Relations

Relations

- Relationships between elements of sets occur very often.
  - (Employee, Salary)
  - (Students, Courses, GPA)
- Relationships between elements of sets are represented using the structure called relation, which is just a subset of the Cartisian product of the sets.
- We use ordered pairs (or *n-tuples*) of elements from the sets to represent relationships.

Binary Relations

- Let A and B be any sets. A *binary relation R* from A to B, (i.e., with signature $R{:}A{\times}B)$ can be identified with a subset of $A{\times}B$.

E.g., let **< : N×N** can be seen as **{(n,m) | $n < m$}**

- $(a,b){\in}R$ means that a is related to b (by R)
- Also written as aRb; also $R(a,b)$
  - E.g., a**<b** and **<** (a,b) both mean (a,b)$\in$ **<**
- A binary relation R corresponds to a characteristic function $P_R{:}A{\times}B{\rightarrow}\{\mathbf{T,F}\}$

Example

A: {students at UNR},   B: {courses offered at UNR}

   R: "relation of students enrolled in courses"

(Jason, CS365), (Mary, CS201)   are in R

If Mary does not take CS365, then (Mary, CS365) is not in R!

If CS480 is not being offered, then (Jason, CS480), (Mary, CS480) are not in R!

Complementary Relations

- Let $R{:}A,B$ be any binary relation.
- Then, $\bar{R}{:}A \times B$, the *complement* of R, is the binary relation defined by
$$\bar{R} :\equiv \{(a,b) \in A \times B \mid (a,b) \notin R\} = (A \times B) - R\}$$
- Note this is just $\overline{R}$ if the universe of discourse is $U = A \times B$; thus the name *complement*.
- Note the complement of $\bar{R}$ is R.

Example: $\not< = \{(a,b) \mid (a,b) \notin <\} = \{(a,b) \mid \neg a{<}b\} = \geq$

Inverse Relations

- Any binary relation $R:A\times B$ has an *inverse* relation $R^{-1}:B\times A$, defined by
$$R^{-1} :\equiv \{(b,a) \mid (a,b)\in R\}.$$

  E.g., $<^{-1} = \{(b,a) \mid a<b\} = \{(b,a) \mid b>a\} = >$.

- *E.g.,* if R:People x Foods is defined by
  $a\ R\ b \Leftrightarrow a\ eats\ b$, then:
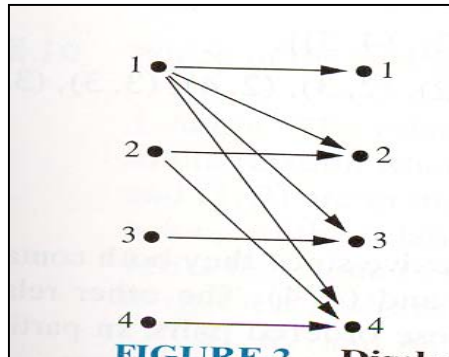 $b\ R^{-1}\ a \Leftrightarrow a\ eats\ b$
  (Compare: *b is eaten by a,* passive voice.)

Functions as Relations

A function f:A➔B is a relation from A to B

A relation from A to B is not always a function f:A➔B (e.g., relations could be one-to-many)

Relations are generalizations of functions!

FIGURE 2

Relations on a Set

- A (binary) relation from a set A to itself is called a relation *on A*. A relation on the set A is a relation from A to A.

- *E.g.*, the "**<**" relation is defined as a relation *on* **N**.

Relations <u>on</u> a Set

A (binary) relation from a set *A* to itself is called a relation *<u>on</u>* the set *A*.

A: {1,2,3,4}

R={(a,b) | a divides b}

FIGURE 2 Displaying the Ordered Pairs in the Relation *R* from Example 4.
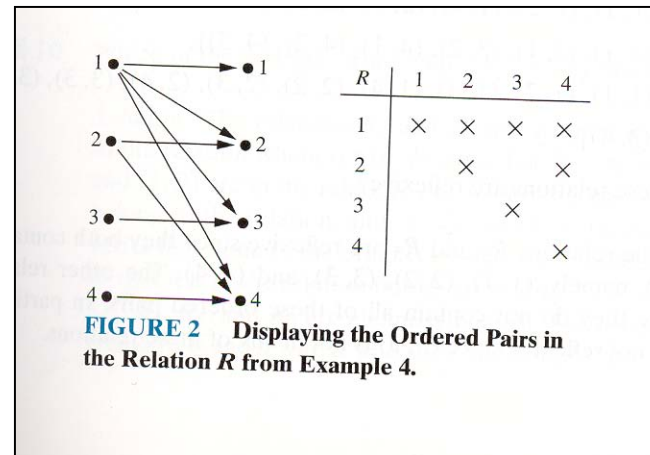
Example

How many relations are there on a set A with n elements?

Reflexivity and relatives

- A relation R on A is *reflexive* iff $\forall a \in A, (aRa)$.
  - *E.g.*, the relation $\geq := \equiv \{(a,b) \mid a \geq b\}$ is reflexive.
- R is *irreflexive* iff $\forall a \in A, (\neg aRa)$
- Note "*irreflexive*" does **NOT** mean "*not reflexive*", which is just $\neg \forall a \in A, (aRa)$.
- E.g., if Adore={(j,m),(b,m),(m,b)(j,j)} then this relation is neither reflexive nor irreflexive

Reflexivity and relatives

- Theorem: A relation R is *irreflexive* iff its *complementary* relation R' is reflexive.
  - Example: $<$ is irreflexive; $\geq$ is reflexive.
  - Proof: trivial

  - Is the "divide" relation on the set of positive integers reflexive?

Some examples

- Reflexive:

  =, 'have same cardinality', $\Leftrightarrow$

  <=, >=, $\Rightarrow$, $\subseteq$, etc.

- Irreflexive:

  <, >, `have different cardinality', $\subset$, 'is logically stronger than'

Symmetry & relatives

- A binary relation R on A is *symmetric* iff $\forall a,b((a,b) \in R \leftrightarrow (b,a) \in R)$.
  - *E.g.,* **=** (equality) is symmetric. **<** is not.
  - "is married to" is symmetric, "likes" is not.
- A binary relation R is *asymmetric* if $\forall a,b((a,b) \in R \rightarrow (b,a) \notin R)$.
  - **Examples: <** is asymmetric, "Adores" is not.
- Let R={(j,m),(b,m),(j,j)}. Is R (a)symmetric?

Symmetry & relatives

- Let R={(j,m),(b,m),(j,j)}.

R is not symmetric (because it does not contain (m,b) and because it does not contain (m,j)).

