Relations

Relations

- Relationships between elements of sets occur very often.
  - (Employee, Salary)
  - (Students, Courses, GPA)
- Relationships between elements of sets are represented using the structure called relation, which is just a subset of the Cartisian product of the sets.
- We use ordered pairs (or *n-tuples*) of elements from the sets to represent relationships.

Binary Relations

- Let A and B be any sets. A *binary relation R* from A to B, (i.e., with signature $R{:}A\times B)$ can be identified with a subset of $A\times B$.

E.g., let **<** : N×N can be seen as $\{(n,m) \mid n < m\}$

- $(a,b)\in R$ means that a is related to b (by R)
- Also written as aRb; also $R(a,b)$
  - E.g., a**<b** and **<** (a,b) both mean (a,b)∈ **<**
- A binary relation R corresponds to a characteristic function $P_R{:}A\times B\rightarrow\{\mathbf{T},\mathbf{F}\}$

Example

A: {students at UNR}, B: {courses offered at UNR}

R: "relation of students enrolled in courses"

(Jason, CS365), (Mary, CS201) <u>are in</u> R

If Mary does not take CS365, then (Mary, CS365) <u>is not</u> in R!

If CS480 is not being offered, then (Jason, CS480), (Mary, CS480) <u>are not</u> in R!

Complementary Relations

- Let $R{:}A,B$ be any binary relation.
- Then, $\cancel{R}{:}A\times B$, the *complement* of R, is the binary relation defined by
$$\cancel{R} :\equiv \{(a,b) \in A\times B \mid (a,b) \notin R\} = (A\times B) - R\}$$
- Note this is just $\overline{R}$ if the universe of discourse is $U = A\times B$; thus the name *complement*.
- Note the complement of $\cancel{R}$ is R.

Example: $\not< \; = \{(a,b) \mid (a,b) \notin <\} = \{(a,b) \mid \neg a<b\} = \,\geq$

Inverse Relations

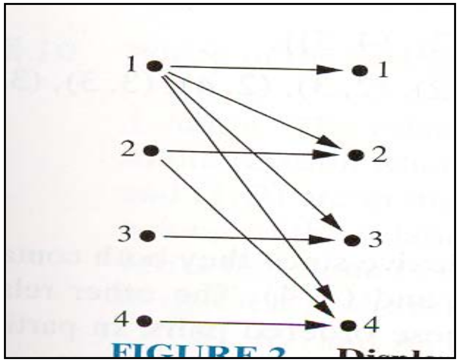
- Any binary relation $R{:}A\times B$ has an *inverse* relation $R^{-1}{:}B\times A$, defined by
  $$R^{-1} :\equiv \{(b,a) \mid (a,b)\in R\}.$$
  E.g., $<^{-1} = \{(b,a) \mid a<b\} = \{(b,a) \mid b>a\} = >$.
- *E.g.*, if R:People x Foods is defined by
  $a\ R\ b \Leftrightarrow a$ *eats* b, then:
  $b\ R^{-1}\ a \Leftrightarrow a$ *eats* b
  (Compare: *b is eaten by a,* passive voice.)

Functions as Relations

A function f:A→B is a relation from A to B

A relation from A to B is not always a function f:A→B (e.g., relations could be one-to-many)

Relations are generalizations of functions!



Relations on a Set

- A (binary) relation from a set A to itself is called a relation *on* A. A relation on the set A is a relation from A to A.

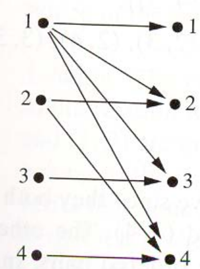
- *E.g.*, the "$<$" relation is defined as a relation *on* $\mathbf{N}$.

Relations <u>on</u> a Set

A (binary) relation from a set *A* to itself is called a relation <u>*on*</u> the set *A*.

A: {1,2,3,4}

R={(a,b) | a divides b}



R	1	2	3	4
1	×	×	×	×
2		×		×
3			×	
4				×

FIGURE 2 **Displaying the Ordered Pairs in the Relation *R* from Example 4.**

Example

How many relations are there on a set A with *n* elements?

Reflexivity and relatives

- A relation R on A is *reflexive* iff $\forall a \in A, (aRa)$.
  - *E.g.*, the relation $\geq :\equiv \{(a,b) \mid a \geq b\}$ is reflexive.
- R is *irreflexive* iff $\forall a \in A, (\neg aRa)$
- Note "*irreflexive*" does **NOT** mean "*not reflexive*", which is just $\neg\forall a \in A, (aRa)$.
- E.g., if Adore={(j,m),(b,m),(m,b)(j,j)} then this relation is neither reflexive nor irreflexive

Reflexivity and relatives

- Theorem: A relation R is *irreflexive* iff its *complementary* relation R' is reflexive.
  - Example: $<$ is irreflexive; $\geq$ is reflexive.
  - Proof: trivial
  - Is the "divide" relation on the set of positive integers reflexive?

Some examples

- Reflexive:

  =, 'have same cardinality', $\Leftrightarrow$

  <=, >=, $\Rightarrow$, $\subseteq$, etc.

- Irreflexive:

  <, >, `have different cardinality', $\subset$, 'is logically stronger than'

Symmetry & relatives

- A binary relation R on A is *symmetric* iff $\forall a,b((a,b)\in R \leftrightarrow (b,a)\in R)$.
  - *E.g.*, = (equality) is symmetric. < is not.
  - "is married to" is symmetric, "likes" is not.
- A binary relation R is *asymmetric* if $\forall a,b((a,b)\in R \rightarrow (b,a)\notin R)$.
  - **Examples:** < is asymmetric, "Adores" is not.
- Let R={(j,m),(b,m),(j,j)}. Is R (a)symmetric?

Symmetry & relatives

- Let R={(j,m),(b,m),(j,j)}.

R is not symmetric (because it does not contain (m,b) and because it does not contain (m,j)).

R is not asymmetric, due to (j,j)

Some direct consequences

Theorems:

1. R is symmetric iff $R = R^{-1}$,
2. R is asymmetric iff $R \cap R^{-1}$ is empty.

Symmetry & its relatives

1. R is symmetric iff $R = R^{-1}$

$\Rightarrow$ Suppose R is symmetric. Then

$(x,y) \in R \Leftrightarrow$

$(y,x) \in R \Leftrightarrow$

$(x,y) \in R^{-1}$

$\Leftarrow$ Suppose $R = R^{-1}$ Then

$(x,y) \in R \Leftrightarrow$

$(x,y) \in R^{-1} \Leftrightarrow$

$(y,x) \in R$

Symmetry & relatives

2. R is asymmetric iff $R \cap R^{-1}$ is empty.

   (Straightforward application of the definitions of asymmetry and R^{-1})

- Question: Can you construct a model in which the relation “son of” is symmetric?
- More precisely: find domain A and relation R on A such that R is symmetric and R(x,y) can reasonably be read as ‘x is a son of y’

- Question: Can you construct a model in which the relation “son_of” is symmetric?
- Solution: any model in which there are no x,y such that son_of(x,y) is true
- E.g., A = {John, Mary, Sarah},
  A**x**A $\supseteq$ R= {}

Antisymmetry

- Consider the relation $x \leq y$
- Is it symmetrical?
- Is it asymmetrical?
- Is it reflexive?
- Is it irreflexive?

