22. Given $r_1 = 3i - 2j + k$, $r_2 = 2i - 4j - 3k$, $r_3 = -i + 2j + 2k$, find the magnitudes of (a) r_3, (b) $r_1 + r_2 + r_3$, (c) $2r_1 - 3r_2 - 5r_3$.

(a) $|r_3| = |-i + 2j + 2k| = \sqrt{(-1)^2 + (2)^2 + (2)^2} = 3$.

(b) $r_1 + r_2 + r_3 = (3i - 2j + k) + (2i - 4j - 3k) + (-i + 2j + 2k) = 4i - 4j + 0k = 4i - 4j$

Then $|r_1 + r_2 + r_3| = |4i - 4j + 0k| = \sqrt{(4)^2 + (-4)^2 + (0)^2} = \sqrt{32} = 4\sqrt{2}$.

(c) $2r_1 - 3r_2 - 5r_3 = 2(3i - 2j + k) - 3(2i - 4j - 3k) - 5(-i + 2j + 2k)$

$= 6i - 4j + 2k - 6i + 12j + 9k + 5i - 10j - 10k = 5i - 2j + k$.

Then $|2r_1 - 3r_2 - 5r_3| = |5i - 2j + k| = \sqrt{(5)^2 + (-2)^2 + (1)^2} = \sqrt{30}$.

23. If $r_1 = 2i - j + k$, $r_2 = i + 3j - 2k$, $r_3 = -2i + j - 3k$ and $r_4 = 3i + 2j + 5k$, find scalars a, b, c such that $r_4 = ar_1 + br_2 + cr_3$.

We require $3i + 2j + 5k = a(2i - j + k) + b(i + 3j - 2k) + c(-2i + j - 3k)$

$= (2a + b - 2c)i + (-a + 3b + c)j + (a - 2b - 3c)k$.

Since i, j, k are non-coplanar we have by Problem 15,

$$2a + b - 2c = 3, \quad -a + 3b + c = 2, \quad a - 2b - 3c = 5.$$

Solving, $a = -2$, $b = 1$, $c = -3$ and $r_4 = -2r_1 + r_2 - 3r_3$.

The vector r_4 is said to be *linearly dependent* on r_1, r_2, and r_3; in other words r_1, r_2, r_3 and r_4 constitute a *linearly dependent* set of vectors. On the other hand any three (or fewer) of these vectors are *linearly independent*.

In general the vectors A, B, C, ... are called linearly dependent if we can find a set of scalars, $a, b, c, \ldots$, not all zero, so that $aA + bB + cC + \ldots = 0$, otherwise they are linearly independent.

24. Find a unit vector parallel to the resultant of vectors $\mathbf{r}_1 = 2\mathbf{i} + 4\mathbf{j} - 5\mathbf{k}$, $\mathbf{r}_2 = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$.

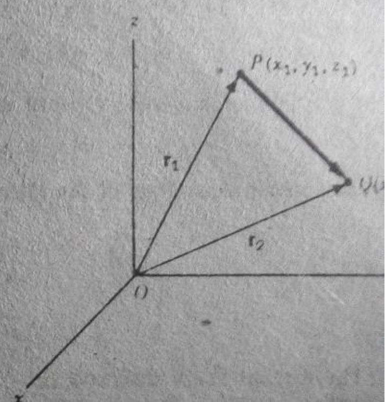
Resultant $\mathbf{R} = \mathbf{r}_1 + \mathbf{r}_2 = (2\mathbf{i} + 4\mathbf{j} - 5\mathbf{k}) + (\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}) = 3\mathbf{i} + 6\mathbf{j} - 2\mathbf{k}$.

$R = |\mathbf{R}| = |3\mathbf{i} + 6\mathbf{j} - 2\mathbf{k}| = \sqrt{(3)^2 + (6)^2 + (-2)^2} = 7$.

Then a unit vector parallel to $\mathbf{R}$ is $\dfrac{\mathbf{R}}{R} = \dfrac{3\mathbf{i} + 6\mathbf{j} - 2\mathbf{k}}{7} = \dfrac{3}{7}\mathbf{i} + \dfrac{6}{7}\mathbf{j} - \dfrac{2}{7}\mathbf{k}$.

Check: $\left|\dfrac{3}{7}\mathbf{i} + \dfrac{6}{7}\mathbf{j} - \dfrac{2}{7}\mathbf{k}\right| = \sqrt{(\frac{3}{7})^2 + (\frac{6}{7})^2 + (-\frac{2}{7})^2} = 1$.

25. Determine the vector having initial point $P(x_1, y_1, z_1)$ and terminal point $Q(x_2, y_2, z_2)$ and find its magnitude.



The position vector of P is $\mathbf{r}_1 = x_1\mathbf{i} + y_1\mathbf{j} + z_1\mathbf{k}$.

The position vector of Q is $\mathbf{r}_2 = x_2\mathbf{i} + y_2\mathbf{j} + z_2\mathbf{k}$.

$\mathbf{r}_1 + \mathbf{PQ} = \mathbf{r}_2$ or

$$\mathbf{PQ} = \mathbf{r}_2 - \mathbf{r}_1 = (x_2\mathbf{i} + y_2\mathbf{j} + z_2\mathbf{k}) - (x_1\mathbf{i} + y_1\mathbf{j} + z_1\mathbf{k})$$
$$= (x_2 - x_1)\mathbf{i} + (y_2 - y_1)\mathbf{j} + (z_2 - z_1)\mathbf{k}.$$

Magnitude of $\mathbf{PQ} = \overline{PQ} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$.

Note that this is the distance between points P and Q.

26. Forces $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ acting on an object are given in terms of their components by the vect tions $\mathbf{A} = A_1\mathbf{i} + A_2\mathbf{j} + A_3\mathbf{k}$, $\mathbf{B} = B_1\mathbf{i} + B_2\mathbf{j} + B_3\mathbf{k}$, $\mathbf{C} = C_1\mathbf{i} + C_2\mathbf{j} + C_3\mathbf{k}$. Find the magnitud resultant of these forces.

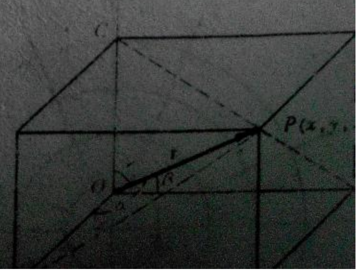
Resultant force $\mathbf{R} = \mathbf{A} + \mathbf{B} + \mathbf{C} = (A_1 + B_1 + C_1)\mathbf{i} + (A_2 + B_2 + C_2)\mathbf{j} + (A_3 + B_3 + C_3)\mathbf{k}$.

Magnitude of resultant $= \sqrt{(A_1 + B_1 + C_1)^2 + (A_2 + B_2 + C_2)^2 + (A_3 + B_3 + C_3)^2}$.

The result is easily extended to more than three forces.

27. Determine the angles α, β and γ which the vector $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ makes with the positive directions of the coordinate axes and show that

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1.$$



Referring to the figure, triangle OAP is a right-angled triangle with right angle at A; then $\cos\alpha = \dfrac{x}{|\mathbf{r}|}$. Similarly from right-angled triangles OBP and OCP, $\cos\beta = \dfrac{y}{|\mathbf{r}|}$ and $\cos\gamma = \dfrac{z}{|\mathbf{r}|}$. Also, $|\mathbf{r}| = r = \sqrt{x^2 + y^2 + z^2}$.

Then $\cos\alpha = \dfrac{x}{r}$, $\cos\beta = \dfrac{y}{r}$, $\cos\gamma = \dfrac{z}{r}$ from is given by $\mathbf{r} = \dfrac{m\mathbf{p} + n\mathbf{q}}{m+n}$ and that this is independent of the origin.

55. If $\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_n$ are the position vectors of masses $m_1, m_2, \ldots, m_n$ respectively relative to an origin O, show that the position vector of the centroid is given by

$$\mathbf{r} = \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2 + \ldots + m_n\mathbf{r}_n}{m_1 + m_2 + \ldots + m_n}$$

and that this is independent of the origin.

56. A quadrilateral $ABCD$ has masses of 1, 2, 3 and 4 units located respectively at its vertices $A(-1,-2,2)$, $B(3,2,-1)$, $C(1,-2,4)$, and $D(3,1,2)$. Find the coordinates of the centroid. *Ans.* (2, 0, 2)

57. Show that the equation of a plane which passes through three given points A, B, C not in the same straight line and having position vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ relative to an origin O, can be written

$$\mathbf{r} = \frac{m\mathbf{a} + n\mathbf{b} + p\mathbf{c}}{m + n + p}$$

where m, n, p are scalars. Verify that the equation is independent of the origin.

58. The position vectors of points P and Q are given by $\mathbf{r}_1 = 2\mathbf{i} + 3\mathbf{j} - \mathbf{k}$, $\mathbf{r}_2 = 4\mathbf{i} - 3\mathbf{j} + 2\mathbf{k}$. Determine PQ in terms of $\mathbf{i}, \mathbf{j}, \mathbf{k}$ and find its magnitude. *Ans.* $2\mathbf{i} - 6\mathbf{j} + 3\mathbf{k}$, 7

59. If $\mathbf{A} = 3\mathbf{i} - \mathbf{j} - 4\mathbf{k}$, $\mathbf{B} = -2\mathbf{i} + 4\mathbf{j} - 3\mathbf{k}$, $\mathbf{C} = \mathbf{i} + 2\mathbf{j} - \mathbf{k}$, find
(*a*) $2\mathbf{A} - \mathbf{B} + 3\mathbf{C}$, (*b*) $|\mathbf{A} + \mathbf{B} + \mathbf{C}|$, (*c*) $|3\mathbf{A} - 2\mathbf{B} + 4\mathbf{C}|$, (*d*) a unit vector parallel to $3\mathbf{A} - 2\mathbf{B} + 4\mathbf{C}$.
Ans. (*a*) $11\mathbf{i} - 8\mathbf{k}$ (*b*) $\sqrt{93}$ (*c*) $\sqrt{398}$ (*d*) $\dfrac{3\mathbf{A} - 2\mathbf{B} + 4\mathbf{C}}{\sqrt{398}}$

60. The following forces act on a particle P: $\mathbf{F}_1 = 2\mathbf{i} + 3\mathbf{j} - 5\mathbf{k}$, $\mathbf{F}_2 = -5\mathbf{i} + \mathbf{j} + 3\mathbf{k}$, $\mathbf{F}_3 = \mathbf{i} - 2\mathbf{j} + 4\mathbf{k}$, $\mathbf{F}_4 = 4\mathbf{i} - 3\mathbf{j} - 2\mathbf{k}$, measured in newtons. Find (*a*) the resultant of the forces, (*b*) the magnitude of the resultant.
Ans. (*a*) $2\mathbf{i} - \mathbf{j}$ (*b*) $\sqrt{5}$

61. In each case determine whether the vectors are linearly independent or linearly dependent:
(*a*) $\mathbf{A} = 2\mathbf{i} + \mathbf{j} - 3\mathbf{k}$, $\mathbf{B} = \mathbf{i} - 4\mathbf{k}$, $\mathbf{C} = 4\mathbf{i} + 3\mathbf{j} - \mathbf{k}$, (*b*) $\mathbf{A} = \mathbf{i} - 3\mathbf{j} + 2\mathbf{k}$, $\mathbf{B} = 2\mathbf{i} - 4\mathbf{j} - \mathbf{k}$, $\mathbf{C} = 3\mathbf{i} + 2\mathbf{j} - \mathbf{k}$.
Ans. (*a*) linearly dependent, (*b*) linearly independent

62. Prove that any four vectors in three dimensions must be linearly dependent.

63. Show that a necessary and sufficient condition that the vectors $\mathbf{A} = A_1\mathbf{i} + A_2\mathbf{j} + A_3\mathbf{k}$, $\mathbf{B} = B_1\mathbf{i} + B_2\mathbf{j} + B_3\mathbf{k}$, $\mathbf{C} = C_1\mathbf{i} + C_2\mathbf{j} + C_3\mathbf{k}$ be linearly independent is that the determinant $\begin{vmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix}$ be different from zero.

64. (*a*) Prove that the vectors $\mathbf{A} = 3\mathbf{i} + \mathbf{j} - 2\mathbf{k}$, $\mathbf{B} = -\mathbf{i} + 3\mathbf{j} + 4\mathbf{k}$, $\mathbf{C} = 4\mathbf{i} - 2\mathbf{j} - 6\mathbf{k}$ can form the sides of a triangle.
(*b*) Find the lengths of the medians of the triangle.
Ans. (*b*) $\sqrt{6}$, $\frac{1}{2}\sqrt{114}$, $\frac{1}{2}\sqrt{150}$

65. Given the scalar field defined by $\phi(x,y,z) = 4yz^3 + 3xyz - z^2 + 2$. Find (*a*) $\phi(1,-1,-2)$, (*b*) $\phi(0,-3,1)$.
Ans. (*a*) 36 (*b*) -11

66. Graph the vector fields defined by
(*a*) $\mathbf{V}(x,y) = x\mathbf{i} - y\mathbf{j}$, (*b*) $\mathbf{V}(x,y) = y\mathbf{i} - x\mathbf{j}$, (*c*) $\mathbf{V}(x,y,z) = \dfrac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{\sqrt{x^2 + y^2 + z^2}}$.

Chapter 2 The DOT and CROSS PRODUCT

THE DOT OR SCALAR PRODUCT of two vectors **A** and **B**, denoted by $\mathbf{A}\cdot\mathbf{B}$ (read **A** *dot* **B**), is defined as the product of the magnitudes of **A** and **B** and the cosine of the angle θ between them. In symbols,

$$\mathbf{A}\cdot\mathbf{B} = AB\cos\theta, \qquad 0 \leqq \theta \leqq \pi$$

Note that $\mathbf{A}\cdot\mathbf{B}$ is a scalar and not a vector.

The following laws are valid:

1. $\mathbf{A}\cdot\mathbf{B} = \mathbf{B}\cdot\mathbf{A}$ — Commutative Law for Dot Products
2. $\mathbf{A}\cdot(\mathbf{B}+\mathbf{C}) = \mathbf{A}\cdot\mathbf{B} + \mathbf{A}\cdot\mathbf{C}$ — Distributive Law
3. $m(\mathbf{A}\cdot\mathbf{B}) = (m\mathbf{A})\cdot\mathbf{B} = \mathbf{A}\cdot(m\mathbf{B}) = (\mathbf{A}\cdot\mathbf{B})m$, where m is a scalar.
4. $\mathbf{i}\cdot\mathbf{i} = \mathbf{j}\cdot\mathbf{j} = \mathbf{k}\cdot\mathbf{k} = 1, \quad \mathbf{i}\cdot\mathbf{j} = \mathbf{j}\cdot\mathbf{k} = \mathbf{k}\cdot\mathbf{i} = 0$
5. If $\mathbf{A} = A_1\mathbf{i} + A_2\mathbf{j} + A_3\mathbf{k}$ and $\mathbf{B} = B_1\mathbf{i} + B_2\mathbf{j} + B_3\mathbf{k}$, then

$$\mathbf{A}\cdot\mathbf{B} = A_1B_1 + A_2B_2 + A_3B_3$$
$$\mathbf{A}\cdot\mathbf{A} = A^2 = A_1^2 + A_2^2 + A_3^2$$
$$\mathbf{B}\cdot\mathbf{B} = B^2 = B_1^2 + B_2^2 + B_3^2$$

6. If $\mathbf{A}\cdot\mathbf{B} = 0$ and **A** and **B** are not null vectors, then **A** and **B** are perpendicular.

THE CROSS OR VECTOR PRODUCT of **A** and **B** is a vector $\mathbf{C} = \mathbf{A}\times\mathbf{B}$ (read **A** *cross* **B**). The magnitude of $\mathbf{A}\times\mathbf{B}$ is defined as the product of the magnitudes of **A** and **B** and the sine of the angle θ between them. The direction of the vector $\mathbf{C} = \mathbf{A}\times\mathbf{B}$ is perpendicular to the plane of **A** and **B** and such that **A**, **B** and **C** form a right-handed system. In symbols,

$$\mathbf{A}\times\mathbf{B} = AB\sin\theta\,\mathbf{u}, \qquad 0 \leqq \theta \leqq \pi$$

where **u** is a unit vector indicating the direction of $\mathbf{A}\times\mathbf{B}$. If $\mathbf{A} = \mathbf{B}$, or if **A** is parallel to **B**, then $\sin\theta = 0$ and we define $\mathbf{A}\times\mathbf{B} = \mathbf{0}$.

The following laws are valid:

1. $\mathbf{A}\times\mathbf{B} = -\mathbf{B}\times\mathbf{A}$ — (Commutative Law for Cross Products Fails.)
2. $\mathbf{A}\times(\mathbf{B}+\mathbf{C}) = \mathbf{A}\times\mathbf{B} + \mathbf{A}\times\mathbf{C}$ — Distributive Law
3. $m(\mathbf{A}\times\mathbf{B}) = (m\mathbf{A})\times\mathbf{B} = \mathbf{A}\times(m\mathbf{B}) = (\mathbf{A}\times\mathbf{B})m$, where m is a scalar.
4. $\mathbf{i}\times\mathbf{i} = \mathbf{j}\times\mathbf{j} = \mathbf{k}\times\mathbf{k} = \mathbf{0}, \quad \mathbf{i}\times\mathbf{j} = \mathbf{k},\ \mathbf{j}\times\mathbf{k} = \mathbf{i},\ \mathbf{k}\times\mathbf{i} = \mathbf{j}$
5. If $\mathbf{A} = A_1\mathbf{i} + A_2\mathbf{j} + A_3\mathbf{k}$ and $\mathbf{B} = B_1\mathbf{i} + B_2\mathbf{j} + B_3\mathbf{k}$, then

$$\mathbf{A}\times\mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{vmatrix}$$

6. The magnitude of $\mathbf{A}\times\mathbf{B}$ is the same as the area of a parallelogram with sides $\mathbf{A}$ and $\mathbf{B}$.

