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Successive Differentiation

Problem 01: If $y = \sin(ax+b)$ or $y = \cos(ax+b)$, Find y_n

Solution: $y = \sin(ax+b)$

Differentiating w.r.to x we get

$$y_1 = \cos(ax+b)\,a$$
$$= a\cos(ax+b) = a\sin\left\{\frac{\pi}{2}+(ax+b)\right\}$$
$$\therefore y_2 = a\cos\left\{\frac{\pi}{2}+(ax+b)\right\}(0+a)$$
$$= a^2\cos\left\{\frac{\pi}{2}+(ax+b)\right\}$$
$$y_2 = a^2\sin\left\{\frac{\pi}{2}+\frac{\pi}{2}(ax+b)\right\} = a^2\sin\left\{\frac{2\pi}{2}+(ax+b)\right\}$$

$$y_n = a^n\sin\left\{\frac{n\pi}{2}+ax+b\right\}$$

Similarly if $y = \cos(ax+b)$ then

$$y_n = a^n\cos\left\{\frac{n\pi}{2}+ax+b\right\}$$

02. If $y = e^{ax}\sin bx$, find y_n.

Solution: Given that, $y = e^{ax}\sin bx$

$$\therefore y_1 = e^{ax}\cos bx\cdot b + e^{ax}a\sin bx$$
$$y_1 = e^{ax}(a\sin bx + b\cos bx)$$

Let $a = r\cos\phi$ and $b = r\sin\phi$, so that

$$r^2 = a^2+b^2 \Rightarrow r = (a^2+b^2)^{\frac{1}{2}}$$

and $\phi = \tan^{-1}\left(\frac{b}{a}\right)$

$\therefore y_1 = e^{ax}(r\cos\phi \sin bx + r\sin\phi \cos bx)$

$y_1 = re^{ax}\sin(bx+\phi)$.

Similarly, $y_2 = r e^{ax}\{a\sin(bx+\phi) + b\cos(bx+\phi)\}$

or, $y_2 = r^2 e^{ax}\sin(bx+2\phi)$

In similar way

$$y_3 = r^3 e^{ax}\sin(bx+3\phi)$$

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$$y_n = r^n e^{ax}\sin(bx+n\phi)$$

$$y_n = (a^2+b^2)^{\frac{n}{2}} e^{ax}\sin\{bx+n\tan^{-1}(\tfrac{b}{a})\}$$ (Answer).

Note That: If $y = e^{ax}\sin(bx+c)$ then

(*) $y_n = (a^2+b^2)^{\frac{n}{2}} e^{ax}\sin\{bx+c+n\tan^{-1}(\tfrac{b}{a})\}$.

or, $y = e^{ax}\cos(bx)$ then

(*) $y_n = (a^2+b^2)^{\frac{n}{2}} e^{ax}\cos\{bx+n\tan^{-1}\tfrac{b}{a}\}$

03. If $y = x^{2n}$, where n is a positive integer, show that
$y_n = 2^n\{1.3.5.\cdots(2n-1)\}x^n$.

Solution: Given that,
$$y = x^{2n}$$
Differentiating w.r to x, we get
$$y_1 = 2n\,x^{2n-1}$$
$$y_2 = 2n(2n-1)x^{2n-2}$$
$$y_3 = 2n(2n-1)(2n-2)x^{2n-3}$$
$$-----------------$$
$$y_n = 2n(2n-1)(2n-2)------\{2n-(n-1)\}x^{2n-n}$$
$$y_n = 2n(2n-1)(2n-2)--------(n+1)x^n$$
$$y_n = \frac{2n(2n-1)(2n-2)-------(n+1)x^n \times n(n-1)(n-2)\cdots3.2.1}{n(n-1)(n-2)----3.2.1}$$
Separating even and odd
$$y_n = \frac{\{2n(2n-2)(2n-4)----4.2\}\{(2n-1)(2n-3)-----5.3.1\}x^n}{n!}$$
$$y_n = \frac{2^n\{n(n-1)(n-2)-----2.1\}\{(2n-1)(2n-3)----5.3.1\}x^n}{n!}$$
$$y_n = 2^n\{(2n-1)(2n-3)-----5.3.1\}x^n.$$
(Showed)