Antisymmetry

- Consider the relation $x \leq y$
- Is it symmetrical? No
- Is it asymmetrical?
- Is it reflexive?
- Is it irreflexive?

Antisymmetry

- Consider the relation $x \leq y$
- Is it symmetrical? No
- Is it asymmetrical? No
- Is it reflexive?
- Is it irreflexive?

Antisymmetry

- Consider the relation $x \leq y$
- Is it symmetrical? No
- Is it asymmetrical? No
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- Is it irreflexive?

Antisymmetry

- Consider the relation $x \leq y$
- Is it symmetrical? No
- Is it asymmetrical? No
- Is it reflexive? Yes
- Is it irreflexive? No

Antisymmetry

- Consider the relation $x \leq y$
  - It is not symmetric. (For instance, $5 \leq 6$ but not $6 \leq 5$)
  - It is not asymmetric. (For instance, $5 \leq 5$)
  - The pattern: the only times when $(a,b) \in \leq$ and $(b,a) \in \leq$ are when $a=b$
- This is called antisymmetry

  Can you say this in predicate logic?

Antisymmetry

- A binary relation R on A is *antisymmetric* iff $\forall a,b((a,b)\in R \wedge (b,a)\in R) \rightarrow a=b)$.
- Examples: $\leq$, $\geq$, $\subseteq$
- Another example: the earlier-defined relation Adore={(j,m),(b,m),(m,b)(j,j)}

- How would you define transitivity of a relation? What are its 'relatives'?

Transitivity & relatives

- A relation R is *transitive* iff (for all a,b,c)
  $((a,b)\in R \wedge (b,c)\in R) \rightarrow (a,c)\in R$.
- A relation is non*transitive* iff it is not transitive.
- A relation R is in*transitive* iff (for all a,b,c)
  $((a,b)\in R \wedge (b,c)\in R) \rightarrow \neg(a,c)\in R$.

Transitivity & relatives

- What about these examples:
  - "x is an ancestor of y"
  - "x likes y"
  - "x is located within 1 mile of y"
  - "x +1 =y"
  - "x beat y in the tournament"
  - "x is stronger than y"

Transitivity & relatives

- What about these examples:
  - “is an ancestor of” is transitive.
  - “likes” is neither trans nor intrans.
  - “is located within 1 mile of”
    is neither trans nor intrans
  - “x +1 =y” is intransitive
  - “x beat y in the tournament” is neither trans nor intrans
  - “x is stronger than y” is transitive.

Exploring the difference between relations and functions

Totality:

- A relation $R{:}A\times B$ is *total* if for every $a\in A$, there is at least one $b\in B$ such that $(a,b)\in R$.
  - N.B., it does not follow that R^{-1} is total
  - It does not follow that R is a function.

Functionality

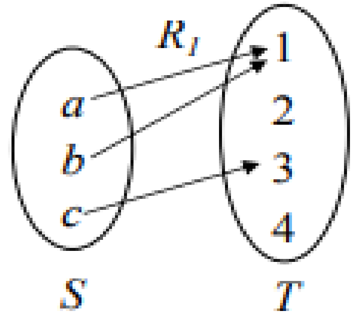
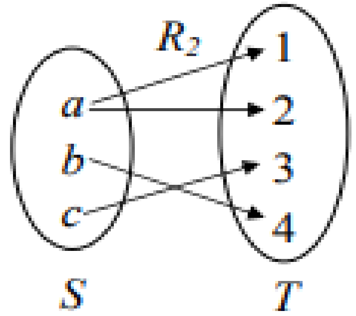
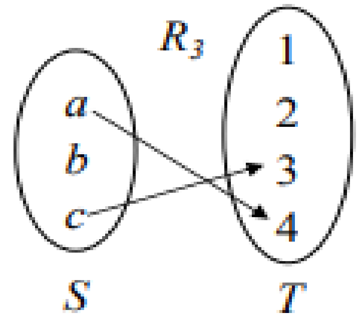
Functionality:

- A relation R: $A \times B$ is *functional* iff, for every $a \in A$, there is *at most one* $b \in B$ such that $(a,b) \in R$.
  - A functional relation R: $A \times B$ does not have to be total (there may be $a \in A$ such that $\neg \exists b \in B\ (aRb)$).
- Say that "R is functional", using predicate logic

Functionality

- R: $A \times B$ is *functional* iff, for every $a \in A$, there is *at most one* $b \in B$ such that $(a,b) \in R$.
  $\forall a \in A$: $\neg\exists\, b_1, b_2 \in B\ (b_1 \neq b_2 \wedge aRb_1 \wedge aRb_2)$.
- If R is functional and total relation, then R can be seen as a function R: $A \rightarrow B$
  Hence one can write $R(a)=b$ *as well as aRb, R(a,b), and (a,b)∈ R. Each of these mean the same.*

Functionality

			
total	*yes*	*yes*	*no*
onto	*no*	*yes*	*no*
functional	*yes*	*no*	*yes*
one-to-one	*no*	*no*	*yes*

R_3 is not total, because the element b is not in the domain.

R_1 is not onto, because the elements 2 and 4 are not in the range.

R_3 is not onto, because the elements 1 and 2 are not in the range.

R_2 is not functional, because the element a has two relatives.

R_1 is not one-to-one, because the element 1 is a relative of two elements in S.

R_2 is not one-to-one, because the element a has two relatives.

Functionality

- *Definition:* R is *antifunctional* iff its inverse relation R^{-1} is functional.

  (**Exercise:** Show that iff R is functional and antifunctional, and both it and its inverse are total, then it is a bijective function.)

Combining what you've learned about functions and relations

Consider the relation $R:\mathbf{N} \rightarrow \mathbf{N}$ defined as

$R = \{(x,y) \mid x \in \mathbf{N} \wedge y \in \mathbf{N} \wedge y = x+1\}$.

Questions:

1. Is R total? Why (not)?
2. Is R functional? Why (not)?
3. Is R an injection? Why (not)?
4. Is R a surjection? Why (not)?

Composite Relations

- Let R:$A\times B$, and S:$B\times C$. Then the *composite* $S\circ R$ of R and S is defined as:

  $S\circ R = \{(a,c) \mid \exists b: aRb \wedge bSc\}$

- Does this remind you of something?