R is not asymmetric, due to (j,j)

Some direct consequences

Theorems:

1. R is symmetric iff $R = R^{-1}$,

2. R is asymmetric iff $R \cap R^{-1}$ is empty.

Symmetry & its relatives

1. R is symmetric iff $R = R^{-1}$

$\Rightarrow$ Suppose R is symmetric. Then
$(x,y) \in R \Leftrightarrow$
$(y,x) \in R \Leftrightarrow$
$(x,y) \in R^{-1}$

$\Leftarrow$ Suppose $R = R^{-1}$ Then
$(x,y) \in R \Leftrightarrow$
$(x,y) \in R^{-1} \Leftrightarrow$
$(y,x) \in R$

Symmetry & relatives

2. R is asymmetric iff $R \cap R^{-1}$ is empty.

   (Straightforward application of the definitions of asymmetry and R^{-1})

- Question: Can you construct a model in which the relation "son of" is symmetric?
- More precisely: find domain A and relation R on A such that R is symmetric and R(x,y) can reasonably be read as 'x is a son of y'

- Question: Can you construct a model in which the relation "son_of" is symmetric?

- Solution: any model in which there are no x,y such that son_of(x,y) is true

- E.g., A = {John, Mary, Sarah}, A**x**A ⊇ R= {}

Antisymmetry

- Consider the relation x≤y
- Is it symmetrical?
- Is it asymmetrical?
- Is it reflexive?
- Is it irreflexive?

Antisymmetry

- Consider the relation x≤y
- Is it symmetrical? No
- Is it asymmetrical?
- Is it reflexive?
- Is it irreflexive?

Antisymmetry

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Antisymmetry

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- Is it irreflexive?

Antisymmetry

- Consider the relation x≤y
- Is it symmetrical? No
- Is it asymmetrical? No
- Is it reflexive? Yes
- Is it irreflexive? No

Antisymmetry

- Consider the relation x≤y
  - It is not symmetric. (For instance, 5≤6 but not 6≤5)
  - It is not asymmetric. (For instance, 5 ≤5)
  - The pattern: the only times when (a,b)∈ ≤ and (b,a)∈ ≤ are when a=b
- This is called antisymmetry
  Can you say this in predicate logic?

Antisymmetry

- A binary relation *R* on *A* is *antisymmetric* iff $\forall a,b((a,b) \in R \wedge (b,a) \in R) \rightarrow a = b)$.

- Examples: $\leq, \geq, \subseteq$

- Another example: the earlier-defined relation Adore={(j,m),(b,m),(m,b)(j,j)}

- How would you define transitivity of a relation? What are its 'relatives'?

Transitivity & relatives

- A relation R is *transitive* iff (for all a,b,c) $$((a,b) \in R \land (b,c) \in R) \rightarrow (a,c) \in R.$$

- A relation is non*transitive* iff it is not transitive.

- A relation R is in*transitive* iff (for all a,b,c) $$((a,b) \in R \land (b,c) \in R) \rightarrow \neg (a,c) \in R.$$

Transitivity & relatives

- What about these examples:
  - "x is an ancestor of y"
  - "x likes y"
  - "x is located within 1 mile of y"
  - "x +1 =y"
  - "x beat y in the tournament"
  - "x is stronger than y"

Transitivity & relatives

- What about these examples:
  - "is an ancestor of" is transitive.
  - "likes" is neither trans nor intrans.
  - "is located within 1 mile of" is neither trans nor intrans
  - "x +1 =y" is intransitive
  - "x beat y in the tournament" is neither trans nor intrans
  - "x is stronger than y" is transitive.

Exploring the difference between relations and functions

Totality:

- A relation $R{:}A{\times}B$ is *total* if for every $a{\in}A$, there is at least one $b{\in}B$ such that $(a,b){\in}R$.
  - N.B., it does not follow that R^{-1} is total
  - It does not follow that R is a function.

Functionality

Functionality:

- A relation R: $A \times B$ is *functional* iff, for every $a \in A$, there is *at most one* $b \in B$ such that $(a,b) \in R$.

  – A functional relation R: $A \times B$ does not have to be total (there may be $a \in A$ such that $\neg \exists b \in B \, (aRb)$).

- Say that "R is functional", using predicate logic

Functionality

- R: $A \times B$ is *functional* iff, for every $a \in A$, there is *at most one* $b \in B$ such that $(a,b) \in R$.
  $\forall a \in A$: $\neg \exists \, b_1, b_2 \in B \, (b_1 \neq b_2 \wedge aRb_1 \wedge aRb_2)$.

- If R is functional and total relation, then R can be seen as a function R: $A \rightarrow B$

  Hence one can write $R(a)=b$ as well as aRb, $R(a,b)$, and $(a,b) \in R$. Each of these mean the same.

Functionality

- *Definition: R* is *antifunctional* iff its inverse relation R^{-1} is functional.

  (**Exercise:** Show that iff *R* is functional and antifunctional, and both it and its inverse are total, then it is a bijective function.)

Combining what you've learned about functions and relations

Consider the relation R:**N**$\rightarrow$**N** defined as

R = {(x,y) | x$\in$**N** $\land$ y$\in$**N** $\land$ y=x+1}.

Questions:

1. Is R total? Why (not)?
2. Is R functional? Why (not)?
3. Is R an injection? Why (not)?
4. Is R a surjection? Why (not)?

Composite Relations

- Let $R{:}A{\times}B$, and $S{:}B{\times}C$. Then the *composite* $S{\circ}R$ of R and S is defined as:

$$S{\circ}R = \{(a,c) \mid \exists b{:}\ aRb \wedge bSc\}$$

- Does this remind you of something?

Composite Relations

- Let $R{:}A{\times}B$, and $S{:}B{\times}C$. Then the *composite* $S{\circ}R$ of R and S is defined as:

  $$S{\circ}R = \{(a,c) \mid \exists b: aRb \land bSc\}$$

- Does this remind you of something?

- Function composition ...

- ... except that $S{\circ}R$ accommodates the fact that S and R may not be functional

Composite Relations

- Function composition is a special case of relation composition: Suppose S and R are functional. Then we have (using the definition above, then switching to function notation)

$S \circ R(a,c)$ iff $\exists b: aRb \land bSc$
iff $R(a)=b$ and $S(b)=c$ iff $S(R(a))=c$

Suppose

- Adore={(a,b),(b,c),(c,c)}
- Detest={(b,d),(c,a),(c,b)}
- Adore°Detest=
- Detest°Adore=

Suppose

- Adore={(a,b),(b,c),(c,c)}
- Detest={(b,d),(c,a),(c,b)}
- Adore°Detest= {(c,b),(c,c)}
- Detest°Adore=

Suppose

- Adore={(a,b),(b,c),(c,c)}
- Detest={(b,d),(c,a),(c,b)}
- Adore°Detest= {(c,b),(c,c)}
- Detest°Adore= {(a,d),(b,a),(b,b),(c,a),(c,b)}

§7.2: *n*-ary Relations

- An *n*-ary relation R on sets $A_1,\ldots,A_n$, is a subset

$$R \subseteq A_1 \times \ldots \times A_n.$$

- This is a straightforward generalisation of a binary relation. For example:

- 3-ary relations:
  - a is between b and c;
  - a gave b to c

§7.2: *n*-ary Relations

- An *n*-ary relation R on sets $A_1,\ldots,A_n$, is a subset

$$R \subseteq A_1 \times \ldots \times A_n.$$

- The sets A_i are called the *domains* of R.

