statistics (section A)   Date : 25.02.201

(i) $\int_0^\infty e^{-x} x^{n-1} dx = \Gamma n$ (गामा)

(ii) $\int_0^\infty e^{-ax} x^{n-1} dx = \frac{\sqrt{n}}{a^n}$

(iii) $\int_0^\infty e^{-ax^2} (x)^{2n-1} dx = \frac{\Gamma n}{2 \cdot a^n}$

(iv) $\int_{-\infty}^{\infty} e^{-a^2x^2} dx = \frac{\sqrt{\pi}}{a}$

$= \frac{\Gamma 1/2}{a}$

(v) $\int_0^1 x^{m-1}(1-x)^{n-1} dx = \beta(m,n)$ [β of 1st kind]

(vi) $\int_0^1 \frac{x^{n-1}}{(1+x)^{m+n}} dx = \beta(m,n)$ [β of 2nd kind]

(vii) $\beta(m,n) = \frac{\Gamma m \, \Gamma n}{\Gamma m+n}$

(viii) $\beta(m,n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{n+m}} dx$

[$\Gamma n = (n-1)\Gamma n-1 = (n-1)(n-2)\Gamma n-2$
$\Gamma n = (n-1)! = (n-1)(n-2)!$]

chi - square distribution
(χ^2 distribution)

$x \sim N(\mu, \sigma^2)$

$\mu \to$ mean, $\sigma \to$ S.D

standard ~~normal~~ normal distribution —

$$z = \frac{x-\mu}{\sigma} \sim N(0,1)$$

χ^2 - variate :

The square of the standard normal variate is known as χ^2 - variate with 1 degrees of freedom (f.d).

These of $x \sim N(\mu, \sigma^2)$. Then

$$z = \frac{x-\mu}{\sigma} \sim N(0,1)$$

and $z^2 = \left(\frac{x-\mu}{\sigma}\right)^2$ is a chi-square variate ~~of~~ with 1 degrees of freedom (f.d).

In general if x_i $(i = 1, 2, \ldots n)$ are 'n' independent normal variate with mean μ_i $(i = 1, 2, \ldots n)$ and variance σ_i^2 $(i = 1, 2, \ldots n)$, Then

$$Z_i^2 = \chi^2 = \sum_{i=1}^{n} \left(\frac{x_i - \mu_i}{\sigma_i}\right)^2$$ is a chi-square variate with n degrees of freedom (d.f).

χ^2 - distribution

A random variable x is said to have a chi-square distribution with n d.f. if its p.d.f. is given by,

$$f(\chi^2) = \frac{1}{2^{n/2}\,\Gamma\frac{n}{2}} e^{-\frac{\chi^2}{2}} (\chi^2)^{\frac{n}{2}-1} \; ; 0 < \chi^2 < \infty$$

where n is the parameter of the χ^2-distribution and also known as degrees of freedom.

$$\boxed{f(x) = \frac{1}{\sqrt{2\pi}\,\sigma} \cdot e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}}$$

Date : 04.03.2017

Properties of the χ^2- distribution:

$$M_{\chi^2}(t) = E(e^{t\chi^2})$$

$$= \int_0^\infty e^{t\chi^2} f(\chi^2)\, d\chi^2$$

$$= \int_0^\infty e^{t\chi^2} \frac{1}{2^{n/2}\Gamma n/2} e^{-\chi^2/2} (\chi^2)^{n/2-1} d\chi^2$$

MGF (Momentary

$$M_x(t) = E(e^{tx})$$

$$= \int_{-\infty}^{\infty} e^{tx} f(x)\, dx$$

$$= \frac{1}{2^{n/2}\Gamma n/2} \int_0^\infty e^{-\frac{\chi^2}{2}(1-2t)} (\chi^2)^{n/2-1} d\chi^2$$

$$= \frac{1}{2^{n/2}\Gamma n/2} \times \frac{\Gamma n/2}{\left(\frac{1-2t}{2}\right)^{n/2}}$$

$$= \frac{1}{2^{n/2}} \times \frac{2^{n/2}}{(1-2t)^{n/2}} = \frac{1}{(1-2t)^{n/2}}$$

$$M_{\chi^2}(t) = (1-2t)^{-n/2}$$

* Find the modulative function of the x^2-distribution.

MGF (momentary Generation function)

$$M_x(t) = E(e^{tx})$$

CGF : (cumulative Genetive function)

$$K_{x}(t) = \log M_x(t)$$

$$K_{x^2}(t) = \log M_{x^2}(t)$$

$$= \log\{(1-2t)^{-n/2}\}$$

$$= -n/2 \log(1-2t)$$

$$= -n/2 \left\{-2t - \frac{(2t)^2}{2} - \frac{(2t)^3}{3} - \frac{(2t)^4}{4} \cdots \infty\right\}$$

$$K_{x^2}(t) = nt + 2n \cdot \frac{t^2}{2} + 4n\frac{t^3}{3} + 8n \cdot \frac{t^4}{4} + \cdots \infty$$

K_1 = co-efficient of $\frac{t}{\lfloor 1}$ = Mean (μ_1')

K_2 = " " $\frac{t^2}{\lfloor 2}$ = Varience (μ_2)

K_3 = " " $\frac{t^3}{\lfloor 3}$ = μ_3

K_4 = " " $\frac{t^4}{\lfloor 4}$

K_1 = co-efficient of $\frac{t}{1!} = n = \mu_1' =$ mean

K_2 = " " " $\frac{t^2}{2!} = 2n = \mu_2 =$ Varience

K_3 = " " " $\frac{t^3}{3!} = 8n = \mu_3$

K_4 = " " " $\frac{t^4}{4!} = 48n$

$$\mu_4 = K_4 + 3K_2^2$$
$$= 48n + 3 \cdot 4n^2$$
$$= 48n + 12n^2$$

Skewness $\beta_1 = \frac{\mu_3^2}{\mu_2^3} = \frac{(8n)^2}{(2n)^2} = \frac{64n^2}{8n^3} = \frac{8}{n}$

Kartosis $\beta_2 = \frac{\mu_4}{\mu_2^2} = \frac{48n + 12n^2}{(2n)^2} = \frac{12}{n} + 3$

mean $= n$

varience $= 2n$

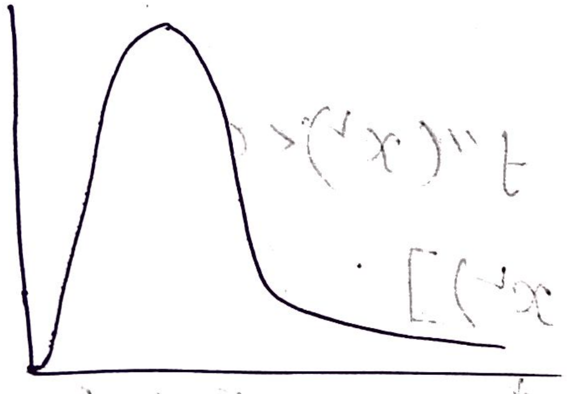
$\beta_1 = \frac{8}{n}$

$\beta_2 = 3 + \frac{12}{n}$

n = degree's of freedom

$\beta_1 = 0$, $\beta_2 = 3$
normal distribution
$\beta_1 > 0 \rightarrow$ positive
$\beta_2 > 3 \rightarrow$ Leptokartic

The χ^2- distribution is positive skewness and leptokartic



Show that χ^2- distribution is always normal distribution.

if $n \rightarrow \infty$ then $\beta_1 \rightarrow 0$
$\beta_2 \rightarrow 3$

Date: 07.03.2017

χ^2

Mode of χ^2 distribution:

$f'(x) = 0 \mid f''(x) < 0$

We know,

$$f(\chi^2) = \frac{1}{2^{n/2}\,\Gamma\frac{n}{2}} e^{-\chi^2/2} (\chi^2)^{\frac{n}{2}-1}$$

Mode is that value of χ^2 which is satifies the equation,

$$f'(\chi^2) = 0 \text{ and } f''(\chi^2) < 0$$

Now $f'(\chi^2) = \frac{\delta}{\delta \chi^2}[f(\chi^2)]$

$$= \frac{1}{2^{n/2}\,\Gamma\frac{n}{2}} \left[e^{-\chi^2/2} \left(\frac{n}{2}-1\right)(\chi^2)^{\frac{n}{2}-1-1} + (\chi^2)^{\frac{n}{2}-1} e^{-\chi^2/2}\left(-\frac{1}{2}\right) \right]$$

$$= \frac{1}{2^{n/2}\,\Gamma\frac{n}{2}} e^{-\chi^2/2} (\chi^2)^{\frac{n}{2}-1} \left[\left(\frac{n}{2}-1\right)(\chi^2)^{-1} - \frac{1}{2} \right]$$

puts $f'(\chi^2) = 0$

$$0 = \frac{1}{2^{\frac{n}{2}} \sqrt{\frac{n}{2}}} e^{-\chi^2/2} (\chi^2)^{\frac{n}{2}-1} \left[\left(\frac{n}{2}-1\right)(\chi^2)^{-1} - \frac{1}{2}\right]$$

$$\Rightarrow e^{-\chi^2/2} (\chi^2)^{\frac{n}{2}-1} \left[\left(\frac{n}{2}-1\right)(\chi^2)^{-1} - \frac{1}{2}\right] = 0$$

Either, $\left(\frac{n}{2}-1\right)(\chi^2)^{-1} - \frac{1}{2} = 0$ or, $e^{-\chi^2/2}(\chi^2)^{n/2-1} \neq 0$

$$\Rightarrow \frac{n-2}{2} \frac{1}{\chi^2} = \frac{1}{2}$$

$$\Rightarrow \chi^2 = n-2$$

It is easy to ~~say that~~ show that at the point $\chi^2 = n-2$,

$$f''(\chi^2) < 0$$

Hence the mode of χ^2-distribution is $(n-2)$ for n degree of freedom.

Additive properties of χ^2-distribution:

~~sum of the χ^2~~

Statement: The sum of independent χ^2-variate is also a χ^2-variate. Let $\chi_1^2, \chi_2^2, \ldots \chi_k^2$ be the k-independent variate with $n_1, n_2 \ldots n_k$ degrees of freedom.

The sum $\sum_{i=1}^{k} x_i^2$ is a x^2- variate with $n_1+n_2+\dots+n_k = \sum_{i=1}^{k} n_i$. df .

$\boxed{\text{state \& prove } x^2\text{- variate.}}$

We know the m.g.f of x^2- variate

$$M_{x_i^2}(t) = (1-2t)^{-n_i/2} \quad i = 1, 2 \dots k$$

since $x_1^2, x_2^2 \dots x_k^2$ be the k - independent x^2- variate with $n_1, n_2 \dots n_k$ df.

The m.g.f of $\sum_{i=1}^{k} x_i^2$ can be written as

$$M_{\sum_{i=1}^{k} x_i^2}(t) = \prod_{i=1}^{k} M_{x_i^2}(t) = M_{x_1^2}(t) \cdot M_{x_2^2}(t) \dots M_{x_k^2}(t)$$

$$= (1-2t)^{n_1/2} \cdot (1-2t)^{n_2/2} \dots (1-2t)^{n_k/2}$$

$$= (1-2t)^{(n_1/2 + n_2/2 + \dots + n_k/2)}$$

$$= (1-2t)^{-\frac{1}{2}\sum_{i=1}^{k} n_i}$$

$$= (1-2t)^{-n/2} \text{ when}$$

which is the momentary function of ~~chi~~ with x^2- variate with $n_1, n_2, \dots, n_k$

So the momentary function of $\sum_{i=1}^{K} x_i^2$ is a x^2- variate with $\sum_{i=1}^{K} n_i$ d.f.

* Application of x^2- distribution:

(i) To test if the hypothetical value of the population varience is $\sigma^2 = \sigma_0^2$ (say)

$H_0 : \sigma^2 = \sigma_0^2$ (say)
$H_1 : \sigma^2 \neq \sigma_0$

(ii) To test the goodness of fit.

(iii) To test the homogenetye of independent estimate of the population varience.

(iv) To test the independent of attributes (charactaristics)

(v) To test the homogenety of independent estimate of the population corrulation coefficient.

* Write down the properties of x^2- distribution?

Properties of the x^2- distribution:

(i) x^2 is a continuous type of distribution and its range 0 to ∞.