Composite Relations

- Let R:$A\times B$, and S:$B\times C$. Then the *composite* $S\circ R$ of R and S is defined as:

  $$S\circ R = \{(a,c) \mid \exists b\text{: } aRb \wedge bSc\}$$

- Does this remind you of something?
- Function composition ...
- ... except that $S\circ R$ accommodates the fact that S and R may not be functional

Composite Relations

- Function composition is a special case of relation composition: Suppose S and R are functional. Then we have (using the definition above, then switching to function notation)

  $S \circ R(a,c)$ *iff* $\exists b$: $aRb \wedge bSc$
  iff $R(a)=b$ *and* $S(b)=c$ *iff* $S(R(a))=c$

Suppose

- $Adore=\{(a,b),(b,c),(c,c)\}$
- $Detest=\{(b,d),(c,a),(c,b)\}$
- $Adore\circ Detest=$
- $Detest\circ Adore=$

Suppose

- $Adore=\{(a,b),(b,c),(c,c)\}$
- $Detest=\{(b,d),(c,a),(c,b)\}$
- $Adore\circ Detest= \{(c,b),(c,c)\}$
- $Detest\circ Adore=$

Suppose

- Adore={(a,b),(b,c),(c,c)}
- Detest={(b,d),(c,a),(c,b)}
- Adore°Detest= {(c,b),(c,c)}
- Detest°Adore= {(a,d),(b,a),(b,b),(c,a),(c,b)}

§7.2: *n*-ary Relations

- An n-ary relation R on sets $A_1,\ldots,A_n$, is a subset

$$R \subseteq A_1 \times \ldots \times A_n.$$

- This is a straightforward generalisation of a binary relation. For example:
- 3-ary relations:
  - a is between b and c;
  - a gave b to c

§7.2: *n*-ary Relations

- An *n*-ary relation *R* on sets $A_1,\ldots,A_n$, is a subset

$$R \subseteq A_1 \times \ldots \times A_n.$$

- The sets A_i are called the *domains* of *R*.
- The *degree* of *R* is *n*.
- *R* is *functional in the domain* A_i if it contains at most one *n*-tuple $(\ldots, a_i, \ldots)$ for any value a_i within domain A_i.

§7.2: n-ary Relations

- R is *functional in the domain* A_i if it contains at most one n-tuple $(\ldots, a_i, \ldots)$ for any value a_i within domain A_i.
- Generalisation: being functional in a combination of two or more domains.

Relational Databases

- A *relational database* is essentially just a set of relations.
- A domain A_i is a (*primary) key* for the database if the relation R is functional in A_i.
- A *composite key* for the database is a set of domains $\{A_i, A_j, \ldots\}$ such that R contains at most 1 n-tuple $(\ldots,a_i,\ldots,a_j,\ldots)$ for each composite value $(a_i, a_j,\ldots)\in A_i\times A_j\times\ldots$

Selection Operators

- Let A be any n-ary domain $A=A_1\times\ldots\times A_n$, and let $C:A\rightarrow\{\mathbf{T},\mathbf{F}\}$ be any *condition* (predicate) on elements (n-tuples) of A.
- The *selection operator* s_C maps any n-ary relation R on A to the relation consisting of all n-tuples from R that satisfy C:

$$s_C(R) = \{a\in R \mid \mathrm{C}(a) = \mathbf{T}\}$$

Selection Operator Example

- Let A = StudentName × Standing × SocSecNos
- Define a condition Upperlevel on A:

  $UpperLevel(name, standing, ssn) \Leftrightarrow ((standing = \text{junior}) \vee (standing = \text{senior}))$
- Then, $s_{UpperLevel}$ takes any relation R on A and produces the subset of R involving of *just* the junior and senior students.

Projection Operators

- Let $A = A_1 \times \ldots \times A_n$ be any *n*-ary domain, and let $\{i_k\}=(i_1,\ldots,i_m)$ be a sequence of indices all falling in the range 1 to n,

- Then the *projection operator* on *n*-tuples

$$P_{\{i_k\}} : A \rightarrow A_{i_1} \ldots \times A_{i_m}$$

is defined by:

$$P_{\{i_k\}}(a_1,\ldots,a_n) = (a_{i_1},\ldots,a_{i_m})$$

Projection Example

- Suppose we have a domain $Cars = Model \times Year \times Color$. (note n=3).
- Consider the index sequence $\{i_k\} = 1,3$. (m=2)
- Then the projection $P_{\{i_k\}}$ maps each tuple $(a_1,a_2,a_3) = (model, year, color)$ to its image:

$$\left(a_{i_1}, a_{i_2}\right) = \left(a_1, a_3\right) = \left(model, color\right)$$

- This operator can be applied to a relation $R \subseteq Cars$ to obtain a list of the model/color combinations available.

Join Operator

- Puts two relations together to form a combined relation which is their composition:
- Iff the tuple (A,B) appears in R_1, and the tuple (B,C) appears in R_2, then the tuple (A,B,C) appears in the join $J(R_1,R_2)$.
  - A, B, and C can also be sequences of elements.

Join Example

- Suppose R_1 is a teaching assignment table, relating *Lecturers* to *Courses*.
- Suppose R_2 is a room assignment table relating *Courses* to *Rooms*, *Times*.
- Then $J(R_1,R_2)$ is like your class schedule, listing (*lecturer*, *course*, *room*, *time*).
- (Joins are similar to *relation composition*. For precise definition, see Rosen, p.486)

Composite Relations

- Let’s see what happens when we compose R with itself ...
- First: different ways to represent relations

§7.3: Representing Relations

- Before saying more about the n-th power of a relation, let's talk about representations
- Some ways to represent *n*-ary relations:
  - With a list of n-tuples.
  - With a function from the (n-ary) domain to **{T,F}**.
- Special ways to represent binary relations:
  - With a zero-one matrix.
  - With a directed graph.

- Why bother with alternative representations? Is one not enough?
- One reason: some calculations are easier using one representation, some things are easier using another
- There are even some basic ideas that are suggested by a particular representation

It’s often worth playing around with different representations

Using Zero-One Matrices

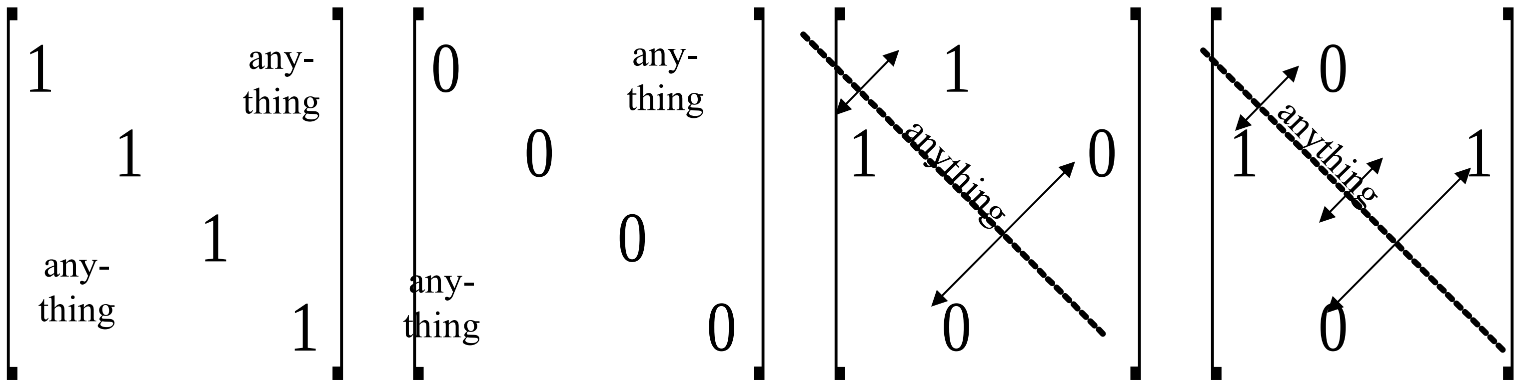
- To represent a binary relation R:$A\times B$ by an $|A|\times|B|$ 0-1 matrix $\mathbf{M}_R = [m_{ij}]$, let $m_{ij} = 1$ iff $(a_i,b_j)\in R$.
- *E.g.*, Suppose Joe likes Susan and Mary, Fred likes Mary, and Mark likes Sally.
- Then the 0-1 matrix representation of the relation Likes:Boys×Girls relation is:

$$
\begin{array}{c}
 \\ \text{Joe} \\ \text{Fred} \\ \text{Mark}
\end{array}
\begin{array}{c}
\begin{array}{ccc} \text{Susan} & \text{Mary} & \text{Sally} \end{array} \\
\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}
\end{array}
$$

- Special case 1-0 matrices for a relation on A (that is, $R{:}A\times A$)
- *Convention*: rows and columns list elements in the same order
- This where 1-0 matrices come into their own!

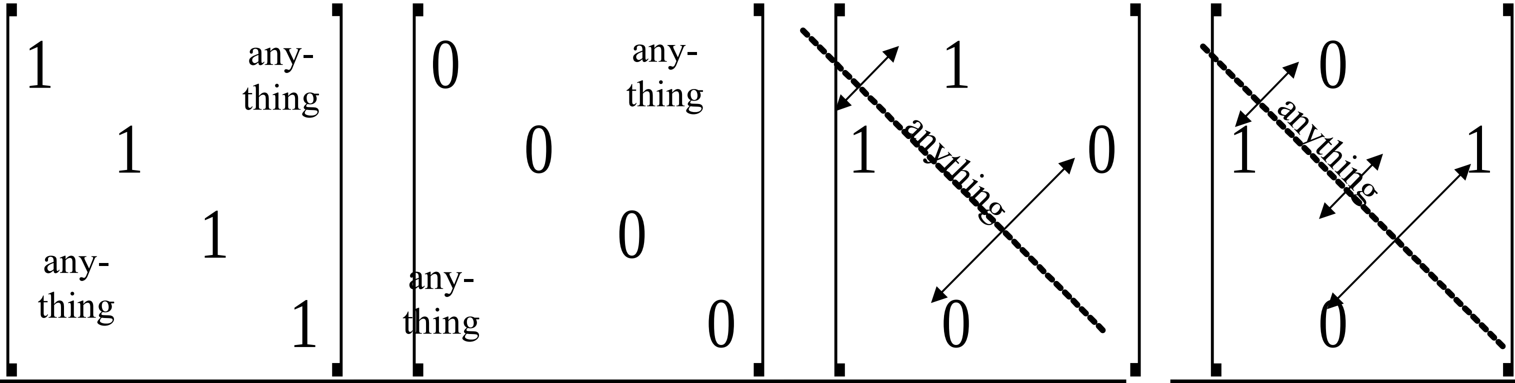
Zero-One Reflexive, Symmetric

- Recall: *Reflexive*, *irreflexive*, s*ymmetric*, *and asymmetric* relations.
  - These relation characteristics are easy to recognize by inspection of the zero-one matrix.



Zero-One Reflexive, Symmetric

- Recall: *Reflexive*, *irreflexive*, s*ymmetric*, *and asymmetric* relations.
  - These relation characteristics are very easy to recognize by inspection of the zero-one matrix.



Reflexive:
only 1’s on diagona

Irreflexive:
only 0’s on diagonal

Symmetric:
all identical
across diagonal

Asymmetric:
all 1’s are across
from 0’s

Matrices

- There exists much mathematical tjeory about graphs
- Some fast algorithms rely on graphs
- More about graphs: Rosen, section 3.8

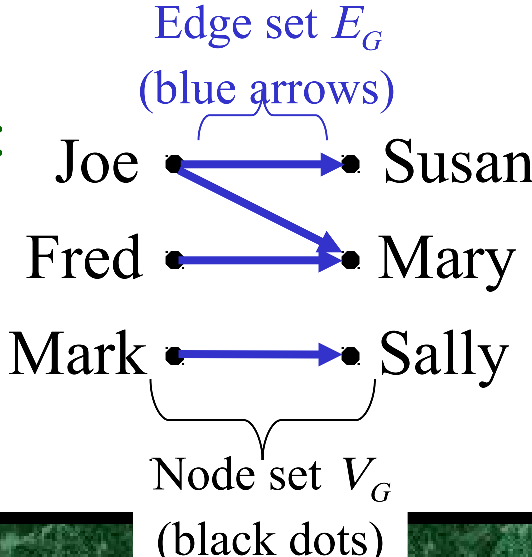
Using Directed Graphs

- A *directed graph* or *digraph* $G=(V_G,E_G)$ is a set V_G of *vertices (nodes)* with a set $E_G \subseteq V_G \times V_G$ of *edges (arcs)*. Visually represented using dots for nodes, and arrows for edges. A relation $R{:}A\times B$ can be represented as a graph $G_R=(V_G=A\cup B,\ E_G=R)$.

Matrix representation $\mathbf{M}_R$:

$$
\begin{array}{l}
 \\
\text{Joe} \\
\text{Fred} \\
\text{Mark}
\end{array}
\begin{array}{c}
\begin{array}{ccc}\text{Susan} & \text{Mary} & \text{Sally}\end{array} \\
\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}
\end{array}
$$

