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Probability

Probability: A numerical measure of an uncertainity of an event of an experiment is called probability.

Experiment: Experiment is an act that can be repeated under given conditions.

Example: Tossing of a coin is a trail and getting head and tail are outcome.

outcome: The result of ~~an~~ random experiment are known as outcome.

Example: in throwing a dice the possible possible outcomes are 1, 2, 3, 4, 5, 6

random experiment: if in each trail of an experiment counted under identical conditions, the outcome is not unique, but may be any one of the possible outcomes, then such an experiment is called ~~an~~ a random experiment.

Example: tossing a coin, throwing a die.

~~Event:~~ **Sample space:** The collection of all possible outcomes of a random experiment is called sample space. it is usually denoted by S.

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Example: if we toss a coin, the sample space $S = \{H, T\}$ where H and T denote the head and tail of a coin

<u>sample point</u>: An element of the sample space is called a sample point. it is denoted by small letter ω.

Example: if we toss a coin, the sample space is $\Omega = \{H, T\} = \{\omega_1, \omega_2\}$, where $H = \omega_1$ and $T = \omega_2$ are the sample points.

~~Event~~: <u>Finite sample space</u>: A sample space is called a finite sample space if it contains a finite number of sample points.

Example: if we toss a coin three times, the sample space of the experiment will contain 8 sample points and the sample space is

$$\Omega = \{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT\}$$

where H and T denote the head and tail of the coin. The sample space is finite. Since it contain a finite number of sample points.

infinite sample space : A sample space is called infinite sample space if it contains infinite number of sample points.

Example : tossing a coin untill we get head.

$A = \{H, TH, TTH, TTTH, \cdots\}$

discrete sample space : A sample is called discrete sample space if it contains finite or infinite number of denumerable sample points

continuous sample space : A sample space is called a continuous sample space if it contains non-denumerable number of sample points.

Example ; Select a number at random from the interval $[0,1]$ of real number's. The sample space is

$$\Omega = \{x \mid x \text{ is a number in } [0,1] \text{ or } 0 \leq x \leq 1\}$$

Event : Collection of out comes is called an Event. An event is a subset of the sample space and it is usually denoted by capital letter A, B, C etc.

There are two types of event as following:

(i) Simple event
(ii) Compound event.

4.

Simple event: An event is called simple event if it contains only one sample point.

Example: Suppose a fair coin is tossed twice let H and T denote the head and tail of the coin respectively. Then the sample space of experiment is $\Omega = \{HH, HT, TH, TT\} = \{w_1, w_2, w_3, w_4\}$

In this example there are four simple events which are $\omega_1 = \{HH\}$ $\omega_2 = \{HT\}$, $w_3 = \{TH\}$ and $w_4 = \{TT\}$

Compound event: An event is called compound event if it contains more than one sample point or it is the union of simple events.

Example: Suppose a fair coin is tossed twice then the sample space of experiment is

$$\Omega = \{HH, HT, TH, TT\}$$

it is a compound event because it contains more than one sample point.

Event space: The class of all events associated with a given experiment is defined to be the event space.

Sure event: An event is called sure event when it is always happens. The probability of sure event is one

impossible event: An event is called impossible event when it is never happens. The probability of an impossible event is zero

Mutually exclusive events: if A and B be two events in A then they are said to be mutually exclusive if $A \cap B = \phi$. That is two events are said to be mutually exclusive if they have no common points.

Complementary event: Let A be only event defined only a sample space Ω. then the complementary of A, denoted by $\bar{A}$, is the event consisting of all the sample points in Ω but not in A.

Probability space: The thriplet $(S, A, P(A))$ is called probability space, where S is a sample space, A is an event and P(A) is the probability function with domain A.

Axioms: A axiom is a statement that is assumed to be true.

<u>Set function</u>: A function with domain collection of set's and counter domain the real line including infinity is defined to be a set of function.

Axioms of probability: Let S be a sample space, let $\mathcal{E}$ be the class of events and let p be a real valued function ~~denot~~ defined on $\mathcal{E}$. The p is called a probability function, and $p(A)$ is called the probability of the event A if the following axioms hold.

