VECTORS

Vector : a one dimensional array of numbers

Examples :

row vector $\begin{bmatrix} 1 & 4 & 2 \end{bmatrix}$ column vector $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$$\textit{Identity vectors}\ \ e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix},\ e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix},\ e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix},\ e_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

MATRICES

Matrix : a two dimensional array of numbers

Examples :

$$\text{zero matrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \text{identity matrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\text{diagonal} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 6 \end{bmatrix}, \quad \text{Tridiagonal} \begin{bmatrix} 1 & 2 & 0 & 0 \\ 3 & 4 & 1 & 0 \\ 0 & 1 & 4 & 1 \\ 0 & 0 & 2 & 1 \end{bmatrix}$$

MATRICES

Examples:

$$\text{symmetric}\begin{bmatrix} 2 & 1 & -1 \\ 1 & 0 & 5 \\ -1 & 5 & 4 \end{bmatrix}, \quad \text{upper triangular}\begin{bmatrix} 1 & 2 & 1 & 3 \\ 0 & 4 & 1 & 0 \\ 0 & 0 & 4 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Determinant of a MATRICES

Defined for square matrices only

Examples:

$$\det\begin{bmatrix} 2 & 3 & -1 \\ 1 & 0 & 5 \\ -1 & 5 & 4 \end{bmatrix} = 2\begin{vmatrix} 0 & 5 \\ 5 & 4 \end{vmatrix} -1\begin{vmatrix} 3 & -1 \\ 5 & 4 \end{vmatrix} -1\begin{vmatrix} 3 & -1 \\ 0 & 5 \end{vmatrix}$$

$$= 2(-25) - 1(12+5) - 1(15-0) = -82$$

Adding and Multiplying Matrices

The addition of two matrices A and B

* Defined only if they have the same size
* $C = A + B \Leftrightarrow c_{ij} = a_{ij} + b_{ij} \quad \forall i, j$

Multiplication of two matrices $A(n \times m)$ and $B(p \times q)$

* The product $C = AB$ is defined only if $m = p$
* $C = A\,B \Leftrightarrow c_{ij} = \sum_{k=1}^{m} a_{ik} b_{kj} \quad \forall i, j$

Systems of Linear Equations

A system of linear equations can be presented in different forms

$$\left.\begin{aligned} 2x_1 + 4x_2 - 3x_3 &= 3 \\ 2.5x_1 - x_2 + 3x_3 &= 5 \\ x_1 \qquad\qquad - 6x_3 &= 7 \end{aligned}\right\} \Leftrightarrow \begin{bmatrix} 2 & 4 & -3 \\ 2.5 & -1 & 3 \\ 1 & 0 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \\ 7 \end{bmatrix}$$

Standard form Matrix form

Solutions of Linear Equations

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \text{is a solution to the following equations:}$$

$$x_1 + x_2 = 3$$

$$x_1 + 2x_2 = 5$$

Solutions of Linear Equations

- A set of equations is **inconsistent** if there exists no solution to the system of equations:

$$x_1 + 2x_2 = 3$$

$$2x_1 + 4x_2 = 5$$

These equations are inconsistent

Solutions of Linear Equations

- Some systems of equations may have **infinite number of solutions**

$$x_1 + 2x_2 = 3$$

$$2x_1 + 4x_2 = 6$$

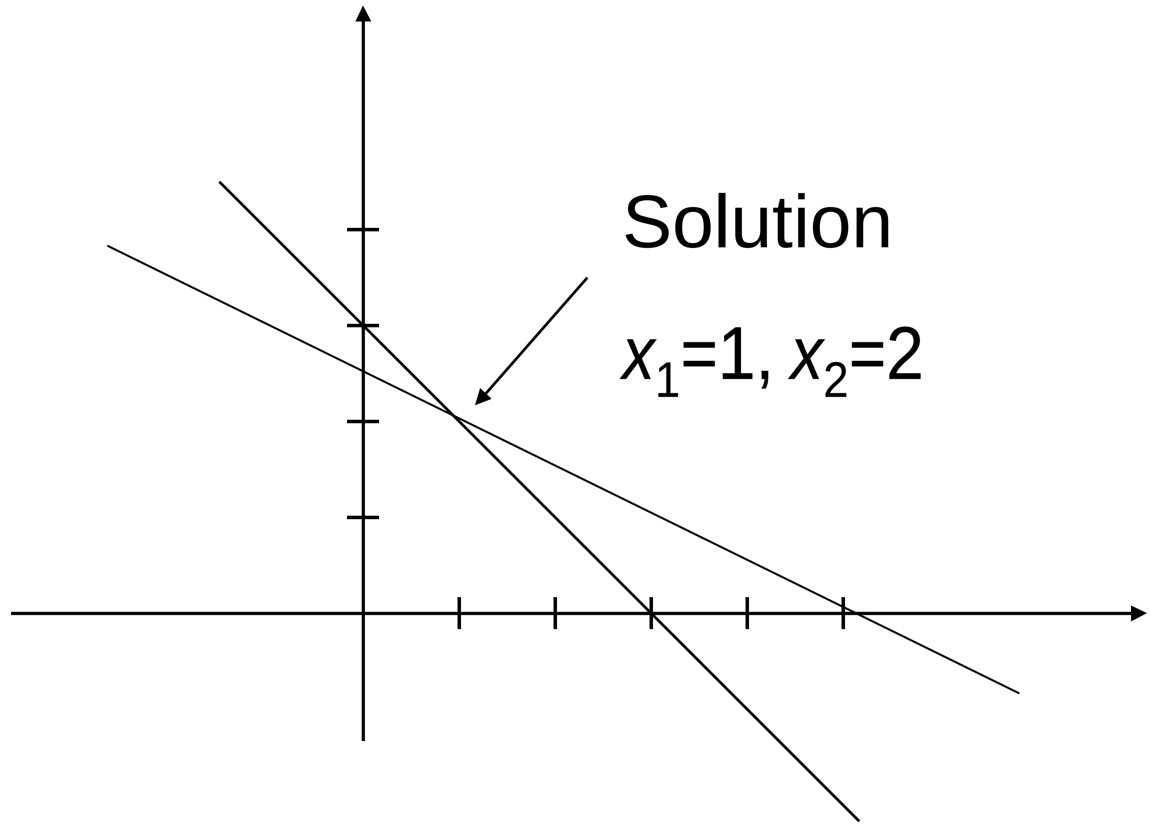
$$\text{have infinite number of solutions}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} a \\ 0.5(3-a) \end{bmatrix} \text{ is a solution for all } a$$