04. If $U = \sin ax + \cos ax$, show that
$U_n = a^n\{1 + (-1)^n \sin 2ax\}^{\frac{1}{2}}$

Solution: Given that, $u = \sin ax + \cos ax$

$\therefore u_1 = a\cos ax - a\sin ax = a\sin\left(\frac{\pi}{2}+ax\right) + a\cos\left(\frac{\pi}{2}+ax\right)$

$u_2 = a\cos\left(\frac{\pi}{2}+ax\right)\cdot a - a\sin\left(\frac{\pi}{2}+ax\right)\cdot a$

$u_2 = a^2\left\{\sin\left(\frac{2\pi}{2}+ax\right) + \cos\left(\frac{2\pi}{2}+ax\right)\right\}$

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$u_n = a^n\left\{\sin\left(\frac{n\pi}{2}+ax\right) + \cos\left(\frac{n\pi}{2}+ax\right)\right\}$

$u_n = a^n\left[\left\{\sin\left(\frac{n\pi}{2}+ax\right) + \cos\left(\frac{n\pi}{2}+ax\right)\right\}^2\right]^{\frac{1}{2}}$

$= a^n\left\{\sin^2\left(\frac{n\pi}{2}+ax\right) + \cos^2\left(\frac{n\pi}{2}+ax\right) + 2\sin\left(\frac{n\pi}{2}+ax\right)\cos\left(\frac{n\pi}{2}+ax\right)\right\}^{\frac{1}{2}}$

$= a^n\left\{1 + \sin 2\left(\frac{n\pi}{2}+ax\right)\right\}^{\frac{1}{2}}$

$= a^n\left\{1 + \sin(n\pi + 2ax)\right\}^{\frac{1}{2}}$

$= a^n\left\{1 + \sin n\pi \cos 2ax + \cos n\pi \cdot \sin 2ax\right\}^{\frac{1}{2}}$

$= a^n\left\{1 + 0 + (-1)^n \sin 2ax\right\}^{\frac{1}{2}}$ | since $\sin n\pi = 0$, $\cos n\pi = (-1)^n$

$u_n = a^n\left\{1 + (-1)^n \sin 2ax\right\}^{\frac{1}{2}}$ (showed)

Do Yourself: If $ax^2 + 2hxy + by^2 = 1$ show that

$$\frac{d^2y}{dx^2} = \frac{h^2 - ab}{(hx+by)^3}.$$

05. Theorem : State and Prove Leibnitz's theorem for nth derivative.

Statement : If U and V are two functions of x, then n-th derivative of their product

i.e $(UV)_n = U_nV + {}^nC_1 U_{n-1}V_1 + {}^nC_2 U_{n-2}V_2 + \cdots + {}^nC_r U_{n-r}V_r + \cdots + UV_n$.

where the suffixes in U and V denote the order of differentiations of U and V with respect to x.

Proof : let $y = UV$

By actual differentiation, we have

$$y_1 = U_1V + UV_1$$

$$y_2 = U_2V + U_1V_1 + U_1V_1 + UV_2 = U_2V + 2U_1V_1 + UV_2$$

$$y_2 = U_2V + 2U_1V_1 + UV_2 = U_2V + {}^2C_1U_1V_1 + UV_2$$

$$y_3 = U_3V + U_2V_1 + 2U_1V_2 + 2U_2V_1 + U_1V_2 + UV_3$$

$$y_3 = U_3V + 3U_2V_1 + 3U_1V_2 + UV_3$$

$$y_3 = U_3V + {}^3C_1U_2V_1 + {}^3C_2U_1V_2 + UV_3$$

The theorem is thus seen to be true when $n = 2$ or 3.

