= (A \cap A^c) \cup (A \cap B)$

$= \varnothing \cup (A \cap B)$

$= A \cap B$

Showing each is a subset of the others

✾ $(A \cap B)^c = A^c \cup B^c$

Proof) Want to prove that

$(A \cap B)^c \subseteq A^c \cup B^c$ and $(A \cap B)^c \supseteq A^c \cup B^c$

$x \in (A \cap B)^c$

$\Rightarrow x \notin (A \cap B)$

$\Rightarrow \neg (x \in A \cap B)$

$\Rightarrow \neg (x \in A \wedge x \in B)$

$\Rightarrow \neg (x \in A) \vee \neg (x \in B)$

$\Rightarrow x \notin A \vee x \notin B$

$\Rightarrow x \in A^c \vee x \in B^c$

$\Rightarrow x \in A^c \cup B^c$

Examples

- Let *A*, *B*, and *C* be sets. Show that:

a) $(A \cup B) \subseteq (A \cup B \cup C)$

b) $(A \cap B \cap C) \subseteq (A \cap B)$

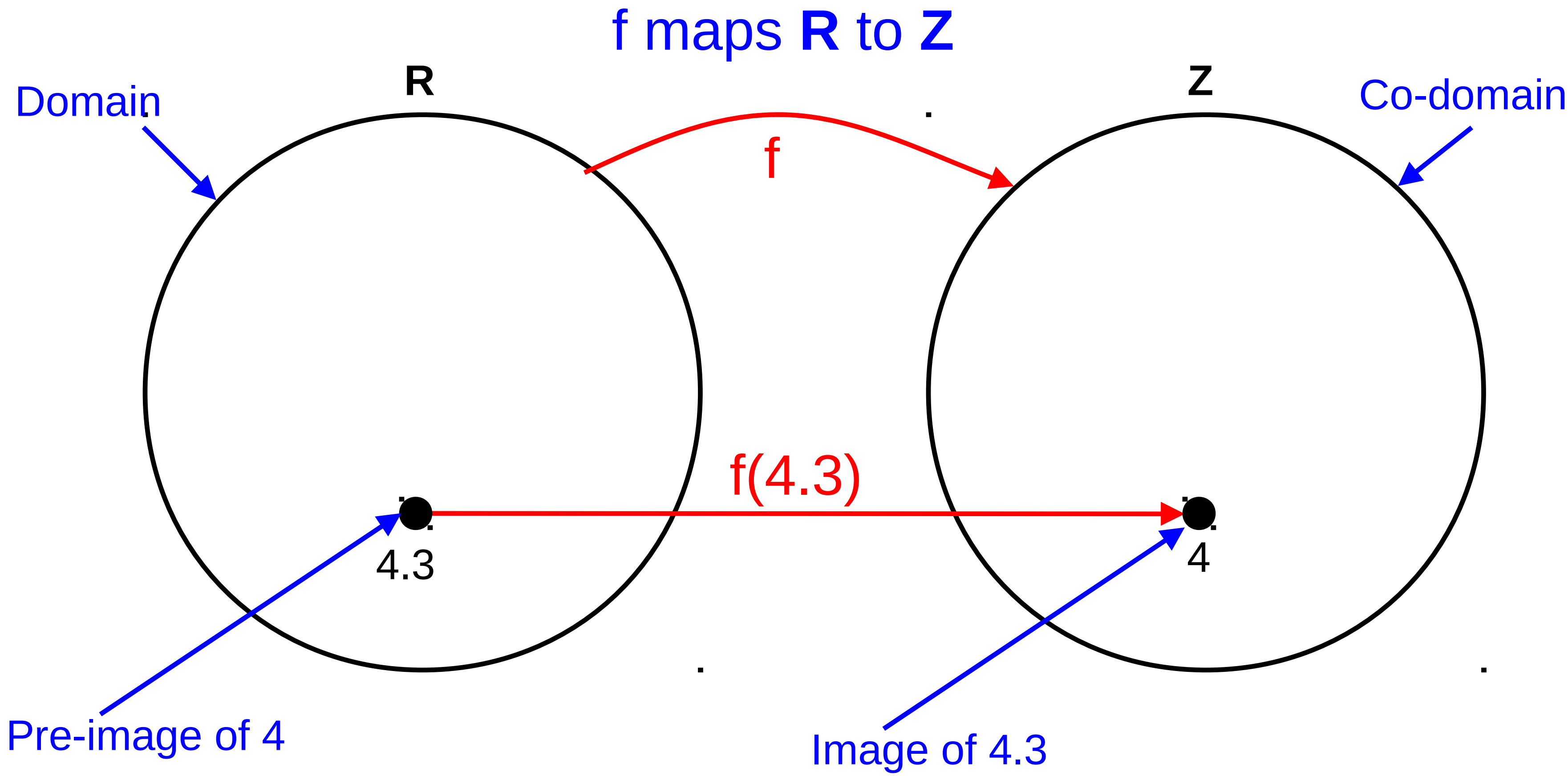
c) $(A - B) - C \subseteq A - C$

d) $(A - C) \cap (C - B) = \varnothing$

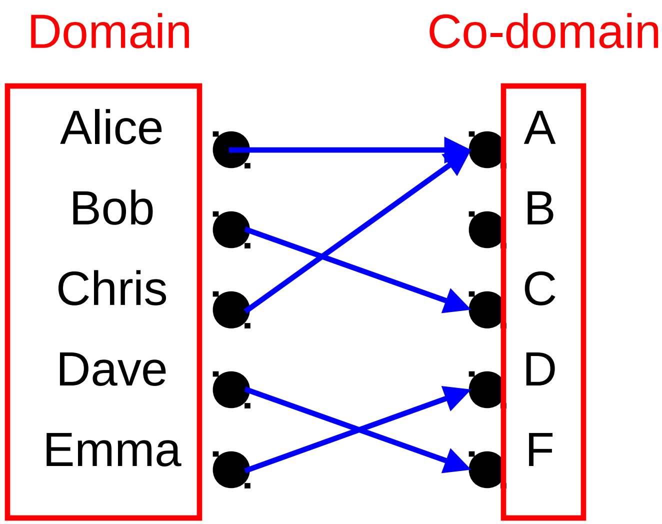
Functions

Definition of a function

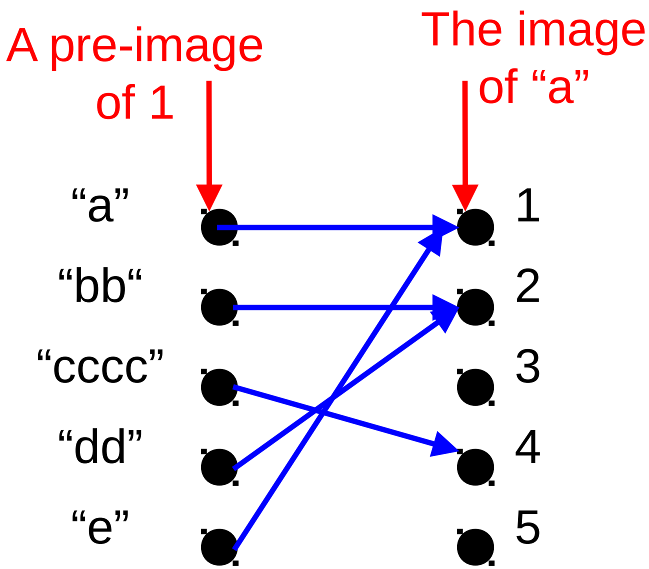
- A function takes an element from a set and maps it to a UNIQUE element in another set



More functions

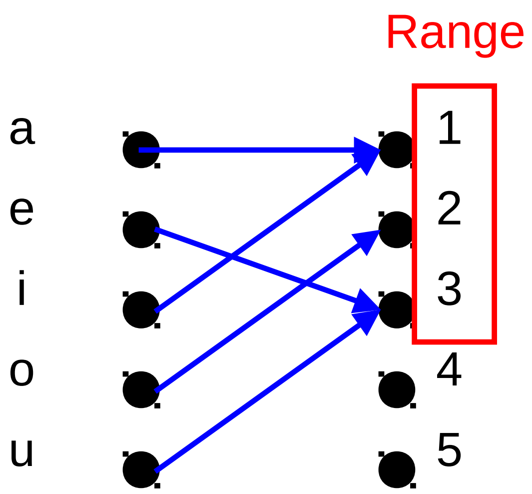


A class grade function

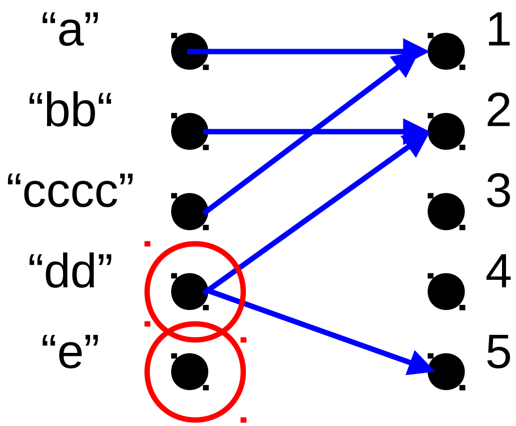


A string length function

Even more functions



Some function…



Not a valid function!
Also not a valid function!

Function arithmetic

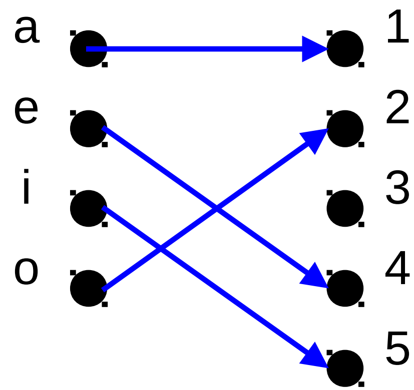
- Let $f_1(x) = 2x$
- Let $f_2(x) = x^2$