- The *degree* of R is *n*.

- R is *functional in the domain A_i* if it contains at most one *n*-tuple $(\ldots, a_i ,\ldots)$ for any value a_i within domain A_i.

§7.2: *n*-ary Relations

- *R* is *functional in the domain A_i* if it contains at most one *n*-tuple $(\ldots, a_i, \ldots)$ for any value a_i within domain A_i.
- Generalisation: being functional in a combination of two or more domains.

Relational Databases

- A *relational database* is essentially just a set of relations.

- A domain A_i is a (*primary) key* for the database if the relation R is functional in A_i.

- A *composite key* for the database is a set of domains $\{A_i, A_j, \ldots\}$ such that R contains at most 1 n-tuple $(\ldots,a_i,\ldots,a_j,\ldots)$ for each composite value $(a_i, a_j,\ldots) \in A_i \times A_j \times \ldots$

Selection Operators

- Let A be any n-ary domain $A = A_1 \times \ldots \times A_n$, and let $C : A \to \{\mathbf{T}, \mathbf{F}\}$ be any *condition* (predicate) on elements (n-tuples) of A.

- The *selection operator* s_C maps any n-ary relation R on A to the relation consisting of all n-tuples from R that satisfy C:

    $$s_C(R) = \{a \in R \mid C(a) = \mathbf{T}\}$$

Selection Operator Example

- Let A = StudentName × Standing × SocSecNos

- Define a condition Upperlevel on A:
  $UpperLevel(name, standing, ssn) \Leftrightarrow$
  $((standing = \text{junior}) \lor (standing = \text{senior}))$

- Then, $s_{UpperLevel}$ takes any relation R on A and produces the subset of R involving of *just* the junior and senior students.

Projection Operators

- Let $A = A_1 \times \ldots \times A_n$ be any n-ary domain, and let $\{i_k\} = (i_1, \ldots, i_m)$ be a sequence of indices all falling in the range 1 to n,

- Then the *projection operator* on n-tuples
$$P_{\{i_k\}} : A \to A_{i_1} \ldots \times A_{i_m}$$
is defined by:
$$P_{\{i_k\}}(a_1, \ldots, a_n) = (a_{i_1}, \ldots, a_{i_m})$$

Projection Example

- Suppose we have a domain *Cars=Model× Year× Color.* (note n=3).

- Consider the index sequence $\{i_k\}$= 1,3. (m=2)

- Then the projection $P_{\{i_k\}}$ maps each tuple $(a_1,a_2,a_3) = (model,year,color)$ to its image:
$$\left(a_{i_1},a_{i_2}\right)=\left(a_1,a_3\right)=\left(model,color\right)$$

- This operator can be applied to a relation $R \subseteq Cars$ to obtain a list of the model/color combinations available.

Join Operator

- Puts two relations together to form a combined relation which is their composition:

- Iff the tuple (A,B) appears in R_1, and the tuple (B,C) appears in R_2, then the tuple (A,B,C) appears in the join $J(R_1,R_2)$.
  - A, B, and C can also be sequences of elements.

Join Example

- Suppose R_1 is a teaching assignment table, relating *Lecturers* to *Courses*.
- Suppose R_2 is a room assignment table relating *Courses* to *Rooms*, *Times*.
- Then $J(R_1,R_2)$ is like your class schedule, listing (*lecturer,course,room,time*).
- (Joins are similar to *relation composition*. For precise definition, see Rosen, p.486)

Composite Relations

- Let's see what happens when we compose R with itself ...

- First: different ways to represent relations

§7.3: Representing Relations

- Before saying more about the n-th power of a relation, let's talk about representations

- Some ways to represent *n*-ary relations:
  - With a list of n-tuples.
  - With a function from the (n-ary) domain to {**T**,**F**}.

- Special ways to represent binary relations:
  - With a zero-one matrix.
  - With a directed graph.

- Why bother with alternative representations? Is one not enough?
- One reason: some calculations are easier using one representation, some things are easier using another
- There are even some basic ideas that are suggested by a particular representation

It's often worth playing around with different representations

Using Zero-One Matrices

- To represent a binary relation $R:A \times B$ by an $|A| \times |B|$ 0-1 matrix $\mathbf{M}_R = [m_{ij}]$, let $m_{ij} = 1$ iff $(a_i, b_j) \in R$.

- *E.g.*, Suppose Joe likes Susan and Mary, Fred likes Mary, and Mark likes Sally.

- Then the 0-1 matrix representation of the relation Likes:Boys×Girls relation is:

	Susan	Mary	Sally
Joe	1	1	0
Fred	0	1	0
Mark	0	0	1

- Special case 1-0 matrices for a relation on A (that is, $R{:}A{\times}A$)
- *Convention*: rows and columns list elements in the same order
- This where 1-0 matrices come into their own!

