Complex Arguments

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* $e^x . e^y = e^{x+y}$

$e^{ix} = \cos x + i \sin x$

$e^{-ix} = \cos x - i \sin x$

$\cos x = \frac{1}{2}(e^{ix} + e^{-ix})$, $\sin x = \frac{1}{2i}(e^{ix} - e^{-ix})$

$e^{2n\pi i} = \cos 2n\pi + i \sin 2n\pi = 1$.

$e^{i\theta} \rightarrow$ principal value

$e^{i\theta} . e^{2n\pi i} \rightarrow$ Genaral value

$a^x = e^{x \log a}$

$\log(xy) = \log x + \log y$, $\log \frac{x}{y} = \log x - \log y$

Ex.1 Evalute $\log(\alpha + i\beta)$, where α and β are real.

Soln Let, $\alpha = r\cos\theta$

$\beta = r \sin\theta$

then, $r = \sqrt{\alpha^2 + \beta^2}$ and $\theta = \tan^{-1}\frac{\beta}{\alpha}$

$\therefore \alpha + i\beta = r\cos\theta + i r \sin\theta$

$= r(\cos\theta + i\sin\theta)$

$\therefore \alpha + i\beta = re^{i\theta}$

$\log(\alpha + i\beta) = \log(re^{i\theta})$

$= \log r + \log e^{i\theta}$

$= \log r + i\theta$

$= \log\sqrt{\alpha^2 + \beta^2} + i \tan^{-1}\frac{\beta}{\alpha}$

Ex.2 Separate into its real and imaginary parts of the ~~expan~~ expression, $(\alpha+i\beta)^{x+iy}$.

Solⁿ Let, $\alpha = r\cos\theta$

$\beta = r\sin\theta$

$\therefore r = \sqrt{\alpha^2+\beta^2}$ and $\theta = \tan^{-1}\frac{\beta}{\alpha}$

$$\therefore \alpha+i\beta = r\cos\theta + ir\sin\theta$$
$$= r(\cos\theta + i\sin\theta)$$
$$= re^{i\theta}$$

$$\therefore \operatorname{Log}(\alpha+i\beta) = \operatorname{Log}(re^{i\theta})$$
$$= \operatorname{Log} r + \log e^{i\theta}$$
$$= \log r + i\theta$$

$$\therefore (\alpha+i\beta)^{x+iy} = e^{(x+iy)\log(\alpha+i\beta)}$$
$$= e^{(x+iy)(\log r + i\theta)}$$
$$= e^{(x\log r - y\theta) + i(y\log r + x\theta)}$$
$$= e^{x\log r - y\theta} \cdot e^{i(y\log r + x\theta)}$$
$$= e^{x\log r - y\theta} \cdot \{\cos(y\log r + x\theta) + i\sin(y\log r + x\theta)\}$$
$$= e^{x\log(\sqrt{\alpha^2+\beta^2}) - y\tan^{-1}\frac{\beta}{\alpha}} \cdot \cos\left\{y\log(\sqrt{\alpha^2+\beta^2}) + x\tan^{-1}\frac{\beta}{\alpha}\right\}$$
$$+ i e^{x\log(\sqrt{\alpha^2+\beta^2}) - y\tan^{-1}\frac{\beta}{\alpha}} \cdot \sin\left\{y\log(\sqrt{\alpha^2+\beta^2}) + x\tan^{-1}\frac{\beta}{\alpha}\right\}$$

Ex. 3 Prove that $i^i = e^{-(4n+1)\frac{\pi}{2}}$

Sol^n we have, $i = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2}$

$$= e^{i\pi/2}$$

$$= e^{i\pi/2} \cdot e^{2n\pi i} \quad [\because e^{2n\pi i} = 1]$$

$$= e^{i\frac{\pi}{2} + 2n\pi i}$$

$$= e^{i\frac{\pi}{2}(4n+1)}$$

Therefore, $\text{Log}\, i = \log e^{i\frac{\pi}{2}(4n+1)}$

$$= i\frac{\pi}{2}(4n+1)$$

Therefore, $i^i = e^{i\log i} = e^{i \cdot i\frac{\pi}{2}(4n+1)} = e^{-\frac{\pi}{2}(4n+1)}$

(Proved)

Ex. 5. If $\tan\log(x+iy) = a+ib$, where $a^2+b^2 \neq 1$, then prove that $\tan\log(x^2+y^2) = \frac{2a}{1-a^2-b^2}$