Let us assume that

$$y_n = U_nV + {}^nC_1U_{n-1}V_1 + {}^nC_2U_{n-2}V_2 + \cdots + {}^nC_rU_{n-r}V_r + \cdots + UV_n. \rightarrow (1)$$

Where n has any particular value.

Differentiating equation (1).

$$y_{n+1} = U_{n+1}V + U_nV_1 + {}^nC_1U_nV_1 + {}^nC_1U_{n-1}V_2 + {}^nC_2U_{n-1}V_2 + {}^nC_2U_{n-2}V$$

$+ \cdots + {}^{n}C_r U_{n-r+1} V_r + {}^{n}C_r U_{n-r} V_{r+1} + \cdots + U_1 V_n + U.$

$$y_{n+1} = U_{n+1}V + ({}^{n}C_1 + 1) U_n V_1 + ({}^{n}C_2 + {}^{n}C_1) U_{n-1} V_2 + \cdots + ({}^{n}C_r + {}^{n}C_{r-1}) U_{n-r}$$
$$+ \cdots + UV_{n+1}.$$

Since ${}^{n}C_r + {}^{n}C_{r-1} = {}^{n+1}C_r$ and ${}^{n}C_1 + 1 = {}^{n+1}C_1$.

$$\therefore y_{n+1} = U_{n+1}V + {}^{n+1}C_1 U_n V_1 + {}^{n+1}C_2 U_{n-1} V_2 + \cdots + {}^{n+1}C_r U_{n-r} V$$
$$+ \cdots + UV_{n+1}.$$

Thus if the theorem holds for n differentiations, it also holds for $n+1$. But it was proved to hold for 2 and 3 differentiations. Hence it holds for four, and so on, hence the theorem is true for every positive integral value of n.

06. If $y = a\cos(\log x) + b\sin(\log x)$, show that
$$x^2 y_2 + x y_1 + y = 0.$$

Solution : Given that, $y = a\cos(\log x) + b\sin(\log x) \longrightarrow (1)$

Differentiating,

$$y_1 = a\{-\sin(\log x)\} \cdot \frac{1}{x} + b\cos(\log x) \cdot \frac{1}{x}$$
$$\therefore x y_1 = -a\sin(\log x) + b\cos(\log x)$$

Differentiating again

$$x y_2 + 1 \cdot y_1 = -a\cos(\log x)\frac{1}{x} - b\sin(\log x) \cdot \frac{1}{x}$$
$$\Rightarrow x^2 y_2 + x y_1 = -\{a\cos(\log x) + b\sin(\log x)\} = -y \quad [\text{using } (1)]$$
$$\therefore x^2 y_2 + x y_2 + y = 0 \quad \text{(Showed)}.$$

(07) If $y = \tan^{-1}x$, then (i) $(1+x^2)y_1 = 1$ and
(ii) $(1+x^2)y_{n+1} + 2nxy_n + n(n-1)y_{n-1} = 0$ Find also the value $(y_n)_0$.

Solution : Given that,

$$y = \tan^{-1}x$$

Differentiating

$$y_1 = \frac{1}{1+x^2}$$

$$\Rightarrow (1+x^2)y_1 = 1 \longrightarrow (1)$$

Differentiating this n-times by Leibnitz's theorem

$$(1+x^2)y_{n+1} + {}^nC_1\, 2x y_n + {}^nC_2\, y_{n-1} \cdot 2 = 0$$

or, $(1+x^2)y_{n+1} + 2nxy_n + \frac{n(n-1)}{2} y_{n-1} \cdot 2 = 0$

or, $(1+x^2)y_{n+1} + 2nxy_n + n(n-1)y_{n-1} = 0 \longrightarrow (2)$ (Proved).