Graphical Solution of Systems of Linear Equations

$$x_1 + x_2 = 3$$

$$x_1 + 2x_2 = 5$$



Cramer's Rule is Not Practical

Cramer's Rule can be used to solve the system

$$x_1 = \frac{\begin{vmatrix} 3 & 1 \\ 5 & 2 \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix}} = 1, \qquad x_2 = \frac{\begin{vmatrix} 1 & 3 \\ 1 & 5 \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix}} = 2$$

Cramer's Rule is not practical for large systems.

To solve N by N system requires $(N+1)(N-1)N!$ multiplications.

To solve a 30 by 30 system, 2.38×10^{35} multiplications are needed.

It can be used if the determinants are computed in efficient way

Naive Gaussian Elimination

- The method consists of two steps:
  - **Forward Elimination**: the system is reduced to upper triangular form. A sequence of elementary operations is used.
  - **Backward Substitution**: Solve the system starting from the last variable.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \Rightarrow \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22}' & a_{23}' \\ 0 & 0 & a_{33}' \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2' \\ b_3' \end{bmatrix}$$

Elementary Row Operations

- Adding a multiple of one row to another
- Multiply any row by a non-zero constant

Example
Forward Elimination

$$\begin{bmatrix} 6 & -2 & 2 & 4 \\ 12 & -8 & 6 & 10 \\ 3 & -13 & 9 & 3 \\ -6 & 4 & 1 & -18 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 16 \\ 26 \\ -19 \\ -34 \end{bmatrix}$$

Part 1: Forward Elimination

Step 1: Eliminate x_1 from equations 2, 3, 4

$$\begin{bmatrix} 6 & -2 & 2 & 4 \\ 0 & -4 & 2 & 2 \\ 0 & -12 & 8 & 1 \\ 0 & 2 & 3 & -14 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 16 \\ -6 \\ -27 \\ -18 \end{bmatrix}$$

Example
Forward Elimination

$$\text{Step2: Eliminate } x_2 \text{ from equations } 3, 4$$

$$\begin{bmatrix} 6 & -2 & 2 & 4 \\ 0 & -4 & 2 & 2 \\ 0 & 0 & 2 & -5 \\ 0 & 0 & 4 & -13 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 16 \\ -6 \\ -9 \\ -21 \end{bmatrix}$$

$$\text{Step3: Eliminate } x_3 \text{ from equation } 4$$

$$\begin{bmatrix} 6 & -2 & 2 & 4 \\ 0 & -4 & 2 & 2 \\ 0 & 0 & 2 & -5 \\ 0 & 0 & 0 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 16 \\ -6 \\ -9 \\ -3 \end{bmatrix}$$

Example
Forward Elimination

Summary of the Forward Elimination:

$$\begin{bmatrix} 6 & -2 & 2 & 4 \\ 12 & -8 & 6 & 10 \\ 3 & -13 & 9 & 3 \\ -6 & 4 & 1 & -18 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 16 \\ 26 \\ -19 \\ -34 \end{bmatrix} \Rightarrow \begin{bmatrix} 6 & -2 & 2 & 4 \\ 0 & -4 & 2 & 2 \\ 0 & 0 & 2 & -5 \\ 0 & 0 & 0 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 16 \\ -6 \\ -9 \\ -3 \end{bmatrix}$$

Example
Backward Substitution

$$\begin{bmatrix} 6 & -2 & 2 & 4 \\ 0 & -4 & 2 & 2 \\ 0 & 0 & 2 & -5 \\ 0 & 0 & 0 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 16 \\ -6 \\ -9 \\ -3 \end{bmatrix}$$