- $f_1+f_2 = (f_1+f_2)(x) = f_1(x)+f_2(x) = 2x+x^2$

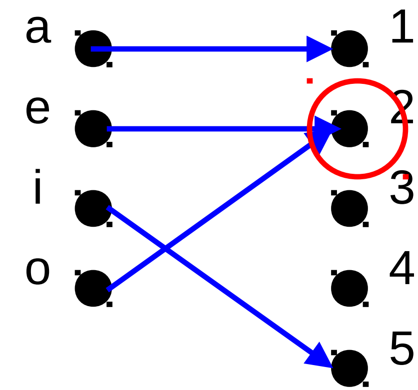
- $f_1*f_2 = (f_1*f_2)(x) = f_1(x)*f_2(x) = 2x*x^2 = 2x^3$

One-to-one functions

- A function is one-to-one if each element in the co-domain has a unique pre-image



A one-to-one function

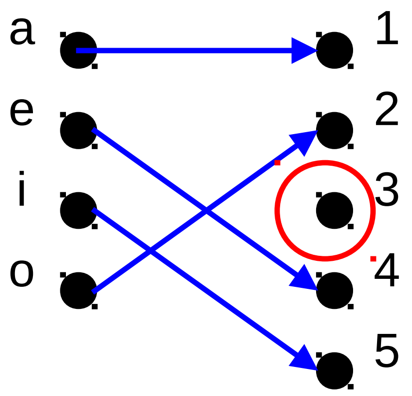


A function that is not one-to-one

More on one-to-one

- Injective is synonymous with one-to-one
  - "A function is injective"
- A function is an injection if it is one-to-one

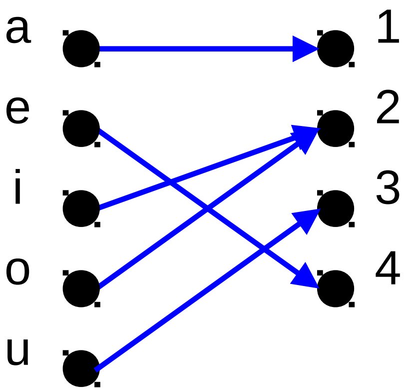
- Note that there can be un-used elements in the co-domain



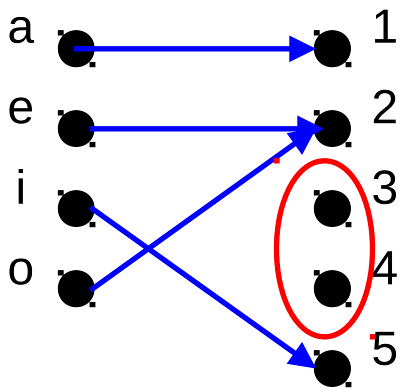
A one-to-one function

Onto functions

- A function is onto if each element in the co-domain is an image of some pre-image



An onto function

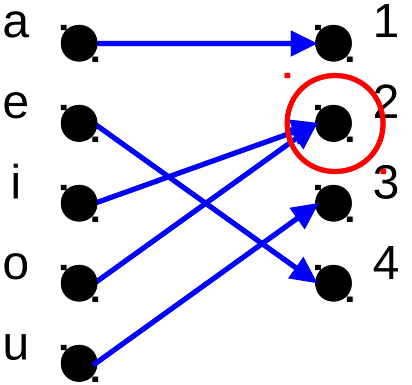


A function that is not onto

More on onto

- Surjective is synonymous with onto
  - “A function is surjective”
- A function is an surjection if it is onto

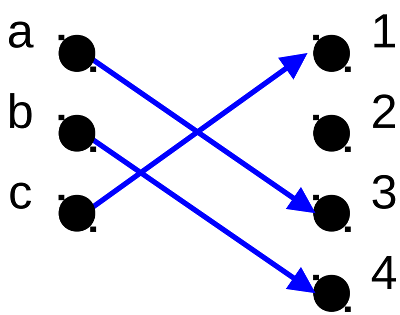
- Note that there can be multiply used elements in the co-domain



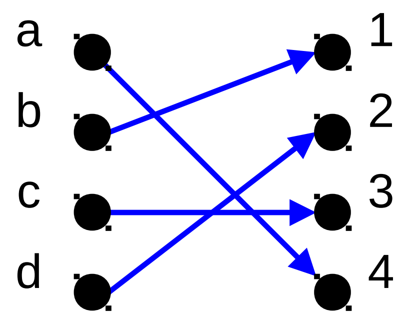
An onto function

Onto vs. one-to-one

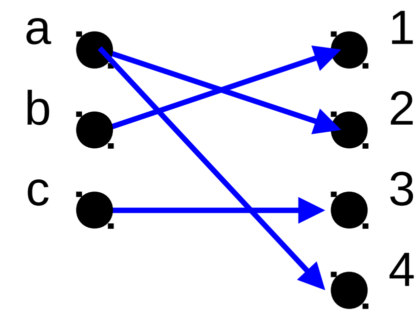
- Are the following functions onto, one-to-one, both, or neither?



