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DE MOIVERE'S THEOREM

Examples - VII

2008(00)

2. If $x+\frac{1}{x}=2\cos\theta$ and $y+\frac{1}{y}=2\cos\phi$, then prove that $\cos(\theta+\phi)$ is one of the values of $\frac{1}{2}\left(xy+\frac{1}{xy}\right)$ and that one of the values of $\frac{x^m}{y^n}+\frac{y^n}{x^m}$ is $2\cos(m\theta-n\phi)$.

Sol^n. Given that

$$x+\frac{1}{x}=2\cos\theta$$

$$x^2-2\cos\theta\, x+1=0$$

$$\therefore x=\frac{2\cos\theta\pm\sqrt{4\cos^2\theta-4}}{2.1}$$

$$=\cos\theta\pm i\sin\theta$$

$\therefore x=\cos\theta+i\sin\theta$ [taking positive sign]

similarly, $y=\cos\phi+i\sin\phi$

$$\therefore xy=(\cos\theta+i\sin\theta)(\cos\phi+i\sin\phi)$$

$$=\cos(\theta+\phi)+i\sin(\theta+\phi)$$

also $\frac{1}{xy}=\frac{1}{x}\cdot\frac{1}{y}=\frac{1}{\cos\theta+i\sin\theta}\cdot\frac{1}{\cos\phi+i\sin\phi}$

$$=(\cos\theta+i\sin\theta)^{-1}\cdot(\cos\phi+i\sin\phi)^{-1}$$

$$=\{\cos(-\theta)+i\sin(-\theta)\}\{\cos(-\phi)+i\sin(-\phi)\}$$

$$=(\cos\theta-i\sin\theta)(\cos\phi-i\sin\phi)$$ [using De moivers Theorem]

$$= \{\cos(\theta+\phi) + i\sin(\theta+\phi)\}^{-1}$$

$$= \cos(\theta+\phi) - i\sin(\theta+\phi)$$

$$\therefore xy + \frac{1}{xy} = 2\cos(\theta+\phi)$$

$$\therefore \frac{1}{2}\left(xy + \frac{1}{xy}\right) = \cos(\theta+\phi)$$

Again, $x^m = \cos m\theta + i\sin m\theta$

and $y^n = \cos n\phi + i\sin n\phi$

$$\therefore \frac{x^m}{y^n} + \frac{y^n}{x^m} = (\cos m\theta + i\sin m\theta)(\cos n\phi - i\sin n\phi)$$

$$+ (\cos n\phi + i\sin n\phi)(\cos m\theta - i\sin m\theta)$$

$$= \cos m\theta \cos n\phi \cancel{- i\cos m\theta \sin n\phi} + \cancel{i\sin m\theta \cos n\phi}$$

$$+ \sin m\theta \sin n\phi + \cos n\phi \cos m\theta \cancel{- i\cos n\phi \sin m\theta}$$

$$\cancel{+ i\sin n\phi \cos m\theta} + \sin n\phi \sin m\theta$$

$$= 2[\cos m\theta \cos n\phi + \sin m\theta \sin n\phi]$$

$$= 2\cos(m\theta - n\phi)$$

2. (ii) If $a = \cos\theta + i\sin\theta$, $b = \cos\phi + i\sin\phi$, find the values of $\cos(\theta+\phi)$ and $\cos(\theta-\phi)$ in terms of a, b

Sol^n: Given that

$$a = \cos\theta + i\sin\theta, \quad b = \cos\phi + i\sin\phi$$

$$\therefore ab = (\cos\theta + i\sin\theta).(\cos\phi + i\sin\phi)$$

$$= \cos\theta\cos\phi + i\cos\theta\sin\phi + i\sin\theta\cos\phi - \sin\theta\sin\phi$$

$$= \cos(\theta+\phi) + i\sin(\theta+\phi)$$

$$\therefore ab^{-1} = \{\cos(\theta+\phi) + i\sin(\theta+\phi)\}^{-1}$$

$$\therefore \frac{1}{ab} = \cos(\theta+\phi) - i\sin(\theta+\phi)$$

$$\therefore ab + \frac{1}{ab} = 2\cos(\theta+\phi)$$

$$\therefore \cos(\theta+\phi) = \frac{1}{2}\left(ab + \frac{1}{ab}\right) \text{ Ans.}$$

Again $\frac{a}{b} + \frac{b}{a} = \frac{\cos\theta + i\sin\theta}{\cos\phi + i\sin\phi} + \frac{\cos\phi + i\sin\phi}{\cos\theta + i\sin\theta}$

$$= \cancel{(\cos\phi + i\sin\phi)(\cos\theta + i\sin\theta)}$$

$$= (\cos\theta + i\sin\theta)(\cos\phi + i\sin\phi)^{-1} + (\cos\phi + i\sin\phi)(\cos\theta + i\sin\theta)$$

$$= (\cos\theta + i\sin\theta)(\cos\phi - i\sin\phi) + (\cos\phi + i\sin\phi)(\cos\theta - i\sin\theta)$$

$$= \cos\theta\cos\phi - \cancel{i\cos\theta\sin\phi} + \cancel{i\sin\theta\cos\phi} + \sin\theta\sin\phi + \cos\theta\cos\phi$$

$$\cancel{- i\cos\phi\sin\theta} + \cancel{i\sin\phi\cos\theta} + \sin\phi\sin\theta$$

$$= 2[\cos\theta\cos\phi + \sin\theta\sin\phi] = 2\cos(\theta-\phi)$$

$$\therefore \cos(\theta-\phi) = \frac{1}{2}\left(\frac{a}{b} + \frac{b}{a}\right) \text{ Ans.}$$