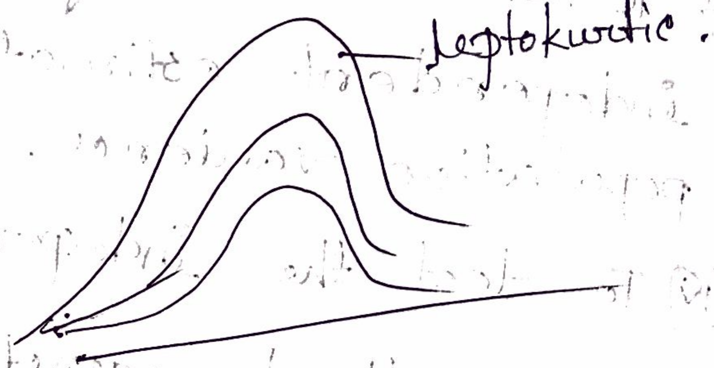
(ii) The distribution contain only one peramater which is the degrees of freedom of the distribution.

(iii) The mean and varience of the χ^2-distribution for of n.d.f. are n and 2n respectively.

(iv) The mode of the χ^2- distribution is (n-2).

(v) χ^2-distribution tends to normal distribution for large degees of freedom.

(vi) χ^2- distribution is positive skewed and leptokurtic.



T - distribution

Date : 11.03.2017

W.S. Grosset (1908) / students'

student's t - distribution

$$t = \frac{\bar{x} - \mu}{s/\sqrt{n}} \quad \Big| \quad E(s^2) = \sigma^2$$

Normal distribution, $D = Z = \dfrac{\bar{x} - \mu}{\sigma/\sqrt{n}}$

n -এর মান large হলে normal distribution.

$n > 30 \rightarrow$ normal distribution

$n \leq 30 \rightarrow$ student's t - distribution

t - distribution : Let $x_1, x_2 \ldots x_n$ be a random sample of size n from a normal distribution with mean μ and varience σ^2 then student's t - in different by the statistic,

$$t = \frac{\bar{x} - \mu}{s/\sqrt{n}} \quad \text{where} \quad \bar{x} = \frac{\sum_{i=1}^{n} x_i}{n}$$

and $s^2 = \dfrac{\sum_{i=1}^{n} (x - \bar{x})^2}{n}$ is an unbiased estimate of population variance σ^2 and ~~it~~ it follow student's t - distribution. with

$v = n-1$ df with probability density function,

$$f(t) = \frac{1}{\sqrt{v}\,\beta\left(\frac{1}{2}, \frac{v}{2}\right)\left(1+\frac{t^2}{v}\right)^{\frac{v+1}{2}}}$$

$$-\infty < t < \infty$$

$$f(t) = \frac{1}{\sqrt{n}\,\beta\left(\frac{1}{2}, \frac{n}{2}\right)\left(1+\frac{t^2}{n}\right)^{\frac{n+1}{2}}} \; ; \; n = v-1$$

Fisher's t-distribution:

$u = \xi = N(0,1)$ = standard normal

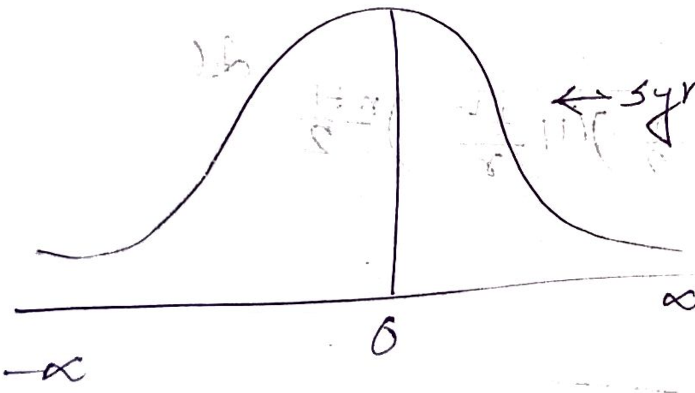
Fisher's t - distribution is the ratio of a standard normal ~~distribution~~ variate to the square-root of an independent chi-square variate divided by ~~their~~ its degrees of freedom, If u or ξ is $N(0,1)$ and χ^2 is an independent chi-square variate with ~~d~~ n d.f then fisher's t-distribution is defined, as,

$$t = \frac{u}{\sqrt{\frac{\chi^2}{n}}} \quad \text{or} \quad t = \frac{\xi}{\sqrt{\frac{\chi^2}{n}}}$$

$$\boxed{t = \frac{\bar{x} - \mu}{s/\sqrt{n}} = \frac{\bar{x} - \mu}{\sqrt{s^2/n}}}$$

$n = \nu$

$-\infty < t < \infty$



← symmetric distribution

Odd raw moments:

$\mu'_{2r+1} = 0 \quad r = 0, 1, 2, 3, \ldots$

$\mu'_{2r} = 0$

Since $f(t)$ is symmetrical about the ~~lize~~ line $t = 0$ all ~~ad~~ add order moment about origin vanish i.e

$\mu'_{2r+1} = 0$, ~~now~~ $r = 0, 1, 2 \ldots$

Now in particular $\mu'_1 = 0$ mean

Hence The central moment ~~con~~ concide with moment about origin. i.e, $\mu_{2r+1}=0$; $r=1,2,3$.

Even ~~odd~~ raw moment:

$$\mu'_{2r} = E(t^{2r}) = \int_{-\infty}^{\infty} t^{2n} f(t)\,dt$$

$$= \int_{-\infty}^{\infty} t^{2n} \frac{1}{\sqrt{n}\,\beta\left(\frac{1}{2}, \frac{n}{2}\right)\left(1+\frac{t^2}{n}\right)^{\frac{n+1}{2}}}\,dt$$

Let, $\frac{t^2}{n} = m$

or, $t^2 = n \cdot m \Rightarrow t = \sqrt{nm}$

$2 \cdot t\, dt = n\, dm$

$$= 2\int_{0}^{\infty} \frac{t^{2n}}{\sqrt{n}\,\beta\left(\frac{1}{2}, \frac{n}{2}\right)\left(1+\frac{t^2}{n}\right)^{\frac{nm}{2}}}\,dt$$

$$= 2\int_{0}^{\infty} \frac{(nm)^{n}}{\sqrt{n}\,\beta\left(\frac{1}{2}, \frac{n}{2}\right)\left(1+\frac{nm}{n}\right)^{\frac{m}{2}}}$$

$$2t\,dt = n\,dm$$

$$2t = \frac{n\,dm}{dt}$$

Date: 14.03.2017

Let, $t^2/n = m$

$$\Rightarrow t^2 = nm$$

$$\Rightarrow t = \sqrt{nm} = (nm)^{1/2}$$

$$\Rightarrow \frac{dt}{dm} = \frac{1}{2}(nm)^{1/2 - 1}$$

$$\Rightarrow \frac{dt}{dm} = \frac{n}{2}\,(nm)^{-1/2}$$

$$= \frac{1}{2\sqrt{nm}}$$

$$\mu_{2r} = \frac{1}{\sqrt{n}\,\beta\left(\frac{1}{2}, \frac{n}{2}\right)} \int_{-\infty}^{\infty} \frac{t^{2r}}{\left(1+\frac{t^2}{n}\right)^{\frac{n+1}{2}}}\,dt$$

$$= \frac{1}{\sqrt{n}\,\beta\left(\frac{1}{2}\cdot\frac{n}{2}\right)} \cdot 2\int_{0}^{\infty} \frac{t^{2r}}{\left(1+\frac{t^2}{n}\right)^{\frac{n+1}{2}}}\,dt$$

when $t=0$, $m=0$

$t=\infty$ $m=\infty$

$$= \frac{2}{\sqrt{n}\,\beta\left(\frac{1}{2}\cdot\frac{n}{2}\right)} \int_{0}^{\infty} \frac{m^{r} n^{r}}{(1+m)^{\frac{n+1}{2}}}\,dm\,\frac{\sqrt{n}}{2\sqrt{m}}$$

$$= \frac{n^{r}}{\beta\left(\frac{1}{2}\cdot\frac{n}{2}\right)} \int_{0}^{\infty} \frac{m^{r-\frac{1}{2}}}{(1+m)^{\frac{n}{2}+\frac{1}{2}}}\,dm$$

$$= \frac{n^{r}}{\beta\left(\frac{1}{2}\,\frac{n}{2}\right)} \int_{0}^{\infty} \frac{m^{(r+\frac{1}{2})-1}\,dm}{(1+m)^{(\frac{n}{2}-r)+(r+\frac{1}{2})}}$$

$$= n^{r}\,\frac{\Gamma\frac{n}{2}-r\;\Gamma r+\frac{1}{2}}{\Gamma\frac{n}{2}-r+r+\frac{1}{2}} \Bigg/ \frac{\Gamma\frac{1}{2}\;\Gamma\frac{n}{2}}{\Gamma\frac{1}{2}+\frac{n}{2}}$$

$$= n^{r}\,\frac{\Gamma\frac{n}{2}-r\;\Gamma r+\frac{1}{2}}{\Gamma\frac{n}{2}+\frac{1}{2}} \times \frac{\Gamma\frac{n}{2}+\frac{1}{2}}{\Gamma\frac{1}{2}\;\Gamma\frac{n}{2}}$$

$$\therefore \mu_{2r} = n^{r}\,\frac{\Gamma n/2-r\;\Gamma r+\frac{1}{2}}{\Gamma 1/2\;\Gamma n/2} \quad ; r=1,2,3..$$

if $r = 1$,

$$\mu'_{2.1} = n \cdot \frac{\Gamma\frac{n}{2}-1\ \Gamma 1+\frac{1}{2}}{\Gamma\frac{1}{2}\ \Gamma\frac{n}{2}}$$

$$= \frac{n\ \Gamma\frac{n}{2}-1\ \Gamma\frac{3}{2}}{\Gamma\frac{1}{2}\ \Gamma\frac{n}{2}}$$

$$= \frac{n\ \Gamma\frac{n}{2}-1\ \left(\frac{3}{2}-1\right)\Gamma\frac{3}{2}-1}{\Gamma\frac{1}{2}\left(\frac{n}{2}-1\right)\Gamma\frac{n}{2}-1}$$

$$= \frac{n\ \frac{1}{2}\ \Gamma\frac{1}{2}}{\Gamma\frac{1}{2}\left(\frac{n}{2}-1\right)}$$

$$= \frac{n}{\cancel{2}} \times \frac{\cancel{2}}{n-2}$$

$$= \frac{n}{n-2}$$

$\therefore \mu_2' = \frac{n}{n-2}$ if $n > 2$

if $r = 2$

$$\mu'_{2.2} = \frac{n^2\ \Gamma\frac{n}{2}-2\ \Gamma 2+\frac{1}{2}}{\Gamma\frac{1}{2}\ \Gamma\frac{n}{2}}$$

$$\mu'_4 = \frac{n^2\ \Gamma\frac{n}{2}-2\ \Gamma\frac{5}{2}}{\Gamma\frac{1}{2}\ \Gamma\frac{n}{2}}$$

$$= \frac{n^2\ \Gamma\frac{n}{2}-2\ \left(\frac{5}{2}-1\right)\left(\frac{5}{2}-2\right)\Gamma\frac{5}{2}-2}{\Gamma\frac{1}{2}\left(\frac{n}{2}-1\right)\left(\frac{n}{2}-2\right)\Gamma\frac{n}{2}-2}$$

$$= \frac{n^2 \frac{3}{2} \cdot \frac{1}{2} \Gamma\frac{1}{2}}{\Gamma\frac{1}{2} \left(\frac{n}{2}-1\right)\left(\frac{n}{2}-2\right)}$$

$$= \frac{n^2 \frac{3}{4}}{\frac{(n-2)(n-4)}{4}}$$

$$= n^2 \cdot \frac{3}{4} \times \frac{4}{(n-2)(n-4)}$$

$$= \frac{3n^2}{(n-2)(n-4)}$$

$\mu_1 = 0$

$\mu_2 = \mu_2' - (\mu_1')^2$

$= \frac{n}{n-2} - (0)^2$

$= \frac{n}{n-2}$ = varience if $n>2$

$\mu_3 = \mu_3' - 3\mu_2' \mu_1' + 2\mu_1'^3$

$= 0 - 3 \cdot \frac{n}{n-2} \cdot 0 + 2(0)^3$

$= 0$

$\mu_4 = \mu_4' - 4\mu_3'\mu_1' + 6\mu_2'\mu_1'^2 - 3\mu_1'^4$

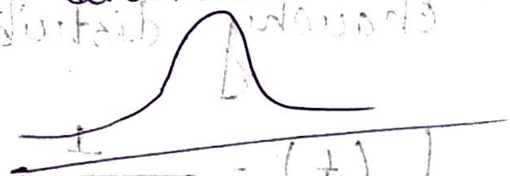
$= \frac{3n^2}{(n-2)(n-4)} - 4 \cdot 0 \cdot 0 + 6 \cdot 0 \cdot (0)^2 - 3 \cdot (0)^4$

$= \frac{3n^2}{(n-2)(n-4)}$ if $n>4$

$\therefore$ Skewness $\beta_1 = \dfrac{\mu_3^2}{\mu_2^3}$

$$= \frac{0^2}{\left(\frac{n}{n-2}\right)^3} = 0$$

If there $\beta_1 = 0$ then the distribution is symmetric.



