11.02.

Study of χ^2 Distribution: (Chi-square)

$X \sim N(\mu, \sigma^2)$ $\quad f(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$; $-\infty < x < \infty$

$z = \frac{x-\mu}{\sigma}$

$f(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}$; $-\infty < z < \infty$

$\left\{ \begin{array}{l} \mu = 0 \\ \sigma = 1 \end{array} \right.$ standard normal distribution

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Chi-square variate:- The square of a standard normal variate is known as chi-square variate with (1) d.f. Thus if $X \sim N(\mu, \sigma^2)$ then $z = \frac{x-\mu}{\sigma}$ is $N(0,1)$ and $z^2 = \left(\frac{x-\mu}{\sigma}\right)^2$ ~~is~~ is a chi-square variate with 1 d.f.

In general if x_i, $(i = 1, 2, \ldots n)$ are n independent normal variate with mean μ_i and variance σ_i^2, $(i = 1, 2, \ldots, n)$, then $\chi^2 = \sum_{i=1}^{n} \left(\frac{x_i - \mu_i}{\sigma_i}\right)^2$ is a chi-square variate with n d.f.

Derivation of the chi square distribution.

Statement:- If $x_i (i = 1, 2, \ldots, n)$ are independent $N(\mu_i, \sigma_i^2)$ variate then the quantity $\chi^2 = \sum_{i=1}^{n} \left(\frac{x_i - \mu_i}{\sigma_i}\right)^2$ is distributed as

P.d.f → probability density function

χ^2 with $\underline{\underline{n}}$ d.f.

Proof: Let, $z_i = \frac{x_i - \mu_i}{\sigma_i}$, $i = 1, 2, \dots, n$.

Since, $x_i \sim N(\mu_i, \sigma_i^2)$; $z_i \sim N(0, 1)$ → mean, variance

$$M_{\chi^2}(t) = M_{\sum_{i=1}^{n} \left(\frac{x_i - \mu_i}{\sigma_i}\right)^2}(t)$$

$$= M_{\sum_{i=1}^{n} z_i^2}(t)$$

Ref

$M_x(t) = E(e^{tx})$

$= \int_{-\infty}^{\infty} e^{tx} f(x)\, dx$

$\int_0^{\infty} e^{-ax^2} x^{2n-1}\, dx = \frac{a^{-n}\Gamma n}{2}$

$\int_0^{\infty} e^{-ax} x^{n-1}\, dx = a^{-n}\Gamma n$

$\int_0^{\infty} e^{-x} x^{n-1}\, dx = \Gamma n$

Now, $M_{z_i^2}(t) = E\left(e^{tz_i^2}\right)$

$$= \int_{-\infty}^{\infty} e^{tz_i^2} f(z_i)\, dz_i$$

$$= \int_{-\infty}^{\infty} e^{tz_i^2} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2} z_i^2}\, dz_i$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\left(\frac{1-2t}{2}\right) z_i^2}\, dz_i$$

$$= \frac{1}{\sqrt{2\pi}}\, 2 \int_0^{\infty} e^{-\left(\frac{1-2t}{2}\right) z_i^2} (z_i)^{2\left(\frac{1}{2}\right)-1}\, dz_i$$

$$= \frac{1}{\sqrt{2\pi}} \, 2 \, \frac{\left(\frac{1-2t}{2}\right)^{-\frac{1}{2}} \Gamma\frac{1}{2}}{2}$$

$$\Gamma(n) = (n-1)!$$

$$= \frac{1}{\sqrt{2\pi}} \left(\frac{2}{1-2t}\right)^{\frac{1}{2}} \sqrt{\pi}$$

$$= \frac{1}{\sqrt{2\pi}} \frac{\sqrt{2}\sqrt{\pi}}{(1-2t)^{1/2}}$$

$$\Rightarrow M_{Z_i^2}(t) = (1-2t)^{-\frac{1}{2}}.$$

$$\therefore M_{\chi^2}(t) = \prod_{i=1}^{n} M_{Z_i^2}(t)$$

$$= \prod_{i=1}^{n} (1-2t)^{-\frac{1}{2}}$$

$$= (1-2t)^{-\frac{n}{2}}.$$

p.d.f gamma $f(x) = \frac{e^{-ax} x^{n-1}}{a^n \Gamma n}$ $0 < x < \infty$

~~Parameter~~ Which is the m.g.f of a Gamma variate with parameter $\frac{1}{2}$ and $\frac{n}{2}$.

[Since, $f(x) = \frac{\lambda^n e^{-\lambda x} x^{n-1}}{\Gamma n}$ is a pdf of a Gamma variate with parameter λ and n.

$$f(\chi^2) = \frac{\left(\frac{1}{2}\right)^{n/2}}{\Gamma(n/2)} e^{-\frac{1}{2}\chi^2} (\chi^2)^{\frac{n}{2}-1}.$$

$$= \frac{e^{-\frac{x^2}{2}} (x^2)^{n/2 - 1}}{2^{\frac{n}{2}} \Gamma \frac{n}{2}} ; \quad 0 \leq x^2 < \alpha$$

which is the p.d.f of a x^2 variate with n df.

(degree of freedom)

<u>Mean:</u> $E(x^2) = \int_0^{\alpha} x^2 f(x) \, dx^2$

$$= \int_0^{\alpha} x^2 \frac{1}{2^{\frac{n}{2}} \Gamma(\frac{n}{2})} e^{-\frac{x^2}{2}} (x^2)^{n/2 - 1} dx^2$$

$$= \frac{1}{2^{\frac{n}{2}} \Gamma \frac{n}{2}} \int_0^{\alpha} e^{-\frac{x^2}{2}} (x^2)^{(\frac{n}{2}+1)-1} d x^2$$

$$= \frac{1}{2^{\frac{n}{2}} \Gamma \frac{n}{2}} \left(\frac{1}{2}\right)^{-(\frac{n}{2}+1)} \Gamma\left(\frac{n}{2}+1\right)$$

$$= \frac{1}{2^{\frac{n}{2} - \frac{n}{2} - 1} \Gamma \frac{n}{2}} \left(\frac{n}{2}\right) \Gamma\left(\frac{n}{2}\right)$$

$= n \rightarrow \boxed{\text{degrees of freedom}}$

1.d.f P.d.f

$$f(\chi^2) = \frac{e^{-\frac{x^2}{2}} (\chi^2)^{\frac{1}{2}-1}}{2^{\frac{1}{2}} \Gamma\frac{1}{2}}$$

$$= \frac{e^{-\frac{x^2}{2}} (x^2)^{-\frac{1}{2}}}{2^{\frac{1}{2}} \sqrt{\pi}}$$

$$= \frac{e^{-\frac{x^2}{2}} (x^2)^{-\frac{1}{2}}}{\sqrt{2\pi}}$$

chi square distribution and its means depends on degrees of freedom.

$$\boxed{\int_0^{\infty} f(x^n)\, d x^2 = 1}$$

$$\text{L.H.S} = \int_0^{\infty} \frac{1}{2^{n/2}\Gamma n/2} e^{-\frac{x^2}{2}} (x^2)^{\frac{n}{2}-1} d x^2 =$$

$$= \frac{1}{2^{\frac{n}{2}} \Gamma\frac{n}{2}} \int_0^{\infty} e^{-\frac{x^2}{2}} (x^2)^{\frac{n}{2}-1} d x^2.$$

2

Ref

$$E(x) = \int_x x f(x)\, dx$$

$$E(x^2) = \int_x x^2 f(x)\, dx$$

$$M_x(t) = E(e^{tx}) = \int_x e^{tx} f(x)\, dx$$

$$\int_0^{\infty} e^{-ax^2} x^{2n-1} dx = \frac{a^{-n}\Gamma n}{2}$$

$$\beta(m,n) = \beta(n,m)$$

$$\Gamma\frac{1}{2} = \sqrt{\pi}$$

$$\beta(m,n) = \frac{\Gamma m \Gamma n}{\Gamma m+n}$$

$$\Gamma n = (n-1)! = (n-1)\Gamma(n-1)$$

$$= (n-1)(n-2)(n-3) \ldots 3.2.1$$

$$\boxed{\lambda = \frac{1}{2}, n = \frac{1}{2}}$$

$$f(x) = \frac{\lambda^n e^{-\lambda x} x^{n-1}}{\Gamma n}$$

p.d.f of gamma variate. λ, n

$$= \frac{(\frac{1}{2})^{1/2} e^{-\frac{1}{2}x} x^{\frac{1}{2}-1}}{\Gamma\frac{1}{2}}$$

$$\int_0^{\infty} e^{-ax} x^{n-1} dx = a^{-n}\Gamma n$$

$$= \frac{1}{2^{\frac{n}{2}} \sqrt{\frac{n}{2}}} \left(\frac{1}{2}\right)^{-\frac{n}{2}} \sqrt{\frac{n}{2}}$$