6. Find the equation whose roots are the 7th powers of the roots of the equation.

$$x^2 - 2x\cos\theta + 1 = 0.$$

Solⁿ: Given that

$$x^2 - 2x\cos\theta + 1 = 0$$

$$\therefore x = \frac{2\cos\theta \pm \sqrt{4\cos^2\theta - 4}}{2}$$

$$= \cos\theta \pm i\sin\theta$$

Let $x_1 = \cos\theta + i\sin\theta$ & $x_2 = \cos\theta - i\sin\theta$

$\therefore x_1^7 = \cos 7\theta + i\sin 7\theta$ & $x_2^7 = \cos 7\theta - i\sin 7\theta$

Now, we form an equation whose roots are x_1^7 & x_2^7

$$\therefore t^2 - t(x_1^7 + x_2^7) + x_1^7 x_2^7 = 0$$

Now, $x_1^7 + x_2^7 = 2\cos 7\theta$

and $x_1^7 x_2^7 = (\cos 7\theta + i\sin 7\theta)(\cos 7\theta - i\sin 7\theta)$

$$= \cos^2 7\theta - \cancel{i\sin 7\theta\cos 7\theta} + \cancel{i\sin 7\theta\cos 7\theta} + \sin^2 7\theta$$

$$= 1$$

Required equation is

$$t^2 - 2\cos 7\theta\, t + 1 = 0 \text{ Ans.}$$

5. If $(a_1+ib_1)(a_2+ib_2)\cdots\cdots(a_n+ib_n) = A+iB$, then show that

(i) $\tan^{-1}\frac{b_1}{a_1} + \tan^{-1}\frac{b_2}{a_2} + \cancel{\tan^{-1}\frac{b_3}{a_3}} + \cdots\cdots + \tan^{-1}\frac{b_n}{a_n} = \tan^{-1}\frac{B}{A}$

(ii) $(a_1^2+b_1^2)(a_2^2+b_2^2)\cdots(a_n^2+b_n^2) = A^2+B^2$

<u>Soln.</u> Let, $a_1 = r_1\cos\theta_1$ and $\cancel{b_2 = r_2\cos\theta_2}$

$b_1 = r_1\sin\theta_1$

$\therefore r_1^2 = a_1^2+b_1^2 \Rightarrow r_1 = \sqrt{a_1^2+b_1^2}$

and $\tan\theta_1 = \frac{b_1}{a_1} \Rightarrow \theta_1 = \tan^{-1}\frac{b_1}{a_1}$

similarly,

$r_2 = \sqrt{a_2^2+b_2^2}$ and $\theta_2 = \tan^{-1}\frac{b_2}{a_2}$

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$r_n = \sqrt{a_n^2+b_n^2}$ and $\theta_n = \tan^{-1}\frac{b_n}{a_n}$

Now $a_1+ib_1 = r_1(\cos\theta_1+i\sin\theta_1)$

$a_2+ib_2 = r_2(\cos\theta_2+i\sin\theta_2)$

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$a_n+ib_n = r_n(\cos\theta_n+i\sin\theta_n)$

$\therefore (a_1+ib_1)(a_2+ib_2)\cdots\cdots(a_n+ib_n) = (r_1 r_2 \cdots\cdots r_n)\{\cos(\theta_1+\theta_2\cdots+\theta_n)$
$+i\sin(\theta_1+\theta_2+\cdots+\theta$

$\therefore A+iB = (r_1 r_2 \cdots r_n)\{\cos(\theta_1+\theta_2+\cdots+\theta_n)+i\sin(\theta_1+\theta_2+\cdots+\theta_n)$

by the given equati

Equating real & imaginary part on both sides

$$r_1 r_2 \cdots\cdots r_n \cos(\theta_1+\theta_2+\cdots\cdots+\theta_n) = A \quad \text{——— (I)}$$

$$r_1 r_2 \cdots\cdots r_n \sin(\theta_1+\theta_2+\cdots\cdots+\theta_n) = B \quad \text{——— (II)}$$

From (II) ÷ (I) we have,

$$\tan(\theta_1+\theta_2+\cdots\cdots+\theta_n) = \frac{B}{A}$$

$$\therefore \theta_1+\theta_2+\cdots\cdots\cdots+\theta_n = \tan^{-1}\frac{B}{A} \quad \text{——— (III)}$$

Now putting the values of $\theta_1, \theta_2 \cdots \theta_n$ in (III) we have,

$$\tan^{-1}\frac{b_1}{a_1} + \tan^{-1}\frac{b_2}{a_2} + \cdots\cdots + \tan^{-1}\frac{b_n}{a_n} = \tan^{-1}\frac{B}{A}$$ (showed)