Kurtosis $\beta_2 = \dfrac{\mu_4}{\mu_2^2}$

$$= \frac{\frac{3n^2}{(n-2)(n-4)}}{\left(\frac{n}{n-2}\right)^2}$$

$$= \frac{3n^2}{(n-2)(n-4)} \times \frac{(n-2)(n-2)}{n^2}$$

$$= \frac{3(n-2)}{(n-4)}$$

$$= \frac{3n-6}{n-4}$$

$$= \frac{3n-12+6}{n-4}$$

$$= \frac{3(n-4)+6}{n-4}$$

$$\therefore \beta_2 = 3 + \frac{6}{n-4} \; ; \; n > 4$$

Here $\beta_2 > 3$ if $n > 4$

$\therefore$ t distribution is symmetrical & leptokartic.

⊛ Show that t distribution is symmetric & leptokartic.

⊛ Under what condition t distribution follow chauchy distribution

$$f(t) = \frac{1}{\sqrt{n}\,\beta\left(\frac{1}{2}\cdot\frac{n}{2}\right)\left(1+\frac{t^2}{n}\right)^{\frac{n+1}{2}}} \quad ; -\infty < t < \infty$$

putting $n = 1$,

$$f(t) = \frac{1}{\sqrt{1}\,\beta\left(\frac{1}{2}\cdot\frac{1}{2}\right)\left(1+\frac{t^2}{1}\right)^{\frac{1+1}{2}}}$$

$$= \frac{1}{\beta\left(\frac{1}{2}\cdot\frac{1}{2}\right)\left(1+t^2\right)}$$

$$= \frac{\Gamma\,1/2+1/2}{\Gamma 1/2\,\Gamma 1/2} \times \frac{1}{(1+t^2)}$$

$$= \frac{\Gamma 1}{\sqrt{\pi}\,\sqrt{\pi}} \times \frac{1}{(1+t^2)}$$

$$= \frac{1}{\sqrt{\pi}\,\sqrt{\pi}} \times \frac{1}{(1+t^2)}$$

$$= \frac{1}{\pi}\,\frac{1}{(1+t^2)}$$

which is the form of the standard chauchy distribution.

⊛ Under what condition t distribution follow standard normal distribution

$\beta_1 = 0$

$\beta_2 = 3 + \frac{6}{n}$

when $n \to \infty$, $\beta_1 \to 0$

$\beta_2 \to 3$

When n is very large t-distribution tends to standard normal distribution.

Date: 18.03.2017

We have, $-\infty < t < \infty$

$$f(t) = \frac{1}{\sqrt{n}\,\beta(\frac{1}{2}, \frac{n}{2})\left(1+\frac{t^2}{n}\right)^{\frac{n+1}{2}}}$$

or, $$f(t) = \frac{\left(1+\frac{t^2}{n}\right)^{-\frac{n}{2}-\frac{1}{2}}}{\sqrt{n}\,\beta(\frac{1}{2}, \frac{n}{2})}$$

$$= \frac{\left(1+\frac{t^2}{n}\right)^{-n/2}\left(1+\frac{t^2}{n}\right)^{-1/2}}{\sqrt{n}\,\beta\left(\frac{1}{2}, \frac{n}{2}\right)}$$

Now taking limit on the both sides we get,

$$\underset{n\to\infty}{Lt} f(t) = \frac{\underset{n\to\infty}{Lt}\left(1+\frac{t^2}{n}\right)^{-n/2}\left(1+\frac{t^2}{n}\right)^{-1/2}}{\underset{n\to\infty}{Lt}\sqrt{n}\,\beta\left(\frac{1}{2}, \frac{n}{2}\right)}$$

(1)

Now $$\underset{n\to\infty}{Lt}\ \sqrt{n}\,\frac{\Gamma\frac{1}{2}\,\Gamma\frac{n}{2}}{\Gamma\left(\frac{1}{2}+\frac{n}{2}\right)} = \sqrt{n}\,\Gamma\tfrac{1}{2}\,\underset{n\to\infty}{Lt}\frac{\Gamma\frac{n}{2}}{\Gamma\left(\frac{1}{2}+\frac{n}{2}\right)}$$

$$\underset{n\to\infty}{Lt}\ \frac{\Gamma(n+k)}{\sqrt{n}} = n^k$$

$$= \left(\sqrt{n}\,\Gamma\tfrac{1}{2}\right)^{-1}\underset{n\to\infty}{Lt}\frac{\Gamma\left(\frac{1}{2}+\frac{n}{2}\right)}{\Gamma\frac{n}{2}}$$

$$= \frac{\sqrt{n}\,\Gamma\frac{1}{2}}{\left(\frac{n}{2}\right)^{1/2}}$$

$$= \frac{\sqrt{n}\,\Gamma\frac{1}{2}}{n^{1/2}\left(\frac{1}{2}\right)^{1/2}}$$

$$= \Gamma\frac{1}{2}\left(\frac{1}{2}\right)^{-\frac{1}{2}}$$

$$= \Gamma\frac{1}{2}\,(2)^{\frac{1}{2}}$$

$$= \sqrt{2}\cdot\Gamma\frac{1}{2} = \sqrt{2}\cdot\sqrt{\pi}$$

$$= \sqrt{2\pi}$$

Again $\underset{n\to\infty}{Lt}\left(1+\frac{t^2}{n}\right)^{-\frac{n}{2}}\left(\underset{n\to\infty}{Lt}\left(1+\frac{t^2}{n}\right)^{-1/2}\right)$

$$\boxed{\underset{n\to\infty}{Lt}\left(1+\frac{m}{n}\right)^{n} = e^{m}}$$

$$= \underset{n\to\infty}{Lt}\left\{\left(1+\frac{t^2}{n}\right)^{n}\right\}^{-1/2}\underset{n\to\infty}{Lt}\left(1+\frac{t^2}{n}\right)^{-1/2}$$

$$= \left(e^{t^2}\right)^{-1/2}(1)^{-1/2}$$

$$= e^{-t^2/2}$$

$$\lim_{n \to \infty} f(t) = \frac{1}{\sqrt{2\pi}} e^{-t^2/2}$$

$$\boxed{f(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-1/2 \left(\frac{x-\mu}{\sigma} \right)^2} = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}} \quad \mu, \sigma^2$$

Application of t - distribution :

1. To test the sample mean $\bar{x}$ differ significantly from the ~~va~~ hypothetical value of mean of μ to observe population mean (single mean test)

2. To test the significance of the difference betn two ~~smple~~ sample means (double mean test).

3. To test the significance of an observe sample correlation coefficient and sample regration coefficient.

(a) To test the significance of an observe partial correlation coefficient.

Properties of the t - distribution :

(1) Student's t - distribution is continuous type and its range $-\infty$ to $+\infty$.

(2) It has only one parameter and it is known as degrees of freedom. (unknown p. = 1 or)

(3) The mean, median & mode are equal to zero and varience $= \frac{n}{n-2}$ if $n > 2$

(4) If $n = 1$ then t - distribution tends to standard cauchy distribution.

(5) t - distribution tends to standard normal distribution when degrees of freedom (n) is very large

(6) All odd moments of t - distribution are zero

(7) The distribution is symmetric & Leptokurtic.

~~t-distribution~~ about

Show that mean deviation ~~of~~ mean of t-distribution $= \dfrac{\sqrt{n}\,\overline{\left|\frac{n-1}{2}\right.}}{\sqrt{\pi}\,\overline{\left|\frac{n}{2}\right.}}$

$\mu_1' = E(t) = 0$

$M.D(\bar{t})$

~~mea~~

M.D. about mean $= E|t - E(t)| = E|t|$

$MD(\bar{x}) = \sum \dfrac{|x_i - \bar{x}|}{n}$

MD (

$E(x) = \int f(x)\,dx$

$= \int |t|$

$= \displaystyle\int_{-\infty}^{\infty} t \cdot \frac{1}{\sqrt{n}\,\beta(1/2, \frac{n}{2})\left(1+\frac{t^2}{n}\right)^{\frac{n+1}{2}}}\,dt$

$= 2\displaystyle\int_0^{\infty} \frac{t\,dt}{\sqrt{n}\,\beta(1/2, \frac{n}{2})\left(1+\frac{t^2}{n}\right)^{\frac{n+1}{2}}}$

Let, $\frac{t^2}{n} = z$

$\Rightarrow t^2 = nz$

$\Rightarrow 2t\,dt = n\,dz$

$\Rightarrow t\,dt = \frac{1}{2} n\,dz$

When $t = 0$, $z = 0$

$t = \infty$, $z = \infty$

$$= \int_0^\infty \frac{n\,dz}{\sqrt{n}\,\beta\left(\frac{1}{2}, \frac{n}{2}\right)(1+z)^{\frac{n+1}{2}}}$$

$$= \frac{\sqrt{n}}{\beta\left(\frac{1}{2}, \frac{n}{2}\right)} \int_0^\infty \frac{dz}{(1+z)^{\frac{n+1}{2}}}$$

$$= \frac{\sqrt{n}}{\beta\left(\frac{1}{2}, \frac{n}{2}\right)} \int_0^\infty \frac{z^{1-1}}{(1+z)^{\left(\frac{n}{2}-\frac{1}{2}\right)+1}}\,dz$$

$$= \frac{\sqrt{n}}{\beta\left(\frac{1}{2}, \frac{n}{2}\right)} \times \beta\left(\frac{n}{2} - \frac{1}{2}; 1\right)$$

$$= \frac{\sqrt{n}}{\frac{\Gamma\frac{1}{2}\,\Gamma\frac{n}{2}}{\Gamma\left(\frac{1}{2}+\frac{n}{2}\right)}} \times \frac{\Gamma\frac{n-1}{2}\,\Gamma 1}{\Gamma\left(\frac{n-1}{2}+1\right)}$$

$$= \frac{\sqrt{n} \times \Gamma\frac{n-1}{2} \times \cancel{\Gamma\frac{n+1}{2}}}{\Gamma\frac{1}{2}\ \Gamma\frac{n}{2}\ \cancel{\Gamma\frac{n+1}{2}}}$$

$$= \frac{\sqrt{n}\ \Gamma\frac{n-1}{2}}{\sqrt{\pi}\ \Gamma\frac{n}{2}}$$

F-distribution Date : 21.03.2017

Def^n: F is defined as the ratio of two independent χ^2 random variable dividing by their respective degges of freedom.

Let χ^2_1 and χ^2_2 are the two independent chi-square variate having n_1 and n_2 degree's of freedom respectively then the F. statistics defined as,

$$F = \frac{\chi_1^2/n_1}{\chi_2^2/n_2} \sim F(n_1, n_2)$$

The probability density function of F is defined as,

$$f(F) = \frac{\left(\frac{n_1}{n_2}\right)^{\frac{n_1}{2}} F^{\frac{n_1}{2}-1}}{\beta\left(\frac{n_1}{2}, \frac{n_2}{2}\right)\left(1+\frac{n_1}{n_2}F\right)^{\frac{n_1+n_2}{2}}}$$

$; 0 \leq F \leq \infty$

where n_1 and n_2 are the degree's of freedom of this distribution.

