Function

Function: Let A and B be sets. A *function from A into B*, denoted by $f: A \to B$, is a "*rule*" that assigns to each element of A exactly one element of B.
We say that f is a *function from A into B*", we simply say "*a function*" and instead of writing $f: A \to B$, we simply write f.
Example $y = x^2 + 1$ is a function. There is only one way to square x and then add 1 to the result. So, no matter what value of x you put into the equation, there is only one possible value of y.
Given $f(x) = -x^2 + 6x - 11$, find $f(4x - 1)$.
Solution:

$$f(4x-1) = -(4x-1)^2 + 6(4x-1) - 11 = -16x^2 + 32x - 18$$

One of the more important ideas about functions is that of the domain and range of a function.
Domain: If $f: A \to B$, The set of all possible values of A is called the domain of f. In simplest terms the domain of a function is the set of all values that can be plugged into a function and have the function exist and have a real number for a value. So, for the domain we need to avoid division by zero, square roots of negative numbers, logarithms of zero and negative numbers *etc.* that is , the domain of f is the set of all real numbers x such that $y = f(x)$ is defined.
Range: If $f: A \to B$, The set of all possible values of B is called the range of f.

Example: Find the domain and range of each of the following functions:

i $y = 5x - 3$
ii $y = \sqrt{4 - 7t}$
iii $y = -2x^2 + 12x + 5$
iv $y = |z - 6| - 3$
v $y = 8$
vi $y = \dfrac{1}{x - 2}$

Solution
i $y = 5x - 3$

Here we can plug any value of x into a polynomial $5x - 3$ and so the domain in this case is also all real numbers or,

$$\text{Domain}: -\infty < x < \infty \quad \text{or} \quad (-\infty, \infty)$$

Again this function $f(x)$ can take on any real numbers and so the range is all real numbers. Using "mathematical" notation this is,

$$\text{Range}: (-\infty, \infty)$$

ii $y = \sqrt{4 - 7t}$.

Here We need to make sure that we don't take square roots of any negative numbers and so we need to require that,

$4 - 7t \geq 0$

$4 \geq 7t$

$\frac{4}{7} \geq t \quad \Rightarrow \quad t \leq \frac{4}{7}$

The domain is then,

$$\text{Dom an}: t \leq \tfrac{4}{7} \quad \text{or} \quad \left(-\infty, \tfrac{4}{7}\right]$$

This is a square root and we know that square roots are always positive or zero and because we can have the square root of zero in this case,

$$g\left(\tfrac{4}{7}\right) = \sqrt{4 - 7\left(\tfrac{4}{7}\right)} = \sqrt{0} = 0$$

We know then that the range will be,

$$\text{Range}: [0, \infty)$$

iii $y = -2x^2 + 12x + 5$

Here we have a quadratic which is a polynomial and so we again know that the domain is all real numbers or,

$$\text{Dom an}: -\infty < x < \infty \quad \text{or} \quad (-\infty, \infty)$$

We know that the graph of this will be a parabola and the coefficient of the x^2 is negative and so the vertex will be the highest point on the graph. If we know the vertex we can then get the range.

Now $y = -2x^2 + 12x + 5$

or, $y = -2(x^2 + 6x) + 5$

or, $y = -2(x^2 + 2.3x + 9) + 18 + 5$

or, $y = -2(x + 3)^2 + 23$

The vertex is, (3,23).

So, we know that this will be the highest point on the graph or the largest value of the function and the parabola will take all values less than this so the range is then.

$$\text{Range}: (-\infty, 23]$$

iv $y = |z - 6| - 3$

This function contains an absolute value and we know that absolute value will be either positive or zero. In this case the absolute value will be zero if $z = 6$ and so the absolute value portion of this function will always be greater than or equal to zero. We are subtracting 3 from the absolute value portion and so we then know that the range will be,

$$\text{Range}: [-3, \infty)$$

We can plug any value into an absolute value and so the domain is all real numbers or,

$$\text{Dom an}: -\infty < x < \infty \quad \text{or} \quad (-\infty, \infty)$$

$(v)\quad y = 8$

This function may seem a little tricky at first but is actually the easiest one in this set of examples. This is a constant function and so an value of x that we plug into the function will give a value of 8. This means that the range is a single value or,

$$\text{Range}: 8$$

The domain is all real numbers,

$$\text{Domain}: -\infty < x < \infty \quad \text{or} \quad (-\infty, \infty)$$

$$vi \quad y = \frac{1}{x-2}$$

Here y is defined for all real number x except 2. The domain of y is

$$x \in \mathbb{R} : x < 2 \text{ or } x > 2 \; = (-\infty, 2) \cup (2, \infty)$$