$$= \frac{1}{2^{\frac{n}{2}-\frac{n}{2}}} = 1.$$

We know that Variance $= E[(x^2)^2] - [E(x^2)]^2$

Now, $E[(x^2)^2] = \int_0^\alpha (x^2)^2 f(x^2)\, dx^2$

$$= \int_0^\alpha (x^2)^2 \frac{1}{2^{\frac{n}{2}} \Gamma \frac{n}{2}} e^{-\frac{x^2}{2}} (x^2)^{\frac{n}{2}-1} dx^2$$

$$= \frac{1}{2^{\frac{n}{2}} \Gamma \frac{n}{2}} \int_0^\alpha e^{-\frac{x^2}{2}} (x^2)^{(\frac{n}{2}+2)-1} dx^2$$

$$= \frac{1}{2^{\frac{n}{2}-\frac{n}{2}-2} \Gamma \frac{n}{2}} \left(\frac{1}{2}\right)^{-(\frac{n}{2}+2)} \Gamma\left(\frac{n}{2}+2\right)$$

$$= \frac{1}{2^{\frac{n}{2}-\frac{n}{2}-2} \Gamma \frac{n}{2}} \left(\frac{n}{2}+1\right) \frac{n}{2} \Gamma \frac{n}{2}.$$

$$= \frac{1}{2^{-2}} \left(\frac{n}{2}+1\right) \frac{n}{2} = 4 \left(\frac{n+2}{2}\right) \frac{n}{2}$$

$$\therefore E[(x^2)^2] = n(n+2)$$

$\therefore Var(\chi^2) = E[(\chi^2)^2] - [E(\chi^2)]^2$

$= n(n+2) - (n)^2$

$= 2n.$

$\boxed{5}$ Mode of the χ^2 distribution:

The mode is that value of x

M.G.F of χ^2 distribution:

$M_{\chi^2}(t) = E[e^{t\chi^2}]$

$= \int_0^\infty e^{t\chi^2} f(\chi^2)\, d\chi^2.$

$=$

$= \frac{1}{2^{\frac{n}{2}} \Gamma\frac{n}{2}} \int_0^\infty e^{-\left(\frac{1-2t}{2}\right)\chi^2} (\chi^2)^{\frac{n}{2}-1} d\chi^2.$

$= \frac{1}{2^{\frac{n}{2}} \Gamma\frac{n}{2}} \left(\frac{1-2t}{2}\right)^{-\frac{n}{2}} \Gamma\frac{n}{2}.$

$\int_0^\infty e^{-ax} x^{n-1} dx = a^{-n} \Gamma n.$

$= (1-2t)^{-\frac{n}{2}}$

which is m.g.f. of χ^2 distribution.

First four raw and central moments are as follows:

$$\mu_1' = \frac{\delta M_{\chi^2}(t)}{\delta t}\Big|_{t=0}$$

$$= \frac{\delta}{\delta t}(1-2t)^{-\frac{n}{2}}\Big|_{t=0}$$

$$= -\frac{n}{2}(1-2t)^{-\frac{n}{2}-1}(-2)\Big|_{t=0}$$

$$= n(1-2t)^{-(\frac{n}{2}+1)}\Big|_{t=0}$$

$$= n.$$

$$\mu_2' = \frac{\delta^2 M_{\chi^2}(t)}{\delta t^2}\Big|_{t=0}$$

$$= \frac{\delta}{\delta t}\left(\frac{\delta M_{\chi^2}(t)}{\delta t}\right)\Big|_{t=0}$$

$$= \frac{\delta}{\delta t}\left[n(1-2t)^{-(\frac{n}{2}+1)}\right]\Big|_{t=0}.$$

$$= -n\left(\frac{n}{2}+1\right)(1-2t)^{-(\frac{n}{2}+2)}(-2)\Big|_{t=0}$$

$$= \cancel{n^2+n}\; n(n+2)(1-2t)^{(-\frac{n}{2}+2)}\Big|_{t=0}$$

$$= n(n+2).$$

μ' - raw moment
μ - central mom

$\therefore \mu_2 = \mu_2' - (\mu_1')^2$.
$= n(n+2) - (n)^2$
$= 2n$;

Now, $\mu_3' = \frac{\delta^3 M_{x^2}(t)}{\delta t^3} \Big|_{t=0}$

$= \frac{\delta}{\delta t} \left(\frac{\delta^2 M_{x^2}(t)}{\delta t^2} \right) \Big|_{t=0}$

$= \frac{\delta}{\delta t} \left[n(n+2)(1-2t)^{-(\frac{n}{2}+2)} \right] \Big|_{t=0}$

$= -n(n+2)(\frac{n}{2}+2)(1-2t)^{-(\frac{n}{2}+3)} \cdot (-2) \Big|_{t=0}$

$= n(n+2)(n+4)(1-2t)^{-(\frac{n}{2}+3)} \Big|_{t=0}$

$= n(n+2)(n+4)$

$\therefore \mu_3 = \mu_3' - 3\mu_2' \mu_1' + 2(\mu_1')^3$.

$= n(n+2)(n+4) - 3n(n+2).n + 2n^3$.

$= n(n^2+6n+8) - 3(n^3+2n^2) + 2n^3$.

$= \underline{n^3} + \underline{6n^2} + 8n - \underline{3n^3} - \underline{6n^2} + \underline{2n^3}$

$= 8n$.

Now, $\mu_4' = \dfrac{\delta^4 M_{x^2}(t)}{\delta t^4}\Bigg|_{t=0}$

$$= \frac{\delta}{\delta t}\left(\frac{\delta^3 M_{x^2}(t)}{\delta t^3}\right)\Bigg|_{t=0}$$

$$= \frac{\delta}{\delta t}\left[n(n+2)(n+4)(1-2t)^{-(\frac{n}{2}+3)}\right]\Bigg|_{t=0}$$

$$= -n(n+2)(n+4)\left(\frac{n+6}{2}\right)(1-2t)^{-(\frac{n}{2}+4)}(-2)\Bigg|_{t=0}$$

$$= n(n+2)(n+4)(n+6)(1-2t)^{-(\frac{n}{2}+4)}\Bigg|_{t=0}$$

$$= n(n+2)(n+4)(n+6)$$

$$\mu_4 = \mu_4' - 4\mu_3'\mu_1' + 6\mu_2'\mu_1' - 3\mu_1'^4$$

$$= n(n+2)(n+4)(n+6) - 4n(n+2)(n+4)n + 6n(n+2)n^2 - 3n^4.$$

$=$

$$= 12n^2 + 48n.$$

Coefficient of skewness

$$\beta_1 = \frac{\mu_3^2}{\mu_2^3} = \frac{(8n)^2}{(2n)^3} = \frac{64n^2}{8n^3} = \frac{8}{n}$$

$$\therefore \beta_2 = \frac{\mu_4}{\mu_2^2} = \frac{12n^2+48n}{(2n)^2} = \frac{12n^2+48n}{4n^2} = 3 + \frac{12}{n} > 3$$

C.G.F. of χ^2 distribution. (cumulant

$$\phi_{\chi^2}(t) = \log M_{\chi^2}(t) = \log (1-2t)^{-\frac{n}{2}}$$

$$= -\frac{n}{2}\log (1-2t)^{-\frac{n}{2}}$$

$$= -\frac{n}{2}\log(1-2t).$$

$$= -\frac{n}{2}\left(-2t - \frac{4t^2}{2} - \frac{8t^3}{3} - \frac{16t^4}{4} - \dots\right).$$

$$= \frac{n}{2}\left(2t + \frac{4t^2}{2} + \frac{8t^3}{3} + \frac{16t^4}{4} + \dots\right).$$

$$= nt + nt^2 + \frac{4}{3}nt^3 + 2nt^4 + \dots$$

kappa K_1 = Coefficient of $\frac{t}{1!} = n$.

K_2 = Co -- of $\frac{t^2}{2!} = 2n$

K_3 = Co -- of $\frac{t^3}{3!} = 8n$

K_4 = Co -- $\frac{t^4}{4!} = 48n$.

$K_1 = \mu_1' = n$

$K_2 = \mu_2 = 2n$

$K_3 = \mu_3 = 8n$.

$K_4 = \mu_4 - 3K_2^2$.

$\Rightarrow \mu_4 = K_4 + 3K_2^2$

$= 48n + 3(2n)^2 = 48n + 12n^2$.

Normal M.G.F

$e^{\mu t + \frac{1}{2}\sigma^2 t^2}$.

mean $\mu = 0$, variance $\sigma = 1$ | $e^{t^2/2}$.

Limiting form of χ^2 distribution for large degrees of freedom or, χ^2 distribution

$$Z = \frac{\chi^2 - n}{\sqrt{2n}} = \sqrt{\frac{n}{2}}$$

$$M_{a+hx}(t) = e^{at} M_x(ht)$$

23.02.