Properties of the F-distribution:

r th. raw moment of F-distribution is

$$\mu_r' = E(F^r)$$

$$= \int_0^\infty F^r f(F)\, dF$$

$\mu_r' = E(x^r) = \int x^r f(x)\, dx$

$$= \int_0^\infty \frac{F^r \left(\frac{n_1}{n_2}\right)^{\frac{n_1}{2}} F^{\frac{n_1}{2}}}{\beta\left(\frac{n_1}{2}, \frac{n_2}{2}\right)\left(1+\frac{n_1}{n_2}F\right)^{\frac{n_1+n_2}{2}}} dF$$

$$= \frac{\left(\frac{n_1}{n_2}\right)^{\frac{n_1}{2}}}{\beta\left(\frac{n_1}{2}, \frac{n_2}{2}\right)} \int_0^\infty \frac{F^{\frac{n_1}{2}+r-1}}{\left(1+\frac{n_1}{n_2}F\right)^{\frac{n_1+n_2}{2}}} dF$$

Let, $\frac{n_1}{n_2} F = U$ when $F=0$, $U=0$

$F=\infty$, $U=\infty$

or $F = \frac{n_2}{n_1} \cdot U$

or $dF = \frac{n_2}{n_1} du$

$$= \frac{\left(\frac{n_1}{n_2}\right)^{\frac{n_1}{2}}}{\beta\left(\frac{n_1}{2}, \frac{n_2}{2}\right)} \int_0^{\infty} \frac{\left(\frac{n_2}{n_1} u\right)^{\frac{n_1}{2}+r-1}}{(1+u)^{\frac{n_1+n_2}{2}}} \left(\frac{n_2}{n_1}\right) du$$

$$= \frac{\left(\frac{n_1}{n_2}\right)^{\frac{n_1}{2}}}{\beta\left(\frac{n_1}{2}, \frac{n_2}{2}\right)} \int_0^{\infty} \frac{\left(\frac{n_2}{n_1}\right)^{\frac{n_1}{2}+r-1} u^{\frac{n_1}{2}+r-1} \cdot \left(\frac{n_2}{n_1}\right)^{1} du}{(1+u)^{\frac{n_1+n_2}{2}}}$$

$$= \frac{\left(\frac{n_1}{n_2}\right)^{\frac{n_1}{2}} \left(\frac{n_2}{n_1}\right)^{\frac{n_1}{2}+r}}{\beta\left(\frac{n_1}{2}, \frac{n_2}{2}\right)} \int_0^{\infty} \frac{u^{\left(\frac{n_1}{2}+r\right)-1} \, du}{(1+u)^{\left(\frac{n_1}{2}+r\right)+\left(\frac{n_2}{2}-r\right)}}$$

$$= \frac{\left(\frac{n_2}{n_1}\right)^{-\frac{n_1}{2}+\frac{n_1}{2}+r}}{\beta\left(\frac{n_1}{2}, \frac{n_2}{n}\right)} \beta\left(\frac{n_1}{2}+r, \frac{n_2}{2}-r\right)$$

$$= \frac{\left(\frac{n_2}{n_1}\right)^{r}}{\frac{\Gamma\frac{n_1}{2}\,\Gamma\frac{n_2}{2}}{\Gamma\frac{n_1+n_2}{2}}} \times \frac{\Gamma\left(\frac{n_1}{2}+r\right)\Gamma\left(\frac{n_2}{2}-r\right)}{\Gamma\left(\frac{n_1}{2}+\cancel{r}+\frac{n_2}{2}-\cancel{r}\right)}$$

$$= \left(\frac{n_2}{n_1}\right)^{r} \frac{\Gamma\left(\frac{n_1}{2}+r\right)\Gamma\left(\frac{n_2}{2}-r\right)}{\Gamma\frac{n_1}{2}\,\Gamma\frac{n_2}{2}}$$

$$\mu_r' = \left(\frac{n_2}{n_1}\right)^r \frac{\Gamma\left(\frac{n_1}{2}+r\right)\Gamma\left(\frac{n_2}{2}-r\right)}{\Gamma\frac{n_1}{2}\,\Gamma\frac{n_2}{2}}$$

Putting $r=1$,

$$\mu_1' = \text{mean} = \left(\frac{n_2}{n_1}\right) \frac{\Gamma\left(\frac{n_1}{2}+1\right)\Gamma\left(\frac{n_2}{2}-1\right)}{\Gamma\frac{n_1}{2}\,\Gamma\frac{n_2}{2}}$$

$$= \frac{\frac{n_2}{n_1}\left(\frac{n_1}{2}+1-1\right)\cancel{\Gamma\left(\frac{n_1}{2}+1-1\right)}\,\cancel{\Gamma\left(\frac{n_2}{2}-1\right)}}{\cancel{\Gamma\frac{n_1}{2}}\,\left(\frac{n_2}{2}-1\right)\cancel{\Gamma\left(\frac{n_2}{2}-1\right)}}$$

$$\boxed{\Gamma n = (n-1)\,\Gamma(n-1)}$$

$$= \frac{\frac{n_2}{2}}{\frac{n_2}{2}-1}$$

$$= \frac{n_2}{\cancel{2}} \times \frac{\cancel{2}}{n_2-2}$$

$$= \frac{n_2}{n_2-2} \quad \text{if } n_2>2$$

Q. Find the mean of the F - distribution.

Putting $r=2$

$$\mu_2' = \left(\frac{n_2}{n_1}\right)^2 \frac{\Gamma\left(\frac{n_1}{2}+2\right)\Gamma\left(\frac{n_2}{2}-2\right)}{\Gamma\frac{n_1}{2}\,\Gamma\frac{n_2}{2}}$$

$$= \left(\frac{n_2}{\cancel{n_1}}\right)^2 \frac{\left(\frac{n_1}{2}+1\right)\left(\frac{n_1}{2}\right)\cancel{\Gamma\frac{n_1}{2}}\;\cancel{\Gamma\left(\frac{n_2}{2}-2\right)}}{\cancel{\Gamma\frac{n_1}{2}}\left(\frac{n_2}{2}-1\right)\left(\frac{n_2}{2}-2\right)\cancel{\Gamma\left(\frac{n_2}{2}-2\right)}}$$

$$= \frac{\frac{n_2^2}{n_1^2} \times \frac{n_1+2}{2} \cdot \frac{n_1}{2}}{\frac{n_2-2}{2} \times \frac{n_2-4}{2}}$$

$= n.$

$$= \frac{n_2^2 (n_1+2)}{n_1(n_2-2)(n_2-4)}$$

We know,

variance $= \mu_2 = \mu_2' - \mu_1'^2$

$$= \frac{n_2^2 (n_1+2)}{n_1(n_2-2)(n_2-4)} - \left(\frac{n_2}{n_2-2}\right)^2$$

$$= \frac{n_2^2(n_1+2)(n_2-2) - n_2^2\{n_1(n_2-4)\}}{n_1(n_2-2)^2(n_2-4)}$$

$$= \frac{n_2^2(n_1n_2-2n_1+2n_2-4) - n_2^2(n_1n_2-4n_1)}{n_1(n_2-2)^2(n_2-4)}$$

$$= \frac{n_2^2\{\cancel{n_1n_2}-2n_1+2n_2-4-\cancel{n_1n_2}+4n_1)}{n_1(n_2-2)^2(n_2-4)}$$

$$= \frac{2 \cdot n_2^2 (n_1+n_2-2)}{n_1(n_2-2)^2(n_2-4)}$$ if $n_2 > 4$

Date: 28.03.2017.

Mode

We have

$$f(F) = \frac{\left(\frac{n_1}{n_2}\right)^{\frac{n_1}{2}} F^{\frac{n_1}{2}-1}}{\beta\left(\frac{n_1}{2}, \frac{n_2}{2}\right)\left(1+\frac{n_1}{n_2}F\right)^{\frac{n_1+n_2}{2}}}$$

$0 < F < \infty$

$$f(F) = c \cdot \frac{F^{\frac{n_1}{2}-1}}{\left(1+\frac{n_1}{n_2}F\right)^{\frac{n_1+n_2}{2}}}$$

where $c = \frac{\left(\frac{n_1}{n_2}\right)^{\frac{n_1}{2}}}{\beta\left(\frac{n_1}{2}, \frac{n_2}{2}\right)}$

Taking log on both sides we get:

$$\log f(F) = \log c + \left(\frac{n_1}{2}-1\right)\log F - \left(\frac{n_1+n_2}{2}\right)\log\left(1+\frac{n_1}{n_2}F\right)$$

$\log\left(\frac{A}{B}\right) = \log A - \log B$

Now differentiating $\log f(F)$ w.r.t F,

$$\frac{\delta \log f(F)}{\delta F} = \left(\frac{n_1}{2}-1\right)\frac{1}{F} - \left(\frac{n_1+n_2}{2}\right)\frac{1}{1+\frac{n_1}{n_2}F} \times \left(\frac{n_1}{n_2}\right)$$

$$\text{or, } \frac{n_1-2}{2F} - \left(\frac{n_1+n_2}{2}\right) \times \frac{1}{\frac{n_1+n_2}{n_2}} \times \frac{n_1}{n_2} = 0$$

$f'(F) = 0$

$$\text{or, } \frac{n_1-2}{F} - \frac{n_1+n_2}{2(n_2+n_1F)} = 0$$

$$\text{or, } \quad \frac{n_1-2}{\cancel{2}F} = \frac{n_1^2 + n_1 n_2}{\cancel{2}(n_2 + n_1 F)}$$

$$\text{or, } \quad (n_1-2)(n_2+n_1F) = (n_1^2 + n_1 n_2)F$$

$$\text{or, } \quad n_1 n_2 + \cancel{n_1^2 F} - 2n_2 - 2n_1F = \cancel{n_1^2 F} + n_1 n_2 F$$

$$\text{or, } \quad F(2n_1 + n_1 n_2) = n_1 n_2 - 2n_2$$

$$\text{or, } \quad F = \frac{n_1 n_2 - 2n_2}{2n_1 + n_1 n_2}$$

$$= \frac{n_2(n_1-2)}{n_1(2+n_2)}$$

$$= \frac{n_1-2}{n_1} \cdot \frac{n_2}{2+n_2} \quad \text{if } n_1 > 2$$

It can be easily varified that at the point $F = \left(\frac{n_1-2}{n_1}\right)\left(\frac{n_2}{2+n_2}\right)$ $f''(F) < 0$. Hence, ~~Mod~~

$$\text{Mode} = \left(\frac{n_1-2}{n_1}\right)\left(\frac{n_2}{2+n_2}\right)$$

Relation between 't' and 'F' distribution :

We have,

$$f(F) = \frac{\left(\frac{n_1}{n_2}\right)^{\frac{n_1}{2}} F^{\frac{n_1}{2}-1}}{\beta\left(\frac{n_1}{2}\cdot\frac{n_2}{2}\right)\left(1+\frac{n_1}{n_2}F\right)^{\frac{n_1+n_2}{2}}}, \quad 0<F<\alpha$$

putting, $n_1 = 1$, $n_2 = n$ (and $t^2 = F$)

or $2t\,dt = dF$

$$f(t) = f(F) = |J|$$

$$\Rightarrow \frac{dF}{dt} = 2t$$

$$\Rightarrow |J| = \left|\frac{dF}{dt}\right| = 2t$$

$$= \frac{\left(\frac{1}{n}\right)^{\frac{1}{2}}\left(t^2\right)^{\frac{1}{2}-1}}{\beta\left(\frac{1}{2}\cdot\frac{n}{2}\right)\left(1+\frac{t^2}{n}\right)^{\frac{1+n}{2}}}$$

$t^2 = F$

$F = 0$, $t=0$

$F=\alpha$, $t=\infty$

$0<t^2<\alpha$

$$f(t) = \frac{2}{\sqrt{n}\,\beta\left(\frac{1}{2}\cdot\frac{n}{2}\right)\left(1+\frac{t^2}{n}\right)^{\frac{n+1}{2}}}$$

$$= \frac{1}{\sqrt{n}\,\beta\left(\frac{1}{2}\cdot\frac{n}{2}\right)\left(1+\frac{t^2}{n}\right)^{n+1}} \quad -\alpha<t<\alpha$$

which is one of the t-distribution with n degrees of freedom.

