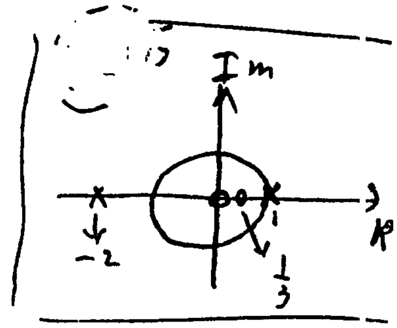
Q1

Solution: Final 2010, ELEC442-6601

$$H(z) = \frac{1-\frac{1}{3}z^{-1}}{1+z^{-1}-2z^{-2}} = \frac{1-\frac{1}{3}z^{-1}}{(1-z^{-1})(1+2z^{-1})}$$

zeros: 0 and $\frac{1}{3}$ Poles: 1 and -2



$$H(z) = \frac{Y(z)}{X(z)} = \frac{1-\frac{1}{3}z^{-1}}{1+z^{-1}-2z^{-2}} \Rightarrow Y(z)+z^{-1}Y(z)-2z^{-2}Y(z) = X(z)-\frac{1}{3}z^{-1}X$$

$$\Rightarrow \boxed{y[n] = -y[n-1]+2y[n-2]+x[n]-\frac{1}{3}x[n-1]}$$

Q2 There are three cases where the system is unstable

i) $\boxed{\text{Roc } |z|>2 \text{ and causal}}$ ii) Roc $1<|z|<2$ not causal iii) Roc $|z|<1$ not causal

We consider Roc $|z|>2$ ~~and~~ since due to being right sided the system is causal

$$H(z) = \frac{1-\frac{1}{3}z^{-1}}{(1-z^{-1})(1+2z^{-1})} = \frac{A}{1-z^{-1}} + \frac{B}{1+2z^{-1}} = \frac{A+2Az^{-1}+B-Bz^{-1}}{(1-z^{-1})(1+2z^{-1})}$$

$$\begin{cases} A+B=1 \\ 2A-B=-\frac{1}{3} \end{cases} \qquad A=\frac{2}{9} \qquad B=\frac{7}{9}$$

$$H(z) = \frac{\frac{2}{9}}{1-z^{-1}} + \frac{\frac{7}{9}}{1+2z^{-1}} \qquad |z|>2$$

$$\boxed{h[n] = \frac{2}{9}u[n] + \frac{7}{9}(-2)^n u[n]}$$

Q3

$$h[n] = h_1[n] * h_2[n] = \{\delta[n-1] + 2\delta[n] + \delta[n+1]\} * h_2[n]$$

$$= \delta[n-1] * h_2[n] + 2\delta[n] * h_2[n] + \delta[n+1] * h_2[n]$$

$$h[n] = h_2[n-1] + 2h_2[n] + h_2[n+1]$$

$$h_2[n] = \delta[n-2] + \delta[n-1] + \delta[n] + \delta[n+1] + \delta[n+2]$$

$$h[n] = \{\delta[n-3] + \delta[n-2] + \delta[n-1] + \delta[n] + \delta[n+1]\} +$$

$$+ 2\{\delta[n-2] + \delta[n-1] + \delta[n] + \delta[n+1] + \delta[n+2]\} +$$

$$+ \{\delta[n-1] + \delta[n] + \delta[n+1] + \delta[n+2] + \delta[n+3]\}$$

$$\boxed{h[n] = \delta[n-3] + 3\delta[n-2] + 4\delta[n-1] + 4\delta[n] + 4\delta[n+1] + 3\delta[n+2] + \delta[n+3]}$$

$$H(e^{j\omega}) = e^{-j3\omega} + 3e^{-j2\omega} + 4e^{-j\omega} + 4 + 4e^{j\omega} + 3e^{j2\omega} + e^{j3\omega}$$