Zero-One Reflexive, Symmetric

- Recall: *Reflexive, irreflexive, symmetric, and asymmetric* relations.
  - These relation characteristics are easy to recognize by inspection of the zero-one matrix.
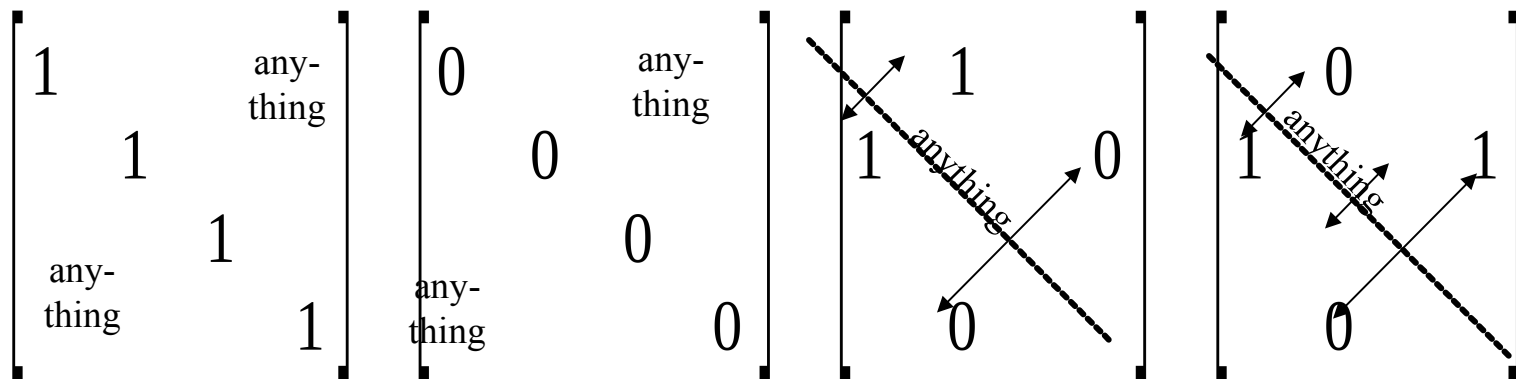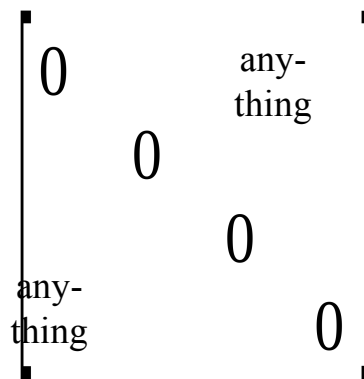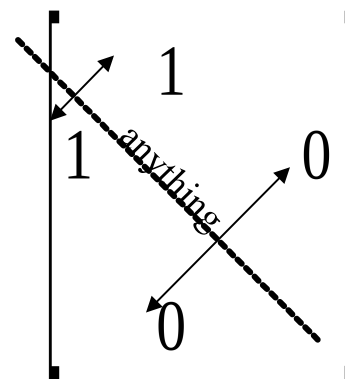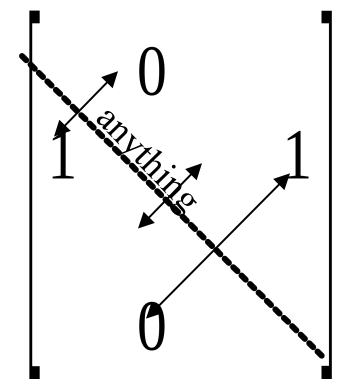
$$\begin{bmatrix} 1 & & & \text{any-thing} \\ & 1 & & \\ & & 1 & \\ \text{any-thing} & & & 1 \end{bmatrix} \quad \begin{bmatrix} 0 & & & \text{any-thing} \\ & 0 & & \\ & & 0 & \\ \text{any-thing} & & & 0 \end{bmatrix} \quad \begin{bmatrix} 1 & & \\ 1 & \text{anything} & 0 \\ & & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & & \\ 1 & \text{anything} & 1 \\ & & 0 \end{bmatrix}$$

Zero-One Reflexive, Symmetric

- Recall: *Reflexive, irreflexive, symmetric, and asymmetric* relations.
  - These relation characteristics are very easy to recognize by inspection of the zero-one matrix.
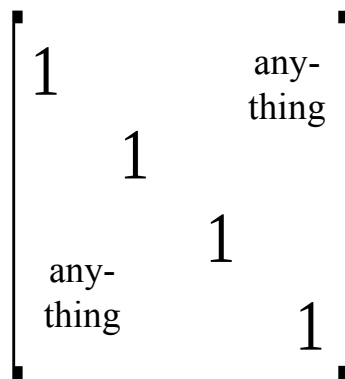
| Reflexive: only 1's on diagonal | Irreflexive: only 0's on diagonal | Symmetric: all identical across diagonal | Asymmetric: all 1's are across from 0's |

Matrices

- There exists much mathematical tjeory about graphs
- Some fast algorithms rely on graphs
- More about graphs: Rosen, section 3.8

Using Directed Graphs

- A *directed graph* or *digraph* $G=(V_G,E_G)$ is a set V_G of *vertices (nodes)* with a set $E_G \subseteq V_G \times V_G$ of *edges (arcs).* Visually represented using dots for nodes, and arrows for edges. A relation $R{:}A \times B$ can be represented as a graph $G_R=(V_G=A \cup B, E_G=R)$.
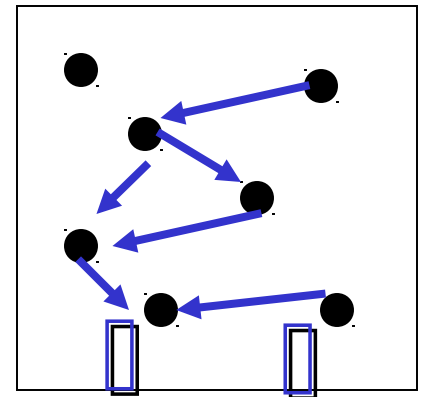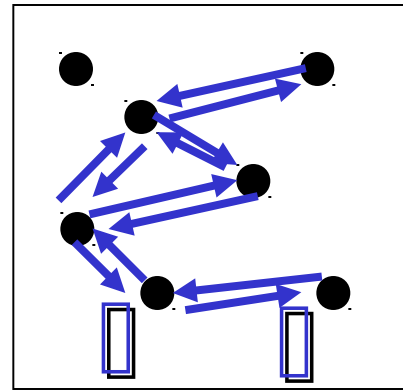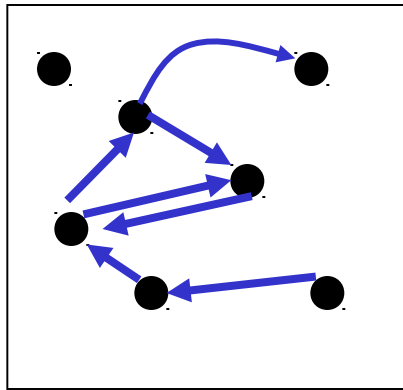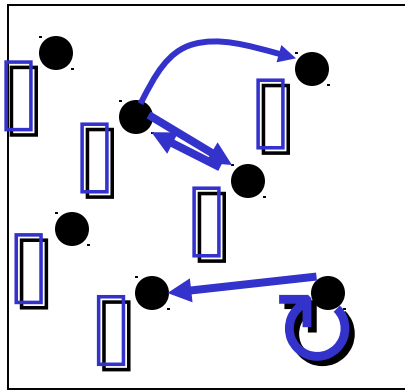
Matrix representation $\mathbf{M}_R$:

	Susan	Mary	Sally
Joe	1	1	0
Fred	0	1	0
Mark	0	0	1

Graph rep. G_R:

Edge set E_G
(blue arrows)

