Tangent & Normal.

M

Rule: If the equation of the curve is $f(x,y)=0$ then

$$\frac{dy}{dx} = -\frac{f_x}{f_y} \quad (f_y \neq 0)$$

** The equation of tangent to the curve $y=f(x)$ at $(x,$

$$(Y-y) = \frac{dy}{dx}(X-x) \text{ or, } (X-x)f_x + (Y-y)f_y = 0$$

** Equation of Normal :

$$(Y-y)\frac{dy}{dx} + (X-x) \text{ or, } (X-x)f_y - (Y-y)f_x = 0$$

01 Problems:

(a) Find the condition that the conics $ax^2+by^2=1$ and $a_1x^2+b_1y^2=1$ shall cut orthogonally

(b) Prove that the curves $\frac{x^2}{a}+\frac{y^2}{b}=1$ and $\frac{x^2}{a'}+\frac{y^2}{b'}=1$ will c orthogonally if $a-b=a'-b'$

(c) Find the condition that the curves $ax^3+by^3=1$ and $a'x^3+b'y$ should cut orthogonally.

Solution: (a) Given that the conics are

$$ax^2+by^2=1 \text{ and } a_1x^2+b_1y^2=1$$

$$\Rightarrow ax^2+by^2-1=0 \text{ and } a_1x^2+b_1y^2-1=0$$

Consider $f(x,y) \equiv ax^2+by^2-1=0 \longrightarrow$ (i)

and $\phi(x,y) \equiv a_1x^2+b_1y^2-1=0 \longrightarrow$ (ii)

Now the slope of (i) is $\frac{dy}{dx} = -\frac{f_x}{f_y}$

And the slope of (ii) is $\frac{dy}{dx} = \frac{-\phi_x}{\phi_y}$

The condition that they should cut orthogonally at $(x,$

$$\therefore \left(-\frac{f_x}{f_y}\right)\cdot\left(-\frac{\phi_x}{\phi_y}\right) = -1.$$

$$\text{or, } f_x\phi_x = -f_y\phi_y$$

$$\text{or, } f_x \phi_x + f_y \phi_y = 0$$

$$\text{or, } 2ax \cdot 2a_1x + 2by \cdot 2b_1y = 0$$

$$\text{i.e. } aa_1x^2 + bb_1y^2 = 0 \longrightarrow \text{(iii)}$$

where
$f_x = 2ax$
$f_y = 2by$
$\phi_x = 2a_1x$
$\phi_y = 2b_1y$

Since the point (x, y) is common to both (i) and (ii). Subtracting (ii) from (i) we get

$$(a - a_1)x^2 + (b - b_1)y^2 = 0 \longrightarrow \text{(iv)}$$

Comparing equation (iii) and (iv), we get

$$\frac{a - a_1}{aa_1} = \frac{b - b_1}{bb_1} \quad \text{or, } \frac{a}{aa_1} - \frac{a_1}{aa_1} = \frac{b}{bb_1} - \frac{b_1}{bb_1}$$

$$\text{or, } \frac{1}{a_1} - \frac{1}{a} = \frac{1}{b_1} - \frac{1}{b}$$

Which is the required condition.

(b) Answer: let $f(x, y) \equiv \frac{x^2}{a} + \frac{y^2}{b} - 1 = 0 \longrightarrow (1)$

& $\phi(x, y) \equiv \frac{x^2}{a_1} + \frac{y^2}{b_1} - 1 = 0 \longrightarrow (2)$

$a_1 = a'$
$b_1 = b'$
just it's symbol

Now the condition that they should cut orthogonally at (x, y)

$$f_x \phi_x + f_y \phi_y = 0$$

$$\text{or, } \frac{2x}{a} \cdot \frac{2x}{a_1} + \frac{2y}{b} \cdot \frac{2y}{b_1} = 0$$

$$\text{or, } \frac{x^2}{aa_1} + \frac{y^2}{bb_1} = 0 \longrightarrow (3)$$

Since the point (x, y) is common to both (1) and (2).
Subtracting (2) from (1) we get

$$\left(\frac{1}{a} - \frac{1}{a_1}\right)x^2 + \left(\frac{1}{b} - \frac{1}{b_1}\right)y^2 = 0 \longrightarrow (4)$$

Comparing equation (3) and (4) we get

$$\frac{\frac{1}{a} - \frac{1}{a_1}}{\frac{1}{aa_1}} = \frac{\frac{1}{b} - \frac{1}{b_1}}{\frac{1}{bb_1}}$$

or, $$\frac{\frac{a_1-a}{aa_1}}{\frac{1}{aa_1}} = \frac{\frac{b_1-b}{bb_1}}{\frac{1}{bb_1}}$$

or, $$\frac{a_1-a}{aa_1}\cdot aa_1 = \frac{b_1-b}{bb_1}\cdot bb_1$$

or $a_1 - a = b_1 - b$ Hence $a-b = a_1 - b_1$

(Proved).

