Graphs

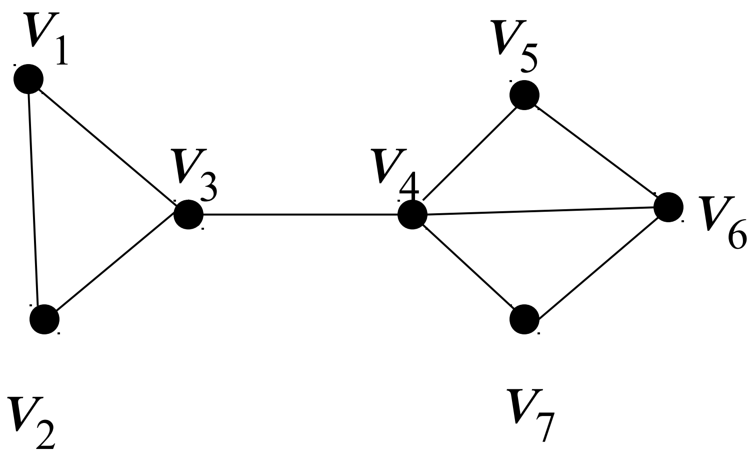
Outline

- Graph and Graph Models
- Graph Terminology and Special Types of Graphs
- Representing Graphs and Graph Isomorphism
- Connectivity
- Euler and Hamiltonian Paths
- Shortest-Path Problems
- Planar Graphs
- Graph Coloring

Introduction to Graphs

Def 1. A graph $G = (V, E)$ consists of V, a nonempty set of vertices (or nodes), and E, a set of edges. Each edge has either one or two vertices associated with it, called its endpoints. An edge is said to connect its endpoints.

eg.



$G=(V, E)$, where

$$V=\{v_1, v_2, \ldots, v_7\}$$

$$E=\{\{v_1,v_2\}, \{v_1,v_3\}, \{v_2,v_3\}, \{v_3,v_4\}, \{v_4,v_5\}, \{v_4,v_6\}, \{v_4,v_7\}, \{v_5,v_6\}, \{v_6,v_7\}\}$$

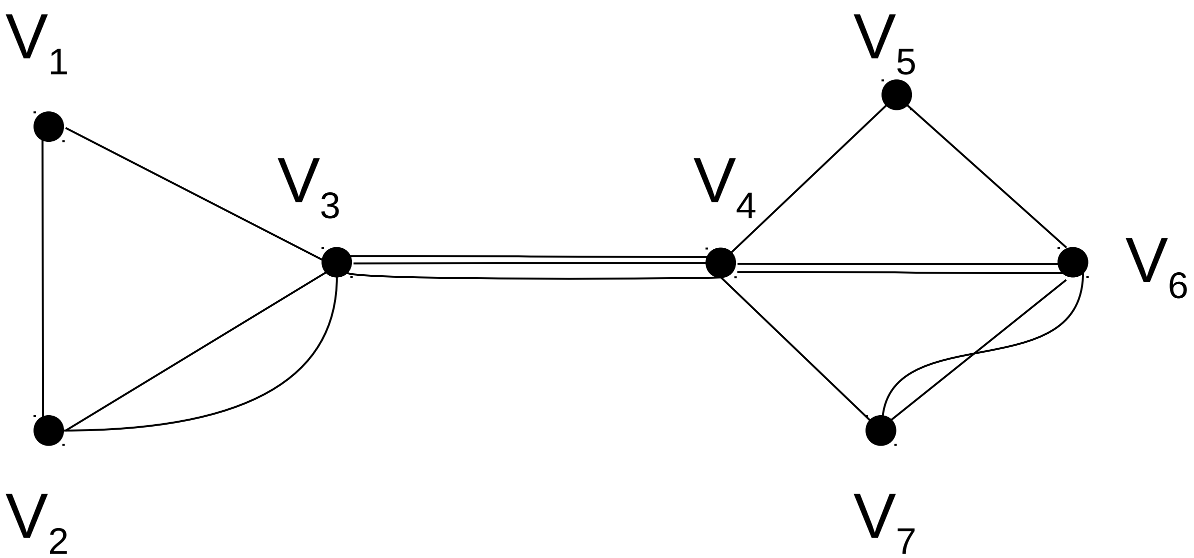
Def A graph in which each edge connects two different vertices and where no two edges connect the same pair of vertices is called a simple graph.

Def Multigraph:

simple graph + multiple edges (multiedges)

(Between two points to allow multiple edges)

eg.



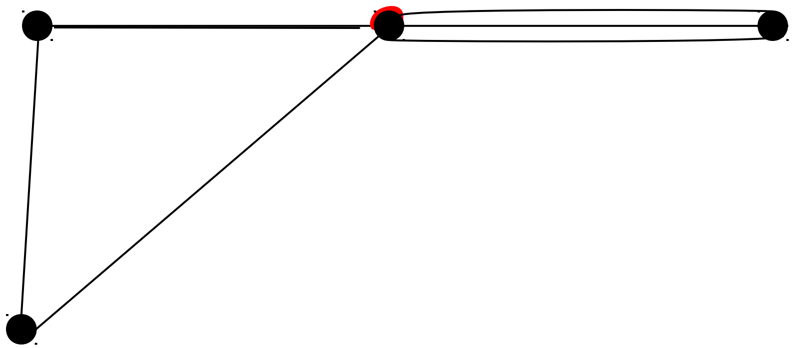
Def. Pseudograph:

simple graph + multiedge

+ loop

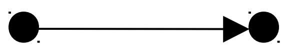
(a loop: )

eg.



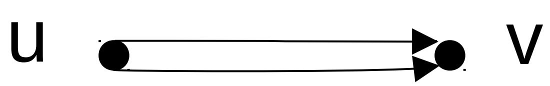
Def 2. Directed graph (digraph):

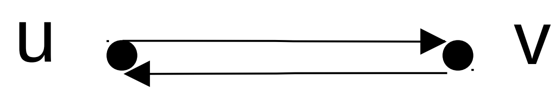
simple graph with each edge directed



Note:  is allowed in a directed graph

Note:





The two edges (u,v), (u,v) are multiedges.

The two edges (u,v), (v,u) are not multiedges.

Def. Directed multigraph: digraph+multiedges

Table 1. Graph Terminology

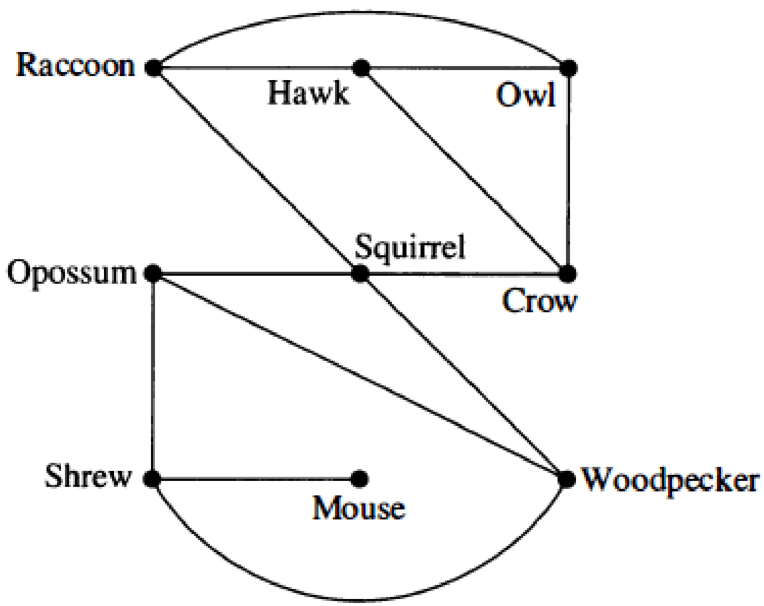
Type	Edges	Multiple Edges	Loops
(simple) graph	undirected edge: {u,v}	✗	✗
Multigraph		✓	✗
Pseudograph		✓	✓
Directed graph	directed edge: (u,v)	✗	✓
Directed multigraph		✓	✓

Graph Models

Example 1. (Niche Overlap graph)

We can use a simple graph to represent interaction of different species of animals. Each animal is represented by a vertex. An undirected edge connects two vertices if the two species represented by these vertices compete.

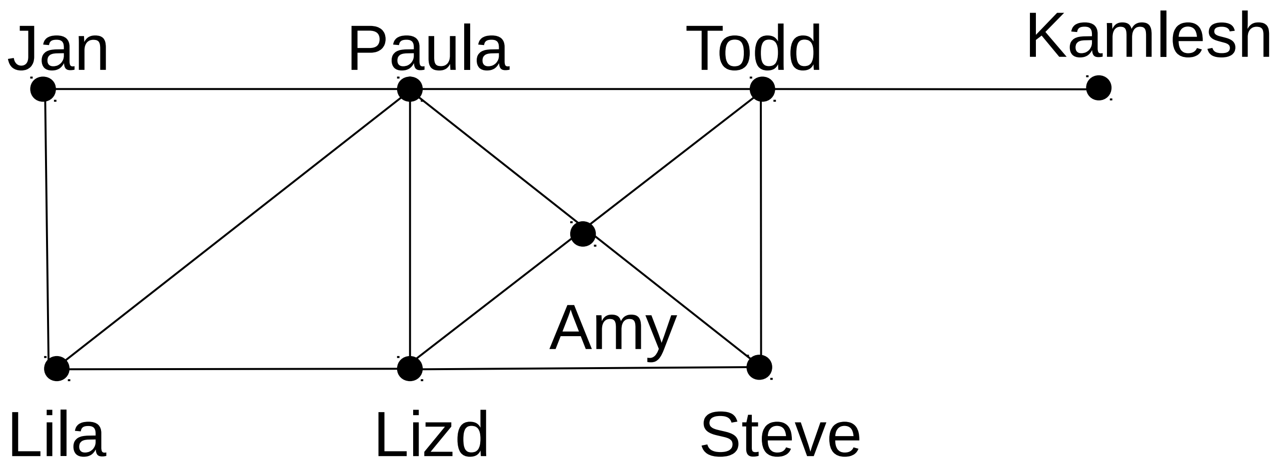
eg



Example 2. (Acquaintanceship graphs)

We can use a simple graph to represent whether two people know each other. Each person is represented by a vertex. An undirected edge is used to connect two people when these people know each other.

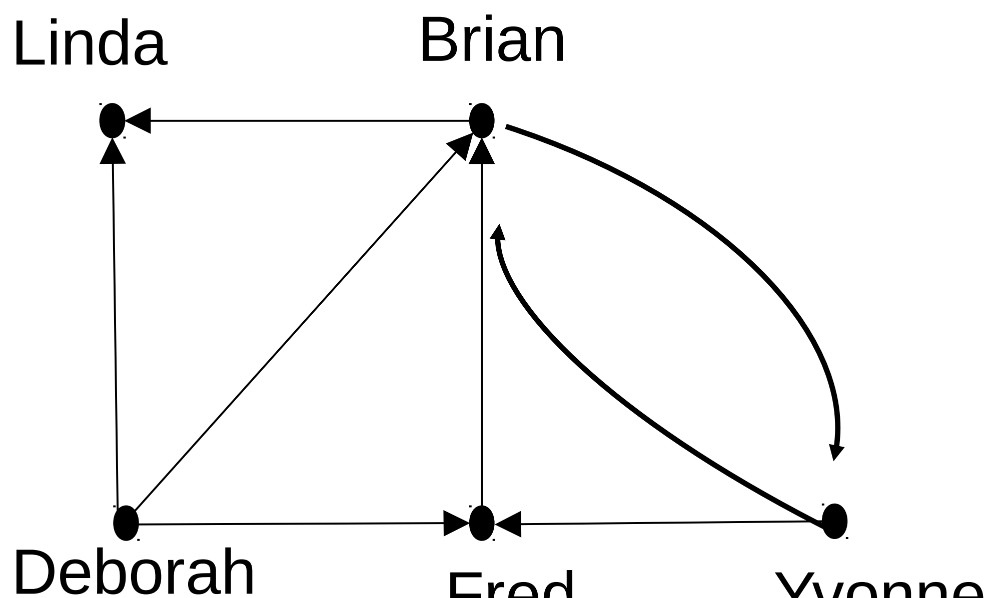
eg



Example 3. (Influence graphs)

In studies of group behavior it is observed that certain people can influence the thinking of others. Simple digraph $\Rightarrow$ Each person of the group is represented by a vertex. There is a directed edge from vertex a to vertex b when the person a influences the person b.

eg



Example 9. (Precedence graphs and concurrent processing)

Computer programs can be executed more rapidly by executing certain statements concurrently. It is important not to execute a statement that requires results of statements not yet executed.

Simple digraph $\Rightarrow$ Each statement is represented by a vertex, and there is an edge from a to b if the statement of b cannot be executed before the statement of a.

eg

S_1: a:=0

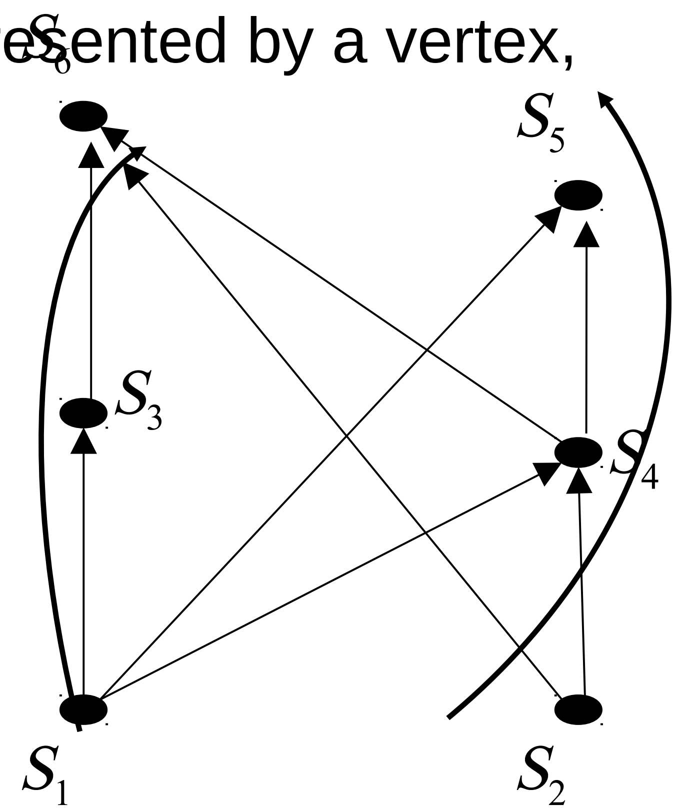
S_2: b:=1

S_3: c:=a+1

S_4: d:=b+a

S_5: e:=d+1

S_6: e:=c+d



Graph Terminology

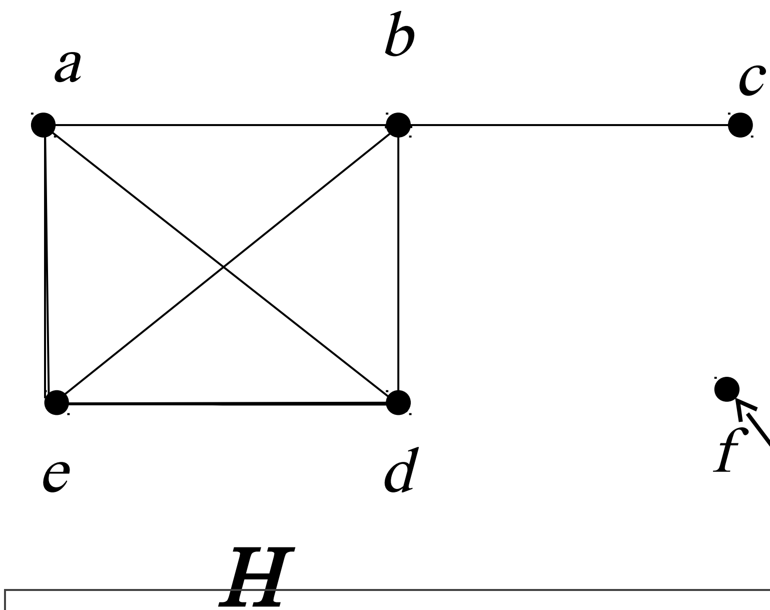
Def 1. Two vertices u and v in a undirected graph G are called adjacent (or neighbors) in G if $\{u, v\}$ is an edge of G.

Note : adjacent: a vertex connected to a vertex
incident: a vertex connected to an edge

Def 2. The degree of a vertex v, denoted by $\deg(v)$, in an undirected graph is the number of edges incident with it.

(Note : A loop adds 2 to the degree.)

Example 1. What are the degrees of the vertices in the graph H?



Sol :

$\deg(a)=4$

$\deg(b)=6$

$\deg(c)=1$

$\deg(d)=5$

$\deg(e)=6$

$\deg(f)=0$

Def. A vertex of degree 0 is called isolated.