Property of χ^2 distribution:

$$\int_0^\infty e^{-ax} x^{n-1}\, dx = a^{-n}\,\Gamma n.$$

If x has a density function $f(x) = e^{-x}$, $x>0$ then $2x$ follows χ^2 distribution with 2 degrees of freedom (d.f). The m.g.f. of $2x$ is $M_{2x}(t) = E(e^{2xt})$

$$= \int_0^\infty e^{2xt} f(x)\, dx.$$

$$= \int_0^\infty e^{2xt} e^{-x}\, dx.$$

$$= \int_0^\infty e^{-x(1-2t)}\, dx.$$

$$= \int_0^\infty e^{-(1-2t)x} x^{1-1}\, dx.$$

$$= (1-2t)^{-1}\,\Gamma 1.$$

$\therefore M_{2x}(t) = (1-2t)^{-1}$ which is the m.g.f of χ_2^2 variate.

Hence $2x$ is distributed as χ^2 distribution with 2 d.f

$$f(\chi^2) = \frac{e^{-\frac{\chi^2}{2}} (\chi^2)^{\frac{2}{2}-1}}{2^{\frac{2}{2}}\,\Gamma\frac{2}{2}} = \frac{e^{-\frac{\chi^2}{2}} (\chi^2)^0}{2\,\Gamma 1}$$

$$= \frac{e^{-\frac{\chi^2}{2}}}{2}, \quad \chi^2 \in (0, \infty)$$

If $X \to N(0,1)$ then $X^2 \to \chi^2$ distribution with 1 d.f.

Proof:

Since $X \to N(0,1)$ $\therefore f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$ $-\infty < X < \infty$

Now the m.g.f of X^2 is $M_{X^2}(t) = E(e^{tX^2})$

$$= \int_{-\infty}^{\infty} e^{tx^2} f(x)\, dx$$

$$= \int_{-\infty}^{\infty} e^{tx^2} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2(\frac{1}{2}-t)} dx$$

$$= \frac{1}{\sqrt{2\pi}} 2 \int_{0}^{\infty} e^{-\left(\frac{1-2t}{2}\right)x^2} x^{2(\frac{1}{2})-1} dx$$

$$= \frac{1}{\sqrt{2\pi}} 2 \frac{\left(\frac{1-2t}{2}\right)^{-\frac{1}{2}} \Gamma(\frac{1}{2})}{2}$$

$$= \frac{1}{\sqrt{2\pi}} (1-2t)^{-\frac{1}{2}} \sqrt{2} \sqrt{\pi}$$

$$M_{X^2}(t) = (1-2t)^{-\frac{1}{2}}$$

which is the m.g.f. of χ_1^2 variate

Hence X^2 is distributed as χ^2 distribution with 1 d.f

$$\therefore f(\chi^2) = \frac{e^{-\frac{\chi^2}{2}} (\chi^2)^{\frac{1}{2}-1}}{2^{\frac{1}{2}} \sqrt{\Gamma(\frac{1}{2})}} = \frac{e^{-\frac{\chi^2}{2}} (\chi^2)^{-\frac{1}{2}}}{\sqrt{2\pi}} \quad 0<\chi<\infty$$

which is the p.d.f of χ^2 distribution with 1 d.f.

Describe the additive property of the chi square distribution

The sum of independent chi square variates is also a chi square variate. More precisely, if $\chi_1^2, \chi_2^2, \ldots, \chi_k^2$ are k independent χ^2 with $n_1, n_2, \ldots, n_k$ d.f then $\chi^2 = \chi_1^2 + \chi_2^2 + \ldots + \chi_k^2$ follows χ^2 distribution with $n = n_1 + n_2 + n_3 + \cdots + n_k$ d.f.

Proof: We know that,

$$M_{\chi^2}(t) = M_{\chi_1^2 + \chi_2^2 + \cdots + \chi_k^2}(t)$$

$$= M_{\chi_1^2}(t) \, M_{\chi_2^2}(t) \ldots M_{\chi_k^2}(t)$$

$$= (1-2t)^{-\frac{n_1}{2}} (1-2t)^{-\frac{n_2}{2}} \ldots (1-2t)^{-\frac{n_k}{2}}$$

$$= (1-2t)^{-\frac{(n_1+n_2+\cdots+n_k)}{2}}$$

$$= (1-2t)^{-\frac{n}{2}}.$$

Since the m.g.f. of χ^2 with n d.f. is

$$(1-2t)^{-\frac{n}{2}}$$

$$\therefore M_{\chi^2}(t) = (1-2t)^{-\frac{(n_1+n_2+\cdots+n_k)}{2}}$$ which

is the m.g.f of a χ^2 variate

with $(n_1+n_2+\cdots+n_k)$ d.f. Hence

$\chi^2 = x_1^2 + x_2^2 + \cdots x_k^2$ is a

χ^2 variate with $n = n_1+n_2+\cdots+n_k$

d.f.

[Mode of χ^2 distribution]

t. distribution,

$F(x) \rightarrow$ cumulative density function.

$$\frac{d}{dx} F(x) = f(x)$$

$$F(x) = \int_{-\infty}^{x} f(x)\,dx$$

$f(x,y) = f(x) \cdot f(y)$

Joint d.f. of x, y — Marginal d.f. of x — y

$$\frac{d}{dx\,dx^2} F(x, x^2) = f(x, x^2)$$

$$dF(x, x^2) = f(x, x^2)\,dx, dx^2.$$

$$= f(x)\,dx \cdot f(x^2)\,dx^2,$$

Derivation of t-distribution

① **Statement:** Let x be normally distributed with mean zero and variance unity and let χ^2 distributed as χ^2 with n. If x and χ^2 are independent then the ratio $t = \frac{x}{\sqrt{\frac{\chi^2}{n}}}$ follows t distribution with d.f. n.

Proof: The joint p.d.f. of χ^2 is given by

$$dF(x, \chi^2) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx \; \frac{1}{2^{n/2}\Gamma\frac{n}{2}} e^{-\frac{\chi^2}{2}} (\chi^2)^{\frac{n}{2}-1} d\chi^2$$

$$= \frac{1}{\sqrt{2\pi}\; 2^{n/2}\Gamma\frac{n}{2}} e^{-\frac{x^2}{2}} e^{-\frac{\chi^2}{2}} (\chi^2)^{\frac{n}{2}-1} dx\, d\chi^2$$

$$\left[-\infty < x < \infty ;\; 0 < \chi^2 < \infty\right]$$

Now, $t = \frac{x}{\sqrt{\chi^2/n}}$ or, $x = t\sqrt{\chi^2/n} \Rightarrow dx = \sqrt{\chi^2/n}\; dt.$

The joint p.d.f. of t and χ^2 is given by

$$dF(t, \chi^2) = \frac{1}{\sqrt{2\pi}\; 2^{n/2}\Gamma(\frac{n}{2})\sqrt{n}} e^{-\frac{t^2\chi^2}{2n}} e^{-\frac{\chi^2}{2}} (\chi^2)^{\frac{n}{2}-1} (\chi^2)^{\frac{1}{2}} dt\, d\chi^2$$

$$= \frac{1}{\sqrt{2\pi}\; 2^{\frac{n}{2}}\Gamma(\frac{n}{2})\sqrt{n}} e^{-\frac{\chi^2}{2}(1+\frac{t^2}{n})} (\chi^2)^{\frac{n+1}{2}-1} dt\, d\chi^2 ;$$

$$-\infty < t < \infty ,\; 0 < \chi^2 < \infty$$

$$\therefore dF(t) = \left[\frac{1}{\sqrt{2\pi}\; 2^{\frac{n}{2}}\Gamma(\frac{n}{2})\sqrt{n}} \int_0^\infty e^{-\frac{\chi^2}{2}(1+\frac{t^2}{n})} (\chi^2)^{\frac{n+1}{2}-1} d\chi^2\right] dt$$

$$= \frac{1}{\sqrt{2\pi}\; 2^{\frac{n}{2}}\Gamma(\frac{n}{2})\sqrt{n}} \left(\frac{1+\frac{t^2}{n}}{2}\right)^{-\frac{n+1}{2}} \Gamma\left(\frac{n+1}{2}\right) dt$$

$$= \frac{\Gamma\left(\frac{n+1}{2}\right) \cdot 2^{\frac{n+1}{2}}}{\sqrt{n}\,\Gamma\left(\frac{n}{2}\right)\Gamma\left(\frac{1}{2}\right)(2)^{\frac{n}{2}+\frac{1}{2}}\left(1+\frac{t^2}{n}\right)^{\frac{n+1}{2}}}\, dt$$

$$= \frac{dt}{\sqrt{n}\,\beta\left(\frac{n}{2},\frac{1}{2}\right)\left(1+\frac{t^2}{n}\right)^{\frac{n+1}{2}}} \quad -\infty < t < \infty$$

which is known as t distribution with d.f. n.

① What is t statistics?