(II) Again, squaring and adding (I) & (II), we have,

$$r_1^2 r_2^2 \cdots r_n^2 \{\cos^2(\theta_1+\theta_2+\cdots+\theta_n) + \sin^2(\theta_1+\theta_2+\cdots\theta_n)\} = A^2+B^2$$

$$\therefore r_1^2 r_2^2 \cdots\cdots r_n^2 = A^2+B^2$$

putting the values of $r_1, r_2 \cdots r_n$ we have

$$(a_1^2+b_1^2)(a_2^2+b_2^2)\cdots\cdots(a_n^2+b_n^2) = A^2+B^2$$ (Proved)

(7.) If $2\cos\alpha = a + a^{-1}$, $2\cos\beta = b + b^{-1}$ etc, then show that, under certain conditions,

(i) $2\cos(\alpha+\beta+\gamma+\cdots) = abc + \cdots + \frac{1}{abc\cdots}$

(ii) $2\cos(p\alpha+q\beta+r\gamma+\cdots) = a^p b^q c^r + \cdots + \frac{1}{a^p b^q c^r \cdots}$.

Soln: we have,

$$2\cos\alpha = a + a^{-1} = a + \frac{1}{a}$$

$$a^2 - 2\cos\alpha\, a + 1 = 0$$

$$\therefore a = \frac{2\cos\alpha \pm \sqrt{4\cos^2\alpha - 4}}{2} = \cos\alpha \pm i\sin\alpha$$

Taking positive sign we have,

$$a = \cos\alpha + i\sin\alpha = e^{i\alpha}$$

and $\frac{1}{a} = a^{-1} = (\cos\alpha + i\sin\alpha)^{-1} = (\cos\alpha - i\sin\alpha) = e^{-i\alpha}$

similarly $b = e^{i\beta}$, & $\frac{1}{b} = e^{-i\beta}$

$c = e^{i\gamma}$ & $\frac{1}{c} = e^{-i\gamma}$

Now, $abc\cdots + \frac{1}{abc\cdots} = (e^{i\alpha}.e^{i\beta}.e^{i\gamma}\cdots) + \frac{1}{e^{i\alpha}.e^{i\beta}.e^{i\gamma}\cdots}(e^{-i\alpha}.e^{-i\beta}.e^{-i\gamma}\cdots)$

$$= e^{i(\alpha+\beta+\gamma+\cdots)} + e^{-i(\alpha+\beta+\gamma+\cdots)}$$

$$= \{\cos(\alpha+\beta+\gamma+\cdots) + i\sin(\alpha+\beta+\gamma+\cdots)\} + \{\cos(\alpha+\beta+\gamma+\cdots) - i\sin(\alpha+\beta+\gamma+\cdots)\}$$

$$= 2\cos(\alpha+\beta+\gamma+\cdots)$$

$$\therefore 2\cos(\alpha+\beta+\gamma+\cdots) = abc\cdots + \frac{1}{abc\cdots}$$

(Showed)

Again we have,

$$a^p = e^{ip\alpha} \text{ and } \frac{1}{a^p} = e^{-ip\alpha}$$

$$b^q = e^{iq\beta} \text{ and } \frac{1}{b^q} = e^{-iq\beta}$$

$$c^r = e^{ir\gamma} \text{ and } \frac{1}{c^r} = e^{-ir\gamma}$$

Now,

$$a^p b^q c^r \ldots\ldots + \frac{1}{a^p b^q c^r \ldots\ldots} = (e^{ip\alpha} \cdot e^{iq\beta} \cdot e^{ir\gamma} \ldots\ldots) + (e^{-ip\alpha} \cdot e^{-iq\beta} \cdot e^{-ir\gamma} \ldots\ldots)$$

$$= e^{i(p\alpha + q\beta + r\gamma + \ldots\ldots)} + e^{-i(p\alpha + q\beta + r\gamma + \ldots\ldots)}$$

$$= \{\cos(p\alpha + q\beta + r\gamma \ldots) + i\sin(p\alpha + q\beta + r\gamma + \ldots\ldots)\}$$

$$+ \{\cos(p\alpha + q\beta + r\gamma + \ldots\ldots) - i\sin(p\alpha + q\beta + r\gamma + \ldots\ldots)\}$$

$$= 2\cos(p\alpha + q\beta + r\gamma + \ldots\ldots)$$

$$\therefore 2\cos(p\alpha + q\beta + r\gamma + \ldots\ldots) = a^p b^q c^r \ldots\ldots + \frac{1}{a^p b^q c^r \ldots\ldots}$$ (Showed)

9. If $(1+x)^n = P_0 + P_1 x + P_2 x^2 + \cdots$, then show that

$$P_0 - P_2 + P_4 - \cdots\cdots = 2^{n/2} \cos \tfrac{1}{4} n\pi,$$

$$P_1 - P_3 + P_5 - \cdots\cdots = 2^{n/2} \sin \tfrac{1}{4} n\pi,$$

Let, $1+i = r(\cos\theta + i\sin\theta)$

$r\cos\theta = 1$ & $r\sin\theta = 1$

$\therefore r = \sqrt{1^2+1^2} = \sqrt{2}$

$\therefore \tan\theta = 1$

$\therefore \theta = \frac{\pi}{4}$.