Ⓒ Answer: ~~Let~~ $f(x,y) \equiv ax^3 + by^3 - 1 = 0 \longrightarrow (1)$

~~And~~ $\phi(x,y) \equiv a'x^3 + b'y^3 - 1 = 0 \longrightarrow (2)$

~~Now the condition that they should cut orthogonally at~~ (x,y)

~~is~~ $f_x\phi_x + f_y\phi_y = 0$

~~i.e.,~~ $3ax^2 \cdot 3a'x^2 + 3by^2 \cdot 3b'y^2 = 0$

~~or,~~ $aa'x^4 + bb'y^4 =$

Given the two curves $ax^3 + by^3 = 1 \longrightarrow (1)$

and $a'x^3 + b'y^3 = 1 \longrightarrow (2)$

Supose the two curves intersect at the point (x_1, y_1). Then $ax_1^3 + by_1^3 = 1 \longrightarrow (A)$

and $a'x_1^3 + b'y_1^3 = 1 \longrightarrow (B)$.

or, $(a-a')x_1^3 + (b-b')y_1^3 = 0$

or $$\frac{x_1^3}{y_1^3} = -\frac{b-b'}{a-a'}$$

or, $$\frac{x_1}{y_1} = -\left(\frac{b-b'}{a-a'}\right)^{\frac{1}{3}} \longrightarrow (3)$$

Now the equation of the tangent at (x_1, y_1) on the eq^n ① is

$$y - y_1 = \frac{dy}{dx}(x - x_1)$$

$$\frac{dy}{dx} = -\frac{3ax^2}{3by^2} = -\frac{ax_1^2}{by_1^2} \text{ at } (x_1, y_1)$$

or, $y - y_1 = -\frac{ax_1^2}{by_1^2}(x - x_1)$

or, $by_1^2 y + ax_1^2 x = ax_1^3 + by_1^3$

or, $ax_1^2 x + by_1^2 y = 1 \longrightarrow$ ④ [using Ⓐ]

Similarly the tangent for the second curve is

$$a' x_1^2 x + b' y_1^2 y = 1 \longrightarrow ⑤$$

Slople of ④ is $= -\frac{ax_1^2}{by_1^2}$

Slople of ⑤ is $= -\frac{a'}{b'}\frac{x_1^2}{y_1^2}$.

Since the curves intersect orthogonally. Then tangent ④ and ⑤ must be perpendicular.

$$-\frac{ax_1^2}{by_1^2} \cdot \left(-\frac{a' x_1^2}{b' y_1^2}\right) = -1$$

or, $aa' x_1^4 = -bb' y_1^4$

or, $\frac{x_1^4}{y_1^4} = -\frac{bb'}{aa'} \Rightarrow \left(\frac{x_1}{y_1}\right)^4 = -\frac{bb'}{aa'}$

or, $\left\{-\left(\frac{b-b'}{a-a'}\right)^{\frac{1}{3}}\right\}^4 = -\frac{bb'}{aa'}$ [using eq^n ③]

or, $aa'\left(\frac{b-b'}{a-a'}\right)^{4/3} = -bb'$

or, $aa'(b-b')^{4/3} + bb'(a-a')^{4/3} = 0$

Which is the required condition.

Problem 02: If $x\cos\alpha + y\sin\alpha = p$ touch the curve $\frac{x^m}{a^m} + \frac{y^m}{b^m} = 1$. show that $(a\cos\alpha)^{\frac{m}{m-1}} + (b\sin\alpha)^{\frac{m}{m-1}} = p$

Solution: Given that, $x\cos\alpha + y\sin\alpha = p \longrightarrow (1)$

and $\frac{x^m}{a^m} + \frac{y^m}{b^m} = 1 \longrightarrow (2)$

Let $f(x,y) \equiv \frac{x^m}{a^m} + \frac{y^m}{b^m} - 1 = 0$

$$\therefore \frac{dy}{dx} = -\frac{f_x}{f_y} = -\left(\frac{m x^{m-1}}{a^m}\right) \Big/ \left(\frac{m y^{m-1}}{b^m}\right)$$

$$= -\frac{b^m}{a^m}\frac{x^{m-1}}{y^{m-1}}$$

Let (x_1, y_1) be any point on a curve (2) and (1). H

$$x_1\cos\alpha + y_1\sin\alpha = p \longrightarrow (T)$$

$$\frac{x_1^m}{a^m} + \frac{y_1^m}{b^m} = 1 \longrightarrow (3)$$

Slope of eqn (2) at (x_1, y_1) is $-\frac{b^m}{a^m}\cdot\frac{x_1^{m-1}}{y_1^{m-1}}$

