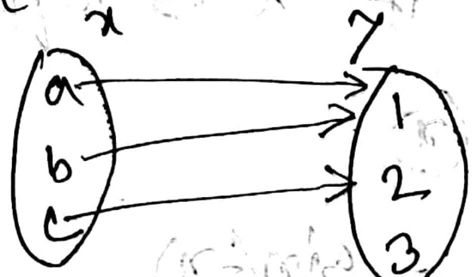
Functions

A function $f: x \to y$ is a correspondence by which each element of x correspond to a unique element of y.



co-domain $= \{1, 2, 3\}$

Domain $= \{a, b, c\}$

Range $= \{1, 2\}$

*** If $y = f(x)$ be a function of x the set of all possible value of x defined as the domain of the function. And the set of all possible values of y i.e. the image point of x is defined as the range of the function.

*** Inverse function:

$y = f(x)$

i.e. $x = g(y)$

$\therefore$ ~~$x = g(f(x))$~~ $\therefore y = f(g(y))$

Odd and even function:

A function is said to be odd if it changes sign with the change of the variable.

Let $f(x)$ be a function if x is replaced by $-x$, then $f(-x) = -f(x)$

Ex: $f(x) = \sin x$, $f(-x) = \sin(-x)$
$= -\sin x = -f(x)$

A function is called an even function ~~of the function~~ if it doesn't change sign with the change of the variable.

Let $f(x)$ be a function if we replace x by $-x$ then $f(-x) = f(x)$

Ex: $f(x) = \cos x$

$\therefore f(-x) = \cos(-x)$
$= \cos x = f(x)$

Continuous and discontinuous function:

A function is said to be continuous for an interval if it has finite value for every value of the variable in the given interval.

Ex: $y = x$

If the function is undefined for any value of the variable in the interval. Then the function is said to be discontinuous for that value of the variable in the interval.

Ex: $y = \frac{\sin x}{x}$

*** Find the domain and range of the function where

$y = f(x) = 1 - x^2$, $x \in R$, $y \in R$

Sol: y is real for any value of x

$\therefore D_f = \{x : x \in R\} = (-\infty, \infty)$

since $x^2 \geq 0$ therefore the maximum value of y is 1.

Here R_f = range of $f = \{y : y \leq 1\} = (-\infty, 1]$

$\boxed{***}$ Find the domain and range of f is given by $y = f(x) = \frac{x-1}{2x-3}$ where $x \in R, y \in R$

Sol: Let $2x-3 = 0$

$2x = 3$

$x = \frac{3}{2}$

$\therefore y = \frac{1/2}{0}$ is not defined

$\therefore D_f = \{x : x \in R$ but $x \neq 3/2\}$

$= (-\infty, 3/2) \cup (3/2, \infty)$

Again solving for x

$y = \frac{x-1}{2x-3} \Rightarrow y(2x-3) = x-1$

$\Rightarrow 2xy - 3y = x - 1$

$\Rightarrow 2xy - x = 3y - 1$

$\Rightarrow x(2y-1) = 3y-1$

$\therefore x = \dfrac{3y-1}{2y-1}$

showing that x is undefined of $2y-1=0$

$\therefore y = \frac{1}{2}$

Hence $R_f = \{y : y \in R$ but $y \neq \frac{1}{2}\}$

$= (-\infty, \frac{1}{2}) \cup (\frac{1}{2}, \infty)$

$= R - \{\frac{1}{2}\}$

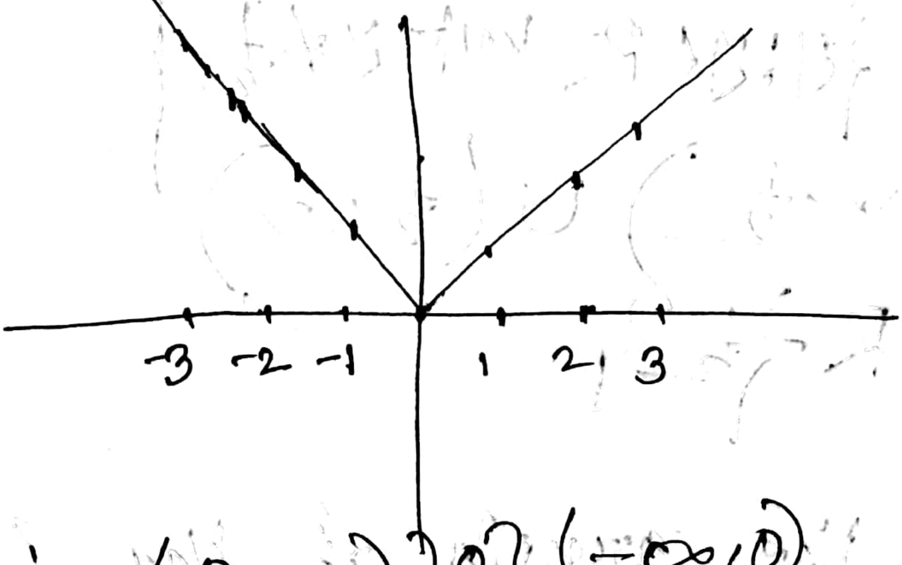
$\boxed{***}$ Draw the graph of the following function $f(x) = \begin{cases} x & \text{when } x > 0 \\ 0 & \text{"} \quad x = 0 \\ -x & \text{"} \quad x < 0 \end{cases}$ $y = |x|$

Sol:

$$y = f(x) = \begin{cases} x, & x > 0 \\ 0, & x = 0 \\ -x, & x < 0 \end{cases} \Bigg\} y = |x|$$

we can form the following table.

x	3	2	1	0	−1	−2	−3
y	3	2	1	0	1	2	3



Domain $(0, \infty) \cup \{0\} \cup (-\infty, 0)$

$= (-\infty, \infty)$

Range $= [0, \infty)$

$\boxed{***}$ $f(x) = \begin{cases} 3+2x & \text{when } -3/2 \le x < 0 \\ 3-2x & \text{when } 0 \le x < 3/2 \\ 3+2x & \text{when } 3/2 \le x \end{cases}$

graph = ?

Find the domain and range of the above function

solve:

Domain of $f = [-3/2, 0) \cup [0, \frac{3}{2}) \cup [3/2, \infty)$

$= [-3/2, \infty)$

Range of $f = [0, 3) \cup [3, 0) \cup [6, \infty)$

$= [0, 3] \cup [6, \infty)$

$\boxed{***}$ Draw the graph of the following function.

$$f(x) = \begin{cases} 2x+6 & \text{when } -3 \le x < 0 \\ 6 & \text{when } 0 < x < 2 \\ 2x-6 & \text{when } 2 \le x < 5 \end{cases}$$

Find the domain and range of the above function.

Domain, $f = [-3, 0) \cup (0, 2) \cup [2, 5]$

$= [-3, 5]$

Range of $f = [0, 6] \cup \{6\} \cup [-2, 4]$

$= [-2, 6]$.