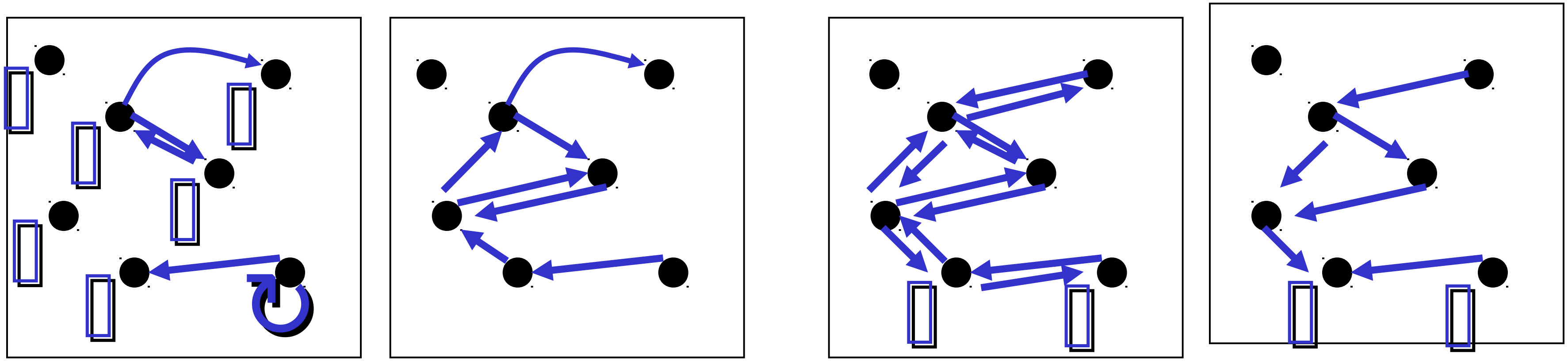
Graph rep. G_R:



Digraph Reflexive, Symmetric

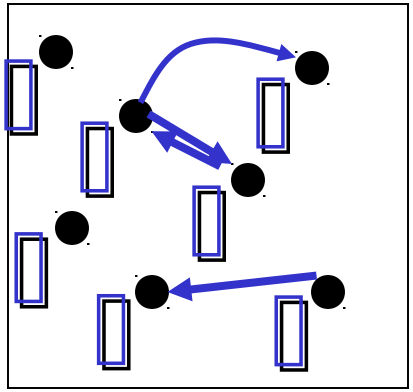
Properties of a relation can determined

by inspection of its graph.

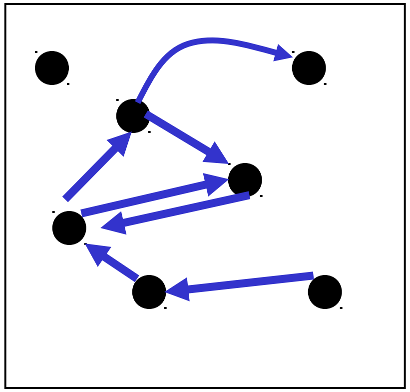


Digraph Reflexive, Symmetric

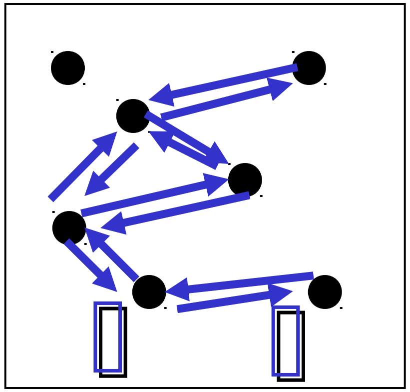
Many properties of a relation can be determined by inspection of its graph.



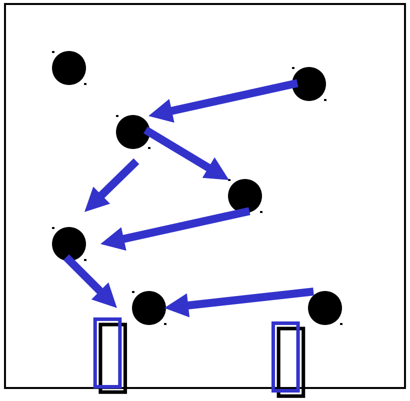
Reflexive:
Every node has a self-loop



Irreflexive:
No node links to itself



Symmetric:
Every link is bidirectional



Antisymmetric:
never (a,b) and (b,a), unless a=b

These are not symmetric & not asymmetric

These are non-reflexive & non-irreflexive

Particularly easy with a graph

- Properties that are somehow 'local' to a given element, e.g.,
  - "does the relation contain any elements that are unconnected to any others?"
- Properties that involve combinations of pairs, e.g.,
  - "does the relation contain any cycles?"
  - things to do with the composition of relations (e.g. the n-th power of R)
- More about graphs: Rosen, chapter 9.

Now: Composing R with itself

- The n^{th} power R^n of a relation R on a set A
  - *The 1st power of R is R itself*
  - *The 2nd power of R is* $R^2 = R \circ R$
  - *The 3rd power of R is* $R^3 = R \circ R \circ R$

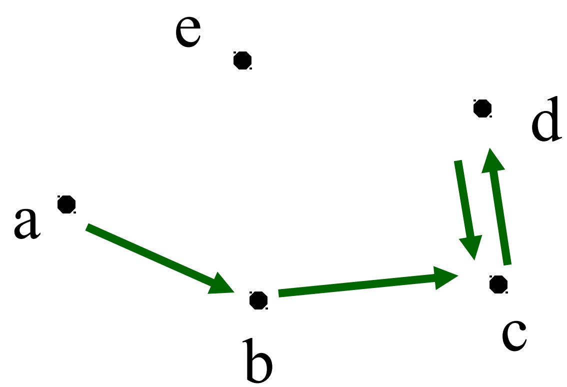
etc.

Composite Relations

- The n^{th} power R^n of a relation R on a set A can be defined recursively by:

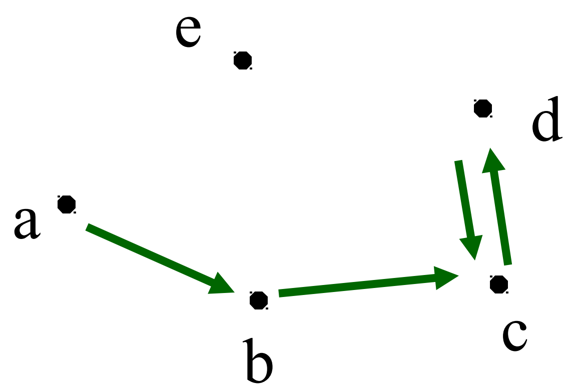
  $R^1 :\equiv R$; $R^{n+1} :\equiv R^n \circ R$ for all $n \geq 1$.

- E.g., $R^2 = R \circ R$; $R^3 = R \circ R \circ R$



Composite Relations

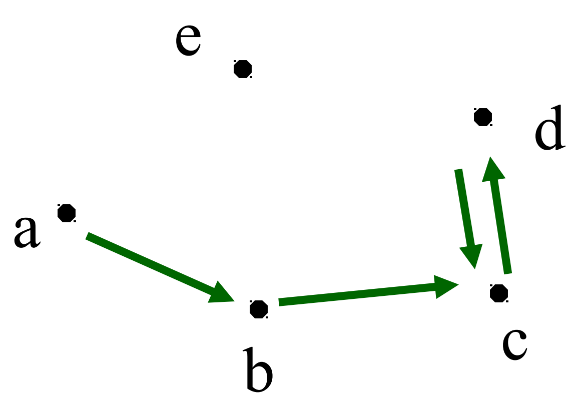
- $R^2 = R \circ R = \ldots$



Composite Relations

- $R^2 = R \circ R = \{(a,c),(b,d),(c,c),(d,d)\}$

- *a: c*   *d: d*
  b: d   *e: -*
  c: c



Back to the n-th power of a relation

- A *path* of length n from node a to b in the directed graph G is a sequence $(a,x_1), (x_1,x_2), \ldots, (x_{n-1},b)$ of n ordered pairs in E_G.
  - Note: there exists a path of length n from a to b in R if and only if $(a,b) \in R^n$.
- A path of length $n \geq 1$ from a to itself is a *cycle*.
- R*: the relation that holds between a and b iff there exists a finite path from a to b using R.
  - Note: R* is transitive!