Relation between F and χ^2 distribution

We have,

$$f(F) = \frac{\left(\frac{n_1}{n_2}\right)^{\frac{n_1}{2}-1} F^{\frac{n_1}{2}-1}}{\beta\left(\frac{n_1}{2},\frac{n_2}{2}\right)\left(1+\frac{n_1}{n_2}F\right)^{\frac{n_1+n_2}{2}}} \qquad 0<F<\infty$$

$$\frac{\Gamma\left(\frac{n_1}{2}+\frac{n_2}{2}\right)\left(\frac{n_1}{n_2}\right)^{\frac{n_1}{2}-1} F^{\frac{n_1}{2}-1}}{\Gamma\frac{n_1}{2}\,\Gamma\frac{n_2}{2}} \qquad \boxed{\beta(m,n) = \frac{\Gamma m\,\Gamma n}{\Gamma m+n}}$$

Now limit $n_2 \to \infty$

$$\lim_{n_2\to\infty} \frac{\Gamma\left(\frac{n_1}{2}+\frac{n_2}{2}\right)}{\Gamma\frac{n_2}{2}\cdot(n_2)^{\frac{n_1}{2}}} \times \frac{(n_1)^{\frac{n_1}{2}}}{\Gamma\frac{n_1}{2}}$$

or, $$\lim_{n_2\to\infty} \frac{\Gamma\left(\frac{n_1}{2}+\frac{n_2}{2}\right)}{\Gamma\frac{n_2}{2}} \times \frac{1}{(n_2)^{\frac{n_1}{2}}} \times \frac{(n_1)^{\frac{n_1}{2}}}{\Gamma\frac{n_1}{2}}$$

$$\boxed{\lim_{n\to\infty} \frac{\Gamma n+k}{\Gamma n} \to n^k}$$

$$= \frac{\left(\frac{n_2}{2}\right)^{\frac{n_1}{2}}}{(n_2)^{\frac{n_1}{2}}} \times \frac{(n_1)^{\frac{n_1}{2}}}{\Gamma\frac{n_1}{2}}$$

$$= \frac{(n_1)^{\frac{n_1}{2}}}{2^{\frac{n_1}{2}} \Gamma \frac{n_1}{2}}$$

Again,

$$\underset{n_2 \to \infty}{Lt} \left(1 + \frac{n_1}{n_2} F\right)^{\frac{n_1}{2} + \frac{n_2}{2}} = \underset{n_2 \to \infty}{Lt} \left(1 + \frac{n_1}{n_2} F\right)^{\frac{n_1}{2}}$$

$$\times \underset{n_2 \to \infty}{Lt} \left(1 + \frac{n_1}{n_2} F\right)^{\frac{n_2}{2}}$$

$$= \underset{n_2 \to \infty}{Lt} \left(1 + \frac{n_1}{n_2} F\right)^{\frac{n_1}{2}} \underset{n_2 \to \infty}{Lt} \left\{ \left(1 + \frac{n_1}{n_2} F\right)^{n_2} \right\}^{\frac{1}{2}}$$

$$\boxed{\left\{\left(1 + \frac{a}{n}\right)^n\right\}^{\frac{1}{2}} = e^{\frac{a}{2}}}$$

1.04.2017

$$\lim_{n\to\infty} \left\{\left(1+\frac{a}{n}\right)^{n}\right\}^{1/2} = e^{a/2}$$

$$\lim_{n\to\infty} \left\{1+\frac{n_1F}{n_2}\right\}^{\frac{n_1}{2}} = 1$$

$$\lim_{n_2\to\infty} \left[\left(1+\frac{n_1F}{n_2}\right)^{n_2}\right]^{1/2} \times \lim_{n_2\to\infty} \left[\left(1+\frac{n_1F}{n_2}\right)^{n_1/2}\right]$$

$$= e^{\frac{n_1F}{2}}$$

Putting all these values in equation (1)

$$f(F) = \frac{\left(\frac{n_1}{\cancel{2}}\right)^{n_1/2} F^{\frac{n_1}{2}-1}}{2^{\frac{n_1}{2}} \cdot \Gamma\frac{n_1}{2}\, e^{\frac{n_1F}{2}}}$$

Again, $\lim$, $n_1F = x^2$

$$\Rightarrow F = \frac{x^2}{n_1}$$

$$\Rightarrow \frac{dF}{dx^2} = \frac{1}{n_1}$$

$$\therefore f(F) = \frac{\left(\frac{n_1}{\cancel{2}}\right)^{\frac{n_1}{2}} \left(\frac{x^2}{n_1}\right)^{\frac{n_1}{2}-1}}{2^{\frac{n_1}{2}}\, \Gamma\frac{n_1}{2}\, e^{\frac{x^2}{2}}}$$

$$= \frac{\left(\frac{n_1}{\cancel{2}}\right)^{\frac{n_1}{2}} \times (n_1^{-1})^{\frac{n_1}{2}-1} (x^2)^{\frac{n_1}{2}-1} (n_1^{-1}) e^{-\frac{x^2}{2}}}{2^{\frac{n_1}{2}}\, \Gamma\frac{n_1}{2}}$$

$$f(\chi^2) = \frac{(n_1)^{\frac{n_1}{2} - \frac{n_1}{2}+1-1} (\chi^2)^{\frac{n_1}{2}-1} e^{-\frac{\chi^2}{2}}}{2^{\frac{n_1}{2}} \Gamma\frac{n_1}{2}}$$

$$= \frac{1}{2^{\frac{n_1}{2}} \Gamma\frac{n_1}{2}} e^{-\frac{\chi^2}{2}} (\chi^2)^{\frac{n_1}{2}-1} \quad 0 < \chi^2 < \infty$$

Which is the form of the ~~χ^2 formula with~~ χ^2- distribution with n_1 degree of freedom

* Relation between F- distribution and β of second ~~prime~~ kind

$$\boxed{\int_0^\infty \frac{x^{\frac{n}{2}-1}}{(1+x)^{\frac{n+m}{2}}} = \beta(m,n)}$$

$$\boxed{f(x) = \frac{1}{\beta(m,n)} \cdot \frac{x^{\frac{n}{2}-1}}{(1+x)^{\frac{n+m}{2}}}}$$

We have, the pdf of β of second kind is,

$$\cancel{f(x) = \frac{x^{\frac{n_1}{2}-1}}{}}$$

$$f(x) = \frac{1}{\beta(n_1, n_2)} \times \frac{x^{\frac{n_1}{2}-1}}{(1+x)^{\frac{n_1+n_2}{2}}} \quad 0 < x < \infty$$

Let, $\frac{n_1}{n_2} F = x$ | $x = 0, F = 0$; $x = \infty, F = \infty$

$$\Rightarrow \frac{dx}{dF} = \frac{n_1}{n_2}$$

$$\therefore f(x) = \frac{1}{\beta(n_1, n_2)} \times \frac{\left(\frac{n_1}{n_2} F\right)^{\frac{n_1}{2}-1}}{\left(1+\frac{n_1}{n_2}F\right)^{\frac{n_1+n_2}{2}}} \times \frac{n_1}{n_2}$$

$$= \frac{1}{\beta\left(\frac{n_1}{2}, \frac{n_2}{2}\right)} \times \frac{\left(\frac{n_1}{n_2}\right)^{\frac{n_1}{2}-1} F^{\frac{n_1}{2}-1} \left(\frac{n_1}{n_2}\right)^{1}}{\left(1+\frac{n_1}{n_2}F\right)^{\frac{n_1+n_2}{2}}}$$

$$\cancel{\frac{\Gamma n_1 \, \Gamma n_2}{\Gamma n_1+n_2}}$$

$$f(F) = \frac{\left(\frac{n_1}{n_2}\right)^{\frac{n_1}{2}} F^{\frac{n_1}{2}-1}}{\beta\left(\frac{n_1}{2}, \frac{n_2}{2}\right)\left(1+\frac{n_1}{n_2}F\right)^{\frac{n_1+n_2}{2}}}$$

$J = \frac{dx}{dt} = \frac{n_1}{n_2}$

$f(F) = f(x)|J|$

* If x_1 and x_2 are two independent random variable ~~be~~ having common density function,

$$f(x) = e^{-x} \quad 0 < x < \infty$$

show that $u = \frac{x_1}{x_2}$ has an ~~t~~ F-distribution.

$$f(x_1) = e^{-x_1}$$
$$f(x_2) = e^{-x_2}$$

The joint distribution of x_1 and x_2 is,

$$f(x_1, x_2) = f(x_1) \cdot f(x_2) = e^{-x_1} \cdot e^{-x_2}$$
$$= e^{-(x_1+x_2)}$$

Let $u = \frac{x_1}{x_2}$

$v = x_1 + x_2$

or, $x_1 = ux_2$

$x_2 = v - x_1$

or, $v = ux_2 + x_2 = (u+1)x_2$

$\Rightarrow x_2 = \frac{v}{u+1}$

$$x_1 = ux_2 = u \cdot \frac{v}{1+u}$$
$$= \frac{uv}{1+u}$$

$$|J| = \left| \frac{\delta(x_1, x_2)}{\delta(u, v)} \right|$$

$$= \begin{vmatrix} \frac{\delta x_1}{\delta u} & \frac{\delta x_1}{\delta v} \\ \frac{\delta x_2}{\delta u} & \frac{\delta x_2}{\delta v} \end{vmatrix}$$

$$= \begin{vmatrix} \dfrac{v(1+u) - uv\cdot 1}{(1+u)^2} & \dfrac{u}{1+u} \\ -\dfrac{v}{(1+u)^2} & \dfrac{1}{1+u} \end{vmatrix}$$

$$= \begin{vmatrix} \dfrac{v + \cancel{uv} - \cancel{uv}}{(1+u)^2} & \dfrac{u}{1+u} \\ -\dfrac{v}{(1+u)^2} & \dfrac{1}{1+u} \end{vmatrix}$$

$$= \frac{v}{(1+u)^3} + \frac{uv}{(1+u)^3}$$

$$= \frac{v+uv}{(1+u)^3} = \frac{v(1+u)}{(1+u)^3} = \frac{v}{(1+u)^2}$$

$$f(x_1, x_2) = e^{-(x_1+x_2)}$$

$$g(u,v) = f(x_1, x_2)\,|J|$$

$$= e^{-v} \frac{v}{(1+u)^2}$$

$0 < u < \infty$

$0 < v < \infty$

$$f(u) = \int_0^\infty e^{-v} \frac{v}{(1+u)^2}\, dv$$

$$= \frac{1}{(1+u)^2} \int_0^\infty v \cdot e^{-v}\, dv$$

$$= \frac{1}{(1+u)^{2}} \int_{0}^{\infty} e^{-v} v^{2-1} \, dv$$

$$= \frac{1}{(1+u)^{2}} \, \Gamma 2$$

$$= \frac{1}{(1+u)^{2}}$$

$$= \frac{1}{(1+u)^{2}}$$

$$= \frac{\left(\frac{2}{2}\right)^{\frac{2}{2}-1} \, u^{\frac{2}{2}-1}}{\beta\left(\frac{2}{2}, \frac{2}{2}\right)\left(1+\frac{2}{2}u\right)^{\frac{2+2}{2}}}$$

which is follow F-ditribution with 2 and 2 degrees of freedom.

Date: 04.04.2017

$$\beta(m,n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{\cancel{m}\, n+m}} dx$$

$$\beta\left(\frac{m}{2}, \frac{n}{2}\right) = \int_0^\infty \frac{x^{\frac{m}{2}-1}}{(1+x)^{\frac{n+m}{2}}} dx$$

We have,

$$f(x) = \frac{1}{\beta\left(\frac{n_1}{2}, \frac{n_2}{2}\right)} \frac{(x)^{\frac{n_1}{2}-1}}{(1+x)^{\frac{n_1+n_2}{2}}}$$

Properties of F- distribution:

(i) ~~A~~ It has a continuous type of distribution and it range 0 to ∞.

(ii) The mean and varience of F- distribution are $\frac{n_2}{n_2-1}$ if $n_2 > 2$ and $\frac{2n_2^2(n_1+n_2-2)}{n_1(n_2-2)^2(n_2-4)}$ if $n_2 > 4$.

(iii) The mode of the F-distribution is $\left(\frac{n_2}{n_2+2}\right)\left(\frac{n_1-2}{n_1}\right)$

(iv) If n_1 and n_2 are very large which are tends to infinite then F-distribution tends to normal distribution.

(v) The distribution is positive skewed and leptokartic.

(vi) If $F \sim F(n_1, n_2)$ then $\frac{n_1}{n_2} F \sim \beta_2\left(\frac{n_1}{2}, \frac{n_2}{2}\right)$

(vii) If $F \sim F(n_1, n_2)$ then $\frac{1}{F} \sim F(n_1, n_2)$

⊞ The application ~~of~~ or uses of F-distribution:

(i) It is used for testing the equility of two population variance $[H_0: \sigma_1^2 = \sigma_2^2]$

(ii) It is used for testing the significance of an observed multiple correlation coefficient.

(iii) It is used for testing the significance of sample correlation ratio. $[\eta^2_{yx}]$

(iv) It is used for testing linearity of regration.

(v) F-distribution is used to test the equility of several means

Date: 8.04.20.

Estimation

Statistical Inference:

[population থেকে sample draw করে parameter estimate করা হয়]

It is the science of drawing valid conclusion from the sample data about the population, from which the sample has been drawn. ~~If~~ i It includes the ~~poe~~ procedures of estimating population parameter and testing wheather a tentative statement about a population parameter is supported by evidance from sample.

Statistical estimation:

The methods of making judgement about a population parameter are called statistical estimation. There are two types of statistical estimation — (i) point estimation
(ii) Interval estimation.

Point estimation: point estimation involves the use of sample data to calculate a single value which is to be ~~such as~~ serve as best estimate of an unknown population paratmeter.

It from the sample value a single value is calculated as an estimate of the population parameter the procedure is termed as paint estimation.