Differentiation

$\boxed{***}$ A function is defined in the following way $f(x) = |x|$

show that $f'(0)$ doesn't exist.

given $f(x) = |x|$

$f(x) = x$ when $x > 0$

$f(x) = -x$ when $x < 0$

$= 0$ " $x = 0$

$$L.H.D = \underset{h \to 0}{Lt} \frac{f(0-h) - f(0)}{-h} = \underset{h \to 0}{Lt} \frac{f(-h) - f(0)}{-h}$$

$$= \underset{h \to 0}{Lt} \frac{h}{-h} = -1$$

$$R.H.D = \underset{h \to 0}{Lt} \frac{f(0+h) - f(0)}{h}$$

$$= \underset{h \to 0}{Lt} \frac{f(h)}{h} = 1$$

$\therefore L.H.D \neq R.H.D$. Hence $f'(0)$ doesn't exist.

7. $f(x) = 3+2x$ for $-3/2 < x \leq 0$

$f(x) = 3-2x$ for $0 < x < 3/2$

show that $f(x)$ is continious at $x=0$ but $f'(0)$ doesn't exist.

L.H.L $\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} 3+2x$

$= 3$

R.H.L. $\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} 3-2x$ $\cancel{3-0}$

$= 3$

F.V when $x=0$

$f(0) = 3+2x = 3$

$\therefore$ L.H.L = R.H.L = F.V

$f(x)$ is continuous.

Let.

$$f'(x) = \underset{h\to 0}{Lt} \frac{f(x+h) - f(x)}{h}$$

R.H.D

$$\therefore f'(0) = \underset{h\to 0}{Lt} \frac{f(0+h) - f(0)}{h}$$

$$= \underset{h\to 0}{Lt} \frac{f(h) - f(0)}{h}$$

$f(x) = 3-2x$
$f(h) = 3-2h$
$f(0) = 3$

$$= \underset{h\to 0}{Lt} \frac{3-2h-3}{h}$$

$$= -2$$

L.H.D

$$\underset{h\to 0}{Lt} \frac{f(0-h) - f(0)}{-h}$$

$f(x) = 3+2x$
$= 3+2(-h)$
$= 3-2h$

$$= \underset{h\to 0}{Lt} \frac{3-2h-3}{-h} = 2$$

$\therefore$ LHD $\neq$ R.H.D

$\therefore f'(0)$ doesnot exist.

8. $f(x) = 1$ for $x < 0$ [confusion আছে]

$= 1 + \sin x$ for $0 \le x < \frac{\pi}{2}$

$= 2 + \left(x - \frac{\pi}{2}\right)^2$ for $\frac{\pi}{2} \le x$

Show that $f'(x)$ ~~doesnot~~ exist at $x = \frac{\pi}{2}$ but doesn't exist at $x = 0$

Sol:

L.H.D $\lim_{h \to 0} \dfrac{f\left(\frac{\pi}{2} - h\right) - f\left(\frac{\pi}{2}\right)}{-h}$

$= \dfrac{1 + \sin\left(\frac{\pi}{2} - h\right) - 2}{-h}$

$\lim_{h \to 0} \dfrac{1 + \cos h - 2}{-h} = \lim_{h \to 0} \dfrac{-1 + \cos h}{-h}$

$= \lim_{h \to 0} \dfrac{2 \sin^2 h/2}{h}$

$= \lim_{h \to 0} 2 \left(\dfrac{\sin h/2}{\frac{h}{2}}\right)^2 \cdot \dfrac{h}{4}$

$= 0$

R.H.D.

$$\underset{h\to 0}{Lt} \frac{f\left(\frac{\pi}{2}+h\right) - f\left(\frac{\pi}{2}\right)}{h}$$

$$= \underset{h\to 0}{Lt} \frac{2+h^2-2}{h} = 0$$

$\therefore$ L.H.D = R.H.D

$\therefore f'(x)$ exist at $x = \frac{\pi}{2}$

again

L.H.D

$$\underset{h\to 0^-}{Lt} \frac{f(0-h) - f(0)}{-h}$$

$$= \underset{h\to 0}{Lt} \frac{1-\sin h - 1}{-h}$$

$$= \underset{h\to 0}{Lt} \frac{1-1}{-h} = 0$$

R.H.D $\underset{h\to 0}{Lt} \frac{f(0+h) - f(0)}{h}$

$$= \underset{h\to 0}{Lt} \frac{1+\sin h - 1}{h}$$

$$= \underset{h\to 0}{Lt} \frac{\sin h}{h} = 1$$

ie L.H.D $\neq$ R.H.D

$\therefore f'(x)$ doesn't exist.

Differentiation

Q.

(i) x^x

Let $y = x^x$

$\log y = \log x^x$

$\log y = x \log x$

$\Rightarrow \frac{1}{y} \cdot \frac{dy}{dx} = x \cdot \frac{1}{x} + \log x$

$\Rightarrow \frac{dy}{dx} = y(1 + \log x)$

$= x^x (1 + \log x)$

(i) x^{e^x}

Let $y = x^{e^x}$

$\Rightarrow \log y = \log x^{e^x}$

$\log y = e^x \log x$

$\log y$

$$\Rightarrow \frac{1}{y}\cdot\frac{dy}{dx} = e^x\cdot\frac{1}{x} + \log x\, e^x$$

$$\Rightarrow \frac{dy}{dx} = y\left(\frac{e^x}{x} + e^x\log x\right)$$

$$= x^{e^x}\left(\frac{e^x}{x} + e^x\log x\right)$$

$$= x^{e^x}\cdot e^x\left(\frac{1}{x} + \log x\right)$$

(xii) x^{x^x}

Let $y = x^{x^x}$

$$\Rightarrow \log y = \log x^{x^x}$$

$$\Rightarrow \log y = x^x \log x$$

$$\Rightarrow \log(\log y) = \log(x^x \log x)$$

$$\Rightarrow \log(\log y) = \log x^x + \log(\log x)$$

$$\Rightarrow \log(\log y) = x\log x + \log(\log x)$$

$$\Rightarrow \frac{1}{\log y}\cdot\frac{1}{y}\cdot\frac{dy}{dx} = x\cdot\frac{1}{x} + \log x + \frac{1}{\log x}\cdot\frac{1}{x}$$

$$= 1 + \log x + \frac{1}{x\log x}$$

$$\Rightarrow \frac{dy}{dx} = y\log y\left(1 + \log x + \frac{1}{x\log x}\right)$$

$= x^{x^x} \cdot x^x \log x \left(1 + \log x + \frac{1}{x \log x}\right)$

(xiii) $(\sin x)^{\cos x} + (\cos x)^{\sin x}$

Let $y = (\sin x)^{\cos x} + (\cos x)^{\sin x}$

$y = u + v$

$\left[\begin{array}{l} u = (\sin x)^{\cos x} \\ v = (\cos x)^{\sin x} \end{array}\right.$

$\therefore \frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}$

Now;

$u = (\sin x)^{\cos x}$

$\Rightarrow \log u = \log (\sin x)^{\cos x}$

$= \cos x \log(\sin x)$

$\Rightarrow \frac{1}{u} \cdot \frac{du}{dx} = \cos x \cdot \frac{1}{\sin x} \cos x + \log(\sin x)(-\sin x)$

$\Rightarrow \frac{1}{u} \cdot \frac{du}{dx} = \cos x \cot x - \sin x \log(\sin x)$

$$\Rightarrow \frac{du}{dx} = (\sin x)^{\cos x}\left[\cos x \cot x - \sin x \log(\sin x)\right]$$