7. If $\mathbf{A}\times\mathbf{B} = \mathbf{0}$, and $\mathbf{A}$ and $\mathbf{B}$ are not null vectors, then $\mathbf{A}$ and $\mathbf{B}$ are parallel.

TRIPLE PRODUCTS. Dot and cross multiplication of three vectors $\mathbf{A}, \mathbf{B}$ and $\mathbf{C}$ may produce meaningful products of the form $(\mathbf{A}\cdot\mathbf{B})\mathbf{C}$, $\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C})$ and $\mathbf{A}\times(\mathbf{B}\times\mathbf{C})$. The following laws are valid:

1. $(\mathbf{A}\cdot\mathbf{B})\mathbf{C} \neq \mathbf{A}(\mathbf{B}\cdot\mathbf{C})$

2. $\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = \mathbf{B}\cdot(\mathbf{C}\times\mathbf{A}) = \mathbf{C}\cdot(\mathbf{A}\times\mathbf{B}) =$ volume of a parallelepiped having $\mathbf{A}, \mathbf{B}$ and $\mathbf{C}$ as edges, or the negative of this volume, according as $\mathbf{A}, \mathbf{B}$ and $\mathbf{C}$ do or do not form a right-handed system. If $\mathbf{A} = A_1\mathbf{i} + A_2\mathbf{j} + A_3\mathbf{k}$, $\mathbf{B} = B_1\mathbf{i} + B_2\mathbf{j} + B_3\mathbf{k}$ and $\mathbf{C} = C_1\mathbf{i} + C_2\mathbf{j} + C_3\mathbf{k}$, then

$$\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = \begin{vmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix}$$

3. $\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) \neq (\mathbf{A}\times\mathbf{B})\times\mathbf{C}$ (Associative Law for Cross Products Fails.)

4. $\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = (\mathbf{A}\cdot\mathbf{C})\mathbf{B} - (\mathbf{A}\cdot\mathbf{B})\mathbf{C}$
   $(\mathbf{A}\times\mathbf{B})\times\mathbf{C} = (\mathbf{A}\cdot\mathbf{C})\mathbf{B} - (\mathbf{B}\cdot\mathbf{C})\mathbf{A}$

The product $\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C})$ is sometimes called the *scalar triple product* or *box product* and may be denoted by $[\mathbf{ABC}]$. The product $\mathbf{A}\times(\mathbf{B}\times\mathbf{C})$ is called the *vector triple product*.

In $\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C})$ parentheses are sometimes omitted and we write $\mathbf{A}\cdot\mathbf{B}\times\mathbf{C}$ (see Problem 41). However, parentheses must be used in $\mathbf{A}\times(\mathbf{B}\times\mathbf{C})$ (see Problems 29 and 47).

RECIPROCAL SETS OF VECTORS. The sets of vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ and $\mathbf{a}', \mathbf{b}', \mathbf{c}'$ are called *reciprocal sets or systems of vectors* if

$$\mathbf{a}\cdot\mathbf{a}' = \mathbf{b}\cdot\mathbf{b}' = \mathbf{c}\cdot\mathbf{c}' = 1$$

$$\mathbf{a}'\cdot\mathbf{b} = \mathbf{a}'\cdot\mathbf{c} = \mathbf{b}'\cdot\mathbf{a} = \mathbf{b}'\cdot\mathbf{c} = \mathbf{c}'\cdot\mathbf{a} = \mathbf{c}'\cdot\mathbf{b} = 0$$

The sets $\mathbf{a}, \mathbf{b}, \mathbf{c}$ and $\mathbf{a}', \mathbf{b}', \mathbf{c}'$ are reciprocal sets of vectors if and only if

$$\mathbf{a}' = \frac{\mathbf{b}\times\mathbf{c}}{\mathbf{a}\cdot\mathbf{b}\times\mathbf{c}}, \qquad \mathbf{b}' = \frac{\mathbf{c}\times\mathbf{a}}{\mathbf{a}\cdot\mathbf{b}\times\mathbf{c}}, \qquad \mathbf{c}' = \frac{\mathbf{a}\times\mathbf{b}}{\mathbf{a}\cdot\mathbf{b}\times\mathbf{c}}$$

where $\mathbf{a}\cdot\mathbf{b}\times\mathbf{c} \neq 0$. See Problems 53 and 54.

$= A_1B_1 + A_2B_2 + A_3B_3$

since $\mathbf{i}\cdot\mathbf{i} = \mathbf{j}\cdot\mathbf{j} = \mathbf{k}\cdot\mathbf{k} = 1$ and all other dot products are zero.

7. If $\mathbf{A} = A_1\mathbf{i} + A_2\mathbf{j} + A_3\mathbf{k}$, show that $A = \sqrt{\mathbf{A}\cdot\mathbf{A}} = \sqrt{A_1^2 + A_2^2 + A_3^2}$.

$\mathbf{A}\cdot\mathbf{A} = (A)(A)\cos 0^\circ = A^2$. Then $A = \sqrt{\mathbf{A}\cdot\mathbf{A}}$.

Also, $\mathbf{A}\cdot\mathbf{A} = (A_1\mathbf{i} + A_2\mathbf{j} + A_3\mathbf{k})\cdot(A_1\mathbf{i} + A_2\mathbf{j} + A_3\mathbf{k})$

$= (A_1)(A_1) + (A_2)(A_2) + (A_3)(A_3) = A_1^2 + A_2^2 + A_3^2$

by Problem 6, taking $\mathbf{B} = \mathbf{A}$.

Then $A = \sqrt{\mathbf{A}\cdot\mathbf{A}} = \sqrt{A_1^2 + A_2^2 + A_3^2}$ is the magnitude of $\mathbf{A}$. Sometimes $\mathbf{A}\cdot\mathbf{A}$ is written $\mathbf{A}^2$.

8. Find the angle between $\mathbf{A} = 2\mathbf{i} + 2\mathbf{j} - \mathbf{k}$ and $\mathbf{B} = 6\mathbf{i} - 3\mathbf{j} + 2\mathbf{k}$.

$\mathbf{A}\cdot\mathbf{B} = AB\cos\theta$, $A = \sqrt{(2)^2 + (2)^2 + (-1)^2} = 3$, $B = \sqrt{(6)^2 + (-3)^2 + (2)^2} = 7$

$\mathbf{A}\cdot\mathbf{B} = (2)(6) + (2)(-3) + (-1)(2) = 12 - 6 - 2 = 4$

Then $\cos\theta = \dfrac{\mathbf{A}\cdot\mathbf{B}}{AB} = \dfrac{4}{(3)(7)} = \dfrac{4}{21} = 0{\cdot}1905$ and $\theta = 79^\circ$ approximately.

9. If $\mathbf{A}\cdot\mathbf{B} = 0$ and if A and B are not zero, show that $\mathbf{A}$ is perpendicular to $\mathbf{B}$.

If $\mathbf{A}\cdot\mathbf{B} = AB\cos\theta = 0$, then $\cos\theta = 0$ or $\theta = 90^\circ$. Conversely, if $\theta = 90^\circ$, $\mathbf{A}\cdot\mathbf{B} = 0$.

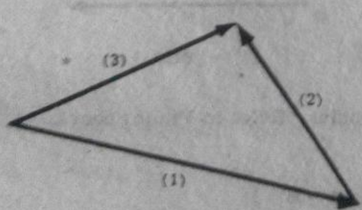
10. Determine the value of a so that $\mathbf{A} = 2\mathbf{i} + a\mathbf{j} + \mathbf{k}$ and $\mathbf{B} = 4\mathbf{i} - 2\mathbf{j} - 2\mathbf{k}$ are perpendicular.

From Problem 9, $\mathbf{A}$ and $\mathbf{B}$ are perpendicular if $\mathbf{A}\cdot\mathbf{B} = 0$.

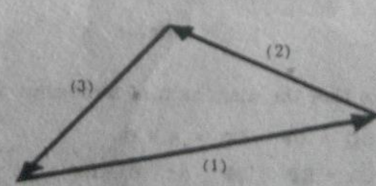
Then $\mathbf{A}\cdot\mathbf{B} = (2)(4) + (a)(-2) + (1)(-2) = 8 - 2a - 2 = 0$ for $a = 3$.

11. Show that the vectors $\mathbf{A} = 3\mathbf{i} - 2\mathbf{j} + \mathbf{k}$, $\mathbf{B} = \mathbf{i} - 3\mathbf{j} + 5\mathbf{k}$, $\mathbf{C} = 2\mathbf{i} + \mathbf{j} - 4\mathbf{k}$ form a right-angled triangle.

We first have to show that the vectors form a triangle.



(a)



(b)

From the figures it is seen that the vectors will form a triangle if

(a) one of the vectors, say (3), is the resultant or sum of (1) and (2),

(b) the sum or resultant of the vectors (1) + (2) + (3) is zero,

according as (a) two vectors have a common terminal point or (b) none of the vectors have a common terminal point. By trial we find $\mathbf{A} = \mathbf{B} + \mathbf{C}$ so that the vectors do form a triangle.

Since $\mathbf{A}\cdot\mathbf{B} = (3)(1) + (-2)(-3) + (1)(5) = 14$, $\mathbf{A}\cdot\mathbf{C} = (3)(2) + (-2)(1) + (1)(-4) = 0$, and $\mathbf{B}\cdot\mathbf{C} = (1)(2) + (-3)(1) + (5)(-4) = -21$, it follows that $\mathbf{A}$ and $\mathbf{C}$ are perpendicular and the triangle is a right-angled triangle.

12. Find the angles which the vector $\mathbf{A} = 3\mathbf{i} - 6\mathbf{j} + 2\mathbf{k}$ makes with the coordinate axes.

Let α, β, γ be the angles which $\mathbf{A}$ makes with the positive x, y, z axes respectively.

$$\mathbf{A}\cdot\mathbf{i} = (A)(1)\cos\alpha = \sqrt{(3)^2+(-6)^2+(2)^2}\ \cos\alpha = 7\cos\alpha$$

$$\mathbf{A}\cdot\mathbf{i} = (3\mathbf{i}-6\mathbf{j}+2\mathbf{k})\cdot\mathbf{i} = 3\mathbf{i}\cdot\mathbf{i} - 6\mathbf{j}\cdot\mathbf{i} + 2\mathbf{k}\cdot\mathbf{i} = 3$$

Then $\cos\alpha = 3/7 = 0.4286$, and $\alpha = 64.6^\circ$ approximately.

Similarly, $\cos\beta = -6/7$, $\beta = 149^\circ$ and $\cos\gamma = 2/7$, $\gamma = 73.4^\circ$.

The cosines of α, β, and γ are called the *direction cosines* of $\mathbf{A}$. (See Prob. 27, Chap. 1).

13. Find the projection of the vector $\mathbf{A} = \mathbf{i} - 2\mathbf{j} + \mathbf{k}$ on the vector $\mathbf{B} = 4\mathbf{i} - 4\mathbf{j} + 7\mathbf{k}$.

A unit vector in the direction $\mathbf{B}$ is $\mathbf{b} = \dfrac{\mathbf{B}}{B} = \dfrac{4\mathbf{i}-4\mathbf{j}+7\mathbf{k}}{\sqrt{(4)^2+(-4)^2+(7)^2}} = \dfrac{4}{9}\mathbf{i} - \dfrac{4}{9}\mathbf{j} + \dfrac{7}{9}\mathbf{k}$.

Projection of $\mathbf{A}$ on the vector $\mathbf{B}$ $= \mathbf{A}\cdot\mathbf{b} = (\mathbf{i}-2\mathbf{j}+\mathbf{k})\cdot(\frac{4}{9}\mathbf{i} - \frac{4}{9}\mathbf{j} + \frac{7}{9}\mathbf{k})$

$$= (1)(\tfrac{4}{9}) + (-2)(-\tfrac{4}{9}) + (1)(\tfrac{7}{9}) = \tfrac{19}{9}.$$

14. Prove the law of cosines for plane triangles.

From Fig.(*a*) below, $\mathbf{B} + \mathbf{C} = \mathbf{A}$ or $\mathbf{C} = \mathbf{A} - \mathbf{B}$.

Then $\qquad \mathbf{C}\cdot\mathbf{C} = (\mathbf{A}-\mathbf{B})\cdot(\mathbf{A}-\mathbf{B}) = \mathbf{A}\cdot\mathbf{A} + \mathbf{B}\cdot\mathbf{B} - 2\mathbf{A}\cdot\mathbf{B}$

and $\qquad C^2 = A^2 + B^2 - 2AB\cos\theta$.

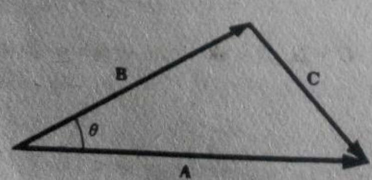


Fig.(*a*)

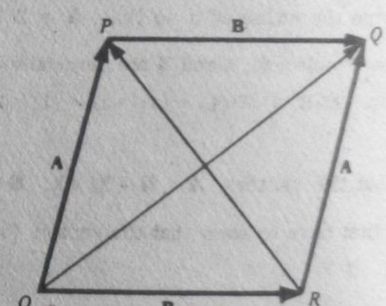


Fig.(*b*)

15. Prove that the diagonals of a rhombus are perpendicular. Refer to Fig.(*b*) above.

$\mathbf{OQ} = \mathbf{OP} + \mathbf{PQ} = \mathbf{A} + \mathbf{B}$

$\mathbf{OR} + \mathbf{RP} = \mathbf{OP}$ or $\mathbf{B} + \mathbf{RP} = \mathbf{A}$ and $\mathbf{RP} = \mathbf{A} - \mathbf{B}$

Then $\mathbf{OQ}\cdot\mathbf{RP} = (\mathbf{A}+\mathbf{B})\cdot(\mathbf{A}-\mathbf{B}) = A^2 - B^2 = 0$, since $A = B$.

Hence $\mathbf{OQ}$ is perpendicular to $\mathbf{RP}$.

16. Determine a unit vector perpendicular to the plane of $\mathbf{A} = 2\mathbf{i} - 6\mathbf{j} - 3\mathbf{k}$ and $\mathbf{B} = 4\mathbf{i} + 3\mathbf{j} - \mathbf{k}$.

Let vector $\mathbf{C} = c_1\mathbf{i} + c_2\mathbf{j} + c_3\mathbf{k}$ be perpendicular to the plane of $\mathbf{A}$ and $\mathbf{B}$. Then $\mathbf{C}$ is perpendicular to $\mathbf{A}$ and also to $\mathbf{B}$. Hence,

$$\mathbf{C}\cdot\mathbf{A} = 2c_1 - 6c_2 - 3c_3 = 0 \quad\text{or}\quad (1)\ \ 2c_1 - 6c_2 = 3c_3$$

$$\mathbf{C}\cdot\mathbf{B} = 4c_1 + 3c_2 - c_3 = 0 \quad\text{or}\quad (2)\ \ 4c_1 + 3c_2 = c_3$$

Solving (1) and (2) simultaneously: $c_1 = \frac{1}{2}c_3$, $c_2 = -\frac{1}{3}c_3$, $\mathbf{C} = c_3(\frac{1}{2}\mathbf{i} - \frac{1}{3}\mathbf{j} + \mathbf{k})$.

Then a unit vector in the direction of $\mathbf{C}$ is $\dfrac{\mathbf{C}}{C} = \dfrac{c_3(\frac{1}{2}\mathbf{i} - \frac{1}{3}\mathbf{j} + \mathbf{k})}{\sqrt{c_3^2[(\frac{1}{2})^2 + (-\frac{1}{3})^2 + (1)^2]}} = \pm(\frac{3}{7}\mathbf{i} - \frac{2}{7}\mathbf{j} + \frac{6}{7}\mathbf{k})$.

17. Find the work done in moving an object along a vector $\mathbf{r} = 3\mathbf{i} + 2\mathbf{j} - 5\mathbf{k}$ if the applied force is $\mathbf{F} = 2\mathbf{i} - \mathbf{j} - \mathbf{k}$. Refer to Fig.(a) below.

$$\begin{aligned}\text{Work done} &= \text{(magnitude of force in direction of motion)(distance moved)}\\ &= (F\cos\theta)(r) \;=\; \mathbf{F}\cdot\mathbf{r}\\ &= (2\mathbf{i} - \mathbf{j} - \mathbf{k})\cdot(3\mathbf{i} + 2\mathbf{j} - 5\mathbf{k}) \;=\; 6 - 2 + 5 \;=\; 9.\end{aligned}$$

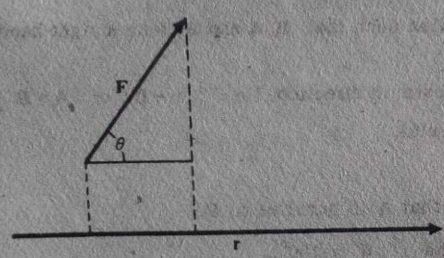


Fig.(a)

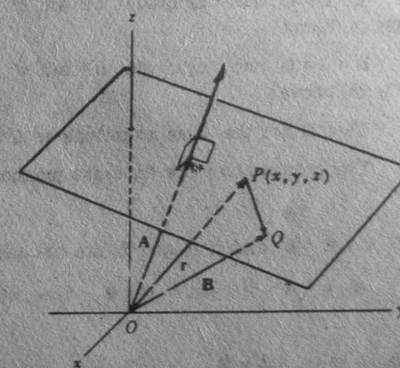


Fig.(b)

18. Find an equation for the plane perpendicular to the vector $\mathbf{A} = 2\mathbf{i} + 3\mathbf{j} + 6\mathbf{k}$ and passing through the terminal point of the vector $\mathbf{B} = \mathbf{i} + 5\mathbf{j} + 3\mathbf{k}$ (see Fig.(b) above).

Let $\mathbf{r}$ be the position vector of point P, and Q the terminal point of $\mathbf{B}$.