P_1: For every event A, $0 \le P(A) \le 1$

P_2: $P(S) = 1$

P_3: If A and B are mutually exclusive events then ~~$P(A_1 \cup A_2 \cup \cdots$~~ $P(A \cup B) = P(A) + P(B)$

P_4: If $A_1, A_2, \ldots,$ is a sequence of mutually exclusive events.

then $P(A_1 \cup A_2 \cup A_3 \cup \cdots) = P(A_1) + P(A_2) + \cdots$

Probability space function: A probability function $P(A)$ is a set function with domain A and counted domain the interval $[0,1]$ which satisfies the following condition.

(i) $0 \le P(A) \le 1$; $\forall A \subset$

(ii) $P(S) = 1$

(iii) Let A_1 and A_2 be mutually exclusive events then

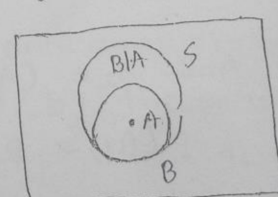
$$P(A_1 \cup A_2) = P(A_1) + P(A_2)$$

iv. Let $A_1, A_2, \ldots, A_n$ be a sequence of mutually exclusive events in S and if $A_i \subset S$ $i = 1, 2, \ldots$

then $P(\bigcup_{i=1} A_i) = \sum_{i=1}^{\infty} P(A_i)$

Theorem: if $A \subset B$ then prove that $P(A) \le P(B)$

Proof: Since $A \subset B$ thus we can say that A and $B|A$ are mutually exclusive events



$\therefore B = A \cup (B|A)$

$\therefore P(B) = P(A) + P(B|A)$

$\Rightarrow P(B) \ge P(A)$ [Since $0 \le P(B|A) \le 1$]

$\therefore P(A) \le P(B)$

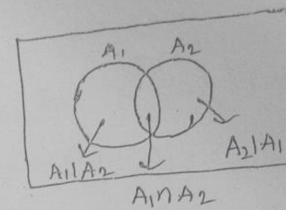
(proved)

The law for addition :

if A_1 and A_2 are two events and are not disjoint

Then, $P(A_1 \cup A_2) = P(A_1) + P(A_2) - P(A_1 \cap A_2)$

Proof : From the vendiagram we get



$(A_1 \cup A_2) = (A_1 | A_2) \cup (A_1 \cap A_2) \cup (A_2 | A_1)$

where $A_1|A_2$, $A_1 \cap A_2$ and $A_2|A_1$ are mutually exclusive

$\therefore P(A_1 \cup A_2) = P(A_1 | A_2) + P(A_1 \cap A_2) + P(A_2 | A_1)$ ... (i)

Now, $A_1 = (A_1 \cap A_2) \cup (A_1 | A_2)$

$P(A_1) = P(A_1 \cap A_2) + P(A_1 | A_2)$ [$(A_1 \cap A_2)$ and $(A_1|A_2)$ mutually exclusive]

$\therefore P(A_1 | A_2) = P(A_1) - P(A_1 \cap A_2)$ ... (ii)

And $A_2 = (A_1 \cap A_2) \cup (A_2 | A_1)$

$P(A_2) = P(A_1 \cap A_2) + P(A_2 | A_1)$

$\therefore P(A_2 | A_1) = P(A_2) - P(A_1 \cap A_2)$ ... (iii)

From (i), (ii) and (iii) we get

$P(A_1 \cup A_2) = P(A_1) - P(A_1 \cap A_2) + P(A_1 \cap A_2) + P(A_2) - P(A_1 \cap A_2)$

$= P(A_1) + P(A_2) - P(A_1 \cap A_2)$

(proved)

Theorem: for any event A, B and C

$$P(A\cup B\cup C) = P(A)+P(B)+P(C) - P(A\cap B) - P(A\cap C) - P(B\cap C) + P(A\cap B\cap C)$$