Then,

$$v = (\cos x)^{\sin x}$$

$$\Rightarrow \log v = \log(\cos x)^{\sin x}$$

$$\Rightarrow \log v = \sin x \log(\cos x)$$

$$\Rightarrow \frac{1}{v}\cdot\frac{dv}{dx} = \sin x \cdot \frac{1}{\cos x}(-\sin x) + \frac{\log(\cos x)}{\cos x}$$

$$\Rightarrow \frac{dv}{dx} = (\cos x)^{\sin x}\left[\cos x \log(\cos x) - \sin x \tan x\right]$$

$$\therefore \frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}$$

$$= (\sin x)^{\cos x}\left[\cos x \cot x - \sin x \log(\sin x)\right] + (\cos x)^{\sin x}\left[\cos x \log(\cos x) - \sin x \tan x\right]$$

(v) Let $y = \log\left[e^x\left(\frac{x-1}{x+1}\right)^{3/2}\right]$

$= \log e^x + \log\left(\frac{x-1}{x+1}\right)^{3/2}$

$\Rightarrow \frac{dy}{dx} = 1 + \frac{3}{2}\log(x-1) - \frac{3}{2}\log(x+1)$

$= 1 + \frac{3}{2}\left(\frac{1}{x-1}\right) - \frac{3}{2}\left(\frac{1}{x+1}\right)$

$\frac{dy}{dx} = \cancel{\frac{x^2+2}{x^2-1}} = \frac{x^2+2}{x^2-1}$

10.

(viii) $x^p y^q = (x+y)^{p+q}$

Taking log on bothsides, we get

$p\log x + q\log y = (p+q)\log(x+y)$

Diff both sides with respect to x,

$p\cdot\frac{1}{x} + q\cdot\frac{1}{y}\cdot\frac{dy}{dx} = (p+q)\cdot\frac{1}{x+y}\left(1+\frac{dy}{dx}\right)$

$\Rightarrow \cancel{\frac{dy}{dx}\left\{\frac{p}{\ }\right.}$

$$\Rightarrow \frac{p}{x} + \frac{q}{y}\cdot\frac{dy}{dx} = \frac{p+q}{x+y} + \frac{p+q}{x+y}\left(\frac{dy}{dx}\right)$$

$$\Rightarrow \frac{dy}{dx}\left(\frac{q}{y} - \frac{p+q}{x+y}\right) = \frac{p+q}{x+y} - \frac{p}{x}$$

$$\Rightarrow \frac{dy}{dx}\left\{\frac{q(x+y)-y(p+q)}{y(x+y)}\right\} = \frac{(p+q)x-(x+y)p}{(x+y)x}$$

$$\Rightarrow \frac{dy}{dx}\left\{\frac{qx+\cancel{qy}-py-\cancel{qy}}{y(x+y)}\right\} = \frac{\cancel{px}+qx-\cancel{px}-py}{(x+y)x}$$

$$\Rightarrow \frac{dy}{dx} = \frac{(qx-py)\,y\,(x+y)}{x(x+y)\,(qx-py)}$$

$$\therefore \frac{dy}{dx} = \frac{y}{x}$$

(XI) $\tan y = \frac{2t}{1-t^2}$, $\sin x = \frac{2t}{1+t^2}$

~~$\Rightarrow y = \tan^{-1}\frac{2t}{1+t^2}$ $\therefore y$~~

$\Rightarrow y = \tan^{-1}\frac{2t}{1-t^2}$

put $t = \tan\theta$

$\therefore y = \tan^{-1} \dfrac{2\tan\theta}{1-\tan^2\theta}$

$y = \tan^{-1} \tan 2\theta$

$\Rightarrow y = 2\theta = 2\tan^{-1} t$

$\therefore \dfrac{dy}{dt} = 2 \cdot \dfrac{1}{1+t^2}$

then, $\sin x = \dfrac{2t}{1+t^2}$

$\Rightarrow x = \sin^{-1} \dfrac{2t}{1+t^2}$

$t = \tan\theta$

$\therefore x = \sin^{-1} \dfrac{2\tan\theta}{1+\tan^2\theta}$

$= \sin^{-1} \sin 2\theta$

$\Rightarrow x = 2\theta$

$\Rightarrow x = 2\tan^{-1} t$

$\therefore \dfrac{dx}{dt} = 2 \cdot \dfrac{1}{1+t^2}$

$$\therefore \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2/1+t^2}{2/1+t^2} = 1$$

13. Differentiate the left-side functions with respect to the right side ones:

(v) $\cos^{-1}\frac{1-x^2}{1+x^2}$ w.r.t $\tan^{-1}\frac{2x}{1-x^2}$

Let

$$y = \cos^{-1}\frac{1-x^2}{1+x^2}$$

put $x = \tan\theta$

$$y = \cos^{-1}\frac{1-\tan^2\theta}{1+\tan^2\theta}$$

$$y = \cos^{-1}\cos 2\theta$$

$$y = 2\theta$$

$$\Rightarrow y = 2\tan^{-1}x$$

$$\therefore \frac{dy}{dx} = 2\cdot\frac{1}{1+x^2}$$

~~put~~

Let $z = \tan^{-1}\frac{2x}{1-x^2}$

put $x = \tan\theta$

$z = \tan^{-1}\frac{2\tan\theta}{1-\tan^2\theta}$

$z = \tan^{-1}\tan 2\theta$

$\Rightarrow z = 2\theta$

$= 2\tan^{-1}x$

$\therefore \frac{dz}{dx} = 2\cdot\frac{1}{1+x^2}$

$\therefore \frac{dy}{dz} = \frac{dy/dx}{dz/dx} = 1$

(vi) $\tan^{-1}\frac{\sqrt{(1+x^2)}-1}{x}$ w.r.t $\tan^{-1}x$

Let

$$y = \tan^{-1}\frac{\sqrt{(1+x^2)}-1}{x}$$

put $x = \tan\theta$

$$y = \tan^{-1}\frac{\sqrt{(1+\tan^2\theta)}-1}{\tan\theta}$$

$$= \tan^{-1}\frac{\sec\theta-1}{\tan\theta}$$

$$= \tan^{-1}\frac{\frac{1}{\cos\theta}-1}{\tan\theta}$$

$$= \tan^{-1}\frac{\frac{1-\cos\theta}{\cos\theta}}{\frac{\sin\theta}{\cos\theta}} = \tan^{-1}\frac{1-\cos\theta}{\cos\theta}\cdot\frac{\cos\theta}{\sin\theta}$$

$$\therefore y = \tan^{-1}\frac{1-\cos\theta}{\sin\theta}$$

$$= \tan^{-1}\frac{2\sin^2\frac{\theta}{2}}{2\sin\frac{\theta}{2}\cdot\cos\frac{\theta}{2}}$$

$$y = \tan^{-1}\left(\tan\frac{\theta}{2}\right)$$

$\Rightarrow y = \frac{\theta}{2} = \frac{1}{2} \cdot \tan^{-1} x$

$\therefore y = \frac{1}{2} \cdot z \qquad [\tan^{-1} x = z]$

$\therefore \frac{dy}{dz} = \frac{1}{2}$ Ans.

14.