Now, put $x = i$, we get

$$(1+i)^n = P_0 + P_1 i - P_2 - iP_3 + P_4 + iP_5 - \cdots\cdots$$

$$= (P_0 + P_2 + P_4 - \cdots\cdots) + i(P_1 - P_3 + P_5 - \cdots\cdots)$$

$$r^n(\cos\theta + i\sin\theta)^n = (P_0 - P_2 + P_4 - \cdots\cdots) + i(P_1 - P_3 + P_5 - \cdots\cdots)$$

Equating real & imaginary parts on both sides we get,

$$P_0 - P_2 + P_4 - \cdots\cdots = r^n \cos n\theta = (\sqrt{2})^n \cos(n \times \tfrac{\pi}{4})$$

$$= 2^{n/2} \cos \frac{n\pi}{4}$$

and $P_1 - P_3 + P_5 - \cdots\cdots = r^n \sin n\theta = 2^{n/2} \sin \frac{n\pi}{4}$ Ans

(Showed)

10. Find all the values of

(i) $(1+i)^{\frac{1}{7}}$ (ii) $(1+i)^{\frac{1}{5}}$ (iii) $(-i)^{\frac{1}{6}}$ (iv) $(-1)^{\frac{1}{5}}$

Sol^n: Here $(1+i) = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$

$$(1+i)^{\frac{1}{7}} = (\sqrt{2})^{\frac{1}{7}}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)^{\frac{1}{7}}$$

$$= 2^{\frac{1}{14}}\left[\cos\frac{1}{7}\left(2n\pi + \frac{\pi}{4}\right) + i\sin\frac{1}{7}\left(2n\pi + \frac{\pi}{4}\right)\right]$$

Now putting $n = 0, 1, 2, \ldots 6$, then the required values are;

$$2^{\frac{1}{14}}\left(\cos\frac{\pi}{28} + i\sin\frac{\pi}{28}\right), \frac{1}{2^{14}}\left(\cos\frac{9\pi}{28} + i\sin\frac{9\pi}{28}\right) \text{ etc.}$$

(ii) Here $(1+i) = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$

$$(1+i)^{\frac{1}{5}} = (\sqrt{2})^{\frac{1}{5}}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)^{\frac{1}{5}}$$

$$= 2^{\frac{1}{10}}\left[\cos\frac{1}{5}\left(2n\pi + \frac{\pi}{4}\right) + i\sin\frac{1}{5}\left(2n\pi + \frac{\pi}{4}\right)\right]$$

Now putting $n = 0, 1, 2, 3$ ~~etc~~, then we have the values are

$$2^{\frac{1}{10}}\left[\cos\frac{\pi}{20} + i\sin\frac{\pi}{20}\right], 2^{\frac{1}{10}}\left(\cos\frac{9\pi}{20} + i\sin\frac{9\pi}{20}\right) \text{ etc}$$

(iii) $(-i)^{\frac{1}{6}}$

$$= \left(\cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2}\right)^{\frac{1}{6}}$$

$$= \left\{\cos\left(2n\pi + \frac{3\pi}{2}\right) + i\sin\left(2n\pi + \frac{3\pi}{2}\right)\right\}^{\frac{1}{6}}$$

$$(-i)^{\frac{1}{6}} = \left\{\cos\frac{1}{6}\left(2n\pi + \frac{3\pi}{2}\right) + i\sin\left(2n\pi + \frac{3\pi}{2}\right)\right\}$$

putting $n = 0, 1, 2, 3, 4, 5$ we get the required values

(IV) $(-1)^{\frac{1}{5}} = \left(\cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2}\right)^{\frac{1}{5}}$

$$= \left\{\cos\left(2n\pi + \frac{3\pi}{2}\right) + i\sin\left(2n\pi + \frac{3\pi}{2}\right)\right\}^{\frac{1}{5}}$$

$$= \cos\frac{1}{5}\left(2n\pi + \frac{3\pi}{2}\right) + i\sin\frac{1}{5}\left(2n\pi + \frac{3\pi}{2}\right)$$

(V) $(-1)^{\frac{2}{5}} = (\cos\pi + i\sin\pi)^{\frac{2}{5}}$

$= \{\cos(2n\pi + \pi) + i\sin(2n\pi + \pi)\}^{\frac{2}{5}}$

$= \cos\frac{2}{5}(2n\pi + \pi) + i\sin\frac{2}{5}(2n\pi + \pi)$

now putting $n = 0, 1, 2, 3 \ldots$ we get the required values.

16. (i) Solve $x^7 = 1$ by the help of De moivre's Theorem.