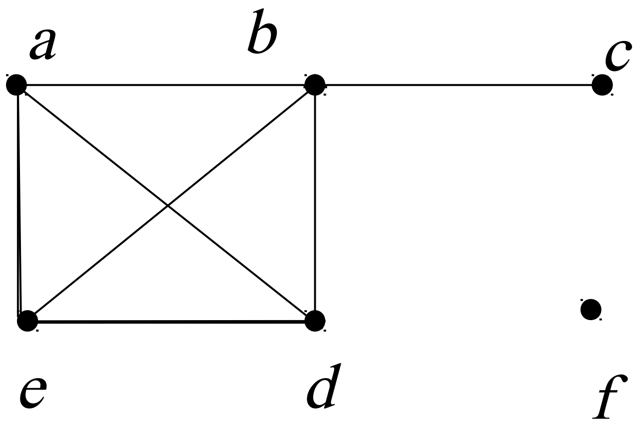
Def. A vertex is **pendant if and only** if it has degree one.

Thm 1. (The Handshaking Theorem)

Let $\boldsymbol{G} = (\boldsymbol{V}, \boldsymbol{E})$ be an undirected graph with e edges (i.e., $|\boldsymbol{E}| = \boldsymbol{e}$). Then

$$\sum_{v \in V} \deg(v) = 2e$$

eg.



H

The graph H has 11 edges, and

$$\sum_{v \in V} \deg(v) = 22$$

Example 3. How many edges are there in a graph with 10 vertices each of degree six?

Sol :

$$10 \cdot 6 = 2e \quad \Rightarrow \quad e = 30$$

Thm 2. An undirected graph $G=(V, E)$ has an even number of vertices of odd degree.

Pf : Let $V_1=\{v\in V \,/\, \deg(v) \text{ is even}\}$,

$V_2=\{v\in V \,/\, \deg(v) \text{ is odd}\}$.

$$2e=\sum_{v\in V_1}\deg(v)+\sum_{v\in V_2}\deg(v) \Rightarrow \sum_{v\in V_2}\deg(v) \text{ is even.}$$

Def 3. $G=(V, E)$: directed graph,

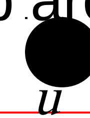
$e=(u, v)\in E$: u is adjacent to v

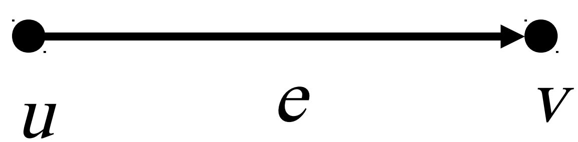
v is adjacent from u

u : initial vertex of e

v : terminal (end) vertex of e

The initial vertex and terminal vertex of a loop are the same





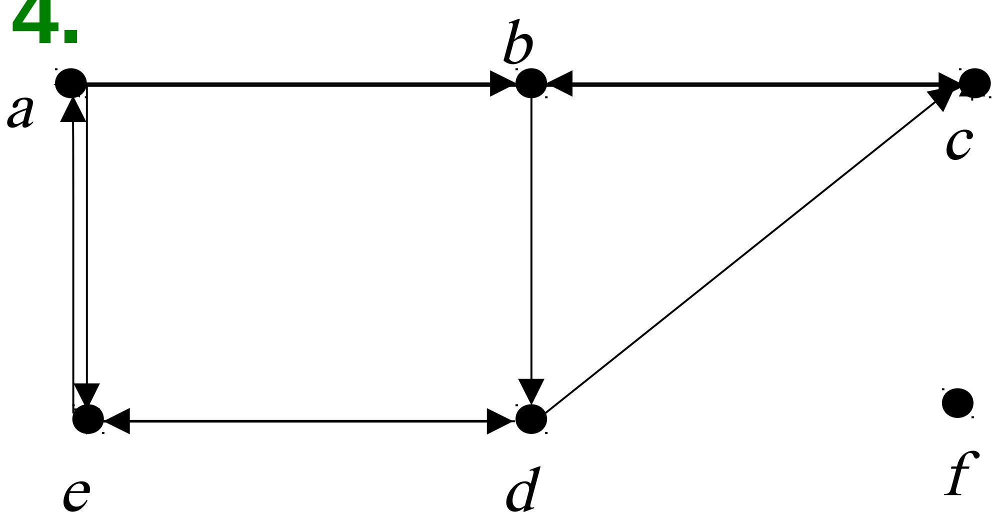
Def 4.

$G=(V,E)$: directed graph, $v\in V$

$\deg^-(v)$: # of edges with v as a terminal.
(in-degree)

$\deg^+(v)$: # of edges with v as a initial vertex
(out-degree)

Example 4.



$\deg^-(a)=2$, $\deg^+(a)=4$
$\deg^-(b)=2$, $\deg^+(b)=1$
$\deg^-(c)=3$, $\deg^+(c)=2$
$\deg^-(d)=2$, $\deg^+(d)=2$
$\deg^-(e)=3$, $\deg^+(e)=3$
$\deg^-(f)=0$, $\deg^+(f)=0$

Thm 3. Let $G = (V, E)$ be a digraph. Then

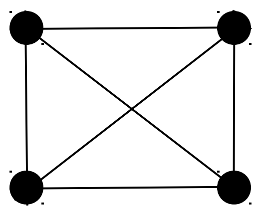
$$\sum_{v \in V} \deg^{-}(v) = \sum_{v \in V} \deg^{+}(v) = |E|$$

Regular Graph

A simple graph $G=(V, E)$ is called regular if every vertex of this graph has the same degree. A regular graph is called n-regular if $\deg(v)=n$, $\forall v\in V$.

eg.

K_4 :

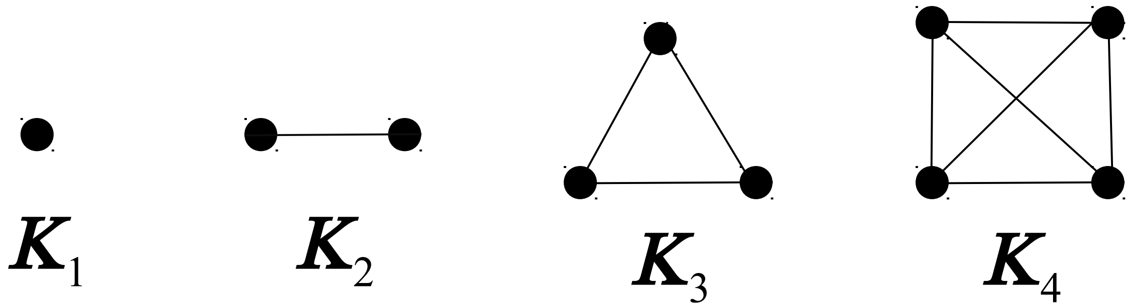


is 3-regular.

Some Special Simple Graphs

Example 5.

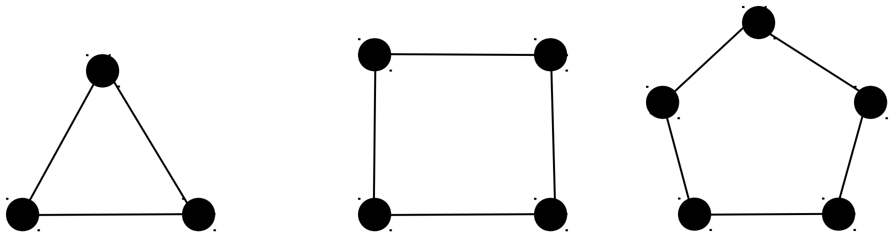
The complete graph on n vertices, denoted by K_n, is the simple graph that contains exactly one edge between each pair of distinct vertices.



Note. K_n is $(n-1)$-regular, $|V(K_n)|=n$,

$$|E(K_n)| = \binom{n}{2}$$

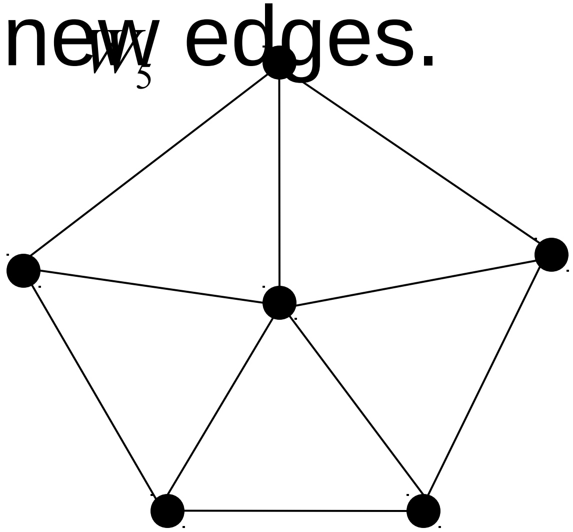
Example 6. The cycle C_n, $n \geq 3$, consists of n vertices $v_1, v_2, \ldots, v_n$ and edges $\{v_1, v_2\}$, $\{v_2, v_3\}$, $\ldots$, $\{v_{n-1}, v_n\}$, $\{v_n, v_1\}$.

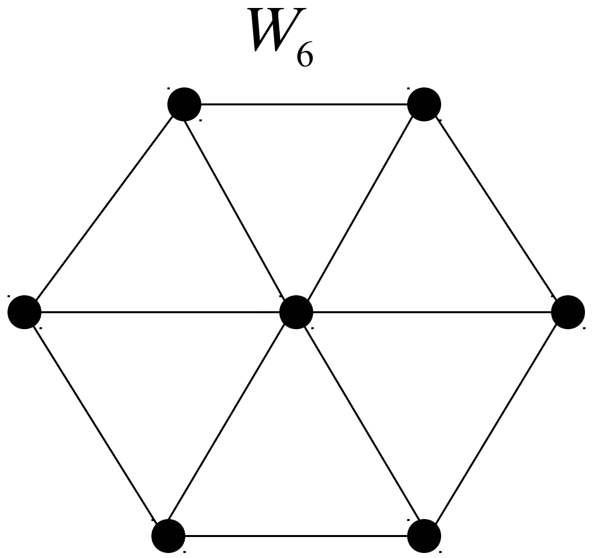


C_3 C_4 C_5

Note C_n is 2-regular, $|V(C_n)| = n$, $|E(C_n)| = n$

Example 7. We obtained the wheel W_n when we add an additional vertex to the cycle C_n, for $n \geq 3$, and connect this new vertex to each of the n vertices in C_n, by new edges.



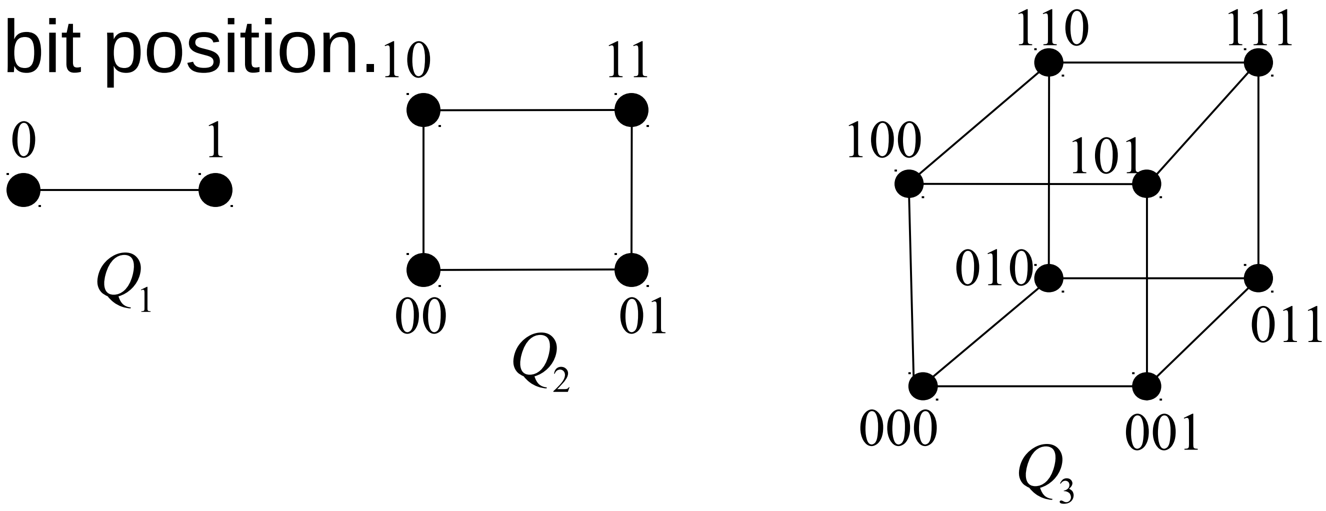


Note. $|V(W_n)| = n + 1$, $|E(W_n)| = 2n$,

W_n is not a regular graph if $n \neq 3$.

Example 8. The n-dimensional hypercube, or n-cube, denoted by Q_n, is the graph that has vertices representing the 2^n bit strings of length n.

Two vertices are adjacent if and only if the bit strings that they represent differ in exactly one bit position.

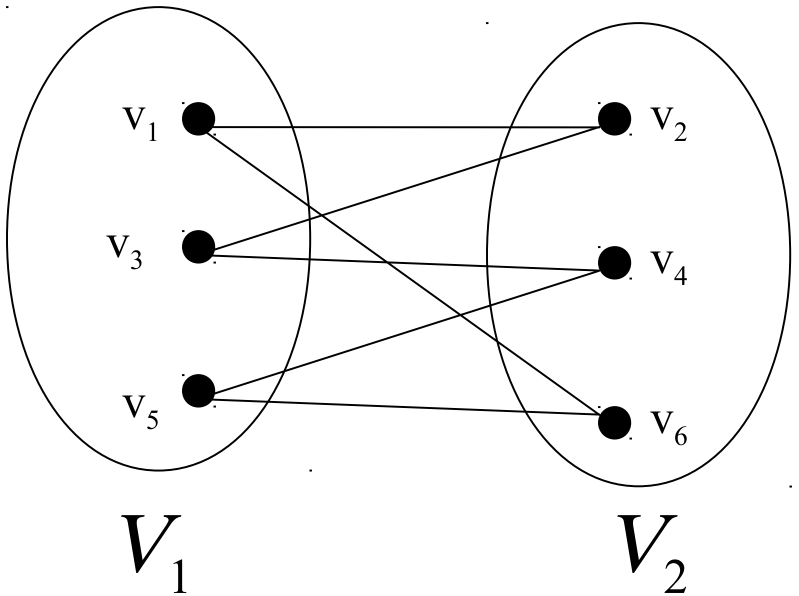


Note. Q_n is n-regular, $|V(Q_n)| = 2^n$, $|E(Q_n)| = (2^n n)/2 = 2^{n-1}n$

Some Special Simple Graphs

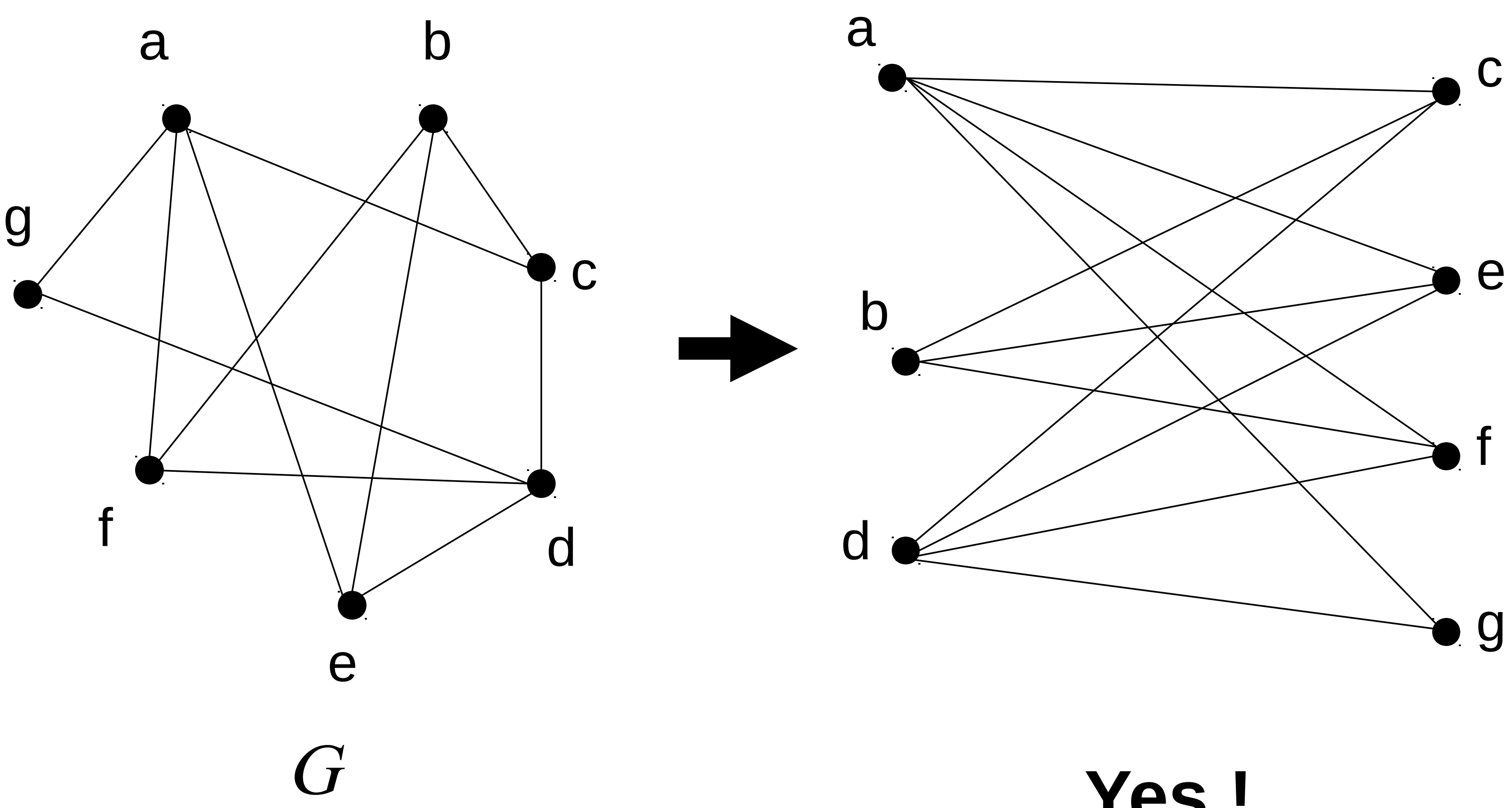
Def 5. A simple graph $G=(V,E)$ is called bipartite if V can be partitioned into V_1 and V_2, $V_1 \cap V_2 = \varnothing$, such that every edge in the graph connect a vertex in V_1 and a vertex in V_2.