(viii) $\tan^{-1} \frac{\cos x}{1+\sin x}$

Let.

$$y = \tan^{-1} \frac{\cos^2\frac{x}{2} - \sin^2\frac{x}{2}}{\sin^2\frac{x}{2} + \cos^2\frac{x}{2} + 2\sin\frac{x}{2}\cdot\cos\frac{x}{2}}$$

$$= \tan^{-1} \frac{\cos^2\frac{x}{2} - \sin^2\frac{x}{2}}{\left(\sin\frac{x}{2} + \cos\frac{x}{2}\right)^2}$$

$$= \tan^{-1} \frac{\cos\frac{x}{2} - \sin\frac{x}{2}}{\sin\frac{x}{2} + \cos\frac{x}{2}}$$

$$= \tan^{-1} \frac{1 - \tan\frac{x}{2}}{1 + \tan\frac{x}{2}}$$

$$= \tan^{-1} \frac{\tan\frac{\pi}{4} - \tan\frac{x}{2}}{1 + \tan\frac{\pi}{4}\cdot\tan\frac{x}{2}}$$

$= \tan^{-1}\left[\tan\left(\frac{\pi}{4} - \frac{x}{2}\right)\right]$

$y = \frac{\pi}{4} - \frac{x}{2}$

$\therefore \frac{dy}{dx} = \frac{d}{dx}\left(\frac{\pi}{4} - \frac{x}{2}\right)$

$= -\frac{1}{2}$ Ans.

successive differentiation

$y = \sin^3 x$, find y_n

sol:

$y = \sin^3 x$

$\left[\sin 3x = 3\sin x - \frac{1}{4}\sin^3 x\right]$

$\sin^3 x = \frac{1}{4}(3\sin x - \sin 3x)$

$\Rightarrow y = \frac{1}{4}(3\sin x - \sin 3x)$

$\therefore y_1 = \frac{1}{4}(3\cos x - 3\cos 3x)$

$= \frac{1}{4}\left[3\sin\left(\frac{\pi}{2} + x\right) - 3\sin\left(3x + \frac{\pi}{2}\right)\right.$

$y_2 = \frac{1}{4}(-3\sin x + 3^2\sin 3x)$

$= \frac{1}{4}\left[3\sin(\pi + x) - 3^2\sin(\pi + 3x)\right]$

$\therefore y_n = \frac{1}{4}\left[3\sin\left(x + \frac{n\pi}{2}\right) - 3^n\sin\left(3x + \frac{n\pi}{2}\right)\right.$

Liebnitth theorem

If u and v are function of x, then the nth derivative of their product is given by

$$(uv)_n = U_n V + {}^nC_1 U_{n-1} V_1 + {}^nC_2 U_{n-2} V_2 + \cdots + U V_n$$

Prove:

Let $y = uv$

$\Rightarrow y_1 = U_1 V + U V_1$

$\Rightarrow y_2 = U_2 V + 2 U_1 V_1 + U V_2$

$= U_2 V + 2C_1 U_1 V_1 + U V_2$

$\therefore y_3 = U_3 V + 3 U_2 V_1 + 3 U_1 V_2 + U V_3$

$= U_3 V + 3C_1 U_2 V_1 + 3C_2 U_1 V_2 + U V_3$

- - - - - - - - - - -

$y_n = U_n V + {}^nC_1 U_{n-1} V + {}^nC_2 U_{n-2} V + \cdots$

~~Ex~~-7. i)

$y = \tan^{-1} x$ then

$n!$
$2!(n-2)!$

(i) $(1+x^2) y_1 = 1$

(ii) $(1+x^2) y_{n+1} + 2nx y_n + n(n-1) y_{n-1} = 0$

Sol: $y_1 = \frac{1}{1+x^2}$

$\Rightarrow y_1 (1+x^2) = 1$

Differentiating n times by liebnitz theorem.

$y_{n+1}(1+x^2) + {}^nC_1 y_n 2x + {}^nC_2 y_{n-1} 2 = 0$

$\Rightarrow (1+x^2) y_{n+1} + 2nx y_n + n(n-1) y_{n-1} = 0$

EX-10

If $\log y = \tan^{-1} x$, show that

(i) $(1+x^2) y_2 + (2x-1) y_1 = 0$

(ii) $(1+x^2) y_{n+2} + (2nx + 2x - 1) y_{n+1} + n(n+1) y_n = 0$

Sol:

$$\log y = \tan^{-1} x$$

$$y_1 \cdot \frac{1}{y} = \frac{1}{1+x^2}$$

$$\Rightarrow (1+x^2) y_1 = y$$

$$\Rightarrow y_1(1+x^2) = y$$

$$\Rightarrow y_1 = y_1 2x + (1+x^2) y_2$$

$$\Rightarrow y_1(2x-1) + (1+x^2) y_2 = 0$$

$$\Rightarrow y_2(1+x^2) + y_1(2x-1) = 0$$

by liebnitz theorem

$$\cancel{y_{n+2}(1+x^2) + n y_{n+1} 2x}$$

$$y_{n+2}(1+x^2) + n y_{n+1} 2x + \frac{n}{2} y_n 2 + y_{n+1}(2x-1) + n y_n 2 = 0$$

$$\Rightarrow y_{n+2}(1+x^2) + 2nx y_{n+1} + n(n-1) y_n + y_{n+1}(2x-1) + y_n 2n = 0$$

$$\Rightarrow y_{n+2}(1+x^2) + y_{n+1}(2nx + 2x - 1) + y_n(n^2+n) = 0$$

13. $y = e^{a\sin^{-1}x}$

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$\Rightarrow y_1 = a e^{a\sin^{-1}x} \cdot \frac{1}{\sqrt{1-x^2}}$

$\Rightarrow y_1 = \frac{ay}{\sqrt{1-x^2}}$

$\Rightarrow y_1^2 (1-x^2) = a^2 y^2$

$\Rightarrow 2y_1 y_2 (1-x^2) - 2x y_1^2 = a^2 2y y_1$

$\Rightarrow y_1 y_2 (1-x^2) - x y_1^2 = a^2 y y_1$

$\Rightarrow y_2 (1-x^2) - x y_1 - a^2 y = 0$

Applying leibnitz theorem

$(1-x^2) y_{n+2} - y_{n+1} 2xn - n(n-1) y_n - y_{n+1} x - n y_n - y_n a^2 = 0$

Ans.

$\Rightarrow (1-x^2) y_{n+2} - (2n+1) y_{n+1} x - (n^2 + a^2) y_n = 0$

Ans.

Indeterminate form

L Hospital's theorem: If $\phi(x)$ and $\psi(x)$ as also their derivatives $\phi'(x)$, $\psi'(x)$ are continuous at $x = a$, and if $\phi(a) = \psi(a) = 0$

Ex.