Method of point estimation:

1. Method of Maximum likelihood Estimation (MLE)
2. Method of moments
3. Method of least square
4. Method of minimum varience
5. Method of minimum chi-squre
6. Method of Inverse probability.

criteria of good estimator:

1. Unbaisedness
2. Consistency
3. Efficiency
4. sufficieny.

⇒ Parameter
Statistic
Estimator
Estimate

Parameter: For drawing valid inference about the population, we in practice deal with sample and obtained the ~~sta~~ estimate of the population characteristics. The unknown characteristic of the population usually known as parameter.

A statistic is generally a function of a set of sample value. clearly statistic is a random variable.

Estimator: If any statistic is used to estimate and unknown paramete $\theta / \lambda /$ $\mu / \sigma^2 / \gamma$ of the distribution, it is ~~alled~~ called an estimator.

Estimate: Any particular value of the estimator say $T_n = T(x_1, x_2 \ldots x_n)$ is called an estimate of θ.

1. Find the momentary generation function of χ^2 distribution.

Ans: $M_{\chi^2}(t) = E(e^{t\chi^2})$

$$= \int_0^\infty e^{t\chi^2} f(\chi^2)\, d\chi^2$$

$$= \int_0^\infty e^{t\chi^2} \frac{1}{2^{n/2}\Gamma\frac{n}{2}} e^{-\chi^2/2} (\chi^2)^{\frac{n}{2}-1} d\chi^2$$

$$= \frac{1}{2^{\frac{n}{2}}\Gamma\frac{n}{2}} \int_0^\infty e^{-\frac{\chi^2}{2}(1-2t)} (\chi^2)^{\frac{n}{2}-1} d\chi^2$$

$$= \frac{1}{2^{\frac{n}{2}}\cancel{\Gamma\frac{n}{2}}} \times \frac{\cancel{\Gamma\frac{n}{2}}}{\left(\frac{1-2t}{2}\right)^{n/2}}$$

$$= (1-2t)^{-n/2}$$

$$= (1-2t)^{-\frac{n}{2}}$$

2. Find the cumulative generation function and show that χ^2-distribution is always positive skewness and leptokurtic.

Ans: $K_{x^2}(t) = \log M_{x^2}(t)$

$$= \log (1-2t)^{-\frac{n}{2}}$$

$$= -\frac{n}{2} \log \cancel{2t}\, 1(-2t)$$

$$= -\frac{n}{2}\left[-2t - \frac{(2t)^2}{2} - \frac{(2t)^3}{3} - \frac{(2t)^4}{4} - \cdots \infty\right]$$

$$= nt + 2n\cdot\frac{t^2}{2} + 4n\frac{t^3}{3} + 8n\frac{t^4}{4} + \cdots \infty$$

coefficien of $K_1 = \frac{t}{1!} = n = \mu_1' = \cancel{\alpha}$ mean

" " $K_2 = \frac{t^2}{2!} = 2n = \mu_2 =$ varience

" " $K_3 = \frac{t^3}{3!} = 8n = \mu_3$

" " $K_4 = \frac{t^4}{4!} = 48n = \mu_4$

$$\therefore \mu_4 = K_4 + 3K_2^2$$

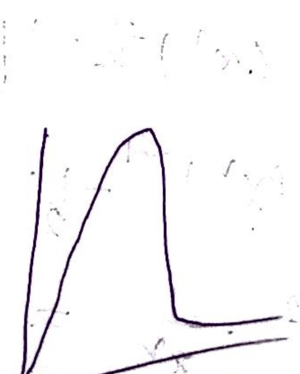
$$= 48n + 3\cdot 4n^2$$

$$= 48n + 12n^2$$

$$\beta_1 = \frac{\cancel{\mu_2^3}}{\cancel{\mu_3^2}}\ \frac{\mu_3^2}{\mu_2^3} = \frac{(8n)^2}{(2n)^3} = \frac{64n^2}{8n^3} = \frac{8}{n}$$

$$\beta_2 = \frac{\mu_4}{\mu_2^2} = \frac{48n+12n^2}{4n^2}$$

$$= 3 + \frac{12}{n}$$



$\beta_1 > 0$, $\beta_2 > 3$, positive skewed & leptokartic

3. Mode of χ^2-distribution:

We know that,

$$f(\chi^2) = \frac{1}{2^{\frac{n}{2}}\sqrt{\frac{n}{2}}} e^{-\frac{\chi^2}{2}} (\chi^2)^{\frac{n}{2}-1}$$

Mode is that value of χ^2-distribution which satisfy the equation,

$$f'(\chi^2) = 0 \quad \text{and} \quad f''(\chi^2) < 0$$

Now,

$$f'(\chi^2) = \frac{\delta}{\delta \chi^2}\left[\frac{1}{2^{n/2}\sqrt{\frac{n}{2}}} e^{-\frac{\chi^2}{2}} (\chi^2)^{\frac{n}{2}-1}\right]$$

$$= \frac{1}{2^{\frac{n}{2}}\sqrt{\frac{n}{2}}}\left[e^{-\frac{\chi^2}{2}}\left(\frac{n}{2}-1\right)(\chi^2)^{\frac{n}{2}-1-1} + (\chi^2)^{\frac{n}{2}-1} e^{-\frac{\chi^2}{2}}\left(-\frac{1}{2}\right)\right]$$

$$= \frac{1}{2^{\frac{n}{2}}\sqrt{\frac{n}{2}}} e^{-\frac{\chi^2}{2}} (\chi^2)^{\frac{n}{2}-1}\left[\left(\frac{n}{2}-1\right)(\chi^2)^{-1} - \frac{1}{2}\right]$$

puts $f'(\chi^2) = 0$

$$e^{-\frac{\chi^2}{2}} (\chi^2)^{\frac{n}{2}-1}\left[\left(\frac{n}{2}-1\right)(\chi^2)^{-1} - \frac{1}{2}\right] = 0$$

$$\left(\frac{n}{2}-1\right)(\chi^2)^{-1} - \frac{1}{2} = 0$$

$$\Rightarrow \frac{n-2}{2} \cdot \frac{1}{\chi^2} = \frac{1}{2}$$

$$\therefore \chi^2 = n-2$$

It is easy to show that at the point $x^2 = n-2$, $f''(x^2) < 0$

Hence the mode of the x^2-distribution is $(n-2)$ ~~with~~ for n degree of freedom.

4 State and prove x^2-variate.

Ans: The sum of independent x^2-variate is also a x^2-~~vat~~ variate. Let $x_1^2, x_2^2 \dots x_k^2$ ~~is~~ be the k independent vat variate with $n_1, n_2, \dots n_k$ degree of freedom the sum $\sum_{i=1}^{k} x_i^2$ is a x^2-variat with $\sum_{i=1}^{k} n_i$ degree of freedom.

We know the momentary generation function of x^2-distribution is

$$M_{x_i^2}(t) = (1-\alpha)^{-\frac{n_i}{2}}$$

since $x_1^2, x_2^2 \dots x_k^2$ be the k independent variate with $n_1, n_2 \dots n_k$ degree of freedom.

The m.g.f of $\sum_{i=1}^{k} x_i^2$ is written as

$$M_{\sum_{i=1}^{k} x_i^2}(t) = \cancel{(1-2t)^{-\frac{n_i}{2}}}$$

$$= \prod_{i=1}^{k} M_{x_i^2}(t)$$

$$= M_{x_1^2}(t) \cdot M_{x_2^2}(t) \cdots M_{x_k^2}(t)$$

$$= (1-2t)^{\frac{n_1}{2}} \cdot (1-2t)^{\frac{n_2}{2}} \cdots (1-2t)^{\frac{n_k}{2}}$$

$$= (1-2t)^{\left(\frac{n_1}{2} + \frac{n_2}{2} + \cdots + \frac{n_k}{2}\right)}$$

$$= (1-2t)^{-\frac{1}{2}\sum_{i=1}^{k} x_i^2}$$

$$= (1-2t)^{-\frac{n}{2}}$$

which is momintary function of x^2 variate with $n_1, n_2 \ldots n_k$ degee of freedom so momentary functon of $\sum_{i=1}^{k} x_i^2$ is also a x^2 variate with ~~n d f~~ $\sum_{i=1} n_i$

$$f(t) = \frac{1}{\sqrt{n}\,\beta\left(\frac{1}{2},\frac{n}{2}\right)\left(1+\frac{t^2}{n}\right)^{\frac{n+1}{2}}}$$

$$\mu'_{2r} = n^r \frac{\Gamma\left(\frac{n}{2}-r\right)\Gamma\left(r+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{n}{2}\right)}$$

$r=1$

$$\mu'_2 = n \frac{\Gamma\left(\frac{n}{2}-1\right)\Gamma\left(1+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{n}{2}\right)}$$

$$= \frac{n\,\cancel{\Gamma\left(\frac{n}{2}-1\right)}\left(\frac{3}{2}-1\right)\Gamma\left(\frac{3}{2}-1\right)}{\Gamma\left(\frac{1}{2}\right)\left(\frac{n}{2}-1\right)\cancel{\Gamma\left(\frac{n}{2}-1\right)}}$$

$$= \frac{n}{\cancel{2}} \cdot \frac{\cancel{2}}{n-2}$$

$$= \frac{n}{n-2}$$

$r=2$

$$\mu'_4 = n^2 \frac{\Gamma\left(\frac{n}{2}-2\right)\Gamma\left(2+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{n}{2}\right)}$$

$$= \frac{n^2\,\cancel{\Gamma\left(\frac{n}{2}-2\right)}\left(2+\frac{1}{2}-1\right)\left(2+\frac{1}{2}-1-1\right)\cancel{\Gamma\left(\frac{1}{2}\right)}}{\cancel{\Gamma\left(\frac{1}{2}\right)}\left(\frac{n}{2}-1\right)\left(\frac{n}{2}-2\right)\cancel{\Gamma\left(\frac{n}{2}-2\right)}}$$

$$= \frac{n^2 \cdot \frac{3}{2}\cdot\frac{1}{2}}{\frac{(n-2)(n-4)}{4}} = \frac{3n^2}{(n-2)(n-4)}$$

$\mu_1 = 0$

$\mu_2 = \mu_2' - (\mu_1')^2$

$\mu_3 = \mu_3' - 2\mu_2'\mu_1' + (\mu_1')^3$

$\mu_4 = \mu_4' - 4\mu_3'\mu_2'^2 + 6\mu_3'\mu_1' - 3\mu_1^4$

for givo

Date : 29.04.2017

Section A

* Show that in estimating for the parameter "μ" is the poision distribution, of

$$f(x,\mu) = \frac{e^{-\mu}, \mu^{x}}{x!} \quad ; \; x = 0,1,2 \ldots \infty.$$

$\bar{x}$ is a sufficient statistic for μ.

■ likelihood function : or density (joint probability or function)

(or pdf)

θ, or μ

$p(\theta$, $p(\mu)$

Let $x_1, x_2 \ldots x_n$ be a random sample of size n from a population with density function $f(x,\theta)$ then the likelihood function of the sample values $x_1, x_2 \ldots x_n$ usually denoted by $L(\theta)$. or $L(x_1, x_2 \ldots x_n; \theta)$ or $(\underset{\sim}{x}, \theta)$ is their joint density function given by

$$L(\theta) = f(x_1,\theta)\, f(x_2,\theta) \ldots f(x_n,\theta)$$

Now the likelihood function is defined as,

$$L(\mu) = f(x_1,\mu)\, f(x_2,\mu) \ldots f(x_n, \mu)$$

$$= \frac{e^{-\mu}\mu^{x_1}}{x_1!} \cdot \frac{e^{-\mu}\mu^{x_2}}{x_2!} \cdots \frac{e^{-\mu}\mu^{x_n}}{x_n!}$$

~~one~~

$$= \frac{e^{-n\mu}\, \mu^{x_1+x_2+\cdots+x_n}}{\prod_{n=1}^{n} x_i!}$$

$$= \frac{e^{-n\mu}\, \mu^{\sum_{i=1}^{n} x_i}}{\prod_{i=1}^{n} x_i!}$$

$$= \frac{e^{-n\mu}\, \mu^{n\bar{x}}}{\prod_{i=1}^{n} x_i!}$$

$$= \frac{e^{-n\mu}\, \mu^{n\bar{x}}}{(n\bar{x})!} \times \frac{(n\bar{x})!}{\prod_{i=1}^{n} x_i!}$$

$$= g(\bar{x}, \mu)\, h(x_1, x_2 \ldots x_n)$$

where

$$g(\bar{x}, \mu) = \frac{e^{-n\mu}\, \mu^{n\bar{x}}}{(n\bar{x})!}$$

$$h(x_1, x_2 \ldots x_n) = \frac{(n\bar{x})!}{\prod_{i=1}^{n} x_i!}$$

$\bar{x}$ is a sufficient ~~estimate~~ statistic for μ.