Solve for x_4, then solve for x_3,... solve for x_1

$$x_4 = \frac{-3}{-3} = 1, \qquad x_3 = \frac{-9+5}{2} = -2$$

$$x_2 = \frac{-6-2(-2)-2(1)}{-4} = 1, \quad x_1 = \frac{16+2(1)-2(-2)-4(1)}{6} = 3$$

Forward Elimination

To eliminate x_1

$$\left.\begin{aligned} a_{ij} &\leftarrow a_{ij} - \left(\frac{a_{i1}}{a_{11}}\right) a_{1j} \quad (1 \le j \le n) \\ b_i &\leftarrow b_i - \left(\frac{a_{i1}}{a_{11}}\right) b_1 \end{aligned}\right\} 2 \le i \le n$$

To eliminate x_2

$$\left.\begin{aligned} a_{ij} &\leftarrow a_{ij} - \left(\frac{a_{i2}}{a_{22}}\right) a_{2j} \quad (2 \le j \le n) \\ b_i &\leftarrow b_i - \left(\frac{a_{i2}}{a_{22}}\right) b_2 \end{aligned}\right\} 3 \le i \le n$$

Forward Elimination

To eliminate x_k
$$\left.\begin{aligned} a_{ij} &\leftarrow a_{ij} - \left(\frac{a_{ik}}{a_{kk}}\right) a_{kj} \quad (k \le j \le n) \\ b_i &\leftarrow b_i - \left(\frac{a_{ik}}{a_{kk}}\right) b_k \end{aligned}\right\} k+1 \le i \le n$$

Continue until x_{n-1} is eliminated.

Backward Substitution

$$x_n = \frac{b_n}{a_{n,n}}$$

$$x_{n-1} = \frac{b_{n-1} - a_{n-1,n}x_n}{a_{n-1,n-1}}$$

$$x_{n-2} = \frac{b_{n-2} - a_{n-2,n}x_n - a_{n-2,n-1}x_{n-1}}{a_{n-2,n-2}}$$

$$x_i = \frac{b_i - \sum_{j=i+1}^{n} a_{i,j}x_j}{a_{i,i}}$$

Naive Gaussian Elimination

- Summary of the Naive Gaussian Elimination
- Example
- Problems with Naive Gaussian Elimination
  - Failure due to zero pivot element
  - Error
- Pseudo-Code

Naive Gaussian Elimination

- The method consists of two steps
  - **Forward Elimination**: the system is reduced to upper triangular form. A sequence of elementary operations is used.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \Rightarrow \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22}' & a_{23}' \\ 0 & 0 & a_{33}' \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2' \\ b_3' \end{bmatrix}$$

  - **Backward Substitution**: Solve the system starting from the last variable. Solve for $x_n, x_{n-1}, \ldots x_1$.

Example 1

Solve using Naive Gaussian Elimination:

Part 1: Forward Elimination ___ Step1: Eliminate x_1 from equations 2, 3

$$x_1 + 2x_2 + 3x_3 = 8 \qquad eq1 \;\; unchanged\,(pivot\;equation)$$

$$2x_1 + 3x_2 + 2x_3 = 10 \qquad eq2 \leftarrow eq2 - \left(\frac{2}{1}\right)eq1$$

$$3x_1 + x_2 + 2x_3 = 7 \qquad eq3 \leftarrow eq3 - \left(\frac{3}{1}\right)eq1$$

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 8 \\ - x_2 - 4x_3 &= -6 \\ -5x_2 - 7x_3 &= -17 \end{aligned}$$

Example 1

Part 1: Forward Elimination Step2: Eliminate x_2 from equation 3

$$x_1 + 2x_2 + 3x_3 = 8 \qquad eq1\ unchanged$$

$$-\ x_2 - 4x_3 = -6 \qquad eq2\ unchanged\ (pivot\ equation)$$

$$-5x_2 - 7x_3 = -17 \qquad eq3 \leftarrow eq3 - \left(\frac{-5}{-1}\right) eq2$$

$$\Rightarrow \begin{cases} x_1 + 2x_2 + 3x_3 = 8 \\ \quad -\ x_2 - 4x_3 = -6 \\ \qquad\qquad 13x_3 = 13 \end{cases}$$

Example 1
Backward Substitution

$$x_3 = \frac{b_3}{a_{3,3}} = \frac{13}{13} = 1$$

$$x_2 = \frac{b_2 - a_{2,3}x_3}{a_{2,2}} = \frac{-6 + 4x_3}{-1} = 2$$

$$x_1 = \frac{b_1 - a_{1,2}x_2 - a_{1,3}x_3}{a_{1,1}} = \frac{8 - 2x_2 - 3x_3}{a_{1,1}} = 1$$

The solution is $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$

Determinant

The elementary operations do not affect the determinant

Example:

$$\mathrm{A} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 2 \\ 3 & 1 & 2 \end{bmatrix} \xrightarrow{\text{Elementary operations}} \mathrm{A}' = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & -4 \\ 0 & 0 & 13 \end{bmatrix}$$

$$\det(A) = \det(A') = -13$$

How Many Solutions Does a System of Equations AX=B Have?