Proof: let $D = B\cup C$

$\therefore P(D) = P(B\cup C)$

$= P(B) + P(C) - P(B\cap C)$

then $A\cap D = A\cap (B\cup C)$

$= (A\cap B)\cup (A\cap C)$

$\therefore P(A\cap D) = P(A\cap B) + P(A\cap C) - P(A\cap B\cap A\cap C)$

$= P(A\cap B) + P(A\cap C) - P(A\cap B\cap C)$

thus $P(A\cup B\cup C) = P(A\cup D)$

$= P(A) + P(D) - P(A\cap D)$

$= P(A) + P(B) + P(C) - P(B\cap C) - P(A\cap B) - P(A\cap C) + P(A\cap B\cap C)$

Independent events: if the occurance of an event is not influenced or affected by the occurance or not occurance of another event, these two events are said to be independent of each other. if two coins are tossed, flower or leaf in one coin and leaf or flower in the other coin are independent events.

Multiplication law: For two events A and B

$$P(A \cap B) = P(A) \cdot P(B|A) \quad ; \; P(A) > 0$$
$$= P(B) \cdot P(A|B) \quad ; \; P(B) > 0$$

where $P(B|A)$ is the conditional probability of occurrence of B when the event A has already happened and $P(A|B)$ is the conditional probability of A given that B has already happened.

Proof: Suppose n occurances contain in the sample space S of which $n(A), n(B)$ and $n(A \cap B)$ occurance contain in the event A, B and $(A \cap B)$ respectively.

$$\therefore P(A) = \frac{n(A)}{n(S)}, \quad P(B) = \frac{n(B)}{n(S)} \text{ and } P(A \cap B) = \frac{n(A \cap B)}{n(S)}.$$

$P(A|B)$ is the probability of $n(A \cap B)$ occurances when $n(B)$ occurances has happened

$$\therefore P(A|B) = \frac{P(A \cap B)}{P(B)}$$

$$= \frac{\frac{n(A \cap B)}{n(S)}}{\frac{n(B)}{n(S)}}$$

$$= \frac{n(A \cap B)}{n(S)} \times \frac{n(S)}{n(B)} = \frac{n(A \cap B)}{n(B)}$$

$\therefore \; P(B|A) = \dfrac{n(A\cap B)}{n(A)}$

We can write,

$$P(A\cap B) = \frac{n(A\cap B)}{n(S)}$$

$$= \frac{n(A)}{n(S)} \cdot \frac{n(A\cap B)}{n(A)}$$

$$= P(A) \cdot P(B|A) \quad \text{---} \; (i)$$

Again,

$$P(A\cap B) = \frac{n(A\cap B)}{n(S)}$$

$$= \frac{n(A\cap B)}{n(B)} \cdot \frac{n(B)}{n(S)}$$

$$= P(A|B) \cdot P(A)$$

So from (i) and (ii) we have

$$P(A\cap B) = P(A) \cdot P(A|B)$$

$$= P(B) \cdot P(A|B)$$

Theorem: For any two event A and B. Show that

$$P(A) + P(B) \geq P(A\cup B) \geq \text{Max}\{P(A), P(B)\} \geq P(A\cap B) \geq P(A) + P(B) - 1$$

Proof: We know that from additive law of probability

$$P(A\cup B) = P(A) + P(B) - P(A\cap B)$$

$$\Rightarrow P(A\cup B) \leq P(A) + P(B) \quad [\text{Since } P(A\cap B) \geq 0]$$