* Let $x_1, x_2, \ldots, x_n$ be a random sample of size 'n' drawn from normal population with mean 'θ' and varience σ^2. show that,

$$t = \frac{\sum x_i}{n} = \bar{x} \text{ is a sufficient statistic for } \theta$$

The pdf is,

$$f(x, \theta, \sigma^2) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2}\left(\frac{x-\theta}{\sigma}\right)^2} \quad -\infty < x < \infty$$

Now the likelihood function is

$$L = L(\theta, \sigma^2) = f(x_1, \theta, \sigma^2) f(x_2 \theta, \sigma^2) \ldots f(x_n \theta, \sigma^2)$$

$$= \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\left(\frac{x_1-\theta}{\sigma}\right)^2} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\left(\frac{x_2-\theta}{\sigma}\right)} \ldots \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\left(\frac{x_n-\theta}{\sigma}\right)}$$

$$= \left(\frac{1}{\sqrt{2\pi}\sigma}\right)^n e^{-\frac{1}{2}\sum_{i=1}^{n}\left(\frac{x_i-\theta}{\sigma}\right)^2}$$

$$= \left(\frac{1}{\sqrt{2\pi}\sigma}\right)^n e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}\{(x_i-\bar{x})+(\bar{x}-\theta)\}^2}$$

$$= \left(\frac{1}{\sqrt{2\pi}\sigma}\right)^n e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}\{(x_i-\bar{x})^2+n(\bar{x}-\theta)^2\}}$$

$\therefore \sum(x_i - \bar{x}) = 0$

$$\sum(x_i-\bar{x})^2 + \sum(\bar{x}-\theta)^2 + 2\sum(x_i-\bar{x})(\bar{x}-\theta)$$

$$= (\bar{x}-\theta)\underbrace{\sum(x_i-\bar{x})}_{0} + \sum\{(x_i-\bar{x})^2 + n(\bar{x}-\theta)^2$$

$$= \sum(x_i-\bar{x})^2 + n(\bar{x}-\theta)^2$$

$$= \left(\frac{1}{\sqrt{2\pi}\sigma}\right)^n e^{-\frac{1}{2\sigma^2} n(\bar{x}-\theta)^2} e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\bar{x})^2}$$

$$= g(\bar{x}, \theta)\, h(x_1, x_2 \ldots x_n)$$

where $g(\bar{x}, \theta) = e^{-\frac{n}{2\sigma^2}(\bar{x}-\theta)^2}$

$$h(x_1, x_2 \ldots x_n) = \left(\frac{1}{\sqrt{2\pi}\sigma}\right)^n e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\bar{x})^2}$$

$\therefore$ $\bar{x}$ is a sufficient statistic for θ.

Unbiasedness

An estimator $T_n = T(x_1, x_2 \ldots x_n)$ is said to be unbiased estimator of $\tau(\theta)$ if $E(T_n) = \tau(\theta)$ for ~~at~~ all $\theta \in \Omega$

T_n is unbiased estimator of $\tau(\theta)$

Exp-1

Let $x_1, x_2 \ldots x_n$ is a random sample from $N(\mu, 1)$ show that $t = \frac{\sum_{i=1}^{n} x_i^2}{n}$ is an ~~estima~~ unbiased estimator of μ^2+1

$E(x) = \mu$

We know, $v(x) = E(x^2) - \{E(x)\}^2$

$E(x_i^2) = v(x) + \{E(x_i)\}^2 = 1 + \mu^2 = \mu^2 + 1$

$$E(t) = \left[E \frac{\sum_{i=1}^{n} x_i^2}{n} \right]$$

$$= \frac{1}{n} \sum_{i=1}^{n} E(x_i^2)$$

$$= \frac{1}{n} \sum_{i=1}^{n} (\mu^2 + 1)$$

$$= \frac{1}{n} n(\mu^2 + 1)$$

$$= (\mu^2 + 1)$$

Date : 30.04.2017

Principle of maximum likelihood method :

$L(\theta) = (x_1, x_2 \dots x_n ; \theta)$

$$\frac{\delta L(\theta)}{\delta \theta} = 0$$

$$\frac{\delta^2 L(\theta)}{\delta \theta^2} < 0$$

$$\frac{\delta \log L(\theta)}{\delta \theta} = 0$$

$$\frac{\delta^2 \log (\theta)}{\delta^2 \theta} < 0$$

The principle of maximum likelihood is taking that value as the estimater of θ which is maximize $L(\theta)$

$\hat{\theta} = d(x_1, x_2 \dots x_n)$ is the value of θ which maximize $L(\theta)$ then $\hat{\theta}$ is the maximum likelihood estimator of θ.

We shall maximize $L(\theta)$ The maximum likelihood estimator is the solution of the equation,

$$\frac{\delta' L(\theta)}{\delta \theta} = 0 \qquad \frac{\delta^2 L(\theta)}{\delta^2 \theta} < 0$$

Also $L(\theta)$ and $\log L(\theta)$ have maximize at the same value of θ. Sometimes it is similar to take $\log L(\theta)$ instead of $L(\theta)$.

$$\frac{\delta \log L(\theta)}{\delta \theta} = 0 \text{ and } \frac{\delta^2 \log L(\theta)}{\delta^2 \theta} < 0.$$

Q. What is the principle of maximum likelihood method.

Example: Let $x_1 x_2 \dots x_n$ be a random sample from a poision distribution with p.d.f is given by,

$$f(x, \lambda) = \frac{e^{-\lambda} \lambda^x}{x!}; \quad x = 0, 1, 2 \dots \infty$$

(i) Find the maximum likelihood estimator (MLE) of λ

(ii) Show that the ~~em~~ estimator is likelihood.

We have $f(x, \lambda) = \dfrac{e^{-\lambda} \lambda^x}{x!}$.

$x = 0, 1 \dots \infty$

$$f(x, \theta) = \frac{e^{-\theta} \theta^x}{x!}$$

$$f(x, \mu) = \frac{e^{-\mu} \mu^x}{x!}$$

Then the likelihood function $L(\theta) = f(x_1, \lambda) f(x_2, \lambda) \dots f(x_n, \lambda)$

$$= \frac{e^{-\lambda} \lambda^{x_1}}{x_1!} \cdot \frac{e^{-\lambda} \lambda^{x_2}}{x_2!} \dots \frac{e^{-\lambda} \lambda^{x_n}}{x_n!}$$

$$= \frac{e^{-n\lambda} \cdot \lambda^{\sum_{i=1}^{n} x_i}}{\prod_{i=1}^{n} x_i!}$$

Taking log on the both sides, we get,

$$\log L(\lambda) = -n\lambda + \frac{\sum x_i \log \lambda}{\log\left(\prod_{i=1}^{n} x_i!\right)}$$

$$= -\lambda\lambda + \sum_{i=1}^{n} x_i \log \lambda - \log\left(\prod_{i=1}^{n} x_i!\right)$$

Now differentiating log $L(\lambda)$ w.r.t. λ and set equal to zero we get,

$$\frac{\delta \log L(\lambda)}{\delta(\lambda)} = 0$$

$$\Rightarrow -n + \frac{\sum_{i=1}^{n} x_i}{\lambda} - 0 = 0$$

$$\Rightarrow -\lambda\lambda + \sum_{i=1}^{n} x_i = 0$$

$$\Rightarrow n\lambda = \sum_{i=1}^{n} x_i$$

$$\Rightarrow \lambda = \frac{\sum_{i=1}^{n} x_i}{n}$$

$$\Rightarrow \hat{\lambda} = \bar{x}$$

Hence the MLE of λ is $\bar{x}$

(ii) $\hat{\lambda} = \bar{x}$ $E(T_n) = \tau(\theta)$

Taking expectation on both side

$$E(\hat{\lambda}) = E(\bar{x}) = E\left(\frac{\sum_{i=1}^{n} x_i}{n}\right)$$

$$= \frac{1}{n} \sum_{i=1}^{n} E(x) = \frac{1}{n} \sum_{i=1}^{n} \lambda$$

$$= \frac{1}{n} \cdot n\lambda = \lambda$$

$\therefore \hat{\lambda}$ is an unbiased estimator of λ.

<u>Example:</u>

Let $x_1, x_2 \ldots x_n$ be a random sample ~~drawn~~ from the distribution having p.d.f is,

$$f(x, p) = {}^{n}c_{x}\, p^{x} q^{n-x} \; ; x = 0, 1 \ldots n$$

(i) Find the MLE of p

(ii) Show that the estimator is likelihood.

We have $f(x, p) = {}^{n}c_{x}\, p^{x} q^{n-x}$; $x = 0, 1 \ldots n$

Now the likelihood function is;

$$L(p) = f(x_1, p)\, f(x_2, p) \ldots f(x_n, p)$$

$$= {}^{n}c_{x_1} p^{x_1} q^{n-x_1} \cdot {}^{n}c_{x_2} p^{x_2} q^{n-x_2} \ldots {}^{n}c_{x_n} p^{x_n} q^{n-x_n}$$

$$= \prod_{i=1}^{n} {}^{n}c_{x_i}\, p^{\sum_{i=1}^{n} x_i} q^{n^2 - \sum_{i=1}^{n} x_i}$$

Taking log on both sides,

$$\log L(p) = \log\left(\prod_{i=1}^{n} {}^{n}c_{x_i}\right) + \sum_{i=1}^{n} x_i \log p + \left(n^2 - \sum_{i=1}^{n} x_i\right) \log q$$

$= \log(\prod {}^{n}c_{x_i}) + \sum x_i \log p + (n - \sum x_i) \log(1-p)$ | $p+q=1$, $\therefore q = 1-p$

Now differentiating both side ~~by p~~ w.r.t. p.

$\frac{\delta \log L(p)}{\delta p} = 0$

$\Rightarrow \frac{\sum_{i=1}^{n} x_i}{p} + \frac{n^2 - \sum_{i=1}^{n} x_i}{1-p} \times (-1) = 0$

$\Rightarrow (1-p) \sum_{i=1}^{n} x_i - p(n^2 - \sum_{i=1}^{n} x_i) = 0$

$\Rightarrow \sum_{i=1}^{n} x_i - p \cancel{\sum_{i=1}^{n} x_i} - pn^2 + \cancel{p \sum_{i=1}^{n} x_i} = 0$

$\Rightarrow \sum_{i=1}^{n} x_i - n^2 p = 0$

$\Rightarrow \hat{p} = \frac{\sum_{i=1}^{n} x_i}{n^2}$

$= \frac{1}{n} \cdot \frac{\sum_{i=1}^{n} x_i}{n}$

$= \frac{\bar{x}}{n}$

Hence $\frac{\bar{x}}{n}$ is the MLE of $\hat{p}$.

(11) Taking expectation on both side,

$$E(\hat{P}) = E\left(\frac{\bar{x}}{n}\right)$$

$$= E\left(\frac{\sum_{i=1}^{n} x_i}{n^2}\right)$$

$$= \frac{1}{n^2} \sum_{i=1}^{n} E(x_i)$$

$$= \frac{1}{n^2} \sum_{i=1}^{n} np$$

$$= \frac{1}{n^2} n \cdot n \cdot p$$

$$= p$$

$\mu = E(x) = np$

varience $v(x) = npq$

which is the unbiased estimator of $\hat{P}$

Date: 02.05.2017

Example:

Let $x_1, x_2, \dots, x_n$ be a random sample of size n from normal distribution with p.d.f is,

$$f(x, \mu, \sigma^2) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \qquad \begin{array}{l} -\infty < x < \infty \\ \sigma^2 > 0 \\ -\infty < \mu < \infty \end{array}$$

Find the MLE of μ and σ^2.