1 (ii)

$$\lim_{x \to 0} \frac{\tan x - x}{x - \sin x}$$

$$= \lim_{x \to 0} \frac{\sec^2 x - 1}{1 - \cos x}$$

$$= \lim_{x \to 0} \frac{2\sec x \cdot \sec x \tan x - 0}{0 + \sin x}$$

$$= \lim_{x \to 0} \frac{2 \cdot \frac{1}{\cos x} \cdot \frac{1}{\cos x} \cdot \frac{\sin x}{\cos x}}{\sin x}$$

$$= \lim_{x \to 0} \frac{2 \sin x}{\cos^3 x \cdot \sin x}$$

$$= \underset{x \to 0}{Lt} \frac{2}{\cos^3 x} = 2 \text{ Ans.}$$

(x). $$\underset{x \to 0}{Lt} \frac{e^x - e^{\sin x}}{x - \sin x}$$

$$= \underset{x \to 0}{Lt} \frac{e^x - e^{\sin x} \cdot \cos x}{1 - \cos x}$$

$$= \underset{x \to 0}{Lt} \frac{e^x - e^{\sin x} \cdot \cos^2 x + e^{\sin x} \sin x}{\sin x}$$

$$= \underset{x \to 0}{Lt} \frac{e^x - e^{\sin x} \cdot \cos^3 x + 2\cos x \sin x \, e^{\sin x} + e^{\sin x} \cos x \sin x + e^{\sin x} \cos x}{\cos x}$$

$$= \frac{1 - 1 + 2 \cdot 0 + 0 + 1}{1}$$

$= 1$ Ans.

4.

(i) $$\underset{x \to \frac{\pi}{2}}{Lt} (\sec x - \tan x) \quad [\infty - \infty]$$

$$= \underset{x \to \frac{\pi}{2}}{Lt} \left(\frac{1}{\cos x} - \frac{\sin x}{\cos x} \right)$$

$$= \underset{x \to \frac{\pi}{2}}{Lt} \frac{1-\sin x}{\cos x}$$

$$= \underset{x \to \frac{\pi}{2}}{Lt} \frac{-\cos x}{-\sin x}$$

$$= \underset{x \to \frac{\pi}{2}}{Lt} \frac{\cos x}{\sin x} = 0 \quad \underline{\text{Ans}}$$

$$\boxed{\star\star}$$

$$\underset{x \to \frac{\pi}{2}}{Lt} (\sin x)^{\tan x}$$

Let $y = (\sin x)^{\tan x}$

$\Rightarrow \log y = \tan x \log(\sin x)$

$\Rightarrow \log y = \underset{x \to \frac{\pi}{2}}{Lt} \tan x \log(\sin x)$

$$= \underset{x \to \frac{\pi}{2}}{Lt} \frac{\log(\sin x)}{\cot x}$$

$$\Rightarrow \log \underset{x \to \frac{\pi}{2}}{Lt} y = \underset{x \to \frac{\pi}{2}}{Lt} \frac{\cot x}{-\operatorname{cosec}^2 x}$$

	1	2	3	4	5	6	7
1				0			
2			0	1	0		
3		0	1	2	1	0	
4	0	1	2	3	2	1	0

$$= \underset{x\to\frac{\pi}{2}}{Lt} \frac{\cos x}{\sin x}\cdot \sin^2 x$$

$$= \underset{x\to\frac{\pi}{2}}{Lt} \cos x \sin x$$

$$= 0 \cancel{= \log 1}$$

$$\therefore \underset{x\to\frac{\pi}{2}}{Lt}\, y = e^0 = 1$$

$$\therefore \underset{x\to\frac{\pi}{2}}{Lt} (\sin x)^{\tan x} = 1 \text{ Ans}$$

(xii) $\underset{x\to 0}{Lt} \left(\frac{\sin x}{x}\right)^{1/x}$

If y be the value of the given limit

$$\log y = \underset{x\to 0}{Lt} \frac{1}{x} \log\left(\frac{\sin x}{x}\right)$$

$$= \underset{x \to 0}{Lt} \frac{\log \frac{\sin x}{x}}{x}$$

$$= \underset{x \to 0}{Lt} \frac{\frac{x}{\sin x}\left(\frac{x\cos x - \sin x}{x^2}\right)}{1}$$

$$= \underset{x \to 0}{Lt} \frac{x\cos x - \sin x}{x \sin x}$$

$$= \underset{x \to 0}{Lt} \frac{\cos x - x\sin x - \cos x}{x\cos x + \sin x}$$

$$= \underset{x \to 0}{Lt} \frac{-\sin x - x\cos x + \cancel{\cos x} + \sin x}{-x\sin x + \cos x + \cos x}$$

$$= \underset{x \to 0}{Lt} \frac{-x\cos x - \sin x}{2\cos x - x\sin x}$$

$$= 0$$

$$\therefore \underset{x \to 0}{Lt}\, y = e^{0} = 1 \ \underline{\text{Ans}}$$

Partial differentiation

The result of differentiating $v = f(x, y)$ with respect to x, treating y as a constant is called the partial derivative of v with respect to x and is denoted by one of the symbols $\frac{\partial v}{\partial x}$, $\frac{\partial f}{\partial x}$, f_x, v_x etc.

Q.
show that

$$\frac{\partial^2 v}{\partial v^2} + \frac{\partial^2 v}{\partial y^2} = 0 \text{ if}$$

$$v = \tan^{-1}\frac{y}{x}$$

$$\Rightarrow \frac{\partial v}{\partial x} = \frac{1}{1+\frac{y^2}{x^2}}\left(\frac{-y}{x^2}\right) = \frac{-y}{x^2+y^2}$$

$$\Rightarrow \frac{\partial^2 v}{\partial x^2} = \frac{(x^2+y^2)\cdot 0 - y\cdot 2x}{(x^2+y^2)^2}$$

$$\frac{\partial^2 v}{\partial x^2} = \frac{-2xy}{(x^2+y^2)^2}$$

Again $\frac{\partial u}{\partial y} = \tan^{-1}\frac{y}{x}$

$$= \frac{1}{1+\frac{y^2}{x^2}} \cdot \left(\frac{1}{x}\right)$$

$$= \frac{x}{x^2+y^2}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{(x^2+y^2)0 - x2y}{(x^2+y^2)^2}$$

$$= \frac{-2xy}{(x^2+y^2)^2}$$

$$\therefore \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

$$\frac{-2xy}{(x^2+y^2)^2} + \frac{2xy}{(x^2+y^2)^2} = 0$$

8. If $u = \log(x^3+y^3+z^3-3xyz)$ show that

(i) $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = \frac{3}{x+y+z}$

(ii) $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = \frac{-3}{(x+y+z)^2}$

Sol:

$$\frac{\partial u}{\partial x} = \frac{3x^2-3yz}{x^3+y^3+z^3-2xyz}$$

$$\frac{\partial u}{\partial y} = \frac{3y^2-3xz}{x^3+y^3+z^3-2xyz}$$

$$\frac{\partial u}{\partial z} = \frac{3z^2-3xy}{x^3+y^3+z^3-2xyz}$$

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = \frac{3(x^2+y^2+z^2-xy-yz-zx)}{(x+y+z)(x^2+y^2+z^2-xy-yz-zx)}$$

$$= \frac{3}{x+y+z}$$ Ans.

<u>Homogeneous function:</u> A function $f(x,y)$ is said to be homogenous of degree n ~~th~~ in the variables x and y if it can be expressed in the form $x^n \phi(y/x)$ is in the form $y^n \phi(x/y)$.