$$\therefore P(A) + P(B) \geq P(A\cup B) \quad \text{--- (i)}$$

We have,

$$A \subset (A\cup B)$$

$$\Rightarrow P(A) \leq P(A\cup B)$$

and $B \subset (A\cup B)$

$$\therefore P(B) \leq P(A\cup B)$$

$$\therefore P(A\cup B) \geq \text{Max}\{P(A), P(B)\} \quad \text{--- (ii)}$$

Again, $A\cap B \subset A$

$$\Rightarrow P(A\cap B) \leq P(A)$$

and $A\cap B \subset B$

$$\Rightarrow P(A\cap B) \leq P(B)$$

$$\therefore \text{Max}\{P(A), P(B)\} \geq P(A\cap B) \quad \text{--- (iii)}$$

And we have

$$A\cup B \subset S$$

$$\therefore P(A\cup B) \leq P(S)$$

$\Rightarrow P(A) + P(B) - P(A \cap B) \leq 1$

$\Rightarrow P(A) + P(B) - 1 \leq P(A \cap B)$

$\Rightarrow P(A \cap B) \geq P(A) + P(B) - 1 \quad \text{--- (iv)}$

Hence from (i), (ii), (iii) and (iv) we get

$P(A) + P(B) \geq P(A \cup B) \geq \text{Max}\{P(A), P(B)\} \geq P(A \cap B) \geq P(A) + P(B) - 1$

Exhaustive event: Outcome of an experiment are said to be exhaustive if they include impossible outcomes.

Example: let $A = \{1, 3, 5\}$ and $B = \{2, 4, 6\}$

$\therefore A \cup B = \{1, 3, 5\} \cup \{2, 4, 6\}$

$= \{1, 2, 3, 4, 5, 6\}$

Question: Show that the probability of a sure event is one.

Ans: let A be a event of sample space.
And also let,
the no. of A event $= m$
the no. of sample space $= n$
if A be a exhastive and sure event then
$m = n$

$\therefore P(A) = \frac{m}{n} = \frac{n}{n} = 1$

$\therefore$ the probability of a sure event is 1.

Question: if A and B are independent events then show that their complement are also independent.

Ans: Since A and B are independent

$$P(A \cap B) = P(A) \cdot P(B)$$

we have to show that

$$P(A^c \cap B^c) = P(A^c) \cdot P(B^c)$$

we know,

$$A^c \cap B^c = (A \cup B)^c$$

$$\therefore P(A^c \cap B^c) = P(A \cup B)^c$$

$$= 1 - P(A \cup B) \qquad [\text{since } P(A) + P(A^c) = 1]$$

$$= 1 - [P(A) + P(B) - P(A \cap B)]$$

$$= 1 - P(A) \cancel{+} - P(B) + P(A \cap B)$$

$$= 1 - P(A) - P(B) + P(A) \cdot P(B)$$

$$\cancel{= p} = 1\{1 - P(A)\} - P(B)\{1 - P(A)\}$$

$$= \{1 - P(A)\}\{1 - P(B)\}$$

$$\therefore P(A^c \cap B^c) = P(A^c)\, P(B^c)$$

$\therefore$ A^c and B^c are independent.

Question: Show that mutually exclusive events can not be independent.

Ans: Let us suppose that A and B two events.

and $P(A) \neq 0$, $P(B) \neq 0$

if A and B are indipendent then

$$P(A \cap B) = P(A) \cdot P(B)$$

$$\neq 0 \qquad \text{--- (i)}$$

Again if A and B are mutually exclusive then

$$A \cap B = \phi$$

$$\therefore P(A \cap B) = P(\phi) = 0 \qquad \text{--- (ii)}$$

from (i) and (ii) we can say that two events can not be independent and also mutually exclusive at a same time.

Problem: If two dice are thrown what is the probability that the sum of the dots is

(i) greater than 8 and

(ii) neither 7 nor 11

Soln. if two dice are thrown, then the sample space be

$$S = \{ (1,1), (1,2), (1,3), (1,4), (1,5), (1,6),$$
$$(2,1), (2,2), (2,3), (2,4), (2,5), (2,6),$$
$$(3,1), (3,2), (3,3), (3,4), (3,5), (3,6),$$
$$(4,1), (4,2), (4,3), (4,4), (4,5), (4,6),$$
$$(5,1), (5,2), (5,3), (5,4), (5,5), (5,6),$$
$$(6,1), (6,2), (6,3), (6,4), (6,5), (6,6) \}$$

The size of sample space is $n(S) = 36$

(i) let A be the event that the sum of the dots is greater than 8

$$\therefore A = \{ (3,6), (4,5), (4,6), (5,4), (5,5), (5,6), (6,3), (6,4), (6,5), (6,6)$$

$$n(A) = 10$$

$$\therefore P(A) = \frac{n(A)}{n(S)} = \frac{10}{36} = \frac{5}{18}$$

(ii) let B be the event that the sum of the dots is neither 7 nor 11.