Since $\mathbf{PQ} = \mathbf{B} - \mathbf{r}$ is perpendicular to $\mathbf{A}$, $(\mathbf{B} - \mathbf{r})\cdot\mathbf{A} = 0$ or $\mathbf{r}\cdot\mathbf{A} = \mathbf{B}\cdot\mathbf{A}$ is the required equation of the plane in vector form. In rectangular form this becomes

$$(x\mathbf{i} + y\mathbf{j} + z\mathbf{k})\cdot(2\mathbf{i} + 3\mathbf{j} + 6\mathbf{k}) = (\mathbf{i} + 5\mathbf{j} + 3\mathbf{k})\cdot(2\mathbf{i} + 3\mathbf{j} + 6\mathbf{k})$$

or

$$2x + 3y + 6z = (1)(2) + (5)(3) + (3)(6) = 35$$

19. In Problem 18 find the distance from the origin to the plane.

The distance from the origin to the plane is the projection of $\mathbf{B}$ on $\mathbf{A}$.

A unit vector in direction $\mathbf{A}$ is $\mathbf{a} = \dfrac{\mathbf{A}}{A} = \dfrac{2\mathbf{i} + 3\mathbf{j} + 6\mathbf{k}}{\sqrt{(2)^2 + (3)^2 + (6)^2}} = \frac{2}{7}\mathbf{i} + \frac{3}{7}\mathbf{j} + \frac{6}{7}\mathbf{k}$.

Then, projection of $\mathbf{B}$ on $\mathbf{A}$ $= \mathbf{B}\cdot\mathbf{a} = (\mathbf{i} + 5\mathbf{j} + 3\mathbf{k})\cdot(\frac{2}{7}\mathbf{i} + \frac{3}{7}\mathbf{j} + \frac{6}{7}\mathbf{k}) = 1(\frac{2}{7}) + 5(\frac{3}{7}) + 3(\frac{6}{7}) = 5$.

20. If $\mathbf{A}$ is any vector, prove that $\mathbf{A} = (\mathbf{A}\cdot\mathbf{i})\mathbf{i} + (\mathbf{A}\cdot\mathbf{j})\mathbf{j} + (\mathbf{A}\cdot\mathbf{k})\mathbf{k}$.

Since $\mathbf{A} = A_1\mathbf{i} + A_2\mathbf{j} + A_3\mathbf{k}$, $\mathbf{A}\cdot\mathbf{i} = A_1\mathbf{i}\cdot\mathbf{i} + A_2\mathbf{j}\cdot\mathbf{i} + A_3\mathbf{k}\cdot\mathbf{i} = A_1$

Similarly, $\mathbf{A}\cdot\mathbf{j} = A_2$ and $\mathbf{A}\cdot\mathbf{k} = A_3$.

Then $\mathbf{A} = A_1\mathbf{i} + A_2\mathbf{j} + A_3\mathbf{k} = (\mathbf{A}\cdot\mathbf{i})\mathbf{i} + (\mathbf{A}\cdot\mathbf{j})\mathbf{j} + (\mathbf{A}\cdot\mathbf{k})\mathbf{k}$.

From Problem 30,

$$\text{area of triangle} = \tfrac{1}{2}|\mathbf{PQ}\times\mathbf{PR}| = \tfrac{1}{2}|(\mathbf{i}-4\mathbf{j}-\mathbf{k})\times(-2\mathbf{i}-\mathbf{j}+\mathbf{k})|$$

$$= \tfrac{1}{2}\left|\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 1 & -4 & -1\\ -2 & -1 & 1\end{vmatrix}\right| = \tfrac{1}{2}|-5\mathbf{i}+\mathbf{j}-9\mathbf{k}| = \tfrac{1}{2}\sqrt{(-5)^2+(1)^2+(-9)^2} = \tfrac{1}{2}\sqrt{107}.$$

32. Determine a unit vector perpendicular to the plane of $\mathbf{A} = 2\mathbf{i}-6\mathbf{j}-3\mathbf{k}$ and $\mathbf{B} = 4\mathbf{i}+3\mathbf{j}-\mathbf{k}$.

$\mathbf{A}\times\mathbf{B}$ is a vector perpendicular to the plane of $\mathbf{A}$ and $\mathbf{B}$.

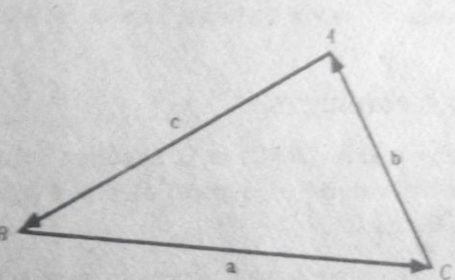
$$\mathbf{A}\times\mathbf{B} = \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 2 & -6 & -3\\ 4 & 3 & -1\end{vmatrix} = 15\mathbf{i}-10\mathbf{j}+30\mathbf{k}$$

A unit vector parallel to $\mathbf{A}\times\mathbf{B}$ is $\dfrac{\mathbf{A}\times\mathbf{B}}{|\mathbf{A}\times\mathbf{B}|} = \dfrac{15\mathbf{i}-10\mathbf{j}+30\mathbf{k}}{\sqrt{(15)^2+(-10)^2+(30)^2}} = \dfrac{3}{7}\mathbf{i}-\dfrac{2}{7}\mathbf{j}+\dfrac{6}{7}\mathbf{k}.$

Another unit vector, opposite in direction, is $(-3\mathbf{i}+2\mathbf{j}-6\mathbf{k})/7$.

Compare with Problem 16.

33. Prove the law of sines for plane triangles.



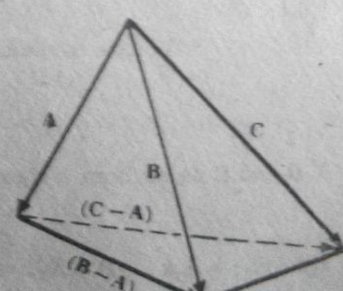
Let $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ represent the sides of triangle ABC as shown in the adjoining figure; then $\mathbf{a}+\mathbf{b}+\mathbf{c} = \mathbf{0}$. Multiplying by $\mathbf{a}\times$, $\mathbf{b}\times$ and $\mathbf{c}\times$ in succession, we find

$$\mathbf{a}\times\mathbf{b} = \mathbf{b}\times\mathbf{c} = \mathbf{c}\times\mathbf{a}$$

i.e. $\qquad ab\sin C = bc\sin A = ca\sin B$

or $\qquad \dfrac{\sin A}{a} = \dfrac{\sin B}{b} = \dfrac{\sin C}{c}.$

34. Consider a tetrahedron with faces F_1, F_2, F_3, F_4. Let $\mathbf{V}_1, \mathbf{V}_2, \mathbf{V}_3, \mathbf{V}_4$ be vectors whose magnitudes are respectively equal to the areas of F_1, F_2, F_3, F_4 and whose directions are perpendicular to these faces in the outward direction. Show that $\mathbf{V}_1+\mathbf{V}_2+\mathbf{V}_3+\mathbf{V}_4 = \mathbf{0}$.



By Problem 30, the area of a triangular face determined by $\mathbf{R}$ and $\mathbf{S}$ is $\frac{1}{2}|\mathbf{R}\times\mathbf{S}|$.

The vectors associated with each of the faces of the tetrahedron are

$$\mathbf{V}_1 = \tfrac{1}{2}\mathbf{A}\times\mathbf{B}, \quad \mathbf{V}_2 = \tfrac{1}{2}\mathbf{B}\times\mathbf{C}, \quad \mathbf{V}_3 = \tfrac{1}{2}\mathbf{C}\times\mathbf{A}, \quad \mathbf{V}_4 = \tfrac{1}{2}(\mathbf{C}-\mathbf{A})\times(\mathbf{B}-\mathbf{A})$$

Then
$$\mathbf{V}_1+\mathbf{V}_2+\mathbf{V}_3+\mathbf{V}_4 = \tfrac{1}{2}[\mathbf{A}\times\mathbf{B}+\mathbf{B}\times\mathbf{C}+\mathbf{C}\times\mathbf{A}+(\mathbf{C}-\mathbf{A})\times(\mathbf{B}-\mathbf{A})]$$
$$= \tfrac{1}{2}[\mathbf{A}\times\mathbf{B}+\mathbf{B}\times\mathbf{C}+\mathbf{C}\times\mathbf{A}+\mathbf{C}\times\mathbf{B}-\mathbf{C}\times\mathbf{A}-\mathbf{A}\times\mathbf{B}+\mathbf{A}\times\mathbf{A}] = \mathbf{0}.$$

This result can be generalized to closed polyhedra and in the limiting case to any closed surface.

Because of the application presented here it is sometimes convenient to assign a direction to area and we speak of the *vector area*.

From Problem 30,

area of triangle $= \frac{1}{2}|\mathbf{PQ}\times\mathbf{PR}| = \frac{1}{2}|(\mathbf{i}-4\mathbf{j}-\mathbf{k})\times(-2\mathbf{i}-\mathbf{j}+\mathbf{k})|$

$$= \frac{1}{2}\left|\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -4 & -1 \\ -2 & -1 & 1 \end{vmatrix}\right| = \frac{1}{2}|-5\mathbf{i}+\mathbf{j}-9\mathbf{k}| = \frac{1}{2}\sqrt{(-5)^2+(1)^2+(-9)^2} = \frac{1}{2}\sqrt{107}.$$

2. Determine a unit vector perpendicular to the plane of $\mathbf{A} = 2\mathbf{i} - 6\mathbf{j} - 3\mathbf{k}$ and $\mathbf{B} = 4\mathbf{i} + 3\mathbf{j} - \mathbf{k}$.

$\mathbf{A}\times\mathbf{B}$ is a vector perpendicular to the plane of $\mathbf{A}$ and $\mathbf{B}$.

$$\mathbf{A}\times\mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -6 & -3 \\ 4 & 3 & -1 \end{vmatrix} = 15\mathbf{i} - 10\mathbf{j} + 30\mathbf{k}$$

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Another unit vector, opposite in direction, is $(-3\mathbf{i}+2\mathbf{j}-6\mathbf{k})/7$.

Compare with Problem 16.

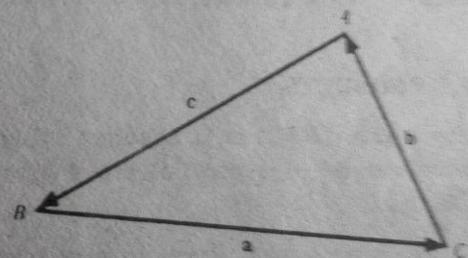
3. Prove the law of sines for plane triangles.

Let $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ represent the sides of triangle ABC as shown in the adjoining figure; then $\mathbf{a}+\mathbf{b}+\mathbf{c} = \mathbf{0}$. Multiplying by $\mathbf{a}\times$, $\mathbf{b}\times$ and $\mathbf{c}\times$ in succession, we find

$$\mathbf{a}\times\mathbf{b} = \mathbf{b}\times\mathbf{c} = \mathbf{c}\times\mathbf{a}$$

i.e. $$ab \sin C = bc \sin A = ca \sin B$$

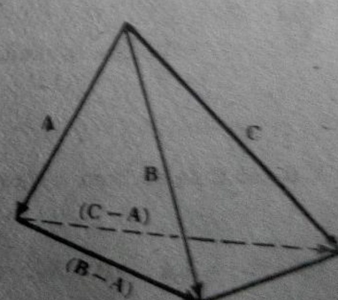
or $$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}.$$



4. Consider a tetrahedron with faces F_1, F_2, F_3, F_4. Let $\mathbf{V}_1, \mathbf{V}_2, \mathbf{V}_3, \mathbf{V}_4$ be vectors whose magnitudes are respectively equal to the areas of F_1, F_2, F_3, F_4 and whose directions are perpendicular to these faces in the outward direction. Show that $\mathbf{V}_1+\mathbf{V}_2+\mathbf{V}_3+\mathbf{V}_4 = \mathbf{0}$.

By Problem 30, the area of a triangular face determined by $\mathbf{R}$ and $\mathbf{S}$ is $\frac{1}{2}|\mathbf{R}\times\mathbf{S}|$.

The vectors associated with each of the faces of the tetrahedron are



$$\mathbf{V}_1 = \tfrac{1}{2}\mathbf{A}\times\mathbf{B}, \quad \mathbf{V}_2 = \tfrac{1}{2}\mathbf{B}\times\mathbf{C}, \quad \mathbf{V}_3 = \tfrac{1}{2}\mathbf{C}\times\mathbf{A}, \quad \mathbf{V}_4 = \tfrac{1}{2}(\mathbf{C}-\mathbf{A})\times(\mathbf{B}-\mathbf{A})$$

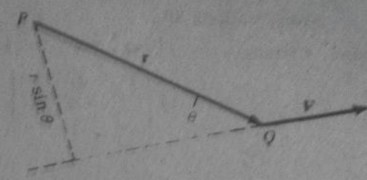
Then $$\mathbf{V}_1+\mathbf{V}_2+\mathbf{V}_3+\mathbf{V}_4 = \tfrac{1}{2}[\mathbf{A}\times\mathbf{B} + \mathbf{B}\times\mathbf{C} + \mathbf{C}\times\mathbf{A} + (\mathbf{C}-\mathbf{A})\times(\mathbf{B}-\mathbf{A})]$$
$$= \tfrac{1}{2}[\mathbf{A}\times\mathbf{B} + \mathbf{B}\times\mathbf{C} + \mathbf{C}\times\mathbf{A} + \mathbf{C}\times\mathbf{B} - \mathbf{C}\times\mathbf{A} - \mathbf{A}\times\mathbf{B} + \mathbf{A}\times\mathbf{A}] = \mathbf{0}.$$

This result can be generalized to closed polyhedra and in the limiting case to any closed surface.

Because of the application presented here it is sometimes convenient to assign a direction to area and we speak of the *vector area*.

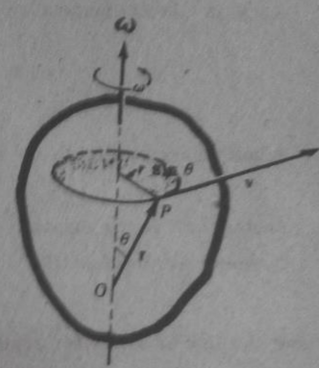
line of action of $\mathbf{F}$. Then if $\mathbf{r}$ is the vector from P to the initial point Q of $\mathbf{F}$,

$$M = F(r\sin\theta) = rF\sin\theta = |\mathbf{r}\times\mathbf{F}|$$



If we think of a right-threaded screw at P perpendicular to the plane of $\mathbf{r}$ and $\mathbf{F}$, then when the force $\mathbf{F}$ acts the screw will move in the direction of $\mathbf{r}\times\mathbf{F}$. Because of this it is convenient to define the moment as the vector $\mathbf{M} = \mathbf{r}\times\mathbf{F}$.

36. A rigid body rotates about an axis through point O with angular speed ω. Prove that the linear velocity $\mathbf{v}$ of a point P of the body with position vector $\mathbf{r}$ is given by $\mathbf{v} = \boldsymbol{\omega}\times\mathbf{r}$, where $\boldsymbol{\omega}$ is the vector with magnitude ω whose direction is that in which a right-handed screw would advance under the given rotation.

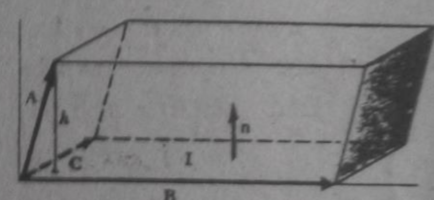


Since P travels in a circle of radius $r\sin\theta$, the magnitude of the linear velocity $\mathbf{v}$ is $\omega(r\sin\theta) = |\boldsymbol{\omega}\times\mathbf{r}|$. Also, $\mathbf{v}$ must be perpendicular to both $\boldsymbol{\omega}$ and $\mathbf{r}$ and is such that $\mathbf{r}, \boldsymbol{\omega}$ and $\mathbf{v}$ form a right-handed system.

Then $\mathbf{v}$ agrees both in magnitude and direction with $\boldsymbol{\omega}\times\mathbf{r}$; hence $\mathbf{v} = \boldsymbol{\omega}\times\mathbf{r}$. The vector $\boldsymbol{\omega}$ is called the *angular velocity*.

TRIPLE PRODUCTS.

37. Show that $\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C})$ is in absolute value equal to the volume of a parallelepiped with sides $\mathbf{A}, \mathbf{B}$ and $\mathbf{C}$.



Let $\mathbf{n}$ be a unit normal to parallelogram I, having the direction of $\mathbf{B}\times\mathbf{C}$, and let h be the height of the terminal point of $\mathbf{A}$ above the parallelogram I.

$$\begin{aligned} \text{Volume of parallelepiped} &= (\text{height } h)(\text{area of parallelogram } I) \\ &= (\mathbf{A}\cdot\mathbf{n})(|\mathbf{B}\times\mathbf{C}|) \\ &= \mathbf{A}\cdot\{|\mathbf{B}\times\mathbf{C}|\,\mathbf{n}\} = \mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) \end{aligned}$$

If $\mathbf{A}, \mathbf{B}$ and $\mathbf{C}$ do not form a right-handed system, $\mathbf{A}\cdot\mathbf{n} < 0$ and the volume $= |\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C})|$.

38. If $\mathbf{A} = A_1\mathbf{i} + A_2\mathbf{j} + A_3\mathbf{k}$, $\mathbf{B} = B_1\mathbf{i} + B_2\mathbf{j} + B_3\mathbf{k}$, $\mathbf{C} = C_1\mathbf{i} + C_2\mathbf{j} + C_3\mathbf{k}$ show that

$$\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = \begin{vmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix}$$

$$\begin{aligned} \mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) &= \mathbf{A}\cdot\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix} \\ &= (A_1\mathbf{i} + A_2\mathbf{j} + A_3\mathbf{k})\cdot[(B_2C_3 - B_3C_2)\mathbf{i} + (B_3C_1 - B_1C_3)\mathbf{j} + (B_1C_2 - B_2C_1)\mathbf{k}] \\ &= A_1(B_2C_3 - B_3C_2) + A_2(B_3C_1 - B_1C_3) + A_3(B_1C_2 - B_2C_1) = \begin{vmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix} \end{aligned}$$

on the vector $\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$.