Solution:

We have $f(x, \mu, \sigma^2) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$

Then the likelihood function is,

$$L(\mu, \sigma^2) = \prod_{i=1}^{n} f(x_i, \mu, \sigma^2)$$

$$= f(x_1, \mu, \sigma^2) f(x_2, \mu, \sigma^2) \dots f(x_n, \mu, \sigma^2)$$

$$= \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2\sigma^2}(x_i-\mu)^2}$$

$$= \cancel{\prod_{i=1}^{n}} \left(\frac{1}{2\pi\sigma^2}\right)^{\frac{n}{2}} e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\mu)^2}$$

Taking log on both sides.

$$\log L(\mu, \sigma^2) = \frac{n}{2}\log\left(\frac{1}{2\pi\sigma^2}\right) - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\mu)^2$$

$$= -\frac{n}{2}\log(2\pi\sigma^2) - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\mu)^2$$

$$= -\frac{n}{2}\log(2\pi) - \frac{n}{2}\log(\sigma^2) - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\mu)^2$$

Differentiating $\log L(\mu, \sigma^2)$ w.r.to 'μ' & σ^2 and set ~~to zero.~~ equal ~~zero.~~ to zero.

$$\frac{\delta \log L(\mu, \sigma^2)}{\delta \mu} = 0$$

$$\Rightarrow -0-0-\frac{1}{2\sigma^2}\, 2\sum_{i=1}^{n}(x_i-\mu)^{2-1} \times (-1) = 0$$

$$\Rightarrow \frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\mu) = 0$$

$$\Rightarrow \sum_{i=1}^{n}(x_i-\mu) = 0$$

$$\Rightarrow \sum_{i=1}^{n}x_i - n\mu = 0$$

$$\Rightarrow n\mu = \sum_{i=1}^{n}x_i$$

$$\hat{\mu} = \bar{x}$$

Hence the MLE of $\hat{\mu}$ is equal to the sample mean $\bar{x}$.

Again differentiating w.r.t σ^2

$$\frac{\delta \log L(\mu, \sigma^2)}{\delta \sigma^2} = 0$$

$$\Rightarrow 0 - \frac{n}{2}\frac{1}{\sigma^2} - (-1)\frac{1}{2\sigma^4}\sum_{i=1}^{n}(x_i-\mu)^2$$

$$\Rightarrow \frac{1}{2\sigma^4}\sum_{i=1}^{n}(x_i-\mu)^2 = \frac{n}{2\sigma^2}$$

$\Rightarrow \sum (x_i - \mu)^2 = n\sigma^2$

$\Rightarrow \hat{\sigma}^2 = \dfrac{\sum (x_i - \hat{\mu})^2}{n}$

Hence the MLE of $\hat{\sigma}^2$ is $\dfrac{\sum (x_i - \hat{\mu})^2}{n}$

* Binomeal, poision, normal.

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Properties of the MLE:

1. MLE are generally not unbiased
2. MLE are consistent
3. MLE are most efficient
4. MLE are sufficient
5. MLE is unique under regularity condition.
6. MLE have the invariance property.

Efficiency | most efficient estimator

$$E(\text{efficiency}) = \frac{V(T_1)}{V(T_2)} = 1 \ (\text{max})$$

most of time

$E < 1$.

~~Consistenty~~ Consistent estimator.

$$T_n \xrightarrow[p]{n \to \infty} \theta \doteq \infty$$

$T_n \doteq T(x_1, x_2 \ldots x_n)$

Consistency:

An estimator $T_n = T(x_1, x_2 \ldots x_n)$ based on a random sample of size n is said to be consistent estimator of $\gamma(\theta)$; $\theta \in \Omega$ the parameter space if T_n converse to $\gamma(\theta)$ in probality.

i.e $T_n \xrightarrow{P} \gamma(\theta)$ as $n \to \infty$.

Efficiency:

If T_1 is the most efficient estimator with varience $v(T_1)$ and T_2 is any other estimator with varience $v(T_2)$ then the efficiency 'E' of T_2 is defined as

$$E = \frac{v(T_1)}{v(T_2)}$$

obviously 'E' can not exceed unity.

Statistics – Sec A   Date: 15.04.2017

Point Estimator

Let $x_1, x_2 \dots x_n$ is a sample from a density function $f(x, \theta)$ or $f(x/\theta)$ where 'θ' is unknown fixed value for which can assume any value is one-dimensional real parameter space Ω.

Let 't' be a function of $x_1, x_2 \dots x_n$. So that 't' is a statistic and hence a random variable. If 't' is said used to estimate 'θ'. Then 't' is called a point estimate of 'θ'.

$$f(x, \theta) = \theta^x (1-\theta)^{x-1}$$

$$t = f(x_1, x_2 \dots x_n)$$

Sufficient Statistic

Let $f(x, \theta)$ be a density of a random variable 'x' where 'θ' is unknown fixed parameter and $\theta \in \Omega$. Let $x_1, x_2 \dots x_n$ be a random sample from this distribution/ density function. Let 't' and t' are two statistics such that t' is not a function of 't'. If the conditional distribution of t' for given t be independent of 'θ'. Then 't' is called a sufficient statistic for 'θ'.

Let the joint distribution of t and t′ be $h(t, t'/\theta)$ as follows

$$h(t, t'/\theta) = g(t, \theta) \cdot h(t'/t)$$

$$\text{or } h(t'/t) = \frac{h(t, t'/\theta)}{g(t, \theta)}$$

Where $g(t, \theta)$ is the marginal distribution of 't' for fixed 'θ' and $h(t'/t)$ is the conditional distribution of t′ given t.

A statistic $t_n = t(x_1, x_2 \ldots x_n)$ is a sufficient estimator of parameter 'θ' if and only if the likelihood function (joint p.d.f. of the sample) can be expressed as,

$$L = \prod_{i=1}^{n} f(x_i, \theta)$$

$$= g(t_n, \theta)\, h(x_1, x_2 \ldots x_n)$$

Where $g(t_n, \theta)$ is the p.d.f of the statistic t_n and $h(x_1, x_2 \ldots x_n)$ is the function of sample observation only.

Example:
Let x_1 and x_2 be independent distributed as $f(x;\theta) = \theta x^{\theta-1}$, $x = 0, 1$, $0 < \theta < 1$. Show that $\prod_{i=1}^{2} x_i$ is a sufficient statistic for θ!

Now the likelihood function

$$L = \prod_{i=1}^{2} f(x_i, \theta) = \theta x_1^{\theta-1} \theta x_2^{\theta-2}$$

$$= \theta^2 \left(\prod_{i=1}^{2} x_i\right)^{\theta-1}$$

$$= \frac{\theta^2 \left(\prod_{i=1}^{2} x_i\right)^{\theta}}{\prod_{i=1}^{2} x_i}$$

$$= \theta^2 \left(\prod_{i=1}^{2} x_i\right)^{\theta} \times \frac{1}{\prod_{i=1}^{2} x_i}$$

$$= g\left(\prod_{i=1}^{2} x_i, \theta\right) \cdot h(x_1, x_2 \ldots x_n)$$

where $g\left(\prod_{i=1}^{2} x_i, \theta\right) = \theta^2 \left(\prod_{i=1}^{2} x_i\right)^{\theta}$; $h(x_1, x_2) = \frac{1}{\prod_{i=1}^{2} x_i}$

$\therefore \prod_{i=1}^{2} x_i$ is a sufficient statistic for 'θ'.

Example:

Let $x_1, x_2, \ldots, x_n$ be a random sample from the density $f(x, \theta) = \theta^{x}(1-\theta)^{1-x}$

$x = 0, 1, \quad 0 < \theta < 1$

Then show that $\sum_{i=1}^{n} x_i$ is a sufficient statistic for θ?

Now the likelihood function or joint density function is,

$$L = \prod_{i=1}^{n} f(x_i, \theta)$$

$$= \theta^{x_1}(1-\theta)^{1-x_1} \cdot \theta^{x_2}(1-\theta)^{1-x_2} \cdots \theta^{x_n}(1-\theta)^{1-x_n}$$

$$= \cancel{\theta^{\sum_{i=1}^{n} x_i}}$$

$$= \theta^{x_1+x_2+\cdots+x_n}(1-\theta)^{(1-x_1)+(1-x_2)+\cdots+(1-x_n)}$$

$$= \theta^{\sum_{i=1}^{n} x_i} \cdot (1-\theta)^{\sum_{i=1}^{n}(1-x_i)}$$

$$= \theta^{\sum_{i=1}^{n} x_i}(1-\theta)^{n-\sum_{i=1}^{n} x_i}$$

$$= g\left(\sum_{i=1}^{n} x_i, \theta\right) \cdot h(x_1, x_2 \ldots x_n)$$

where $g\left(\sum_{i=1}^{n} x_i, \theta\right) = \theta^{\sum_{i=1}^{n} x_i}(1-\theta)^{n-\sum_{i=1}^{n} x_i}$

and $h(x_1, x_2 \ldots x_n) = 1$

$\therefore \sum_{i=1}^{n} x_i$ is a sufficient statistic for θ

$$L = \theta^{\sum_{i=1}^{n} x_i} (1-\theta)^{n - \sum_{i=1}^{n} x_i}$$

Now the marginal distribution of $\sum_{i=1}^{n} x_i$ is,

$$g\left(\sum_{i=1}^{n} x_i, \theta\right) = \left({}^{n}C_{\sum_{i=1}^{n} x_i}\right) \theta^{\sum_{i=1}^{n} x_i} (1-\theta)^{n - \sum_{i=1}^{n} x_i}$$

$$\boxed{f(x,p) = {}^{n}C_{x}\, p^{x} q^{n-x}}$$

$$h(t'/t) = \frac{h(t\,t'/\theta)}{g(t,\theta)}$$

$$= \frac{\theta^{\sum x_i} (1-\theta)^{n-\sum x_i}}{\left({}^{n}C_{\sum x_i}\right) \theta^{\sum x_i} (1-\theta)^{n-\sum x_i}}$$

$$= \frac{1}{{}^{n}C_{\sum x_i}}$$

Section B

2nd tutorial will be held on April 24, 2017 at 8:15 am.

Test about correlation coefficient

correlation Analysis:

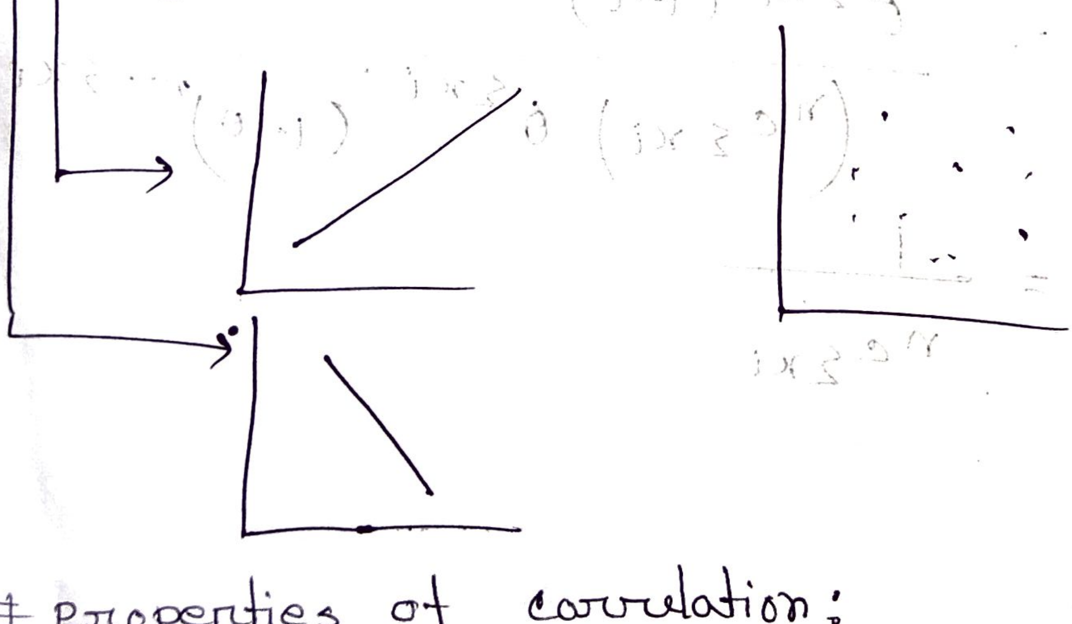
$$y = \alpha + \beta x$$

scatter plots:

(I) positive relationship

(II) Negative "

(III) No "



properties of correlation:

Limitation.

$\Rightarrow r = 0$

$\Rightarrow 0 < r < 0.25 =$ weak

$\Rightarrow 0.25 \leq r < 0.75 =$ intermediate

$\Rightarrow 0.75 \leq r < 1 =$ strong

$\Rightarrow r = 1 =$ perfect

Example :

9.7 Test about correlation coefficients.

$$t = \frac{r\sqrt{n-2}}{\sqrt{1-r^2}}$$

$df = 4$

$\alpha = 0.05$

- test procedure
- type - I and Type - II error (difference)
- def^n (& hypothetical test, . . .
- Best critical region
- Math correlation - coefficient on test

Problem : The coefficient of correlation obtained from a random sample of 18 pairs of values is 0.45. Is this significant ~~___~~ of correlation in the

Test of significance?

$\boxed{\text{* rank ; \& run test}}$ * * *