Sol^n. $x^7 = 1 = \cos 0^\circ + i\sin 0^\circ$

$= \cos 2n\pi + i\sin 2n\pi$

$\therefore$ ~~$x^7 = \cos 0$~~

$$\therefore x = (\cos 2n\pi + i\sin 2n\pi)^{\frac{1}{7}}$$

$$= \cos\frac{2n\pi}{7} + i\sin\frac{2n\pi}{7}$$

putting $n = 0, 1, 2, 3, 4, 5$ and 6, we get the seven roots of x. are,

$(\cos 0^\circ + i\sin 0^\circ)$, $\left(\cos\frac{2\pi}{7} + i\sin\frac{2\pi}{7}\right)$, $\left(\cos\frac{4\pi}{7} + i\sin\frac{4\pi}{7}\right)$

$\left(\cos\frac{6\pi}{7} + i\sin\frac{6\pi}{7}\right)$, $\left(\cos\frac{8\pi}{7} + i\sin\frac{8\pi}{7}\right)$ $\left(\cos\frac{10\pi}{7} + i\sin\frac{10\pi}{7}\right)$

$\left(\cos\frac{12\pi}{7} + i\sin\frac{12\pi}{7}\right)$

12. using De Moivre's Theorem, solve the equations.

(1) $x^7+x^4+x^3+1=0$

Sol^n: Given that, $x^7+x^4+x^3+1=0$

$\Rightarrow x^4(x^3+1)+(x^3+1)=0$

$\Rightarrow (x^4+1)(x^3+1)=0$

i.e either

$x^4+1=0$ or $x^3+1=0$

$\therefore x^3+1=0$

$\Rightarrow x^3=-1$

$\Rightarrow x^3=-1=(\cos\pi+i\sin\pi)$

$=\cos(2n+1)\pi+i\sin(2n+1)\pi$

$x=\{\cos(2n+1)\pi+i\sin(2n+1)\pi\}^{\frac{1}{3}}$

$=\cos\frac{(2n+1)}{3}\pi+i\sin\frac{(2n+1)\pi}{3}$

putting $n=0,1,2,$

$x=-1,\ \cos\frac{\pi}{3}+i\sin\frac{\pi}{3},\ \cos\frac{5\pi}{3}+i\sin\frac{5\pi}{3}$ Ans.

and $x^4+1=0$

$x^4=-1$

$\Rightarrow x^4=i^2$

$\Rightarrow (x^2)^2-i^2=0$

$\Rightarrow (x^2+i)(x^2-i)=0$

$x^2+i=0$

$x^2=-i=\cos\frac{3\pi}{2}+i\sin\frac{3\pi}{2}$

$\therefore x=\{\cos\frac{3\pi}{2}+i\sin\frac{3\pi}{2}\}^{\frac{1}{2}}$

$=\cos\frac{3\pi}{4}+i\sin\frac{3\pi}{4}$

$x^2 + i = 0$

$x^2 = -i = \cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2}$

$= \cancel{\cos(2n+1)\frac{3\pi}{2} + i\sin(2n+1)\frac{3\pi}{2}}$

$= \cos(2n\pi + \frac{3\pi}{2}) + i\sin(2n\pi + \frac{3\pi}{2})$

$\therefore x = \left\{\cos(2n\pi + \frac{3\pi}{2}) + i\sin(2n\pi + \frac{3\pi}{2})\right\}^{\frac{1}{2}}$

$= \cos\frac{1}{2}(2n\pi + \frac{3\pi}{2}) + i\sin\frac{1}{2}(2n\pi + \frac{3\pi}{2})$

put $n = 0, 1$

$x = \cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}, \ \cos\frac{7\pi}{4} + i\sin\frac{7\pi}{4}$

and $x^2 - i = 0$

$x^2 = i = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2}$

$= \cos(2n\pi + \frac{\pi}{2}) + i\sin(2n\pi + \frac{\pi}{2})$

$\therefore x = \left\{\cos(2n\pi + \frac{\pi}{2}) + i\sin(2n\pi + \frac{\pi}{2})\right\}^{\frac{1}{2}}$

$= \cos\frac{1}{2}(2n\pi + \frac{\pi}{2}) + i\sin\frac{1}{2}(2n\pi + \frac{\pi}{2})$

put $n = 0, 1$

$\therefore x = \cos\frac{\pi}{4} + i\sin\frac{\pi}{4}, \ \cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}$

13. Prove that $\sqrt[n]{a+ib}+\sqrt[n]{a-ib}$ has n real values and find those of $\sqrt[3]{1+\sqrt{-3}}+\sqrt[3]{1-\sqrt{-3}}$.

Sol^n. Let $a = r\cos\theta$ and $b = r\sin\theta$

$$\therefore r^2 = a^2+b^2 \Rightarrow r = \sqrt{a^2+b^2} \quad \text{——— (I)}$$

$$\& \ \theta = \tan^{-1}\frac{b}{a} \quad \text{——— (II)}$$

Now $\sqrt[n]{a+ib} = (a+ib)^{\frac{1}{n}} = r^{\frac{1}{n}}(\cos\theta + i\sin\theta)^{\frac{1}{n}}$

$$= r^{\frac{1}{n}}\left\{\cos\frac{2m\pi+\theta}{n} + i\sin\frac{2m\pi+\theta}{n}\right\}$$

similarly $\sqrt[n]{a-ib} = r^{\frac{1}{n}}\left\{\cos\frac{2m\pi+\theta}{n} - i\sin\frac{2m\pi+\theta}{n}\right\}$.

$$\therefore \sqrt[n]{a+ib}+\sqrt[n]{a-ib} = r^{\frac{1}{n}}\left\{2\cos\frac{2m\pi+\theta}{n}\right\} \quad \text{——— (III)}$$