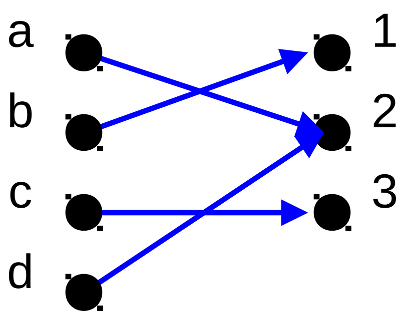
1-to-1, not onto



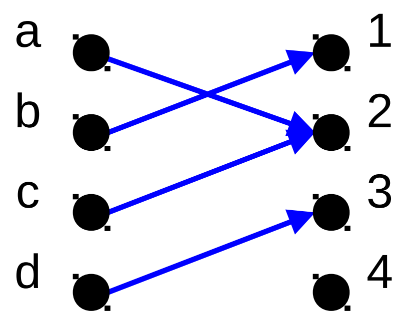
Both 1-to-1 and onto



Not a valid function



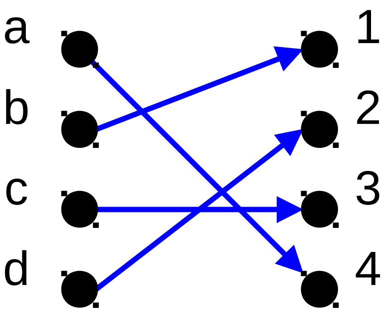
Onto, not 1-to-1



Neither 1-to-1 nor onto

Bijections

- Consider a function that is both one-to-one and onto:



- Such a function is a one-to-one correspondence, or a bijection

Identity functions

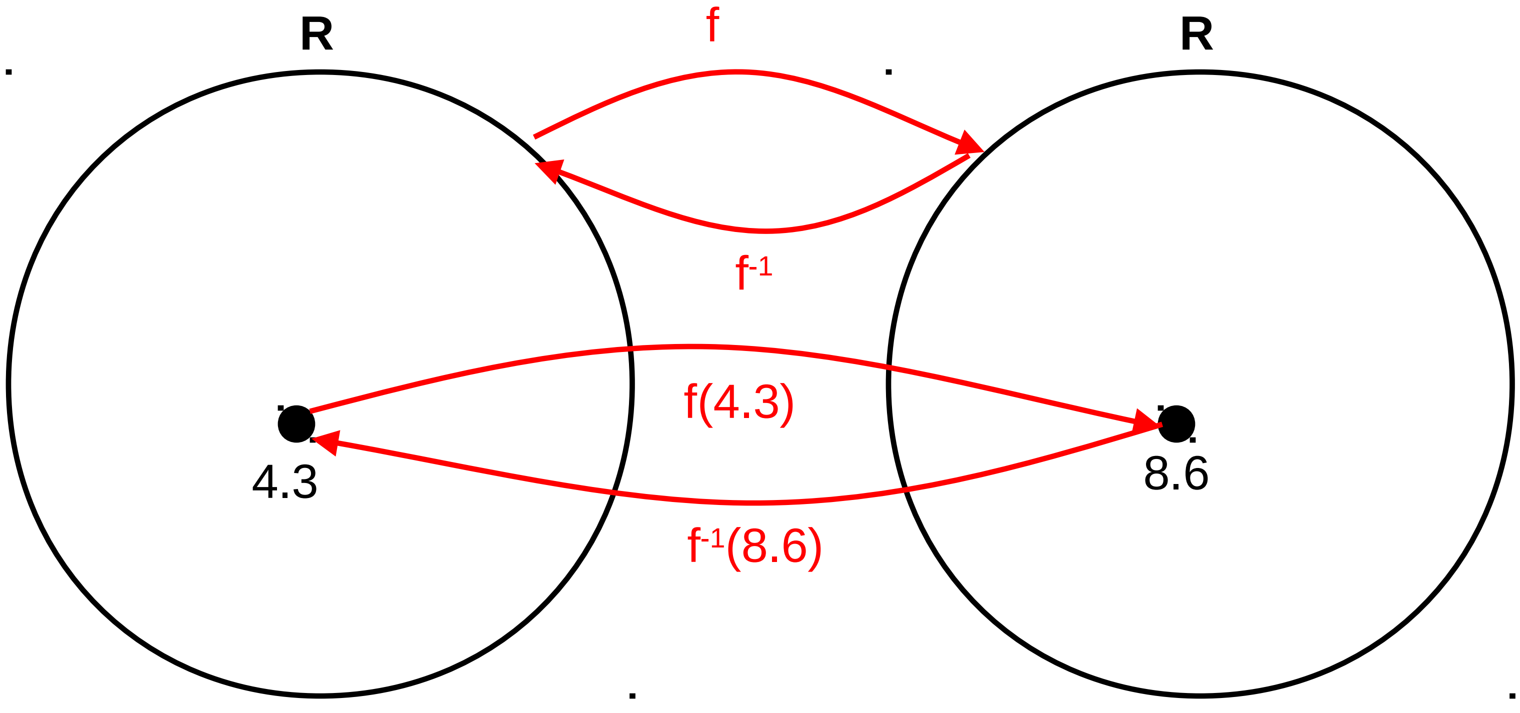
- A function such that the image and the pre-image are ALWAYS equal