63. Find the projection of the vector $2\mathbf{i} - 3\mathbf{j} + 6\mathbf{k}$ on the line passing through the points $(2,3,-1)$ and $(-2,-4,3)$.

64. Find the projection of the vector $4\mathbf{i} - 3\mathbf{j} + \mathbf{k}$ to both **A** and **B**.
Ans. 1

65. If $\mathbf{A} = 4\mathbf{i} - \mathbf{j} + 3\mathbf{k}$ and $\mathbf{B} = -2\mathbf{i} + \mathbf{j} - 2\mathbf{k}$, find a unit vector perpendicular
Ans. $\pm(\mathbf{i} - 2\mathbf{j} - 2\mathbf{k})/3$ *Ans.* arc cos 1/3 or $70°32'$

66. Find the acute angle formed by two diagonals of a cube. *Ans.* $\pm(3\mathbf{i} + 4\mathbf{j})/5$

67. Find a unit vector parallel to the xy plane and perpendicular to the vector $4\mathbf{i} - 3\mathbf{j} + \mathbf{k}$.

68. Show that $\mathbf{A} = (2\mathbf{i} - 2\mathbf{j} + \mathbf{k})/3$, $\mathbf{B} = (\mathbf{i} + 2\mathbf{j} + 2\mathbf{k})/3$ and $\mathbf{C} = (2\mathbf{i} + \mathbf{j} - 2\mathbf{k})/3$ are mutually orthogonal unit vectors.

69. Find the work done in moving an object along a straight line from $(3,2,-1)$ to $(2,-1,4)$ in a force field given by $\mathbf{F} = 4\mathbf{i} - 3\mathbf{j} + 2\mathbf{k}$. *Ans.* 15

70. Let **F** be a constant vector force field. Show that the work done in moving an object around any closed polygon in this force field is zero.

71. Prove that an angle inscribed in a semi-circle is a right angle.

72. Let $ABCD$ be a parallelogram. Prove that $\overline{AB}^2 + \overline{BC}^2 + \overline{CD}^2 + \overline{DA}^2 = \overline{AC}^2 + \overline{BD}^2$.

73. If $ABCD$ is any quadrilateral and P and Q are the midpoints of its diagonals, prove that

$$\overline{AB}^2 + \overline{BC}^2 + \overline{CD}^2 + \overline{DA}^2 = \overline{AC}^2 + \overline{BD}^2 + 4\overline{PQ}^2$$

This is a generalization of the preceding problem.

74. (*a*) Find an equation of a plane perpendicular to a given vector **A** and distant p from the origin.
(*b*) Express the equation of (*a*) in rectangular coordinates.
Ans. (*a*) $\mathbf{r}\cdot\mathbf{n} = p$, where $\mathbf{n} = \mathbf{A}/A$; (*b*) $A_1x + A_2y + A_3z = Ap$

75. Let $\mathbf{r}_1$ and $\mathbf{r}_2$ be unit vectors in the xy plane making angles α and β with the positive x-axis.
(*a*) Prove that $\mathbf{r}_1 = \cos\alpha\,\mathbf{i} + \sin\alpha\,\mathbf{j}$, $\mathbf{r}_2 = \cos\beta\,\mathbf{i} + \sin\beta\,\mathbf{j}$.
(*b*) By considering $\mathbf{r}_1\cdot\mathbf{r}_2$ prove the trigonometric formulas

$$\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta, \qquad \cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

76. Let **a** be the position vector of a given point (x_1, y_1, z_1), and **r** the position vector of any point (x, y, z). Describe the locus of **r** if (*a*) $|\mathbf{r} - \mathbf{a}| = 3$, (*b*) $(\mathbf{r} - \mathbf{a})\cdot\mathbf{a} = 0$, (*c*) $(\mathbf{r} - \mathbf{a})\cdot\mathbf{r} = 0$.
Ans. (*a*) Sphere, centre at (x_1, y_1, z_1) and radius 3.
(*b*) Plane perpendicular to **a** and passing through its terminal point.
(*c*) Sphere with centre at $(x_1/2, y_1/2, z_1/2)$ and radius $\frac{1}{2}\sqrt{x_1^2 + y_1^2 + z_1^2}$, or a sphere with **a** as diameter.

77. Given that $\mathbf{A} = 3\mathbf{i} + \mathbf{j} + 2\mathbf{k}$ and $\mathbf{B} = \mathbf{i} - 2\mathbf{j} - 4\mathbf{k}$ are the position vectors of points P and Q respectively.
(*a*) Find an equation for the plane passing through Q and perpendicular to line PQ.
(*b*) What is the distance from the point $(-1,1,1)$ to the plane ?
Ans. (*a*) $(\mathbf{r} - \mathbf{B})\cdot(\mathbf{A} - \mathbf{B}) = 0$ or $2x + 3y + 6z = -28$; (*b*) 5

78. Evaluate each of the following:
(*a*) $2\mathbf{j}\times(3\mathbf{i} - 4\mathbf{k})$, (*b*) $(\mathbf{i} + 2\mathbf{j})\times\mathbf{k}$, (*c*) $(2\mathbf{i} - 4\mathbf{k})\times(\mathbf{i} + 2\mathbf{j})$, (*d*) $(4\mathbf{i} + \mathbf{j} - 2\mathbf{k})\times(3\mathbf{i} + \mathbf{k})$, (*e*) $(2\mathbf{i} + \mathbf{j} - \mathbf{k})\times(3\mathbf{i} - 2\mathbf{j} + 4\mathbf{k})$.
Ans. (*a*) $-8\mathbf{i} - 6\mathbf{k}$, (*b*) $2\mathbf{i} - \mathbf{j}$, (*c*) $8\mathbf{i} - 4\mathbf{j} + 4\mathbf{k}$, (*d*) $\mathbf{i} - 10\mathbf{j} - 3\mathbf{k}$, (*e*) $2\mathbf{i} - 11\mathbf{j} - 7\mathbf{k}$

79. If $\mathbf{A} = 3\mathbf{i} - \mathbf{j} - 2\mathbf{k}$ and $\mathbf{B} = 2\mathbf{i} + 3\mathbf{j} + \mathbf{k}$, find: (*a*) $|\mathbf{A}\times\mathbf{B}|$, (*b*) $(\mathbf{A} + 2\mathbf{B})\times(2\mathbf{A} - \mathbf{B})$, (*c*) $|(\mathbf{A} + \mathbf{B})\times(\mathbf{A} - \mathbf{B})|$.
Ans. (*a*) $\sqrt{195}$, (*b*) $-25\mathbf{i} + 35\mathbf{j} - 55\mathbf{k}$, (*c*) $2\sqrt{195}$

80. If $\mathbf{A} = \mathbf{i} - 2\mathbf{j} - 3\mathbf{k}$, $\mathbf{B} = 2\mathbf{i} + \mathbf{j} - \mathbf{k}$ and $\mathbf{C} = \mathbf{i} + 3\mathbf{j} - 2\mathbf{k}$, find:
(*a*) $|(\mathbf{A}\times\mathbf{B})\times\mathbf{C}|$, (*c*) $\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C})$, (*e*) $(\mathbf{A}\times\mathbf{B})\times(\mathbf{B}\times\mathbf{C})$
(*b*) $|\mathbf{A}\times(\mathbf{B}\times\mathbf{C})|$, (*d*) $(\mathbf{A}\times\mathbf{B})\cdot\mathbf{C}$, (*f*) $(\mathbf{A}\times\mathbf{B})(\mathbf{B}\cdot\mathbf{C})$
Ans. (*a*) $5\sqrt{26}$, (*b*) $3\sqrt{10}$, (*c*) -20, (*d*) -20, (*e*) $-40\mathbf{i} - 20\mathbf{j} + 20\mathbf{k}$, ((*f*) $35\mathbf{i} - 35\mathbf{j} + 35\mathbf{k}$

81. Show that if $\mathbf{A} \neq \mathbf{0}$ and both of the conditions (*a*) $\mathbf{A}\cdot\mathbf{B} = \mathbf{A}\cdot\mathbf{C}$ and (*b*) $\mathbf{A}\times\mathbf{B} = \mathbf{A}\times\mathbf{C}$ hold simultaneously then $\mathbf{B} = \mathbf{C}$, but if only one of these conditions holds then $\mathbf{B} \neq \mathbf{C}$ necessarily.

82. Find the area of a parallelogram having diagonals $\mathbf{A} = 3\mathbf{i} + \mathbf{j} - 2\mathbf{k}$ and $\mathbf{B} = \mathbf{i} - 3\mathbf{j} + 4\mathbf{k}$. *Ans.* $5\sqrt{3}$

The DOT and CROSS PRODUCT

83. Find the area of a triangle with vertices at $(3,-1,2)$, $(1,-1,-3)$ and $(4,-3,1)$. *Ans.* $\frac{1}{2}\sqrt{165}$

84. If $\mathbf{A} = 2\mathbf{i}+\mathbf{j}-3\mathbf{k}$ and $\mathbf{B} = \mathbf{i}-2\mathbf{j}+\mathbf{k}$, find a vector of magnitude 5 perpendicular to both $\mathbf{A}$ and $\mathbf{B}$.
Ans. $\pm\,\frac{5\sqrt{3}}{3}(\mathbf{i}+\mathbf{j}+\mathbf{k})$

85. Use Problem 75 to derive the formulas
$$\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta, \qquad \sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$

86. A force given by $\mathbf{F} = 3\mathbf{i}+2\mathbf{j}-4\mathbf{k}$ is applied at the point $(1,-1,2)$. Find the moment of $\mathbf{F}$ about the point $(2,-1,3)$. *Ans.* $2\mathbf{i}-7\mathbf{j}-2\mathbf{k}$

87. The angular velocity of a rotating rigid body about an axis of rotation is given by $\boldsymbol{\omega} = 4\mathbf{i}+\mathbf{j}-2\mathbf{k}$. Find the linear velocity of a point P on the body whose position vector relative to a point on the axis of rotation is $2\mathbf{i}-3\mathbf{j}+\mathbf{k}$. *Ans.* $-5\mathbf{i}-8\mathbf{j}-14\mathbf{k}$

88. Simplify $(\mathbf{A}+\mathbf{B})\cdot(\mathbf{B}+\mathbf{C})\times(\mathbf{C}+\mathbf{A})$. *Ans.* $2\mathbf{A}\cdot\mathbf{B}\times\mathbf{C}$

89. Prove that $$(\mathbf{A}\cdot\mathbf{B}\times\mathbf{C})(\mathbf{a}\cdot\mathbf{b}\times\mathbf{c}) = \begin{vmatrix} \mathbf{A}\cdot\mathbf{a} & \mathbf{A}\cdot\mathbf{b} & \mathbf{A}\cdot\mathbf{c} \\ \mathbf{B}\cdot\mathbf{a} & \mathbf{B}\cdot\mathbf{b} & \mathbf{B}\cdot\mathbf{c} \\ \mathbf{C}\cdot\mathbf{a} & \mathbf{C}\cdot\mathbf{b} & \mathbf{C}\cdot\mathbf{c} \end{vmatrix}$$

90. Find the volume of the parallelepiped whose edges are represented by $\mathbf{A} = 2\mathbf{i}-3\mathbf{j}+4\mathbf{k}$, $\mathbf{B} = \mathbf{i}+2\mathbf{j}-\mathbf{k}$, $\mathbf{C} = 3\mathbf{i}-\mathbf{j}+2\mathbf{k}$. *Ans.* 7

91. If $\mathbf{A}\cdot\mathbf{B}\times\mathbf{C} = 0$, show that either (*a*) $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ are coplanar but no two of them are collinear, or (*b*) two of the vectors $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ are collinear, or (*c*) all of the vectors $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ are collinear.

92. Find the constant a such that the vectors $2\mathbf{i}-\mathbf{j}+\mathbf{k}$, $\mathbf{i}+2\mathbf{j}-3\mathbf{k}$ and $3\mathbf{i}+a\mathbf{j}+5\mathbf{k}$ are coplanar. *Ans.* $a=-4$

93. If $\mathbf{A} = x_1\mathbf{a}+y_1\mathbf{b}+z_1\mathbf{c}$, $\mathbf{B} = x_2\mathbf{a}+y_2\mathbf{b}+z_2\mathbf{c}$ and $\mathbf{C} = x_3\mathbf{a}+y_3\mathbf{b}+z_3\mathbf{c}$, prove that
$$\mathbf{A}\cdot\mathbf{B}\times\mathbf{C} = \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} (\mathbf{a}\cdot\mathbf{b}\times\mathbf{c})$$

94. Prove that a necessary and sufficient condition that $\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = (\mathbf{A}\times\mathbf{B})\times\mathbf{C}$ is $(\mathbf{A}\times\mathbf{C})\times\mathbf{B} = \mathbf{0}$. Discuss the cases where $\mathbf{A}\cdot\mathbf{B}=0$ or $\mathbf{B}\cdot\mathbf{C}=0$.

95. Let points P, Q and R have position vectors $\mathbf{r}_1 = 3\mathbf{i}-2\mathbf{j}-\mathbf{k}$, $\mathbf{r}_2 = \mathbf{i}+3\mathbf{j}+4\mathbf{k}$ and $\mathbf{r}_3 = 2\mathbf{i}+\mathbf{j}-2\mathbf{k}$ relative to an origin O. Find the distance from P to the plane OQR. *Ans.* 3

96. Find the shortest distance from $(6,-4,4)$ to the line joining $(2,1,2)$ and $(3,-1,4)$. *Ans.* 3

97. Given points $P(2,1,3)$, $Q(1,2,1)$, $R(-1,-2,-2)$ and $S(1,-4,0)$, find the shortest distance between lines PQ and RS. *Ans.* $3\sqrt{2}$

98. Prove that the perpendiculars from the vertices of a triangle to the opposite sides (extended if necessary) meet in a point (the *orthocentre* of the triangle).

99. Prove that the perpendicular bisectors of the sides of a triangle meet in a point (the *circumcentre* of the triangle).

100. Prove that $(\mathbf{A}\times\mathbf{B})\cdot(\mathbf{C}\times\mathbf{D}) + (\mathbf{B}\times\mathbf{C})\cdot(\mathbf{A}\times\mathbf{D}) + (\mathbf{C}\times\mathbf{A})\cdot(\mathbf{B}\times\mathbf{D}) = 0$.

101. Let PQR be a spherical triangle whose sides p, q, r are arcs of great circles. Prove the *law of cosines for spherical triangles*,
$$\cos p = \cos q\,\cos r + \sin q\,\sin r\,\cos P$$

SOLVED PROBLEMS

1. If $\mathbf{R}(u) = x(u)\mathbf{i} + y(u)\mathbf{j} + z(u)\mathbf{k}$, where x, y and z are differentiable functions of a scalar u, prove that $\dfrac{d\mathbf{R}}{du} = \dfrac{dx}{du}\mathbf{i} + \dfrac{dy}{du}\mathbf{j} + \dfrac{dz}{du}\mathbf{k}$.

$$\frac{d\mathbf{R}}{du} = \lim_{\Delta u \to 0} \frac{\mathbf{R}(u+\Delta u) - \mathbf{R}(u)}{\Delta u}$$

$$= \lim_{\Delta u \to 0} \frac{[x(u+\Delta u)\mathbf{i} + y(u+\Delta u)\mathbf{j} + z(u+\Delta u)\mathbf{k}] - [x(u)\mathbf{i} + y(u)\mathbf{j} + z(u)\mathbf{k}]}{\Delta u}$$

$$= \lim_{\Delta u \to 0} \frac{x(u+\Delta u) - x(u)}{\Delta u}\mathbf{i} + \frac{y(u+\Delta u) - y(u)}{\Delta u}\mathbf{j} + \frac{z(u+\Delta u) - z(u)}{\Delta u}\mathbf{k}$$

$$= \frac{dx}{du}\mathbf{i} + \frac{dy}{du}\mathbf{j} + \frac{dz}{du}\mathbf{k}$$

2. Given $\mathbf{R} = \sin t\,\mathbf{i} + \cos t\,\mathbf{j} + t\mathbf{k}$, find (a) $\dfrac{d\mathbf{R}}{dt}$, (b) $\dfrac{d^2\mathbf{R}}{dt^2}$, (c) $\left|\dfrac{d\mathbf{R}}{dt}\right|$, (d) $\left|\dfrac{d^2\mathbf{R}}{dt^2}\right|$.

(a) $\dfrac{d\mathbf{R}}{dt} = \dfrac{d}{dt}(\sin t)\mathbf{i} + \dfrac{d}{dt}(\cos t)\mathbf{j} + \dfrac{d}{dt}(t)\mathbf{k} = \cos t\,\mathbf{i} - \sin t\,\mathbf{j} + \mathbf{k}$

(b) $\dfrac{d^2\mathbf{R}}{dt^2} = \dfrac{d}{dt}\left(\dfrac{d\mathbf{R}}{dt}\right) = \dfrac{d}{dt}(\cos t)\mathbf{i} - \dfrac{d}{dt}(\sin t)\mathbf{j} + \dfrac{d}{dt}(1)\mathbf{k} = -\sin t\,\mathbf{i} - \cos t\,\mathbf{j}$

(c) $\left|\dfrac{d\mathbf{R}}{dt}\right| = \sqrt{(\cos t)^2 + (-\sin t)^2 + (1)^2} = \sqrt{2}$

(d) $\left|\dfrac{d^2\mathbf{R}}{dt^2}\right| = \sqrt{(-\sin t)^2 + (-\cos t)^2} = 1$

3. A particle moves along a curve whose parametric equations are $x = e^{-t}$, $y = 2\cos 3t$, $z = 2\sin 3t$, where t is the time.
(a) Determine its velocity and acceleration at any time.
(b) Find the magnitudes of the velocity and acceleration at $t = 0$.