Now putting $m = 0, 1, 2 \ldots\ldots (n-1)$ we have n values of the given expression and it is clear that all value are real. (Proved)

Again, comparing $\sqrt[n]{a+ib}+\sqrt[n]{a-ib}$ and $\sqrt[3]{1+\sqrt{-3}}+\sqrt[3]{1-\sqrt{-3}}$

we have $a = 1$, $b = \sqrt{3}$

From (I) & (2) we have

$r = 2$ and $\theta = \frac{\pi}{3}$ also $n = 3$

putting this values in (III) we have

$$\sqrt[3]{1+\sqrt{-3}}+\sqrt[3]{1-\sqrt{-3}} = 2^{\frac{1}{3}}\left\{2\cos\left(\frac{2m\pi+\frac{\pi}{3}}{3}\right)\right\}$$

$$= 2^{\frac{1}{3}}\left\{2\cos\left(\frac{6m+1}{9}\right)\right\}$$

where, $m = 0, 1, 2 \ldots$

14. If n be a positive integer, then prove that,

$$(1+i)^n + (1-i)^n = 2^{\frac{n}{2}+1} \cdot \cos \frac{1}{4} n\pi$$

Sol^n. there, $1+i = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$

$$\therefore (1+i)^n = (\sqrt{2})^n \left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)^n$$

$$= 2^{\frac{n}{2}}\left(\cos\frac{n\pi}{4} + i\sin\frac{n\pi}{4}\right)$$

similarly $(1-i)^n = 2^{n/2}\left(\cos\frac{n\pi}{4} - i\sin\frac{n\pi}{4}\right)$

$$\therefore (1+i)^n + (1-i)^n = 2^{n/2} \cdot 2\cos\frac{n\pi}{4}$$

$$= 2^{n/2+1}\cos\frac{n\pi}{4} \text{ (Proved)}$$

15 (i) If $x = \cos\alpha + i\sin\alpha$, $y = \cos\beta + i\sin\beta$ and $z = \cos\gamma + i\sin\gamma$ ~~then~~ and if $x+y+z=0$, then $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 0$

Sol^n. $x = \cos\alpha + i\sin\alpha$, $y = \cos\beta + i\sin\beta$, $z = \cos\gamma + i\sin\gamma$

$$\therefore \frac{1}{x} = \bar{x}' = \cos\alpha - i\sin\alpha$$

$$\frac{1}{y} = \bar{y}' = \cos\beta - i\sin\beta$$

$$\frac{1}{z} = \bar{z}' = \cos\gamma - i\sin\gamma$$

since, $x+y+z=0$

$$\therefore \cos\alpha + i\sin\alpha + \cos\beta + i\sin\beta + \cos\gamma + i\sin\gamma = 0$$

$$\therefore (\cos\alpha + \cos\beta + \cos\gamma) + i(\sin\alpha + \sin\beta + \sin\gamma) = 0$$

$\therefore \cos\alpha+\cos\beta+\cos\gamma=0$ & $\sin\alpha+\sin\beta+\sin\gamma=0$ —— (1)

$\therefore \frac{1}{x}+\frac{1}{y}+\frac{1}{z} = (\cos\alpha - i\sin\alpha)+(\cos\beta - i\sin\beta)+(\cos\gamma - i\sin\gamma)$

$= (\cos\alpha+\cos\beta+\cos\gamma) - i(\sin\alpha+\sin\beta+\sin\gamma)$

$= 0 - i.0$ [From (1)]

$\therefore \frac{1}{x}+\frac{1}{y}+\frac{1}{z} = 0$ ~~show~~

Hence we have, $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=0$

(15) (ii) If $\sin\alpha+\sin\beta+\sin\gamma = \cos\alpha+\cos\beta+\cos\gamma=0$, then $\Sigma\sin^2\alpha = \Sigma\cos^2\alpha = \frac{3}{2}$.

Solⁿ: Let, $x=\cos\alpha+i\sin\alpha$, $y=$ ~~sinα~~ $\cos\beta+i\sin\beta$

& $z=\cos\gamma+i\sin\gamma$

$\therefore x+y+z = (\cos\alpha+\cos\beta+\cos\gamma)+i(\sin\alpha+\sin\beta+\sin\gamma)$

$= 0+i.0$

$\therefore x+y+z=0$ —— (1)

Again, $\frac{1}{x}+\frac{1}{y}+\frac{1}{z} = x^{-1}+y^{-1}+z^{-1} = (\cos\alpha - i\sin\alpha)+(\cos\beta - i\sin\beta)+(\cos\gamma - i\sin\gamma)$

$= (\cos\alpha+\cos\beta+\cos\gamma) - i(\sin\alpha+\sin\beta+\sin\gamma)$

$= 0 - i.0$

$\therefore \frac{1}{x}+\frac{1}{y}+\frac{1}{z}=0$ —— (11)

Now, $(x+y+z)^2 = x^2+y^2+z^2+2(xy+yz+zx)$

$\Rightarrow (x+y+z)^2 = x^2+y^2+z^2+2xyz\left(\frac{1}{z}+\frac{1}{x}+\frac{1}{y}\right)$

$\Rightarrow 0 = x^2+y^2+z^2+0$ [From (I) & (II)]