Example 9.



∴ C_6 is bipartite.

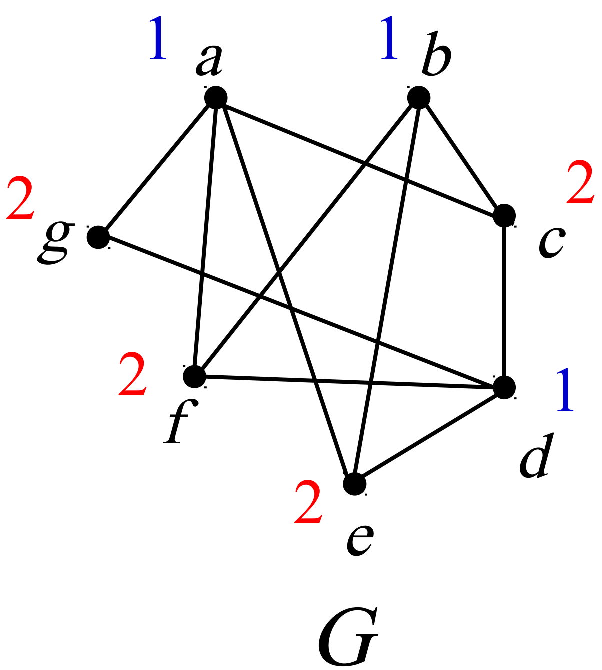
Example 10. Is the graph G bipartite ?



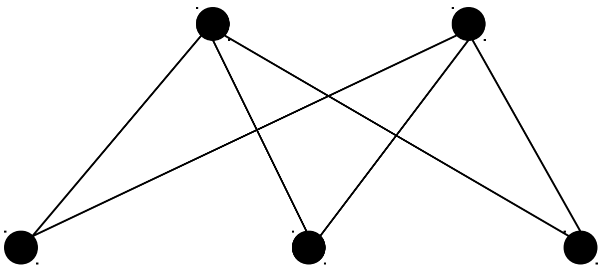
Yes !

Thm 4. A simple graph is bipartite if and only if it is possible to assign one of two different colors to each vertex of the graph so that no two adjacent vertices are assigned the same color.

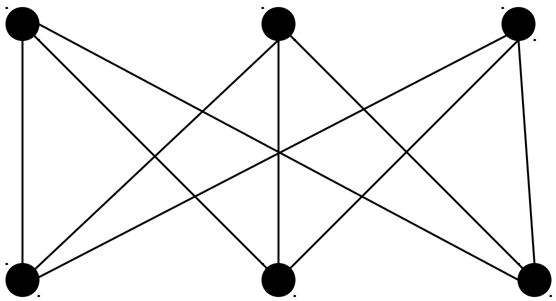
Example 12. Use Thm 4 to show that G is bipartite.



- Example 11. Complete Bipartite graphs ($K_{m,n}$)



$K_{2,3}$



$K_{3,3}$

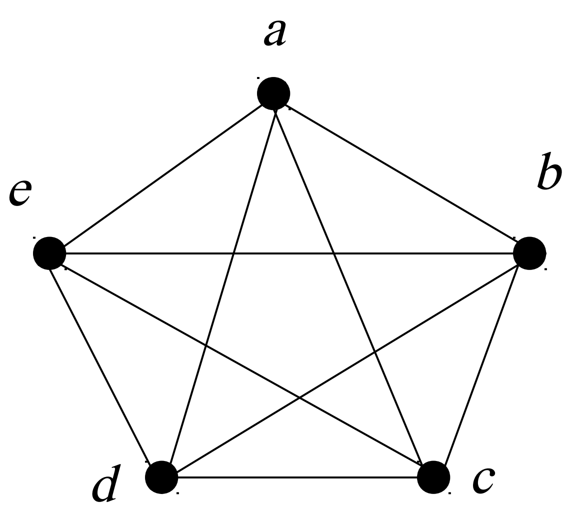
Note. $|V(K_{m,n})| = m+n$, $|E(K_{m,n})| = mn$,

$K_{m,n}$ is regular if and only if $m=n$.

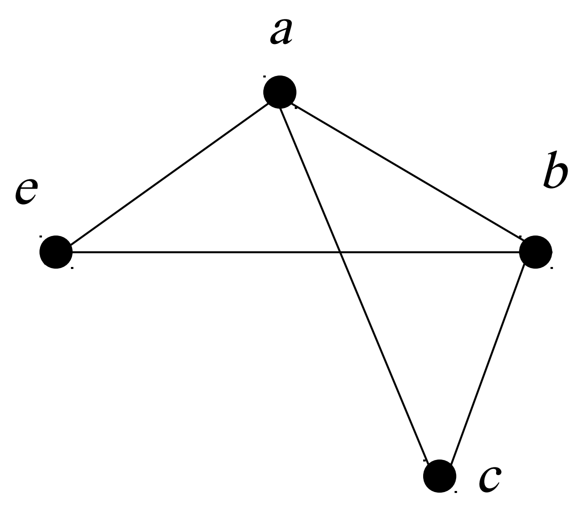
New Graphs from Old

Def 6. A subgraph of a graph $G=(V, E)$ is a graph $H=(W, F)$ where $W \subseteq V$ and $F \subseteq E$. (Notice the f point w to connect)

Example 14. A subgraph of K_5



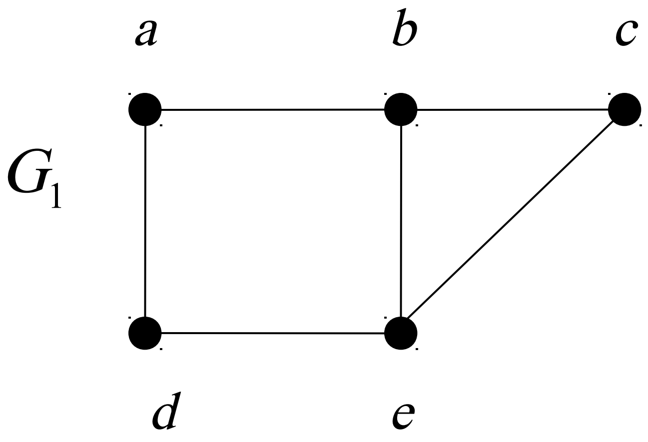
K_5

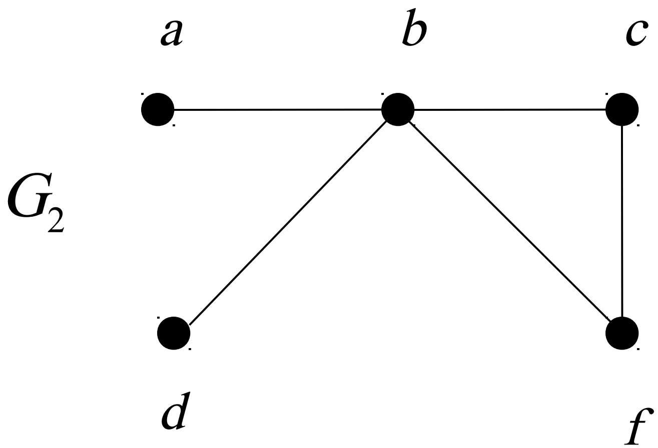


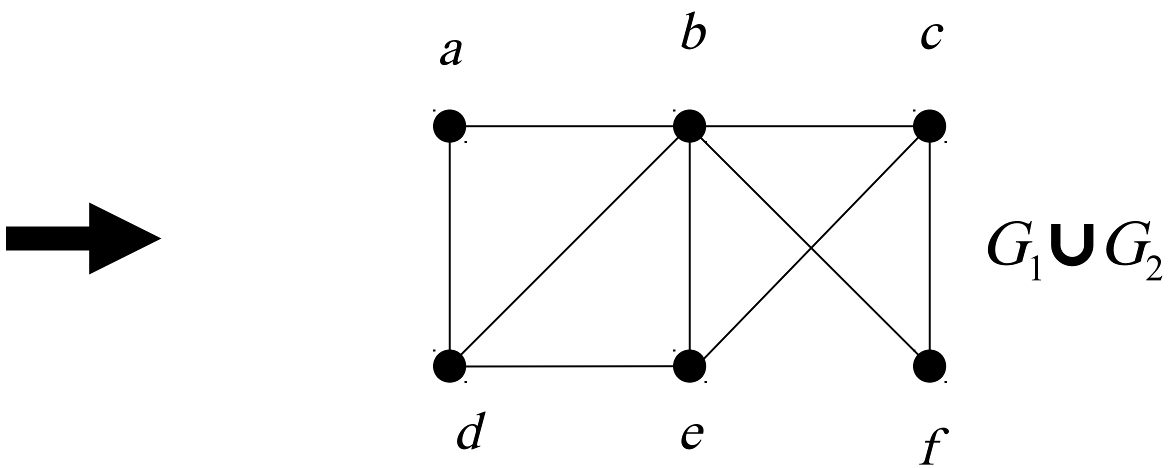
subgraph of K_5

Def 7. The union of two simple graphs $G_1=(V_1, E_1)$ and $G_2=(V_2, E_2)$ is the simple graph $G_1 \cup G_2=(V_1 \cup V_2, E_1 \cup E_2)$

Example 15.



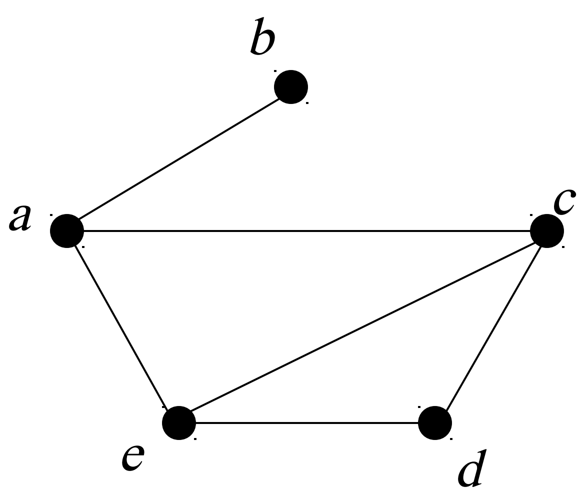




Representing Graphs and Graph Isomorphism

※Adjacency list

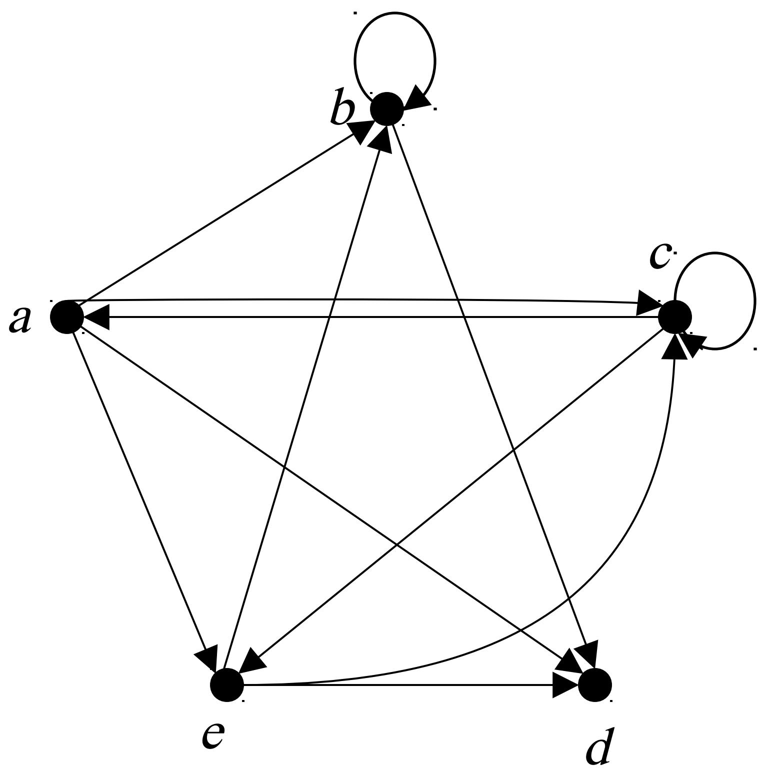
Example 1. Use adjacency lists to describe the simple graph given below.



Sol :

Vertex	Adjacent Vertices
a	b,c,e
b	a
c	a,d,e
d	c,e
e	a,c,d

Example 2. (digraph)



Initial vertex	Terminal vertices
a	*b,c,d,e*
b	*b,d*
c	*a,c,e*
d	
e	*b,c,d*

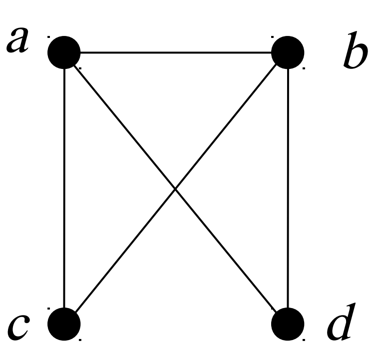
※Adjacency Matrices

Def. $G=(V, E)$: simple graph, $V=\{v_1, v_2, \ldots, v_n\}$.

A matrix A is called the adjacency matrix of G

if $A=[a_{ij}]_{n\times n}$, where $a_{ij}=\begin{cases} 1, \text{ if } \{v_i, v_j\}\in E, \\ 0, \text{ otherwise.} \end{cases}$

Example 3.

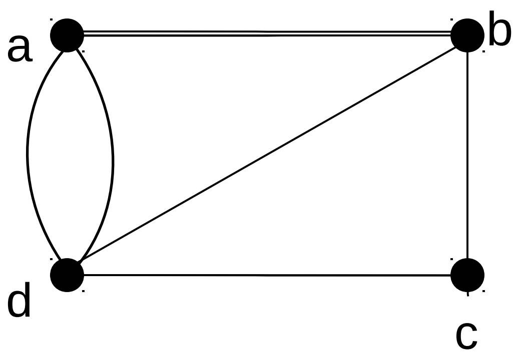


$$A_1 = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix} \end{array} \qquad A_2 = \begin{array}{c} \\ b \\ d \\ c \\ a \end{array} \begin{array}{c} \begin{array}{cccc} b & d & c & a \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

Note:

1. There are n! different adjacency matrices for a graph with n vertices.
2. The adjacency matrix of an undirected graph is symmetric.
3. $a_{ii} = 0$ (simple matrix has no loop)

Example 5. (Pseudograph) (Matrix may not be 0,1 matrix.)



$$A = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{bmatrix} 0 & 3 & 0 & 2 \\ 3 & 0 & 1 & 1 \\ 0 & 1 & 1 & 2 \\ 2 & 1 & 2 & 0 \end{bmatrix} \end{array}$$

Def. If $A=[a_{ij}]$ is the adjacency matrix for the directed graph, then

$$a_{ij} = \begin{cases} 1 & \text{, if } V_i \longrightarrow V_j \\ 0 & \text{, otherwise} \end{cases}$$

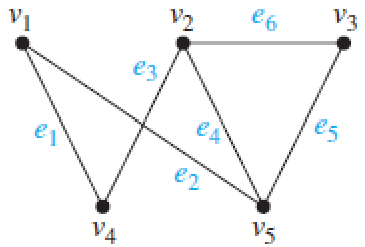
(So the matrix is not necessarily symmetrical)

Incidence Matrices

Def. Let $G=(V, E)$: be an undirected graph. Suppose that $v_1, v_2, \ldots, v_n$ are the vertices and $e_1, e_2, \ldots, e_n$ are the edges of G . Then the incidence matrix with respect to this ordering of V and E is the n x m matrix $M=[m_{ij}]$, *where*

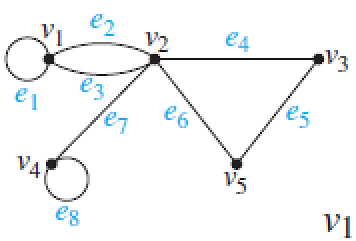
$$m_{i,j} = \begin{cases} 1 \text{ when edge } e_j \text{ is incident with } v_i, \\ 0 \text{ otherwise}. \end{cases}$$

Example 6.