Why is R* of interest?

- Suppose an infectious disease is transmitted by shaking hands (Shake(x,y))
- To know who is infected by John, you need to think about two things:

1. Determine $\{x \in \text{person}: \text{Shake}(\text{John},x)\}$
   This gives you the direct infectees
2. Everyone infected by someone infected by John. Note: this is recursive

- Suppose S(hake) = {(a,b), (b,c), (c,d)}. We want to compute S*.
- $S \subseteq S^*$, so S*(a,b), S*(b,c), S*(c,d)
- Infer S*(a,c) and S*(b,d)
  (using the following rule twice:
  $S(x,y)$ & $S(y,z) \rightarrow S(x,z)$)
- Are we done?

Who is infected?

- Suppose S(hake)= {(a,b), S(b,c), S(c,d)}. We want to compute S*.
- $S \subseteq S^*$, so $S^*(a,b)$, $S^*(b,c)$, $S^*(c,d)$
- Infer $S^*(a,c)$ and $S^*(b,d)$
(using the following rule twice:
$S(x,y)$ & $S(y,z) \rightarrow S(x,z)$)
- Second step: $S^*(a,d)$

We don’t always know R* …

- We often don’t know the exact extension of a relation (i.e., which pairs are elements of the relation)
- Presumably, you’ve never shook hands with the president of Mongolia: $\neg S(you,PM)$
- How about $S^*(you,PM)$ …?

Other examples of R*

- R(a,b) $\Leftrightarrow$ there's a direct bus service from a to b.
- R(p,q) $\Leftrightarrow$ there exists an inference rule that allows you to infer q from p

What is R* in each of these cases?

Other examples of R*

- R(a,b) ⇔ there's a direct bus service from a to b.
- R(p,q) ⇔ there exists an inference rule that allows you to infer q from p

What is R* in each of these cases?

- R(a,b) ⇔ one can go by bus from a to b.
- R(p,q) ⇔ there exists a proof
  that q follows from p

R*

How would you formally define R*?

R*

How would you formally define R*?

Here's a safe bet

$$R^i$$

R*

How would you formally define R*?

Here's a finite variant, where $n = |A|$ *(proof in book* that n is large enough*)*

$$R^i$$

- So much (for the moment!) about graphs
- Let's return to properties like reflexivity, symmetry and transitivity
- Informally: How can we "make" a relation transitive (etc.)?

§7.4: Closures of Relations

- For any property X, the <u>X closure</u> of a set A is defined as the "smallest" superset of A that has property X. More specifically,
  - The *reflexive closure* of a relation R on A is the smallest superset of R that is reflexive.
  - The *symmetric closure* of R is the smallest superset of R that is symmetric
  - The *transitive closure* of R is the smallest superset of R that is transitive

Calculating closures

- The *reflexive closure* of a relation R on A is obtained by "adding" (a,a) to R for each $a \in A$. *I.e.*, it is $R \cup I_A$ (Check that this is the r.c.)
- The *symmetric closure* of R is obtained by "adding" (b,a) to R for each (a,b) in R. *I.e.*, it is $R \cup R^{-1}$ (Check that this is the s.c.)
- The *transitive closure* of R is obtained by *"repeatedly"* adding (a,c) to R for each $(a,b),(b,c)$ in R …

Calculating closures

- Adore={(a,b),(b,c),(c,c)}
- Detest={(b,d),(c,a),(c,b)}
- The *symmetric closure* of ...

  ... Adore=

  ... Detest=

Calculating closures

- Adore={(a,b),(b,c),(c,c)}
- Detest={(b,d),(c,a),(c,b)}
- The *symmetric closure* of ...

  ... Adore={(a,b),(b,c),(c,c),(b,a), (c,b)}

  ... Detest={(b,d),(c,a),(c,b), (d,b), (a,c),(b,c)}

Calculating closures

- Adore={(a,b),(b,c),(c,c)}
- Detest={(b,d),(c,a),(c,b)}
- The *transitive closure* of ...

  ... Adore=

  ... Detest=

Calculating closures

- Adore={(a,b),(b,c),(c,c)}
- Detest={(b,d),(c,a),(c,b)}
- The *transitive closure* of ...

  ... Adore={(a,b),(b,c),(c,c),(a,c)}

  ... Detest={(b,d),(c,a),(c,b),(c,d)}

TC(R)

- A more precise definition of the transitive closure of R (abbr: TC(R)) is:
- TC(R)= the intersection of all transitive supersets of R.
- Let's check that this matches our earlier definition
  - It follows from the new definition that there exists no smaller transitive superset of R than TC(R).
  - TC(R) itself is a transitive superset of R.
    Proof:

TC(R)

TC(R) is a transitive superset of R. Proof:

- If A and B are transitive supersets of R then $A \cap B$ is a transitive superset of R

1. $A \cap B$ is a superset of R.
2. $A \cap B$ is a transitive. (Suppose (x,y) and (y,z) are elements of $A \cap B$. Then (x,z) is an element of $A \cap B$.)

TC(R)

- So TC(R) is a transitive superset of R.
- Since it is the intersection of all transitive supersets of R, TC(R) is <u>the smallest</u> transitive superset of R.
  - Suppose X is a transitive superset of R and X$\subset$TC(R). Then (TC(R) $\cap$X) $\subset$ TC(R). But TC(R) is the intersection of all trans. supersets of X, hence (TC(R) $\cap$X) = TC(R). Contradiction.
- Now we relate TC(R) with the graph-theoretic concept R*:

Theorem: R*=TC(R)

Theorem: R* = the transitive closure of R

We need to prove that R* is the smallest transitive superset of R.

1. Proof that R* is transitive:

Suppose xR*y and yR*z.
E.g., $xR^{n}y$ and $yR^{m}z$
Then $xR^{n+m}z$, hence xR*z

Proof ctd.

2. Evidently, $R \subseteq R^*$, so R^* is a superset of R.

We now know that R^* is a transitive superset of R.