Put $x=0$ in equation (1) & (2) we get

$$(y_1)_0 = 1$$

$$(y_{n+1})_0 + n(n-1)(y_{n-1})_0 = 0$$

$$(y_{n+1})_0 = -n(n-1)(y_{n-1})_0 \longrightarrow (3)$$

Again, from (1) $\quad 2xy_1 + (1+x^2)y_2 = 0$.

Also put $x=0$, $\quad 0 + (y_2)_0 = 0$

$$\therefore (y_2)_0 = 0.$$

Put $n = 3, 5, 7$ -------- in equation (3), we get

$(y_4)_0 = -n(n-1)(y_2)_0 = 0$, since $(y_2)_0 = 0$

$(y_6)_0 = -n(n-1)(y_4)_0 = 0$

Thus $(y_n)_0 = 0$ when n is even.

Again, Put $n = 2, 3, 5$ in eq^n (3) we get

$(y_3)_0 = -2\cdot 1\,(y_1)_0 = -2\cdot 1\cdot 1$

$(y_5)_0 = -4\cdot 3\,(y_3)_0 = (-1)^2\, 4\cdot 3\cdot 2\cdot 1$

$= (-1)^{\frac{5-1}{2}}\, 4\cdot 3\cdot 2\cdot 1 = (-1)^{\frac{5-1}{2}}(5-1)(5-2)\cdots$

Thus $(y_n)_0 = (-1)^{\frac{n-1}{2}}(n-1)(n-2)\cdots 3\cdot 2\cdot 1$

when n is odd.

Q8. If $y = \sin^{-1}x$ then (i) $(1-x^2)y_2 - xy_1 = 0$ and (ii) $(1-x^2)y_{n+2} - (2n+1)xy_{n+1} - n^2 y_n = 0$. Find also the value of $(y_n)_0$.

Solution: Given that,

$$y = \sin^{-1}x$$

Differentiating, $y_1 = \dfrac{1}{\sqrt{1-x^2}}$

$$\sqrt{1-x^2}\, y_1 = 1$$

or, $(1-x^2)\, y_1^2 = 1 \rightarrow (1)$ [squaring both sides]

Again, differentiating

$$(1-x^2)\, 2y_1 y_2 - 2x\, y_1^2 = 0$$

$$(1-x^2)\, y_2 - xy_1 = 0 \longrightarrow (2)$$

Applying Leibnitz's theorem for n-th derivatives.

$$(1-x^2)\, y_{n+2} + {}^nC_1\, y_{n+1}\,(-2x) + {}^nC_2\, y_n\cdot(-2) - xy_{n+1} - {}^nC_1\, y_n = 0$$

$$(1-x^2)y_{n+2} - 2nxy_{n+1} - \frac{n(n-1)}{2}y_n \cdot 2 - xy_{n+1} - ny_n = 0$$

or, $(1-x^2)y_{n+2} - 2nxy_{n+1} - n^2y_n + ny_n - xy_{n+1} - ny_n = 0$

or, $(1-x^2)y_{n+2} - (2n+1)xy_{n+1} - n^2y_n = 0 \longrightarrow ③$

(Proved)