- f(x) = 1*x
- f(x) = x + 0

- The domain and the co-domain must be the same set

Inverse functions

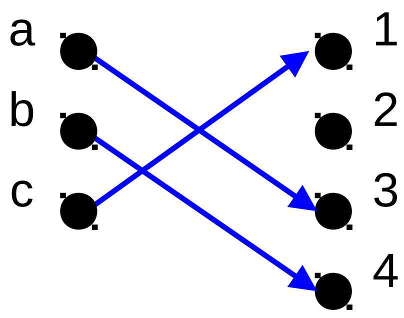
Let f(x) = 2*x



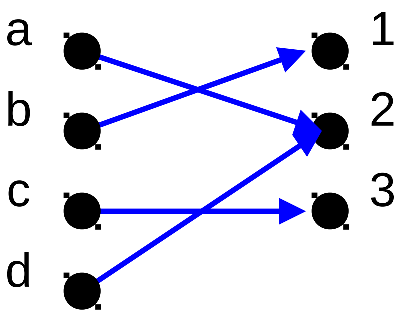
Then $f^{-1}(x) = x/2$

More on inverse functions

- Can we define the inverse of the following functions?



What is $f^{-1}(2)$?
Not onto!



What is $f^{-1}(2)$?
Not 1-to-1!

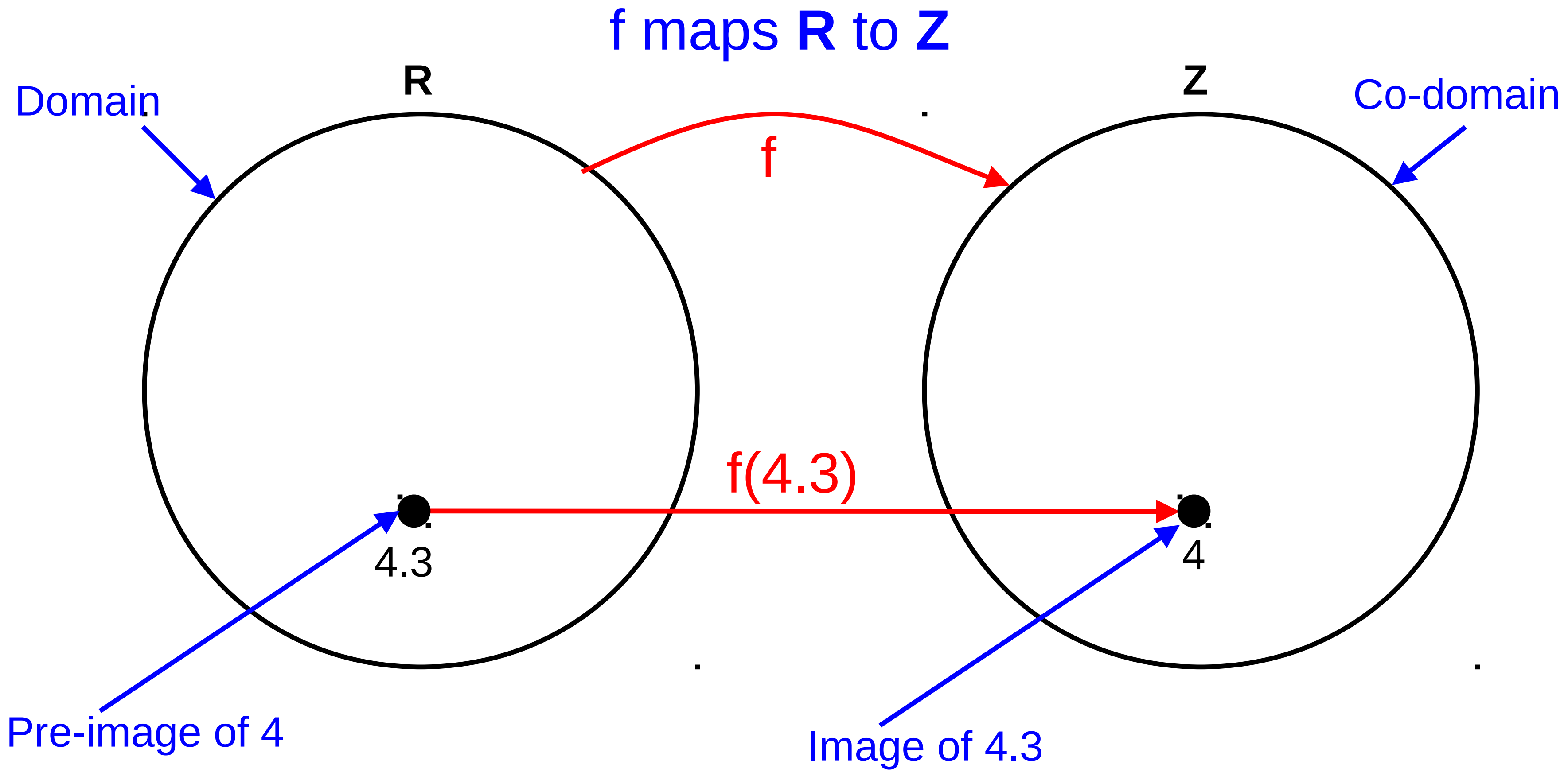
- An inverse function can ONLY be done defined on a bijection