The equation of the tangent to the curve (2) at is

$$y - y_1 = -\frac{b^m}{a^m}\frac{x_1^{m-1}}{y_1^{m-1}}(x - x_1)$$

$$\text{or, } y_1^{m-1}(y - y_1) = -\frac{b^m}{a^m}(x - x_1)x_1^{m-1}$$

$$\Rightarrow \frac{y_1^{m-1}y}{b^m} + \frac{x_1^{m-1}x}{a^m} = \frac{x_1^m}{a^m} + \frac{y_1^m}{b^m}$$

$$\Rightarrow \frac{y_1^{m-1}y}{b^m} + \frac{x_1^{m-1}x}{a^m} = 1 \quad [\text{using } (3)]$$

$$\Rightarrow \frac{x_1^{m-1}}{a^m}x + \frac{y_1^{m-1}}{b^m}y = 1 \longrightarrow (4)$$

If equation (I) touch the given curve, then eq^n (I) and (4) must be identical.

Hence $\dfrac{x_1^{m-1}/a^m}{\cos\alpha} = \dfrac{y_1^{m-1}/b^m}{\sin\alpha} = \dfrac{1}{p}$.

$\Rightarrow \dfrac{x_1^{m-1}/a^{m-1}}{a\cos\alpha} = \dfrac{y_1^{m-1}/b^{m-1}}{b\sin\alpha} = \dfrac{1}{p}$

i.e $\left(\dfrac{x_1}{a}\right)^{m-1} = \dfrac{a\cos\alpha}{p}$, $\left(\dfrac{y_1}{b}\right)^{m-1} = \dfrac{b\sin\alpha}{p}$

or, $\left(\dfrac{x_1}{a}\right) = \left(\dfrac{a\cos\alpha}{p}\right)^{\frac{1}{m-1}}$, $\left(\dfrac{y_1}{b}\right) = \left(\dfrac{b\sin\alpha}{p}\right)^{\frac{1}{m-1}}$

$\therefore \left(\dfrac{x_1}{a}\right)^{m} = \left(\dfrac{a\cos\alpha}{p}\right)^{\frac{m}{m-1}}$, $\left(\dfrac{y_1}{b}\right)^{m} = \left(\dfrac{b\sin\alpha}{p}\right)^{\frac{m}{m-1}}$

Adding above two equation

$$\frac{x_1^m}{a^m} + \frac{y_1^m}{b^m} = \left(\frac{a\cos\alpha}{p}\right)^{\frac{m}{m-1}} + \left(\frac{b\sin\alpha}{p}\right)^{\frac{m}{m-1}}$$

or, $1 = \left(\dfrac{a\cos\alpha}{p}\right)^{\frac{m}{m-1}} + \left(\dfrac{b\sin\alpha}{p}\right)^{\frac{m}{m-1}}$ [Using (3)]

$\therefore (a\cos\alpha)^{\frac{m}{m-1}} + (b\sin\alpha)^{\frac{m}{m-1}} = p^{\frac{m}{m-1}}$.

(Showed)

03. If $lx+my=1$ touches the curve $(ax)^n+(by)^n=1$, show that $\left(\frac{l}{a}\right)^{\frac{n}{n-1}}+\left(\frac{m}{b}\right)^{\frac{n}{n-1}}=1$.

Solution: Given that,

$$lx+my=1 \longrightarrow (1)$$

$$\cancel{\text{and } (l/a)^{\frac{n}{n-1}} + m/b}\ (ax)^n+(by)^n=1 \longrightarrow (2)$$

Let (x_1, y_1) be the any point on the curve (2). Then

$$(ax_1)^n+(by_1)^n=1 \longrightarrow (3)$$

Now the slope of (2) at (x_1, y_1) is $= -\frac{n a^n x_1^{n-1}}{n b^n y_1^{n-1}}$

$$= -\frac{a^n}{b^n}\cdot\frac{x_1^{n-1}}{y_1^{n-1}}$$

The equation of tangent to the curve (2) at the point (x_1, y_1) is

$$y-y_1 = -\frac{a^n}{b^n}\frac{x_1^{n-1}}{y_1^{n-1}}(x-x_1)$$

$$\Rightarrow b^n y_1^{n-1}(y-y_1)+a^n x_1^{n-1}(x-x_1)=0$$

$$\Rightarrow a^n x_1^{n-1}x+b^n y_1^{n-1}y = a^n x_1^n + b^n x^n$$

$$\Rightarrow a^n x_1^{n-1}x+b^n y_1^{n-1}y = 1 \rightarrow (4) \quad [\text{using eq}^n (3)]$$

If eq^n (1) touch the curve (2) then eq^n (1) and (4) must be identical.

