Advanced Counting Techniques

Outline

- Recurrence Relations
- Solving Linear Recurrence Relations
- Generating Functions

Recurrence Relations

- We specified sequences by providing explicit formulas for their terms. There are many other ways to specify a sequence. For example, another way to specify a sequence is to provide one or more initial terms together with a rule for determining subsequent terms from those that precede them
- A rule of the latter sort is known as **Recurrence Relation**

DEFINITION 1: A *recurrence relation* for the sequence $\{a_n\}$ is an equation that expresses a_n in terms of one or more of the previous terms of the sequence, namely, $a_0, a_1, \ldots, a_{n-1}$, for all integers n with $n \geq n_0$, where n_0 is a nonnegative integer.

A sequence is called a *solution* of a recurrence relation if its terms satisfy the recurrence relation. (A recurrence relation is said to *recursively define* a sequence.)

Recurrence Relations Contd

- **EXAMPLE 1:** Let $\{a_n\}$ be a sequence that satisfies the recurrence relation $a_n = a_{n-1} + 3$ for $n = 1, 2, 3, \ldots,$ and suppose that $a_0 = 2$. What are a_1, a_2, and a_3?

Sol: 5, 8, 11

EXAMPLE 2: Let $\{a_n\}$ be a sequence that satisfies the recurrence relation $a_n = a_{n-1} - a_{n-2}$ for $n = 2, 3, 4, \ldots,$ and suppose that $a_0 = 3$ and $a_1 = 5$. What are a_2 and a_3?

Sol: 2, -3

The **initial conditions** for a recursively defined sequence specify the terms that precede the first term where the recurrence relation takes effect. For instance, the initial condition in Example 1 is $a_0 = 2$, *and the initial conditions in Example 2 are* $a_0 = 3$ *and* $a_1 = 5$.

Fibonacci sequence

DEFINITION 2: The *Fibonacci sequence*, $f_0, f_1, f_2, \ldots,$ is defined by the initial conditions $f_0 = 0, f_1 = 1$, and the recurrence relation $f_n = f_{n-1} + f_{n-2}$ for $n = 2, 3, 4, \ldots$.

EXAMPLE 3: Find the Fibonacci numbers $f_2, f_3, f_4, f_5,$ *and* f_6.

Sol: 1, 2, 3, 5, 8

We say that we have solved the recurrence relation together with the initial conditions when we find an explicit formula, called a **closed formula,** for the terms of the sequence.

EXAMPLE 4: Determine whether the sequence $\{a_n\}$, where $a_n = 3n$ for every nonnegative integer n, is a solution of the recurrence relation $a_n = 2a_{n-1} - a_{n-2}$ for $n = 2, 3, 4, \ldots$. Answer the same question where $a_n = 2n$ and where $a_n = 5$.

EXAMPLE 5: Solve the recurrence relation and initial condition in Example 1.
i.starting with the initial condition $a_1 = 2$, *and working upward until we reach* a_n *to deduce a closed formula for* the sequence.
ii. starting with the term a_n *and working downward until we reach the initial condition* $a_1 = 2$ *to deduce this same* formula.

The technique used in Example 5 is called **iteration.** We have iterated, or repeatedly used, the recurrence relation.

- The first approach is called **forward substitution –** we found successive terms beginning with the initial condition and ending with a_n.
- *The second approach is called* **backward substitution**, because we began with a_n *and iterated to express it in terms of falling* terms of the sequence until we found it in terms of a_1.

EXAMPLE 6: Compound Interest
Suppose that a person deposits \$10,000 in a savings account at a bank yielding 11% per year with interest compounded annually. How much will be in the account after 30 years?

Sol: \$228*,922.97.*

EXAMPLE 6

- **a)** Find a recurrence relation for the number of ways to climb *n* stairs if the person climbing the stairs can take one stair or two stairs at a time.
- **b)** What are the initial conditions?
- **c)** In how many ways can this person climb a flight of eight stairs?

- **EXAMPLE 7:** A factory makes custom sports cars at an increasing rate. In the first month only one car is made, in the second month two cars are made, and so on, with n cars made in the nth month.

  a) Set up a recurrence relation for the number of cars produced in the first n months by this factory.

  b) How many cars are produced in the first year?

  c) Find an explicit formula for the number of cars produced in the first n months by this factory

Exercises

- Find the first five terms of the sequence defined by each of these recurrence relations and initial conditions.

  a) $a_n = 6a_{n-1}$, $a_0 = 2$

  b) $a_n = a^2_{n-1}$, $a_1 = 2$

  c) $a_n = a_{n-1} + 3a_{n-2}$, $a_0 = 1$, $a_1 = 2$
- Find the first six terms of the sequence defined by each of these recurrence relations and initial conditions.

  a) $a_n = -2a_{n-1}$, $a_0 = -1$

  b) $a_n = a_{n-1} - a_{n-2}$, $a_0 = 2$, $a_1 = -1$

  c) $a_n = 3a^2_{n-1}$, $a_0 = 1$

  d) $a_n = na_{n-1} + a^2_{n-2}$, $a_0 = -1$, $a_1 = 0$
- Show that the sequence $\{a_n\}$ is a solution of the recurrence relation $a_n = -3a_{n-1} + 4a_{n-2}$ if

  a) $a_n = 0$. **b)** $a_n = 1$.

  c) $a_n = (-4)^n$. **d)** $a_n = 2(-4)^n + 3$.

Exercises Contd

- Is the sequence $\{a_n\}$ a solution of the recurrence relation $a_n = 8a_{n-1} - 16a_{n-2}$ if

  a) $a_n = 0$? **b)** $a_n = 1$?

  c) $a_n = 2^n$? **d)** $a_n = 4^n$?

  e) $a_n = n4^n$? **f)** $a_n = 2.4^n + 3n4^n$?

- A person deposits \$1000 in an account that yields 9% interest compounded annually.

  a) Set up a recurrence relation for the amount in the account at the end of n years.

  b) Find an explicit formula for the amount in the account at the end of n years.

  c) How much money will the account contain after 100 years?

- Suppose that the number of bacteria in a colony triples every hour.

  a) Set up a recurrence relation for the number of bacteria after n hours have elapsed.

  b) If 100 bacteria are used to begin a new colony, how many bacteria will be in the colony in 10 hours?

Exercises Contd

- Assume that the population of the world in 2010 was 6.9 billion and is growing at the rate of 1.1% a year.

  a) Set up a recurrence relation for the population of the world *n* years after 2010.

  b) Find an explicit formula for the population of the world *n* years after 2010.

  c) What will the population of the world be in 2030?

- A factory makes custom sports cars at an increasing rate. In the first month only one car is made, in the second month two cars are made, and so on, with *n* cars made in the *n*th month.

  a) Set up a recurrence relation for the number of cars produced in the first *n* months by this factory.

  b) How many cars are produced in the first year?

  c) Find an explicit formula for the number of cars produced in the first *n* months by this factory.

- An employee joined a company in 2009 with a starting salary of $50,000. Every year this employee receives a raise of $1000 plus 5% of the salary of the previous year.

  a) Set up a recurrence relation for the salary of this employee *n* years after 2009.

  b) What will the salary of this employee be in 2017?

  c) Find an explicit formula for the salary of this employee *n* years after 2009.

Recurrence Relations

- **Example 1.** Let $\{a_n\}$ be a sequence that satisfies the recurrence relation $a_n=a_{n-1}-a_{n-2}$ for $n=2,3,\ldots,$ and suppose that $a_0=3$,and $a_1=5$.

  Here $a_0=3$ and $a_1=5$ are the initial conditions.

  By the recurrence relation,

$$a_2 = a_1-a_0 = 2$$

$$a_3 = a_2-a_1 = -3$$

$$a_4 = a_3-a_2 = -5$$

$$\vdots$$

Q1: Applications ?

Q2: Are there better ways for computing the terms of $\{a_n\}$?

Applications of Recurrence Relations

- We will show that recurrence relations can be used to study and to solve counting problems.

EXAMPLE 2: For example, suppose that the number of bacteria in a colony doubles every hour. If a colony begins with five bacteria, how many will be present in *n* hours?

Let a_n be the number of bacteria at the end of n hours. Because the number of bacteria doubles every hour, the relationship $a_n = 2a_{n-1}$ holds whenever n is a positive integer and the initial condition $a_0 = 5$, uniquely determines *an* for all nonnegative integers n.

We can find a formula for *an* using the iterative approach $a_n = 5 \cdot 2^n$ for all nonnegative integers n

※Modeling with Recurrence Relations

We can use recurrence relations to model (describe) a wide variety of problems.

Such as finding:

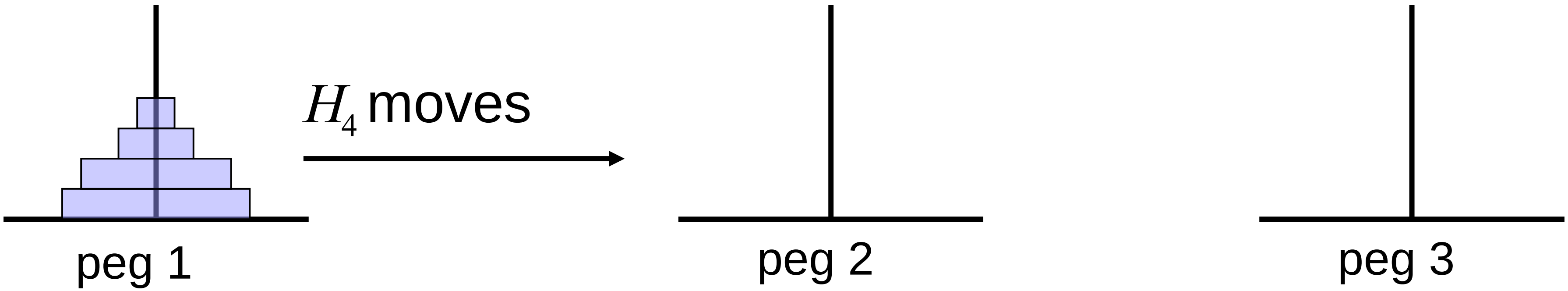
- ❑compound interest
- ❑counting rabbits on an island
- ❑determining the number of moves in the Tower of Hanoi puzzle and
- ❑counting bit strings with certain properties.

Example 3. (The Tower of Hanoi)

The rules of the puzzle allow disks to be moved one at a time from one peg to another as long as a disk is never placed on top of a smaller disk. Let H_n denote the number of moves needed to solve the Tower of Hanoi problem with n disks.

Set up a recurrence relation for the sequence $\{H_n\}$.

Target: n Generic classifier **disk** From **peg** 1 Move to **peg** 2



Sol : $H_n = 2H_{n-1} + 1$, $H_1 = 1$

($n-1$ Generic classifier disk From peg $1 \rightarrow$ peg 3, Subsection n Generic classifier disk From peg $1 \rightarrow$ peg 2,

$n-1$ Generic classifier disk From peg $3 \rightarrow$ peg 2)

In the previous example: $H_n = 2H_{n-1} + 1, \quad H_1 = 1$

Sol: $2^n - 1$

Example 4. (Bit Counting)

Find a recurrence relation and give initial conditions for the number of bit strings of length n that do not have two consecutive 0s. How many such bit strings are there of length 5 ?

Sol :

Let a_n be the number of bit strings of length n that do not have two consecutive 0s.

1	2	…	n-3	n-2	n-1	n

$a_{n\text{-}1}$ ways ______ 1

$a_{n\text{-}2}$ ways ______ 1 0

$$\therefore\ a_n = a_{n-1} + a_{n-2},\quad n \geq 3$$

$$a_1 = 2 \text{ (string : 0,1)}$$

$$a_2 = 3 \text{ (string : 01,10,11)}$$

$$\therefore\ a_3 = a_2 + a_1 = 5,\ a_4 = 8,\ a_5 = 13$$

Example 5. (Codeword enumeration)

A computer system considers a string of decimal digits a valid codeword if it contains an even number of 0 digits. Let a_n be the number of valid n-digit codewords. Find a recurrence relation for a_n.

Sol :

$\boxed{1}\ \boxed{2}\ \boxed{3}\ \cdots\ \boxed{n\text{-}1}\ \boxed{n}$

$\underline{a_{n-1} \text{ ways}}$ $1\sim 9$

$\underline{10^{n-1} - a_{n-1} \text{ ways}}$ 0

$$\begin{aligned} \therefore a_n &= 9a_{n-1} + (10^{n-1} - a_{n-1}) \\ &= 8a_{n-1} + 10^{n-1} \qquad , n \geq 2 \\ a_1 &= 9 \end{aligned}$$

Exercises

- **a)** Find a recurrence relation for the number of bit strings of length *n* that contain a pair of consecutive 0s.
- **b)** What are the initial conditions?
- **c)** How many bit strings of length seven contain two consecutive 0s?

- **a)** Find a recurrence relation for the number of bit strings of length *n* that do not contain three consecutive 0s.
- **b)** What are the initial conditions?
- **c)** How many bit strings of length seven do not contain three consecutive 0s?

- **a)** Find a recurrence relation for the number of ways to climb *n* stairs if the person climbing the stairs can take one stair or two stairs at a time.
- **b)** What are the initial conditions?
- **c)** In how many ways can this person climb a flight of eight stairs?

- **a)** Find a recurrence relation for the number of ways to climb *n* stairs if the person climbing the stairs can take one, two, or three stairs at a time.
- **b)** What are the initial conditions?
- **c)** In many ways can this person climb a flight of eight stairs?

- **a)** Find a recurrence relation for the number of ternary strings of length *n* that do not contain two consecutive 0s.
- **b)** What are the initial conditions?
- **c)** How many ternary strings of length six do not contain two consecutive 0s?

Solving Recurrence Relations

Def 1. A linear homogeneous recurrence relation of degree k (i.e., k terms) with constant coefficients is a recurrence relation of the form

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k}$$

where $c_i \in \mathbf{R}$ and $c_k \neq 0$

Example 1 and 2.

$f_n = f_{n-1} + f_{n-2}$	(True, deg=2)
$a_n = a_{n-5}$	(True, deg=5)
$a_n = a_{n-1} + a_{n-2}^2$	(False, not linear)
$a_n = n a_{n-1}$	(False , not linear)
$H_n = 2H_{n-1} + 1$	(False, not homogeneous)

Solving Linear Homogeneous Recurrence Relations with Constant Coefficients

Theorem 1.

Let $a_n = c_1 a_{n-1} + c_2 a_{n-2}$ be a recurrence relation

with $c_1, c_2 \in \mathbf{R}$.

If $r^2 - c_1 r - c_2 = 0$ (Known as the characteristic equation) has two distinct roots r_1 and r_2.

Then the solution of a_n is $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$,

for $n=0,1,2,\ldots,$ where α_1, α_2 are constants.

(α_1, α_2 Available a_0, a_1 Work out)

Example 3.

What's the solution of the recurrence relation

$$a_n = a_{n-1} + 2a_{n-2}$$

with $a_0=2$ and $a_1=7$?

Sol :

$$a_n = 3\times 2^n - (-1)^n.$$

Example 4. Find an explicit formula for the Fibonacci numbers.

Sol :

$$f_n = \frac{1}{\sqrt{5}} \times \left(\frac{1+\sqrt{5}}{2}\right)^n + \frac{-1}{\sqrt{5}} \left(\frac{1-\sqrt{5}}{2}\right)^n$$

Thm 2.

Let $a_n = c_1 a_{n-1} + c_2 a_{n-2}$ be a recurrence relation with $c_1, c_2 \in \mathbf{R}$.

If $r^2 - c_1 r - c_2 = 0$ has only one root r_0.

Then the solution of a_n is

$$a_n = \alpha_1 \cdot r_0^n + \alpha_2 \cdot n \cdot r_0^n$$

for $n=0,1,2,\ldots,$ where α_1 and α_2 are constants.

Example 5.

What's the solution of $a_n = 6a_{n-1} - 9a_{n-2}$ with $a_0=1$ and $a_1=6$?

Sol : $a_n = 3^n + n \times 3^n$

Thm 3.

Let $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k}$ be a recurrence relation with $c_1, c_2, \ldots, c_k \in \mathbf{R}$.

If $r^k - c_1 r^{k-1} - c_2 r^{k-2} - \ldots - c_k = 0$ has k distinct roots $r_1, r_2, \ldots, r_k$.

Then the solution of a_n is

$a_n = \alpha_1 r_1^{\,n} + \alpha_2 r_2^{\,n} + \ldots + \alpha_k r_k^{\,n}$, for $n = 0, 1, 2, \ldots$

where $\alpha_1, \alpha_2, \ldots \alpha_k$ are constants.

Example 6 (k = 3)

Find the solution of $a_n = 6a_{n-1} - 11a_{n-2} + 6a_{n-3}$ with initial conditions a_0=2, a_1=5 and a_2=15 .

Sol : $a_n = 1 - 2^n + 2 \times 3^n$

Thm 4.

Let $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k}$ be a recurrence relation with $c_1, c_2, \ldots, c_k \in \mathbf{R}$.

If $r^k - c_1 r^{k-1} - c_2 r^{k-2} - \ldots - c^k = 0$ has
t distinct roots $r_1, r_2, \ldots, r_t$
with multiplicities $m_1, m_2, \ldots, m_t$ respectively,
where $m_i \geq 1, \forall i$, and $m_1 + m_2 + \ldots + m_t = k$,

then

$$
\begin{aligned}
a_n = & \left(\alpha_{1,0} + \alpha_{1,1} \cdot n + \ldots + \alpha_{1,m_1-1} \cdot n^{m_1-1}\right) r_1^n \\
& + \left(\alpha_{2,0} + \alpha_{2,1} \cdot n + \ldots + \alpha_{2,m_2-1} \cdot n^{m_2-1}\right) \cdot r_2^n \\
& + \ldots + \left(\alpha_{t,0} + \alpha_{t,1} \cdot n + \ldots + \alpha_{t,m_t-1} \cdot n^{m_t-1}\right) \cdot r_t^n
\end{aligned}
$$

where $\alpha_{i,j}$ are constants for $1 \leq i \leq t$ and $0 \leq j \leq m_i - 1$.