Joe → Susan
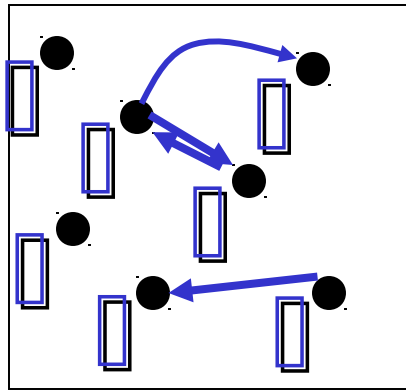Fred → Mary
Mark → Sally

Node set V_G
(black dots)
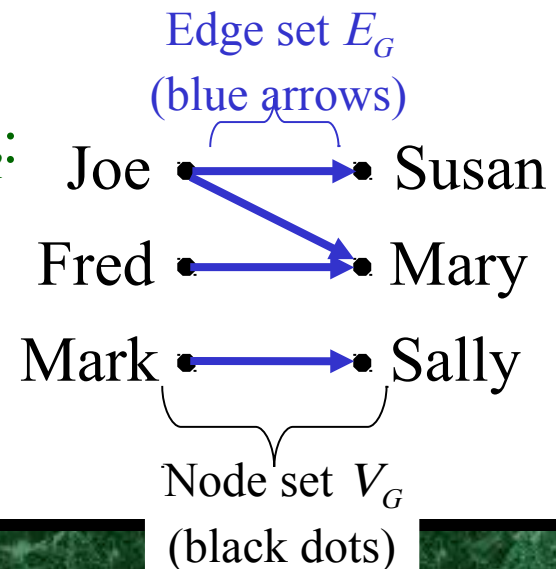
Digraph Reflexive, Symmetric

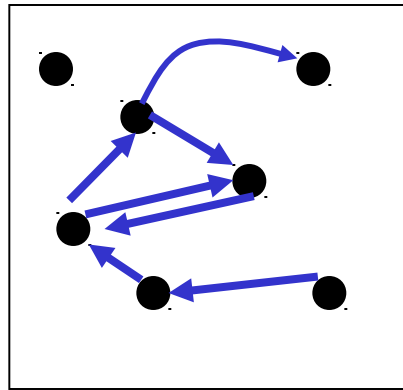Properties of a relation can determined
by inspection of its graph.
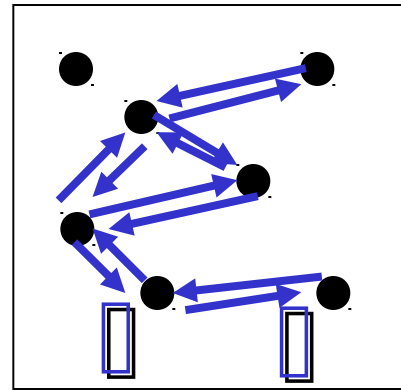
Digraph Reflexive, Symmetric

Many properties of a relation can be determined

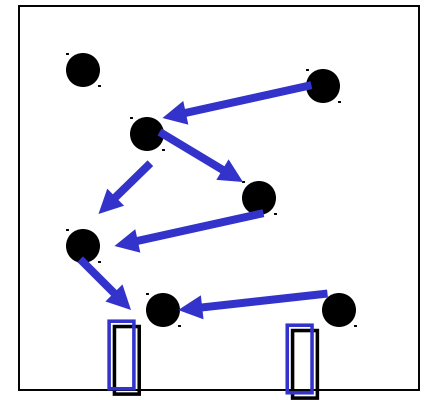by inspection of its graph.

Reflexive:
Every node
has a self-loop

Irreflexive:
No node
links to itself

Symmetric:
Every link is
bidirectional

Antisymmetric:
never (a,b) and
(b,a), unless a=b

These are not symmetric & not asymmetric

These are non-reflexive & non-irreflexive

Particularly easy with a graph

- Properties that are somehow 'local' to a given element, e.g.,
  - "does the relation contain any elements that are unconnected to any others?"
- Properties that involve combinations of pairs, e.g.,
  - "does the relation contain any cycles?"
  - things to do with the composition of relations (e.g. the n-th power of R)
- More about graphs: Rosen, chapter 9.

Now: Composing R with itself

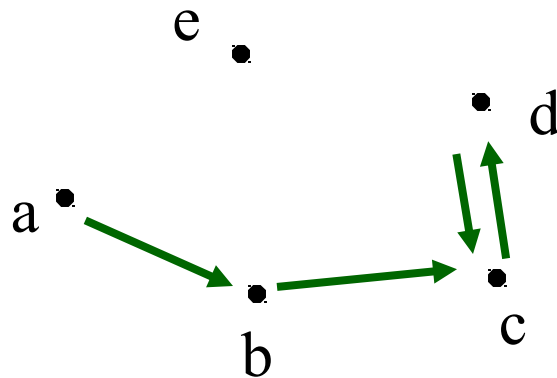- The *n*th power R^n of a relation R on a set A
  - *The 1st power of R is R itself*
  - *The 2nd power of R is $R^2 = R \circ R$*
  - *The 3rd power of R is $R^3 = R \circ R \circ R$*

etc.

Composite Relations

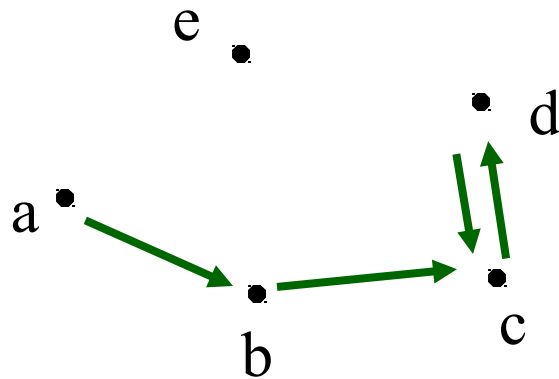- The nth power R^n of a relation R on a set A can be defined recursively by:

$$R^1 :\equiv R; \quad R^{n+1} :\equiv R^n \circ R \quad \text{for all } n \geq 1.$$

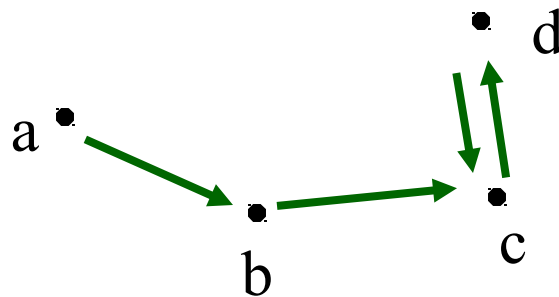- E.g., $R^2 = R \circ R; \quad R^3 = R \circ R \circ R$

Composite Relations

- $R^2 = R \circ R = \dots$

e •

d •

a •

b •

c •

Composite Relations

- $R^2 = R \circ R = \{(a,c),(b,d),(c,c),(d,d)\}$

- a: c   d: d
  b: d   e: -
  c: c

e

d

a

b

c

Back to the n-th power of a relation

- A *path* of length *n* from node *a* to *b* in the directed graph *G* is a sequence $(a, x_1), (x_1, x_2), \ldots, (x_{n-1}, b)$ of *n* ordered pairs in E_G.
  - Note: there exists a path of length *n* from *a* to *b* in *R* if and only if $(a, b) \in R^n$.
- A path of length $n \geq 1$ from *a* to itself is a *cycle*.
- R*: the relation that holds between a and b iff there exists a finite path from a to b using R.
  - Note: R* is transitive!