Hence $\frac{a^n x_1^{n-1}}{l} = \frac{b^n y_1^{n-1}}{m} = \frac{1}{1}$

$$\Rightarrow \frac{a^{n-1}x_1^{n-1}}{l/a} = \frac{b^{n-1}y_1^{n-1}}{m/b} = 1$$

$\therefore (ax_1)^{n-1} = l/a$ and $(by_1)^{n-1} = \frac{m}{b}$

$ax_1 = \left(\frac{l}{a}\right)^{\frac{1}{n-1}}, \quad by_1 = \left(\frac{m}{b}\right)^{\frac{1}{n-1}}$

$$(ax_1)^n = \left(\frac{l}{a}\right)^{\frac{n}{n-1}}, \quad (by_1)^n = \left(\frac{m}{b}\right)^{\frac{n}{n-1}}$$

Adding above two we get

$$\left(\frac{l}{a}\right)^{\frac{n}{n-1}} + \left(\frac{m}{b}\right)^{\frac{n}{n-1}} = (ax_1)^n + (by_1)^n$$

$$\therefore \left(\frac{l}{a}\right)^{\frac{n}{n-1}} + \left(\frac{m}{b}\right)^{\frac{n}{n-1}} = 1 \quad [\text{using } (3)]$$

(Showed)

Q6. If $lx + my = 1$ is normal to the parabola $y^2 = 4ax$, then $al^3 + 2alm^2 = m^2$.

Solution : Given that $lx + my = 1 \longrightarrow (1)$

& $y^2 = 4ax \longrightarrow (2)$

Let (x_1, y_1) be the any point on (1) and (2) then

$y_1^2 = 4ax_1 \rightarrow (A)$ & $y_1^2 = 4ax_1 \longrightarrow (3)$

From (2) $2y\frac{dy}{dx} = 4a \Rightarrow \frac{dy}{dx} = \frac{2a}{y}$

Slope of (2) at (x_1, y_1) is $\frac{2a}{y_1}$.

$\therefore$ Equation of the normal at the point (x_1, y_1) is

$$(x - x_1) + \frac{2a}{y_1}(y - y_1) = 0$$

$$\text{or, } xy_1 - x_1y_1 + 2ay - 2ay_1 = 0$$

$$\Rightarrow y_1x + 2ay = y_1(2a + x_1) \longrightarrow (4)$$

Equation (1) and (4) must be identical

$$\frac{y_1}{l} = \frac{2a}{m} = \frac{y_1(2a + x_1)}{1}$$

$\therefore \frac{y_1}{l} = \frac{2a}{m}$ and $\frac{y_1}{l} = \frac{y_1(2a+x_1)}{1}$.

or, $y_1 = \frac{2al}{m}$, $\frac{1}{l} = (2a+x_1) \Rightarrow x_1 = \frac{1}{l} - 2a$

Putting these values in equation (A) we get

$$\left(\frac{2al}{m}\right)^2 = 4a\left(\frac{1}{l} - 2a\right)$$

$$\Rightarrow 4a^2l^2 = \frac{4am^2}{l} - 8a^2m^2$$

$$\Rightarrow a^2l^2 = \frac{am^2 - 2a^2m^2l}{l} = \frac{a(m^2 - 2am^2l)}{l}$$

$$\Rightarrow al^3 = m^2 - 2alm^2$$

Hence $al^3 + 2alm^2 = m^2$. (Showed)

Problem 05: (a) Show that the tangent at (a, b) to t curve $(x/a)^3 + (y/b)^3 = 2$ is $\frac{x}{a} + \frac{y}{b} = 2$.