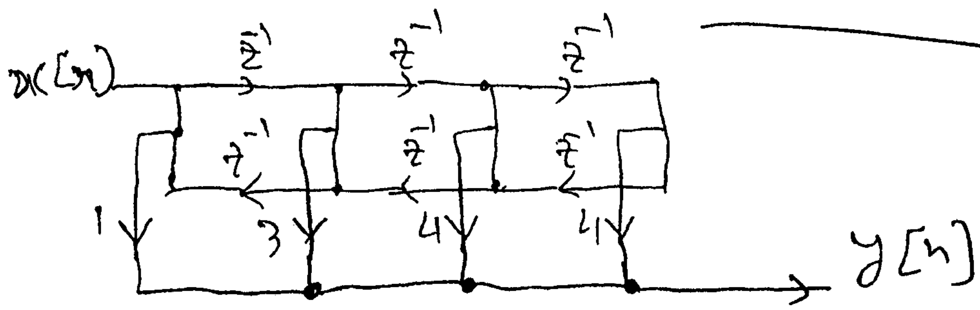
$$H(e^{j\omega}) = 4 + 4(e^{j\omega} + e^{-j\omega}) + 3(e^{j2\omega} + e^{-j2\omega}) + (e^{j3\omega} + e^{-j3\omega})$$

$$\boxed{H(e^{j\omega}) = 4 + 8\cos\omega + 6\cos 2\omega + 2\cos 3\omega}$$

Q4 We shift the impulse response by three samples to have $\underbrace{\text{no}}_{\text{non-zero}}$ component ~~[illegible]~~ for $n<0$.

$$\bar{h}[n] = \delta[n] + 3\delta[n-1] + 4\delta[n-2] + 4\delta[n-3] + 4\delta[n-4] + 3\delta[n-5] + \delta[n-6]$$

$\bar{h}[n]$ is causal



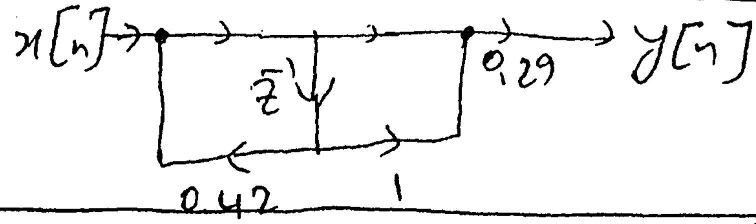
Q5 $H(j\Omega)=\dfrac{a}{j\Omega+a}$ $\quad |H(j\Omega)|=\dfrac{|a|}{\sqrt{\Omega^2+a^2}}$ $\quad |H(j0)|=1$, $|H(ja)|=0.707 \Rightarrow \Omega_c=a$

$$\Omega_c=\frac{2}{T}\tan\frac{\omega_c}{2} \Rightarrow a=\frac{2}{T}\tan\frac{0.25\pi}{2}=\frac{0.828}{T}$$

$$H(s)=\frac{0.828/T}{s+0.828/T} \qquad s=\frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right) \Rightarrow H(z)=\frac{0.29(1+z^{-1})}{1-0.42z^{-2}}$$

$$|H(e^{j0})|=1 \quad \& \quad |H(e^{j0.25\pi})|=\frac{0.29\,|1+\cos\frac{\pi}{4}-j\sin\frac{\pi}{4}|}{|1-0.42\cos\frac{\pi}{4}+j0.42\sin\frac{\pi}{4}|}=0.7$$

therefore 0.25π is cutoff



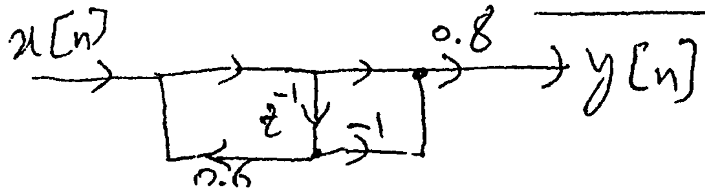
Q6 $\theta_p=0.25\pi$ $\quad \omega_p=0.15\pi$ $\quad \alpha=-\dfrac{\cos\left(\frac{0.25\pi+0.15\pi}{2}\right)}{\cos\left(\frac{0.25\pi-0.15\pi}{2}\right)}=-0.82$

$$z^{-1}\to -\frac{z^{-1}+\alpha}{1+\alpha z^{-1}}=-\frac{z^{-1}-0.82}{1-0.82z^{-1}} \qquad H(z)=\frac{0.29\left(1-\frac{z^{-1}-0.82}{1-0.82z^{-1}}\right)}{1+0.42\frac{z^{-1}-0.82}{1-0.82z^{-1}}}$$

$$H(z)=\frac{0.8(1-z^{-1})}{1-0.6z^{-1}}$$

$$|H(e^{j\pi})|=\frac{0.8(1-e^{-j\pi})}{1-0.6e^{-j\pi}}=1 \qquad |H(e^{j0.15\pi})|=0.7$$