Now solve for y,

$$y = \frac{1}{x-2}$$

$$\text{or,} \quad x - 2 = \frac{1}{y}$$

$$\text{or,} \quad x = 2 + \frac{1}{y}$$

Note that x can be solved if and only if $y \neq 0$. The range of y is

$$y \in \mathbb{R} : y < 0 \text{ or } y > 0 \; = (-\infty, 0) \cup (0, \infty).$$

Example: Find the domain and range of

$$y = \frac{x-1}{2x-3} \quad \text{where } x \in \mathbb{R}, y \in \mathbb{R}$$

If $2x - 3 = 0$ or, $x = 3/2$ then y is not defined. So the domain of the function is

$$x \in \mathbb{R}: x \neq 3/2 \; = (-\infty, 3/2) \cup (3/2, \infty)$$

Again solving for y,

$$y = \frac{x-1}{2x-3}$$

$$\text{or,} \quad y(2x-3) = x - 1$$

$$\text{or,} \quad x = \frac{3y-1}{2y-1}$$

If $2y - 1 = 0$ or, $y = 1/2$ then x is not defined. So the range of the function is

$$y \in \mathbb{R}: y \neq 1/2 \; = (-\infty, 1/2) \cup (1/2, \infty)$$

Example: Find the domain and range of the function

$$y = \frac{x^2 + 1}{x^2 - 5x + 6}$$

The denominator $x^2 - 5x + 6 = (x-2)(x-3)$ which is zero when $x = 2$ and $x = 3$.
Therefore the function is not defined for $x = 2$ and $x = 3$.
Hence the domain is $(-\infty, 2) \cup (2,3) \cup (3, \infty)$.
Now, solving for y,

$$y = \frac{x^2 + 1}{x^2 - 5x + 6}$$

or, $\quad y(x^2 - 5x + 6) = x^2 + 1$

or, $\quad (y-1)x^2 - 5yx + (6y - 1) = 0$

This is a quadratic equation in x then

$$x = \frac{5y \pm \sqrt{(-5y)^2 - 4(y-1)(6y-1)}}{2(y-1)}$$

We get real solution for x, if the discriminant $(-5y)^2 - 4(y-1)(6y-1) \geq 0$

$$y^2 + 28y - 4 \geq 0$$

Now the roots of $y^2 + 28y - 4 = 0$ are given by

$$y = \frac{-28 \pm \sqrt{28^2 - 4.1(-4)}}{2}$$

$$\boxed{y = -14 \pm 10\sqrt{2}}$$

Hence $y^2 + 28y - 4$ becomes $y^2 + 28y - 4 = (y - (-14 - 10\sqrt{2}))(y - (-14 + 10\sqrt{2}))$

	$y < (-14 - 10\sqrt{2})$	$y = (-14 - 10\sqrt{2})$	$(-14 - 10\sqrt{2}) < y > (-14 + 10\sqrt{2})$	$y = (-14 + 10\sqrt{2})$	$y > (-14 + 10\sqrt{2})$
$y - (-14 - 10\sqrt{2})$	-	0	+	+	+
$y - (-14 + 10\sqrt{2})$	-	-	-	0	+
$y^2 + 28y - 4$	+	+	-	0	+

The range of the function is

$$y \in \mathbb{R}: y \leq -14 - 10\sqrt{2} \text{ or } y \geq -14 + 10\sqrt{2}$$

$$= (-\infty, -14 - 10\sqrt{2}] \cup [-14 + 10\sqrt{2}, 1) \cup (1, \infty)$$

Example Let $f(x) = \sqrt{x+7} - \sqrt{x^2+2x-15}$. Find the domain of f.