$\therefore x^2+y^2+z^2=0$

$(\cos\alpha+i\sin\alpha)^2+(\cos\beta+i\sin\beta)^2+(\cos\gamma+i\sin\gamma)^2=0$

$\Rightarrow \cos^2\alpha+2i\sin\alpha\cos\alpha-\sin^2\alpha+\cos^2\beta+2i\sin\beta\cos\beta-\sin^2\beta$
$+\cos^2\gamma+2i\sin\gamma\cos\gamma-\sin^2\gamma=0$

$\Rightarrow (\cos^2\alpha+\cos^2\beta+\cos^2\gamma)-(\sin^2\alpha+\sin^2\beta+\sin^2\gamma)+i(\sin2\alpha+\sin2\beta+\sin2\gamma)=0$

$\Rightarrow (\Sigma\cos^2\alpha-\Sigma\sin^2\alpha)+i(\sin2\alpha+\sin2\beta+\sin2\gamma)=0$

Now equating real & imaginary part, we have,

$\Sigma\cos^2\alpha-\Sigma\sin^2\alpha=0$

$\Rightarrow \Sigma\cos^2\alpha=\Sigma\sin^2\alpha$

$\boxed{\Sigma\cos^2\alpha+\Sigma\cos^2\alpha=\Sigma(\cos^2\alpha+\sin^2\alpha)}$

$\Rightarrow 2\Sigma\cos^2\alpha=\frac{1}{2}\Sigma(\cos^2\alpha+\sin^2\alpha)$

and each $=\frac{1}{2}\Sigma(\cos^2\alpha+\sin^2\alpha)=\frac{1}{2}(1+1+1)=\frac{3}{2}$

$\therefore \Sigma\sin^2\alpha=\Sigma\cos^2\alpha=\frac{3}{2}$ (Proved)

17. If $x = \cos\theta + i\sin\theta$ and $1+\sqrt{1-a^2} = na$, then prove that

$$1 + a\cos\theta = \frac{a}{2n}(1+nx)\left(1+\frac{n}{x}\right)$$

Sol^n.

~~1+a co~~

$$1+\sqrt{1-a^2} = na$$

$$\sqrt{1-a^2} = na-1$$

$$1-a^2 = n^2a^2 - 2na + 1$$

$$\Rightarrow 2na = n^2a^2 + a^2$$

$$\Rightarrow a(1+n^2) = 2n$$

$$\frac{a}{2n} = \frac{1}{1+n^2}$$

and $x = \cos\theta + i\sin\theta$

$$\therefore \frac{1}{x} = x^{-1} = \cos\theta - i\sin\theta$$

$$x + \frac{1}{x} = 2\cos\theta$$

$$R.H.S = \frac{a}{2n}(1+nx)\left(1+\frac{n}{x}\right)$$

$$= \frac{a}{2n}\left(1+\frac{n}{x}+nx+n^2\right)$$

$$= \frac{a}{2n}\left\{(1+n^2)+n\left(x+\frac{1}{x}\right)\right\}$$

$$= \frac{a}{2n}(1+n^2) + \frac{a}{2n}\cdot n\left(x+\frac{1}{x}\right)$$

$$= \frac{1}{(1+n^2)}\cdot(1+n^2) + \frac{a}{2}\cdot\left(x+\frac{1}{x}\right)$$

$$= 1 + \frac{a}{2}\cdot 2\cos\theta$$

$$= 1 + a\cos\theta$$ (Proved)

(18.) If $\alpha = \cos\frac{2r\pi}{n} + i\sin\frac{2r\pi}{n}$, and if r and p are prime to n, then prove that, $1+\alpha^p+\alpha^{2p}+\cdots+\alpha^{(n-1)p}=0$

Soln. we see ~~that~~ that,

$1+\alpha^p+\alpha^{2p}+\cdots+\alpha^{(n-1)p}$ are in G.P

$$\therefore 1+\alpha^p+\alpha^{2p}+\cdots+\alpha^{(n-1)p} = 1.\frac{1-\alpha^{np}}{1-\alpha^p} = \frac{1-\alpha^{np}}{1-\alpha^p} \quad \cdots (1)$$

since, $\alpha = \cos\frac{2r\pi}{n} + i\sin\frac{2r\pi}{n}$

$$\therefore \alpha^{np} = \left(\cos\frac{2r\pi}{n} + i\sin\frac{2r\pi}{n}\right)^{np}$$

$$= \cos 2rp\pi + i\sin 2rp\pi$$

Now from (1)

$$1+\alpha^p+\alpha^{2p}+\cdots+\alpha^{(n-1)p} = \frac{1-(\cos 2rp\pi + i\sin 2rp\pi)}{1-\alpha^p}$$

$$= \frac{1-1}{1-\alpha^p} = 0 \text{ (Proved)}$$

10. If $(1+x)^n = a_0 + a_1 x + a_2 x^2 + \cdots\cdots$ (n being a positive integer), then prove that.

$$a_0 + a_4 + a_8 + \cdots\cdots = 2^{n-2} + 2^{\frac{1}{2}n-1} \cos\frac{1}{4} n\pi$$