Unique	No solution	Infinite
$\det(A) \neq 0$	$\det(A) = 0$	$\det(A) = 0$
reduced matrix has no zero rows	reduced matrix has one or more zero rows	reduced matrix has one or more zero rows
	corresponding B elements $\neq 0$	corresponding B elements $= 0$

Examples

Unique

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} X = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\downarrow$$

$$\begin{bmatrix} 1 & 2 \\ 0 & -2 \end{bmatrix} X = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$solution:$

$$X = \begin{bmatrix} 0 \\ 0.5 \end{bmatrix}$$

No solution

$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} X = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$$\downarrow$$

$$\begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} X = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

$No\ solution$

$$0 = -1\ impossible!$$

infinte # of solutions

$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} X = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$

$$\downarrow$$

$$\begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} X = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

$Infinite\ \#\ solutions$

$$X = \begin{bmatrix} \alpha \\ 1 - .5\alpha \end{bmatrix}$$

Pseudo-Code: Forward Elimination

Do k = 1 to n-1

  Do i = k+1 to n

    factor = $a_{i,k}$ / $a_{k,k}$

    Do j = k+1 to n

      $a_{i,j} = a_{i,j} – \text{factor} * a_{k,j}$

    End Do

    $b_i = b_i – \text{factor} * b_k$

  End Do

End Do

Pseudo-Code: Back Substitution

$x_n = b_n / a_{n,n}$

Do i = n-1 downto 1

  sum = b_i

  Do j = i+1 to n

    sum = sum – $a_{i,j}$ * x_j

  End Do

  x_i = sum / $a_{i,i}$

End Do

Lectures 15-16:
Gaussian Elimination with Scaled Partial Pivoting

- Problems with Naive Gaussian Elimination
- Definitions and Initial step
- Forward Elimination
- Backward substitution
- Example

Problems with Naive Gaussian Elimination

- The Naive Gaussian Elimination may fail for very simple cases. (**The pivoting element is zero).**

$$\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

- **Very small pivoting element may result in serious computation errors**

$$\begin{bmatrix} 10^{-10} & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Example 2

Solve the following system using Gaussian Elimination with Scaled Partial Pivoting:

$$\begin{bmatrix} 1 & -1 & 2 & 1 \\ 3 & 2 & 1 & 4 \\ 5 & 8 & 6 & 3 \\ 4 & 2 & 5 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix}$$

Example 2
Initialization step

$$\begin{bmatrix} 1 & -1 & 2 & 1 \\ 3 & 2 & 1 & 4 \\ 5 & 8 & 6 & 3 \\ 4 & 2 & 5 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix}$$

Scale vector:

disregard sign

find largest in magnitude in each row

$$\text{Scale vector} \quad S = \begin{bmatrix} 2 & 4 & 8 & 5 \end{bmatrix}$$

$$\text{Index Vector} \quad L = \begin{bmatrix} 1 & 2 & 3 & 4 \end{bmatrix}$$

Why Index Vector?

- Index vectors are used because it is much easier to exchange a single index element compared to exchanging the values of a complete row.
- In practical problems with very large N, exchanging the contents of rows may not be practical.

Example 2
Forward Elimination-- Step 1: eliminate x1

Selection of the pivot equation

$$\begin{bmatrix} 1 & -1 & 2 & 1 \\ 3 & 2 & 1 & 4 \\ 5 & 8 & 6 & 3 \\ 4 & 2 & 5 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix} \Rightarrow \begin{cases} S = [2 \quad 4 \quad 8 \quad 5] \\ L = [1 \quad 2 \quad 3 \quad 4\ ] \end{cases}$$

$$Ratios \ = \left\{ \frac{\left|a_{l_i,1}\right|}{S_{l_i}} \ i = 1,2,3,4 \right\} = \left\{ \frac{|1|}{2}, \frac{|3|}{4}, \frac{|5|}{8}, \boxed{\frac{|4|}{5}} \right\} \Rightarrow \text{max corresponds to } l_4$$

equation 4 is the first pivot equation Exchange l_4 *and* l_1

$L = [\,4 \quad 2 \quad 3 \quad 1\,]$

Example 2

Forward Elimination-- Step 1: eliminate x1

Update A and B

$$\begin{bmatrix} 1 & -1 & 2 & 1 \\ 3 & 2 & 1 & 4 \\ 5 & 8 & 6 & 3 \\ \boxed{4 \quad 2 \quad 5 \quad 3} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ \boxed{-1} \end{bmatrix}$$

First pivot equation

$$\Rightarrow \begin{bmatrix} 0 & -1.5 & 0.75 & 0.25 \\ 0 & 0.5 & -2.75 & 1.75 \\ 0 & 5.5 & -0.25 & -0.75 \\ \boxed{4 \quad 2 \quad 5 \quad 3} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1.25 \\ 1.75 \\ 2.25 \\ \boxed{-1} \end{bmatrix}$$