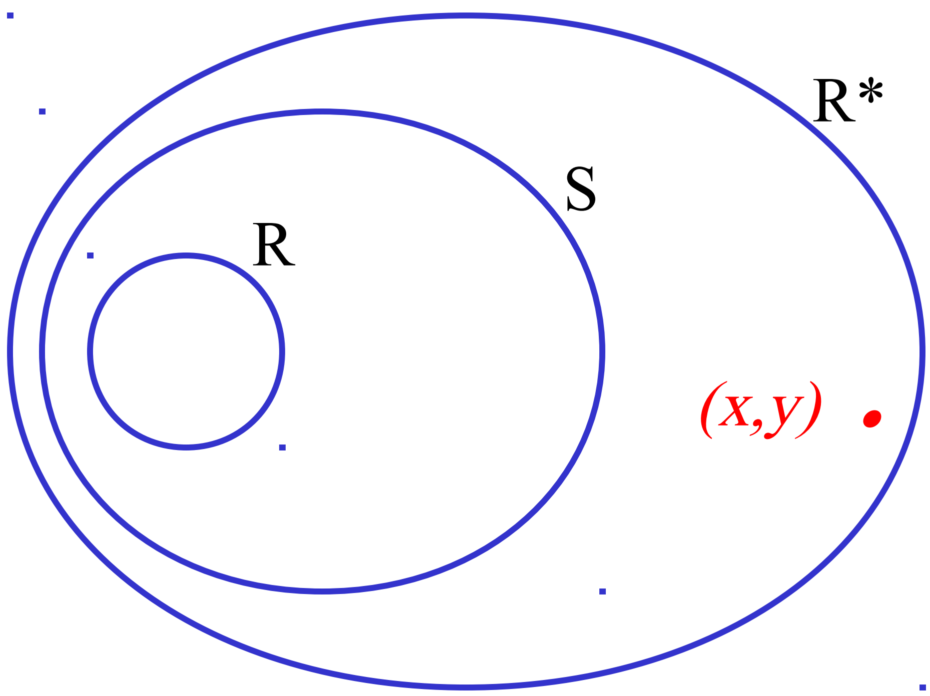
3. R cannot have a <u>smaller</u> transitive superset than R^*.

Proof: Suppose such a transitive superset S of R existed. This would mean that there exists a pair (x,y) such that xR^*y while $\neg xSy$. But xR^*y means $\exists n$ such that xR^ny. But since $R \subseteq S$, it would follow that xS^ny; but because S is transitive, this would imply that xSy. Contradiction.

(Compare Rosen p.500 (5th ed.), p.548 (6th ed.))

An Euler diagram might help ...

Suppose there existed a transitive superset of R that's smaller than R* ...



- Fast algorithms are available for calculating R*, especially *Warshall's algorithm* (also called Roy-Warshall algorithm)
- FYI: this algorithm uses a matrix representation.

§7.5: Equivalence Relations

- Definition: An *equivalence relation* on a set *A* is any binary relation on *A* that is *reflexive, symmetric, and transitive.*

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  - *E.g.*, = is an equivalence relation.
  - But many other relations follow this pattern too

§7.5: Equivalence Relations

- Definition: An *equivalence relation* on a set A is any binary relation on A that is *reflexive, symmetric, and transitive.*
  - *E.g.*, $=$ is an equivalence relation.
  - For any function $f{:}A\rightarrow B$, the relation "have the same f value", or $=_f :\equiv \{(a_1,a_2) \mid f(a_1)=f(a_2)\}$ is an equivalence relation,
    - *e.g.*, let m="mother of" then $=_m$ = "have the same mother" is an equivalence relation

Equivalence Relation Examples

- "Strings *a* and *b* are the same length."
- "Integers *a* and *b* have the same absolute value."

Equivalence Relation Examples

Let's talk about relations between functions:

1. How about: $R(f,g) \Leftrightarrow f(2)=g(2)$?
2. How about: $R(f,g) \Leftrightarrow f(1)=g(1) \vee f(2)=g(2)$?

Equivalence Relation Examples

1. How about: $R(f,g) \Leftrightarrow f(2)=g(2)$?

   Yes. Reflexivity: $f(2)=f(2)$, for all f

   Sym: $f(2)=g(2)$ implies $g(2)=f(2)$

   Trans: $f(2)=g(2)$ and $g(2)=h(2)$ implies $f(2)=h(2)$.

2. How about: $R(f,g) \Leftrightarrow f(1)=g(1) \vee f(2)=g(2)$?

Equivalence Relation Examples

How about $R(f,g) \Leftrightarrow f(1)=g(1) \vee f(2)=g(2)$?

- No. Counterexample against transitivity:

$f(1)=a$, $f(2)=b$

$g(1)=a$, $g(2)=c$

$h(1)=b$, $h(2)=c$

Equivalence Classes

- Let R be any equivalence relation.
- The *equivalence class* of *a under R*,
  $[a]_R :\equiv \{ x \mid aRx \}$ (optional subscript R)
  - Intuitively, this is the set of all elements that are "equivalent" to a according to R.
  - Each such b (including a itself) can be seen as a *representative* of $[a]_R$.

Equivalence Classes

- Why can we talk so loosely about elements being equivalent to each other (as if the relation didn't have a direction)?
- In some sense, it does not matter which representative of an equivalence class you take as your starting point:

If aRb then $\{ x \mid aRx \} = \{ x \mid bRx \}$

Equivalence Classes

If aRb then aRx $\Leftrightarrow$ bRx **Proof:**

1. Suppose aRb while bRx.

Then aRx follows directly by *transitivity.*

2. Suppose aRb while aRx.

aRb implies bRa *(symmetry)*. But bRa and aRx imply bRx by *transitivity*

Equivalence Classes

We now know that

If aRb then $\{ x \mid aRx \} = \{ x \mid bRx \}$

Equally,

If aRb then $\{ x \mid xRa \} = \{ x \mid xRb \}$
(due to symmetry)

In other words, an equivalence class based on R is simply a maximal set of things related by R

Equivalence Class Examples

- "(Strings *a* and *b*) *have the same length*."
  - Suppose a has length 3. Then $[a]$ = the set of all strings of length 3.
- "(Integers *a* and *b*) *have the same absolute value*."
  - $[a]$ = the set $\{a, -a\}$

Equivalence Class Examples

- "Formulas φ and ψ contain the same number of brackets" (e.g. for formulas of propositional logic, using the strict syntax)
- Now what is $[((p \wedge q) \vee r)]$?

Equivalence Class Examples

- Consider the equivalence relation $\Leftrightarrow$ (i.e., logical equivalence, for example between formulas of propositional logic)
- What is $[p \wedge q]$?

Partitions

- A *partition* of a set *A* is a collection of disjoint nonempty subsets of A that have A as their union.

- Intuitively: a partition of A divides A into separate parts (in such a way that there is no remainder).