Put $x=0$ in equation ①, ② and ③ we get

$(y_1)_0 = 1, \quad (y_2)_0 = 0$

$(y_{n+2})_0 - n^2(y_n)_0 = 0$

or, $(y_{n+2})_0 = n^2(y_n)_0 \longrightarrow ④$

Put $n = 1, 3, 5$ in ④ we get

$(y_3)_0 = 1^2 (y_1)_0 = 1^2 \cdot 1$

$(y_5)_0 = 3^2 (y_3)_0 = 3^2 \cdot 1^2 \cdot 1$

$(y_7)_0 = 5^2 (y_5)_0 = 5^2 \cdot 3^2 \cdot 1^2 \cdot 1$

$= (7-2)^2 (7-4)^2 \cdot (7-5)^2 \cdot 1$

Hence ~~n is~~ when n is odd

$(y_n)_0 = (n-2)^2 (n-4)^2 (n-5)^2 \cdots 3^2 \cdot 1^2 \cdot 1.$

Again putting $n = 2, 4, 6$ in equation ④ we get

$(y_4)_0 = 2^2 (y_2)_0 = 0$

$(y_6)_0 = 4^2 (y_4)_0 = 0$

Thus when n is even $(y_n)_0 = 0.$

9. If $y = (\sin^{-1}x)^2$, then $(1-x^2)y_{n+2} - (2n+1)xy_{n+1} - n^2 y_n =$

Answer: Given that, $y = (\sin^{-1}x)^2 \longrightarrow (1)$

Differentiating,

$y_1 = 2\sin^{-1}x \dfrac{1}{\sqrt{1-x^2}}$

$\Rightarrow \sqrt{1-x^2}\, y_1 = 2\sin^{-1}x$

or, $(1-x^2)y_1^2 = 4(\sin^{-1}x)^2$

or, $(1-x^2)y_1^2 = 4y$ [Using eq^n (1)]

Again, differentiating

$(1-x^2)2y_1y_2 - 2xy_1^2 = 4y_1$

$2y_1\{(1-x^2)y_2 - xy_1\} = 4y_1$

$\therefore (1-x^2)y_2 - xy_1 - 2 = 0$

Applying leibnitz's theorem

$(1-x^2)y_{n+2} + {}^nC_1 y_{n+1}(-2x) + {}^nC_2 y_n(-2) - xy_{n+1} - {}^nC_1 y_n$

or, $(1-x^2)y_{n+2} - 2nxy_{n+1} - 2\dfrac{n(n-1)}{2}y_n - xy_{n+1} - ny_n = 0$

or, $(1-x^2)y_{n+2} - 2nxy_{n+1} - n^2y_n + ny_n - xy_{n+1} - ny_n = 0$

or, $(1-x^2)y_{n+2} - (2n+1)xy_{n+1} - n^2y_n = 0$ (Proved)

10. If $\log y = \tan^{-1}x$, show that

$(1+x^2)y_{n+2} + (2nx+2x-1)y_{n+1} + n(n+1)y_n = 0$

Solution: Given that, $\log y = \tan^{-1}x \longrightarrow (1)$

or, $\dfrac{1}{y}y_1 = \dfrac{1}{1+x^2}$

or, $(1+x^2)y_1 = y$

Again,

$(1+x^2)y_2 + 2xy_1 = y_1$

or, $(1+x^2)y_2 + (2x-1)y_1 = 0$

Appling leibnitz's theorem

$(1+x^2)y_{n+2} + {}^nC_1 y_{n+1}\cdot 2x + {}^nC_2 y_n \cdot 2 + (2x-1)y_{n+1} + {}^nC_1 y_n = 0$

$\Rightarrow (1+x^2)y_{n+2} + 2nx\,y_{n+1} + \frac{n(n-1)}{2}\cdot 2y_n + (2x-1)y_{n+1} + 2ny_n = 0$

or, $(1+x^2)y_{n+2} + (2nx + 2x - 1)y_{n+1} + (n^2 - n + 2n)y_n = 0$

or $(1+x^2)y_{n+2} + (2nx + 2x - 1)y_{n+1} + (n^2 + n)y_n = 0$

Hence $(1+x^2)y_{n+1} + (2nx + 2x - 1)y_{n+1} + n(n+1)y_n = 0$. (Showed)

(11). If $y = a\cos(\log x) + b\sin(\log x)$ then prove that $x^2y_{n+2} + (2n+1)xy_{n+1} + (n^2+1)y_n = 0$

Sol^n:

1st You Do Problem (6)