Example 2
Forward Elimination-- Step 2: eliminate x2

Selection of the second pivot equation

$$\begin{bmatrix} 0 & -1.5 & 0.75 & 0.25 \\ 0 & 0.5 & -2.75 & 1.75 \\ 0 & 5.5 & -0.25 & -0.75 \\ 4 & 2 & 5 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1.25 \\ 1.75 \\ 2.25 \\ -1 \end{bmatrix}$$

$$S = [2 \ \ 4 \ \ 8 \ \ 5\ ] \qquad L = [\ \ 4 \quad 2 \quad 3 \quad 1\ ]$$

$$\text{Ratios}: \left\{ \frac{\left|a_{l_i,2}\right|}{S_{l_i}} \ i = 2,3,4 \right\} = \left\{ \frac{0.5}{4} \quad \frac{5.5}{8} \quad \boxed{\frac{1.5}{2}} \right\} \Rightarrow L = [\ 4\ 1\ \ 3\ \ 2\ \ ]$$

Example 2

Forward Elimination-- Step 3: eliminate x3

$$\begin{bmatrix} 0 & -1.5 & 0.75 & 0.25 \\ 0 & 0 & \boxed{-2.5 \quad 1.8333} \\ 0 & 0 & 0.25 & 1.6667 \\ 4 & 2 & 5 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1.25 \\ 2.1667 \\ 6.8333 \\ -1 \end{bmatrix}$$

Third pivot equation

$L = [\ 4\ 1\ 2\ \ 3]$

$$\begin{bmatrix} 0 & -1.5 & 0.75 & 0.25 \\ 0 & 0 & -2.5 & 1.8333 \\ 0 & 0 & 0 & 2 \\ 4 & 2 & 5 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1.25 \\ 2.1667 \\ 9 \\ -1 \end{bmatrix}$$

Example 2
Backward Substitution

$$\begin{bmatrix} 0 & -1.5 & 0.75 & 0.25 \\ 0 & 0 & -2.5 & 1.8333 \\ 0 & 0 & 0 & 2 \\ 4 & 2 & 5 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1.25 \\ 2.1667 \\ 9 \\ -1 \end{bmatrix} \qquad L = [\ 4\ 1\ 2\ \ 3]$$

$$x_4 = \frac{b_3}{a_{3,4}} = \frac{9}{2} = 4.5, \quad x_3 = \frac{b_2 - a_{2,4}x_4}{a_{2,3}} = \frac{2.1667 - 1.8333x_4}{-2.5} = 2.4327$$

$$x_2 = \frac{b_1 - a_{1,4}x_4 - a_{1,3}x_3}{a_{1,2}} = \frac{1.25 - 0.25x_4 - 0.75x_3}{-1.5} = 1.1333$$

$$x_1 = \frac{b_4 - a_{4,4}x_4 - a_{4,3}x_3 - a_{4,2}x_2}{a_{l_1,1}} = \frac{-1 - 3x_4 - 5x_3 - 2x_2}{4} = -7.2333$$

Example 3

Solve the following sytstem using Gaussian Elimination with Scaled Partial Pivoting

$$\begin{bmatrix} 1 & -1 & 2 & 1 \\ 3 & 2 & 1 & 4 \\ 5 & -8 & 6 & 3 \\ 4 & 2 & 5 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix}$$

Example 3
Initialization step

$$\begin{bmatrix} 1 & -1 & 2 & 1 \\ 3 & 2 & 1 & 4 \\ 5 & -8 & 6 & 3 \\ 4 & 2 & 5 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix}$$

$$\text{Scale vector} \quad S = \begin{bmatrix} 2 & 4 & 8 & 5 \end{bmatrix}$$

$$\text{Index Vector} \quad L = \begin{bmatrix} 1 & 2 & 3 & 4 \end{bmatrix}$$

Example 3
Forward Elimination-- Step 1: eliminate x1

Selection of the pivot equation

$$\begin{bmatrix} 1 & -1 & 2 & 1 \\ 3 & 2 & 1 & 4 \\ 5 & -8 & 6 & 3 \\ 4 & 2 & 5 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix} \Rightarrow \begin{cases} S = [2 \quad 4 \quad 8 \quad 5] \\ L = [1 \quad 2 \quad 3 \quad 4\ ] \end{cases}$$

$$Ratios = \left\{ \frac{\left|a_{l_i,1}\right|}{S_{l_i}} \; i = 1,2,3,4 \right\} = \left\{ \frac{|1|}{2}, \frac{|3|}{4}, \frac{|5|}{8}, \boxed{\frac{|4|}{5}} \right\} \Rightarrow \text{max corresponds to } l_4$$

equation 4 is the first pivot equation Exchange l_4 *and* l_1

$L = [\,4 \quad 2 \quad 3 \quad 1\,]$

Example 3

Forward Elimination-- Step 1: eliminate x1

Update A and B

$$\begin{bmatrix} 1 & -1 & 2 & 1 \\ 3 & 3 & 1 & 4 \\ 5 & -8 & 6 & 3 \\ 4 & 2 & 5 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 0 & -1.5 & 0.75 & 0.25 \\ 0 & 0.5 & -2.75 & 1.75 \\ 0 & -10.5 & -0.25 & -0.75 \\ 4 & 2 & 5 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1.25 \\ 1.75 \\ 2.25 \\ -1 \end{bmatrix}$$