(a) The position vector $\mathbf{r}$ of the particle is $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} = e^{-t}\mathbf{i} + 2\cos 3t\,\mathbf{j} + 2\sin 3t\,\mathbf{k}$.

Then the velocity is $\mathbf{v} = \dfrac{d\mathbf{r}}{dt} = -e^{-t}\mathbf{i} - 6\sin 3t\,\mathbf{j} + 6\cos 3t\,\mathbf{k}$

and the acceleration is $\mathbf{a} = \dfrac{d^2\mathbf{r}}{dt^2} = e^{-t}\mathbf{i} - 18\cos 3t\,\mathbf{j} - 18\sin 3t\,\mathbf{k}$

(b) At $t = 0$, $\dfrac{d\mathbf{r}}{dt} = -\mathbf{i} + 6\mathbf{k}$ and $\dfrac{d^2\mathbf{r}}{dt^2} = \mathbf{i} - 18\mathbf{j}$. Then

magnitude of velocity at $t = 0$ is $\sqrt{(-1)^2 + (6)^2} = \sqrt{37}$

magnitude of acceleration at $t = 0$ is $\sqrt{(1)^2 + (-18)^2} = \sqrt{325}$.

4. A particle moves along the curve $x = 2t^2$, $y = t^2 - 4t$, $z = 3t - 5$, where t is the time. Find the components of its velocity and acceleration at time $t = 1$ in the direction $\mathbf{i} - 3\mathbf{j} + 2\mathbf{k}$.

$$\text{Velocity} = \frac{d\mathbf{r}}{dt} = \frac{d}{dt}\left[2t^2\mathbf{i} + (t^2-4t)\mathbf{j} + (3t-5)\mathbf{k}\right]$$
$$= 4t\,\mathbf{i} + (2t-4)\mathbf{j} + 3\mathbf{k} = 4\mathbf{i} - 2\mathbf{j} + 3\mathbf{k} \quad \text{at } t = 1.$$

Unit vector in direction $\mathbf{i} - 3\mathbf{j} + 2\mathbf{k}$ is $\dfrac{\mathbf{i} - 3\mathbf{j} + 2\mathbf{k}}{\sqrt{(1)^2+(-3)^2+(2)^2}} = \dfrac{\mathbf{i} - 3\mathbf{j} + 2\mathbf{k}}{\sqrt{14}}$.

Then the component of the velocity in the given direction is

$$\frac{(4\mathbf{i} - 2\mathbf{j} + 3\mathbf{k})\cdot(\mathbf{i} - 3\mathbf{j} + 2\mathbf{k})}{\sqrt{14}} = \frac{(4)(1) + (-2)(-3) + (3)(2)}{\sqrt{14}} = \frac{16}{\sqrt{14}} = \frac{8\sqrt{14}}{7}$$

$$\text{Acceleration} = \frac{d^2\mathbf{r}}{dt^2} = \frac{d}{dt}\left(\frac{d\mathbf{r}}{dt}\right) = \frac{d}{dt}\left[4t\,\mathbf{i} + (2t-4)\mathbf{j} + 3\mathbf{k}\right] = 4\mathbf{i} + 2\mathbf{j} + 0\mathbf{k}.$$

Then the component of the acceleration in the given direction is

$$\frac{(4\mathbf{i} + 2\mathbf{j} + 0\mathbf{k})\cdot(\mathbf{i} - 3\mathbf{j} + 2\mathbf{k})}{\sqrt{14}} = \frac{(4)(1) + (2)(-3) + (0)(2)}{\sqrt{14}} = \frac{-2}{\sqrt{14}} = \frac{-\sqrt{14}}{7}$$

5. A curve C is defined by parametric equations $x = x(s)$, $y = y(s)$, $z = z(s)$, where s is the arc length of C measured from a fixed point on C. If $\mathbf{r}$ is the position vector of any point on C, show that $d\mathbf{r}/ds$ is a unit vector tangent to C.

The vector $\dfrac{d\mathbf{r}}{ds} = \dfrac{d}{ds}(x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) = \dfrac{dx}{ds}\mathbf{i} + \dfrac{dy}{ds}\mathbf{j} + \dfrac{dz}{ds}\mathbf{k}$ is tangent to the curve $x = x(s)$, $y = y(s)$, $z = z(s)$. To show that it has unit magnitude we note that

$$\left|\frac{d\mathbf{r}}{ds}\right| = \sqrt{\left(\frac{dx}{ds}\right)^2 + \left(\frac{dy}{ds}\right)^2 + \left(\frac{dz}{ds}\right)^2} = \sqrt{\frac{(dx)^2 + (dy)^2 + (dz)^2}{(ds)^2}} = 1$$

since $(ds)^2 = (dx)^2 + (dy)^2 + (dz)^2$ from the calculus.

6. (a) Find the unit tangent vector to any point on the curve $x = t^2+1$, $y = 4t-3$, $z = 2t^2-6t$.
(b) Determine the unit tangent at the point where $t=2$.

(a) A tangent vector to the curve at any point is

$$\frac{d\mathbf{r}}{dt} = \frac{d}{dt}\left[(t^2+1)\mathbf{i} + (4t-3)\mathbf{j} + (2t^2-6t)\mathbf{k}\right] = 2t\,\mathbf{i} + 4\mathbf{j} + (4t-6)\mathbf{k}$$

The magnitude of the vector is $\left|\dfrac{d\mathbf{r}}{dt}\right| = \sqrt{(2t)^2 + (4)^2 + (4t-6)^2}$.

Then the required unit tangent vector is $\mathbf{T} = \dfrac{2t\,\mathbf{i} + 4\mathbf{j} + (4t-6)\mathbf{k}}{\sqrt{(2t)^2 + (4)^2 + (4t-6)^2}}$

Note that since $\left|\dfrac{d\mathbf{r}}{dt}\right| = \dfrac{ds}{dt}$, $\mathbf{T} = \dfrac{d\mathbf{r}/dt}{ds/dt} = \dfrac{d\mathbf{r}}{ds}$.

(b) At $t=2$, the unit tangent vector is $\mathbf{T} = \dfrac{4\mathbf{i} + 4\mathbf{j} + 2\mathbf{k}}{\sqrt{(4)^2 + (4)^2 + (2)^2}} = \dfrac{2}{3}\mathbf{i} + \dfrac{2}{3}\mathbf{j} + \dfrac{1}{3}\mathbf{k}$.

7. If $\mathbf{A}$ and $\mathbf{B}$ are differentiable functions of a scalar u, prove:

(a) $\dfrac{d}{du}(\mathbf{A}\cdot\mathbf{B}) = \mathbf{A}\cdot\dfrac{d\mathbf{B}}{du} + \dfrac{d\mathbf{A}}{du}\cdot\mathbf{B}$, (b) $\dfrac{d}{du}(\mathbf{A}\times\mathbf{B}) = \mathbf{A}\times\dfrac{d\mathbf{B}}{du} + \dfrac{d\mathbf{A}}{du}\times\mathbf{B}$

$\ldots \cos t)\mathbf{i} - (t^3 \cos t + 3t^2 \sin t)\mathbf{j} + (5t^2 \sin t - \sin t - 11t \cos t)\mathbf{k}$

Another Method.

$$\mathbf{A}\times\mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 5t^2 & t & -t^3 \\ \sin t & -\cos t & 0 \end{vmatrix} = -t^3 \cos t\,\mathbf{i} - t^3 \sin t\,\mathbf{j} + (-5t^2 \cos t - t \sin t)\mathbf{k}$$

Then $\dfrac{d}{dt}(\mathbf{A}\times\mathbf{B}) = (t^3 \sin t - 3t^2 \cos t)\mathbf{i} - (t^3 \cos t + 3t^2 \sin t)\mathbf{j} + (5t^2 \sin t - 11t \cos t - \sin t)\mathbf{k}$

(c) $\dfrac{d}{dt}(\mathbf{A}\cdot\mathbf{A}) = \mathbf{A}\cdot\dfrac{d\mathbf{A}}{dt} + \dfrac{d\mathbf{A}}{dt}\cdot\mathbf{A} = 2\mathbf{A}\cdot\dfrac{d\mathbf{A}}{dt}$

$= 2(5t^2\mathbf{i} + t\mathbf{j} - t^3\mathbf{k})\cdot(10t\,\mathbf{i} + \mathbf{j} - 3t^2\mathbf{k}) = 100t^3 + 2t + 6t^5$

Another Method. $\mathbf{A}\cdot\mathbf{A} = (5t^2)^2 + (t)^2 + (-t^3)^2 = 25t^4 + t^2 + t^6$

Then $\dfrac{d}{dt}(25t^4 + t^2 + t^6) = 100t^3 + 2t + 6t^5$.

9. If $\mathbf{A}$ has constant magnitude show that $\mathbf{A}$ and $d\mathbf{A}/dt$ are perpendicular provided $|d\mathbf{A}/dt| \neq 0$.

Since $\mathbf{A}$ has constant magnitude, $\mathbf{A}\cdot\mathbf{A}$ = constant.

Then $\dfrac{d}{dt}(\mathbf{A}\cdot\mathbf{A}) = \mathbf{A}\cdot\dfrac{d\mathbf{A}}{dt} + \dfrac{d\mathbf{A}}{dt}\cdot\mathbf{A} = 2\mathbf{A}\cdot\dfrac{d\mathbf{A}}{dt} = 0$.

Thus $\mathbf{A}\cdot\dfrac{d\mathbf{A}}{dt} = 0$ and $\mathbf{A}$ is perpendicular to $\dfrac{d\mathbf{A}}{dt}$ provided $\left|\dfrac{d\mathbf{A}}{dt}\right| \neq 0$.

10. Prove $\dfrac{d}{du}(\mathbf{A}\cdot\mathbf{B}\times\mathbf{C}) = \mathbf{A}\cdot\mathbf{B}\times\dfrac{d\mathbf{C}}{du} + \mathbf{A}\cdot\dfrac{d\mathbf{B}}{du}\times\mathbf{C} + \dfrac{d\mathbf{A}}{du}\cdot\mathbf{B}\times\mathbf{C}$, where $\mathbf{A},\mathbf{B},\mathbf{C}$ are differentiable functions of a scalar u.

By Problems 7(*a*) and 7(*b*), $\dfrac{d}{du}\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = \mathbf{A}\cdot\dfrac{d}{du}(\mathbf{B}\times\mathbf{C}) + \dfrac{d\mathbf{A}}{du}\cdot\mathbf{B}\times\mathbf{C}$

$= \mathbf{A}\cdot\left[\mathbf{B}\times\dfrac{d\mathbf{C}}{du} + \dfrac{d\mathbf{B}}{du}\times\mathbf{C}\right] + \dfrac{d\mathbf{A}}{du}\cdot\mathbf{B}\times\mathbf{C}$

$= \mathbf{A}\cdot\mathbf{B}\times\dfrac{d\mathbf{C}}{du} + \mathbf{A}\cdot\dfrac{d\mathbf{B}}{du}\times\mathbf{C} + \dfrac{d\mathbf{A}}{du}\cdot\mathbf{B}\times\mathbf{C}$

11. Evaluate $\dfrac{d}{dt}\left(\mathbf{V}\cdot\dfrac{d\mathbf{V}}{dt}\times\dfrac{d^2\mathbf{V}}{dt^2}\right)$.

By Problem 10, $\dfrac{d}{dt}\left(\mathbf{V}\cdot\dfrac{d\mathbf{V}}{dt}\times\dfrac{d^2\mathbf{V}}{dt^2}\right) = \mathbf{V}\cdot\dfrac{d\mathbf{V}}{dt}\times\dfrac{d^3\mathbf{V}}{dt^3} + \mathbf{V}\cdot\dfrac{d^2\mathbf{V}}{dt^2}\times\dfrac{d^2\mathbf{V}}{dt^2} + \dfrac{d\mathbf{V}}{dt}\cdot\dfrac{d\mathbf{V}}{dt}\times\dfrac{d^2\mathbf{V}}{dt^2}$

$= \mathbf{V}\cdot\dfrac{d\mathbf{V}}{dt}\times\dfrac{d^3\mathbf{V}}{dt^3} + 0 + 0 = \mathbf{V}\cdot\dfrac{d\mathbf{V}}{dt}\times\dfrac{d^3\mathbf{V}}{dt^3}$

12. A particle moves so that its position vector is given by $\mathbf{r} = \cos\omega t\,\mathbf{i} + \sin\omega t\,\mathbf{j}$ where ω is a constant. Show that (*a*) the velocity $\mathbf{v}$ of the particle is perpendicular to $\mathbf{r}$, (*b*) the acceleration $\mathbf{a}$ is directed toward the origin and has magnitude proportional to the distance from the origin, (*c*) $\mathbf{r}\times\mathbf{v}$ = a constant vector.

(a) $\mathbf{v} = \dfrac{d\mathbf{r}}{dt} = -\omega \sin \omega t\, \mathbf{i} + \omega \cos \omega t\, \mathbf{j}$

Then $\mathbf{r}\cdot\mathbf{v} = [\cos \omega t\, \mathbf{i} + \sin \omega t\, \mathbf{j}] \cdot [-\omega \sin \omega t\, \mathbf{i} + \omega \cos \omega t\, \mathbf{j}]$

$= (\cos \omega t)(-\omega \sin \omega t) + (\sin \omega t)(\omega \cos \omega t) = 0$

and $\mathbf{r}$ and $\mathbf{v}$ are perpendicular.

(b) $\dfrac{d^2\mathbf{r}}{dt^2} = \dfrac{d\mathbf{v}}{dt} = -\omega^2 \cos \omega t\, \mathbf{i} - \omega^2 \sin \omega t\, \mathbf{j}$

$= -\omega^2 [\cos \omega t\, \mathbf{i} + \sin \omega t\, \mathbf{j}] = -\omega^2 \mathbf{r}$

Then the acceleration is opposite to the direction of $\mathbf{r}$, i.e. it is directed toward the origin. Its magnitude is proportional to $|\mathbf{r}|$ which is the distance from the origin.

(c) $\mathbf{r}\times\mathbf{v} = [\cos \omega t\, \mathbf{i} + \sin \omega t\, \mathbf{j}] \times [-\omega \sin \omega t\, \mathbf{i} + \omega \cos \omega t\, \mathbf{j}]$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos \omega t & \sin \omega t & 0 \\ -\omega \sin \omega t & \omega \cos \omega t & 0 \end{vmatrix} = \omega(\cos^2 \omega t + \sin^2 \omega t)\mathbf{k} = \omega \mathbf{k}, \text{ a constant vector.}$$

Physically, the motion is that of a particle moving on the circumference of a circle with constant angular speed ω. The acceleration, directed toward the centre of the circle, is the *centripetal acceleration.*

13. Prove: $\mathbf{A}\times\dfrac{d^2\mathbf{B}}{dt^2} - \dfrac{d^2\mathbf{A}}{dt^2}\times\mathbf{B} = \dfrac{d}{dt}(\mathbf{A}\times\dfrac{d\mathbf{B}}{dt} - \dfrac{d\mathbf{A}}{dt}\times\mathbf{B})$.

$$\frac{d}{dt}(\mathbf{A}\times\frac{d\mathbf{B}}{dt} - \frac{d\mathbf{A}}{dt}\times\mathbf{B}) = \frac{d}{dt}(\mathbf{A}\times\frac{d\mathbf{B}}{dt}) - \frac{d}{dt}(\frac{d\mathbf{A}}{dt}\times\mathbf{B})$$

$$= \mathbf{A}\times\frac{d^2\mathbf{B}}{dt^2} + \frac{d\mathbf{A}}{dt}\times\frac{d\mathbf{B}}{dt} - [\frac{d\mathbf{A}}{dt}\times\frac{d\mathbf{B}}{dt} + \frac{d^2\mathbf{A}}{dt^2}\times\mathbf{B}] = \mathbf{A}\times\frac{d^2\mathbf{B}}{dt^2} - \frac{d^2\mathbf{A}}{dt^2}\times\mathbf{B}$$

14. Show that $\mathbf{A}\cdot\dfrac{d\mathbf{A}}{dt} = A\dfrac{dA}{dt}$.

Let $\mathbf{A} = A_1\mathbf{i} + A_2\mathbf{j} + A_3\mathbf{k}$. Then $A = \sqrt{A_1^2 + A_2^2 + A_3^2}$.

$$\frac{dA}{dt} = \frac{1}{2}(A_1^2 + A_2^2 + A_3^2)^{-1/2}(2A_1\frac{dA_1}{dt} + 2A_2\frac{dA_2}{dt} + 2A_3\frac{dA_3}{dt})$$

$$= \frac{A_1\dfrac{dA_1}{dt} + A_2\dfrac{dA_2}{dt} + A_3\dfrac{dA_3}{dt}}{(A_1^2 + A_2^2 + A_3^2)^{1/2}} = \frac{\mathbf{A}\cdot\dfrac{d\mathbf{A}}{dt}}{A}, \quad \text{i.e. } A\frac{dA}{dt} = \mathbf{A}\cdot\frac{d\mathbf{A}}{dt}.$$

Another Method.

Since $\mathbf{A}\cdot\mathbf{A} = A^2$, $\dfrac{d}{dt}(\mathbf{A}\cdot\mathbf{A}) = \dfrac{d}{dt}(A^2)$.