All odd order moments of t distribution are zero

$$\mu'_{2r+1} = E\left(t^{2r+1}\right); \quad r = 0, 1, 2, 3, \ldots$$

$$= \int_{-\infty}^{\infty} t^{2r+1} f(t)\, dt.$$

$$= \int_{-\infty}^{\infty} t^{2r+1} \frac{1}{\sqrt{n}\,\beta\left(\frac{n}{2},\frac{1}{2}\right)\left(1+\frac{t^2}{n}\right)^{\frac{n+1}{2}}}\, dt.$$

$$= \frac{1}{\sqrt{n}\,\beta\left(\frac{n}{2},\frac{1}{2}\right)} \int_{-\infty}^{\infty} \frac{t^{2r+1}}{\left(1+\frac{t^2}{n}\right)^{\frac{n+1}{2}}}\, dt$$

$$= \frac{1}{\sqrt{n}\,\beta\left(\frac{n}{2},\frac{1}{2}\right)} \int_{-\infty}^{\infty} g(t)\, dt. \quad \text{where, } g(t) = \frac{t^{2r+1}}{\left(1+\frac{t^2}{n}\right)^{\frac{n+1}{2}}};$$

$r = 0, 1, 2, 3, \ldots$

Now, $g(-t) = \dfrac{(-t)^{2r+1}}{\left(1+\frac{(-t)^2}{n}\right)^{\frac{n+1}{2}}} = \dfrac{-t^{2r+1}}{\left(1+\frac{t^2}{n}\right)^{\frac{n+1}{2}}}$

$$\Rightarrow -g(-t) = \frac{t^{2r+1}}{\left(1+\frac{t^2}{n}\right)^{\frac{n+1}{2}}}$$

$\because$ $g(t) = -g(-t)$ $\therefore$ $g(t)$ is an odd function.

So, $\int_{-\infty}^{\infty} g(t)\, dt = 0$.

$$\therefore \mu'_{2r+1} = \frac{1}{\sqrt{n}\, \beta\left(\frac{n}{2}, \frac{1}{2}\right)} (0) = 0; \quad r = 0, 1, \ldots$$

$\Rightarrow$ All odd order raw moments of t distribution are zero.

$\because$ $\mu'_{2r+1} = 0$ $\therefore$ $\mu_{2r+1} = 0$, for $r = 0, 1, 2 \ldots$

$\Rightarrow$ All odd order raw and central moments of t distribution are zero.

Even order moments of t distribution.

Formula: $\beta(m,n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\,dx$

$\beta(m,n) = \frac{\Gamma m\,\Gamma n}{\Gamma(n+m)}$

We know that, $\mu'_{2r} = E(t^{2r}) = \int_{-\infty}^{\infty} t^{2r} f(t)\,dt$. $r = 1, 2, 3$

$$= \int_{-\infty}^{\infty} t^{2r} \frac{1}{\sqrt{n}\,\beta\left(\frac{n}{2},\frac{1}{2}\right)\left(1+\frac{t^2}{n}\right)^{\frac{n+1}{2}}}\,dt.$$

$$= \frac{1}{\sqrt{n}\,\beta\left(\frac{n}{2},\frac{1}{2}\right)} \int_{-\infty}^{\infty} \frac{t^{2r}}{\left(1+\frac{t^2}{n}\right)^{\frac{n+1}{2}}}\,dt$$

$$= \frac{1}{\sqrt{n}\,\beta\left(\frac{n}{2},\frac{1}{2}\right)} \int_{-\infty}^{\infty} g(t)\,dt \quad \text{where } g(t) = \frac{t^{2r}}{\left(1+\frac{t^2}{n}\right)^{\frac{n+1}{2}}} \; ; r = 1, 2, \dots$$

Now, $g(-t) = \frac{(-t)^{2r}}{\left(1+\frac{(-t)^2}{n}\right)^{\frac{n+1}{2}}} = \frac{t^{2r}}{\left(1+\frac{t^2}{n}\right)^{\frac{n+1}{2}}}$

$$\Rightarrow g(-t) = \frac{t^{2r}}{\left(1+\frac{t^2}{n}\right)^{\frac{n+1}{2}}}$$

$\therefore g(t) = g(-t) \therefore g(t)$ is even fun.

So, $\int_{-\infty}^{\infty} g(t)\,dt = 2\int_0^{\infty} g(t)\,dt$

$$\therefore \mu'_{2r} = \frac{2}{\sqrt{n}\,\beta\left(\frac{n}{2},\frac{1}{2}\right)} \int_0^{\infty} g(t)\,dt$$

Now putting the value of $g(t)$ in the above equation, we obtain

$$\mu'_{2r} = \frac{1}{\sqrt{n}\,\beta\left(\frac{n}{2},\frac{1}{2}\right)}\int_0^{\infty} \frac{t^{2r}}{\left(1+\frac{t^2}{n}\right)^{\frac{n+1}{2}}}\,dt.$$

$$\mu'_{2r} = \frac{2}{\sqrt{n}\,\beta\left(\frac{n}{2},\frac{1}{2}\right)}\int_0^{\infty} \frac{\left(\frac{t^2}{n}\right)^{r} n^{r}}{\left(1+\frac{t^2}{n}\right)^{\frac{n+1}{2}}}\,dt.$$

$$= \frac{n^{r}}{\sqrt{n}\,\beta\left(\frac{n}{2},\frac{1}{2}\right)}\int_0^{\infty} \frac{\left(\frac{t^2}{n}\right)^{r} n t^{-1}\,\frac{2t}{n}\,dt}{\left(1+\frac{t^2}{n}\right)^{\frac{n+1}{2}}}$$

$$= \frac{n^{r-\frac{1}{2}}}{\beta\left(\frac{n}{2},\frac{1}{2}\right)}\int_0^{\infty} \frac{\left(\frac{t^2}{n}\right)^{(r+1)-1} n t^{-1}\,d\left(\frac{t^2}{n}\right)}{\left(1+\frac{t^2}{n}\right)^{\frac{n+1}{2}}} \quad \Big|\ d\left(\frac{t^2}{n}\right) = \frac{2t\,dt}{n}.$$

$$\mu'_{2r} = \frac{n^{r}}{\beta\left(\frac{n}{2},\frac{1}{2}\right)}\int_0^{\infty} \frac{\left(\frac{t^2}{n}\right)^{(r+1)-1}\left(\frac{t^2}{n}\right)^{-\frac{1}{2}} d\left(\frac{t^2}{n}\right)}{\left(1+\frac{t^2}{n}\right)^{\frac{n+1}{2}}}$$

$$= \frac{n^{r}}{\beta\left(\frac{n}{2},\frac{1}{2}\right)}\int_0^{\infty} \frac{\left(\frac{t^2}{n}\right)^{\left(r+\frac{1}{2}\right)-1} d\left(\frac{t^2}{n}\right)}{\left(1+\frac{t^2}{n}\right)^{\left(r+\frac{1}{2}\right)+\frac{n+1}{2}-r-\frac{1}{2}}} \quad \Big|\ \beta(m,n) = \int_0^{\infty} \frac{x^{m-1}}{(1+x)^{m+n}}$$

$$= \frac{n^{r}}{\beta\left(\frac{n}{2},\frac{1}{2}\right)}\,\beta\left(r+\frac{1}{2},\frac{n}{2}-r\right). \qquad \beta(m,n) = \frac{\Gamma m \Gamma n}{\Gamma(m+n)}$$

$$= \frac{\Gamma\left(\frac{n+1}{2}\right) n^r}{\Gamma\left(\frac{n}{2}\right)\Gamma\left(\frac{1}{2}\right)} \cdot \frac{\Gamma\left(r+\frac{1}{2}\right)\Gamma\left(\frac{n}{2}-r\right)}{\Gamma\left(\frac{n+1}{2}\right)}$$

$$\therefore \mu'_{2r} = \frac{\Gamma\left(\frac{n}{2}-r\right)\Gamma\left(r+\frac{1}{2}\right) n^r}{\Gamma\left(\frac{n}{2}\right)\Gamma\left(\frac{1}{2}\right)} \quad \boxed{r = 1, 2, 3.}$$

$\mu_{2r} =$

04.03.18

Student's t distribution:

Statement: If $x_1, x_2, \ldots, x_n$ is a random sample from normal population with mean μ and variance σ^2 then $t = \frac{\bar{x}-\mu}{s/\sqrt{n}}$ will be distributed as student's t with d.f $(n-1)$, where $\bar{x} = \sum x_i/n$ is the simple mean and $s^2 = \frac{\sum(x_i-\bar{x})^2}{n-1}$ is an unbiased estimate of the population variance σ^2.

$\Longrightarrow$ property of t distribution.

(i) State.

(ii) odd order mm.

(iii) even "

(iv) student's t

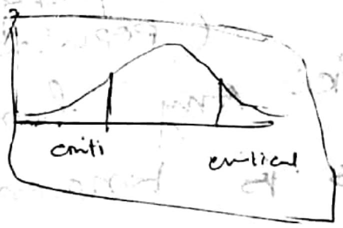
(v) when $n=1$ t-dis is cocci dist

(v) Skewness, kurtosis,

(vi) Mean, median, variance.

$$F = \frac{\chi_1^2/n_1}{\chi_2^2/n_2} = \frac{n_2\,\chi_1^2}{n_1\,\chi_2^2} \sim F_{n_1, n_2}\ , \quad 0 < F < \infty$$

Null hypothesis H_0
Alternative " H_1



13.02.18 gupta kapur

Parameter: Parameter is an unknown constant which uniquely specify the distribution (population)

Statistic: Any function of a set of sample values is known as a statistic.

Statistical Hypothesis: A statistical hypothesis is an assertion about the distribution of one or more random variable. From definition it follows that not every hypothesis is a statistical hypothesis. There are two kinds of statistical hypothesis, (i) Simple hypothesis. (ii) Composite hypothesis.

Some definitions

Statistical Hypothesis: A statistical hypothesis is an assertion about the distribution of one or more random variable.

Types of Hypothesis [Null Hypothesis
Alternative "]

Null Hypothesis: A statistical hypothesis which is picked up for testing is known as null hypothesis. It is denoted by H_0.

Alternative Hypothesis: Any hypothesis other than null hypothesis that we might consider as alternative to null hypothesis is known as Alternative hypothesis. It is denoted by H_1.

Simple Hypothesis: If the statistical hypothesis completely specifies the distribution then it is called simple hypothesis.

Composite hypothesis: If the statistical hypothesis does not completely specify the distribution then it is called a composite hypothesis.

Test statistic: The statistic used to provide evidence about the null hypothesis is called the ~~the~~ test statistic.

First kind of Error: Rejection of null hypothesis (H_0) when it is true is an incorrect decision or an error. This error is known as first kind of error or Type-I error.

Second kind of Error: Acceptance of null hypothesis (H_0) when it is false i.e. H_1 is true is an error. Which is known as second kind of error or Type-II error.

~~test~~ Level of significant: The probability of null hypothesis (H_0) when it is true is known as the level of significance of the test.

It is also called size of the test.
It is denoted by α.

Critical Region: To test any hypothesis on the basis of a random sample of observations, we divide the sample space into two regions. If the observed sample point falls into one of these regions, say ω then we reject the null hypothesis this region is called critical region.

Acceptance Region: The region consisting of points belonging to sample space but not to the critical region is called acceptance region. It is denoted by $W-\omega$.

Type of test

- One Tailed
  - Left Tailed
    - $H_0 : \mu = \mu_0$
    - $H_1 : \mu < \mu_0$
  - Right Tailed
    - $H_0 : \mu = \mu_0$
    - $H_1 : \mu > \mu_0$
- Two Tailed
  - $H_0 = \mu = \mu_0$
  - $H_1 : \mu \neq \mu_0$

Decision Rule

If $p < 0.05$ then H_0 is rejected at 5% level of significance

If $p < 0.01$ then H_0 is rejected at 1% level of significance.

Mean Test :

Type of Mean	Variance Status	Test statistic
Single Mean Test	with known variance	Z
	with unknown variance	t
Double Mean Test	with known variance	Z
	with unknown variance	t
Several mean test	—	F

Corrected Term (CT) $= \frac{T^2}{n}$

Sum of square (SS) $= \sum_{i=1} \sum_{j} y_{ij}^2 - CT$

Sum of square Between (SSB)

$$SS.B = \sum_{i=1} \left\{ \frac{T_i^2}{n_i} \right\} - CT$$

Sum of square within (SSW)

Gupta & kapur

		The null hypothesis is true	false
Decision	We decide to reject null Hypothesis	Type I error (rejecting a true null hypothesis	Cor
	We fail to reject the null hypothesis	Correct De	Type II error (failing to reject a false null hypothesis) β

Power of a Test:

The power of a hypothesis test is the probability $(1-\beta)$ of rejecting a false null hypothesis, which is computed by using a particular significance level α and a particular value of the population parameters that is an alternative to the value assumed true in the null hypothesis. That is the power of the hypothesis test is the probability of supporting an alternative hypothesis that is true

RND

single mean test (σ known)

$Z_1 = \frac{x-\mu}{\sigma}$

Normal

ZA

Nurul Islam
Chapter 11

Methods of Statistics
Dr. M. G. Mostofa
9

Dependent variable → Response variable
Independent → Explanatory.

~~(indep)~~

At 5% level of significance the tabulated value of d are ± 1.96 but our calculated of d is -1.14 which falles in the acceptance region so we may accept our null hypothesis. Hence our test is insignificant.

2·03·18

$y = x$

$\frac{dy}{dx}$

$\mu = \mu_0 = 65.0$ inch is calculated

$(\bar{x}_1 - \bar{x}_2) - (\mu_1 - \mu_2)$

$\cancel{(\bar{x}_i - \mu_1) - (x}$

$$s_1^2 = \frac{\sum (x_{1i} - \bar{x}_1)^2}{n_1 - 1}$$

$$s_2^2 = \frac{\sum (x_{2i} - \bar{x}_2)^2}{n_2 - 1}$$

$$s_1^2 (n_1 - 1) = \sum (x_{1i} - \bar{x}_1)^2$$

$$s_2^2 (n_2 - 1) = \sum (x_{2i} - \bar{x}_2)^2$$

$$s^2 = \frac{\sum (x_{1i} - \bar{x}_1)^2 + \sum (x_{2i} - \bar{x}_2)^2}{n_1 + n_2 - 2}$$

$$= \frac{s_1^2 (n_1 - 1) + s_2^2 (n_2 - 1)}{n_1 + n_2 - 2}$$

M.L.E. Parameter estimate. Gupta and Kapur

2/10

$\left\{ \begin{matrix} \chi^2 \\ t \\ F \end{matrix} \right\}$ several test — proportion, correlation, mean, variance — contingency table

$$f(\chi^2) = \frac{e^{-\frac{\chi^2}{2}} (\chi^2)^{\frac{n}{2}-1}}{2^{n/2} \Gamma \frac{n}{2}} \qquad 0 \le \chi^2 \le \infty$$

M.G.F of χ^2 distribution is $M_{\chi^2}(t) = (1-2t)^{-n/2}$

$$\mu_1' = \frac{\delta}{\delta t} (1-2t)^{-n/2} \Big|_{t=0}$$

$\mu_2 = \mu_2' - (\mu_1')^2$

$\mu_3 = \mu_3' - 3\mu_2'\mu_1' + 2(\mu_1')^3$

$\mu_4 = \mu_4' - 4\mu_3'\mu_1' + 6\mu_2'\mu_1'^2 - 3(\mu_1')^4$

$$M_{x^2}(t) = M_{x_1^2 + x_2^2 + \cdots}(t)$$

$$= M_{x_1^2}(t)\, M_{x_2^2}(t)$$

p.d.f $\quad f(x^2) = \dfrac{e^{-\frac{x^2}{2}} (x^2)^{n/2 - 1}}{2^{n/2} \Gamma\frac{n}{2}}$

$$M_x(t) = M_{x_1^2}(t)\, M_{x_2^2}(t) \cdots$$

$$= (1-2t)^{-\frac{n_1}{2}} \cdot (1-2t)^{-\frac{n_2}{2}}$$

$$= (1-2t)^{-\left(\frac{n_1 + n_2 + \cdots}{2}\right)}$$

$$= (1-2t)^{-\frac{n}{2}}$$

M.G.F. $2x$

$$M_{2x}(t) = E\left(e^{2xt}\right)$$

$$= \int_0^{\infty} e^{2xt} f(x)\, dx.$$

$$= \int_0^{\infty} e^{2xt} e^{-x}\, dx.$$

$$= \int_0^{\infty} e^{-(1-2t)x}\, dx$$

$$= \int_0^{\infty} e^{-(1-2t)x} x^{1-1}\, dx,$$

$$= (1-2t)^{-1} \Gamma 1.$$

$$M_{2X}(t) = (1-2t)^{-1}$$

$$(1-2t)^{-\frac{2}{2}}$$

If x_i $(i = 1 - n)$ are n ind. $N(\mu, \sigma^2)$ v.a. then the gam

$$\chi^2 = \sum_{i=1}^{n} \left(\frac{x_i - \mu}{\sigma}\right)^2 \text{ is }$$

Let $z_1 = \frac{x_1 - \mu_1}{\sigma_1}$ $i = 1 \_\_ n$.

Since $x_i \sim N(\mu, \sigma^2)$;

$z_i \sim N(0, 1)$.

$$M_{\chi^2}(t) = M_{\sum_{i=1}^{n} \left(\frac{x_i - \mu_1}{\sigma_1}\right)^2}(t)$$

$$= M_{\sum_{i=1}^{n} z_i}(t)$$

$$M_{\sum z_i^2}(t) \quad \cancel{M_{z_i}}$$

~~Mor~~

Now, $M_{z_i^2}(t) = E\left(e^{t z_i^2}\right)$

$$= \int_{-\infty}^{\infty} e^{t z_i^2} f(z_i)\, dz_i$$

$$= \int_{-\infty}^{\infty} e^{t z_i^2} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2} z_i^2} dz_i$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\left(\frac{1}{2} - t\right) z_i^2} dz_i$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\left(\frac{1-2t}{2}\right) z_i^2} (z_i^2)^{2 \cdot \frac{1}{2} - 1} \, dz_i$$

$$= \frac{1}{\sqrt{2\pi}} 2 \int_{0}^{\infty} e^{-\left(\frac{1-2t}{2}\right) z_i^2} (z_i^2)^{2 \cdot \frac{1}{2} - 1} \, dz_i$$

$$= \frac{1}{\sqrt{2\pi}} \cancel{2} \frac{\left(\frac{1-2t}{2}\right)^{-\frac{1}{2}} \Gamma\frac{1}{2}}{\cancel{2}}$$

$$= \frac{1}{\sqrt{2\pi}} \left(\frac{2}{1-2t}\right)^{\frac{1}{2}} \Gamma\frac{1}{2}$$

$$= \frac{1}{\cancel{\sqrt{2\pi}}} \frac{\cancel{\sqrt{2}}}{(1-2t)^{1/2}} \cancel{\sqrt{\pi}}$$

$$= (1-2t)^{-1/2}$$

q.d.f.

$$M_{x^2}(t) = \prod_{i=1}^{n} M_{z_i^2}(t)$$

$$= \prod_{i=1}^{n} M_{z_i^2}(t)$$

$$= \prod_{i=1}^{n} (1-2t)^{-1/2}$$

$M_{x^2}(t) = (1-2t)^{-n/2}$, m.g.f.

Since $f(x) = \dfrac{\lambda^n e^{-\lambda x} x^{n-1}}{\Gamma n}$.

$E(x^2) = \int_0^\infty x^2 f(x^2)\, dx^2$,

$e^{-\frac{x^2}{2}} (x^2)^{\frac{n}{2}-1} d x^2$.

$V(x^2) = \cancel{E(x^2)^2}$

$= E((x^2)^2) - [E(x^2)]^2$

$\int_0^\infty (x^2)^2 \dfrac{1}{2^{n/2}\Gamma\frac{n}{2}} e$

$E(x^2) = \int_0^\infty x^2 f(x)\, dx$

If x follow $N(0, 1)$

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} \cancel{dx}$$

α

24.01.018.

F Distribution:

Statement: If χ_1^2 and χ_2^2 are independent χ^2 varia... with d.f n_1 and n_2 respectively, then the ratio $F = \frac{\frac{\chi_1^2}{n_1}}{\frac{\chi_2^2}{n_2}} : \frac{n_2 \chi_1^2}{n_1 \chi_2^2}$ will posses F distribution with numerator d.f n_1 and denominator d.f. n_2 and the distribution of F will be of the form.

$$d.F(F) = \frac{n_1^{\frac{n_1}{2}} n_2^{\frac{n_2}{2}} F^{\frac{n_1}{2}-1}}{\beta\left(\frac{n_1}{2}, \frac{n_2}{2}\right)(n_1 F + n_2)^{\frac{n_1+n_2}{2}}} dF$$

$0 \leq F \infty$

$n_1 > 0, n_2 >$

$\frac{n_1}{2} - 1$

$)^2$

$\infty)$.

Proof:

Since x_1^2 and x_2^2 are independently distributed x^2 variate with d.f n_1 and n_2 respectively; there the joint p.d.f of x_1^2 and x_2^2 is

$$f(x_1^2, x_2^2) = \frac{e^{-\frac{x_1^2}{2}}(x_1^2)^{\frac{n_1}{2}-1}}{2^{\frac{n_1}{2}}\Gamma\frac{n_1}{2}} \cdot \frac{e^{-\frac{x_2^2}{2}}(x_2^2)^{\frac{n_2}{2}-1}}{2^{\frac{n_2}{2}}\Gamma\frac{n_2}{2}}$$

$$= \frac{e^{-\frac{x_1^2}{2}} e^{-\frac{x_2^2}{2}} (x_1^2)^{\frac{n_1}{2}-1}(x_2^2)^{\frac{n_2}{2}-1}}{2^{\frac{n_1+n_2}{2}}\Gamma\frac{n_1}{2}\Gamma\frac{n_2}{2}}$$

$$0 < x_1^2 < \infty$$

$$0 < x_2^2 < \infty$$

Now, $F = \frac{n_2}{n_1} \frac{x_1^2}{x_2^2}$ and let $\omega = x_2^2$

$\therefore x_1^2 = \frac{n_1}{n_2} x_2^2 F$ and $x_2^2 = \omega$.

$\Rightarrow x_1^2 = \frac{n_1}{n_2} F\omega$.

So $|J| = \begin{vmatrix} \frac{d x_1^2}{dF} & \frac{d x_2^2}{dF} \\ \frac{d x_1^2}{d\omega} & \frac{d x_2^2}{d\omega} \end{vmatrix} = \begin{vmatrix} \frac{n_1}{n_2}\omega & 0 \\ \frac{n_1}{n_2} F & 1 \end{vmatrix}$

$= \frac{n_1}{n_2} \omega$.

$\therefore$ The joint pdf of F and ω is

$$f(F,\omega) = \frac{1}{2^{\frac{n_1+n_2}{2}} \Gamma\frac{n_1}{2} \Gamma\frac{n_2}{2}} e^{-\left(\frac{n_1}{2n_2} F\omega\right)} e^{-\frac{\omega}{2}} \left(\frac{n_1}{n_2} F\omega\right)^{\frac{n_1}{2}-1} (\omega)^{\frac{n_2}{2}-1} . |J|$$

$$= \frac{1}{2^{\frac{n_1+n_2}{2}} \Gamma\frac{n_1}{2} \Gamma\frac{n_2}{2}} e^{-\frac{\omega}{2}\left(1+\frac{n_1}{n_2}F\right)} \left(\frac{n_1}{n_2}F\right)^{\frac{n_1}{2}-1} (\omega)^{\frac{n_1}{2}+\frac{n_2}{2}-1} \left(\frac{n_1}{n_2}\omega\right).$$

$$= \frac{1}{2^{\frac{n_1+n_2}{2}} \Gamma\frac{n_1}{2} \Gamma\frac{n_2}{2}} e^{-\frac{\omega}{2}\left(1+\frac{n_1}{n_2}F\right)} \left(\frac{n_1}{n_2}\right)^{\frac{n_1}{2}} (F)^{\frac{n_1}{2}-1}$$

$$(\omega)^{\frac{n_1+n_2}{2}-1}.$$

$0 < F < \infty$
$0 < \omega < \infty.$

Now, $f(F) = \int_{\omega} f(F, \omega)$

$$= \frac{1}{2^{\frac{n_1+n_2}{2}} \Gamma\frac{n_1}{2} \Gamma\frac{n_2}{2}} \left(\frac{n_1}{n_2}\right)^{\frac{n_1}{2}} (F)^{\frac{n_1}{2}-1}$$

$$\int_0^{\infty} e^{-\frac{\omega}{2}\left(1+\frac{n_1}{n_2}F\right)} (\omega)^{\frac{n_1+n_2}{2}-1} d\omega$$

$$= \frac{1}{2^{\frac{n_1+n_2}{2}} \Gamma\frac{n_1}{2} \Gamma\frac{n_2}{2}} \left(\frac{n_1}{n_2}\right)^{\frac{n_1}{2}} (F)^{\frac{n_1}{2}-1}.$$

$$\left\{\frac{1}{2}\left(1+\frac{n_1}{n_2}F\right)\right\}^{-\left\{\frac{n_1+n_2}{2}\right\}} \Gamma\left(\frac{n_1+n_2}{2}\right)$$

$$= \frac{\left(\frac{n_1}{n_2}\right)^{\frac{n_1}{2}} (F)^{\frac{n_1}{2}-1}}{2^{\frac{n_1+n_2}{2}}} \left(\frac{2}{\frac{n_2+n_1F}{n_2}}\right)^{\frac{n_1+n_2}{2}} \frac{\Gamma\left(\frac{n_1+n_2}{2}\right)}{\Gamma\frac{n_1}{2} \Gamma\frac{n_2}{2}}$$

$$= \frac{1}{2^{\frac{n_1+n_2}{2}} \Gamma\frac{n_1}{2} \Gamma\frac{n_2}{2}} e^{-\frac{\omega}{2}\left(1+\frac{n_1}{n_2}F\right)} \left(\frac{n_1}{n_2}\right)^{\frac{n_1}{2}} (F)^{\frac{n_1}{2}-1} (\omega)^{\frac{n_1+n_2}{2}-1}.$$

$$0 < F < \alpha$$
$$0 < \omega < \alpha.$$

Now, $f(F) = \int_{\omega} f(F, \omega)$

$$= \frac{1}{2^{\frac{n_1+n_2}{2}} \Gamma\frac{n_1}{2} \Gamma\frac{n_2}{2}} \left(\frac{n_1}{n_2}\right)^{\frac{n_1}{2}} (F)^{\frac{n_1}{2}-1} \int_0^{\alpha} e^{-\frac{\omega}{2}\left(1+\frac{n_1}{n_2}F\right)} (\omega)^{\frac{n_1+n_2}{2}-1} d\omega$$