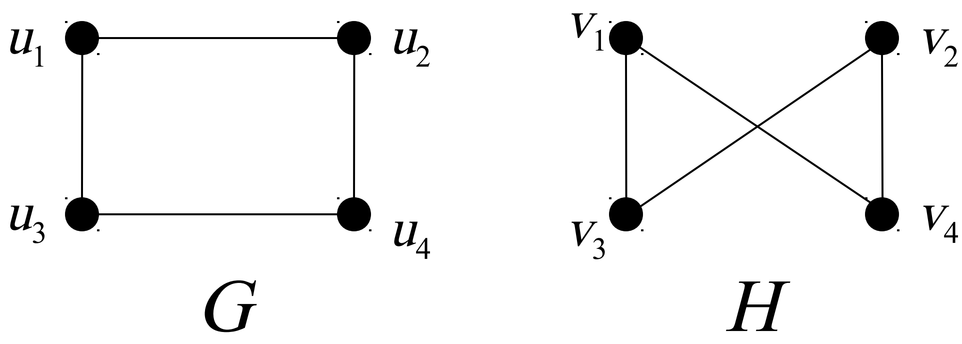
$$\begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{array} \begin{array}{c} \begin{array}{cccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \end{array} \\ \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 \end{bmatrix} \end{array}.$$

Example 7.



$$\begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{array} \begin{array}{c} \begin{array}{cccccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 & e_8 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

※Isomorphism of Graphs



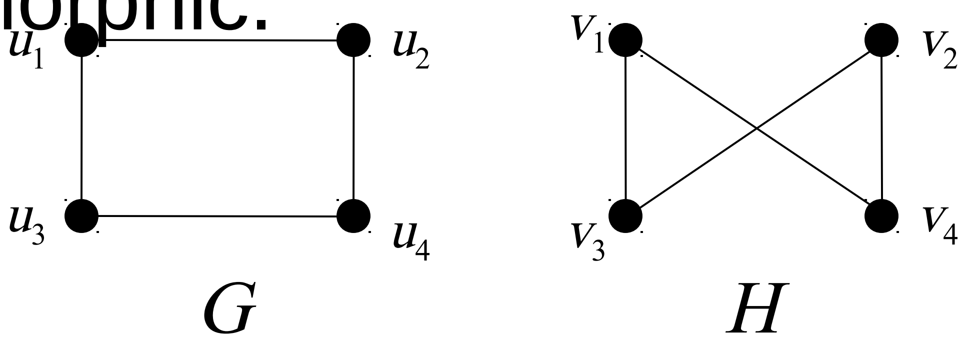
G is isomorphic to H

Def 1.

The simple graphs $G_1=(V_1,E_1)$ and $G_2=(V_2,E_2)$ are isomorphic if there is an one-to-one and onto function f from V_1 to V_2 with the property that $a\sim b$ in G_1 iff $f(a)\sim f(b)$ in G_2, $\forall a,b\in V_1$

f is called an isomorphism.

Example 8. Show that G and H are isomorphic.



Sol.

The function f with $f(u_1) = v_1$, $f(u_2) = v_4$, $f(u_3) = v_3$, and $f(u_4) = v_2$ is a one-to-one correspondence between $V(G)$ and $V(H)$.

❇Isomorphism graphs there will be:

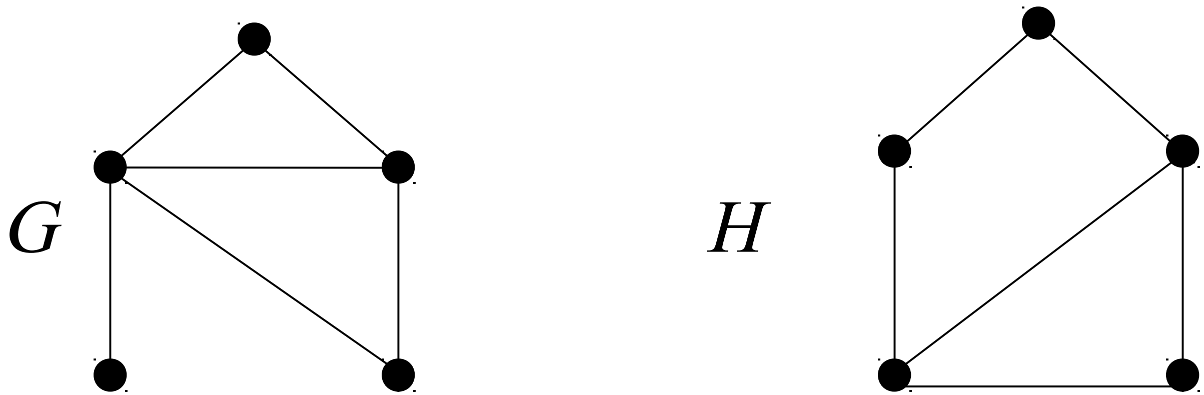
(1) The same number of points (vertices)

(2) The same number of edges

(3) The same number of degree

※Given figures, judging whether they are isomorphic in general is not an easy task.

Example 9. Show that G and H are not isomorphic.

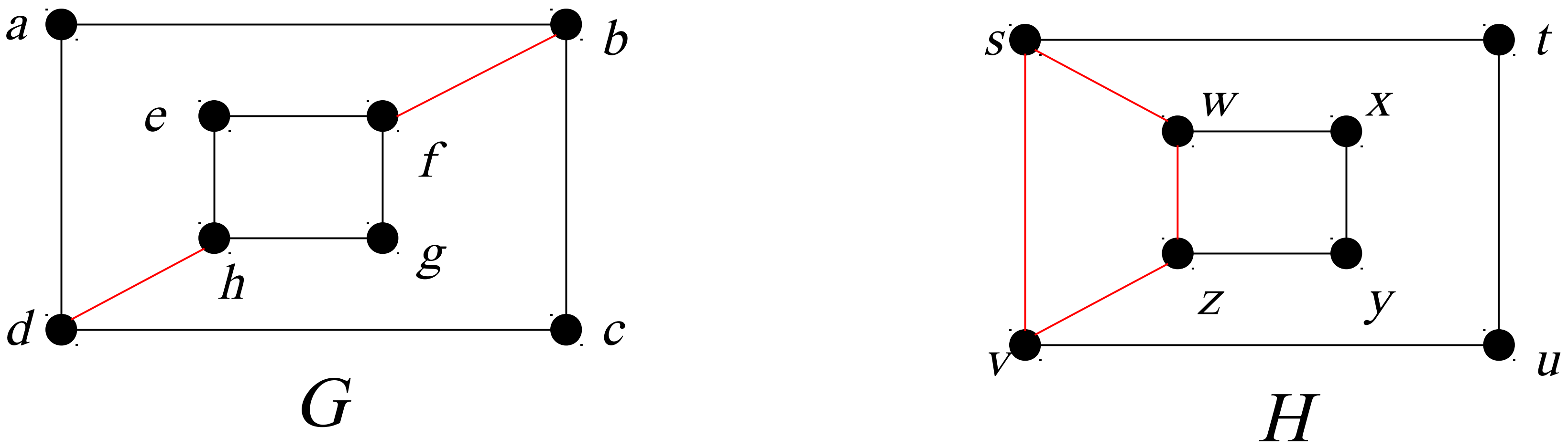


Sol :

G has a vertex of degree $= 1$, H don't

Example 10.

Determine whether G and H are isomorphic.



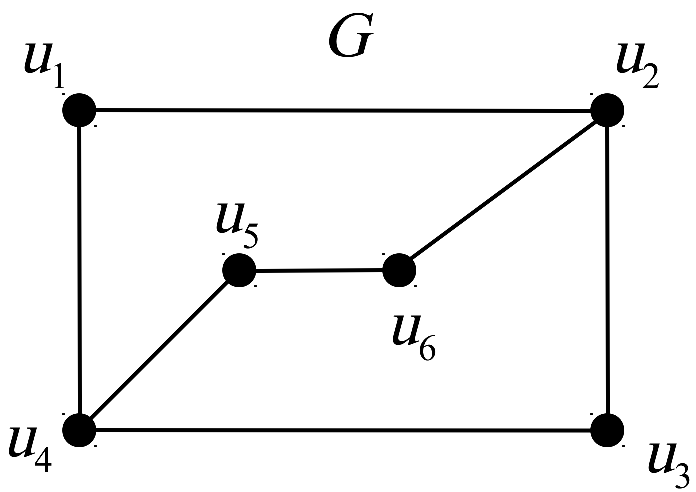
Sol : ∵ In G, deg(a)=2, which must correspond to either t, u, x, or y in H degree

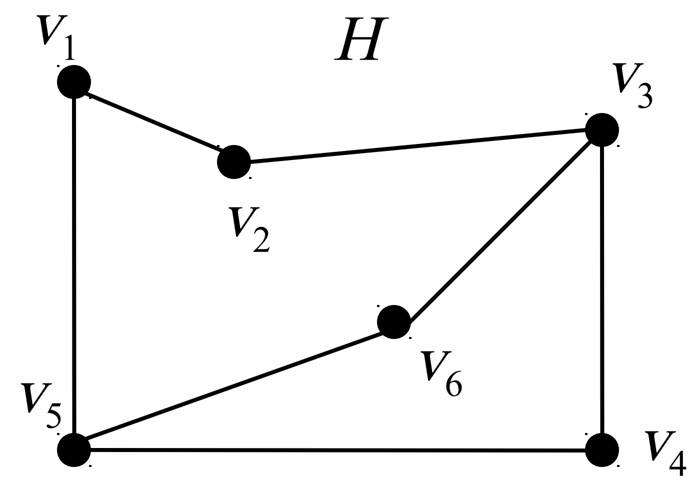
Each of these four vertices in H is adjacent to another vertex of degree two in H,

which is not true for a in G

∴ G and H are not isomorphic.

Example 11. Determine whether the graphs G and H are isomorphic.





Sol:

$f(u_1)=v_6,\ f(u_2)=v_3,\ f(u_3)=v_4,\ f(u_4)=v_5,\ f(u_5)=v_1,\ f(u_6)=v_2$

$\Rightarrow$ Yes

Graph - Isomorphism

G1 = (V1, E2) and G2 = (V2, E2) are isomorphic if:

There is a one-to-one and onto function f from V1 to V2 with the property that

a and b are adjacent in G1 if and only if f (a) and f (b) are adjacent in G2, for all a and b in V1.

Function f is called isomorphism

Application Example:

In chemistry, to find if two compounds have the same structure

Graph - Isomorphism

Representation example: G1 = (V1, E1) , G2 = (V2, E2)

$f(u_1) = v_1$, $f(u_2) = v_4$, $f(u_3) = v_3$, $f(u_4) = v_2$,

Connectivity

Basic Idea: In a Graph Reachability among vertices by traversing the edges

Application Example:

- In a city to city road-network, if one city can be reached from another city.

- Problems if determining whether a message can be sent between two

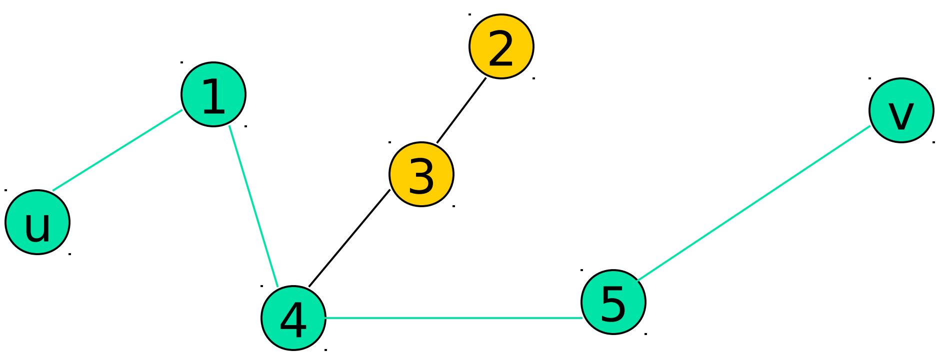
computer using intermediate links

- Efficiently planning routes for data delivery in the Internet

Connectivity – Path

A **Path** is a sequence of edges that begins at a vertex of a graph and travels along edges of the graph, always connecting pairs of adjacent vertices.

Representation example: G = (V, E), Path P represented, from u to v is {{u, 1}, {1, 4}, {4, 5}, {5, v}}



Connectivity – Path

Definition for Directed Graphs

A **Path** of length n (> 0) from u to v in G is a sequence of n edges e_1, e_2 , e_3, …, e_n of G such that f (e_1) = (x_o, x_1), f (e_2) = (x_1, x_2), …, f (e_n) = (x_{n-1}, x_n), where x_0 = u and x_n = v. A path is said to pass through x_0, x_1, …, x_n or traverse e_1, e_2 , e_3, …, e_n

For Simple Graphs, sequence is x_0, x_1, …, x_n

In directed multigraphs when it is not necessary to distinguish between their edges, we can use sequence of vertices to represent the path

Circuit/Cycle: u = v, length of path > 0

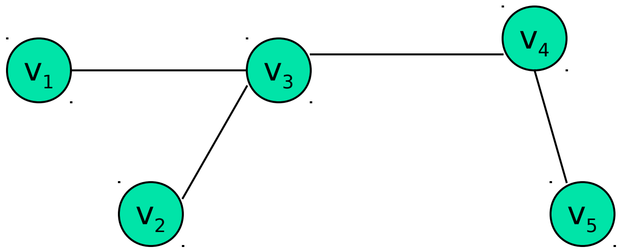
Simple Path: does not contain an edge more than once

Connectivity – Connectedness

Undirected Graph

An undirected graph is connected if there exists is a simple path between every pair of vertices

Representation Example: G (V, E) is connected since for V = {v_1, v_2, v_3, v_4, v_5}, there exists a path between {v_i, v_j}, $1 \leq i, j \leq 5$



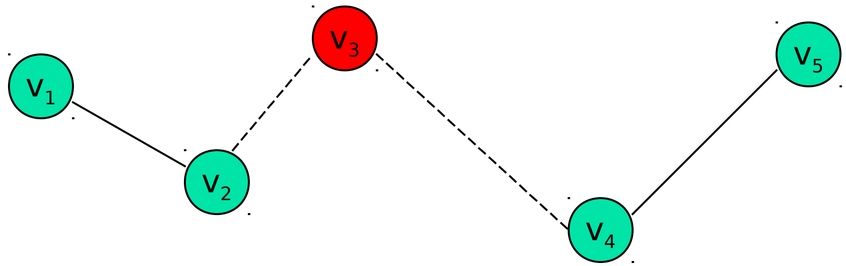
Connectivity – Connectedness

Undirected Graph

Articulation Point (Cut vertex): removal of a vertex produces a subgraph with more connected components than in the original graph. The removal of a cut vertex from a connected graph produces a graph that is not connected

Cut Edge: An edge whose removal produces a subgraph with more connected components than in the original graph.