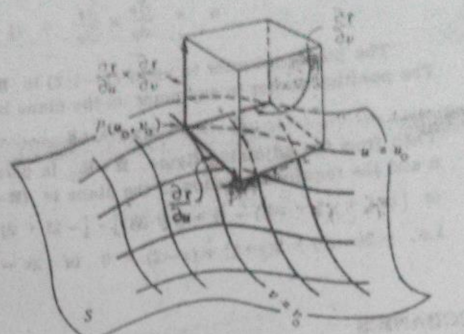
$\dfrac{d}{dt}(\mathbf{A}\cdot\mathbf{A}) = \mathbf{A}\cdot\dfrac{d\mathbf{A}}{dt} + \dfrac{d\mathbf{A}}{dt}\cdot\mathbf{A} = 2\mathbf{A}\cdot\dfrac{d\mathbf{A}}{dt}$ and $\dfrac{d}{dt}(A^2) = 2A\dfrac{dA}{dt}$

Then $2\mathbf{A}\cdot\dfrac{d\mathbf{A}}{dt} = 2A\dfrac{dA}{dt}$ or $\mathbf{A}\cdot\dfrac{d\mathbf{A}}{dt} = A\dfrac{dA}{dt}$.

...nstant vector $\mathbf{A}\cdot\dfrac{d\mathbf{A}}{dt} = 0$ as in Problem 9.

bers (u,v) as defining the *curvilinear coordinates* on the surface. If all the curves u = constant and v = constant are perpendicular at each point of intersection, we call the curvilinear coordinate system *orthogonal*. For further discussion of curvilinear coordinates see Chapter 7.

(*b*) Consider point P having coordinates (u_0, v_0) on a surface S, as shown in the adjacent diagram. The vector $\partial \mathbf{r}/\partial u$ at P is obtained by differentiating $\mathbf{r}$ with respect to u, keeping v = constant = v_0. From the theory of space curves, it follows that $\partial \mathbf{r}/\partial u$ at P represents a vector tangent to the curve $v = v_0$ at P, as shown in the adjoining figure. Similarly, $\partial \mathbf{r}/\partial v$ at P represents a vector tangent to the curve u = constant = u_0. Since $\partial \mathbf{r}/\partial u$ and $\partial \mathbf{r}/\partial v$ represent vectors at P tangent to curves which lie on the surface S at P, it follows that these vectors are tangent to the surface at P. Hence it follows that $\dfrac{\partial \mathbf{r}}{\partial u} \times \dfrac{\partial \mathbf{r}}{\partial v}$ is a vector normal to S at P.



(*c*) $$\frac{\partial \mathbf{r}}{\partial u} = -a \sin u \sin v \, \mathbf{i} + a \cos u \sin v \, \mathbf{j}$$

$$\frac{\partial \mathbf{r}}{\partial v} = a \cos u \cos v \, \mathbf{i} + a \sin u \cos v \, \mathbf{j} - a \sin v \, \mathbf{k}$$

Then $$\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -a \sin u \sin v & a \cos u \sin v & 0 \\ a \cos u \cos v & a \sin u \cos v & -a \sin v \end{vmatrix}$$

$$= -a^2 \cos u \sin^2 v \, \mathbf{i} - a^2 \sin u \sin^2 v \, \mathbf{j} - a^2 \sin v \cos v \, \mathbf{k}$$

represents a vector normal to the surface at any point (u,v).

A unit normal is obtained by dividing $\dfrac{\partial \mathbf{r}}{\partial u} \times \dfrac{\partial \mathbf{r}}{\partial v}$ by its magnitude, $\left| \dfrac{\partial \mathbf{r}}{\partial u} \times \dfrac{\partial \mathbf{r}}{\partial v} \right|$, given by

$$\sqrt{a^4 \cos^2 u \sin^4 v + a^4 \sin^2 u \sin^4 v + a^4 \sin^2 v \cos^2 v}$$

$$= \sqrt{a^4 (\cos^2 u + \sin^2 u) \sin^4 v + a^4 \sin^2 v \cos^2 v}$$

$$= \sqrt{a^4 \sin^2 v (\sin^2 v + \cos^2 v)} = \begin{cases} a^2 \sin v & \text{if } \sin v > 0 \\ -a^2 \sin v & \text{if } \sin v < 0 \end{cases}$$

Then there are two unit normals given by

$$\pm (\cos u \sin v \, \mathbf{i} + \sin u \sin v \, \mathbf{j} + \cos v \, \mathbf{k}) = \pm \mathbf{n}$$

It should be noted that the given surface is defined by $x = a \cos u \sin v$, $y = a \sin u \sin v$, $z = a \cos v$ from which it is seen that $x^2 + y^2 + z^2 = a^2$, which is a sphere of radius a. Since $\mathbf{r} = a\mathbf{n}$, it follows that

$$\mathbf{n} = \cos u \sin v \, \mathbf{i} + \sin u \sin v \, \mathbf{j} + \cos v \, \mathbf{k}$$

is the *outward drawn unit normal* to the sphere at the point (u,v).

26. Find an equation for the tangent plane to the surface $z = x^2 + y^2$ at the point $(1, -1, 2)$.

Let $x = u$, $y = v$, $z = u^2 + v^2$ be parametric equations of the surface. The position vector to any point on the surface is

$$\mathbf{r} = u\mathbf{i} + v\mathbf{j} + (u^2 + v^2)\mathbf{k}$$

Then $\frac{\partial \mathbf{r}}{\partial u} = \mathbf{i} + 2u\,\mathbf{k} = \mathbf{i} + 2\mathbf{k}$, $\frac{\partial \mathbf{r}}{\partial v} = \mathbf{j} + 2v\,\mathbf{k} = \mathbf{j} - 2\mathbf{k}$ at the point (1,−1,2), where $u=1$ and $v=-1$.

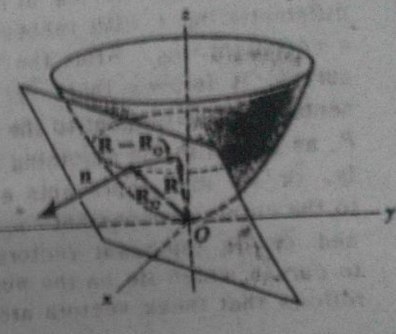
By Problem 25, a normal $\mathbf{n}$ to the surface at this point is

$$\mathbf{n} = \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} = (\mathbf{i} + 2\mathbf{k}) \times (\mathbf{j} - 2\mathbf{k}) = -2\mathbf{i} + 2\mathbf{j} + \mathbf{k}$$

The position vector to point (1,−1,2) is $\mathbf{R}_0 = \mathbf{i} - \mathbf{j} + 2\mathbf{k}$.

The position vector to any point on the plane is

$$\mathbf{R} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$$



Then from the adjoining figure, $\mathbf{R} - \mathbf{R}_0$ is perpendicular to $\mathbf{n}$ and the required equation of the plane is $(\mathbf{R} - \mathbf{R}_0)\cdot\mathbf{n} = 0$

or $[(x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) - (\mathbf{i} - \mathbf{j} + 2\mathbf{k})]\cdot[-2\mathbf{i} + 2\mathbf{j} + \mathbf{k}] = 0$

i.e. $-2(x-1) + 2(y+1) + (z-2) = 0$ or $2x - 2y - z = 2$.

ECHANICS

7. Show that the acceleration $\mathbf{a}$ of a particle which travels along a space curve with velocity given by

$$\mathbf{a} = \frac{dv}{dt}\mathbf{T} + \frac{v^2}{\rho}\mathbf{N}$$

where $\mathbf{T}$ is the unit tangent vector to the space curve, $\mathbf{N}$ is its unit principal normal, and ρ radius of curvature.

Velocity $\mathbf{v}$ = magnitude of $\mathbf{v}$ multiplied by unit tangent vector $\mathbf{T}$

or $\mathbf{v} = v\mathbf{T}$

Differentiating, $\mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{d}{dt}(v\mathbf{T}) = \frac{dv}{dt}\mathbf{T} + v\frac{d\mathbf{T}}{dt}$

But by Problem 18(*a*), $\frac{d\mathbf{T}}{dt} = \frac{d\mathbf{T}}{ds}\frac{ds}{dt} = \kappa\mathbf{N}\frac{ds}{dt} = \kappa v\mathbf{N} = \frac{v\mathbf{N}}{\rho}$

Then $\mathbf{a} = \frac{dv}{dt}\mathbf{T} + v\left(\frac{v\mathbf{N}}{\rho}\right) = \frac{dv}{dt}\mathbf{T} + \frac{v^2}{\rho}\mathbf{N}$

This shows that the component of the acceleration is dv/dt in a direction tangent to the path and direction of the principal normal to the path. The latter acceleration is often called the *centrip* *ration*. For a special case of this problem see Problem 12.

$\mathbf{r}$ is the position vector of a particle of mass m relative to point O and $\mathbf{F}$ is the exte the particle, then $\mathbf{r}\times\mathbf{F} = \mathbf{M}$ is the torque or moment of $\mathbf{F}$ about O. Show that $\mathbf{M} = d\mathbf{H}$ $= \mathbf{r}\times m\mathbf{v}$ and $\mathbf{v}$ is the velocity of the particle.

$$\mathbf{M} = \mathbf{r}\times\mathbf{F} = \mathbf{r}\times\frac{d}{dt}(m\mathbf{v}) \quad \text{by Newton's law}$$

$\nabla \cdot A = 0$
$\nabla \cdot \vec{A} = 0$
$\nabla \times (\nabla \times A) = -\nabla^2 A + \nabla(\nabla \cdot \vec{A})$

Chapter 4

GRADIENT, DIVERGENCE and CURL

THE VECTOR DIFFERENTIAL OPERATOR DEL, written ∇, is defined by

$$\nabla = \frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k} = \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}$$

This vector operator possesses properties analogous to those of ordinary vectors. It is useful in defining three quantities which arise in practical applications and are known as the *gradient*, the *divergence* and the *curl*. The operator ∇ is also known as *nabla*.

THE GRADIENT. Let $\phi(x,y,z)$ be defined and differentiable at each point (x,y,z) in a certain region of space (i.e. ϕ defines a differentiable scalar field). Then the *gradient* of ϕ, written $\nabla\phi$ or grad ϕ, is defined by

$$\nabla\phi = \left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}\right)\phi = \frac{\partial\phi}{\partial x}\mathbf{i} + \frac{\partial\phi}{\partial y}\mathbf{j} + \frac{\partial\phi}{\partial z}\mathbf{k}$$

Note that $\nabla\phi$ defines a vector field.

The component of $\nabla\phi$ in the direction of a unit vector $\mathbf{a}$ is given by $\nabla\phi\cdot\mathbf{a}$ and is called the *directional derivative* of ϕ in the direction $\mathbf{a}$. Physically, this is the rate of change of ϕ at (x,y,z) in the direction $\mathbf{a}$.

THE DIVERGENCE. Let $\mathbf{V}(x,y,z) = V_1\mathbf{i} + V_2\mathbf{j} + V_3\mathbf{k}$ be defined and differentiable at each point (x,y,z) in a certain region of space (i.e. $\mathbf{V}$ defines a differentiable vector field). Then the *divergence* of $\mathbf{V}$, written $\nabla\cdot\mathbf{V}$ or div $\mathbf{V}$, is defined by

$$\nabla\cdot\mathbf{V} = \left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}\right)\cdot(V_1\mathbf{i} + V_2\mathbf{j} + V_3\mathbf{k})$$

$$= \frac{\partial V_1}{\partial x} + \frac{\partial V_2}{\partial y} + \frac{\partial V_3}{\partial z}$$

Note the analogy with $\mathbf{A}\cdot\mathbf{B} = A_1B_1 + A_2B_2 + A_3B_3$. Also note that $\nabla\cdot\mathbf{V} \neq \mathbf{V}\cdot\nabla$.

THE CURL. If $\mathbf{V}(x,y,z)$ is a differentiable vector field then the *curl* or *rotation* of $\mathbf{V}$, written $\nabla\times\mathbf{V}$, curl $\mathbf{V}$ or rot $\mathbf{V}$, is defined by

$$\nabla\times\mathbf{V} = \left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}\right)\times(V_1\mathbf{i} + V_2\mathbf{j} + V_3\mathbf{k})$$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_1 & V_2 & V_3 \end{vmatrix}$$

$$= \mathbf{i}\left\{\frac{n}{2}(x^2+y^2+z^2)^{n/2-1}\,2x\right\} + \mathbf{j}\left\{\frac{n}{2}(x^2+y^2+z^2)^{n/2-1}\,2y\right\} + \mathbf{k}\left\{\frac{n}{2}(x^2+y^2+z^2)^{n/2-1}\,2z\right\}$$

$$= n(x^2+y^2+z^2)^{n/2-1}(x\mathbf{i} + y\mathbf{j} + z\mathbf{k})$$

$$= n(r^2)^{n/2-1}\,\mathbf{r} = nr^{n-2}\,\mathbf{r}$$

Note that if $\mathbf{r} = r\mathbf{r}_1$ where $\mathbf{r}_1$ is a unit vector in the direction $\mathbf{r}$, then $\nabla r^n = nr^{n-1}\mathbf{r}_1$.

5. Show that $\nabla\phi$ is a vector perpendicular to the surface $\phi(x,y,z) = c$ where c is a constant.

Let $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ be the position vector to any point $P(x,y,z)$ on the surface. Then $d\mathbf{r} = dx\,\mathbf{i} + dy\,\mathbf{j} + dz\,\mathbf{k}$ lies in the tangent plane to the surface at P.

But $d\phi = \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz = 0$ or $\left(\frac{\partial\phi}{\partial x}\mathbf{i} + \frac{\partial\phi}{\partial y}\mathbf{j} + \frac{\partial\phi}{\partial z}\mathbf{k}\right)\cdot(dx\,\mathbf{i} + dy\,\mathbf{j} + dz\,\mathbf{k}) = 0$

i.e. $\nabla\phi\cdot d\mathbf{r} = 0$ so that $\nabla\phi$ is perpendicular to $d\mathbf{r}$ and therefore to the surface.

6. Find a unit normal to the surface $x^2y + 2xz = 4$ at the point $(2,-2,3)$.

$\nabla(x^2y + 2xz) = (2xy + 2z)\mathbf{i} + x^2\mathbf{j} + 2x\,\mathbf{k} = -2\mathbf{i} + 4\mathbf{j} + 4\mathbf{k}$ at the point $(2,-2,3)$.

Then a unit normal to the surface $= \dfrac{-2\mathbf{i} + 4\mathbf{j} + 4\mathbf{k}}{\sqrt{(-2)^2 + (4)^2 + (4)^2}} = -\frac{1}{3}\mathbf{i} + \frac{2}{3}\mathbf{j} + \frac{2}{3}\mathbf{k}.$

Another unit normal is $\frac{1}{3}\mathbf{i} - \frac{2}{3}\mathbf{j} - \frac{2}{3}\mathbf{k}$ having direction opposite to that above.

7. Find an equation for the tangent plane to the surface $2xz^2 - 3xy - 4x = 7$ at the point $(1,-1,2)$.

$$\nabla(2xz^2 - 3xy - 4x) = (2z^2 - 3y - 4)\mathbf{i} - 3x\,\mathbf{j} + 4xz\,\mathbf{k}$$

Then a normal to the surface at the point $(1,-1,2)$ is $7\mathbf{i} - 3\mathbf{j} + 8\mathbf{k}$.

The equation of a plane passing through a point whose position vector is $\mathbf{r}_0$ and which is perpendicular to the normal $\mathbf{N}$ is $(\mathbf{r} - \mathbf{r}_0)\cdot\mathbf{N} = 0$. (See Chap.2, Prob.18.) Then the required equation is

$$[(x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) - (\mathbf{i} - \mathbf{j} + 2\mathbf{k})]\cdot(7\mathbf{i} - 3\mathbf{j} + 8\mathbf{k}) = 0$$

or

$$7(x-1) - 3(y+1) + 8(z-2) = 0.$$

8. Let $\phi(x,y,z)$ and $\phi(x+\Delta x,\ y+\Delta y,\ z+\Delta z)$ be the temperatures at two neighbouring points $P(x,y,z)$ and $Q(x+\Delta x,\ y+\Delta y,\ z+\Delta z)$ of a certain region.

(a) Interpret physically the quantity $\dfrac{\Delta\phi}{\Delta s} = \dfrac{\phi(x+\Delta x,\ y+\Delta y,\ z+\Delta z) - \phi(x,y,z)}{\Delta s}$ where Δs is the distance between points P and Q.

(b) Evaluate $\lim\limits_{\Delta s\to 0}\dfrac{\Delta\phi}{\Delta s} = \dfrac{d\phi}{ds}$ and interpret physically.

(c) Show that $\dfrac{d\phi}{ds} = \nabla\phi\cdot\dfrac{d\mathbf{r}}{ds}$.

(b) From the calculus,

$$\Delta\phi = \frac{\partial\phi}{\partial x}\Delta x + \frac{\partial\phi}{\partial y}\Delta y + \frac{\partial\phi}{\partial z}\Delta z + \text{infinitesimals of order higher than } \Delta x, \Delta y \text{ and } \Delta z$$

Then
$$\lim_{\Delta s\to 0}\frac{\Delta\phi}{\Delta s} = \lim_{\Delta s\to 0}\frac{\partial\phi}{\partial x}\frac{\Delta x}{\Delta s} + \frac{\partial\phi}{\partial y}\frac{\Delta y}{\Delta s} + \frac{\partial\phi}{\partial z}\frac{\Delta z}{\Delta s}$$

or
$$\frac{d\phi}{ds} = \frac{\partial\phi}{\partial x}\frac{dx}{ds} + \frac{\partial\phi}{\partial y}\frac{dy}{ds} + \frac{\partial\phi}{\partial z}\frac{dz}{ds}$$

$\frac{d\phi}{ds}$ represents the rate of change of temperature with respect to distance at point P in a direction toward Q. This is also called the *directional derivative* of ϕ.