$$= \frac{1}{2^{\frac{n_1+n_2}{2}} \Gamma\frac{n_1}{2} \Gamma\frac{n_2}{2}} \left(\frac{n_1}{n_2}\right)^{\frac{n_1}{2}} (F)^{\frac{n_1}{2}-1} \left\{\frac{1}{2}\left(1+\frac{n_1}{n_2}F\right)\right\}^{-\left\{\frac{n_1+n_2}{2}\right\}} \Gamma\left(\frac{n_1+n_2}{2}\right)$$

$$= \frac{\left(\frac{n_1}{n_2}\right)^{\frac{n_1}{2}} (F)^{\frac{n_1}{2}-1}}{2^{\frac{n_1+n_2}{2}} \cancel{\Gamma\frac{n_1}{2}\Gamma\frac{n_2}{2}}} \left(\frac{2}{\frac{n_2+n_1F}{n_2}}\right)^{\frac{n_1+n_2}{2}} \frac{\Gamma\left(\frac{n_1}{2}+\frac{n_2}{2}\right)}{\Gamma\frac{n_1}{2}\Gamma\frac{n_2}{2}}$$

$$= \frac{\left(\frac{n_1}{n_2}\right)^{\frac{n_1}{2}} (F)^{\frac{n_1}{2}-1}}{2^{\left(\frac{n_1+n_2}{2}\right)}} \cdot \frac{(2)^{\frac{n_1+n_2}{2}}}{\frac{(n_2+n_1F)^{\frac{n_1+n_2}{2}}}{(n_2)^{\frac{n_1+n_2}{2}}}} \cdot \frac{1}{\beta(m,n)}$$

$$= \frac{\left(\frac{n_1}{n_2}\right)^{\frac{n_1}{2}} (n_2)^{\frac{n_1}{2}} (n_2)^{\frac{n_2}{2}} (F)^{\frac{n_1}{2}-1}}{\beta(m,n)\,(n_2+n_1F)^{\frac{n_1+n_2}{2}}}$$

$$= \frac{(n_1)^{\frac{n_1}{2}} (n_2)^{\frac{n_2}{2}} (F)^{\frac{n_1}{2}-1}}{\beta(m,n)\,(n_2+n_1F)^{\frac{n_1+n_2}{2}}} \qquad 0 \le F < \infty$$

which is the form of F distribution with

d.f.

13.05

many

Formula $\displaystyle\int_0^{\infty} \frac{x^{m-1}}{(1+x)^{m+n}}\,dx$

$= \beta(m,n)$

Property of F distribution:

Establish F distribution from Beta distribution of First kind.

$$f(x) = \frac{1}{\beta\left(\frac{n_1}{2}, \frac{n_2}{2}\right)} x^{\frac{n_1}{2}-1} (1-x)^{\frac{n_2}{2}-1}$$

let $x = \frac{n_1 F}{n_2 + n_1 F}$;

$$\therefore \frac{dx}{dF} = \frac{(n_2 + n_1 F) n_1 - n_1 F (0 + n_1)}{(n_2 + n_1 F)^2}$$

$$= \frac{n_1 n_2 + n_1^2 F - n_1^2 F}{(n_2 + n_1 F)^2}$$

$$= \frac{n_1 n_2}{(n_2 + n_1 F)}$$

i.e $|J| = \left|\frac{dx}{dF}\right| = \frac{n_1 n_2}{(n_2 + n_1 F)^2}$.

$$\therefore f(F) = \frac{1}{\beta\left(\frac{n_1}{2}, \frac{n_2}{2}\right)} \left(\frac{n_1 F}{n_2 + n_1 F}\right)^{\frac{n_1}{2}-1} \left(1 - \frac{n_1 F}{n_2 + n_1 F}\right)^{\frac{n_2}{2}-1} \cdot J.$$

$$= \frac{1}{\beta\left(\frac{n_1}{2}, \frac{n_2}{2}\right)} \left(\frac{n_1 F}{n_2 + n_1 F}\right)^{\frac{n_1}{2}-1} \left(1 - \frac{n_1 F}{n_2 + n_1 F}\right)^{\frac{n_2}{2}-1} \frac{n_1 n_2}{(n_2 + n_1 F)^2}$$

$$\Rightarrow f(F) = \frac{n_1^{\frac{n_1}{2}}\, n_2^{\frac{n_2}{2}}\, F^{\frac{n_1}{2}-1}}{\beta\left(\frac{n_1}{2}, \frac{n_2}{2}\right)(n_2+n_1F)^{\left(\frac{n_1+n_2}{2}\right)}}\,; \quad 0 < F < \infty$$

$n_1 > 0$ and $n_2 > 0$

<u>Moments of F distribution</u>

r^{th} moment about origin is given by

$$\mu'_r = E(F^r)$$

$$= \int_0^\infty F^r \frac{\left(\frac{n_1}{n_2}\right)^{\frac{n_1}{2}} (F)^{\frac{n_1}{2}-1}}{\beta\left(\frac{n_1}{2}, \frac{n_2}{2}\right)\left(1+\frac{n_1}{n_2}F\right)^{\left(\frac{n_1+n_2}{2}\right)}}\, dF$$

$$= \frac{\left(\frac{n_1}{n_2}\right)^{\frac{n_1}{2}}}{\beta\left(\frac{n_1}{2}, \frac{n_2}{2}\right)} \int_0^\infty \frac{(F)^{\frac{n_1+2r}{2}-1}}{\left(1+\frac{n_1}{n_2}F\right)^{\left(\frac{n_1+n_2}{2}\right)}}\, dF$$

$$= \frac{\left(\frac{n_1}{n_2}\right)^{\frac{n_1}{2}}}{\beta\left(\frac{n_1}{2}, \frac{n_2}{2}\right)} \int_0^\infty \frac{\left(\frac{n_1}{n_2}F\right)^{\frac{n_1+2r}{2}-1} \left(\frac{n_2}{n_1}\right)^{\frac{n_1+2r}{2}-1}}{\left(1+\frac{n_1}{n_2}F\right)^{\left(\frac{n_1+n_2}{2}\right)}} \left(\frac{n_2}{n_1}\right) d\left(\frac{n_1}{n_2}F\right)$$

$$= \frac{\left(\frac{n_1}{n_2}\right)^{\frac{n_1}{2}} \left(\frac{n_2}{n_1}\right)^{\frac{n_1+2r}{2}}}{\beta\left(\frac{n_1}{2}, \frac{n_2}{2}\right)} \int_0^\infty \frac{\left(\frac{n_1}{n_2}F\right)^{\frac{n_1+2r}{2}-1}}{\left(1+\frac{n_1}{n_2}F\right)^{\frac{n_1+n_2}{2}}}\, d\left(\frac{n_1}{n_2}F\right).$$

$$=\frac{\left(\frac{n_1}{n_2}\right)^{\frac{n_1}{2}}\left(\frac{n_2}{n_1}\right)^{\frac{n_1+2r}{2}}}{\beta\left(\frac{n_1}{2},\frac{n_2}{2}\right)}\int_0^{\infty}\frac{\left(\frac{n_1}{n_2}F\right)^{\frac{n_1+2r}{2}-1}}{\left(1+\frac{n_1}{n_2}F\right)^{\left(\frac{n_1+2r}{2}\right)+\left(\frac{n_2-2r}{2}\right)}}\,d\left(\frac{n_1}{n_2}F\right)$$

$$=\frac{\left(\frac{n_1}{n_2}\right)^{\frac{n_1}{2}}\left(\frac{n_2}{n_1}\right)^{\frac{n_1+2r}{2}}}{\beta\left(\frac{n_1}{2},\frac{n_2}{2}\right)}\,\beta\left(\frac{n_1+2r}{2},\frac{n_2-2r}{2}\right)$$

$$=\frac{\left(\frac{n_2}{n_1}\right)^{r}}{\frac{\Gamma\frac{n_1}{2}\,\Gamma\frac{n_2}{2}}{\Gamma\left(\frac{n_1+n_2}{2}\right)}}\;\frac{\Gamma\left(\frac{n_1+2r}{2}\right)\Gamma\left(\frac{n_2-2r}{2}\right)}{\Gamma\left(\frac{n_1+2r}{2}+\frac{n_2-2r}{2}\right)}$$

$$=\left(\frac{n_2}{n_1}\right)^{r}\frac{\Gamma\left(\frac{n_1+n_2}{2}\right)}{\Gamma\frac{n_1}{2}\,\Gamma\frac{n_2}{2}}\;\frac{\Gamma\left(\frac{n_1+2r}{2}\right)\Gamma\left(\frac{n_2-2r}{2}\right)}{\Gamma\left(\frac{n_1+n_2}{2}\right)}$$

$$=\left(\frac{n_2}{n_1}\right)^{r}\frac{\Gamma\left(\frac{n_1+2r}{2}\right)\Gamma\left(\frac{n_2-2r}{2}\right)}{\Gamma\frac{n_1}{2}\,\Gamma\frac{n_2}{2}}$$

$$\left(\frac{n_2}{n_1}\right)^{r}\frac{\Gamma\left(\frac{n_1}{2}+r\right)\Gamma\left(\frac{n_2}{2}-r\right)}{\Gamma\frac{n_1}{2}\,\Gamma\frac{n_2}{2}};\quad r=1,2,3,\ldots$$

when $r=1$ then.