Soln Given that,

$$(1+x)^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$$

putting $x = 1$ and -1 we have

$$(1+1)^n = a_0 + a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + a_7 + a_8 + \cdots$$

$$2^n = a_0 + a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + a_7 + a_8 + \cdots \quad \text{—— (1)}$$

and $(1-1)^n = a_0 - a_1 + a_2 - a_3 + \cdots$

$$0 = a_0 - a_1 + a_2 - a_3 + a_4 + a_5 + a_6 - a_7 + a_8 - \cdots \quad \text{(2)}$$

Adding (1) & (2) we have,

$$2^n = 2(a_0 + a_2 + a_4 + a_6 + a_8 + \cdots) \quad \text{—— (A)}$$

Again putting $x = i$ and $-i$, we have

$$(1+i)^n = a_0 + a_1 i - a_2 - a_3 i + a_4 + a_5 i - \cdots \quad \text{(3)}$$

$$(1-i)^n = a_0 - a_1 i - a_2 + a_3 i + a_4 - a_5 i - \cdots\cdots \quad \text{(4)}$$

Adding (4) and (3)

$$(1+i)^n + (1-i)^n = 2(a_0 - a_2 + a_4 - a_6 + a_8 - \cdots) \quad \text{—— B}$$

Again, Adding (A) and (B), we have,

$$2^n + (1+i)^n + (1-i)^n = 2 \times 2(a_0 + a_4 + a_8 + \cdots)$$

$$\Rightarrow 2^n + (1+i)^n + (1-i)^n = 2^2 (a_0 + a_4 + a_8 + \cdots) \quad \text{—— (c)}$$

Now, Let $1 = r\cos\theta$, $1 = r\sin\theta$

$\therefore r = \sqrt{2}$ and $\theta = \frac{\pi}{4}$

$1 + i = r(\cos\theta + i\sin\theta)$

$= \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$

$\therefore (1+i)^n = (\sqrt{2})^n\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$

$= 2^{n/2}\left(\cos\frac{n\pi}{4} + i\sin\frac{n\pi}{4}\right)$

Similarly, $(1-i)^n = 2^{n/2}\left(\cos\frac{n\pi}{4} - i\sin\frac{n\pi}{4}\right)$

$(1+i)^n + (1-i)^n = 2^{n/2} \cdot 2\cos\frac{n\pi}{4} = 2^{\frac{n}{2}+1}\cos\frac{n\pi}{4}$

Hence from (c) we get

$a_0 + a_4 + a_8 + \cdots = 2^{n-2} + 2^{\frac{n}{2}+1}\cos\frac{n\pi}{4} \cdot \frac{1}{2^2}$

$= 2^{n-2} + 2^{\frac{n}{2}-1}\cos\frac{n\pi}{4}$ (Proved)

Ex-4 Prove that the roots of the equation $x^{10}+11x^5-1$ are the values of $\frac{\pm\sqrt{5}-1}{2}\left(\cos\frac{2r\pi}{5}\pm i\sin\frac{2r\pi}{5}\right)$ where r is an integer.

Solⁿ $x^{10}+11x^5-1=0$

$(x^5)^2+11x^5-1=0$

$\therefore x^5=\frac{-11\pm\sqrt{121+4}}{2}$

$=\frac{-11\pm5\sqrt{5}}{2}=\frac{-176\pm80\sqrt{5}}{32}=\left(\frac{\pm\sqrt{5}-1}{2}\right)^5$

$=\left(\frac{\pm\sqrt{5}-1}{2}\right)^5\cdot1=\left(\frac{\pm\sqrt{5}-1}{2}\right)^5(\cos0^\circ+i\sin0^\circ)$

$=\left(\frac{\pm\sqrt{5}-1}{2}\right)^5(\cos2r\pi+i\sin2r\pi)$

$x=\left(\frac{\pm\sqrt{5}-1}{2}\right)(\cos2r\pi+i\sin2r\pi)^{\frac{1}{5}}$

$=\frac{\pm\sqrt{5}-1}{2}\left(\cos\frac{2r\pi}{5}+i\sin\frac{2r\pi}{5}\right)$ where r is any integer (Proved)

(12) (ii) Sol^n. we have,

$$x^6+x^5+x^4+x^3+x^2+x+1=0$$

Multiplying both sides by $(x-1)$ we have

$$(x-1)(x^6+x^5+x^4+x^3+x^2+x+1)=0$$

$$\Rightarrow x^7+x^6+x^5+x^4+x^3+x^2+x-x^6-x^5-x^4-x^3-x^2-x-1=0$$

$$\Rightarrow x^7-1=0$$

$$\Rightarrow x^7=1=\cos 0^\circ+i\sin 0^\circ=\cos 2n\pi+i\sin 2n\pi$$

$$x=(\cos 2n\pi+i\sin 2n\pi)^{\frac{1}{7}}$$

$$\therefore x=\cos\frac{2n\pi}{7}+i\sin\frac{2n\pi}{7} \quad \text{——— (1)}$$

when we put $n=0$ then we get $x=1$ i.e $x-1=0$ which is disallowed because we have multiplied give eqn. with $(x-1)$. So putting $n=1,2,3,4,5,6$ in (1) we get the values which are the # solution of the given equation. Ans: