Chapter 5

VECTOR INTEGRATION

ORDINARY INTEGRALS OF VECTORS. Let $\mathbf{R}(u) = R_1(u)\mathbf{i} + R_2(u)\mathbf{j} + R_3(u)\mathbf{k}$ be a vector depending on a single scalar variable u, where $R_1(u)$, $R_2(u)$, $R_3(u)$ are supposed continuous in a specified interval. Then

$$\int \mathbf{R}(u)\,du = \mathbf{i}\int R_1(u)\,du + \mathbf{j}\int R_2(u)\,du + \mathbf{k}\int R_3(u)\,du$$

is called an *indefinite integral* of $\mathbf{R}(u)$. If there exists a vector $\mathbf{S}(u)$ such that $\mathbf{R}(u) = \frac{d}{du}(\mathbf{S}(u))$, then

$$\int \mathbf{R}(u)\,du = \int \frac{d}{du}(\mathbf{S}(u))\,du = \mathbf{S}(u) + \mathbf{c}$$

where $\mathbf{c}$ is an *arbitrary constant vector* independent of u. The *definite integral* between limits $u=a$ and $u=b$ can in such case be written

$$\int_a^b \mathbf{R}(u)\,du = \int_a^b \frac{d}{du}(\mathbf{S}(u))\,du = \mathbf{S}(u) + \mathbf{c}\Big|_a^b = \mathbf{S}(b) - \mathbf{S}(a)$$

This integral can also be defined as a limit of a sum in a manner analogous to that of elementary integral calculus.

LINE INTEGRALS. Let $\mathbf{r}(u) = x(u)\mathbf{i} + y(u)\mathbf{j} + z(u)\mathbf{k}$, where $\mathbf{r}(u)$ is the position vector of (x,y,z), define a curve C joining points P_1 and P_2, where $u=u_1$ and $u=u_2$ respectively.

We assume that C is composed of a finite number of curves for each of which $\mathbf{r}(u)$ has a continuous derivative. Let $\mathbf{A}(x,y,z) = A_1\mathbf{i} + A_2\mathbf{j} + A_3\mathbf{k}$ be a vector function of position defined and continuous along C. Then the integral of the tangential component of $\mathbf{A}$ along C from P_1 to P_2, written as

$$\int_{P_1}^{P_2} \mathbf{A}\cdot d\mathbf{r} = \int_C \mathbf{A}\cdot d\mathbf{r} = \int_C A_1\,dx + A_2\,dy + A_3\,dz$$

is an example of a *line integral*. If $\mathbf{A}$ is the force $\mathbf{F}$ on a particle moving along C, this line integral represents the work done by the force. If C is a closed curve (which we shall suppose is a *simple closed curve*, i.e. a curve which does not intersect itself anywhere) the integral around C is often denoted by

$$\oint \mathbf{A}\cdot d\mathbf{r} = \oint A_1\,dx + A_2\,dy + A_3\,dz$$

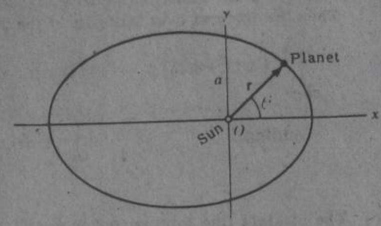
In aerodynamics and fluid mechanics this integral is called the *circulation* of $\mathbf{A}$ about C, where $\mathbf{A}$ represents the velocity of a fluid.

In general, any integral which is to be evaluated along a curve is called a line integral. Such integrals can be defined in terms of limits of sums as are the integrals of elementary calculus.

For methods of evaluation of line integrals, see the Solved Problems.

The following theorem is important.

From analytical geometry, the polar equation of a conic section with focus at the origin and eccentricity ϵ is $r = \dfrac{a}{1+\epsilon\cos\theta}$ where a is a constant. Comparing this with the equation derived, it is seen that the required orbit is a conic section with eccentricity $\epsilon = p/GM$. The orbit is an ellipse, parabola or hyperbola according as ϵ is less than, equal to or greater than one. Since orbits of planets are closed curves it follows that they must be ellipses.



Ellipse $r = \dfrac{a}{1+\epsilon\cos\theta}$

LINE INTEGRALS

6. If $\mathbf{A} = (3x^2+6y)\mathbf{i} - 14yz\,\mathbf{j} + 20xz^2\,\mathbf{k}$, evaluate $\int_C \mathbf{A}\cdot d\mathbf{r}$ from $(0,0,0)$ to $(1,1,1)$ along the following paths C:

(a) $x = t,\ y = t^2,\ z = t^3$.

(b) the straight lines from $(0,0,0)$ to $(1,0,0)$, then to $(1,1,0)$, and then to $(1,1,1)$.

(c) the straight line joining $(0,0,0)$ and $(1,1,1)$.

$$\int_C \mathbf{A}\cdot d\mathbf{r} = \int_C \left[(3x^2+6y)\mathbf{i} - 14yz\,\mathbf{j} + 20xz^2\mathbf{k}\right]\cdot(dx\,\mathbf{i} + dy\,\mathbf{j} + dz\,\mathbf{k})$$
$$= \int_C (3x^2+6y)\,dx - 14yz\,dy + 20xz^2\,dz$$

(a) If $x=t,\ y=t^2,\ z=t^3$, points $(0,0,0)$ and $(1,1,1)$ correspond to $t=0$ and $t=1$ respectively. Then

$$\int_C \mathbf{A}\cdot d\mathbf{r} = \int_{t=0}^{1} (3t^2+6t^2)\,dt - 14(t^2)(t^3)\,d(t^2) + 20(t)(t^3)^2\,d(t^3)$$
$$= \int_{t=0}^{1} 9t^2\,dt - 28t^6\,dt + 60t^9\,dt$$
$$= \int_{t=0}^{1} (9t^2 - 28t^6 + 60t^9)\,dt = 3t^3 - 4t^7 + 6t^{10}\Big|_0^1 = 5$$

Another Method.

Along C, $\mathbf{A} = 9t^2\mathbf{i} - 14t^5\mathbf{j} + 20t^7\mathbf{k}$ and $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} = t\mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k}$ and $d\mathbf{r} = (\mathbf{i} + 2t\mathbf{j} + 3t^2\mathbf{k})\,dt$.

Then $$\int_C \mathbf{A}\cdot d\mathbf{r} = \int_{t=0}^{1} (9t^2\,\mathbf{i} - 14t^5\,\mathbf{j} + 20t^7\,\mathbf{k})\cdot(\mathbf{i} + 2t\,\mathbf{j} + 3t^2\,\mathbf{k})\,dt$$
$$= \int_0^1 (9t^2 - 28t^6 + 60t^9)\,dt = 5$$

(b) Along the straight line from $(0,0,0)$ to $(1,0,0)$ $y=0,\ z=0,\ dy=0,\ dz=0$ while x varies from 0 to 1. Then the integral over this part of the path is

$$\int_{x=0}^{1} \left(3x^2+6(0)\right)dx - 14(0)(0)(0) + 20x(0)^2(0) = \int_{x=0}^{1} 3x^2\,dx = x^3\Big|_0^1 = 1$$

Along the straight line from $(1,0,0)$ to $(1,1,0)$ $x=1,\ z=0,\ dx=0,\ dz=0$ while y varies from 0 to 1. Then the integral over this part of the path is

$$\int_{y=0}^{1} (3(1)^2+6y)\,0 - 14y(0)\,dy + 20(1)(0)^2\,0 = 0$$

Along the straight line from (1,1,0) to (1,1,1) $x=1,\ y=1,\ dx=0,\ dy=0$ while z varies from 0 to 1. Then the integral over this part of the path is

$$\int_{z=0}^{1} (3(1)^2+6(1))\,0 - 14(1)\,z(0) + 20(1)\,z^2.dz = \int_{z=0}^{1} 20z^2\,dz = \frac{20z^3}{3}\Big|_0^1 = \frac{20}{3}$$

Adding, $$\int_C \mathbf{A}\cdot d\mathbf{r} = 1 + 0 + \frac{20}{3} = \frac{23}{3}$$

(c) The straight line joining (0,0,0) and (1,1,1) is given in parametric form by $x=t,\ y=t,\ z=t$. Then

$$\int_C \mathbf{A}\cdot d\mathbf{r} = \int_{t=0}^{1} (3t^2+6t)\,dt - 14(t)(t)\,dt + 20(t)(t)^2\,dt$$

$$= \int_{t=0}^{1} (3t^2+6t-14t^2+20t^3)\,dt = \int_{t=0}^{1} (6t-11t^2+20t^3)\,dt = \frac{13}{3}$$

7. Find the total work done in moving a particle in a force field given by $\mathbf{F} = 3xy\,\mathbf{i} - 5z\,\mathbf{j} + 10x\,\mathbf{k}$ along the curve $x = t^2+1,\ y = 2t^2,\ z = t^3$ from $t=1$ to $t=2$.

$$\text{Total work} = \int_C \mathbf{F}\cdot d\mathbf{r} = \int_C (3xy\,\mathbf{i} - 5z\,\mathbf{j} + 10x\,\mathbf{k})\cdot(dx\,\mathbf{i} + dy\,\mathbf{j} + dz\,\mathbf{k})$$

$$= \int_C 3xy\,dx - 5z\,dy + 10x\,dz$$

$$= \int_{t=1}^{2} 3(t^2+1)(2t^2)\,d(t^2+1) - 5(t^3)\,d(2t^2) + 10(t^2+1)\,d(t^3)$$

$$= \int_1^2 (12t^5 + 10t^4 + 12t^3 + 30t^2)\,dt = 303$$

8. If $\mathbf{F} = 3xy\,\mathbf{i} - y^2\,\mathbf{j}$, evaluate $\int_C \mathbf{F}\cdot d\mathbf{r}$ where C is the curve in the xy plane, $y = 2x^2$, from (0,0) to (1,2).

Since the integration is performed in the xy plane ($z=0$), we can take $\mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j}$. Then

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_C (3xy\,\mathbf{i} - y^2\,\mathbf{j})\cdot(dx\,\mathbf{i} + dy\,\mathbf{j})$$

$$= \int_C 3xy\,dx - y^2\,dy$$

First Method. Let $x=t$ in $y=2x^2$. Then the parametric equations of C are $x=t,\ y=2t^2$. Points (0,0) and (1,2) correspond to $t=0$ and $t=1$ respectively. Then

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_{t=0}^{1} 3(t)(2t^2)\,dt - (2t^2)^2\,d(2t^2) = \int_{t=0}^{1} (6t^3-16t^5)\,dt = -\frac{7}{6}$$

Second Method. Substitute $y = 2x^2$ directly, where x goes from 0 to 1. Then

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_{x=0}^{1} 3x(2x^2)\,dx - (2x^2)^2\,d(2x^2) = \int_{x=0}^{1} (6x^3-16x^5)\,dx = -\frac{7}{6}$$

Note that if the curve were traversed in the opposite sense, i.e. from (1,2) to (0,0), the value of the integral would have been 7/6 inst[illegible] 7/6.

9. Find the work done in moving a particle once around a circle C in the xy plane, if the circle has centre at the origin and radius 3 and if the force field is given by

$$\mathbf{F} = (2x - y + z)\mathbf{i} + (x + y - z^2)\mathbf{j} + (3x - 2y + 4z)\mathbf{k}$$

In the plane $z=0$, $\mathbf{F} = (2x-y)\mathbf{i} + (x+y)\mathbf{j} + (3x-2y)\mathbf{k}$ and $d\mathbf{r} = dx\,\mathbf{i} + dy\,\mathbf{j}$ so that the work done is

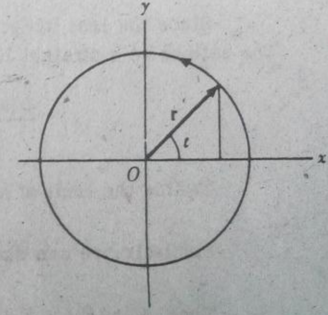
$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_C [(2x-y)\mathbf{i} + (x+y)\mathbf{j} + (3x-2y)\mathbf{k}]\cdot[dx\,\mathbf{i} + dy\,\mathbf{j}]$$

$$= \int_C (2x-y)\,dx + (x+y)\,dy$$

Choose the parametric equations of the circle as $x = 3\cos t$, $y = 3\sin t$ where t varies from 0 to 2π (see adjoining figure). Then the line integral equals

$$\int_{t=0}^{2\pi} [2(3\cos t) - 3\sin t][-3\sin t]\,dt + [3\cos t + 3\sin t][3\cos t]\,dt$$

$$= \int_0^{2\pi} (9 - 9\sin t\cos t)\,dt = 9t - \frac{9}{2}\sin^2 t\Big|_0^{2\pi} = 18\pi$$



$\mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j}$
$= 3\cos t\,\mathbf{i} + 3\sin t\,\mathbf{j}$

In traversing C we have chosen the counterclockwise direction indicated in the adjoining figure. We call this the *positive* direction, or say that C has been traversed in the positive sense. If C were traversed in the clockwise (negative) direction the value of the integral would be -18π.

10. (*a*) If $\mathbf{F} = \nabla\phi$, where ϕ is single-valued and has continuous partial derivatives, show that the work done in moving a particle from one point $P_1 \equiv (x_1, y_1, z_1)$ in this field to another point $P_2 \equiv (x_2, y_2, z_2)$ is independent of the path joining the two points.

(*b*) Conversely, if $\int_C \mathbf{F}\cdot d\mathbf{r}$ is independent of the path C joining any two points, show that there exists a function ϕ such that $\mathbf{F} = \nabla\phi$.

(*a*) Work done $= \displaystyle\int_{P_1}^{P_2} \mathbf{F}\cdot d\mathbf{r} = \int_{P_1}^{P_2} \nabla\phi\cdot d\mathbf{r}$

$$= \int_{P_1}^{P_2} \left(\frac{\partial\phi}{\partial x}\mathbf{i} + \frac{\partial\phi}{\partial y}\mathbf{j} + \frac{\partial\phi}{\partial z}\mathbf{k}\right)\cdot(dx\,\mathbf{i} + dy\,\mathbf{j} + dz\,\mathbf{k})$$

$$= \int_{P_1}^{P_2} \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz$$

$$= \int_{P_1}^{P_2} d\phi = \phi(P_2) - \phi(P_1) = \phi(x_2, y_2, z_2) - \phi(x_1, y_1, z_1)$$

Then the integral depends only on points P_1 and P_2 and not on the path joining them. This is true of course only if $\phi(x,y,z)$ is single-valued at all points P_1 and P_2.

23. If $\mathbf{F} = 4xz\,\mathbf{i} - y^2\,\mathbf{j} + yz\,\mathbf{k}$, evaluate $\iint_S \mathbf{F}\cdot\mathbf{n}\, dS$ where S is the surface of the cube bounded by $x=0$, $x=1$, $y=0$, $y=1$, $z=0$, $z=1$.

Face DEFG: $\mathbf{n} = \mathbf{i}$, $x=1$. Then

$$\iint_{DEFG} \mathbf{F}\cdot\mathbf{n}\, dS = \int_0^1\int_0^1 (4z\,\mathbf{i} - y^2\,\mathbf{j} + yz\,\mathbf{k})\cdot\mathbf{i}\, dy\, dz$$

$$= \int_0^1\int_0^1 4z\, dy\, dz = 2$$

Face ABCO: $\mathbf{n} = -\mathbf{i}$, $x = 0$. Then

$$\iint_{ABCO} \mathbf{F}\cdot\mathbf{n}\,dS = \int_0^1\int_0^1 (-y^2\,\mathbf{j} + yz\,\mathbf{k})\cdot(-\mathbf{i})\,dy\,dz = 0$$

Face ABEF: $\mathbf{n} = \mathbf{j}$, $y = 1$. Then

$$\iint_{ABEF} \mathbf{F}\cdot\mathbf{n}\,dS = \int_0^1\int_0^1 (4xz\,\mathbf{i} - \mathbf{j} + z\,\mathbf{k})\cdot\mathbf{j}\,dx\,dz = \int_0^1\int_0^1 -dx\,dz = -1$$

Face OGDC: $\mathbf{n} = -\mathbf{j}$, $y = 0$. Then

$$\iint_{OGDC} \mathbf{F}\cdot\mathbf{n}\,dS = \int_0^1\int_0^1 (4xz\,\mathbf{i})\cdot(-\mathbf{j})\,dx\,dz = 0$$

Face BCDE: $\mathbf{n} = \mathbf{k}$, $z = 1$. Then

$$\iint_{BCDE} \mathbf{F}\cdot\mathbf{n}\,dS = \int_0^1\int_0^1 (4x\,\mathbf{i} - y^2\,\mathbf{j} + y\,\mathbf{k})\cdot\mathbf{k}\,dx\,dy = \int_0^1\int_0^1 y\,dx\,dy = \tfrac{1}{2}$$

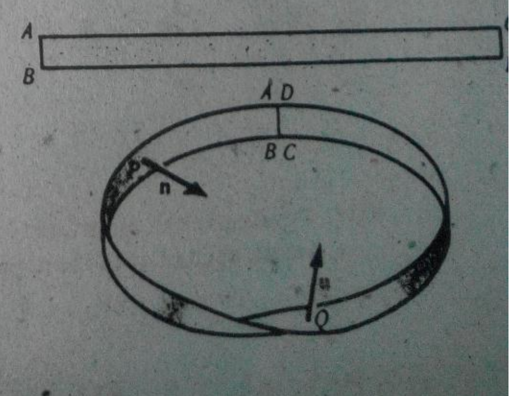
Face AFGO: $\mathbf{n} = -\mathbf{k}$, $z = 0$. Then

$$\iint_{AFGO} \mathbf{F}\cdot\mathbf{n}\,dS = \int_0^1\int_0^1 (-y^2\,\mathbf{j})\cdot(-\mathbf{k})\,dx\,dy = 0$$

Adding, $$\iint_S \mathbf{F}\cdot\mathbf{n}\,dS = 2 + 0 + (-1) + 0 + \tfrac{1}{2} + 0 = \tfrac{3}{2}.$$

24. In dealing with surface integrals we have restricted ourselves to surfaces which are two-sided. Give an example of a surface which is not two-sided.

Take a strip of paper such as $ABCD$ as shown in the adjoining figure. Twist the strip so that points A and B fall on D and C respectively, as in the adjoining figure. If $\mathbf{n}$ is the positive normal at point P of the surface, we find that as $\mathbf{n}$ moves around the surface it reverses its original direction when it reaches P again. If we tried to colour only one side of the surface we would find the whole thing coloured. This surface, called a *Moebius strip*, is an example of a one-sided surface. This is sometimes called a *non-orientable* surface. A two-sided surface is *orientable*.



41. Evaluate $\oint_C \mathbf{F}\cdot d\mathbf{r}$ where $\mathbf{F} = (x-3y)\mathbf{i} + (y-2x)\mathbf{j}$ and C is the closed curve in the xy plane, $x = 2\cos t$, $y = 3\sin t$ from $t=0$ to $t=2\pi$. *Ans.* 6π, if C is traversed in the positive (counterclockwise) direction.

42. If $\mathbf{T}$ is a unit tangent vector to the curve C, $\mathbf{r}=\mathbf{r}(u)$, show that the work done in moving a particle in a force field $\mathbf{F}$ along C is given by $\int_C \mathbf{F}\cdot\mathbf{T}\,ds$ where s is the arc length.

3. If $\mathbf{F} = (2x+y^2)\mathbf{i} + (3y-4x)\mathbf{j}$, evaluate $\oint_C \mathbf{F}\cdot d\mathbf{r}$ around the triangle C of Figure 1, (*a*) in the indicated direction, (*b*) opposite to the indicated direction. *Ans.* (*a*) $-14/3$ (*b*) $14/3$

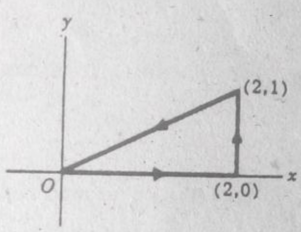


Fig. 1

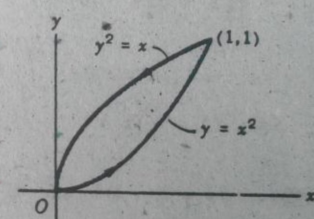


Fig. 2

Evaluate $\oint_C \mathbf{A}\cdot d\mathbf{r}$ around the closed curve C of Fig. 2 above if $\mathbf{A} = (x-y)\mathbf{i} + (x+y)\mathbf{j}$. *Ans.* 2/3

If $\mathbf{A} = (y-2x)\mathbf{i} + (3x+2y)\mathbf{j}$, compute the circulation of $\mathbf{A}$ about a circle C in the xy plane with centre at the origin and radius 2, if C is traversed in the positive direction. *Ans.* 8π

(*a*) If $\mathbf{A} = (4xy-3x^2z^2)\mathbf{i} + 2x^2\mathbf{j} - 2x^3z\,\mathbf{k}$, prove that $\int_C \mathbf{A}\cdot d\mathbf{r}$ is independent of the curve C joining wo given points. (*b*) Show that there is a differentiable function ϕ such that $\mathbf{A} = \nabla\phi$ and find it.
ins. (*b*) $\phi = 2x^2y - x^3z^2 + \text{constant}$

) Prove that $\mathbf{F} = (y^2\cos x + z^3)\mathbf{i} + (2y\sin x - 4)\mathbf{j} + (3xz^2+2)\mathbf{k}$ is a conservative force field.
) Find the scalar potential for $\mathbf{F}$.
) Find the work done in moving an object in this field from $(0,1,-1)$ to $(\pi/2,-1,2)$.
ns. (*b*) $\phi = y^2\sin x + xz^3 - 4y + 2z + \text{constant}$ (*c*) $15 + 4\pi$

ove that $\mathbf{F} = r^2\mathbf{r}$ is conservative and find the scalar potential. *Ans.* $\phi = \frac{r^4}{4} + \text{constant}$

etermine whether the force field $\mathbf{F} = 2xz\,\mathbf{i} + (x^2-y)\mathbf{j} + (2z-x^2)\mathbf{k}$ is conservative or non-conservative.
s. non-conservative

ow that the work done on a particle in moving it from A to B equals its change in kinetic energies at se points whether the force field is conservative or not.

Chapter 6 The Divergence Theorem, Stokes' Theorem, and Related Integral Theorems

THE DIVERGENCE THEOREM OF GAUSS states that if V is the volume bounded by a closed surface S and $\mathbf{A}$ is a vector function of position with continuous derivatives, then

$$\iiint_V \nabla\cdot\mathbf{A}\,dV = \iint_S \mathbf{A}\cdot\mathbf{n}\,dS = \oiint_S \mathbf{A}\cdot d\mathbf{S}$$

where $\mathbf{n}$ is the positive (outward drawn) normal to S.

STOKES' THEOREM states that if S is an open, two-sided surface bounded by a closed, non-intersecting curve C (simple closed curve) then if $\mathbf{A}$ has continuous derivatives

$$\oint_C \mathbf{A}\cdot d\mathbf{r} = \iint_S (\nabla\times\mathbf{A})\cdot\mathbf{n}\,dS = \iint_S (\nabla\times\mathbf{A})\cdot d\mathbf{S}$$

where C is traversed in the positive direction. The direction of C is called *positive* if an observer, walking on the boundary of S in this direction, with his head pointing in the direction of the positive normal to S, has the surface on his left.

GREEN'S THEOREM IN THE PLANE. If R is a closed region of the xy plane bounded by a simple closed curve C and if M and N are continuous functions of x and y having continuous derivatives in R, then

$$\oint_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

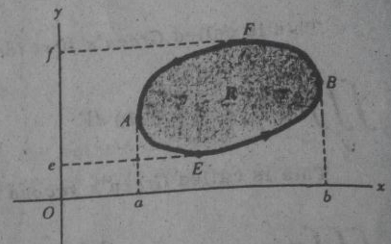
where C is traversed in the positive (counterclockwise) direction. Unless otherwise stated we shall always assume $\oint$ to mean that the integral is described in the positive sense.

Green's theorem in the plane is a special case of Stokes' theorem (see Problem 4). Also, it is of interest to notice that Gauss' divergence theorem is a generalization of Green's theorem in the plane where the (plane) region R and its closed boundary (curve) C are replaced by a (space) region V and its closed boundary (surface) S. For this reason the divergence theorem is often called *Green's theorem in space* (see Problem 4).

Green's theorem in the plane also holds for regions bounded by a finite number of simple closed curves which do not intersect (see Problems 10 and 11).

GREEN'S THEOREM IN THE PLANE

1. Prove Green's theorem in the plane if C is a closed curve which has the property that any straight line parallel to the coordinate axes cuts C in at most two points.



Let the equations of the curves AEB and AFB (see adjoining figure) be $y = Y_1(x)$ and $y = Y_2(x)$ respectively. If R is the region bounded by C, we have

$$\iint_R \frac{\partial M}{\partial y}\,dx\,dy = \int_{x=a}^{b}\left[\int_{y=Y_1(x)}^{Y_2(x)} \frac{\partial M}{\partial y}\,dy\right]dx = \int_{x=a}^{b} M(x,y)\Big|_{y=Y_1(x)}^{Y_2(x)}\,dx = \int_a^b \Big[M(x,Y_2) - M(x,Y_1)\Big]dx$$

$$= -\int_a^b M(x,Y_1)\,dx - \int_b^a M(x,Y_2)\,dx = -\oint_C M\,dx$$

Then

$$(1) \qquad \oint_C M\,dx = -\iint_R \frac{\partial M}{\partial y}\,dx\,dy$$

Similarly let the equations of curves EAF and EBF be $x = X_1(y)$ and $x = X_2(y)$ respectively. Then

$$\iint_R \frac{\partial N}{\partial x}\,dx\,dy = \int_{y=e}^{f}\left[\int_{x=X_1(y)}^{X_2(y)} \frac{\partial N}{\partial x}\,dx\right]dy = \int_e^f \Big[N(X_2,y) - N(X_1,y)\Big]dy$$

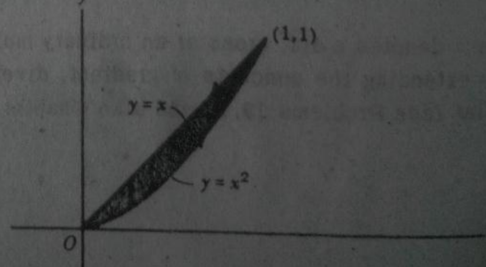
$$= \int_f^e N(X_1,y)\,dy + \int_e^f N(X_2,y)\,dy = \oint_C N\,dy$$