Representation example: G (V, E), v_3 is the articulation point or edge $\{v_2, v_3\}$, the number of connected components is 2 (> 1)



Connectivity – Connectedness

Directed Graph

A directed graph is **strongly connected** if there is a path from a to b and from b to a whenever a and b are vertices in the graph

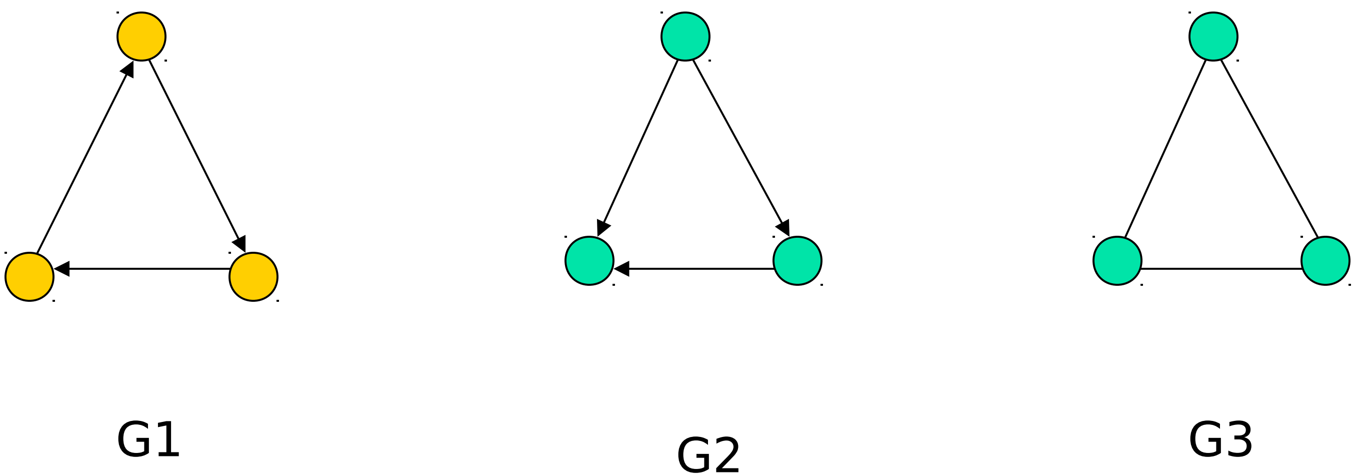
A directed graph is **weakly connected** if there is a (undirected) path between every two vertices in the underlying undirected path

A strongly connected Graph can be weakly connected but the vice-versa is not true (why?)

Connectivity – Connectedness

Directed Graph

Representation example: G1 (Strong component), G2 (Weak Component), G3 is undirected graph representation of G2 or G1

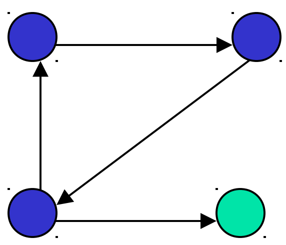


Connectivity – Connectedness

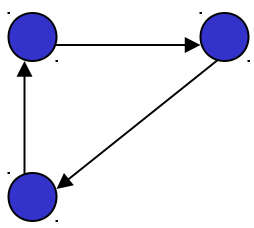
Directed Graph

Strongly connected Components: subgraphs of a Graph G that are strongly connected

Representation example: G1 is the strongly connected component in G



G

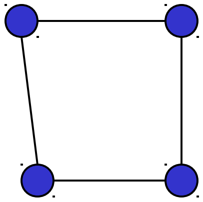


G1

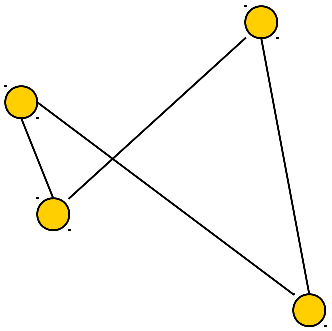
Isomorphism - revisited

A isomorphic invariant for simple graphs is the existence of a simple circuit of length k , k is an integer > 2 (why ?)

Representation example: G1 and G2 are isomorphic since we have the invariants, similarity in degree of nodes, number of edges, length of circuits



G1



G2

Counting Paths

Theorem: Let G be a graph with adjacency matrix A with respect to the ordering v_1, v_2, …, V_n (with directed on undirected edges, with multiple edges and loops allowed). The number of different paths of length r from Vi to Vj, where r is a positive integer, equals the (i, j)th entry of (adjacency matrix) $A^{r.}$

Proof: By Mathematical Induction.

Base Case: For the case N = 1, a_{ij} =1 implies that there is a path of length 1. This is true since this corresponds to an edge between two vertices.

We assume that theorem is true for N = r and prove the same for N = r +1. Assume that the (i, j)th entry of A^r is the number of different paths of length r from v_i to v_j. By induction hypothesis, b_{ik} is the number of paths of length r from v_i to v_k.

Counting Paths

Case r +1: In $A^{r+1} = A^r . A$,

The $(i, j)^{th}$ entry in A^{r+1} , $b_{i1}a_{1j} + b_{i2}\, a_{2j} + \ldots + b_{in}\, a_{nj}$

where b_{ik} is the $(i, j)^{th}$ entry of A^r.

By induction hypothesis, b_{ik} is the number of paths of length r from v_i to v_k.

The $(i, j)^{th}$ entry in A^{r+1} corresponds to the length between i and j and the length is r+1. This path is made up of length r from v_i to v_k and of length from v_k to vj. By product rule for counting, the number of such paths is $b_{ik^*}\, a_{kj}$ The result is $b_{i1}a_{1j} + b_{i2}\, a_{2j} + \ldots + b_{in}\, a_{nj}$, the desired result.

Connectivity

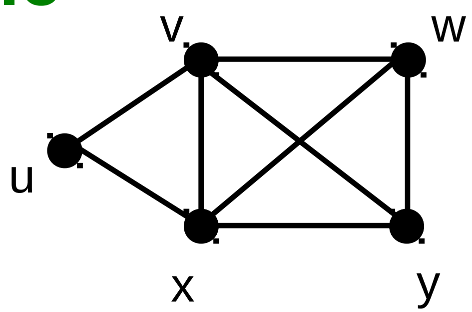
Def. 1 :

In an undirected graph, a path of length n from u to v is a sequence of $n+1$ adjacent vertices going from vertex u to vertex v.
(e.g., P: $u=x_0, x_1, x_2, \ldots, x_n=v$.) (P has n edges.)

Def. 2:

path: Points and edges in unrepeatable
trail: Allows duplicate path (not repeatable)
walk: Allows point and duplicate path

Example



path: u, v, y
trail: u, v, w, y, v, x, y
walk: u, v, w, v, x, v, y

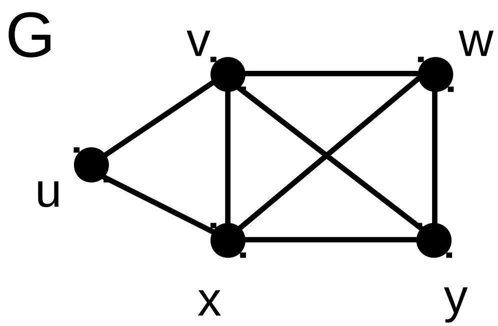
Def:

cycle: path with $u=v$

circuit: trail with $u=v$

closed walk: walk with $u=v$

Example



cycle: u, v, y, x, u

trail: u, v, w, y, v, x, u

walk: u, v, w, v, x, v, y, x, u

Paths in Directed Graphs

The same as in undirected graphs, but the path must go in the direction of the arrows.

Connectedness in Undirected Graphs

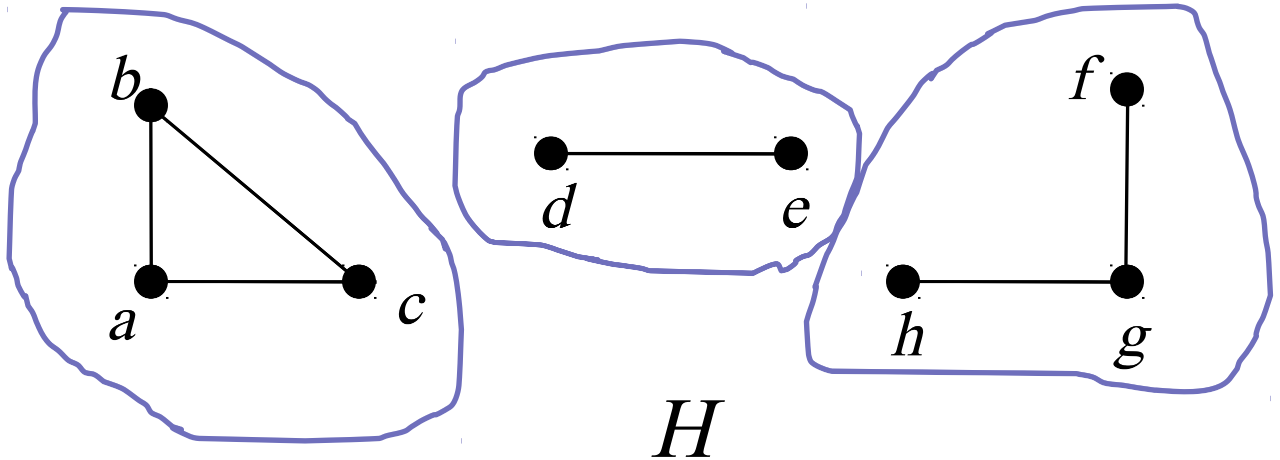
Def. 3:

An undirected graph is *connected* if there is a path between every pair of distinct vertices in the graph.

Def:

Connected component: maximal connected subgraph. (An unconnected graph will have several component)

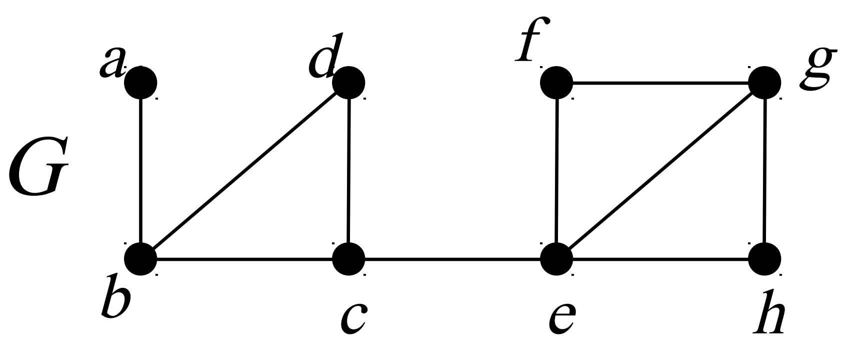
Example 6 What are the connected components of the graph H?



Def:

A *cut vertex* separates one connected component into several components if it is removed.
A *cut edge* separates one connected component into two components if it is removed.

Example 8. Find the cut vertices and cut edges in the graph G.



Sol:

cut vertices: b, c, e

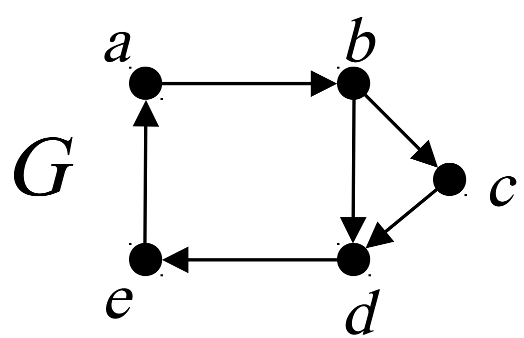
cut edges:

$\{a, b\}, \{c, e\}$

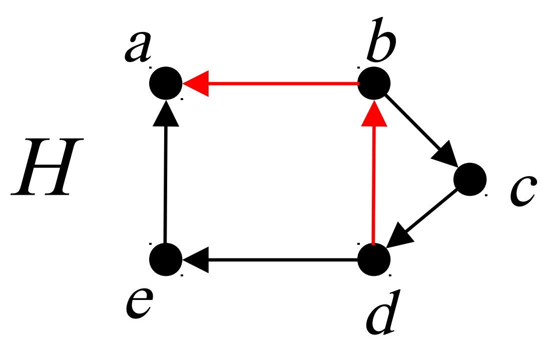
Connectedness in Directed Graphs

Def. 4: A directed graph is *strongly connected* if there is a path from a to b for any two vertices a, b. A directed graph is *weakly connected* if there is a path between every two vertices in the underlying undirected graphs.

Example 9 Are the directed graphs G and H strongly connected or weakly connected?



strongly connected

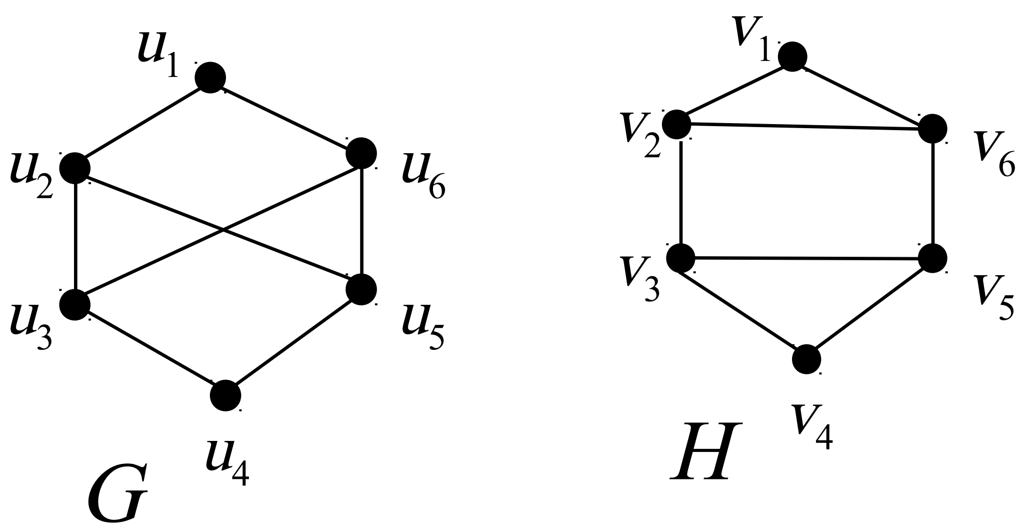


weakly connected

Paths and Isomorphism

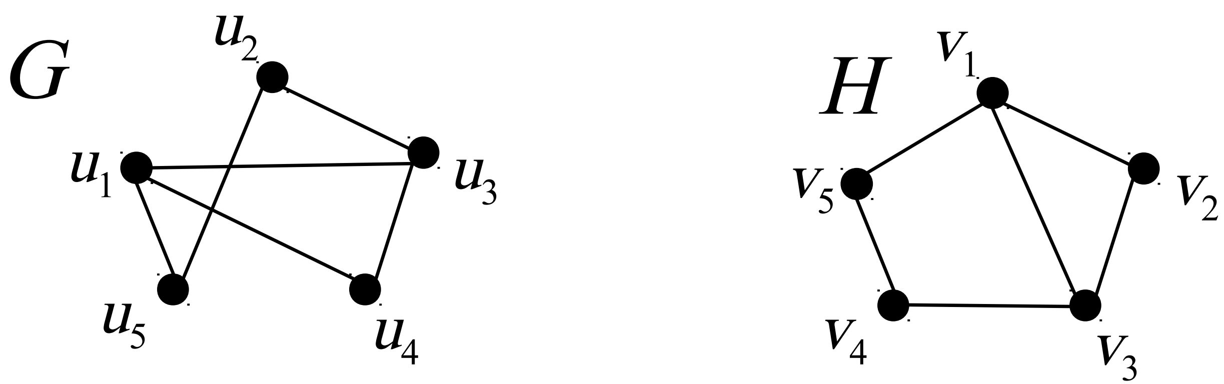
Note that connectedness, and the existence of a circuit or simple circuit of length k are graph invariants with respect to isomorphism.

Example 12. Determine whether the graphs G and H are isomorphic.



Sol: No, Because G has no simple circuit of length three, but H does

Example 13. Determine whether the graphs G and H are isomorphic.



Sol.

Both G and H have 5 vertices, 6 edges, two vertices of deg 3, three vertices of deg 2, a 3-cycle, a 4-cycle, and a 5-cycle. $\Rightarrow$ G and H may be isomorphic.

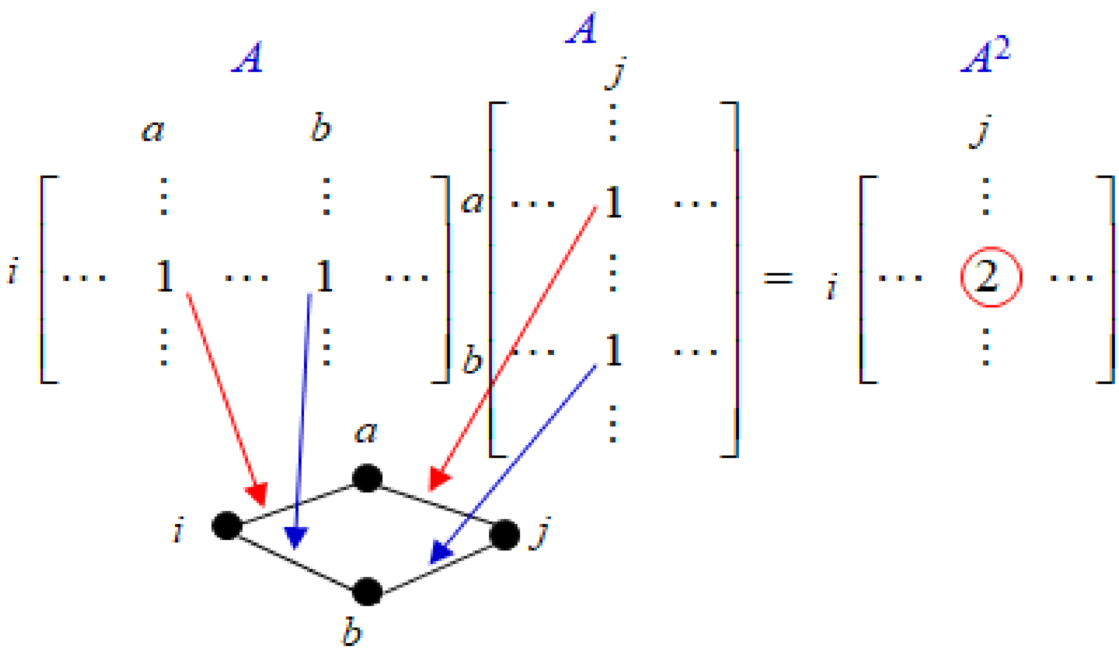
The function f with $f(u_1) = v_1,\ f(u_2) = v_4,\ f(u_3) = v_3,$ $f(u_4) = v_2$ and $f(u_5) = v_5$ is a one-to-one correspondence between $V(G)$ and $V(H)$. $\Rightarrow$ G and H are isomorphic.

Counting Paths between Vertices

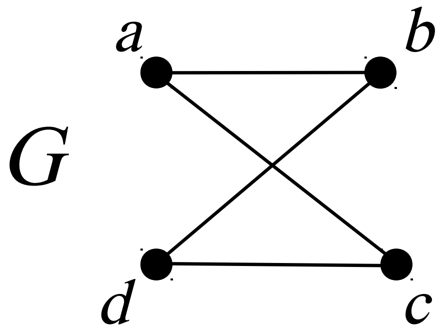
Theorem 2:

Let G be a graph with adjacency matrix A with respect to the ordering $v_1, v_2, \ldots, v_n$. The number of walks of length r from v_i to v_j is equal to $(A^r)_{i,j}$.

Proof (Only simple examples**)**



Example 14. How many walks of length 4 are there from a to d in the graph G?



Sol.

The adjacency matrix of G (ordering as a, b, c, d) is

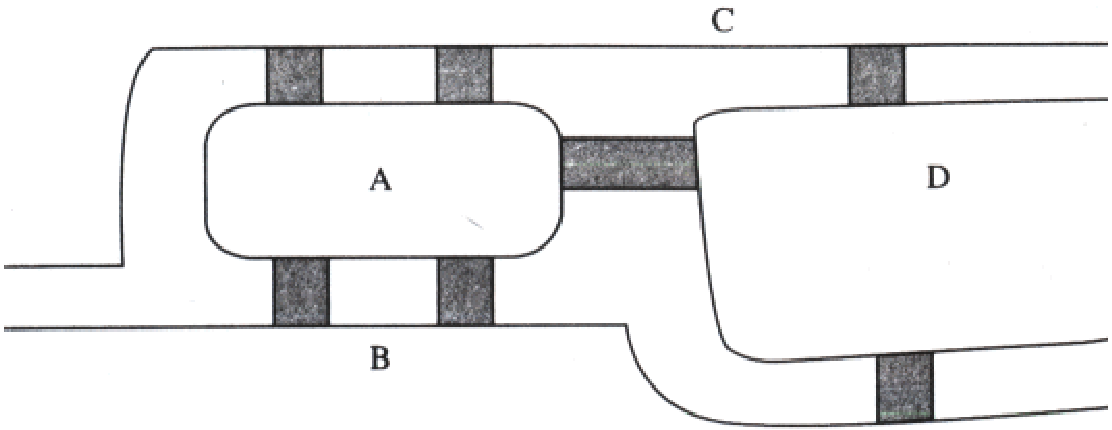
$$A = \begin{array}{c c} & \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{array}{c} a \\ b \\ c \\ d \end{array} & \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \end{array} \Rightarrow \quad A^4 = \begin{bmatrix} 8 & 0 & 0 & 8 \\ 0 & 8 & 8 & 0 \\ 0 & 8 & 8 & 0 \\ 8 & 0 & 0 & 8 \end{bmatrix} \Rightarrow 8$$

a-b-a-b-d, a-b-a-c-d, a-c-a-b-d, a-c-a-c-d,
a-b-d-b-d, a-b-d-c-d, a-c-d-b-d, a-c-d-c-d

Euler & Hamilton Paths

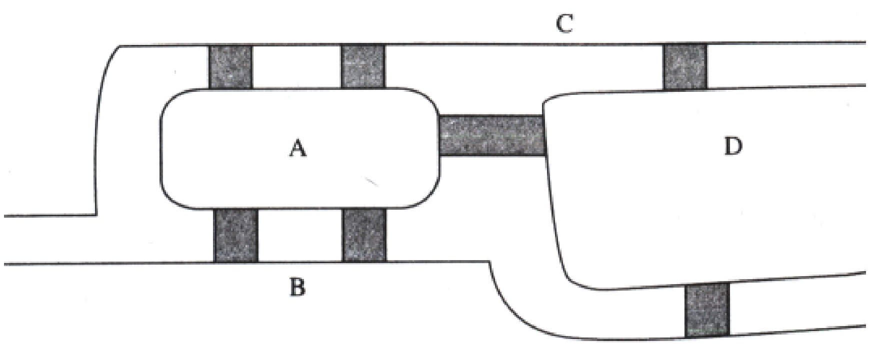
Graph Theory

- 1736, Euler solved the Königsberg Bridge Problem (Seven bridges problem)



Q:Is there a way can each bridge once, and return to the starting point?

Königsberg Bridge Problem



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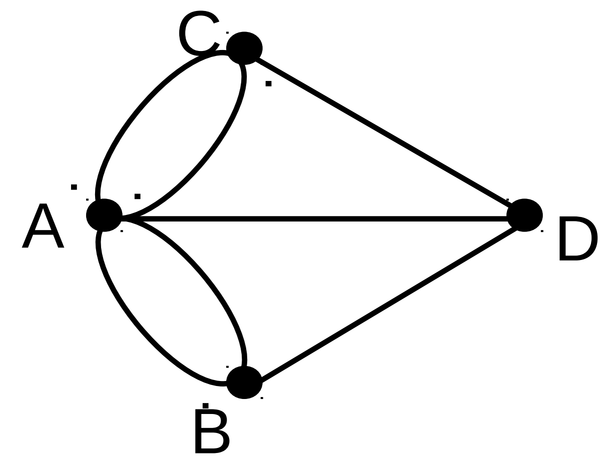


Q: Is there a way, you can walk down each side, and back to the starting point?

Ans: (Because each time a point is required from one side to the point, then the other side out, so after each time you want to use a pair of side.

- connection must be an even number of sides on each point
- the move does not exist

Def 1:

An *Euler circuit* in a graph *G* is a simple circuit containing every edge of *G*.
An *Euler path* in *G* is a simple path containing every edge of *G*.

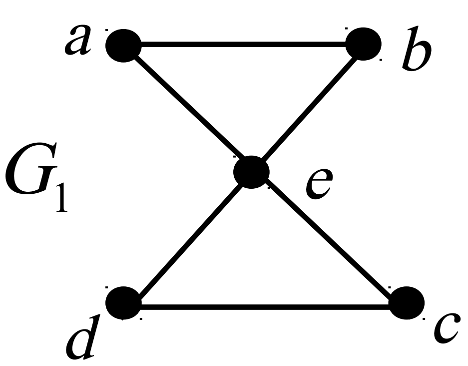
Thm. 1:

A connected multigraph with at least two vertices has an Euler circuit if and only if each of its vertices has even degree.

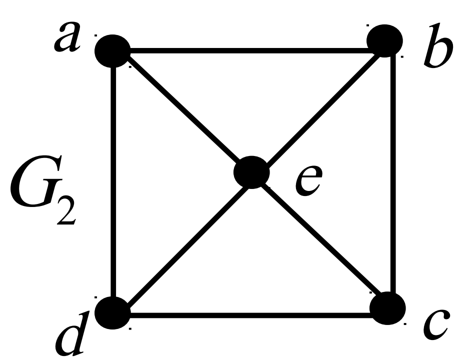
Thm. 2:

A connected multigraph has an Euler path (but not an Euler circuit) if and only if it has exactly 2 vertices of odd degree.

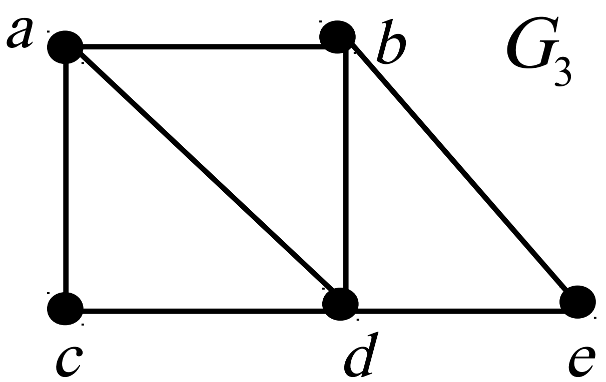
Example 1. Which of the following graphs have an Euler circuit or an Euler path?



Euler circuit

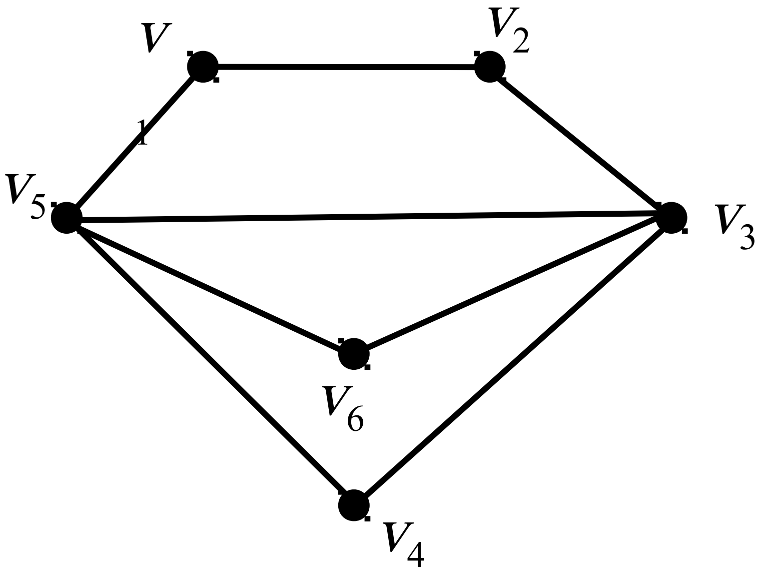


none



Euler path

Example



Step 1: find the 1st circuit

C: $v_1, v_2, v_3, v_4, v_5, v_1$

Step 2: $H = G - C \neq \varnothing$,

find subcircuit

SC: v_3, v_5, v_6, v_3

Step 3:

$C = C \cup SC$,

$H = G - C = \varnothing$, stop

C: $v_1, \underbrace{v_2, v_3, v_5, v_6, v_3}_{SC \text{ embedded}}, v_4, v_5, v_1$

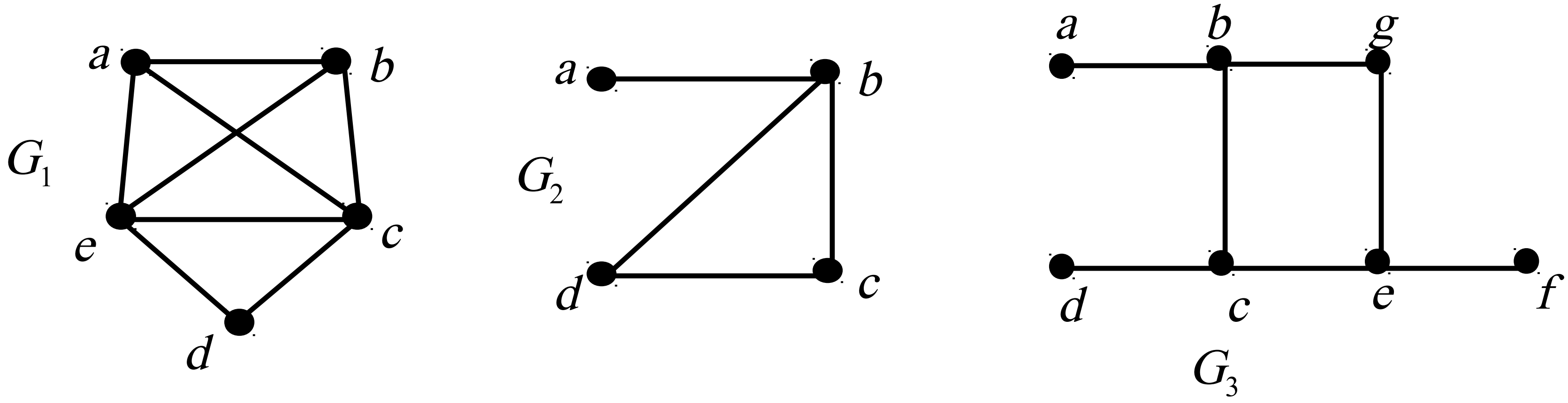
APPLICATIONS OF EULER PATHS AND CIRCUITS

- Euler paths and circuits can be used to solve many practical problems
  - traversing each street in a neighborhood
  - each road in a transportation network
  - each connection in a utility grid, or
  - each link in a communications network exactly once
- Among the other areas where Euler circuits and paths are applied is in
  - the layout of circuits,
  - in network multicasting, and
  - in molecular biology, where Euler paths are used in the sequencing of DNA

Hamilton Paths and Circuits

Def. 2: A *Hamilton path* is a path that traverses each vertex in a graph G exactly once.
A *Hamilton circuit* is a circuit that traverses each vertex in G exactly once.

Example 1. Which of the following graphs have a Hamilton circuit or a Hamilton path?

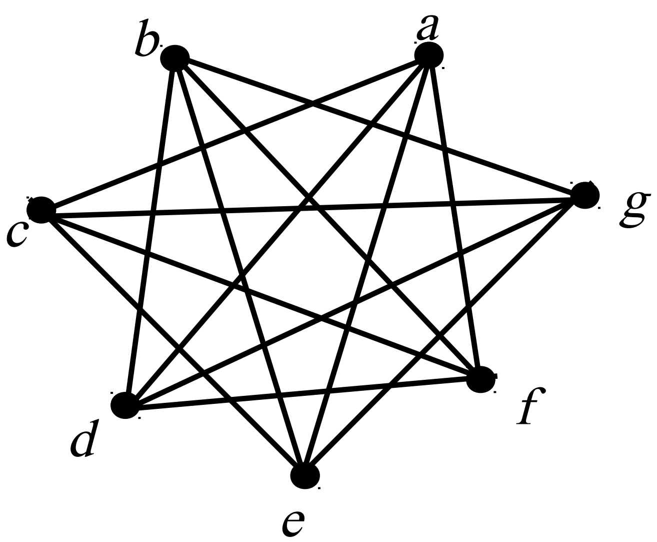


Hamilton circuit: G_1 Hamilton path: G_1, G_2

Thm. 3 (Dirac's Thm.):

If (but <u>not</u> only if) G is a simple graph with $n \geq 3$ vertices such that the degree of every vertex in G is at least $n/2$, then G has a Hamilton circuit.

Example



each vertex has deg $\geq n/2 = 3.5$

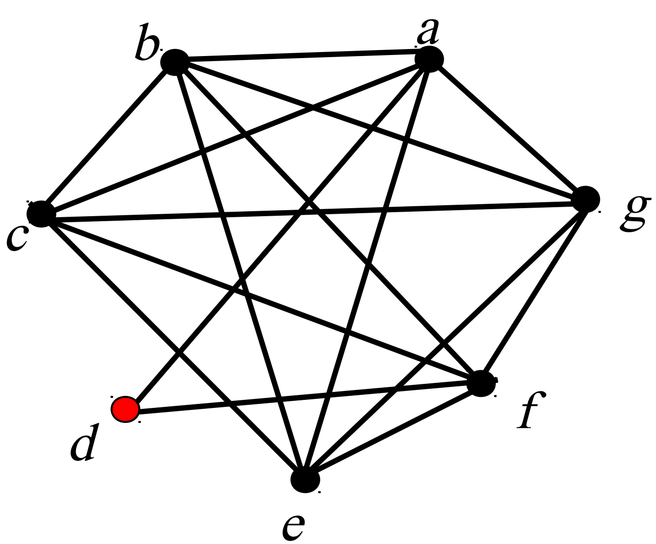
$\Rightarrow$Hamilton circuit exists

Such as: a, c, e, g, b, d, f, a

Thm. 4 (Ore's Thm.):

If G is a simple graph with $n \geq 3$ vertices such that $\deg(u)+\deg(v) \geq n$ for every pair of nonadjacent vertices u and v, then G has a Hamilton circuit.

Example



each nonadjacent vertex pair has deg sum $\geq n = 7$

$\Rightarrow$Hamilton circuit exists

Such as: a, d, f, e, c, b, g, a

Applications of Hamilton Circuits

- The famous **traveling salesperson problem** or **TSP** (also known in older literature as the **traveling salesman problem**)

Shortest-Path Problems

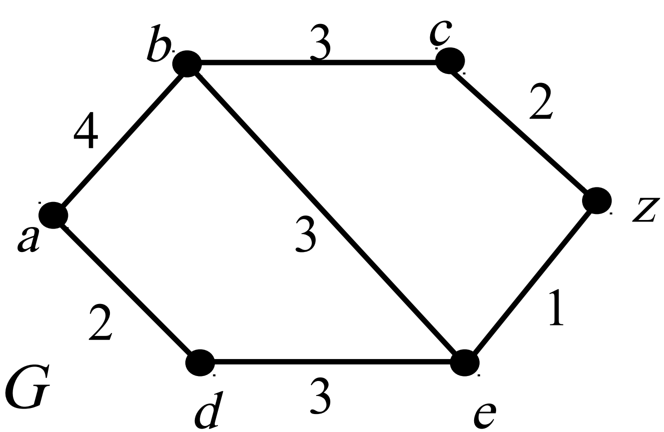
Def:

1. Graphs that have a number assigned to each edge are called *weighted graphs*.
2. The length of a path in a weighted graph is the sum of the weights of the edges of this path.

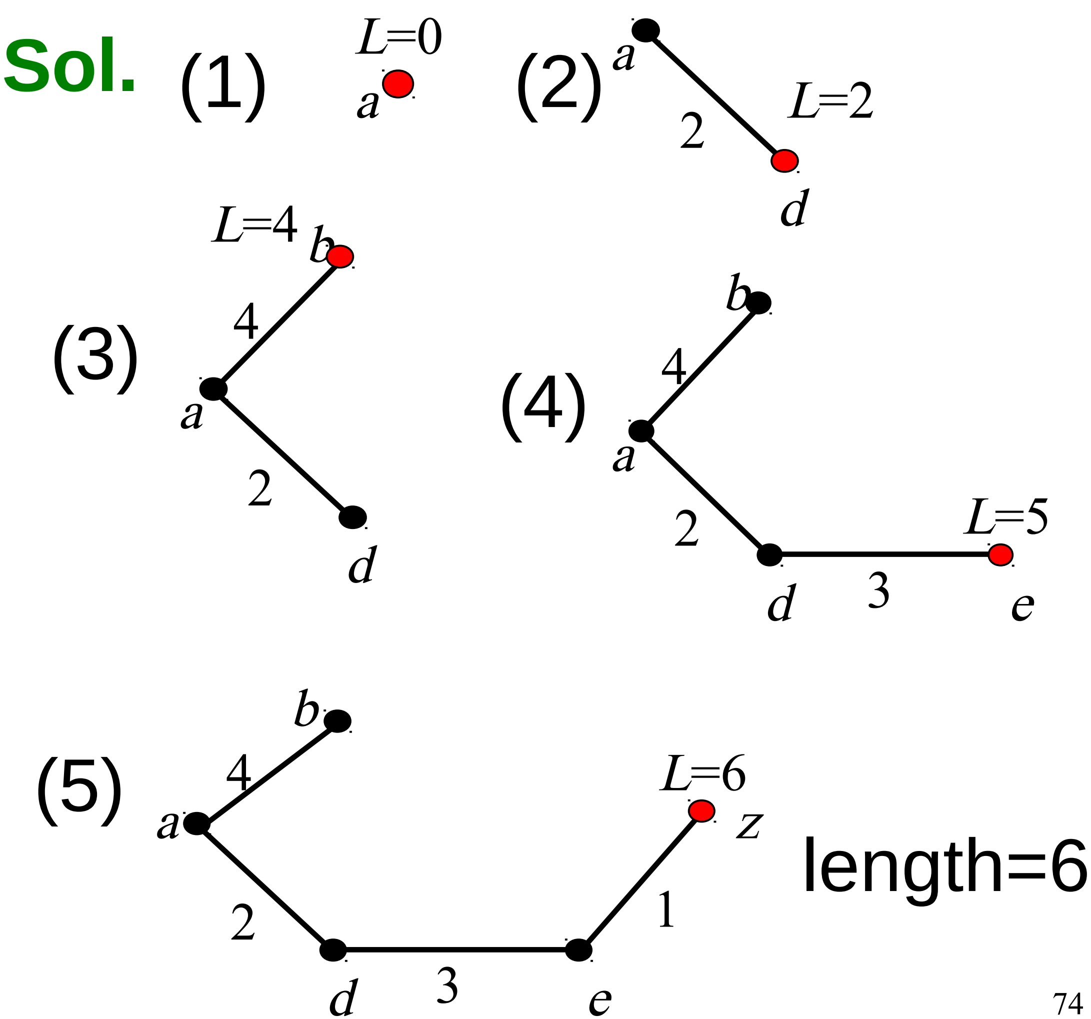
Shortest path Problem:

Determining the path of least sum of the weights between two vertices in a weighted graph.

Example 1. What is the length of a shortest path between a and z in the weighted graph G?



Sol.



Dijkstra's Algorithm (find the length of a shortest path from a to z)

Procedure *Dijkstra*(G: weighted connected simple graph, with all weights positive)

{G has vertices $a = v_0, v_1, \ldots, v_n = z$ and weights $w(v_i, v_j)$ where $w(v_i, v_j) = \infty$ if $\{v_i, v_j\}$ is not an edge in G}

for $i := 1$ **to** n
    $L(v_i) := \infty$
$L(a) := 0$
$S := \varnothing$
while $z \notin S$
begin
    $u :=$ a vertex not in S with $L(u)$ minimal
    $S := S \cup \{u\}$
    for all vertices v not in S
        if $L(u) + w(u, v) < L(v)$ **then** $L(v) := L(u) + w(u, v)$

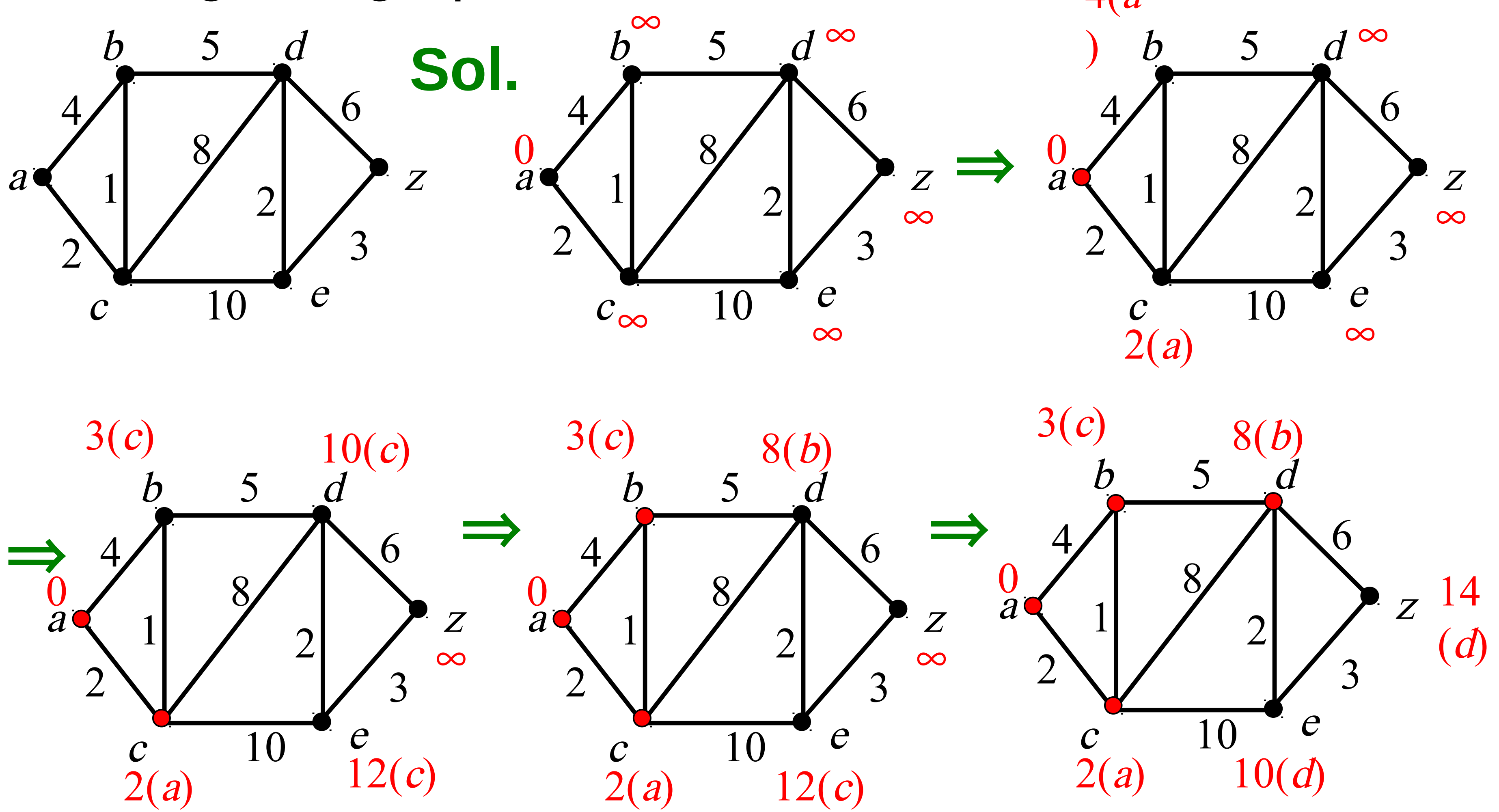
end {$L(z)$ = length of a shortest path from a to z}

This algorithm can be extended to construct a shortest path.

trace(We add a variable record thing is u before v previous (v): Finally, going on from z = u algorithm trace)

Example 2. Use Dijkstra's algorithm to find the length of a shortest path between a and z in the weighted graph.

Sol.



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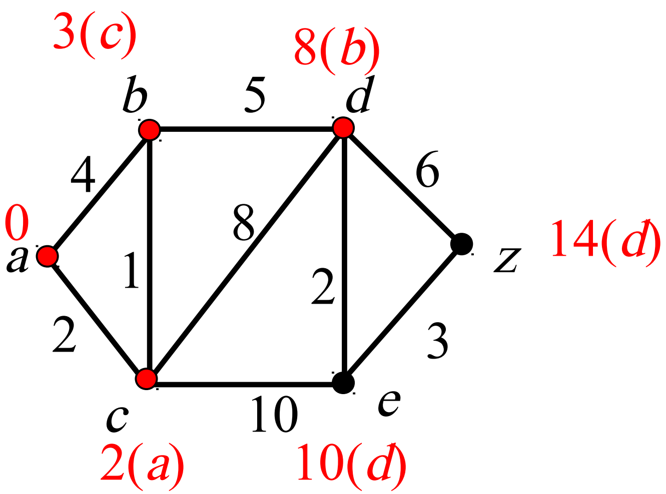
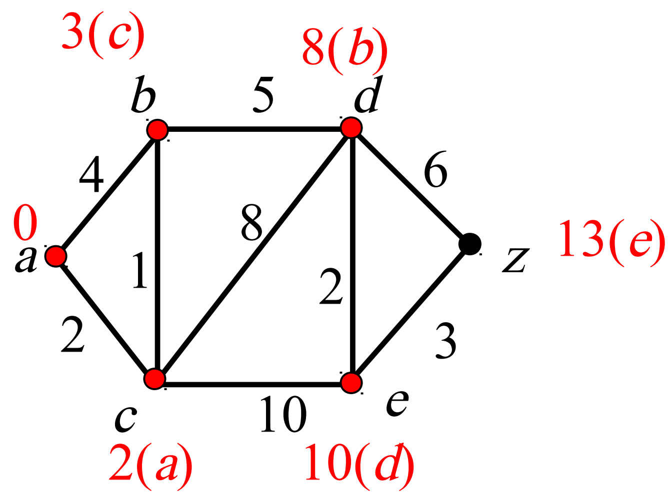
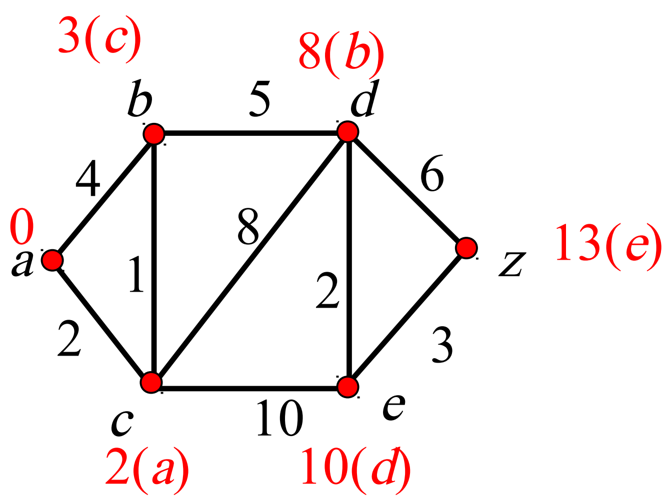


⇒



⇒



⇒ path: *a, c, b, d, e, z*
length: 13

Thm. 1

Dijkstra's algorithm finds the length of a shortest path between two vertices in a connected simple undirected weighted graph.

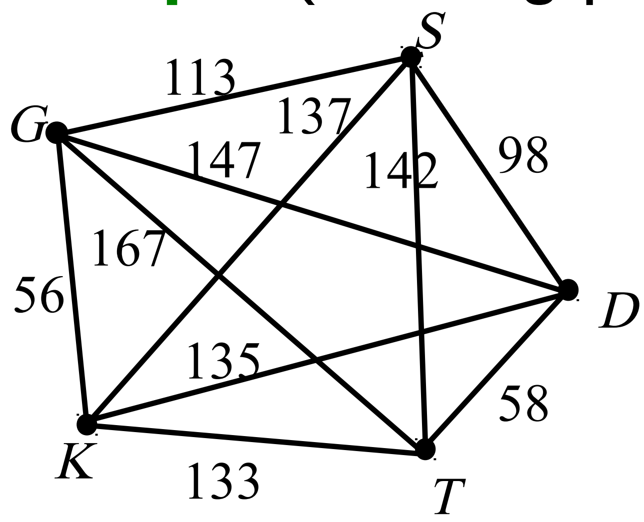
Thm. 2

Dijkstra's algorithm uses $O(n^2)$ operations (additions and comparisons) to find the length of a shortest path between two vertices in a connected simple undirected weighted graph with n vertices.

The Traveling Salesman Problem:

A traveling salesman wants to visit each of n cities exactly once and return to his starting point. In which order should he visit these cities to travel the minimum total distance?

Example (starting point D)



$D \to T \to K \to G \to S \to D$: 458

$D \to T \to S \to G \to K \to D$: 504

$D \to T \to S \to K \to G \to D$: 540

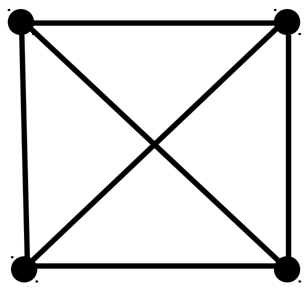
...

Planar Graphs

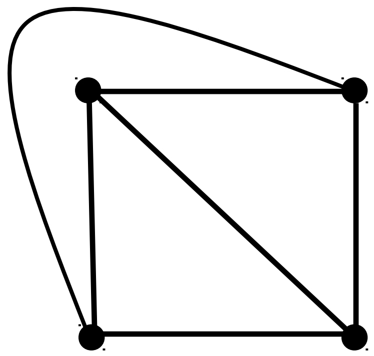
Def 1.

A graph is called *planar* if it can be drawn in the plane without any edge crossing. Such a drawing is called a *planar representation* of the graph.

Example 1: Is K_4 planar?



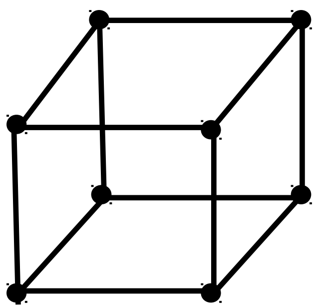
K_4



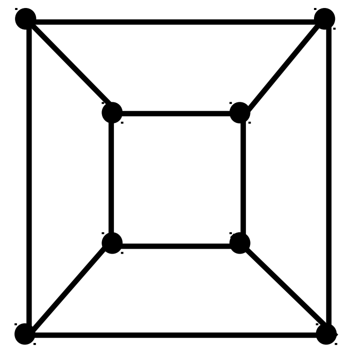
K_4 drawn with no crossings

$\therefore$ K_4 is planar

Example 2: Is Q_3 planar?



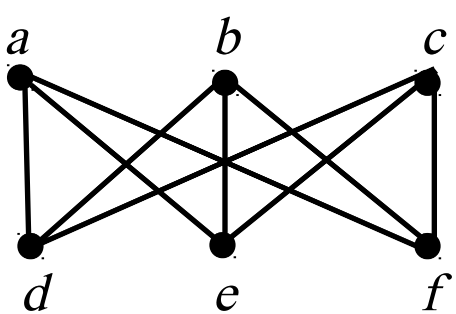
Q_3



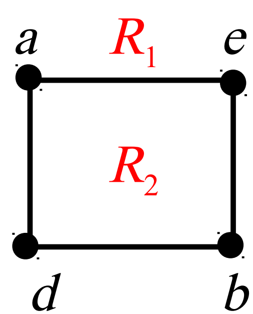
Q_3 drawn with no crossings

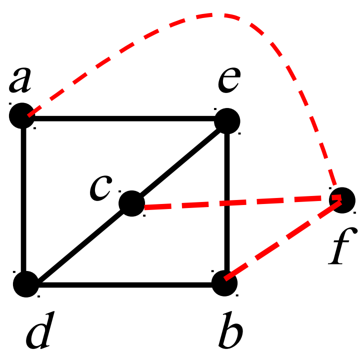
$\therefore$ Q_3 is planar

Example 3: Show that $K_{3,3}$ is nonplanar.



Sol.





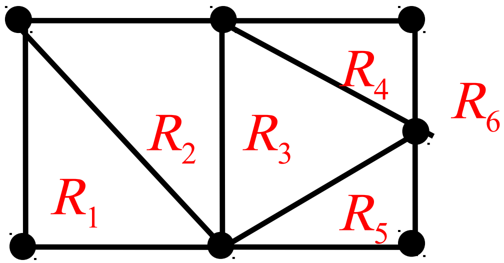
In any drawing, aebd is cycle, and will cut the plane into two region

Regardless of which region c, could no longer put the f in that side staggered

Euler's Formula

A planar representation of a graph splits the plane into regions, including an unbounded region.

Example : How many regions are there in the following graph?



Sol. 6

Thm 1 (Euler's Formula)

Let G be a connected planar simple graph with e edges and v vertices. Let r be the number of regions in a planar representation of G. Then $r = e - v + 2$.

Example 4: Suppose that a connected planar graph has 20 vertices, each of degree 3. Into how many regions does a representation of this planar graph split the plane?

Sol.

$$v = 20,\ 2e = 3 \times 20 = 60,\ e = 30$$

$$r = e - v + 2 = 30 - 20 + 2 = 12$$

Corollary 1

If G is a connected planar simple graph with e edges and v vertices, where $v \geq 3$, then $e \leq 3v - 6$.

Example 5: Show that K_5 is nonplanar.

Sol.

$$v = 5,\ e = 10,\ \text{but } 3v - 6 = 9.$$

Corollary 2

If G is a connected planar simple graph, then G has a vertex of degree ≤ 5.

pf: Let G be a planar graph of v vertices and e edges.

If $\deg(v) \geq 6$ for every $v \in V(G)$

$$\Rightarrow \sum_{v \in V(G)} \deg(v) \geq 6v$$

$$\Rightarrow 2e \geq 6v \qquad \rightarrow\leftarrow (e \leq 3v - 6)$$

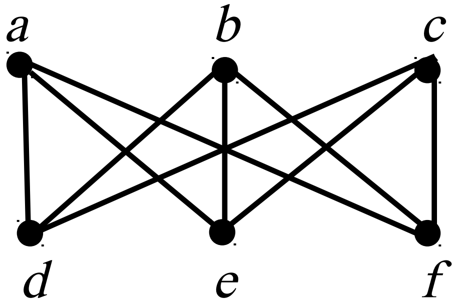
Corollary 3

If a connected planar simple graph has e edges and v vertices with $v \geq 3$ and <u>no circuits of length three</u>, then $e \leq 2v - 4$.

Example 6: Show that $K_{3,3}$ is nonplanar by Cor. 3.

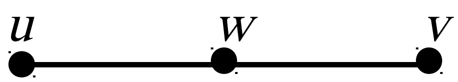
Sol.

Because $K_{3,3}$ has no circuits of length three, and $v = 6,\ e = 9,$ but $e = 9 > 2v - 4$.



Kuratowski's Theorem

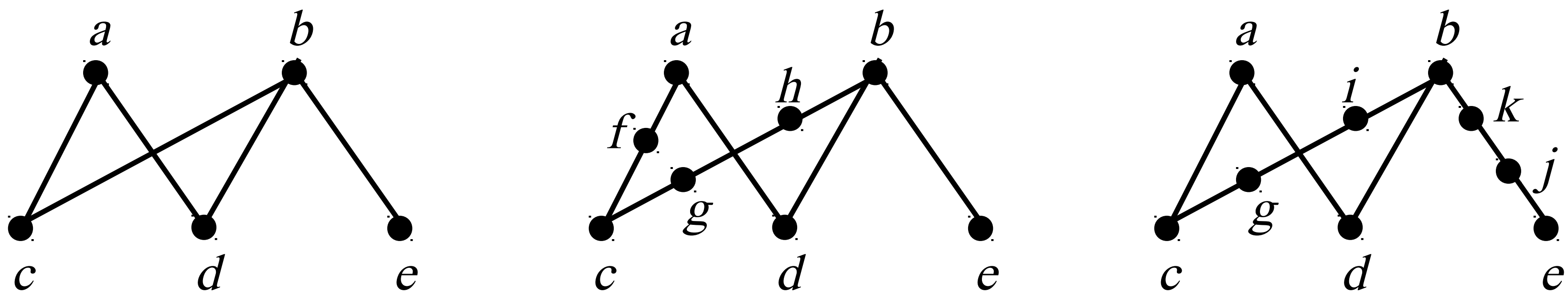
If a graph is planar, so will be any graph obtained by removing an edge $\{u, v\}$ and adding a new vertex w together with edges $\{u, w\}$ and $\{v, w\}$.



Such an operation is called an elementary subdivision.

Two graphs $G_1 = (V_1, E_1)$, $G_2 = (V_2, E_2)$ are called homeomorphic if they can be obtained from the same graph by a sequence of elementary subdivisons.

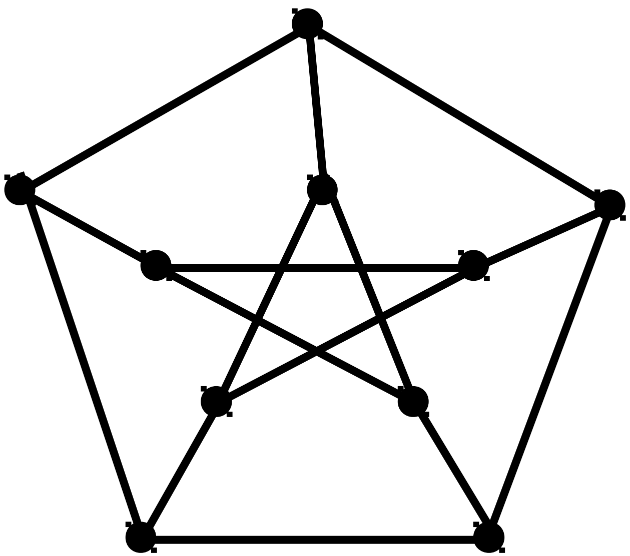
Example 7: Show that the graphs G_1, G_2, and G_3 are all homeomorphic.



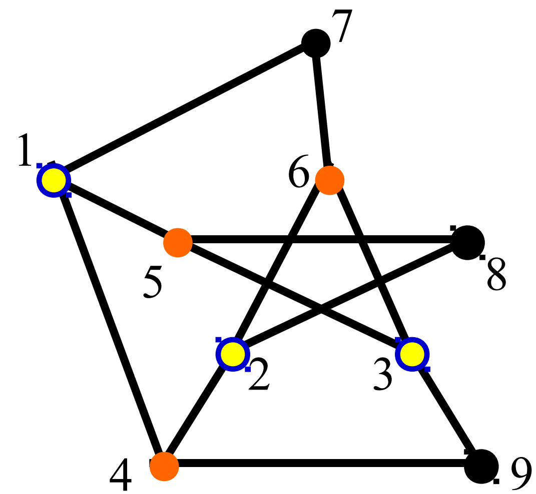
Sol: all three can be obtained from G_1

Thm 2. (Kuratowski Theorem)
A graph is nonplanar if and only if it contains a subgraph homeomorphic to $K_{3,3}$ or K_5.

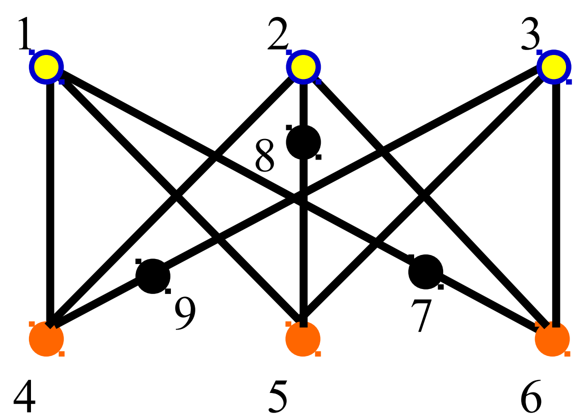
Example 9: Show that the Petersen graph is not planar.



Sol:



It is homeomorphic to $K_{3,3}$.

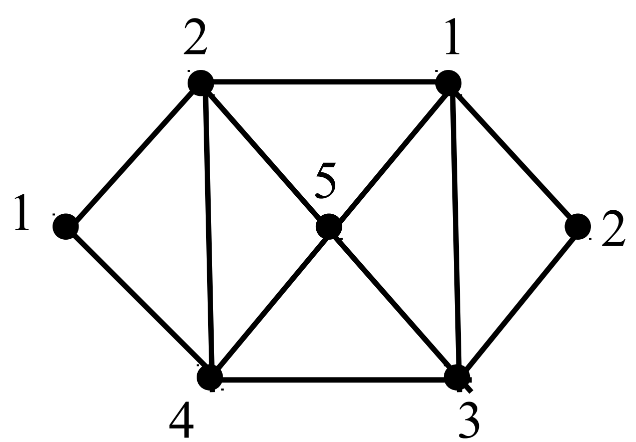


Graph Coloring

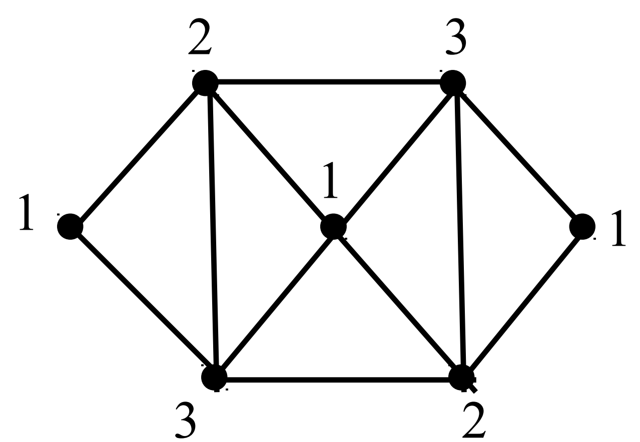
Def. 1:

A *coloring* of a simple graph is the assignment of a color to each vertex of the graph so that no two adjacent vertices are assigned the same color.

Example:



5-coloring



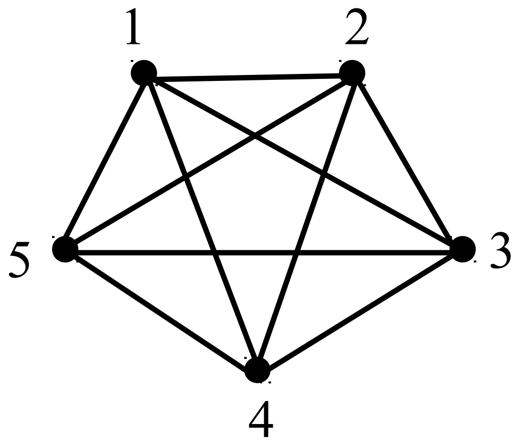
3-coloring

Less the number of colors, the better

Def. 2:

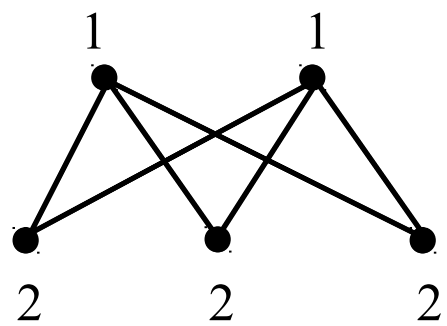
The *chromatic number* of a graph is the least number of colors needed for a coloring of this graph. (denoted by $\chi(G)$)

Example 2: $\chi(K_5)=5$



Note: $\chi(K_n)=n$

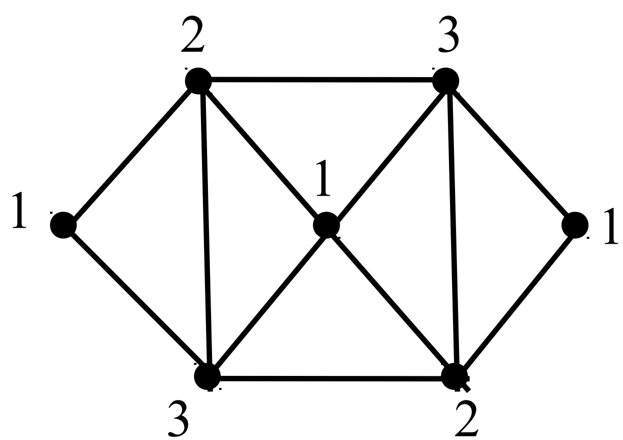
Example: $\chi(K_{2,3}) = 2$.



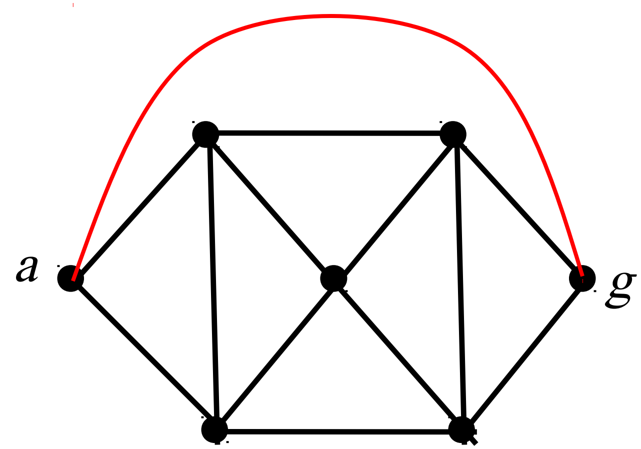
Note: $\chi(K_{m,n}) = 2$

Note: If G is a bipartite graph, $\chi(G) = 2$.

Example 1: What are the chromatic numbers of the graphs G and H?



G



H

Sol: G has a 3-cycle

$\Rightarrow \chi(G) \geq 3$

G has a 3-coloring

$\Rightarrow \chi(G) \leq 3$

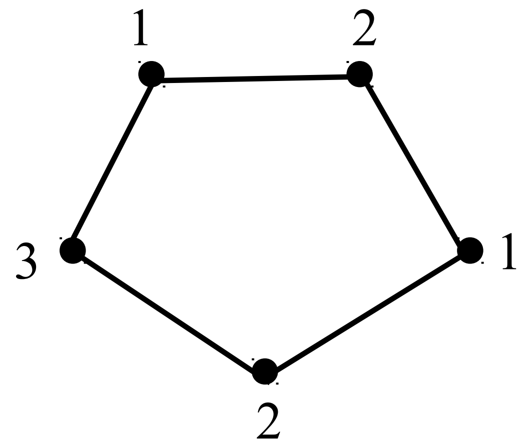
$\Rightarrow \chi(G) = 3$

Sol: any 3-coloring for $H - \{(a,g)\}$ gives the same color to a and g

$\Rightarrow \chi(H) > 3$

4-coloring exists $\Rightarrow \chi(H) = 4$

Example 4: $\chi(C_n) = \begin{cases} 2 \text{ if } n \text{ is even,} \\ 3 \text{ if } n \text{ is odd.} \end{cases}$



C_n is bipartite when n is even.

Thm 1. (The Four Color Theorem)

The chromatic number of a planar graph is no greater than four.

Corollary

Any graph with chromatic number >4 is nonplanar.