$\therefore B = \{(1,1), (1,2), (1,3), (1,4), (1,5), (2,1), (2,2), (2,3), (2,4), (2,6), (3,1), (3,2), (3,3), (3,5), (3,6), (4,1), (4,2), (4,4), (4,5), (5,1), (5,3), (5,4), (5,5), (6,2), (6,3), (6,4), (6,6)\}$

$\therefore n(B) = 28$

$\therefore P(B) = \frac{n(B)}{n(S)} = \frac{28}{36} = \frac{7}{8}$

Problem: 2 Two balanced dice, one black and one red are thrown and the numbers of dots on the upper faces are noted.

(a) List of a sample space of the experiments.

(b) What is the probability of throwing a double?

(c) What is the probability that the sum is five?

(d) What is the probability that at least one is a six.

(e) What is the probability that the number on the red die is at least 4 greater than the number on the black die.

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Sol^n: if two dice are thrown then the sample space be

$$S = \{ (1,1), (1,2), (1,3), (1,4), (1,5), (1,6),$$
$$(2,1), (2,1), (2,3), (2,4), (2,5), (2,6),$$
$$(3,1), (3,2), (3,3), (3,4), (3,5), (3,6),$$
$$(4,1), (4,2), (4,3), (4,4), (4,5), (4,6),$$
$$(5,1), (5,2), (5,3), (5,4), (5,5), (5,6),$$
$$(6,1), (6,2), (6,3), (6,4), (6,5), (6,6), \}$$

The size of sample space is $n(S) = 36$

(b) Let A be the event of throwing a double.
Then A will contain the following 6 sample points

$$A = \{ (1,1), (2,2), (3,3), (4,4), (5,5), (6,6) \}$$

$n(A) = 6$

$$\therefore \quad P(A) = \frac{n(A)}{n(S)} = \frac{6}{36} = \frac{1}{6}$$

(c) Let B be the event that the sum is 5
Then B will contain the sample points.

$$B = \{ (1,4), (2,3), (3,2), (4,1) \}$$

i.e. $n(B) = 4$

$$\therefore \quad P(B) = \frac{n(B)}{n(S)} = \frac{4}{36} = \frac{1}{9}$$

(d) Let c be the event that at least one is six.

Then c will contain the sample points

$$c = \{(1,6), (2,6), (3,6), (4,6), (5,6), (6,1), (6,2), (6,3), (6,4), (6,5), (6,6)\}$$

$$n(c) = 11$$

$$\therefore P(c) = \frac{n(c)}{n(S)} = \frac{11}{36}$$

(e) Let E be events that the number on the red die is at least 4 greater than the number on the black die.

Then E will contains the sample points

$$E = \{(1,5), (2,6), (1,6)\}$$

$$n(E) = 3$$

$$\therefore P(E) = \frac{n(E)}{n(S)} = \frac{3}{36} = \frac{1}{12}$$

Problem: if $P(A) = 0.7$, $P(B) = 0.2$ $P(A \cap B) = 0.1$

Find $P(A|B)$ and $P(A \cup B)$

are the event A and B independent ?

Solⁿ: Given that $P(A) = 0.7$

$P(B) = 0.2$ and

$P(A \cap B) = 0.1$

$\therefore P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$= 0.7 + 0.2 - 0.1$

$= 0.8$

and $P(A|B) = \dfrac{P(A \cap B)}{P(B)} = \dfrac{0.1}{0.2}$

Again $P(A) \cdot P(B) = 0.7 \times 0.2$

$= 0.14$

$\therefore P(A) \cdot P(B) \neq P(A \cap B)$

$\therefore$ The event A and B are not Independent.

Problem 4: The probability that A can solve a problem is 0.9 and that B can solve problem 0.75. Both A and B try to solve the problem independently. What is the probability that the problem will be solved.

Solⁿ: Given that $P(A) = 0.9$

$P(B) = 0.75$

The probability that both A and B can solve.