Solution Note that $f(x)$ is defined if and only if $x+7 \geq 0$ and $x^2+2x-15 \geq 0$. Solve the two inequalities separately:

- $$\begin{aligned} x+7 &\geq 0 \\ x &\geq -7; \end{aligned}$$

- $$\begin{aligned} x^2+2x-15 &\geq 0 \\ (x+5)(x-3) &\geq 0 \end{aligned}$$

	$x < -5$	$x = -5$	$-5 < x < 3$	$x = 3$	$x > 3$
$x-3$	$-$	$-$	$-$	0	+
$x+5$	$-$	0	+	+	+
$(x-3)(x+5)$	+	0	$-$	0	+

thus, $x \leq -5$ or $x \geq 3$.

Therefore, we have
$$\begin{aligned} \operatorname{dom}(f) &= \{x \in \mathbb{R} : x \geq -7 \text{ and } (x \leq -5 \text{ or } x \geq 3)\} \\ &= \{x \in \mathbb{R} : (x \geq -7 \text{ and } x \leq -5) \text{ or } (x \geq -7 \text{ and } x \geq 3)\} \\ &= \{x \in \mathbb{R} : -7 \leq x \leq -5 \text{ or } x \geq 3\} \\ &= [-7, -5] \cup [3, \infty). \end{aligned}$$

Example Let $f(x) = \dfrac{2x+1}{x^2+1}$. Find the range of f.

Solution

Put $y = f(x) = \dfrac{2x+1}{x^2+1}$.

Solve for x.

$$\begin{aligned} y &= \frac{2x+1}{x^2+1} \\ yx^2 + y &= 2x+1 \\ \underline{yx^2 - 2x + (y-1)} &= 0 \\ x &= \frac{2 \pm \sqrt{4 - 4y(y-1)}}{2y} \text{ if } y \neq 0. \qquad x = \frac{-1}{2} \text{ if } y = 0. \\ &= \frac{1 \pm \sqrt{1 - y^2 + y}}{y} \text{ if } y \neq 0. \end{aligned}$$

Combining the two cases, we see that x can be solved if and only if $1 - y^2 + y \geq 0$, that is, $y^2 - y - 1 \leq 0$.

The range of f is $\{y \in \mathbb{R} : y^2 - y - 1 \leq 0\}$.

To solve the inequality $y^2 - y - 1 \leq 0$, first we find the zero of the left-side by quadratic formula to get $\dfrac{1 \pm \sqrt{5}}{2}$

By the factor theorem and comparing coefficient of y^2, we see that $y^2 - y - 1 = \left(y - \dfrac{1-\sqrt{5}}{2}\right)\left(y - \dfrac{1+\sqrt{5}}{2}\right)$

	$y < \frac{1-\sqrt{5}}{2}$	$y = \frac{1-\sqrt{5}}{2}$	$\frac{1-\sqrt{5}}{2} < y < \frac{1+\sqrt{5}}{2}$	$y = \frac{1+\sqrt{5}}{2}$	$y > \frac{1+\sqrt{5}}{2}$
$y - \frac{1-\sqrt{5}}{2}$	$-$	0	+	+	+
$y - \frac{1+\sqrt{5}}{2}$	$-$	$-$	$-$	0	+
$y^2 - y - 1$	+	0	$-$	0	+

From the table, we see that

$$\begin{aligned} \operatorname{ran}(f) &= \{y \in \mathbb{R} : \frac{1-\sqrt{5}}{2} \leq y \leq \frac{1+\sqrt{5}}{2}\} \\ &= [\frac{1-\sqrt{5}}{2}, \frac{1+\sqrt{5}}{2}] \end{aligned}$$

Inverse Function

Let $f : X \to Y$ be an one to one function and let Y_1 be the range of . The *inverse (function)* of f, denoted by f^{-1}, is the function from Y_1 to X such that for every $y \in Y_1$, $f^{-1}(y)$ is the unique element of X satisfying $f(f^{-1}(y)) = y$

The following figure indicates a function f from a set X to a set Y. Assuming that f is injective, for each y belonging to the range of f, there is one and only one element x of X such that $f(x) = y$. This element x is defined to be $f^{-1}(y)$.