$\therefore x^2y_2 \pm xy_1 + y = 0$

Applying leibntz's theorem

$x^2y_{n+2} + {}^nC_1 y_{n+1}\cdot 2x + {}^nC_2 y_n \cdot 2 + xy_{n+1} + {}^nC_1 y_n \cdot 1 + y_n = 0$

or, $x^2y_{n+2} + 2nxy_{n+1} + \frac{n(n-1)}{2}\cdot y_n \cdot 2 \pm xy_{n+1} + ny_n + y_n = 0$

or, $x^2y_{n+2} + 2nx\,y_{n+1} + (n^2 - n + n + 1)y_n + xy_{n+1} = 0$

or, $x^2y_{n+2} + (2n+1)y_{n+1} \mp (n^2+1)y_n = 0$

(Proved)

Problem 12: If $y = (x^2-1)^n$ then
$(x^2-1)y_{n+2} + 2xy_{n+1} - n(n+1)y_n = 0$.

Solution: Given that, $y = (x^2-1)^n \longrightarrow (1)$

By Differentiating

$y_1 = n(x^2-1)^{n-1} \cdot 2x$

or, $(x^2-1)y_1 = n(x^2-1)^{n-1} \cdot 2x(x^2-1)$

or, $(x^2-1)y_1 = 2nx(x^2-1)^n = 2nxy$ [using (1)]

Again diff. w.r to x

$(x^2-1)y_2 + 2xy_1 = 2nxy_1 + 2n \cdot y$

or, $(x^2-1)y_2 + (2x - 2nx)y_1 - 2ny = 0$

or $(x^2-1)y_2 + 2(1-n)xy_1 - 2ny = 0$

Applying Leibnitz's theorem we get

$(x^2-1)y_{n+2} + {}^nC_1 y_{n+1} 2x + {}^nC_2 y_n \cdot 2 + 2(1-n)xy_{n+1} + 2(1-n){}^nC_1 y_n \cdot 1 - 2ny_n = 0$

$\Rightarrow (x^2-1)y_{n+2} + 2nxy_{n+1} + \frac{n(n-1)}{2} \cdot 2y_n + (2-2n)xy_{n+1} + 2n(1-n)y_n - 2ny_n = 0$

$\Rightarrow (x^2-1)y_{n+2} + (2n+2-2n)xy_{n+1} + (n^2 - n + 2n - 2n^2 - 2n)y_n = 0$

$\Rightarrow (x^2-1)y_{n+2} + 2xy_{n+1} + (-n^2-n)y_n = 0$

$\therefore (x^2-1)y_{n+2} + 2xy_{n+1} - n(n+1)y_n = 0$

(Proved)

13. If $y = e^{a\sin^{-1}x}$, then $(1-x^2)y_{n+2} - (2n+1)xy_{n+1} - (n^2+a^2)y_n = 0$.

Soln: Given that, $y = e^{a\sin^{-1}x} \longrightarrow (1)$

By actual differentiating

$$y_1 = e^{a\sin^{-1}x} \cdot \frac{a}{\sqrt{1-x^2}}$$

$$\Rightarrow \sqrt{1-x^2}\, y_1 = a e^{a\sin^{-1}x}$$

$$\Rightarrow (1-x^2) y_1^2 = a^2 \left(e^{a\sin^{-1}x}\right)^2 = a^2 y^2 \quad [\text{using } (1)]$$

$$\therefore 2y_1(1-x^2)y_2 + y_1^2(-2x) = a^2 \cdot 2y y_1$$

$$\Rightarrow 2(1-x^2)y_1 y_2 - 2xy_1^2 - 2a^2 y y_1 = 0$$

$$\Rightarrow (1-x^2)y_2 - xy_1 - a^2 y = 0 \quad \text{dividing by } 2y_1$$

Applying Leibnitz's theorem

$$(1-x^2)y_{n+2} + {}^nC_1 y_{n+1}(-2x) + {}^nC_2 y_n(-2) - (xy_{n+1} + {}^nC_1 y_n \cdot 1) - a^2 y_n = 0$$