(c) $$\frac{d\phi}{ds} = \frac{\partial\phi}{\partial x}\frac{dx}{ds} + \frac{\partial\phi}{\partial y}\frac{dy}{ds} + \frac{\partial\phi}{\partial z}\frac{dz}{ds} = \left(\frac{\partial\phi}{\partial x}\mathbf{i} + \frac{\partial\phi}{\partial y}\mathbf{j} + \frac{\partial\phi}{\partial z}\mathbf{k}\right)\cdot\left(\frac{dx}{ds}\mathbf{i} + \frac{dy}{ds}\mathbf{j} + \frac{dz}{ds}\mathbf{k}\right)$$

$$= \nabla\phi\cdot\frac{d\mathbf{r}}{ds}.$$

Note that since $\frac{d\mathbf{r}}{ds}$ is a unit vector, $\nabla\phi\cdot\frac{d\mathbf{r}}{ds}$ is the component of $\nabla\phi$ in the direction of this unit vector.

9. Show that the greatest rate of change of ϕ, i.e. the maximum directional derivative, takes place in the direction of, and has the magnitude of, the vector $\nabla\phi$.

By Problem 8(c), $\frac{d\phi}{ds} = \nabla\phi\cdot\frac{d\mathbf{r}}{ds}$ is the projection of $\nabla\phi$ in the direction $\frac{d\mathbf{r}}{ds}$. This projection will be a maximum when $\nabla\phi$ and $\frac{d\mathbf{r}}{ds}$ have the same direction. Then the maximum value of $\frac{d\phi}{ds}$ takes place in the direction of $\nabla\phi$ and its magnitude is $|\nabla\phi|$.

10. Find the directional derivative of $\phi = x^2yz + 4xz^2$ at $(1,-2,-1)$ in the direction $2\mathbf{i} - \mathbf{j} - 2\mathbf{k}$.

$$\begin{aligned}\nabla\phi &= \nabla(x^2yz + 4xz^2) = (2xyz + 4z^2)\mathbf{i} + x^2z\mathbf{j} + (x^2y + 8xz)\mathbf{k}\\ &= 8\mathbf{i} - \mathbf{j} - 10\mathbf{k} \quad \text{at} \quad (1,-2,-1).\end{aligned}$$

The unit vector in the direction of $2\mathbf{i} - \mathbf{j} - 2\mathbf{k}$ is

$$\mathbf{a} = \frac{2\mathbf{i} - \mathbf{j} - 2\mathbf{k}}{\sqrt{(2)^2 + (-1)^2 + (-2)^2}} = \frac{2}{3}\mathbf{i} - \frac{1}{3}\mathbf{j} - \frac{2}{3}\mathbf{k}$$

Then the required directional derivative is

$$\nabla\phi\cdot\mathbf{a} = (8\mathbf{i} - \mathbf{j} - 10\mathbf{k})\cdot\left(\frac{2}{3}\mathbf{i} - \frac{1}{3}\mathbf{j} - \frac{2}{3}\mathbf{k}\right) = \frac{16}{3} + \frac{1}{3} + \frac{20}{3} = \frac{37}{3}$$

Since this is positive, ϕ is increasing in this direction.

11. (a) In what direction from the point $(2,1,-1)$ is the directional derivative of $\phi = x^2yz^3$ a maximum?
(b) What is the magnitude of this maximum?

$$\begin{aligned}\nabla\phi &= \nabla(x^2yz^3) = 2xyz^3\,\mathbf{i} + x^2z^3\,\mathbf{j} + 3x^2yz^2\,\mathbf{k}\\ &= -4\mathbf{i} - 4\mathbf{j} + 12\mathbf{k} \quad \text{at} \quad (2,1,-1).\end{aligned}$$

Then by Problem 9,

(*a*) the directional derivative is a maximum in the direction $\nabla\phi = -4\mathbf{i} - 4\mathbf{j} + 12\mathbf{k}$,

(*b*) the magnitude of this maximum is $|\nabla\phi| = \sqrt{(-4)^2 + (-4)^2 + (12)^2} = \sqrt{176} = 4\sqrt{11}$.

12. Find the angle between the surfaces $x^2+y^2+z^2 = 9$ and $z = x^2+y^2-3$ at the point $(2,-1,2)$.

The angle between the surfaces at the point is the angle between the normals to the surfaces at point.

A normal to $x^2+y^2+z^2 = 9$ at $(2,-1,2)$ is

$$\nabla\phi_1 = \nabla(x^2+y^2+z^2) = 2x\,\mathbf{i} + 2y\,\mathbf{j} + 2z\,\mathbf{k} = 4\mathbf{i} - 2\mathbf{j} + 4\mathbf{k}$$

A normal to $z = x^2+y^2-3$ or $x^2+y^2-z = 3$ at $(2,-1,2)$ is

$$\nabla\phi_2 = \nabla(x^2+y^2-z) = 2x\,\mathbf{i} + 2y\,\mathbf{j} - \mathbf{k} = 4\mathbf{i} - 2\mathbf{j} - \mathbf{k}$$

$(\nabla\phi_1)\cdot(\nabla\phi_2) = |\nabla\phi_1|\,|\nabla\phi_2|\cos\theta$, where θ is the required angle. Then

$$(4\mathbf{i} - 2\mathbf{j} + 4\mathbf{k})\cdot(4\mathbf{i} - 2\mathbf{j} - \mathbf{k}) = |4\mathbf{i} - 2\mathbf{j} + 4\mathbf{k}|\,|4\mathbf{i} - 2\mathbf{j} - \mathbf{k}|\cos\theta$$

$$16 + 4 - 4 = \sqrt{(4)^2+(-2)^2+(4)^2}\,\sqrt{(4)^2+(-2)^2+(-1)^2}\,\cos\theta$$

and $\cos\theta = \dfrac{16}{6\sqrt{21}} = \dfrac{8\sqrt{21}}{63} = 0.5819$; thus the acute angle is $\theta = \text{arc}\cos 0.5819 = 54^\circ 25'$.

13. Let R be the distance from a fixed point $A(a,b,c)$ to any point $P(x,y,z)$. Show that ∇R is a vector in the direction $\mathbf{AP} = \mathbf{R}$.

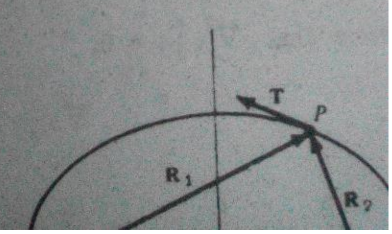
If $\mathbf{r}_A$ and $\mathbf{r}_P$ are the position vectors $a\,\mathbf{i} + b\,\mathbf{j} + c\,\mathbf{k}$ and $x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$ of A and P respectively, $\mathbf{R} = \mathbf{r}_P - \mathbf{r}_A = (x-a)\mathbf{i} + (y-b)\mathbf{j} + (z-c)\mathbf{k}$, so that $R = \sqrt{(x-a)^2+(y-b)^2+(z-c)^2}$. Then

$$\nabla R = \nabla\left(\sqrt{(x-a)^2 + (y-b)^2 + (z-c)^2}\right) = \frac{(x-a)\mathbf{i} + (y-b)\mathbf{j} + (z-c)\mathbf{k}}{\sqrt{(x-a)^2 + (y-b)^2 + (z-c)^2}} = \frac{\mathbf{R}}{R}$$

is a unit vector in the direction $\mathbf{R}$.

14. Let P be any point on an ellipse whose foci are at points A and B, as shown in the figure b Prove that lines AP and BP make equal angles with the tangent to the ellipse at P.

Let $\mathbf{R}_1 = \mathbf{AP}$ and $\mathbf{R}_2 = \mathbf{BP}$ denote vectors drawn respectively from foci A and B to point P on the ellipse, and let $\mathbf{T}$ be a unit tangent to the ellipse at P.



Since an ellipse is the locus of all points P the sum of whose distances from two fixed points A and B is a constant p, it is seen that the equation of the ellipse is

This is true exactly only in the limit as the parallelepiped shrinks to P, i.e. as $\Delta x, \Delta y$ and Δz approach zero. If there is no gain of fluid anywhere, then $\nabla \cdot \mathbf{v} = 0$. This is called the *continuity equation* for an incompressible fluid. Since fluid is neither created nor destroyed at any point, it is said to have no sources or sinks. A vector such as $\mathbf{v}$ whose divergence is zero is sometimes called *solenoidal*.

22. Determine the constant a so that the vector $\mathbf{V} = (x+3y)\mathbf{i} + (y-2z)\mathbf{j} + (x+az)\mathbf{k}$ is solenoidal.

A vector $\mathbf{V}$ is solenoidal if its divergence is zero (Problem 21).

$$\nabla \cdot \mathbf{V} = \frac{\partial}{\partial x}(x+3y) + \frac{\partial}{\partial y}(y-2z) + \frac{\partial}{\partial z}(x+az) = 1 + 1 + a$$

Then $\nabla \cdot \mathbf{V} = a + 2 = 0$ when $a = -2$.

THE CURL

23. If $\mathbf{A} = xz^3\,\mathbf{i} - 2x^2yz\,\mathbf{j} + 2yz^4\,\mathbf{k}$, find $\nabla \times \mathbf{A}$ (or curl $\mathbf{A}$) at the point $(1,-1,1)$.

$$\nabla \times \mathbf{A} = \left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}\right) \times (xz^3\,\mathbf{i} - 2x^2yz\,\mathbf{j} + 2yz^4\,\mathbf{k})$$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xz^3 & -2x^2yz & 2yz^4 \end{vmatrix}$$

$$= \left[\frac{\partial}{\partial y}(2yz^4) - \frac{\partial}{\partial z}(-2x^2yz)\right]\mathbf{i} + \left[\frac{\partial}{\partial z}(xz^3) - \frac{\partial}{\partial x}(2yz^4)\right]\mathbf{j} + \left[\frac{\partial}{\partial x}(-2x^2yz) - \frac{\partial}{\partial y}(xz^3\right.$$

$$= (2z^4 + 2x^2y)\mathbf{i} + 3xz^2\,\mathbf{j} - 4xyz\,\mathbf{k} = 3\mathbf{j} + 4\mathbf{k} \quad \text{at } (1,-1,1).$$

. If $\mathbf{A} = x^2y\,\mathbf{i} - 2xz\,\mathbf{j} + 2yz\,\mathbf{k}$, find curl curl $\mathbf{A}$.

$$\text{curl curl } \mathbf{A} = \nabla \times (\nabla \times \mathbf{A})$$

$$= \nabla \times \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2y & -2xz & 2yz \end{vmatrix} = \nabla \times [(2x+2z)\mathbf{i} - (x^2+2z)\mathbf{k}]$$

$(\mathbf{A}\times\mathbf{r})$ if $\nabla\times\mathbf{A} = 0$.

Let $\mathbf{A} = A_1\mathbf{i} + A_2\mathbf{j} + A_3\mathbf{k}$, $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$.

Then
$$\mathbf{A}\times\mathbf{r} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ A_1 & A_2 & A_3 \\ x & y & z \end{vmatrix}$$
$$= (zA_2 - yA_3)\mathbf{i} + (xA_3 - zA_1)\mathbf{j} + (yA_1 - xA_2)\mathbf{k}$$

and
$$\nabla\cdot(\mathbf{A}\times\mathbf{r}) = \frac{\partial}{\partial x}(zA_2 - yA_3) + \frac{\partial}{\partial y}(xA_3 - zA_1) + \frac{\partial}{\partial z}(yA_1 - xA_2)$$
$$= z\frac{\partial A_2}{\partial x} - y\frac{\partial A_3}{\partial x} + x\frac{\partial A_3}{\partial y} - z\frac{\partial A_1}{\partial y} + y\frac{\partial A_1}{\partial z} - x\frac{\partial A_2}{\partial z}$$
$$= x(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z}) + y(\frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x}) + z(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y})$$
$$= [x\mathbf{i} + y\mathbf{j} + z\mathbf{k}]\cdot[(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z})\mathbf{i} + (\frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x})\mathbf{j} + (\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y})\mathbf{k}]$$

$= \mathbf{r}\cdot(\nabla\times\mathbf{A}) = \mathbf{r}\cdot\text{curl }\mathbf{A}$. If $\nabla\times\mathbf{A} = 0$ this reduces to zero.

27. Prove: (a) $\nabla\times(\nabla\phi) = 0$ (curl grad $\phi = 0$), (b) $\nabla\cdot(\nabla\times\mathbf{A}) = 0$ (div curl $\mathbf{A} = 0$).

(a)
$$\nabla\times(\nabla\phi) = \nabla\times(\frac{\partial\phi}{\partial x}\mathbf{i} + \frac{\partial\phi}{\partial y}\mathbf{j} + \frac{\partial\phi}{\partial z}\mathbf{k})$$
$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial\phi}{\partial x} & \frac{\partial\phi}{\partial y} & \frac{\partial\phi}{\partial z} \end{vmatrix}$$
$$= [\frac{\partial}{\partial y}(\frac{\partial\phi}{\partial z}) - \frac{\partial}{\partial z}(\frac{\partial\phi}{\partial y})]\mathbf{i} + [\frac{\partial}{\partial z}(\frac{\partial\phi}{\partial x}) - \frac{\partial}{\partial x}(\frac{\partial\phi}{\partial z})]\mathbf{j} + [\frac{\partial}{\partial x}(\frac{\partial\phi}{\partial y}) - \frac{\partial}{\partial y}(\frac{\partial\phi}{\partial x})]\mathbf{k}$$
$$= (\frac{\partial^2\phi}{\partial y\,\partial z} - \frac{\partial^2\phi}{\partial z\,\partial y})\mathbf{i} + (\frac{\partial^2\phi}{\partial z\,\partial x} - \frac{\partial^2\phi}{\partial x\,\partial z})\mathbf{j} + (\frac{\partial^2\phi}{\partial x\,\partial y} - \frac{\partial^2\phi}{\partial y\,\partial x})\mathbf{k} = 0$$

provided we assume that ϕ has continuous second partial derivatives so that the order of differentiation i immaterial.

(b)
$$\nabla\cdot(\nabla\times\mathbf{A}) = \nabla\cdot\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_1 & A_2 & A_3 \end{vmatrix}$$
$$= \nabla\cdot[(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z})\mathbf{i} + (\frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x})\mathbf{j} + (\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y})\mathbf{k}]$$
$$= \frac{\partial}{\partial x}(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z}) + \frac{\partial}{\partial y}(\frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x}) + \frac{\partial}{\partial z}(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y})$$

$$= \frac{\partial^2 A_3}{\partial x\,\partial y} - \frac{\partial^2 A_2}{\partial x\,\partial z} + \frac{\partial^2 A_1}{\partial y\,\partial z} - \frac{\partial^2 A_3}{\partial y\,\partial x} + \frac{\partial^2 A_2}{\partial z\,\partial x} - \frac{\partial^2 A_1}{\partial z\,\partial y} = 0$$

assuming that $\mathbf{A}$ has continuous second partial derivatives.

Note the similarity between the above results and the results $(\mathbf{C}\times\mathbf{C}m) = (\mathbf{C}\times\mathbf{C})m = 0$, where m is a scalar and $\mathbf{C}\cdot(\mathbf{C}\times\mathbf{A}) = (\mathbf{C}\times\mathbf{C})\cdot\mathbf{A} = 0$.

28. Find $\operatorname{curl}(\mathbf{r}\,f(r))$ where $f(r)$ is differentiable.

$$\operatorname{curl}(\mathbf{r}\,f(r)) = \nabla\times(\mathbf{r}\,f(r))$$

$$= \nabla\times(x\,f(r)\mathbf{i} + y\,f(r)\mathbf{j} + z\,f(r)\mathbf{k})$$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x\,f(r) & y\,f(r) & z\,f(r) \end{vmatrix}$$

$$= (z\frac{\partial f}{\partial y} - y\frac{\partial f}{\partial z})\mathbf{i} + (x\frac{\partial f}{\partial z} - z\frac{\partial f}{\partial x})\mathbf{j} + (y\frac{\partial f}{\partial x} - x\frac{\partial f}{\partial y})\mathbf{k}$$

But $\dfrac{\partial f}{\partial x} = (\dfrac{\partial f}{\partial r})(\dfrac{\partial r}{\partial x}) = \dfrac{\partial f}{\partial r}\dfrac{\partial}{\partial x}(\sqrt{x^2+y^2+z^2}) = \dfrac{f'(r)\,x}{\sqrt{x^2+y^2+z^2}} = \dfrac{f'x}{r}$. Similarly, $\dfrac{\partial f}{\partial y} = \dfrac{f'y}{r}$ and $\dfrac{\partial f}{\partial z} = \dfrac{f'z}{r}$.

Then the result $= (z\dfrac{f'y}{r} - y\dfrac{f'z}{r})\mathbf{i} + (x\dfrac{f'z}{r} - z\dfrac{f'x}{r})\mathbf{j} + (y\dfrac{f'x}{r} - x\dfrac{f'y}{r})\mathbf{k} = \mathbf{0}$.