$$\mu_1' = \frac{n_2}{n_1}\,\frac{\Gamma\left(\frac{n_1}{2}+1\right)\Gamma\left(\frac{n_2}{2}-1\right)}{\Gamma\left(\frac{n_1}{2}\right)\Gamma\left(\frac{n_2}{2}\right)}$$

$$= \frac{n_2}{n_1}\,\frac{\frac{n_1}{2}\Gamma\left(\frac{n_1}{2}\right)\Gamma\left(\frac{n_2}{2}-1\right)}{\Gamma\left(\frac{n_1}{2}\right)\left(\frac{n_2}{2}-1\right)\Gamma\left(\frac{n_2}{2}-1\right)}$$

$$= \frac{n_2}{n_2-2} = \text{mean} \quad \text{where } n_2 > 2.$$

$$\mu_2' = \left(\frac{n_2}{n_1}\right)^2 \frac{\Gamma\left(\frac{n_1}{2}+2\right)\Gamma\left(\frac{n_2}{2}-2\right)}{\Gamma\left(\frac{n_1}{2}\right)\Gamma\left(\frac{n_2}{2}\right)}$$

$$= \left(\frac{n_2}{n_1}\right)^2 \frac{\left(\frac{n_1}{2}+1\right)\frac{n_1}{2}\,\cancel{\Gamma\left(\frac{n_1}{2}\right)}\,\Gamma\left(\frac{n_2}{2}-2\right)}{\cancel{\Gamma\left(\frac{n_1}{2}\right)}\left(\frac{n_2}{2}-1\right)\left(\frac{n_2}{2}-2\right)\Gamma\left(\frac{n_2}{2}-2\right)}$$

$$= \cancel{\left(\frac{n_2}{n_1}\right)^2}\ \frac{n_2^2}{n_1}\ \frac{\frac{n_1+2}{4}}{\frac{(n_2-2)}{2}\,\frac{(n_2-4)}{2}}$$

$$= \frac{n_2^2\,(n_1+2)}{n_1\,(n_2-2)(n_2-4)}\ ; \quad n_2 > 4.$$

$$\mu_2 = \mu_2' - \mu_1'^2$$

$$= \frac{n_2^2}{n_1}\frac{(n_1+2)}{(n_2-2)(n_2-4)} - \frac{n_2^2}{(n_2-2)^2}$$

$$= \frac{n_2^2(n_1+2)(n_2-2) - n_1 n_2^2(n_2-4)}{n_1(n_2-2)^2(n_2-4)}$$

$$= \frac{n_2^2(n_1n_2 + 2n_2 - 2n_1 - 4) - n_2^2\{n_1(n_2-4)\}}{n_1(n_2-2)^2(n_2-4)}$$

$$= \frac{n_2^2}{(n_2-2)^2}\left\{\frac{n_1n_2 + 2n_2 - 2n_1 - 4 - n_1n_2 + 4n_1}{n_1(n_2-4)}\right\}$$

$$= \frac{n_2^2}{(n_2-2)^2}\left[2n_1 + 2n_2 - 4\right.$$

$$= \frac{n_2^2}{(n_2-2)^2}\left[\frac{2(n_1+n_2-2)}{n_1(n_2-4)}\right];$$

Establish Beta distribution of first kind from F distribution.

Soln: According to the question, the pdf of F distribution is given by,

$$f(F) = \frac{n_1^{n_1/2}\, n_2^{n_2/2}\, F^{n_1/2 - 1}}{\beta\left(\frac{n_1}{2}, \frac{n_2}{2}\right)(n_2 + n_1 F)^{\frac{n_1+n_2}{2}}} \qquad \begin{array}{l} 0 \le F < \infty \\ n_1 > 0 \text{ and } n_2 > 0 \end{array}$$

Let, $X = \dfrac{n_1 F}{n_2 + n_1 F}$

or $x n_2 + x n_1 F = n_1 F$

or, $x n_2 = n_1 F(1-x)$

$\therefore F = \dfrac{n_2 x}{n_1(1-x)}$

$$\therefore \frac{dF}{dx} = \frac{n_2}{n_1} \frac{(1-x)1 - x(0-1)}{(1-x)^2}$$

$$= \frac{n_2}{n_1} \frac{1-x+x}{(1-x)^2}$$

$$= \frac{n_2}{n_1(1-x)^2}$$

i.e. $|J| = \left|\dfrac{dF}{dx}\right| = \dfrac{n_2}{n_1(1-x)^2}$

$$\therefore f(x) = \frac{n_1^{\frac{n_1}{2}} n_2^{\frac{n_2}{2}} \left(\frac{n_2 x}{n_1(1-x)}\right)^{\frac{n_1}{2}-1}}{\beta\left(\frac{n_1}{2}, \frac{n_2}{2}\right)\left(n_2 + n_1 \frac{n_2 x}{n_1(1-x)}\right)^{\frac{n_1+n_2}{2}}} |J|$$

$$= \frac{n_1\, n_2^{\frac{n_1+n_2}{2}-1} \left(\frac{x}{1-x}\right)^{\frac{n_1}{2}-1}}{\beta\left(\frac{n_1}{2}, \frac{n_2}{2}\right)\left(n_2 + \frac{n_2 x}{(1-x)}\right)^{\left(\frac{n_1+n_2}{2}\right)}} \cdot \frac{n_2}{n_1(1-x)^2}$$

$$= \frac{n_2^{\frac{n_1+n_2}{2}} x^{\frac{n_1}{2}-1}}{\beta\left(\frac{n_1}{2}, \frac{n_2}{2}\right)\left(\frac{n_2(1-x)+n_2 x}{(1-x)}\right)^{\frac{n_1+n_2}{2}}} \cdot \frac{1}{(1-x)^{\frac{n_1}{2}-1+2}}$$

$$= \frac{n_2^{\frac{n_1+n_2}{2}} x^{\frac{n_1}{2}-1}}{\beta\left(\frac{n_1}{2}, \frac{n_2}{2}\right)\left(\frac{n_2 - n_2 x + n_2 x}{1-x}\right)^{\frac{n_1+n_2}{2}}} \cdot \frac{1}{(1-x)^{\frac{n_1}{2}+1}}$$

$$= \frac{n_2^{\frac{n_1+n_2}{2}} x^{\frac{n_1}{2}-1}}{\beta\left(\frac{n_1}{2}, \frac{n_2}{2}\right)\left(\frac{n_2}{1-x}\right)^{\frac{n_1+n_2}{2}}} \cdot \frac{1}{(1-x)^{\frac{n_1}{2}+1}}$$

$$= \frac{x^{\frac{n_1}{2}-1} (1-x)^{\frac{n_1+n_2}{2}}}{\beta\left(\frac{n_1}{2}, \frac{n_2}{2}\right)} \cancel{(1-x)} \frac{1}{(1-x)^{\frac{n_1}{2}+1}}$$

Mathemetical statistics. chapter 13 (Large sample theory) gupta kapur

(17) statistical inference | (18) definition.

$$f(x) = \frac{1}{\beta\left(\frac{n_1}{2}, \frac{n_2}{2}\right)} x^{\frac{n_1}{2}-1} (1-x)^{\frac{n_2}{2}-1} \quad 0 < x < 1$$

which is the pdf of beta distribution (first kind) with parameter $\frac{n_1}{2}$ and $\frac{n_2}{2}$.

Table 12 → Table 17

$\frac{0.05}{2}$, 0.025

1 - 0.025 = .975

Test definition several

- mean
- variance
- proportion
- correlation coefficient

4, 4 6, 6 10, 12, 18, 22, 23, 25, 28, 33 44,

− − − + + − − + + + − − − + + + +

+ − − + +

$N_1 = 11$ (−)

$N_2 = 11$ (+).

$\boxed{\text{Chi-square}}$

Gupta
Kapur

f → mode, Harmonic mean
resiprocal of arithmetic mean,

$E(\frac{1}{F}) =$

properties.

r.th order moment
mean varian
$n \to \infty$ norm
beta first kind ⟷ f
mean, variance
skewness / kurtosis

$f \to \chi^2$

① Define point estimation with example.

(ii) What are the methods of point estimation.

(iii) What are the desirable critaria of a good estimator (4 ta crit

(1) unbiased
consitency
efficiency
sufficience

29.

(iv) What is M.L.E (maximum likelihood est)

(v) Let $x_1, x_2, \ldots x_n$ be a random sample from a poison distribution with pdf given by $f(x|\lambda) = \frac{e^{-\lambda} \lambda^{x}}{x!}$; $x = 0, 1, 2, \ldots \infty$.

Find the maximum likelihood estimator of λ. Show that the estimator is unbiased.

⇒ attribute $\frac{r \times c}{\text{theory}}$ + math. $\boxed{\text{tables connection}}$ } non-para

non-para

mean, several, corelation → theory.

$$M = 0.43429 \left[1 + \frac{1}{3(k-1)} \left\{ \sum \frac{1}{v_i} - \frac{1}{v} \right\} \right]$$