Eulers theorem on homogeneous function:

<u>Statement:</u> If $f(x,y)$ be a homogeneous function of x and y of degree n, then

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nf(x,y)$$

Ex-1 If $u = \tan^{-1}\dfrac{x^3+y^3}{x-y}$ show that

$$\frac{x\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \sin 2u$$

Sol:

For the given relation we get

$$\tan u = \frac{x^3+y^3}{x-y}$$

$$= \frac{x^3\left(1+\frac{y^3}{x^3}\right)}{x(1-y/x)}$$

$$= x^2\phi(y/x)$$

$\therefore \tan u$ is a homogenous function of degree 2.

$\therefore \tan u = v$

$$\Rightarrow x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} = 2v$$

$$\Rightarrow x\sec^2 u\frac{\partial u}{\partial x} + y\sec^2 u\frac{\partial u}{\partial y} = 2\tan u$$

$$\Rightarrow x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2\cdot\frac{\tan u}{\sec^2 u}$$

$$\Rightarrow x\frac{\partial u}{\partial x} + y\cdot\frac{\partial u}{\partial y} = 2\cdot\frac{\sin u}{\cos u}\times\cos^2 u$$

$$\therefore x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \sin 2u$$

3. (ii) If $u = \cos^{-1}\left\{(x+y)/\sqrt{x}+\sqrt{y}\right\}$

show that

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + \frac{1}{2}\cot u = 0$$

$$u = \cos^{-1}\left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right)$$

$$\Rightarrow \cos u = \frac{x\left(1+\frac{y}{x}\right)}{\sqrt{x}\left(1+\frac{\sqrt{y}}{\sqrt{x}}\right)}$$

$$= x^{\frac{1}{2}}\phi(y/x)$$

$\cos u$ is a homogenous function of degree $\frac{1}{2}$.

$\cos u = v$

$$\Rightarrow x \cdot \frac{\partial V}{\partial x} + y \frac{\partial V}{\partial y} = \frac{1}{2} V$$

$$\Rightarrow x(-\sin u) \frac{\partial u}{\partial x} + y(-\sin u) \frac{\partial u}{\partial y} = \frac{1}{2} \cos u$$

$$\Rightarrow x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} + \frac{1}{2} \cot u = 0$$

Maxima and Minima

A function $f(x)$ is said to have a maximum value for $x = c$, provided we can get a positive quantity δ such that for all values of x in the interval

$c - \delta < x < c + \delta$ $(x \neq c)$ $f(c) > f(x)$

i.e. if $f(c+h) - f(c) < 0$ for $|h|$ sufficiently small. Simillarly, the function $f(x)$ has a ~~maxim~~ minimum value for $x = d$ provided we can get an interval $d - \delta' < x < d + \delta'$ within which $f(d) < f(x)$ $(x \neq d)$

i.e. if $f(a+h) - f(a) > 0$ for $|h|$ sufficiently small.

Ex-1 Find for what values of x the following expression is maximum and minimum respectively

$2x^3 - 21x^2 + 36x - 20$

Find also the maximum and minimum values of the expression.

Let $f(x) = 2x^3 - 21x^2 + 36x - 20$

$\therefore f'(x) = 6x^2 - 42x + 36$

For maximum and minimum value

$f'(x) = 0$

$6x^2 - 42x + 36 = 0$

$\Rightarrow (x-1)(x-6) = 0$

$\therefore x = 1, 6$

Again

$$f''(x) = 12x - 42$$
$$= 6(2x - 7)$$

when $x=1$ $f''(x) = -30$ which -ve

when $x=6$ $f''(x) = 30$ " +ve

Hence the given expression is maximum for $x=1$ and minimum for $x=6$.

$\therefore$ The max and min values of the given expression are respectively,

$\therefore f(1) = -3$ and $f(6) = -128$

Ex-6 Examine whether $x^{\frac{1}{x}}$ posses a maximum or a minimum and determine the same

Let $y = x^{\frac{1}{x}}$

$$\log y = \frac{1}{x} \log x$$

$$\Rightarrow \frac{1}{y}\cdot\frac{dy}{dx} = \frac{1}{x^2} - \frac{1}{x^2}\log x$$

$$= \frac{1}{x^2}(1-\log x) \longrightarrow (1)$$

when $\frac{dy}{dx} = 0$

$\therefore \; 1-\log x = 0$

$\log x = 1$

$\log x = \log e$

$\therefore \; x = e$

Again differentiating (1) with respect to x;

$$-\frac{1}{y^2}\left(\frac{dy}{dx}\right)^2 + \frac{1}{y}\cdot\frac{d^2y}{dx^2} = \frac{x^2\left(-\frac{1}{x}\right) - (1-\log x)2x}{x^4}$$

$$= \frac{-x-2x+\log x \cdot 2x}{x^4}$$

$$= \frac{x(-1-2+2\log x)}{x^4}$$

$$= \frac{-3+2\log x}{x^3}$$

when $x = e$,

$$\left(\frac{1}{e^2}(1-\log e)\right)^2 + \frac{d^2y}{dx^2} = e^{\frac{1}{e}}\,\frac{-3+2}{e^3}$$

$$\Rightarrow 0 + \frac{d^2y}{dx^2} = \frac{-e^{\frac{1}{e}}}{e^3}$$ which is negative

$\therefore$ for $\left(x = e, \frac{dy}{dx} = 0\right)$

For $x = e$, the funct

17. (i) Given $\frac{x}{2} + \frac{y}{3} = 1$ find the maximum value of xy and min value of x^2+y^2

sol:

$$\frac{x}{2} + \frac{y}{3} = 1$$

$$\Rightarrow 3x + 2y = 6$$

$$\Rightarrow y = \frac{1}{2}(6-3x)$$

$$\Rightarrow y = \frac{3}{2}(2-x)$$

$$\therefore xy = \frac{3}{2}(2x - x^2)$$

Let $p = xy = \frac{3}{2}(2x - x^2)$

for max and min value

$$\frac{dP}{dx} = 0$$

$$\Rightarrow \frac{3}{2}(2-2x) = 0$$

$$\therefore x = 1$$

$$\therefore \frac{d^2P}{dx^2} = \frac{3}{2}(0-2)$$

$$= -3 \text{ which is } -ve$$

$\therefore P = xy$ is maximum at $x=1$

$\therefore$ maximum value of xy

$$xy = \frac{3}{2}(2\cdot1 - 1^2)$$

$$= \frac{3}{2} \text{ Ans.}$$

Again.