Few Examples

- f: $\mathbf{Z} \rightarrow \mathbf{Z}$
  - $f(x) = x$
  - $f(x) = 2x$
  - $f(x) = x+1$
- f: $\mathbf{R} \rightarrow \mathbf{R}$
  - $f(x) = 2x$
  - $f(x) = x^2$
  - $f(x) = x^3$
- f: $\mathbf{R} \rightarrow \mathbf{R}^+ \cup \{0\}$
  - $f(x) = x^2$

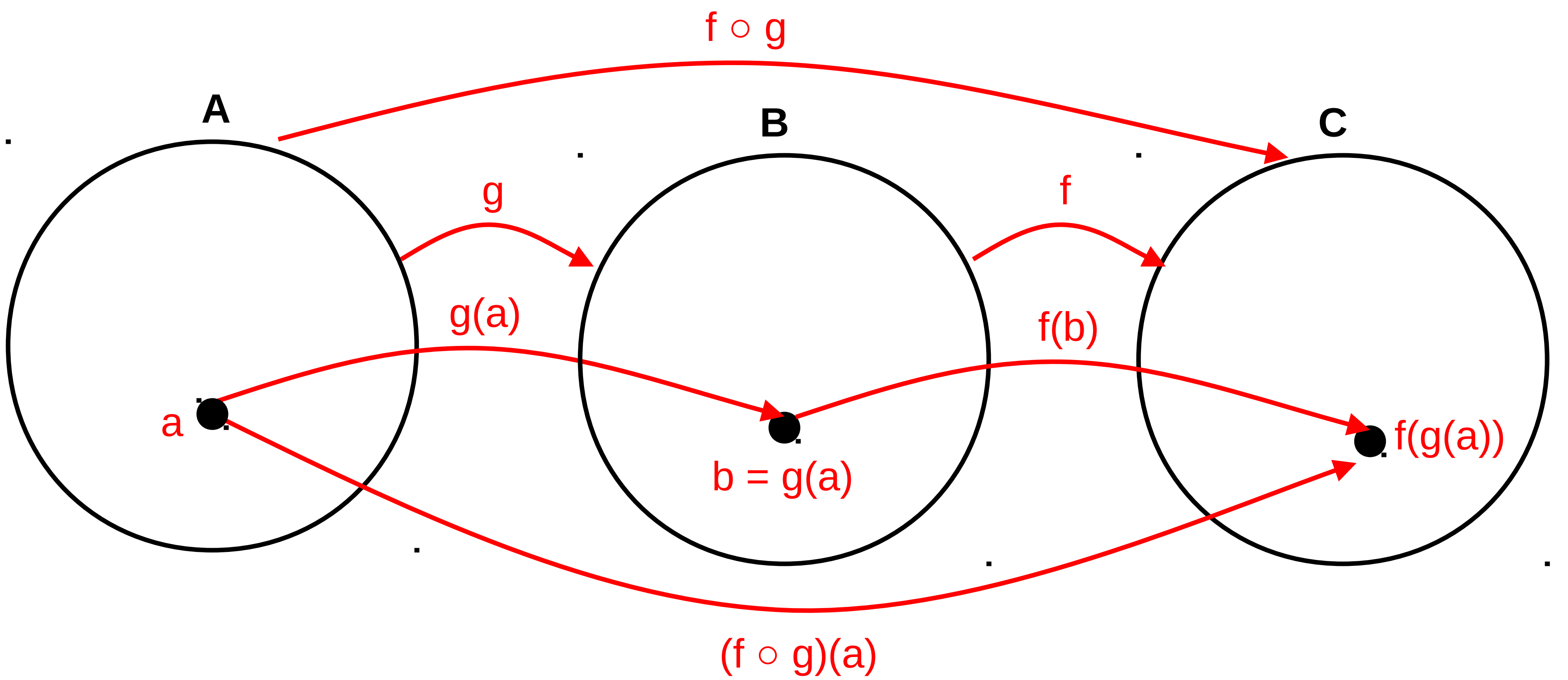
Definition of a function

- A function takes an element from a set and maps it to a UNIQUE element in another set