Supplementary explanation ：

Feature program root for: 1 (2 Multiple root),
–2 (3 Multiple root),
3 (And no double lazing root)

The general solution of the above theorem is ：

$a_n = (\alpha_{1,1} + \alpha_{1,2} \cdot n) \cdot 1^n + (\alpha_{2,1} + \alpha_{2,2} \cdot n + \alpha_{2,3} \cdot n^2) \cdot (-2)^n + \alpha_{3,1} \cdot 3^n$

(Variable α Subscripts can be obtained from 1 Start lining up, as long as you don't repeat well)

$a_n = (\alpha_1 + \alpha_2 \cdot n) \cdot 1^n + (\alpha_3 + \alpha_4 \cdot n + \alpha_5 \cdot n^2) \cdot (-2)^n + \alpha_6 \cdot 3^n$

Example 7. Find the solution to the recurrence relation $a_n = -3a_{n-1} - 3a_{n-2} - a_{n-3}$ with initial conditions $a_0 = 1$, $a_1 = -2$ and $a_2 = -1$.

Sol : $a_n = (1+3n-2n^2) \times (-1)^n$

Linear Nonhomogeneous Recurrence Relations with Constant Coefficients

Example: $a_n = 3a_{n-1} + 2n$

A recurrence relation of the form

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k} + F(n),$$

where $c_1, c_2, \ldots, c_k$ are real numbers and $F(n)$ is a function not identically zero depending only on n.

The recurrence relation

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k}$$

is called the associated homogeneous recurrence relation.

Example 8:

$a_n = a_{n-1} + 2^n$, associated h.r.r $\Rightarrow a_n = a_{n-1}$

$a_n = a_{n-1} + a_{n-2} + n^2+1$, associated h.r.r $\Rightarrow a_n = a_{n-1} + a_{n-2}$

$a_n = 3a_{n-1} + n3^n$, associated h.r.r $\Rightarrow a_n = 3a_{n-1}$

$a_n = a_{n-1} + a_{n-3} + n!$, associated h.r.r $\Rightarrow a_n = a_{n-1} + a_{n-3}$

Theorem 5. If $\{a_n^{(p)}\}$ is a particular solution of $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k} + F(n)$,

then every solution is of the form $\{a_n^{(p)} + a_n^{(h)}\}$, where $\{a_n^{(h)}\}$ is a solution of

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k}$$

Proof. If $\{a_n^{(p)}\}$ and $\{b_n\}$ are both solutions of

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k} + F(n),$$

then $a_n^{(p)} = c_1 a_{n-1}^{(p)} + c_2 a_{n-2}^{(p)} + \ldots + c_k a_{n-k}^{(p)} + F(n)$,

and $b_n = c_1 b_{n-1} + c_2 b_{n-2} + \ldots + c_k b_{n-k} + F(n)$.

$$\Rightarrow a_n^{(p)} - b_n = c_1(a_{n-1} - b_{n-1}) + c_2(a_{n-2} - b_{n-2}) + \ldots + c_k(a_{n-k} - b_{n-k})$$

$\Rightarrow \{a_n^{(p)} - b_n\}$ is a solution of $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k}$

$\Rightarrow b_n = a_n^{(p)} + a_n^{(h)}$

Example 9. Find all solutions of the recurrence relation $a_n = 3a_{n-1} + 2n$. What is the solution with $a_1 = 3$?

Sol : $a_n = (11/6) \times 3^n - n - 3/2$

Example 10. Find all solutions of the recurrence relation $a_n = 5a_{n-1} - 6a_{n-2} + 7^n$.

Sol : $a_n = a_n^{(h)} + a_n^{(p)} = \alpha_1 \times 3^n + \alpha_2 \times 2^n + (49/20) \cdot 7^n$

Theorem 6.

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k} + F(n),$$

where $F(n) = (b_t n^t + b_{t-1} n^{t-1} + \ldots + b_1 n + b_0) s^n$.

When s is not a root of the characteristic equation of the associated linear homogeneous recurrence relation, there is a particular solution of the form

$$(p_t n^t + p_{t-1} n^{t-1} + \ldots + p_1 n + p_0) s^n.$$

When s is a root of the characteristic equation and its multiplicity is m, there is a particular solution of the form

$$n^m (p_t n^t + p_{t-1} n^{t-1} + \ldots + p_1 n + p_0) s^n.$$

Example 11. What form does a particular solution of the linear nonhomogeneous recurrence relation $a_n = 6a_{n-1} - 9a_{n-2} + F(n)$ have when $F(n) = 3^n$, $F(n) = n3^n$, $F(n) = n^2 2^n$, and $F(n) = (n^2+1)3^n$.