Let $q = x^2 + y^2$

putting the value of y

$$q = x^2 + \frac{9}{4}(2-x)^2 \Rightarrow \frac{dq}{dx} = 2x - \frac{9}{2}(2-x)$$

for max and min value

$$\frac{dq}{dx} = 0$$

$$\Rightarrow 2x - \frac{9}{2}(2-x) = 0$$

$$\Rightarrow 2x - \frac{(18-9x)}{2} = 0$$

$$\Rightarrow \frac{4x - 18 + 9x}{2} = 0$$

$$\Rightarrow 13x - 18 = 0$$

$$\therefore x = \frac{18}{13}$$

Now, $\frac{d^2q}{dx^2} = 2 + \frac{9}{2}$, which is +ve

$\therefore$ q is min at $x = \frac{18}{13}$

$\therefore$ min value of $x^2 + y^2$

$$= \left(\frac{18}{13}\right)^2 + \frac{9}{4}\left(2 - \frac{18}{13}\right)^2$$

$$= \frac{324}{169} + \frac{144}{169}$$

$$= \frac{36}{13}$$

19. Given $xy = 4$. Find the max and min value of $4x + 9y$

Sol: $xy = 4$

$y = \frac{4}{x}$

Let $f(x) = 4x + 9y$

$= 4x + \frac{36}{x}$

$\therefore f'(x) = 4 - \frac{36}{x^2} = 0$

$\therefore x = 3, -3$

Now, $f''(x) = 0 + \frac{72}{x^3}$

$\therefore f''(3) = +\text{ve value}$

$\therefore$ Hence, $f(x)$ is min at $x = 3$

$\therefore$ minimum value

$\therefore 4 \cdot 3 + \frac{36}{3}$

$= 24$

Now, $f''(-3) = -ve$

$\therefore f(x)$ is max, at $x = -3$

$\therefore$ The maximum value

$$f(-3) = 4(-3) + \frac{36}{-3} = -24$$

$\therefore$ maximum value is -24.

6. Show that $x^5 - 5x^4 + 5x^3 - 1$ is a maximum when $x = 1$, a minimum when $x = 3$, neither $x = 0$.

Sol:

$$f(x) = x^5 - 5x^4 + 5x^3 - 1$$

$$\therefore f'(x) = 5x^4 - 20x^3 + 15x^2$$

$$= 5x^2(x^2 - 4x + 3)$$

$$= 5x^2(x-1)(x-3)$$

$\therefore f'(x) = 0$ when $x = 0, 1, 3$

Again $f''(x) = 20x^3 - 60x^2 + 30x$

$$f'''(x) = 60x^2 - 120x + 30$$

when $x = 0$, $f''(x)$ is zero and also $f'''(x) \neq 0$

so neither max nor min at $x=0$.
when $x=1$, $f''(x)$ is negative, hence $f(x)$ is max. when $x=1$ and at $x=3$, $f''(x)$ is positive, hence $f(x)$ is min when $x=3$.

Tangent and Normal

**Find the equation of the curve is

$$f(x,y) = \left(\frac{x}{a}\right)^{2/3} + \left(\frac{y}{b}\right)^{2/3} - 1 = 0$$

The equation of the tangent is

$$(X-x) f_x + (Y-y) f_y = 0$$

$$(X-x) \frac{2}{3} \frac{x^{-1/3}}{a^{2/3}} + (Y-y) \frac{2}{3} \frac{y^{-1/3}}{b^{2/3}} = 0$$

$$\Rightarrow \frac{2}{3} \frac{X x^{-1/3}}{a^{2/3}} + \frac{2}{3} \frac{Y y^{-1/3}}{a^{2/3}} = \frac{2}{3}\left(\frac{x}{a}\right)^{2/3} + \frac{2}{3}\left(\frac{y}{b}\right)^{2/3}$$

$$\Rightarrow \frac{X x^{-1/3}}{a^{2/3}} + \frac{Y y^{-1/3}}{b^{2/3}} = 1 \text{ (proved)}$$

Q (iii) show that the tangent at (a,b) to the curve

$$\left(\frac{x}{a}\right)^3+\left(\frac{y}{b}\right)^3=2 \text{ is } \frac{x}{a}+\frac{y}{b}=2$$

<u>sol:</u> given a curve

$$\left(\frac{x}{a}\right)^3+\left(\frac{y}{b}\right)^3=2$$

$$\Rightarrow x^3b^3+y^3a^3=2a^3b^3$$

$$\Rightarrow f(x,y)=b^3x^3+a^3y^3-2a^3b^3=0$$

$$\therefore \cancel{f(x)}\, f_x=3x^2b^3 \text{ and } f_y=3y^2a^3$$

$$\therefore \frac{dy}{dx}=-\frac{f_x}{f_y}=\frac{-3a^2b^3}{3b^2a^3}=-\frac{b}{a}$$

Now the equation of tangent at (a,b) is

$$y-b=-b/a\,(x-a)$$

$$\Rightarrow ay-ab=-bx+ab$$

$$\Rightarrow xb+ay=2ab$$

$$\Rightarrow \frac{x}{a}+\frac{y}{b}=2 \text{ (proved)}$$

(iii) Find the equation of the tangent at the point (x, y) on each of the following curves.

$$\frac{x^m}{a^m} + \frac{y^m}{b^m} = 1$$

Sol:

Let

$$f(x,y) = \frac{x^m}{a^m} + \frac{y^m}{b^m} - 1 = 0$$

The equation of the tangent is

$$(X-x) f_x + (Y-y) f_y = 0$$

$$\Rightarrow (X-x) \frac{m x^{m-1}}{a^m} + (Y-y) \frac{m y^{m-1}}{b^m} = 0$$

$$\Rightarrow \frac{m X x^{m-1}}{a^m} + \frac{Y m y^{m-1}}{b^m} = \frac{m x^m}{a^m} + \frac{y^m}{b^m}$$

$$\Rightarrow \frac{X x^{m-1}}{a^m} + \frac{Y y^{m-1}}{b^m} = \frac{x^m}{a^m} + \frac{y^m}{b^m}$$

$$\Rightarrow \frac{X x^{m-1}}{a^m} + \frac{Y y^{m-1}}{b^m} = \cancel{\frac{x^m}{a^m}} \; 1$$

2. Find the equation of the tangent at the point θ on each of the following curves,

(i) $x = a(\theta + \sin\theta),\ y = a(1 - \cos\theta)$

$\therefore \frac{dx}{d\theta} = a(1+\cos\theta),\ \frac{dy}{d\theta} = a\sin\theta$

$\therefore \frac{dy}{dx} = \frac{a\sin\theta}{a(1+\cos\theta)} = \frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\cos^2\frac{\theta}{2}}$

$= \frac{\sin\theta/2}{\cos\theta/2}$

Equation of tangent at θ is:

$Y - y = \frac{dy}{dx}(X - x)$

$\Rightarrow Y - a(1-\cos\theta) = \frac{\sin\theta/2}{\cos\theta/2}\left(X - a(\theta + \sin\theta)\right)$

$\Rightarrow Y\cos\theta/2 - a\cos\theta/2 + a\cos\theta\cos\theta/2 = X\sin\theta/2 - a\theta\sin\theta/2 - a\sin\theta\sin\frac{\theta}{2}$

$\Rightarrow -X\sin\frac{\theta}{2} + Y\cos\frac{\theta}{2} = a\cos\frac{\theta}{2} - a\theta\sin\theta/2 - a\left\{\cos\theta\cos\frac{\theta}{2} + \sin\theta\sin\frac{\theta}{2}\right\}$

$\Rightarrow -X\sin\frac{\theta}{2} + Y\cos\frac{\theta}{2} = a\cos\frac{\theta}{2} - a\theta\sin\frac{\theta}{2} - a\cos\left(\theta - \frac{\theta}{2}\right)$

$\Rightarrow -X\sin\frac{\theta}{2} + Y\cos\frac{\theta}{2} = \frac{a\cos\frac{\theta}{2} - a\theta\sin\frac{\theta}{2} - a\cos\theta/2}{}$