Why is R* of interest?

- Suppose an infectious disease is transmitted by shaking hands (Shake(x,y))

- To know who is infected by John, you need to think about two things:

  1. Determine $\{x \in$ person: Shake(John,x)$\}$ This gives you the direct infectees

  2. Everyone infected by someone infected by John. Note: this is recursive

- Suppose S(hake) = {(a,b), (b,c), (c,d)}. We want to compute S*.
- S ⊆ S*, so S*(a,b), S*(b,c), S*(c,d)
- Infer S*(a,c) and S*(b,d) (using the following rule twice: S(x,y) & S(y,z) → S(x,z))
- Are we done?

Who is infected?

- Suppose S(hake)= {(a,b), S(b,c), S(c,d)}. We want to compute S*.

- S $\subseteq$ S*, so S*(a,b), S*(b,c), S*(c,d)

- Infer S*(a,c) and S*(b,d) (using the following rule twice: S(x,y) & S(y,z) $\rightarrow$ S(x,z))

- Second step: S*(a,d)

We don't always know R* …

- We often don't know the exact extension of a relation (i.e., which pairs are elements of the relation)
- Presumably, you've never shook hands with the president of Mongolia: ¬S(you,PM)
- How about S*(you,PM) …?

Other examples of R*

- R(a,b) $\Leftrightarrow$ there's a direct bus service from a to b.
- R(p,q) $\Leftrightarrow$ there exists an inference rule that allows you to infer q from p

What is R* in each of these cases?

Other examples of R*

- R(a,b) ⟺ there's a direct bus service from a to b.
- R(p,q) ⟺ there exists an inference rule that allows you to infer q from p

What is R* in each of these cases?

- R(a,b) ⟺ one can go by bus from a to b.
- R(p,q) ⟺ there exists a proof that q follows from p

R*

How would you formally define R*?

R*

How would you formally
  define R*?

Here's a safe bet

R^l_i

R*

How would you formally define R*?

Here's a finite variant, where $n = |A|$ *(proof in book* that n is large enough*)*

R^i_i

- So much (for the moment!) about graphs
- Let's return to properties like reflexivity, symmetry and transitivity
- Informally: How can we "make" a relation transitive (etc.)?

§7.4: Closures of Relations

- For any property *X*, the <u>*X* closure</u> of a set *A* is defined as the "smallest" superset of *A* that has property *X*. More specifically,
  - The *reflexive closure* of a relation *R* on *A* is the smallest superset of R that is reflexive.
  - The *symmetric closure* of *R* is the smallest superset of R that is symmetric
  - The *transitive closure* of *R* is the smallest superset of R that is transitive

Calculating closures

- The *reflexive closure* of a relation R on A is obtained by "adding" (a,a) to R for each $a \in A$. I.e., it is $R \cup I_A$ (Check that this is the r.c.)

- The *symmetric closure* of R is obtained by "adding" (b,a) to R for each (a,b) in R. I.e., it is $R \cup R^{-1}$ (Check that this is the s.c.)

- The *transitive closure* of R is obtained by *"repeatedly"* adding (a,c) to R for each $(a,b),(b,c)$ in R …

Calculating closures

- Adore={(a,b),(b,c),(c,c)}
- Detest={(b,d),(c,a),(c,b)}
- The *symmetric closure* of ...

  ... Adore=

  ... Detest=

Calculating closures

- Adore={(a,b),(b,c),(c,c)}
- Detest={(b,d),(c,a),(c,b)}
- The *symmetric closure* of ...

  ... Adore={(a,b),(b,c),(c,c),(b,a), (c,b)}

  ... Detest={(b,d),(c,a),(c,b), (d,b), (a,c),(b,c)}

Calculating closures

- Adore={(a,b),(b,c),(c,c)}
- Detest={(b,d),(c,a),(c,b)}
- The *transitive closure* of ...

  ... Adore=

  ... Detest=

Calculating closures

- Adore={(a,b),(b,c),(c,c)}
- Detest={(b,d),(c,a),(c,b)}
- The *transitive closure* of ...

  ... Adore={(a,b),(b,c),(c,c),(a,c)}

  ... Detest={(b,d),(c,a),(c,b),(c,d)}

TC(R)

- A more precise definition of the transitive closure of R (abbr: TC(R)) is:

- TC(R)= the intersection of all transitive supersets of R.

- Let's check that this matches our earlier definition
  - It follows from the new definition that there exists no smaller transitive superset of R than TC(R).
  - TC(R) itself is a transitive superset of R.
    Proof:

TC(R)

TC(R) is a transitive superset of R. Proof:

- If A and B are transitive supersets of R then A∩B is a transitive superset of R

1. A∩B is a superset of R.
2. A∩B is a transitive. (Suppose (x,y) and (y,z) are elements of A∩B. Then (x,z) is an element of A∩B.)

TC(R)

- So TC(R) is a transitive superset of R.
- Since it is the intersection of all transitive supersets of R, TC(R) is <u>the smallest</u> transitive superset of R.
  - Suppose X is a transitive superset of R and $X \subset TC(R)$. Then $(TC(R) \cap X) \subset TC(R)$. But $TC(R)$ is the intersection of all trans. supersets of X, hence $(TC(R) \cap X) = TC(R)$. Contradiction.
- Now we relate TC(R) with the graph-theoretic concept R*:

Theorem: R*=TC(R)

Theorem: R* = the transitive closure of R

We need to prove that R* is the smallest transitive superset of R.

1. Proof that R* is transitive:

Suppose xR*y and yR*z.
E.g., xR^{n}y and yR^{m}z
Then xR^{n+m}z, hence xR*z

Proof ctd.

2. Evidently, R⊆R*, so R* is a superset of R.

We now know that R* is a transitive superset of R.

3. R cannot have a <u>smaller</u> transitive superset than R*.

Proof: Suppose such a transitive superset S of R existed. This would mean that there exists a pair (x,y) such that xR*y while $\neg xSy$. But xR*y means $\exists n$ such that $xR^n y$. But since R⊆S, it would follow that $xS^n y$; but because S is transitive, this would imply that xSy. Contradiction.
 (Compare Rosen p.500 (5th ed.), p.548 (6th ed.)

An Euler diagram might help ...

Suppose there existed a transitive superset of R that's smaller than R* ...