Sol^n Given, $\tan\log(x+iy) = a+ib$

$$\log(x+iy) = \tan^{-1}(a+ib)$$

Therefore, $\log(x-iy) = \tan^{-1}(a-ib)$

Therefore, $\log(x^2+y^2) = \log\{(x+iy)(x-iy)\}$

$= \log(x+iy) + \log(x-iy)$

$= \tan^{-1}(a+ib) + \tan^{-1}(a-ib)$

$= \tan^{-1} \dfrac{(a+ib)+(a-ib)}{1-(a+ib)(a-ib)}$

$= \tan^{-1} \dfrac{2a}{1-(a^2+b^2)}$

$\therefore \log(x^2+y^2) = \tan^{-1} \dfrac{2a}{1-a^2-b^2}$

$\therefore \tan\log(x^2+y^2) = \dfrac{2a}{1-a^2-b^2}$ (Proved)

Ex. 8 Express $\text{Log}\{\log(\cos\theta + i\sin\theta)\}$ in the form $A+iB$

Soln. $\text{Log}(\cos\theta + i\sin\theta) = \text{Log}\, e^{i\theta}$

$= \text{Log}\, e^{i\theta} \cdot e^{2n\pi i}$

$= \log e^{i(2n\pi+\theta)}$

$= (2n\pi+\theta)i$

$= (2n\pi+\theta)(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2})$

$= (2n\pi+\theta)e^{i\pi/2}$.

Therefore, $\text{Log}\{\text{Log}(\cos\theta + i\sin\theta)\} = \log\{(2n\pi+\theta) \cdot e^{i\pi/2}\}$

$= \log(2n\pi+\theta) + \log e^{i\pi/2}$

$= \log(2n\pi+\theta) + \log e^{i\pi/2} \cdot e^{2K\pi i}$

$= \log(2n\pi+\theta) + \log e^{2K\pi i + i\pi/2}$

$= \log(2n\pi + \theta) + 2K\pi i + i\pi/2$

$= \log(2n\pi + \theta) + i\pi(2K + \frac{1}{2})$

$\therefore \operatorname{Log}\{\operatorname{Log}(\cos\theta + i\sin\theta)\} = \log(2n\pi + \theta) + i\pi(2K + \frac{1}{2})$

(V) Show that, $\tan\left(i \log \frac{a-ib}{a+ib}\right) = \frac{2ab}{a^2-b^2}$

<u>Soln</u> Let, $a = r\cos\theta$

$b = r\sin\theta$

$\therefore r = \sqrt{a^2+b^2}, \quad \theta = \tan^{-1}\frac{b}{a}$

$\therefore \operatorname{Log}(a+ib) = \log(r\cos\theta + ir\sin\theta)$

$= \operatorname{Log}(re^{i\theta})$

$= \log r + \log e^{i\theta}$

$= \log r + i\theta$

$= \log\sqrt{a^2+b^2} + i\tan^{-1}\frac{b}{a}$

Similarly, $\operatorname{Log}(a-ib) = \log\sqrt{a^2+b^2} - i\tan^{-1}\frac{b}{a}$

$\therefore \log\frac{a-ib}{a+ib} = \log(a-ib) - \log(a+ib)$

$= \log\sqrt{a^2+b^2} - i\tan^{-1}\frac{b}{a} - \log\sqrt{a^2+b^2} - i\tan^{-1}\frac{b}{a}$

$= -2i\tan^{-1}\frac{b}{a}$

$$\therefore \tan\left(i\log\frac{a-ib}{a+ib}\right) = \tan\left(-i\cdot 2i\tan^{-1}\frac{b}{a}\right)$$

$$= \tan\left(2\tan^{-1}\frac{b}{a}\right)$$

$$= \frac{2\tan\left(\tan^{-1}\frac{b}{a}\right)}{1-\left\{\tan\left(\tan^{-1}\frac{b}{a}\right)\right\}^2}$$