(b) Prove that $\frac{x}{a} + \frac{y}{b} = 1$ touches the curve

$$\frac{x}{a} + \log(y/b) = 0.$$

Solution: (a) Given that $\left(\frac{x}{a}\right)^3 + \left(\frac{y}{b}\right)^3 = 2 \longrightarrow (1)$

or, $x^3b^3 + y^3a^3 = 2a^3b^3$

Differentiating w.r to x we have

$$3x^2b^3 + 3y^2a^3\frac{dy}{dx} = 0$$

$$\therefore \frac{dy}{dx} = -\frac{b^3x^2}{a^3y^2}.$$

Value of $\frac{dy}{dx}$ at (a,b) is $= -\frac{b^3}{a^3} \cdot \frac{a^2}{b^2} = -\frac{b}{a}$

The equation of tangent to the curve ① at (a,b) is

$$y - b = -\frac{b}{a}(x-a)$$

or, $\frac{y-b}{b} = -\left(\frac{x-a}{a}\right)$

or, $\frac{y}{b} - 1 = -\frac{x}{a} + 1$

$\Rightarrow \frac{x}{a} + \frac{y}{b} = 2$

(b) Answer:- Given that $\frac{x}{a} + \log(y/b) = 0 \rightarrow$ ①

Differentiating w.r. to x we get

$$\frac{1}{a} + \frac{1}{y/b} \cdot \frac{1}{b}\frac{dy}{dx} = 0$$

or, $\frac{1}{a} + \frac{b}{y} \cdot \frac{1}{b}\frac{dy}{dx} = 0$

$\therefore \frac{dy}{dx} = -\frac{y}{a}$

Let (x_1, y_1) be the any point on ①. Then

$\frac{dy}{dx}$ at (x_1, y_1) is $-y_1/a$

The eq^n of tangent to the ① at (x_1, y_1) is

$$y - y_1 = -y_1/a\,(x - x_1)$$

or, $\frac{y}{y_1} - 1 = -\left(\frac{x}{a} - \frac{x_1}{a}\right)$

or, $\frac{y}{y_1} + \frac{x}{a} = 1 + \frac{x_1}{a} \longrightarrow$ ②

This becomes identical with $\frac{x}{a}+\frac{y}{b}=1$ when $x_1=0$ and $y_1=b$ which point clearly satisfies the given equation ①. i.e. $\frac{0}{a}+\log(b/b)=0$

Hence $\frac{x}{a}+\frac{y}{b}$ touches the given curve.

(Proved)

6. Show that the curve $\left(\frac{x}{a}\right)^n+\left(\frac{y}{b}\right)^n=2$ touches the straight line $\frac{x}{a}+\frac{y}{b}=2$ at the point (a,b), what be the value of n.

Solution: The given curve is $\left(\frac{x}{a}\right)^n+\left(\frac{y}{b}\right)^n=2$

$$\therefore \frac{dy}{dx}=\frac{-nx^{n-1}/a^n}{ny^{n-1}/b^n}=\frac{-b^n x^{n-1}}{a^n y^{n-1}}$$

Value of $\frac{dy}{dx}$ at $(a,b)=-\frac{b^n}{a^n}\cdot\frac{a^{n-1}}{b^{n-1}}=-\frac{b}{a}$

Equation of tangent at (a, b) is

$$y-b=-\frac{b}{a}(x-a)$$

$$\text{or, } \frac{y}{b}-1=-\left(\frac{x}{a}-1\right)$$

$$\text{or, } \frac{x}{a}+\frac{y}{b}=2$$

(Showed)

* Problem 07 : Prove that all points of the curve $y^2 = 4a\{x + a\sin(x/a)\}$ at which the tangent is parallel to the x-axis lie on a parabola.

Sol^n : Given $y^2 = 4a\{x + a\sin(x/a)\} \longrightarrow (1)$

Let (x_1, y_1) be the point on the curve at which the tangent is parallel to x-axis. $\therefore y_1^2 = 4a\{x_1 + a\sin(\frac{x_1}{a})\} \longrightarrow (2)$

Differentiating (1) w.r.to x

$$2y\frac{dy}{dx} = 4a\left\{1 + a\cos\left(\frac{x}{a}\right)\cdot\frac{1}{a}\right\}$$

$$\therefore \frac{dy}{dx} = \frac{2a}{y}\left\{1 + \cos\left(\frac{x}{a}\right)\right\}$$

Therefore $\frac{dy}{dx}$ at the point (x_1, y_1) is $\frac{2a}{y_1}\left\{1 + \cos\left(\frac{x_1}{a}\right)\right\}$

But $\frac{dy}{dx}$ at the point (x_1, y_1) should be zero.

$$\therefore \frac{2a}{y_1}\left(1 + \cos\frac{x_1}{a}\right) = 0$$

$$1 + \cos\frac{x_1}{a} = 0$$

or, $\cos\frac{x_1}{a} = -1$ or, $\cos^2\frac{x_1}{a} = 1$

or, $1 - \sin^2\frac{x_1}{a} = 1$

or, $\sin\frac{x_1}{a} = 0$

Putting this value in eq^n (2), we have $y_1^2 = 4ax_1$

Locus of the point (x_1, y_1) is $y^2 = 4ax$ which is parabola. (Proved)