Sol :

The associated linear homogeneous recurrence relation is $a_n = 6a_{n-1} - 9a_{n-2}$.

characteristic equation: $r^2 - 6r + 9 = 0 \Rightarrow r = 3$ (Multiple root)

$F(n) = 3^n$, and 3 is a root $\Rightarrow a_n^{(p)} = p_0 n^2 3^n$

$F(n) = n3^n$, and 3 is a root $\Rightarrow a_n^{(p)} = n^2(p_1 n + p_0)\, 3^n$

$F(n) = n^2 2^n$, and 2 is not a root $\Rightarrow a_n^{(p)} = (p_2 n^2 + p_1 n + p_0) 2^n$

$F(n) = (n^2+1)3^n$, and 3 is a root

$\Rightarrow a_n^{(p)} = n^2\,(p_2 n^2 + p_1 n + p_0)\, 3^n$

Example 12. Find the solutions of the recurrence relation $a_n = a_{n-1} + n$ with $a_1=1$.

Sol :

$$a_n = a_n^{(p)} + a_n^{(h)} = (n^2+n)/2$$

Exercises

- Solve these recurrence relations together with the initial conditions given.

  a) $a_n = 2a_{n-1}$ for $n \geq 1$, $a_0 = 3$

  b) $a_n = a_{n-1}$ for $n \geq 1$, $a_0 = 2$

  c) $a_n = 5a_{n-1} - 6a_{n-2}$ for $n \geq 2$, $a_0 = 1$, $a_1 = 0$

  d) $a_n = 4a_{n-1} - 4a_{n-2}$ for $n \geq 2$, $a_0 = 6$, $a_1 = 8$

  e) $a_n = -4a_{n-1} - 4a_{n-2}$ for $n \geq 2$, $a_0 = 0$, $a_1 = 1$

  f) $a_n = 4a_{n-2}$ for $n \geq 2$, $a_0 = 0$, $a_1 = 4$

  g) $a_n = a_{n-2}/4$ for $n \geq 2$, $a_0 = 1$, $a_1 = 0$

- Find the solution to $a_n = 7a_{n-2} + 6a_{n-3}$ with $a_0 = 9$, $a_1 = 10$, and $a_2 = 32$.

- Find the solution to $a_n = 5a_{n-2} - 4a_{n-4}$ with $a_0 = 3$, $a_1 = 2$, $a_2 = 6$, and $a_3 = 8$.

- Find the solution to $a_n = 2a_{n-1} + 5a_{n-2} - 6a_{n-3}$ with $a_0 = 7$, $a_1 = -4$, and $a_2 = 8$.

- Solve the recurrence relation $a_n = -3a_{n-1} - 3a_{n-2} - a_{n-3}$ with $a_0 = 5$, $a_1 = -9$, and $a_2 = 15$.

Exercises Contd

- Consider the nonhomogeneous linear recurrence relation $a_n = 3a_{n-1} + 2^n$.

  a) Show that $a_n = -2^{n+1}$ is a solution of this recurrence relation.

  b) Use Theorem 5 to find all solutions of this recurrence relation.

  c) Find the solution with $a_0 = 1$.

- What is the general form of the particular solution guaranteed to exist by Theorem 6 of the linear nonhomogeneous recurrence relation

  $a_n = 8a_{n-2} - 16a_{n-4} + F(n)$ if

  a) $F(n) = n^3$? **b)** $F(n) = (-2)^n$?

  c) $F(n) = n2^n$? **d)** $F(n) = n^2 4^n$?

  e) $F(n) = (n^2 - 2)(-2)^n$? **f)** $F(n) = n^4 2^n$?

  g) $F(n) = 2$?

- **a)** Find all solutions of the recurrence relation $a_n = 2a_{n-1} + 3^n$.
- **b)** Find the solution of the recurrence relation in part (a) with initial condition $a_1 = 5$.

Generating Functions.

Def 1. The generating function for the sequence $a_0, a_1, a_2, \ldots$ of real numbers is the infinite series

$$G(x) = a_0 + a_1 x + \ldots + a_n x^n + \ldots$$

$$= \sum_{k=0}^{\infty} a_k x^k$$

(If the series $\{a_n\}$ Is a finite, can be seen as Infinite, but the latter is equal to 0)

Example 1. Find the generating functions for the sequences $\{a_k\}$ with

(1) $a_k = 3$

(2) $a_k = k+1$

(3) $a_k = 2^k$

Sol :

(1) $G(x) = \sum_{k=0}^{\infty} a_k x^k = \sum_{k=0}^{\infty} 3x^k$

(2) $G(x) = \sum_{k=0}^{\infty} a_k x^k = \sum_{k=0}^{\infty} (k+1)x^k$

(3) $G(x) = \sum_{k=0}^{\infty} a_k x^k = \sum_{k=0}^{\infty} 2^k x^k$

Example 2. What is the generating function for the sequence 1,1,1,1,1,1 ?

a_0 a_1 a_2 a_3 a_4 a_5 $a_6=0$

Sol :

$$G(x) = \sum_{k=0}^{\infty} a_k x^k = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots$$

$$= 1 + x + x^2 + \ldots + x^5 \qquad \text{(expansion)}$$

$$= \frac{x^6 - 1}{x - 1} \qquad \text{(closed form)}$$

Example 3.