Then

$$(2) \qquad \oint_C N\,dy = \iint_R \frac{\partial N}{\partial x}\,dx\,dy$$

Adding (1) and (2), $\displaystyle\oint_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)dx\,dy.$

2. Verify Green's theorem in the plane for $\displaystyle\oint_C (xy + y^2)\,dx + x^2\,dy$ where C is the closed curve of the region bounded by $y = x$ and $y = x^2$.



$y = x$ and $y = x^2$ intersect at (0,0) and (1,1). The positive direction in traversing C is as shown in the adjacent diagram.

Along $y = x^2$, the line integral equals

$$\int_0^1 \left((x)(x^2) + x^4\right) dx \quad + \quad (x^2)(2x)\, dx \quad = \quad \int_0^1 (3x^3 + x^4)\, dx \quad = \quad \frac{19}{20}$$

Along $y = x$ from $(1,1)$ to $(0,0)$ the line integral equals

$$\int_1^0 \left((x)(x) + x^2\right) dx \quad + \quad x^2\, dx \quad = \quad \int_1^0 3x^2\, dx \quad = \quad -1$$

Then the required line integral $= \frac{19}{20} - 1 = -\frac{1}{20}$.

$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\, dy \quad = \quad \iint_R \left[\frac{\partial}{\partial x}(x^2) - \frac{\partial}{\partial y}(xy + y^2)\right] dx\, dy$$

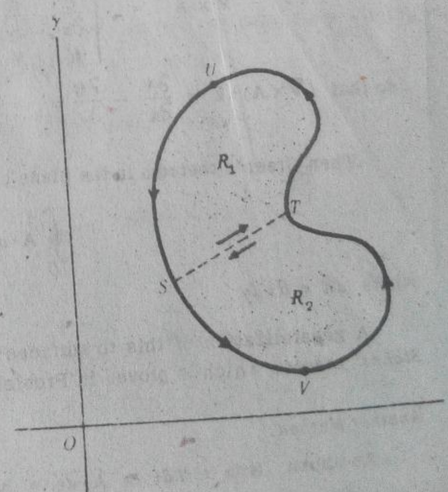
$$= \iint_R (x - 2y)\, dx\, dy \quad = \quad \int_{x=0}^1 \int_{y=x^2}^x (x - 2y)\, dy\, dx$$

$$= \int_0^1 \left[\int_{x^2}^x (x - 2y)\, dy\right] dx \quad = \quad \int_0^1 (xy - y^2)\Big|_{x^2}^x\, dx$$

$$= \int_0^1 (x^4 - x^3)\, dx \quad = \quad -\frac{1}{20}$$

so that the theorem is verified.

3. Extend the proof of Green's theorem in the plane given in Problem 1 to the curves C for which lines parallel to the coordinate axes may cut C in more than two points.



Consider a closed curve C such as shown in the adjoining figure, in which lines parallel to the axes may meet C in more than two points. By constructing line ST the region is divided into two regions R_1 and R_2 which are of the type considered in Problem 1 and for which Green's theorem applies, i.e.,

$$(1) \qquad \int_{STUS} M\, dx + N\, dy \quad = \quad \iint_{R_1} \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\, dy$$

$$(2) \qquad \int_{SVTS} M\, dx + N\, dy \quad = \quad \iint_{R_2} \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\, dy$$

Adding the left hand sides of (1) and (2), we have, omitting the integrand $M\,dx + N\,dy$ in each case,

$$\int_{STUS} + \int_{SVTS} \quad = \quad \int_{ST} + \int_{TUS} + \int_{SVT} + \int_{TS} \quad = \quad \int_{TUS} + \int_{SVT} \quad = \quad \int_{TUSVT}$$