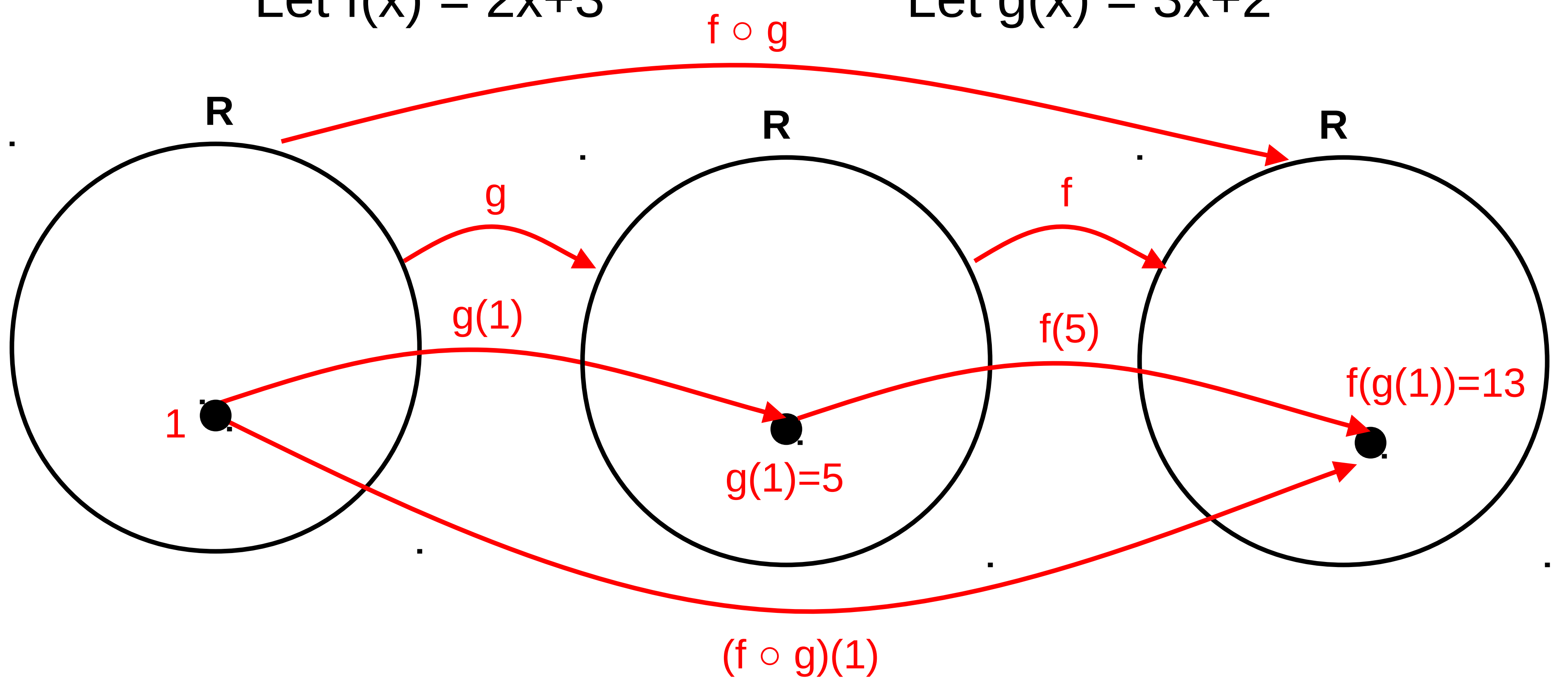
Compositions of functions

$$(f \circ g)(x) = f(g(x))$$



Compositions of functions

Let f(x) = 2x+3 Let g(x) = 3x+2



f(g(x)) = 2(3x+2)+3 = 6x+7

Compositions of functions

Does $f(g(x)) = g(f(x))$?

Let $f(x) = 2x+3$

Let $g(x) = 3x+2$

$f(g(x)) = 2(3x+2)+3 = 6x+7$

$g(f(x)) = 3(2x+3)+2 = 6x+11$

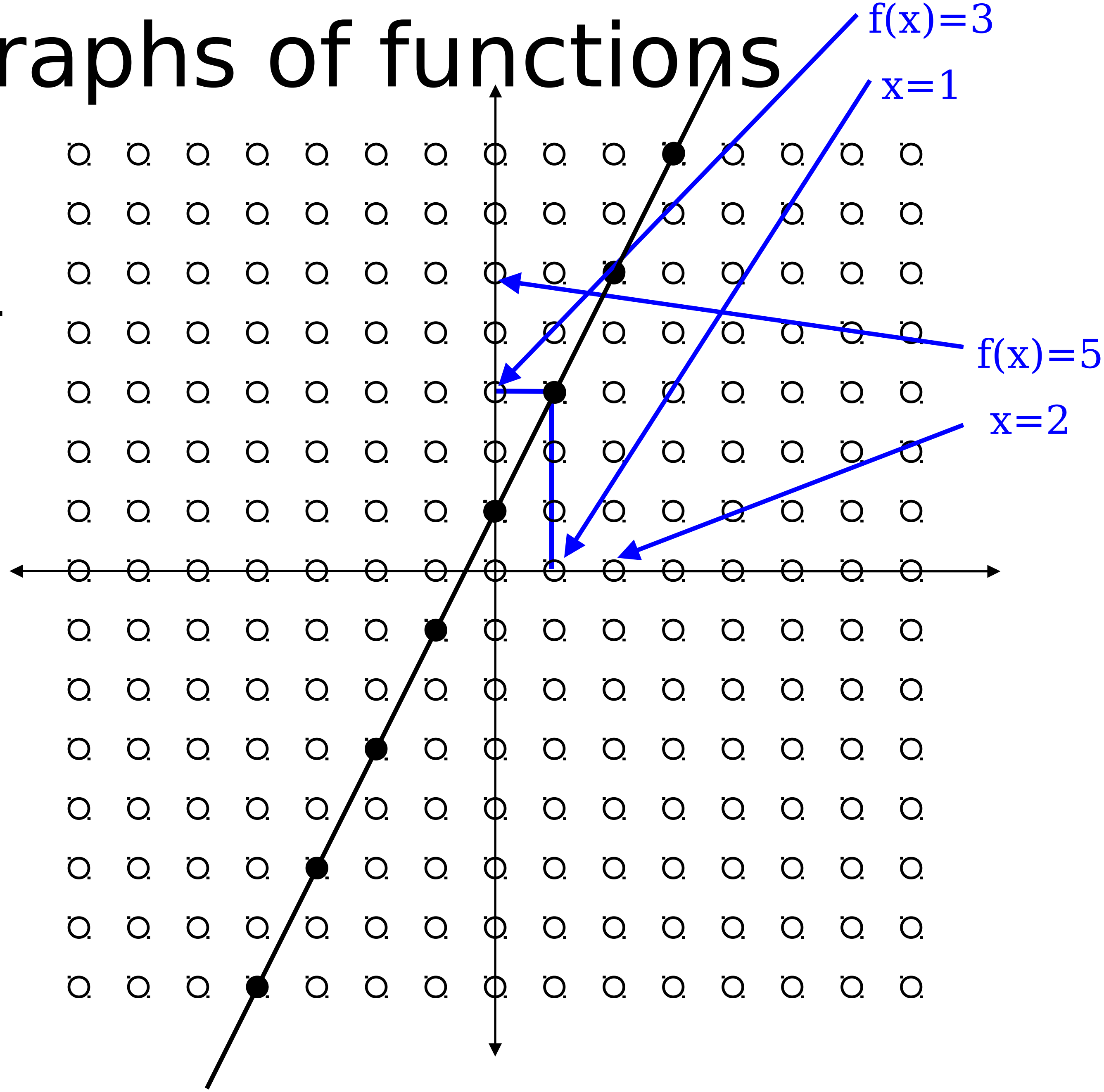
Not equal!