$\therefore P(A \cap B) = P(A) \cdot P(B)$ [Since A and B are independent]

$= 0.9 \times 0.75$

$= 0.675$

Problem 5: Suppose A and B are two mutually exclusive event with $P(A) = 0.35$ and $P(B) = 0.15$

Find (i) $P(A \cup B)$ (ii) $P(\bar{A})$ (iii) $P(A \cap B)$

(iv) $P(\bar{A} \cup \bar{B})$ (v) $P(\bar{A} \cap \bar{B})$

Solⁿ: (i) $P(A \cup B) = P(A) + P(B)$

$= 0.35 + 0.15$

$= 0.5$

(ii) $P(\bar{A}) = 1 - P(A)$

$= 1 - 0.35$

$= 0.65$

(iii) $P(A \cap B) = P(\phi) = 0$ Since A and B are mutually exclusive.

Problem 6: Three events A, B, C are mutually exclusive events and their union is the sample space Ω. if $P(A) = \frac{3}{2} P(B)$, $P(B) = 2P(C)$ find the probability of A, B, and C.

Soln: Three events A, B, C are mutually exclusive and their union is the sample space Ω.

then $A \cup B \cup C = \Omega$ thus $P(A \cup B \cup C) = P(\Omega) = 1$

Now $P(A \cup B \cup C) = P(A) + P(B) + P(C)$

$\therefore P(A) + P(B) + P(C) = 1$

$\Rightarrow \frac{3}{2} P(B) + P(B) + \frac{1}{2} P(B) = 1$

$\Rightarrow 3 P(B) = 1$

$\therefore P(B) = 1/3$

$\therefore P(A) = \frac{3}{2} \times P(B)$

$= \frac{3}{2} \times \frac{1}{3} = \frac{1}{2}$

$\therefore P(B) = 2P(C)$

$\therefore P(C) = \frac{1}{2} P(B) = \frac{1}{2} \times \frac{1}{3} = \frac{1}{6}$

Hence $P(A) = \frac{1}{2}$, $P(B) = \frac{1}{3}$ and $P(C) = \frac{1}{6}$

Problem: 7. in a class of 100 students 75 play football, 50 play crickets, 40 play both of them. A student is selected at random from the class. what is the probability that the selected student

(i) plays only cricket but not fottball.

(ii) plays at least one of the game. and

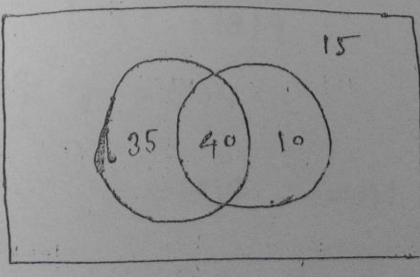
does not play any of the two games.

Soln: Let A be the event that A student plays football and B be the event that the student plays cricket.

Then $n(A) = 75$ $n(B) = 50$ and

$n(A \cap B) = 40$

$n(\Omega) = 100$



$A \cup B = 85$

it is clear that

$(A \cap \bar{B})$ is the event that a student plays only football where $n(A \cap \bar{B}) = 35$

$(A \cap B)$ is the event that student plays both game $n(A \cap B) = 40$

$(\bar{A} \cap B)$ is the event that student plays only cricket where $n(\bar{A} \cap B) = 10$

(i) $p(\bar{A} \cap B) = \dfrac{n(\bar{A} \cap B)}{n(\Omega)} = \dfrac{10}{100} = \dfrac{1}{10}$

(ii) $p(A \cup B) = p(A \cap \bar{B}) + p(A \cap B) + p(\bar{A} \cap B)$

$$= \frac{n(A \cap \bar{B})}{n(\Omega)} + \frac{n(A \cap B)}{n(\Omega)} + \frac{n(\bar{A} \cap B)}{n(\Omega)}$$

$$= \frac{35}{100} + \frac{40}{100} + \frac{10}{100}$$

$$= \frac{85}{100} = 0.85$$

(iii) $p(\bar{A} \cap \bar{B}) = p(\overline{A \cup B}) = 1 - p(A \cup B)$

$$= 1 - 0.85$$

$$= 0.15$$