$$= \frac{2\frac{b}{a}}{1-\frac{b^2}{a^2}}$$

$$= \frac{\frac{2b}{a}}{\frac{a^2-b^2}{a^2}}$$

$$= \frac{2b}{a}\times\frac{a^2}{a^2-b^2}$$

$$= \frac{2ab}{a^2-b^2} \text{ (showed)}$$

Ex4 If $A+iB = \log(x+iy)$, then show that

$$B = \tan^{-1}\frac{y}{x} \text{ and } A = \frac{1}{2}\log(x^2+y^2)$$

Soln Given, $A+iB = \log(x+iy)$

$$e^{A+iB} = x+iy$$

$$x+iy = e^{A}\cdot e^{iB}$$

$$= e^{A}(\cos B + i\sin B)$$

$$x+iy = e^{A}\cos B + i e^{A}\sin B$$

Equating real and imaginary parts, we have

$$x = e^{A} \cdot \cos B \quad , \quad y = e^{A} \sin B$$

$$\therefore \frac{y}{x} = \frac{e^{A} \sin B}{e^{A} \cos B}$$

$$y/x = \tan B$$

$$\therefore B = \tan^{-1}(y/x) \text{ (proved)}$$

Again,

$$x^2 + y^2 = (e^{A} \cos B)^2 + (e^{A} \sin B)^2$$

$$= e^{2A} (\cos^2 B + \sin^2 B)$$

$$= e^{2A}$$

$$\operatorname{Log}(x^2 + y^2) = \operatorname{Log} e^{2A}$$

$$= 2A$$

$$\therefore A = \frac{1}{2} \log(x^2 + y^2) \text{ (proved)}$$

6. If $i^{i^{i^{\cdots \text{ad. inf}}}} = A+iB$, then prove that

$$\tan\frac{\pi A}{2} = \frac{B}{A} \text{ and } A^2+B^2 = e^{-\pi B}$$

Sol^n Given,

$$i^{i^{i^{\cdots \text{ad. inf.}}}} = A+iB$$

$$i^{(i^{i^{\cdots \text{ad. inf.}}})} = A+iB$$

$$\therefore i^{A+iB} = A+iB \quad \text{—— (1)}$$

But, $i = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} = e^{i\frac{\pi}{2}}$

$$\therefore i^{A+iB} = e^{i\frac{\pi}{2}(A+iB)} = e^{\frac{\pi}{2}(iA-B)} = e^{-\frac{B\pi}{2}} \cdot e^{\frac{i\pi A}{2}}$$

Now, from (1), we have

$$A+iB = e^{-\frac{B\pi}{2}} \cdot e^{\frac{i\pi A}{2}}$$

$$= e^{-\frac{B\pi}{2}}\left[\cos\frac{A\pi}{2} + i\sin\frac{A\pi}{2}\right]$$

Equating real and imaginary parts, we have

$$A = e^{-\frac{B\pi}{2}}\cos\frac{A\pi}{2}$$

$$B = e^{-\frac{B\pi}{2}}\sin\frac{A\pi}{2}$$

Dividing, we get $\frac{B}{A} = \tan\left(\frac{A\pi}{2}\right)$

$$\therefore \tan\frac{\pi A}{2} = B/A$$

and squaring and adding, we get

$$A^2+B^2 = e^{-B\pi}$$

8. Show that,

$$\log\log(x+iy) = \frac{1}{2}\log(p^2+q^2) + i\tan^{-1}\frac{q}{p}.$$

where, $p = \log\sqrt{x^2+y^2}$ and $q = \tan^{-1}\frac{y}{x}$

Soln. Let, $x = r\cos\theta$

$y = r\sin\theta$

Squaring and adding, we get

$$x^2+y^2 = r^2$$

$$\therefore r = \sqrt{x^2+y^2}$$

Dividing, we have

$$\frac{y}{x} = \tan\theta$$

$$\therefore \theta = \tan^{-1}\frac{y}{x}$$

$$\therefore x+iy = r\cos\theta + ir\sin\theta$$
$$= re^{i\theta}$$

$$\log(x+iy) = \log(re^{i\theta})$$
$$= \log r + \log e^{i\theta}$$
$$= \log r + i\theta$$
$$= \log\sqrt{x^2+y^2} + i\tan^{-1}\frac{y}{x}$$

$$\therefore \operatorname{Log}(x+iy) = p + iq,$$

where $p = \log\sqrt{x^2+y^2}$ and $q = \tan^{-1}\frac{y}{x}$

$\therefore \log\log(x+iy) = \log(p+iq)$

Let, $p = r\cos\theta$ and $q = r\sin\theta$

squaring and adding,

$$r^2 = p^2+q^2$$
$$r = \sqrt{p^2+q^2}$$

Dividing, we have

$$\frac{q}{p} = \tan\theta$$
$$\therefore \theta = \tan^{-1}\frac{q}{p}$$

$$\begin{aligned}\therefore \log\log(x+iy) &= \log(p+iq)\\ &= \log(r\cos\theta + ir\sin\theta)\\ &= \log\{r(\cos\theta + i\sin\theta)\}\\ &= \log(re^{i\theta})\\ &= \log r + \log e^{i\theta}\\ &= \log r + i\theta\\ &= \log\sqrt{p^2+q^2} + i\tan^{-1}\frac{q}{p}\end{aligned}$$

$$\therefore \log\log(x+iy) = \frac{1}{2}\log(p^2+q^2) + i\tan^{-1}\frac{q}{p}$$

(Proved)

where $p = \log\sqrt{x^2+y^2}$

$q = \tan^{-1}\frac{y}{x}$

2. prove that the principal value of $(\alpha+i\beta)^{x+iy}$ is wholly real or wholly imaginary accordings as

$$\frac{1}{2}y\log(\alpha^2+\beta^2)+x\tan^{-1}\frac{\beta}{\alpha}$$

is an even or odd multiple of $\frac{1}{2}\pi$.