That is,

$f^{-1}(y) = x$ if and only if $f(x) = y$

Given two one-to-one functions $f(x)$ and $g(x)$ if

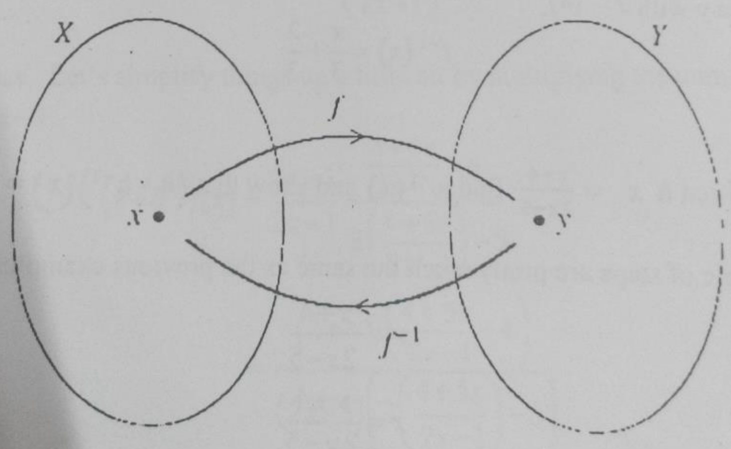
$$(f \circ g)(x) = x \quad \text{and} \quad (g \circ f)(x) = x$$

then we say that $f(x)$ and $g(x)$ are **inverses** of each other. More specifically we will say that $g(x)$ is the **inverse** of $f(x)$ and denote it by

$$g(x) = f^{-1}(x)$$

Likewise we could also say that $f(x)$ is the **inverse** of $g(x)$ and denote it by

$$f(x) = g^{-1}(x)$$



Example Given $f(x) = 3x - 2$ find $f^{-1}(x)$.

Solution

Now, we already know what the inverse to this function is as we've already done some work with it. However, it would be nice to actually start with this since we know what we should get. This will work as a nice verification of the process.

So, let's get started. We'll first replace $f(x)$ with y.

$$y = 3x - 2$$

Next, replace all x's with y and all y's with x.

$$x = 3y - 2$$

Now, solve for y.

$$x + 2 = 3y$$

$$\frac{1}{3}(x+2) = y$$

$$\frac{x}{3} + \frac{2}{3} = y$$

Finally replace y with $f^{-1}(x)$.

$$f^{-1}(x) = \frac{x}{3} + \frac{2}{3}$$

Example: Given $h\ x\ = \frac{x+4}{2x-5}$ find $h^{-1}(x)$ and show that $(h \circ h^{-1})(x) = x$

Solution

The first couple of steps are pretty much the same as the previous examples so here they are,

$$y = \frac{x+4}{2x-5}$$

$$x = \frac{y+4}{2y-5}$$

Now, be careful with the solution step. With this kind of problem it is very easy to make a mistake here.

$$
\begin{aligned}
x(2y-5) &= y+4 \\
2xy-5x &= y+4 \\
2xy-y &= 4+5x \\
(2x-1)y &= 4+5x \\
y &= \frac{4+5x}{2x-1}
\end{aligned}
$$

So, if we've done all of our work correctly the inverse should be,

$$h^{-1}(x) = \frac{4+5x}{2x-1}$$

Finally we'll need to do the verification. This is also a fairly messy process and it doesn't really matter which one we work with.

$$
\begin{aligned}
\left(h \circ h^{-1}\right)(x) &= h\left[h^{-1}(x)\right] \\
&= h\left[\frac{4+5x}{2x-1}\right] \\
&= \frac{\frac{4+5x}{2x-1}+4}{2\left(\frac{4+5x}{2x-1}\right)-5}
\end{aligned}
$$

Okay, this is a mess. Let's simplify things up a little bit by multiplying the numerator and denominator by $2x-1$.

$$
\begin{aligned}
\left(h \circ h^{-1}\right)(x) &= \frac{2x-1}{2x-1}\,\frac{\frac{4+5x}{2x-1}+4}{2\left(\frac{4+5x}{2x-1}\right)-5} \\
&= \frac{(2x-1)\left(\frac{4+5x}{2x-1}+4\right)}{(2x-1)\left(2\left(\frac{4+5x}{2x-1}\right)-5\right)} \\
&= \frac{4+5x+4(2x-1)}{2(4+5x)-5(2x-1)} \\
&= \frac{4+5x+8x-4}{8+10x-10x+5} \\
&= \frac{13x}{13} = x
\end{aligned}
$$