Example 3

Forward Elimination-- Step 2: eliminate x2

Selection of the second pivot equation

$$\begin{bmatrix} 0 & -1.5 & 0.75 & 0.25 \\ 0 & 0.5 & -2.75 & 1.75 \\ 0 & -10.5 & -0.25 & -0.75 \\ 4 & 2 & 5 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1.25 \\ 1.75 \\ 2.25 \\ -1 \end{bmatrix}$$

$$S = [2 \ \ 4 \ \ 8 \ \ 5\ ] \qquad L = [\ \ 4 \quad 2 \quad 3 \quad 1\ ]$$

$$\text{Ratios}: \left\{ \frac{\left| a_{l_i,2} \right|}{S_{l_i}} \ i = 2,3,4 \right\} = \left\{ \frac{0.5}{4} \quad \boxed{\frac{10.5}{8}} \quad \frac{1.5}{2} \right\} \Rightarrow L = [\ 4\ 3\ 2\ 1]$$

Example 3

Forward Elimination-- Step 2: eliminate x2

Updating A and B

$$\begin{bmatrix} 0 & -1.5 & 0.75 & 0.25 \\ 0 & 0.5 & -2.75 & 1.75 \\ 0 & \boxed{-10.5 & -0.25 & -0.75} \\ 4 & 2 & 5 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1.25 \\ 1.75 \\ 2.25 \\ -1 \end{bmatrix}$$

$L = [\ 4\ 1\ 3\ 2]$

$$\begin{bmatrix} 0 & 0 & 0.7857 & 0.3571 \\ 0 & 0 & -2.7619 & 1.7143 \\ 0 & -10.5 & -0.25 & -0.75 \\ 4 & 2 & 5 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0.9286 \\ 1.8571 \\ 2.25 \\ -1 \end{bmatrix}$$

Example 3

Forward Elimination-- Step 3: eliminate x3

Selection of the third pivot equation

$$\begin{bmatrix} 0 & 0 & 0.7857 & 0.3571 \\ 0 & 0 & -2.7619 & 1.7143 \\ 0 & -10.5 & -0.25 & -0.75 \\ 4 & 2 & 5 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0.9286 \\ 1.8571 \\ 2.25 \\ -1 \end{bmatrix}$$

$$S = [2\ \ 4\ \ 8\ \ 5\ ] \qquad L = [\ \ 4\quad 3\quad 2\quad 1\ ]$$

$$\text{Ratios}: \left\{ \frac{\left| a_{l_i,3} \right|}{S_{l_i}} \ i = 3,4 \right\} = \left\{ \quad \boxed{\frac{2.7619}{4}} \quad \frac{0.7857}{2} \right\} \Rightarrow L = [\ 4\ 3\ 2\ 1]$$

Example 3
Forward Elimination-- Step 3: eliminate x3

$$\begin{bmatrix} 0 & 0 & 0.7857 & 0.3571 \\ 0 & 0 & -2.7619 & 1.7143 \\ 0 & -10.5 & -0.25 & -0.75 \\ 4 & 2 & 5 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0.9286 \\ 1.8571 \\ 2.25 \\ -1 \end{bmatrix}$$

$$L = [\ 4\ 3\ 2\ \ 1]$$

$$\begin{bmatrix} 0 & 0 & 0 & 0.8448 \\ 0 & 0 & -2.7619 & 1.7143 \\ 0 & -10.5 & -0.25 & -0.75 \\ 4 & 2 & 5 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1.4569 \\ 1.8571 \\ 2.25 \\ -1 \end{bmatrix}$$

Example 3
Backward Substitution

$$\begin{bmatrix} 0 & 0 & 0 & 0.8448 \\ 0 & 0 & -2.7619 & 1.7143 \\ 0 & -10.5 & -0.25 & -0.75 \\ 4 & 2 & 5 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1.4569 \\ 1.8571 \\ 2.25 \\ -1 \end{bmatrix} \qquad L = [\ 4\ 3\ 2\ \ 1]$$

$$x_4 = \frac{b_{l_4}}{a_{l_4,4}} = \frac{1.4569}{0.8448} = 1.7245, \quad x_3 = \frac{b_{l_3} - a_{l_3,4}x_4}{a_{l_3,3}} = \frac{1.8571 - 1.7143x_4}{-2.7619} = 0.3980$$

$$x_2 = \frac{b_{l_2} - a_{l_2,4}x_4 - a_{l_2,3}x_3}{a_{l_2,2}} = -0.3469$$

$$x_1 = \frac{b_{l_1} - a_{l_1,4}x_4 - a_{l_1,3}x_3 - a_{l_1,2}x_2}{a_{l_1,1}} = \frac{-1 - 3x_4 - 5x_3 - 2x_2}{4} = -1.8673$$

How Do We Know If a Solution is Good or Not

Given AX=B

- X is a solution if AX-B=0
- Compute the residual vector R= AX-B
- Due to rounding error, R may not be zero

$$\text{The solution is acceptable if } \max_i \left| r_i \right| \leq \varepsilon$$

How Good is the Solution?

$$\begin{bmatrix} 1 & -1 & 2 & 1 \\ 3 & 2 & 1 & 4 \\ 5 & -8 & 6 & 3 \\ 4 & 2 & 5 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix} \quad solution \quad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -1.8673 \\ -0.3469 \\ 0.3980 \\ 1.7245 \end{bmatrix}$$

$$\text{Residues}: \; R = \begin{bmatrix} 0.005 \\ 0.002 \\ 0.003 \\ 0.001 \end{bmatrix}$$

Remarks:

- We use index vector to avoid the need to move the rows which may not be practical for large problems.
- If we order the equation as in the last value of the index vector, we have a triangular form.
- Scale vector is formed by taking maximum in magnitude in each row.
- Scale vector does not change.
- The original matrices A and B are used in checking the residuals.

Lecture 17
Tridiagonal & Banded Systems and Gauss-Jordan Method

- Tridiagonal Systems
- Diagonal Dominance
- Tridiagonal Algorithm
- Examples
- Gauss-Jordan Algorithm

Tridiagonal Systems

Tridiagonal Systems:

- The non-zero elements are in the main diagonal, super diagonal and subdiagonal.
- $a_{ij}=0$ if $|i-j| > 1$

$$\begin{bmatrix} 5 & 1 & 0 & 0 & 0 \\ 3 & 4 & 1 & 0 & 0 \\ 0 & 2 & 6 & 2 & 0 \\ 0 & 0 & 1 & 4 & 1 \\ 0 & 0 & 0 & 1 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \end{bmatrix}$$

Tridiagonal Systems

- Occur in many applications
- Needs less storage (4n-2 compared to n^2 +n for the general cases)
- Selection of pivoting rows is unnecessary (under some conditions)
- Efficiently solved by Gaussian elimination

Algorithm to Solve Tridiagonal Systems

- Based on Naive Gaussian elimination.
- As in previous Gaussian elimination algorithms
  - Forward elimination step
  - Backward substitution step
- Elements in the super diagonal are not affected.
- Elements in the main diagonal, and B need updating

Tridiagonal System

All the a elements will be zeros, need to update the d and b elements

The c elements are not updated

$$\begin{bmatrix} d_1 & c_1 & & & \\ a_1 & d_2 & c_2 & & \\ & a_2 & d_3 & \ddots & \\ & & \ddots & \ddots & c_{n-1} \\ & & & a_{n-1} & d_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_n \end{bmatrix} \Rightarrow \begin{bmatrix} d_1 & c_1 & & & \\ & d_2' & c_2 & & \\ & & d_3' & \ddots & \\ & & & \ddots & c_{n-1} \\ & & & & d_n' \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2' \\ b_3' \\ \vdots \\ b_n' \end{bmatrix}$$

Diagonal Dominance

A matrix A is diagonally dominant if

$$|a_{ii}| > \sum_{\substack{j=1,\\ j\neq i}}^{n} |a_{ij}| \quad for\ (1 \le i \le n)$$

The magnitude of each diagonal element is larger than the sum of elements in the corresponding row.

Diagonal Dominance

Examples :

$$\begin{bmatrix} 3 & 0 & 1 \\ 1 & 6 & 1 \\ 1 & 2 & -5 \end{bmatrix}$$

Diagonally dominant

$$\begin{bmatrix} -3 & 0 & 1 \\ 2 & 3 & 2 \\ 1 & 2 & 1 \end{bmatrix}$$

Not Diagonally dominant

Diagonally Dominant Tridiagonal System

- A tridiagonal system is diagonally dominant if

$$\left|d_i\right| > \left|c_i\right| + \left|a_{i-1}\right| \qquad (1 \le i \le n)$$

- Forward Elimination preserves diagonal dominance

Solving Tridiagonal System

Forward Elimination

$$d_i \leftarrow d_i - \left(\frac{a_{i-1}}{d_{i-1}}\right)c_{i-1}$$

$$b_i \leftarrow b_i - \left(\frac{a_{i-1}}{d_{i-1}}\right)b_{i-1} \qquad 2 \le i \le n$$

Backward Substitution

$$x_n = \frac{b_n}{d_n}$$

$$x_i = \frac{1}{d_i}\left(b_i - c_i x_{i+1}\right) \qquad \text{for } i = n-1, n-2, \ldots, 1$$

Example

Solve

$$\begin{bmatrix} 5 & 2 & & \\ 1 & 5 & 2 & \\ & 1 & 5 & 2 \\ & & 1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 12 \\ 9 \\ 8 \\ 6 \end{bmatrix} \quad \Rightarrow D = \begin{bmatrix} 5 \\ 5 \\ 5 \\ 5 \end{bmatrix}, A = \begin{bmatrix} 1 \\ 1 \\ 1 \\ \ \end{bmatrix}, C = \begin{bmatrix} 2 \\ 2 \\ 2 \\ \ \end{bmatrix}, \ B = \begin{bmatrix} 12 \\ 9 \\ 8 \\ 6 \end{bmatrix}$$

Forward Elimination

$$d_i \leftarrow d_i - \left(\frac{a_{i-1}}{d_{i-1}} \right) c_{i-1}, \quad b_i \leftarrow b_i - \left(\frac{a_{i-1}}{d_{i-1}} \right) b_{i-1} \qquad 2 \le i \le 4$$

Backward Substitution

$$x_n = \frac{b_n}{d_n}, \quad x_i = \frac{1}{d_i} \left(b_i - c_i x_{i+1} \right) \qquad \text{for } \ i = 3,2,1$$