$\Rightarrow -X\sin\frac{\theta}{2} + Y\cos\frac{\theta}{2} = -a\theta\sin\theta/2$

$\Rightarrow X\sin\theta/2 - Y\cos\theta/2 = a\theta\sin\theta/2$

(proved)

10. (i) If $lx + my = 1$ touches the curve $(ax)^n + (by)^n = 1$ show that —

$$\left(l/a\right)^{n/n-1} + \left(\frac{m}{b}\right)^{n/n-1} = 1$$

Let $f(x,y) = (ax)^n + (by)^n - 1 = 0$

$= a^n x^n + b^n y^n - 1 = 0$

$\therefore f_x = n x^{n-1} a^n$

$f_y = n y^{n-1} b^n$

$$\therefore \frac{dy}{dx} = \frac{-f_x}{f_y} = -\frac{n a^n x^{n-1}}{\cancel{n} y^{n-1} b^n} = -\frac{a^n x^{n-1}}{y^{n-1} b^n}$$

So, the equation of tangent at (x,y)

$\cancel{(X-x)} = (Y-y) = \frac{dy}{dx}(X-x)$

$$\Rightarrow Y - y = \frac{-a^n x^{n-1}}{b^n y^{n-1}}(X - x)$$

$$\Rightarrow Y y^{n-1} b^n - b^n y^n = -X a^n x^{n-1} + a^n x^n$$

$$\Rightarrow X a^n x^{n-1} + Y y^{n-1} b^n = a^n x^n + b^n y^n$$

$$= (ax)^n + (by)^n = 1$$

$$\therefore X a^n x^{n-1} + Y y^{n-1} b^n = 1 \quad \text{——(i)}$$

$$lx + my = 1 \quad \text{——(ii)}$$

(i) and (ii) are identical.

$$\frac{a^n x^{n-1}}{l} = \frac{y^{n-1} b^n}{m} = 1$$

$$\therefore a^n x^{n-1} = l \quad \text{and} \quad y^{n-1} b^n = m$$

$$a^{n-1} x^{n-1} = \frac{l}{a}$$

$$\Rightarrow (ax)^{n-1} = l/a$$

$$\Rightarrow (ax) = (l/a)^{1/n-1}$$

$$\Rightarrow (ax)^n = (l/a)^{n/n-1} \quad \text{——(iii)}$$

$$y^{n-1} b^{n-1} = \frac{m}{b}$$

$$(by)^{n-1} = m/b$$

$$by = (m/b)^{1/n-1}$$

$$(by)^n = (m/b)^{n/n-1} \quad \text{——(iv)}$$

adding (iii), (iv)

$$\left(\frac{la}{a}\right)^{n/n-1} + \left(\frac{m}{b}\right)^{n/n-1} = (ax)^n + (by)^n$$

$$\Rightarrow \left(\frac{l}{a}\right)^{n/n-1} + \left(\frac{m}{b}\right)^{n/n-1} = 1.$$

X (B)

7.
(i) Show that the following curves cut orthogonally.

$r = a(1+\cos\theta)$

$r = b(1-\cos\theta)$

$\boxed{\text{No Need}}$

Asymptotes

Rule-1: Asymtotes not parallel y axis.

$y = mx + c$

To find $m \rightarrow$ put $x = 1$, $y = m$ in the given equation $\phi_n(m) = 0$

To find $c \rightarrow$ $c = -\dfrac{\phi_{n-1}(m)}{\phi_n'(m)}$ [n = degree of the equation.

Ex-3

Find the asymtotes of the curve

Soln: $x^3 + 3x^2y - xy^2 - 3y^3 + x^2 - 2xy + 2xy + 3y^2 + 4x + 5 = 0$

~~Clearly the~~

This is a 3rd degree equation and both x^3 and y^3 are present. So, there is no asymtotes parallel to either the x axis or the y axis.

For oblique asymptotes putting $x = 1$, $y = m$ in third and second degree terms, we get

$$\phi_3(m) = 1+3m - m^2 - 3m^3$$
$$= (1+3m) - m^2(1+3m)$$
$$= (1+3m)(1+m)(1-m)$$

$$\phi_2(m) = 1 - 2m + 3m^2$$

Now, $\phi_3(m) = 0$

$\therefore m = 1, -1, -\frac{1}{3}$

Now $c = \dfrac{-\phi_2(m)}{\phi_3'(m)}$

$$c = \frac{-(1-2m+3m^2)}{3-2m-9m^2}$$

when $m = 1$, $c = \frac{1}{4}$

$m = -1$, $c = \frac{3}{2}$

$m = -\frac{1}{3}$, $c = \frac{3}{4}$

Required asymptotes are

$$y = x + \frac{1}{4}, \quad y = -x + \frac{3}{2} \text{ and } y = -\frac{1}{3}x - \frac{3}{4}$$

$$\Rightarrow 4y - 4x - 1 = 0; \quad 2y + 2x - 3 = 0;$$

$$y = \frac{-4x - 9}{12}$$

$$\Rightarrow -4x - 9 = 12y$$

$$\Rightarrow 12y + 4x + 9 = 0$$

Rule-2: Factor the highest degree terms

The possible asymptotes are parallel to the real linear factors when $x \to \infty$ and y values come from the factors.

Ex-8:

$$x^4 - y^4 + 3x^2y + 3xy^2 + xy = 0$$

$$(x^2 - y^2)(x^2 + y^2) + 3xy(x + y) + xy = 0$$

$$\Rightarrow (x - y)(x + y)(x^2 + y^2) + 3xy(x + y) + xy = 0$$

The linear factors of $x^2 + y^2$ are imaginary. Hence there are only two real asymptotes.

First asymptotes is

$$x-y+\underset{x\to\infty}{Lt}\frac{3xy(x+y)+xy}{(x+y)(x^2+y^2)}=0$$

$$\Rightarrow x-y+\underset{\substack{x\to\infty\\ y=x}}{Lt}\frac{xy(3x+3y+1)}{(x+y)(x^2+y^2)}=0$$

$$\Rightarrow x-y+\underset{x\to\infty}{Lt}\frac{x^2(3x+3x+1)}{(x+x)(x^2+x^2)}=0$$

$$\Rightarrow x-y+\underset{x\to\infty}{Lt}\frac{6x^3+x^2}{4x^3}=0$$

$$\Rightarrow x-y+\underset{x\to\infty}{Lt}\frac{6x+1}{4x}=0$$

$$\Rightarrow x-y+\frac{3}{2}=0$$

second asymptotes is

$$x+y+\underset{x\to\infty}{Lt}\frac{3xy(x+y)+xy}{(x-y)(x^2+y^2)}=0$$

$$\lim_{\substack{x+y+x\to\infty \\ y=-x}} \frac{xy\{3x+3y+1\}}{(x-y)(x^2+y^2)} = 0$$

$$\Rightarrow \lim_{x+y+x\to\infty} \frac{-x^2(3x-3x+1)}{2x\cdot 2x^2} = 0$$

$$\Rightarrow \lim_{x+y+x\to\infty} \frac{-x^2}{4x^3}$$

$$\Rightarrow x+y=0$$

$\therefore$ Required asymptotes are $x+y=0$;

$2x-2y+3=0$