$$\Rightarrow (1-x^2)y_{n+2} + n y_{n+1}(-2x) + \frac{n(n-1)}{2} \cdot y_n(-2) - xy_{n+1} - ny_n - a^2 y_n = 0$$

$$\Rightarrow (1-x^2)y_{n+2} - 2xn y_{n+1} + (-n^2+n)y_n - xy_{n+1} - ny_n - a^2 y_n = 0$$

$$\Rightarrow (1-x^2)y_{n+2} - (2n+1)xy_{n+1} + (-n^2+n-n-a^2)y_n = 0$$

$$\Rightarrow (1-x^2)y_{n+2} - (2n+1)xy_{n+1} - (n^2+a^2)y_n = 0$$

14. If $y = \sin(m\sin^{-1}x)$, then $(1-x^2)y_{n+2} - (2n+1)xy_{n+1} + (m^2-n^2)y_n = 0$

Solution: Given that,

$$y = \sin(m\sin^{-1}x)$$

By actual differentiating

$y_1 = \cos(m \sin^{-1}x) \cdot m \frac{1}{\sqrt{1-x^2}}$

$\Rightarrow (\sqrt{1-x^2})\, y_1 = m \cos(m \sin^{-1}x)$

$\Rightarrow (1-x^2)\, y_1^2 = m^2 \cos^2(m \sin^{-1}x) = m^2 \{1 - \sin^2(m \sin^{-1}x)\}$

$(1-x^2)\, y_1^2 = m^2 - m^2 y^2$ [using eqn ①]

$\therefore (1-x^2)\, y_1^2 + m^2 y^2 = m^2$

Again, differentiating

$(1-x^2)\, 2y_1 y_2 + y_1^2 (-2x) + m^2 \cdot 2y y_1 = 0$

$(1-x^2)\, y_2 - x y_1 + m^2 y = 0$

Applying Leibnitz's theorem

$(1-x^2)\, y_{n+2} + {}^nC_1\, y_{n+1} (-2x) + {}^nC_2\, y_n (-2) - (x y_{n+1} + {}^nC_1\, y_n \cdot 1) + m^2 y_n = 0$

$\Rightarrow (1-x^2)\, y_{n+2} + n\, y_{n+1} (-2x) + \frac{n(n-1)}{2} y_n (-2) - x y_{n+1} - n y_n + m^2 y_n = 0$

$\Rightarrow (1-x^2)\, y_{n+2} - 2nx\, y_{n+1} - n(n-1) y_n - x y_{n+1} - n y_n + m^2 y_n = 0$

$\Rightarrow (1-x^2)\, y_{n+2} - (2n+1) x\, y_{n+1} + (m^2 - n^2 + n - n)\, y_n = 0.$

$\therefore (1-x^2)\, y_{n+2} - (2n+1) x\, y_{n+1} + (m^2 - n^2)\, y_n = 0$ (showed)

15) If $y = x^{n-1} \log x$, then prove that $y_n = \frac{(n-1)!}{x}$

Soln: Given that, $y = x^{n-1} \log x \longrightarrow ①$

By actual differentiating we have

$$y_1 = x^{n-1}\,\cancel{\log}\,\frac{1}{x} + (n-1)\,x^{n-2}\log x$$

$$y_1 = (n-1)\,x^{n-2}\log x + x^{n-2}$$

$$y_2 = (n-1)(n-2)\,x^{n-3}\log x + (n-1)\,x^{n-2}\cdot\frac{1}{x} + (n-2)\,x^{n-3}$$

$$y_2 = (n-1)(n-2)\,x^{n-3}\log x + (n-1)\,x^{n-3} + (n-2)\,x^{n-3}$$

- -

- -

$$y_{n-1} = (n-1)(n-2)(n-3)\cdots 3.2.1\,\log x + \text{constent}$$

$$y_n = (n-1)(n-2)(n-3)\cdots 3.2.1\,\frac{1}{x} + 0$$

$$y_n = \frac{(n-1)!}{x}$$ (Proved)