R*

S

R

(x,y)
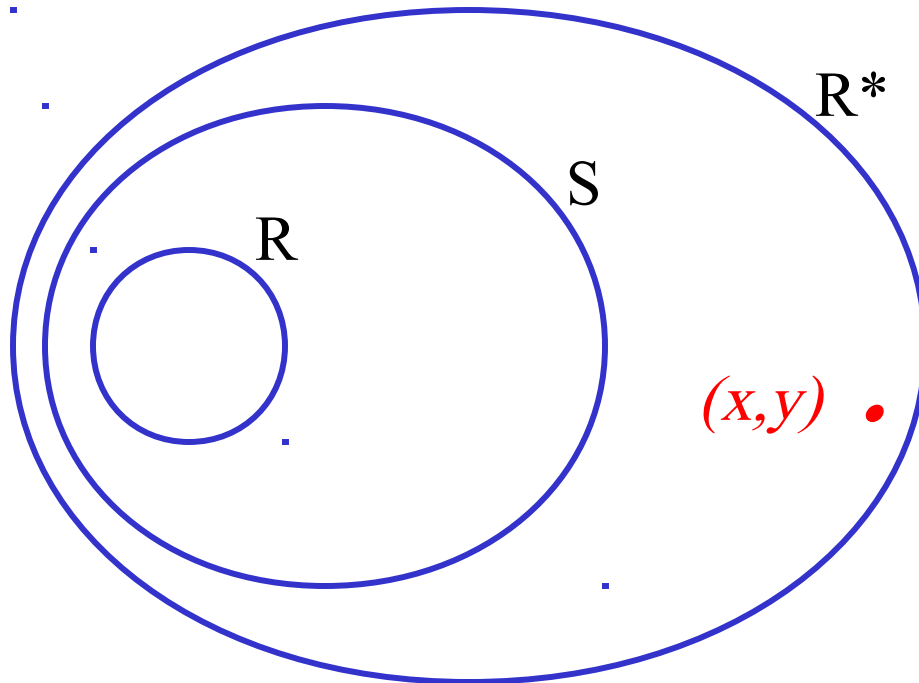
- Fast algorithms are available for calculating R*, especially *Warshall's algorithm* (also called Roy-Warshall algorithm)
- FYI: this algorithm uses a matrix representation.

§7.5: Equivalence Relations

- Definition: An *equivalence relation* on a set *A* is any binary relation on *A* that is *reflexive, symmetric, and transitive.*

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  - *E.g.,* **=** is an equivalence relation.
  - But many other relations follow this pattern too

§7.5: Equivalence Relations

- Definition: An *equivalence relation* on a set *A* is any binary relation on *A* that is *reflexive, symmetric, and transitive.*
  - *E.g.,* **=** is an equivalence relation.
  - For any function $f:A \rightarrow B$, the relation "have the same f value", or $=_f :\equiv \{(a_1, a_2) \mid f(a_1) = f(a_2)\}$ is an equivalence relation,
    - *e.g.,* let m="mother of" then $=_m$ = "have the same mother" is an equivalence relation

Equivalence Relation Examples

- "Strings *a* and *b* are the same length."
- "Integers *a* and *b* have the same absolute value."

Equivalence Relation Examples

Let's talk about relations between functions:

1. How about: R(f,g) ⟺ f(2)=g(2) ?

2. How about: R(f,g) ⟺ f(1)=g(1)∨f(2)=g(2) ?

Equivalence Relation Examples

1. How about: R(f,g) ⟺ f(2)=g(2) ?
   Yes. Reflexivity: f(2)=f(2), for all f
       Sym: f(2)=g(2) implies g(2)=f(2)
       Trans: f(2)=g(2) and g(2)=h(2)
              implies f(2)=h(2).

2. How about: R(f,g) ⟺ f(1)=g(1)∨f(2)=g(2) ?

Equivalence Relation Examples

How about $R(f,g) \Leftrightarrow f(1)=g(1) \lor f(2)=g(2)$?

- No. Counterexample against transitivity:

$f(1)=a$, $f(2)=b$

$g(1)=a$, $g(2)=c$

$h(1)=b$, $h(2)=c$

Equivalence Classes

- Let R be any equivalence relation.
- The *equivalence class* of *a under R*,

  $$[a]_R :\equiv \{\, x \mid aRx \,\}$$ (optional subscript R)

  - Intuitively, this is the set of all elements that are "equivalent" to *a* according to *R*.
  - Each such *b* (including *a* itself) can be seen as a *representative* of $[a]_R$.

Equivalence Classes

- Why can we talk so loosely about elements being equivalent to each other (as if the relation didn't have a direction)?

- In some sense, it does not matter which representative of an equivalence class you take as your starting point:

  If aRb then $\{\ x \mid aRx\ \} = \{\ x \mid bRx\ \}$

Equivalence Classes

If aRb   then   aRx ⟺ bRx   **Proof:**

1. Suppose aRb while bRx.
Then aRx follows directly by *transitivity.*

2. Suppose aRb while aRx.

aRb implies bRa *(symmetry).* But
bRa and aRx imply bRx by *transitivity*

Equivalence Classes

We now know that
 If aRb then $\{\ x \mid aRx\ \} = \{\ x \mid bRx\ \}$

Equally,
 If aRb then $\{\ x \mid xRa\ \} = \{\ x \mid xRb\ \}$
 (due to symmetry)

In other words, an equivalence class based on R is simply a maximal set of things related by R

Equivalence Class Examples

- "(Strings *a* and *b*) *have the same length.*"
  - Suppose a has length 3. Then $[a] =$ the set of all strings of length 3.
- "(Integers *a* and *b*) *have the same absolute value.*"
  - $[a] =$ the set $\{a, -a\}$

Equivalence Class Examples

- "Formulas φ and ψ contain the same number of brackets" (e.g. for formulas of propositional logic, using the strict syntax)
- Now what is [((p∧q)∨r)]?

Equivalence Class Examples

- Consider the equivalence relation ⇔ (i.e., logical equivalence, for example between formulas of propositional logic)
- What is [p∧q]?

Partitions

- A *partition* of a set *A* is a collection of disjoint nonempty subsets of A that have A as their union.

- Intuitively: a partition of A divides A into separate parts (in such a way that there is no remainder).

Partitions and equivalence classes

- Consider a *partition* of a set A into $A_1, ..A_n$
  - The A_i's are all disjoint : For all x and for all i, if $x \in A_i$ and $x \in A_j$ then $A_i = A_j$
  - The union of the A_i's = A

Partitions and equivalence classes

- A *partition* of a set A can be viewed as the set of all the equivalence classes $\{A_1, A_2, \dots \}$ for some equivalence relation on A.