Function composition is not commutative!

Graphs of functions

Let f(x)=2x+1

Plot (x, f(x))



This is a plot of f(x)

Useful functions

- Floor: $\lfloor x \rfloor$ means take the greatest integer less than or equal to the number
- Ceiling: $\lceil x \rceil$ means take the lowest integer greater than or equal to the number
- round(x) = $\lfloor x+0.5 \rfloor$

Floor, Ceiling Examples

- Find these values

- $\lfloor 1.1 \rfloor$ 1
- $\lceil 1.1 \rceil$ 2
- $\lfloor -0.1 \rfloor$ -1
- $\lceil -0.1 \rceil$ 0

Ceiling and floor properties

Let n be an integer

(1a) $\lfloor x \rfloor = n$ if and only if $n \le x < n+1$

(1b) $\lceil x \rceil = n$ if and only if $n-1 < x \le n$

(1c) $\lfloor x \rfloor = n$ if and only if $x-1 < n \le x$

(1d) $\lceil x \rceil = n$ if and only if $x \le n < x+1$

(2) $x-1 < \lfloor x \rfloor \le x \le \lceil x \rceil < x+1$

(3a) $\lfloor -x \rfloor = -\lceil x \rceil$

(3b) $\lceil -x \rceil = -\lfloor x \rfloor$

(4a) $\lfloor x+n \rfloor = \lfloor x \rfloor + n$

(4b) $\lceil x+n \rceil = \lceil x \rceil + n$

Ceiling property proof

- Prove rule 4a: $\lfloor x+n \rfloor = \lfloor x \rfloor + n$
  - Where n is an integer
  - Will use rule 1a: $\lfloor x \rfloor = n$ if and only if $n \leq x < n+1$
- Direct proof!
  - Let $m = \lfloor x \rfloor$
  - Thus, $m \leq x < m+1$ (by rule 1a)
  - Add n to both sides: $m+n \leq x+n < m+n+1$
  - By rule 4a, $m+n = \lfloor x+n \rfloor$
  - Since $m = \lfloor x \rfloor$, $m+n$ also equals $\lfloor x \rfloor + n$
  - Thus, $\lfloor x \rfloor + n = m+n = \lfloor x+n \rfloor$

Factorial

- Factorial is denoted by n!
- n! = n * (n-1) * (n-2) * … * 2 * 1
- Thus, 6! = 6 * 5 * 4 * 3 * 2 * 1 = 720
- Note that 0! is defined to equal 1

Proving Function problems

- Let *f* be an invertible function from Y to Z
- Let *g* be an invertible function from X to Y
- Show that the inverse of $f \bigcirc g$ is:
  - $(f \bigcirc g)^{-1} = g^{-1} \bigcirc f^{-1}$

(Pf) Thus, we want to show, for all $z \in Z$ and $x \in X$

$((f \circ g) \circ (g^{-1} \circ f^{-1}))\ (x) = x$ and $((f^{-1} \circ g^{-1}) \circ (g \circ f))\ (z) = z$

$$
\begin{aligned}
((f \circ g) \circ (g^{-1} \circ f^{-1}))\ (x) &= (f \circ g)\ ((g^{-1} \circ f^{-1}))\ (x)) \\
&= (f \circ g)\ (g^{-1}\ (f^{-1}(x))) \\
&= (f\ (g\ (g^{-1}\ (f^{-1}(x))))) \\
&= (f\ (f^{-1}(x)) \\
&= x
\end{aligned}
$$

The second equality is similar