Therefor 0.15π is cut-off



Q7

$$y[n] + \frac{3}{4} y[n-1] + \frac{1}{8} y[n-2] = \frac{1}{4} x[n] + x[n-1]$$

$$Y(z) + \frac{3}{4} z^{-1} Y(z) + \frac{1}{8} z^{-2} Y(z) = \frac{1}{4} X(z) + z^{-1} X(z)$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\frac{1}{4} + z^{-1}}{1 + \frac{3}{4} z^{-1} + \frac{1}{8} z^{-2}} = \frac{\frac{1}{4} z(z+4)}{(z+\frac{1}{2})(z+\frac{1}{4})}$$

Zeros: 0 & -4

Poles: $-\frac{1}{2}$ & $-\frac{1}{4}$

For a minimum phase system, the system and its inverse should be stable and causal. Therefore, all poles and zeros should be inside unit circle.

This system is not min phase since zero at -4 is outside unit circle.

Q8

$$H(z) = \frac{\frac{1}{4} + z^{-1}}{(1+\frac{1}{2} z^{-1})(1 + \frac{1}{4} z^{-1})} = G_{\text{allpass}}(z) \cdot H_{\min}(z)$$

we choose $G(z) = \frac{z^{-1} + \frac{1}{4}}{1 + \frac{1}{4} z^{-1}}$ to remove the zero of -4 from $H(z)$. We know that $|G(e^{j\Omega})| = 1$.

$$\frac{\frac{1}{4} + z^{-1}}{(1+\frac{1}{2} z^{-1})(1+\frac{1}{4} z^{-1})} = \frac{z^{-1} + \frac{1}{4}}{1 + \frac{1}{4} z^{-1}} \cdot H_{\min}(z)$$

$$H_{\min}(z) = \frac{1}{1 + \frac{1}{2} z^{-1}} \rightarrow \boxed{h_{\min}[n] = \left(-\frac{1}{2}\right)^n u[n]}$$

$-\frac{1}{2} < |z|$

Question 9:

$$x_d[n] = x_c(nT) \xleftrightarrow{FT} X_d\left(e^{j\omega}\right) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c\left(\frac{\omega}{T} - j\frac{2k\pi}{T}\right)$$

$$x[n] = \begin{cases} x_d\left[\frac{n}{7}\right] & n = \pm 7k \\ 0 & otherwise \end{cases} \xleftrightarrow{FT} X\left(e^{j\omega}\right) = X_d\left(e^{j7\omega}\right)$$

$$y[n] = x[n] * h[n] \xleftrightarrow{FT} Y\left(e^{j\omega}\right) = H\left(e^{j\omega}\right)X\left(e^{j\omega}\right)$$

$$y_d[n] = y[4n] \xleftrightarrow{FT} Y_d\left(e^{jw}\right) = \frac{1}{4}\sum_{k=0}^{3} Y\left(e^{j\left(\frac{\omega-2k\pi}{4}\right)}\right)$$

$$y_c(t) = \left\{\sum_{n=-\infty}^{\infty} y_d[n]\delta\left(t - \frac{4nT}{7}\right)\right\} * h_{DC}(t) \xleftrightarrow{FT} Y_c(j\Omega) = \begin{cases} \frac{4T}{7} Y_d\left(e^{\frac{j4\Omega T}{7}}\right) & |\Omega| < \frac{\pi}{10T} \\ 0 & otherwise \end{cases}$$

Question 10:

$$y_c(t) = \frac{7}{4T} x_c(t) \xleftrightarrow{FT} Y_c(j\Omega) = \frac{7}{4T} X_c(j\Omega)$$