Let $m \in \mathbf{Z}^+$ and $a_k = \binom{m}{k}$, for $k = 0, 1, \ldots, m$.

What is the generating function for the sequence $a_0, a_1, \ldots, a_m$?

Sol :

$$G(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_m x^m$$

$$= \binom{m}{0} + \binom{m}{1} x + \binom{m}{2} x^2 + \cdots + \binom{m}{m} x^m$$

$$= (1+x)^m$$

$$(x+y)^n = \binom{n}{0} x^n + \binom{n}{1} x^{n-1} y + \ldots + \binom{n}{n-1} x y^{n-1} + \binom{n}{n} y^n = \sum_{j=0}^{n} \binom{n}{j} x^{n-j} y^j$$

Useful Facts About Power Series

Example 4.

The function $f(x) = \dfrac{1}{1-x}$ is the generating function of the sequence $1, 1, 1, \ldots,$ because

$$\sum_{k=0}^{\infty} x^k = 1 + x + x^2 + \ldots = \frac{1}{1-x} \text{ when } |x| < 1.$$

Example 5.

The function $f(x) = \dfrac{1}{1-ax}$ is the generating function of the sequence $1, a, a^2, \ldots,$ because

$$\sum_{k=0}^{\infty} (ax)^k = 1 + ax + a^2x^2 + \ldots = \frac{1}{1-ax} \text{ when } |ax| < 1 \text{ for } a \neq 0.$$

Theorem 1.

Let $f(x) = \sum_{k=0}^{\infty} a_k x^k$ and $g(x) = \sum_{k=0}^{\infty} b_k x^k$.

Then $f(x) + g(x) = \sum_{k=0}^{\infty} (a_k + b_k) x^k$.

$$f(x)\, g(x) = (a_0 + a_1 x + a_2 x^2 + \ldots)(b_0 + b_1 x + b_2 x^2 + \ldots)$$

$$= (a_0 b_0) + (a_0 b_1 + a_1 b_0)\, x + (a_0 b_2 + a_1 b_1 + a_2 b_0)\, x^2 + \ldots$$

$$= \sum_{k=0}^{\infty} \left(\sum_{j=0}^{k} a_j b_{k-j} \right) x^k$$

Example 6.

Let $f(x) =$ $f(x) = \frac{1}{(1-x)^2}$. Use Example 4 to find the coefficients $a_0, a_1, a_2, \ldots$ in the expansion $f(x) = \sum_{k=0}^{\infty} a_k x^k$.

Sol : $a_k = k+1$

Def 2.

Let $u \in \mathbf{R}$ and $k \in \mathbf{N}$. Then the extended binomial coefficient $\binom{u}{k}$ is defined by

$$\binom{u}{k} = \begin{cases} u(u-1)(u-2)\ldots(u-k+1)/k!, & \text{if } k > 0 \\ 1 & \text{if } k = 0 \end{cases}$$

Example 7. Find $\binom{-2}{3}$ and $\binom{1/2}{3}$

Sol : $\dfrac{1}{16}$

Example 8

When the top parameter is a negative integer, the extended binomial coefficient can be expressed in terms of an ordinary binomial coefficient.

$$\binom{-n}{r} = \frac{(-n)(-n-1)\ldots(-n-r+1)}{r!}$$

$$= \frac{(-1)^r (n)(n+1)\ldots(n+r-1)}{r!}$$

$$= \frac{(-1)^r (n+r-1)!}{r!(n-1)!}$$

$$= (-1)^r \binom{n+r-1}{r}$$

Thm 2. (The Extended Binomial Theorem)

Let $x \in \mathbf{R}$ with $|x|<1$ and let $u \in \mathbf{R}$, then

$$(1+x)^u = \sum_{k=0}^{\infty} \binom{u}{k} x^k$$

Example 9.