29. Prove $\nabla\times(\nabla\times\mathbf{A}) = -\nabla^2\mathbf{A} + \nabla(\nabla\cdot\mathbf{A})$.

$$\nabla\times(\nabla\times\mathbf{A}) = \nabla\times\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ A_1 & A_2 & A_3 \end{vmatrix}$$

$$= \nabla\times\left[(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z})\mathbf{i} + (\frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x})\mathbf{j} + (\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y})\mathbf{k}\right]$$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \dfrac{\partial A_3}{\partial y} - \dfrac{\partial A_2}{\partial z} & \dfrac{\partial A_1}{\partial z} - \dfrac{\partial A_3}{\partial x} & \dfrac{\partial A_2}{\partial x} - \dfrac{\partial A_1}{\partial y} \end{vmatrix}$$

$$= \left[\frac{\partial}{\partial y}(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y}) - \frac{\partial}{\partial z}(\frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x})\right]\mathbf{i}$$

$$+ \left[\frac{\partial}{\partial z}(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z}) - \frac{\partial}{\partial x}(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y})\right]\mathbf{j}$$

$$+ \left[\frac{\partial}{\partial x}(\frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x}) - \frac{\partial}{\partial y}(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z})\right]\mathbf{k}$$

$$= \left(-\frac{\partial^2 A_1}{\partial y^2} - \frac{\partial^2 A_1}{\partial z^2}\right)\mathbf{i} + \left(-\frac{\partial^2 A_2}{\partial z^2} - \frac{\partial^2 A_2}{\partial x^2}\right)\mathbf{j} + \left(-\frac{\partial^2 A_3}{\partial x^2} - \frac{\partial^2 A_3}{\partial y^2}\right)\mathbf{k}$$

$$+ \left(\frac{\partial^2 A_2}{\partial y\,\partial x} + \frac{\partial^2 A_3}{\partial z\,\partial x}\right)\mathbf{i} + \left(\frac{\partial^2 A_3}{\partial z\,\partial y} + \frac{\partial^2 A_1}{\partial x\,\partial y}\right)\mathbf{j} + \left(\frac{\partial^2 A_1}{\partial x\,\partial z} + \frac{\partial^2 A_2}{\partial y\,\partial z}\right)\mathbf{k}$$

$$= \left(-\frac{\partial^2 A_1}{\partial x^2} - \frac{\partial^2 A_1}{\partial y^2} - \frac{\partial^2 A_1}{\partial z^2}\right)\mathbf{i} + \left(-\frac{\partial^2 A_2}{\partial x^2} - \frac{\partial^2 A_2}{\partial y^2} - \frac{\partial^2 A_2}{\partial z^2}\right)\mathbf{j} + \left(-\frac{\partial^2 A_3}{\partial x^2} - \frac{\partial^2 A_3}{\partial y^2} - \frac{\partial^2 A_3}{\partial z^2}\right)\mathbf{k}$$

$$+ \left(\frac{\partial^2 A_1}{\partial x^2} + \frac{\partial^2 A_2}{\partial y\,\partial x} + \frac{\partial^2 A_3}{\partial z\,\partial x}\right)\mathbf{i} + \left(\frac{\partial^2 A_1}{\partial x\,\partial y} + \frac{\partial^2 A_2}{\partial y^2} + \frac{\partial^2 A_3}{\partial z\,\partial y}\right)\mathbf{j} + \left(\frac{\partial^2 A_1}{\partial x\,\partial z} + \frac{\partial^2 A_2}{\partial y\,\partial z} + \frac{\partial^2 A_3}{\partial z^2}\right)\mathbf{k}$$

$$= -\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)(A_1\mathbf{i} + A_2\mathbf{j} + A_3\mathbf{k})$$

$$+ \mathbf{i}\frac{\partial}{\partial x}\left(\frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z}\right) + \mathbf{j}\frac{\partial}{\partial y}\left(\frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z}\right) + \mathbf{k}\frac{\partial}{\partial z}\left(\frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z}\right)$$

$$= -\nabla^2\mathbf{A} + \nabla\left(\frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z}\right)$$

$$= -\nabla^2\mathbf{A} + \nabla(\nabla\cdot\mathbf{A})$$

If desired, the labour of writing can be shortened in this as well as other derivations by writing only the $\mathbf{i}$ components since the others can be obtained by symmetry.

The result can also be established formally as follows. From Problem 47(a), Chapter 2,

$$(1) \qquad \mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = \mathbf{B}(\mathbf{A}\cdot\mathbf{C}) - (\mathbf{A}\cdot\mathbf{B})\mathbf{C}$$

Placing $\mathbf{A} = \mathbf{B} = \nabla$ and $\mathbf{C} = \mathbf{F}$,

$$\nabla\times(\nabla\times\mathbf{F}) = \nabla(\nabla\cdot\mathbf{F}) - (\nabla\cdot\nabla)\mathbf{F} = \nabla(\nabla\cdot\mathbf{F}) - \nabla^2\mathbf{F}$$

Note that the formula (1) must be written so that the operators **A** and **B** precede the operand **C**, otherwise the formalism fails to apply.

30. If $\mathbf{v} = \boldsymbol{\omega}\times\mathbf{r}$, prove $\boldsymbol{\omega} = \frac{1}{2}\,\text{curl}\,\mathbf{v}$ where $\boldsymbol{\omega}$ is a constant vector.

$$\text{curl}\,\mathbf{v} = \nabla\times\mathbf{v} = \nabla\times(\boldsymbol{\omega}\times\mathbf{r}) = \nabla\times\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ \omega_1 & \omega_2 & \omega_3\\ x & y & z\end{vmatrix}$$

$$= \nabla\times\left[(\omega_2 z - \omega_3 y)\mathbf{i} + (\omega_3 x - \omega_1 z)\mathbf{j} + (\omega_1 y - \omega_2 x)\mathbf{k}\right]$$

$$= \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z}\\ \omega_2 z - \omega_3 y & \omega_3 x - \omega_1 z & \omega_1 y - \omega_2 x\end{vmatrix} = 2(\omega_1\mathbf{i} + \omega_2\mathbf{j} + \omega_3\mathbf{k}) = 2\boldsymbol{\omega}.$$

Then $\omega = \frac{1}{2}\nabla \times \mathbf{v} = \frac{1}{2}\,\text{curl}\,\mathbf{v}$.

This problem indicates that the curl of a vector field has something to do with rotational properties of the field. This is confirmed in Chapter 6. If the field $\mathbf{F}$ is that due to a moving fluid, for example, then a paddle wheel placed at various points in the field would tend to rotate in regions where $\text{curl}\,\mathbf{F} \neq \mathbf{0}$, while if $\text{curl}\,\mathbf{F} = \mathbf{0}$ in the region there would be no rotation and the field $\mathbf{F}$ is then called *irrotational*. A field which is not irrotational is sometimes called a *vortex field*.

31. If $\nabla\cdot\mathbf{E} = 0$, $\nabla\cdot\mathbf{H} = 0$, $\nabla\times\mathbf{E} = -\frac{\partial \mathbf{H}}{\partial t}$, $\nabla\times\mathbf{H} = \frac{\partial \mathbf{E}}{\partial t}$, show that $\mathbf{E}$ and $\mathbf{H}$ satisfy $\nabla^2 u = \frac{\partial^2 u}{\partial t^2}$.

$$\nabla\times(\nabla\times\mathbf{E}) = \nabla\times\left(-\frac{\partial \mathbf{H}}{\partial t}\right) = -\frac{\partial}{\partial t}(\nabla\times\mathbf{H}) = -\frac{\partial}{\partial t}\left(\frac{\partial \mathbf{E}}{\partial t}\right) = -\frac{\partial^2 \mathbf{E}}{\partial t^2}$$

By Problem 29, $\nabla\times(\nabla\times\mathbf{E}) = -\nabla^2\mathbf{E} + \nabla(\nabla\cdot\mathbf{E}) = -\nabla^2\mathbf{E}$. Then $\nabla^2\mathbf{E} = \frac{\partial^2 \mathbf{E}}{\partial t^2}$.

Similarly, $\nabla\times(\nabla\times\mathbf{H}) = \nabla\times\left(\frac{\partial \mathbf{E}}{\partial t}\right) = \frac{\partial}{\partial t}(\nabla\times\mathbf{E}) = \frac{\partial}{\partial t}\left(-\frac{\partial \mathbf{H}}{\partial t}\right) = -\frac{\partial^2 \mathbf{H}}{\partial t^2}$.

But $\nabla\times(\nabla\times\mathbf{H}) = -\nabla^2\mathbf{H} + \nabla(\nabla\cdot\mathbf{H}) = -\nabla^2\mathbf{H}$. Then $\nabla^2\mathbf{H} = \frac{\partial^2 \mathbf{H}}{\partial t^2}$.

The given equations are related to *Maxwell's equations of electromagnetic theory*. The equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = \frac{\partial^2 u}{\partial t^2}$ is called the *wave equation*.

MISCELLANEOUS PROBLEMS.

32. (*a*) A vector $\mathbf{V}$ is called irrotational if $\text{curl}\,\mathbf{V} = \mathbf{0}$ (see Problem 30). Find constants a,b,c so that

$$\mathbf{V} = (x + 2y + az)\mathbf{i} + (bx - 3y - z)\mathbf{j} + (4x + cy + 2z)\mathbf{k}$$

is irrotational.

(*b*) Show that $\mathbf{V}$ can be expressed as the gradient of a scalar function.

$$(a)\ \text{curl}\,\mathbf{V} = \nabla\times\mathbf{V} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x+2y+az & bx-3y-z & 4x+cy+2z \end{vmatrix} = (c+1)\mathbf{i} + (a-4)\mathbf{j} + (b-2)\mathbf{k}$$

This equals zero when $a = 4$, $b = 2$, $c = -1$ and

$$\mathbf{V} = (x + 2y + 4z)\mathbf{i} + (2x - 3y - z)\mathbf{j} + (4x - y + 2z)\mathbf{k}$$

(*b*) Assume $\mathbf{V} = \nabla\phi = \frac{\partial\phi}{\partial x}\mathbf{i} + \frac{\partial\phi}{\partial y}\mathbf{j} + \frac{\partial\phi}{\partial z}\mathbf{k}$.

Then (*1*) $\frac{\partial\phi}{\partial x} = x + 2y + 4z$, (*2*) $\frac{\partial\phi}{\partial y} = 2x - 3y - z$, (*3*) $\frac{\partial\phi}{\partial z} = 4x - y + 2z$.

Integrating (*1*) partially with respect to x, keeping y and z constant,

$$(4)\qquad \phi = \frac{x^2}{2} + 2xy + 4xz + f(y,z)$$

where $f(y,z)$ is an arbitrary function of y and z. Similarly from (2) and (3),

(5) $$\phi = 2xy - \frac{3y^2}{2} - yz + g(x,z)$$

(6) $$\phi = 4xz - yz + z^2 + h(x,y).$$

Comparison of (4), (5) and (6) shows that there will be a common value of ϕ if we choose

$$f(y,z) = -\frac{3y^2}{2} + z^2, \quad g(x,z) = \frac{x^2}{2} + z^2, \quad h(x,y) = \frac{x^2}{2} - \frac{3y^2}{2}$$

so that

$$\phi = \frac{x^2}{2} - \frac{3y^2}{2} + z^2 + 2xy + 4xz - yz$$

Note that we can also add any constant to ϕ. In general if $\nabla \times \mathbf{V} = \mathbf{0}$, then we can find ϕ so that $\mathbf{V} = \nabla\phi$. A vector field $\mathbf{V}$ which can be derived from a scalar field ϕ so that $\mathbf{V} = \nabla\phi$ is called a *conservative vector field* and ϕ is called the *scalar potential*. Note that conversely if $\mathbf{V} = \nabla\phi$, then $\nabla \times \mathbf{V} = \mathbf{0}$ (see Prob.27*a*).

33. Show that if $\phi(x,y,z)$ is any solution of Laplace's equation, then $\nabla\phi$ is a vector which is both solenoidal and irrotational.

By hypothesis, ϕ satisfies Laplace's equation $\nabla^2\phi = 0$, i.e. $\nabla\cdot(\nabla\phi) = 0$. Then $\nabla\phi$ is solenoidal (see Problems 21 and 22).

From Problem 27*a*, $\nabla \times (\nabla\phi) = \mathbf{0}$ so that $\nabla\phi$ is also irrotational.

34. Give a possible definition of grad B.

Assume $\mathbf{B} = B_1\mathbf{i} + B_2\mathbf{j} + B_3\mathbf{k}$. Formally, we can define grad B as

$$\begin{aligned}\nabla\mathbf{B} &= (\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k})(B_1\mathbf{i} + B_2\mathbf{j} + B_3\mathbf{k}) \\ &= \frac{\partial B_1}{\partial x}\mathbf{ii} + \frac{\partial B_2}{\partial x}\mathbf{ij} + \frac{\partial B_3}{\partial x}\mathbf{ik} \\ &\quad + \frac{\partial B_1}{\partial y}\mathbf{ji} + \frac{\partial B_2}{\partial y}\mathbf{jj} + \frac{\partial B_3}{\partial y}\mathbf{jk} \\ &\quad + \frac{\partial B_1}{\partial z}\mathbf{ki} + \frac{\partial B_2}{\partial z}\mathbf{kj} + \frac{\partial B_3}{\partial z}\mathbf{kk}\end{aligned}$$

The quantities $\mathbf{ii}$, $\mathbf{ij}$, etc., are called *unit dyads*. (Note that $\mathbf{ij}$, for example, is not the same as $\mathbf{ji}$.) A quantity of the form

$$a_{11}\mathbf{ii} + a_{12}\mathbf{ij} + a_{13}\mathbf{ik} + a_{21}\mathbf{ji} + a_{22}\mathbf{jj} + a_{23}\mathbf{jk} + a_{31}\mathbf{ki} + a_{32}\mathbf{kj} + a_{33}\mathbf{kk}$$

is called a *dyadic* and the coefficients $a_{11}, a_{12}, \ldots$ are its *components*. An array of these nine components in the form

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

is called a 3 by 3 *matrix*. A dyadic is a generalization of a vector. Still further generalization leads to

42. If $\phi = 2xz^4 - x^2y$, find $\nabla\phi$ and $|\nabla\phi|$ at the point $(2,-2,-1)$. *Ans.* $10\mathbf{i} - 4\mathbf{j} - 16\mathbf{k}$, $2\sqrt{93}$

43. If $\mathbf{A} = 2x^2\mathbf{i} - 3yz\mathbf{j} + xz^2\mathbf{k}$ and $\phi = 2z - x^3y$, find $\mathbf{A}\cdot\nabla\phi$ and $\mathbf{A}\times\nabla\phi$ at the point $(1,-1,1)$.
Ans. 5, $7\mathbf{i} - \mathbf{j} - 11\mathbf{k}$

44. If $F = x^2z + e^{y/x}$ and $G = 2z^2y - xy^2$, find (a) $\nabla(F+G)$ and (b) $\nabla(FG)$ at the point $(1,0,-2)$.
Ans. (a) $-4\mathbf{i} + 9\mathbf{j} + \mathbf{k}$, (b) $-8\mathbf{j}$

45. Find $\nabla|\mathbf{r}|^3$. *Ans.* $3r\mathbf{r}$

46. Prove $\nabla f(r) = \dfrac{f'(r)\mathbf{r}}{r}$.

47. Evaluate $\nabla(3r^2 - 4\sqrt{r} + \dfrac{6}{\sqrt[3]{r}})$. *Ans.* $(6 - 2r^{-3/2} - 2r^{-7/3})\mathbf{r}$

48. If $\nabla U = 2r^4\mathbf{r}$, find U. *Ans.* $r^6/3$ + constant

49. Find $\phi(r)$ such that $\nabla\phi = \dfrac{\mathbf{r}}{r^5}$ and $\phi(1) = 0$. *Ans.* $\phi(r) = \dfrac{1}{3}(1 - \dfrac{1}{r^3})$

50. Find $\nabla\psi$ where $\psi = (x^2+y^2+z^2)\,e^{-\sqrt{x^2+y^2+z^2}}$. *Ans.* $(2-r)\,e^{-r}\,\mathbf{r}$

51. If $\nabla\phi = 2xyz^3\mathbf{i} + x^2z^3\mathbf{j} + 3x^2yz^2\mathbf{k}$, find $\phi(x,y,z)$ if $\phi(1,-2,2) = 4$. *Ans.* $\phi = x^2yz^3 + 20$

52. If $\nabla\psi = (y^2 - 2xyz^3)\mathbf{i} + (3 + 2xy - x^2z^3)\mathbf{j} + (6z^3 - 3x^2yz^2)\mathbf{k}$, find ψ.
Ans. $\psi = xy^2 - x^2yz^3 + 3y + (3/2)z^4$ + constant

53. If U is a differentiable function of x,y,z, prove $\nabla U\cdot d\mathbf{r} = dU$.

54. If F is a differentiable function of x,y,z,t where x,y,z are differentiable functions of t, prove that

$$\frac{dF}{dt} = \frac{\partial F}{\partial t} + \nabla F\cdot\frac{d\mathbf{r}}{dt}$$

55. If $\mathbf{A}$ is a constant vector, prove $\nabla(\mathbf{r}\cdot\mathbf{A}) = \mathbf{A}$.

56. If $\mathbf{A}(x,y,z) = A_1\mathbf{i} + A_2\mathbf{j} + A_3\mathbf{k}$, show that $d\mathbf{A} = (\nabla A_1\cdot d\mathbf{r})\mathbf{i} + (\nabla A_2\cdot d\mathbf{r})\mathbf{j} + (\nabla A_3\cdot d\mathbf{r})\mathbf{k}$.

57. Prove $\nabla(\dfrac{F}{G}) = \dfrac{G\nabla F - F\nabla G}{G^2}$ if $G \neq 0$.

58. Find a unit vector which is perpendicular to the surface of the paraboloid of revolution $z = x^2 + y^2$ at the point $(1,2,5)$. *Ans.* $\dfrac{2\mathbf{i} + 4\mathbf{j} - \mathbf{k}}{\pm\sqrt{21}}$

59. Find the unit outward drawn normal to the surface $(x-1)^2 + y^2 + (z+2)^2 = 9$ at the point $(3,1,-4)$.
Ans. $(2\mathbf{i} + \mathbf{j} - 2\mathbf{k})/3$

60. Find an equation for the tangent plane to the surface $xz^2 + x^2y = z - 1$ at the point $(1,-3,2)$.
Ans. $2x - y - 3z + 1 = 0$

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61. Find equations for the tangent plane and normal line to the surface $z = x^2 + y^2$ at the point $(2,-1,5)$.
Ans. $4x - 2y - z = 5$, $\dfrac{x-2}{4} = \dfrac{y+1}{-2} = \dfrac{z-5}{-1}$ or $x = 4t+2$, $y = -2t-1$, $z = -t+5$

62. Find the directional derivative of $\phi = 4xz^3 - 3x^2y^2z$ at $(2,-1,2)$ in the direction $2\mathbf{i} - 3\mathbf{j} + 6\mathbf{k}$.
Ans. $376/7$

63. Find the directional derivative of $P = 4e^{2x-y+z}$ at the point $(1,1,-1)$ in a direction toward the point $(-3,5,6)$. *Ans.* $-20/9$