Example

$$D = \begin{bmatrix} 5 \\ 5 \\ 5 \\ 5 \end{bmatrix}, A = \begin{bmatrix} 1 \\ 1 \\ 1 \\ \end{bmatrix}, C = \begin{bmatrix} 2 \\ 2 \\ 2 \\ \end{bmatrix}, \ B = \begin{bmatrix} 12 \\ 9 \\ 8 \\ 6 \end{bmatrix}$$

Forward Elimination

$$d_2 = d_2 - \left(\frac{a_1}{d_1}\right) c_1 = 5 - \frac{1 \times 2}{5} = 4.6, \quad b_2 = b_2 - \left(\frac{a_1}{d_1}\right) b_1 = 9 - \frac{1 \times 12}{5} = 6.6$$

$$d_3 = d_3 - \left(\frac{a_2}{d_2}\right) c_2 = 5 - \frac{1 \times 2}{4.6} = 4.5652, \quad b_3 = b_3 - \left(\frac{a_2}{d_2}\right) b_2 = 8 - \frac{1 \times 6.6}{4.6} = 6.5652$$

$$d_4 = d_4 - \left(\frac{a_3}{d_3}\right) c_3 = 5 - \frac{1 \times 2}{4.5652} = 4.5619, \quad b_4 = b_4 - \left(\frac{a_3}{d_3}\right) b_3 = 6 - \frac{1 \times 6.5652}{4.5652} = 4.5619$$

Example
Backward Substitution

- After the Forward Elimination:

$$D^T = \begin{bmatrix} 5 & 4.6 & 4.5652 & 4.5619 \end{bmatrix},\ B^T = \begin{bmatrix} 12 & 6.6 & 6.5652 & 4.5619 \end{bmatrix}$$

- Backward Substitution:

$$x_4 = \frac{b_4}{d_4} = \frac{4.5619}{4.5619} = 1,$$

$$x_3 = \frac{b_3 - c_3 x_4}{d_3} = \frac{6.5652 - 2\times 1}{4.5652} = 1$$

$$x_2 = \frac{b_2 - c_2 x_3}{d_2} = \frac{6.6 - 2\times 1}{4.6} = 1$$

$$x_1 = \frac{b_1 - c_1 x_2}{d_1} = \frac{12 - 2\times 1}{5} = 2$$

Gauss-Jordan Method

- The method reduces the general system of equations AX=B to IX=B where I is an identity matrix.

- Only Forward elimination is done and no backward substitution is needed.

- It has the same problems as Naive Gaussian elimination and can be modified to do partial scaled pivoting.

- It takes 50% more time than Naive Gaussian method.

Gauss-Jordan Method
Example

$$\begin{bmatrix} 2 & -2 & 2 \\ 4 & 2 & -1 \\ 2 & -2 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 7 \\ 2 \end{bmatrix}$$

Step 1 Eleminate x_1 from *equations* 2*and* 3

$$\left. \begin{aligned} eq1 &\leftarrow eq1/2 \\ eq2 &\leftarrow eq2 - \left(\frac{4}{1}\right) eq1 \\ eq3 &\leftarrow eq3 - \left(\frac{2}{1}\right) eq1 \end{aligned} \right\} \Rightarrow \begin{bmatrix} 1 & -1 & 1 \\ 0 & 6 & -5 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 7 \\ 2 \end{bmatrix}$$

Gauss-Jordan Method
Example

$$\begin{bmatrix} 1 & -1 & 1 \\ 0 & 6 & -5 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 7 \\ 2 \end{bmatrix}$$

Step 2 Eleminate x_2 from *equations* 1 *and* 3

$$\left.\begin{aligned} eq2 &\leftarrow eq2/6 \\ eq1 &\leftarrow eq1 - \left(\frac{-1}{1}\right) eq2 \\ eq3 &\leftarrow eq3 - \left(\frac{0}{1}\right) eq2 \end{aligned}\right\} \Rightarrow \begin{bmatrix} 1 & 0 & 0.1667 \\ 0 & 1 & -0.8333 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1.1667 \\ 1.1667 \\ 2 \end{bmatrix}$$

Gauss-Jordan Method
Example

$$\begin{bmatrix} 1 & 0 & 0.1667 \\ 0 & 1 & -0.8333 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1.1667 \\ 1.1667 \\ 2 \end{bmatrix}$$

$\textit{Step}\ 3\ \ \text{Eleminate}\ \ \mathrm{x}_3\ \text{from}\ \ \textit{equations}\ 1\ \textit{and}\ 2$

$$\left.\begin{aligned} eq3 &\leftarrow eq3/2 \\ eq1 &\leftarrow eq1 - \left(\frac{0.1667}{1}\right) eq3 \\ eq2 &\leftarrow eq2 - \left(\frac{-0.8333}{1}\right) eq3 \end{aligned}\right\} \Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

Gauss-Jordan Method
Example

$$\begin{bmatrix} 2 & -2 & 2 \\ 4 & 2 & -1 \\ 2 & -2 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 7 \\ 2 \end{bmatrix}$$

is transformed to

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \quad \Rightarrow \textit{solution is} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$