- For example, consider the set A=$\{1,2,3,4,5,6\}$ and its partition $\{\{1,2,3\},\{4\},\{5,6\}\}$

- R = $\{$ (1,1),(2,2),(3,3),(1,2),(1,3),(2,3),(2,1),(3,1), (3,2) ,(4,4), (5,5),(6,6),(5,6),(6,5) $\}$

Partitions and equivalence classes

- We sometimes say:
  - A partition of A *induces* an equivalence relation on A
  - An equivalence relation on A *induces* a partition of A

§7.6: Partial Orderings

- A relation *R* on *A* is called a *partial ordering* or *partial order* iff it is reflexive, antisymmetric, and transitive.
  - We often use a symbol looking something like ≼ (or analogous shapes) for such relations.
  - Examples: ≤, ≥ on real numbers, ⊆, ⊇ on sets.
  - Another example: the "divides" relation | on **Z**⁺.
    - It is not necessarily the case that either *a*≼*b* or *b*≼*a*.
- A set *A* together with a partial order ≼ on *A* is called a *partially ordered set* or *poset* and is denoted (*A*, ≼).
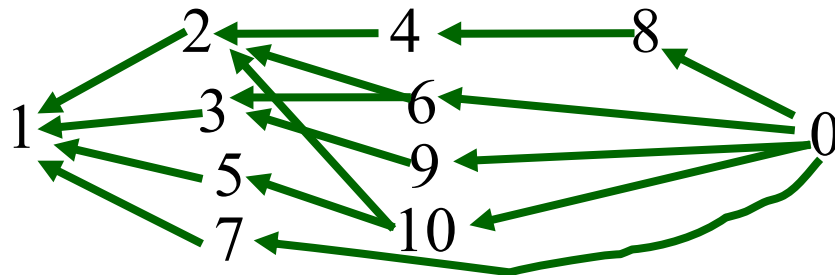
Posets as Noncyclical Digraphs

- If a set S is partially ordered by a relation R then its graph can be simplified:
  - Looping edges need not be drawn, because they can be inferred
  - Instead of drawing edges for R(a,b), R(b,c) and R(a,c) , the latter can be omitted (because it can be inferred)
  - If direction of arrows is represented as left-to-right (or top-down) order then it's called a Hasse diagram (We won't do that here)

Posets as Noncyclical Digraphs

- There is a one-to-one correspondence between posets and the reflexive+transitive closures of noncyclical digraphs.

- Example: consider the poset ({0,…,10}, |)
  - Its "minimal" digraph:

Posets as Noncyclical Digraphs

- Prove: a graph for a partial order cannot contain cycles

Posets as Noncyclical Digraphs

- **Theorem**: a graph for a partial order cannot contain cycles with length > 1.

- **Proof**: suppose there is a cycle $a_1 R a_2 R \dots R a_n R a_1$ (with n>1). Then, with n-1 applications of transitivity, we have $a_1 R a_n$. But also $a_n R a_1$, which conflicts with antisymmetry.

Posets do not have cycles

- **Proof**: suppose there is a cycle $a_1Ra_2R\ldots Ra_nRa_1$. Then, with n-1 applications of transitivity, we have a_1Ra_n. But also a_nRa_1, which conflicts with antisymmetry.

$a_1 \longrightarrow a_2 \longrightarrow \quad \longrightarrow a_n$
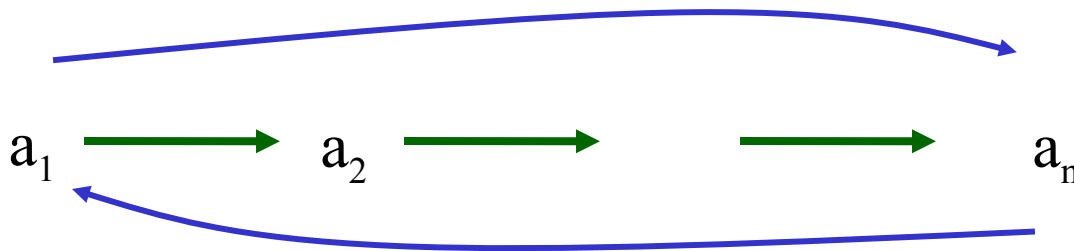
- Can something be both a poset and an equivalence relation?

- Can something be both a poset and an equivalence relation?
  - Equiv: ref, sym, trans
  - Poset: ref, antisym, trans
- Can a relation (that is reflexive and transitive) be both sym and antisym?

- Can a relation that is reflexive and transitive be both sym and antisym?
- Yes: the empty relation R={} is an example
- But any relation R ⊆ {(x,x): x∈A} will also qualify.
  - It's reflexive
  - It's symmetric and antisymmetric
  - It's transitive
- Other relations cannot qualify. (Prove at home)

Some other types of orderings

1. A **lattice** is a poset in which every pair of elements has a least upper bound (LUB) and a greatest lower bound (GLB).
   Formally: (done in exercise)
   Example: $(Z^+, |)$ In this case,
      LUB = Least Common Multiple
      GLB = Greatest Common Denominator
   Non-example: $(\{1,2,3\}, |)$

Some other types of orderings

2. **Linearly ordered** sets (also: **totally** ordered sets): posets in which
    all elements are "comparable" (i.e., related by R).
    Formally: $\forall x, y \in A(xRy \vee yRx)$.

    Example:
    Non-example:

Some other types of orderings

Linearly ordered sets (also: **totally** ordered sets): posets in which *all elements are comparable*. Formally:
$\forall x,y \in A(xRy \lor yRx)$.
Example: $(\mathbf{N}, \leq)$
Non-example: $(\mathbf{N}, | \,)$ (where $|$ is 'divides')
Non-example: $\subseteq$

An application of posets

- Consider $(A, \leq)$, where A is a set of project tasks and $a < b$ means "a must be completed before b can be completed"

- (Sometimes it's easier to define $<$ than $\leq$)

- Note that $(A, \leq)$ is a poset: ref, antisym, trans

An application of posets

- A common problem: Given $(A, \leq)$, find a *linear* order $(A, \preccurlyeq)$ that is *compatible* with $(A, \leq)$. (That is, $(A, \leq) \subseteq (A, \preccurlyeq)$)

- (We're assuming that tasks cannot be carried out in parallel)

- Algorithm for finding a compatible linear order given a finite partial order: p.526.

Some other types of orderings

2. **Well-ordered** sets: linearly ordered sets in which *every nonempty subset has a least element* (that is, an element a such that $\forall x \in A(aRx)$)

    Example: …
    Non-example: …

Some other types of orderings

2. **Well-ordered** sets: linearly ordered sets in which *every nonempty subset has a least element* (that is, an element a such that $\forall x \in A(aRx)$)

   Example: $(\mathbf{N}, \leq)$
   Non-examples:
   $(\mathbf{Z}, \leq)$,
   (non-negative elements of $\mathbf{R}, \leq$)

Some other types of orderings

2. Non-examples: $(\mathbf{Z}, \leq)$, $(\mathbf{R}^+, \leq)$

- $(\mathbf{Z}, \leq)$: $\mathbf{Z}$ itself has no least element.

- $(\text{Non-negative } \mathbf{R}, \leq)$:

  Nonnegative $\mathbf{R}$ itself does have a least element , but

  $\mathbf{R}^+ \subseteq \text{Nonnegative } \mathbf{R}$ has no least element.

- Well-orderings are behind one of the most general proof techniques that exist: *mathematical induction*.

- The last 30 slides were a tiny crash course in the theory of *mathematical structures*

- Compare Rosen, chapter 7.6.