Find the generating functions for $(1+x)^{-n}$ and $(1-x)^{-n}$ where $n\in \mathbf{Z}^{+}$

Sol : By the Extended Binomial Theorem.

$$(1+x)^{-n}=\sum_{k=0}^{\infty}\binom{-n}{k}x^{k}=\sum_{k=0}^{\infty}\frac{1}{k!}(-n)(-n-1)\ldots(-n-k+1)x^{k}$$

$$=\sum_{k=0}^{\infty}\frac{(-1)^{k}}{k!}(n)(n+1)\ldots(n+k-1)x^{k}$$

$$=\sum_{k=0}^{\infty}(-1)^{k}\binom{n+k-1}{k}x^{k}$$

$$\boxed{\text{Note}:\binom{-n}{k}=(-1)^{k}\binom{n+k-1}{k}}$$

By replacing x by $-x$ we have

$$(1-x)^{-n}=\sum_{k=0}^{\infty}\binom{n+k-1}{k}x^{k}$$

Counting Problems and Generating Functions

Generating functions can be used to count the number of combinations of various types.

Example 10.

Find the number of solutions of $e_1 + e_2 + e_3 = 17$, where e_1, e_2, e_3 are integers with $2 \le e_1 \le 5$, $3 \le e_2 \le 6$, and $4 \le e_3 \le 7$.

Sol : $(4, 6, 7), (5, 5, 7), (5, 6, 6)$ A total of 3

Example 11.

In how many different ways can eight identical cookies be distributed among three distinct children if each child receives at least two cookies and no more than four cookies?

Sol : The number of solutions is the coefficient of x^8 in the expansion of

$$(x^2 + x^3 + x^4)^3$$

$$\therefore (c_1, c_2, c_3) = (2, 2, 4), (2, 3, 3), (2, 4, 2), (3, 2, 3), (3, 3, 2), (4, 2, 2)$$

※Using Generating Functions to solve Recurrence Relations.

Example 12.

Solving the recurrence relation $a_k = 3a_{k-1}$ for $k=1,2,3,\ldots$ and initial condition $a_0 = 2$.

Sol :

Another method ： (by 7.2 From Thm 1)

$$r - 3 = 0 \Rightarrow r = 3 \Rightarrow a_n = \alpha \cdot 3^n$$

$$\because a_0 = 2 = \alpha$$

$$\therefore a_n = 2 \cdot 3^n$$

Alternatively, use generating functions

Example 13.

Solving the recurrence relation

$a_n = 5a_{n-1}$ - $4a_{n-2}$ for $n \geq 0$, and initial condition $a_0 = 1$, $a_1 = 2$.

Sol : $$a_n = \frac{4^n + 2}{3}$$

Example 14

Solving $a_k = 8a_{k-1} + 10^{k-1}$ for $k = 1, 2, 3, \ldots$ and initial condition $a_1 = 9$.

Sol : $a_k = (10^k + 8^k)/2$

Exercises

- Find the generating function for the finite sequence 1, 4, 16, 64, 256.

- Find a closed form for the generating function for the sequence $\{a_n\}$, where

  a) $a_n = 5$ for all $n = 0, 1, 2, \ldots.$

  b) $a_n = 3^n$ for all $n = 0, 1, 2, \ldots.$

- Find the coefficient of x^{10} in the power series of each of these functions.

  a) $1/(1 - 2x)$ **b)** $1/(1 + x)^2$

  c) $1/(1 - x)^3$ **d)** $1/(1 + 2x)^4$ **e)** $x^4/(1 - 3x)^3$

- **a)** What is the generating function for $\{a_k\}$, where a_k is the number of solutions of $x_1 + x_2 + x_3 = k$ when x_1, x_2, and x_3 are integers with $x_1 \geq 2$, $0 \leq x_2 \leq 3$, and $2 \leq x_3 \leq 5$?

  b) Use your answer to part (a) to find a_6.

Exercises Contd

- Find a closed form for the generating function for the sequence $\{a_n\}$, where
- **a)** $a_n = 5$ for all $n = 0, 1, 2, \ldots$.
- **b)** $a_n = 3^n$ for all $n = 0, 1, 2, \ldots$.
- **c)** $a_n = 2$ for $n = 3, 4, 5, \ldots$ and $a_0 = a_1 = a_2 = 0$.
- **d)** $a_n = 2n + 3$ for all $n = 0, 1, 2, \ldots$.

- For each of these generating functions, provide a closed formula for the sequence it determines.

  a) $(3x - 4)3$ **b)** $(x^3 + 1)^3$

  c) $1/(1 - 5x)$ **d)** $x^3/(1 + 3x)$

  e) $x^2 + 3x + 7 + (1/(1 - x^2))$ **f)** $(x^4/(1 - x^4)) - x^3 - x^2 - x - 1$
- **g)** $x^2/(1 - x)^2$ **h)** $2e^{2x}$
- Use generating functions to solve the recurrence relation $a_k = 3a_{k-1} + 2$ with the initial condition $a_0 = 1$.