Sol^n: Let, $\alpha = r\cos\theta$

$\beta = r\sin\theta$

squaring and adding, we get

$$r^2=\alpha^2+\beta^2$$

$$\therefore r=\sqrt{\alpha^2+\beta^2}$$

Dividing, we have,

$$\tan\theta=\frac{\beta}{\alpha}$$

$$\therefore \theta=\tan^{-1}\frac{\beta}{\alpha}$$

$$\therefore (\alpha+i\beta)^{x+iy} = e^{(x+iy)\log(\alpha+i\beta)}$$

$$= e^{(x+iy)\operatorname{Log}(r\cos\theta+ir\sin\theta)}$$

$$= e^{(x+iy)\log(re^{i\theta})}$$

$$= e^{(x+iy)(\log r+\log e^{i\theta})}$$

$$= e^{(x+iy)(\log r+i\theta)}$$

$$= e^{x\log r+ix\theta+iy\log r-y\theta}$$

$$= e^{x\log r-y\theta+i(y\log r+x\theta)}$$

$$= e^{x\log r-y\theta}\cdot e^{i(y\log r+x\theta)}$$

$$= e^{x\log r-y\theta}\left[\cos(y\log r+x\theta)+i\sin(y\log r+x\theta)\right]$$

$$= e^{x\log\sqrt{\alpha^2+\beta^2} - y\tan^{-1}\frac{\beta}{\alpha}} \cdot \left[\cos\left(y\log\sqrt{\alpha^2+\beta^2} + x\tan^{-1}\frac{\beta}{\alpha}\right)\right.$$

$$\left. + i\sin\left(y\log\sqrt{\alpha^2+\beta^2} + x\tan^{-1}\frac{\beta}{\alpha}\right)\right]$$

The principal value of $(\alpha+i\beta)^{x+iy}$ is wholly real if imaginary part vanishes,

$$\boxed{\begin{array}{l}\sin\theta = 0\\ \theta = n\pi\end{array}}$$

$$\sin\left(y\log\sqrt{\alpha^2+\beta^2} + x\tan^{-1}\frac{\beta}{\alpha}\right) = 0$$

$$y\log\sqrt{\alpha^2+\beta^2} + x\tan^{-1}\frac{\beta}{\alpha} = n\pi \quad , n = 0, 1, 2, 3, \cdots$$

$$\therefore \frac{1}{2}y\log(\alpha^2+\beta^2) + x\tan^{-1}\frac{\beta}{\alpha} = 2n\cdot\frac{\pi}{2}$$

$$\frac{1}{2}y\log(\alpha^2+\beta^2) + x\tan^{-1}\frac{\beta}{\alpha} = \text{even multiple of } \frac{\pi}{2} \text{ (Proved)}$$

The principal value of $(\alpha+i\beta)^{x+iy}$ is wholly imaginary if real part vanishes.

$$\boxed{\begin{array}{l}\cos\theta = 0\\ \theta = (2n+1)\frac{\pi}{2}\end{array}}$$

$$\cos\left(y\log\sqrt{\alpha^2+\beta^2} + x\tan^{-1}\frac{\beta}{\alpha}\right) = 0$$

$$y\log\sqrt{\alpha^2+\beta^2} + x\tan^{-1}\frac{\beta}{\alpha} = (2n+1)\frac{\pi}{2}$$

$$\therefore \frac{1}{2}y\log(\alpha^2+\beta^2) + x\tan^{-1}\frac{\beta}{\alpha} = \text{odd multiple of } \frac{\pi}{2} \text{ (Proved)}$$

10. If $\tan(x+iy) = u+iv$, then prove that $u^2+v^2+2u\cot 2x = 1$.

Soln: we have, $\tan(x+iy) = u+iv$

$$\tan(x-iy) = u-iv$$

Now, $\tan 2x = \tan\{(x+iy)+(x-iy)\}$

$$= \frac{\tan(x+iy)+\tan(x-iy)}{1-\tan(x+iy).\tan(x-iy)}$$

$$= \frac{u+iv+u-iv}{1-(u+iv)(u-iv)}$$

$$= \frac{2u}{1-(u^2+v^2)}$$

$$\therefore \tan 2x = \frac{2u}{1-u^2-v^2}$$

$$1-u^2-v^2 = \frac{2u}{\tan 2x}$$

$$1-u^2-v^2 = 2u\cot 2x$$

$$1 = u^2+v^2+2u\cot 2x$$

$\therefore u^2+v^2+2u\cot 2x = 1$ (proved)