Partitions and equivalence classes

- Consider a *partition* of a set A into A_1, ..A_n
  - The A_i's are all disjoint : For all x and for all i, if $x \in A_i$ and $x \in A_j$ then $A_i = A_j$
  - The union of the A_i's = A

Partitions and equivalence classes

- A *partition* of a set A can be viewed as the set of all the equivalence classes $\{A_1, A_2, \ldots\}$ for some equivalence relation on A.
- For example, consider the set A={1,2,3,4,5,6} and its partition {{1,2,3},{4},{5,6}}
- R = { (1,1),(2,2),(3,3),(1,2),(1,3),(2,3),(2,1),(3,1), (3,2) ,(4,4), (5,5),(6,6),(5,6),(6,5) }

Partitions and equivalence classes

- We sometimes say:
  - A partition of A *induces* an equivalence relation on A
  - An equivalence relation on A *induces* a partition of A

§7.6: Partial Orderings

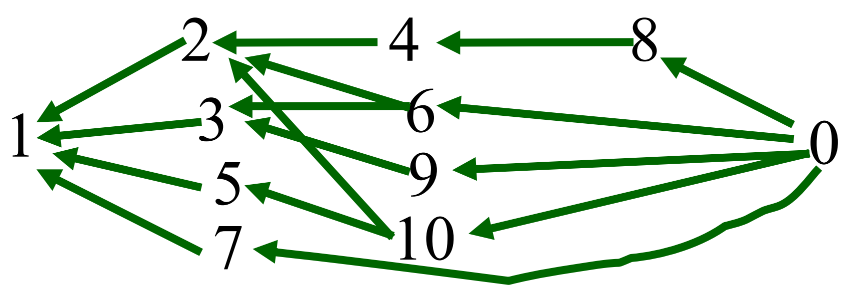
- A relation R on A is called a *partial ordering* or *partial order* iff it is reflexive, antisymmetric, and transitive.
  - We often use a symbol looking something like $\preccurlyeq$ (or analogous shapes) for such relations.
  - Examples: $\leq$, $\geq$ on real numbers, $\subseteq$, $\supseteq$ on sets.
  - Another example: the "divides" relation $|$ on $\mathbf{Z}^+$.
    - It is not necessarily the case that either $a \preccurlyeq b$ or $b \preccurlyeq a$.
- A set A together with a partial order $\preccurlyeq$ on A is called a *partially ordered set* or *poset* and is denoted $(A, \preccurlyeq)$.

Posets as Noncyclical Digraphs

- If a set S is partially ordered by a relation R then its graph can be simplified:
  - Looping edges need not be drawn, because they can be inferred
  - Instead of drawing edges for R(a,b), R(b,c) and R(a,c) , the latter can be omitted (because it can be inferred)
  - If direction of arrows is represented as left-to-right (or top-down) order then it's called a Hasse diagram (We won't do that here)

Posets as Noncyclical Digraphs

- There is a one-to-one correspondence between posets and the reflexive+transitive closures of noncyclical digraphs.
- Example: consider the poset ({0,…,10}, |)
  - Its "minimal" digraph:



Posets as Noncyclical Digraphs

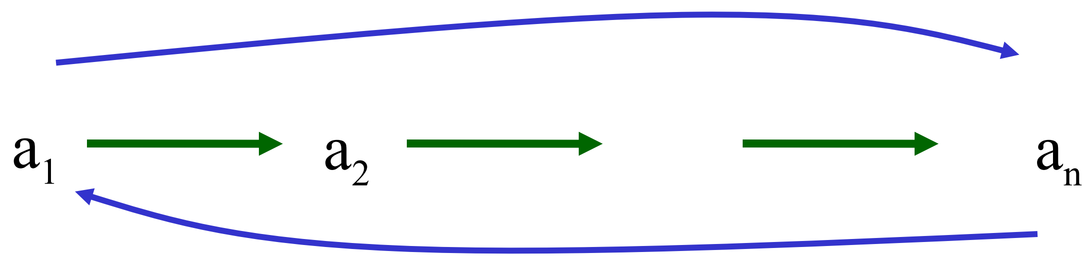
- Prove: a graph for a partial order cannot contain cycles

Posets as Noncyclical Digraphs

- **Theorem**: a graph for a partial order cannot contain cycles with length > 1.
- **Proof**: suppose there is a cycle $a_1Ra_2R\ldots Ra_nRa_1$ (with n>1). Then, with n-1 applications of transitivity, we have a_1Ra_n. But also a_nRa_1, which conflicts with antisymmetry.

Posets do not have cycles

- **Proof**: suppose there is a cycle $a_1Ra_2R\ldots R\mathbf{a_nRa_1}$. Then, with n-1 applications of transitivity, we have $\mathbf{a_1Ra_n}$. But also $\mathbf{a_nRa_1}$, which conflicts with antisymmetry.



- Can something be both a poset and an equivalence relation?

- Can something be both a poset and an equivalence relation?
  - Equiv: ref, sym, trans
  - Poset: ref, antisym, trans
- Can a relation (that is reflexive and transitive) be both sym and antisym?

- Can a relation that is reflexive and transitive be both sym and antisym?
- Yes: the empty relation R={} is an example
- But any relation $R \subseteq \{(x,x): x \in A\}$ will also qualify.
  - It's reflexive
  - It's symmetric and antisymmetric
  - It's transitive
- Other relations cannot qualify. (Prove at home)

Some other types of orderings

1. A **lattice** is a poset in which every pair of elements has a least upper bound (LUB) and a greatest lower bound (GLB).

   Formally: (done in exercise)

   Example: $(Z^+, |)$ In this case,

   LUB = Least Common Multiple

   GLB = Greatest Common Denominator

   Non-example: $(\{1,2,3\}, |)$

Some other types of orderings

2. **Linearly ordered** sets (also: **totally** ordered sets): posets in which *all elements are "comparable" (i.e., related by R)*.

   Formally: $\forall x,y \in A(xRy \vee yRx)$.

   Example:

   Non-example:

Some other types of orderings

Linearly ordered sets (also: **totally** ordered sets): posets in which *all elements are comparable*. Formally:

$\forall x,y \in A(xRy \vee yRx)$.

Example: $(\mathbf{N},\leq)$

Non-example: $(\mathbf{N}, |\ )$ (where | is 'divides')

Non-example: $\subseteq$

An application of posets

- Consider $(A,\leq)$, where A is a set of project tasks and $a<b$ means "a must be completed before b can be completed"
- (Sometimes it's easier to define $<$ than $\leq$)
- Note that $(A,\leq)$ is a poset: ref, antisym, trans

An application of posets

- A common problem: Given $(A,\leq)$, find a *linear* order $(A,\preccurlyeq)$ that is *compatible* with $(A,\leq)$. (That is, $(A,\leq) \subseteq (A,\preccurlyeq)$)
- (We're assuming that tasks cannot be carried out in parallel)
- Algorithm for finding a compatible linear order given a finite partial order: p.526.

Some other types of orderings

2. **Well-ordered** sets: linearly ordered sets in which *every nonempty subset has a least element* (that is, an element a such that $\forall x \in A(aRx)$)

   Example: …
   Non-example: …

Some other types of orderings

2. **Well-ordered** sets: linearly ordered sets in which *every nonempty subset has a least element* (that is, an element a such that $\forall x \in A (aRx)$)

Example: ($\mathbf{N}$,≤)
Non-examples:
($\mathbf{Z}$,≤) ,
(non-negative elements of $\mathbf{R}$, ≤)

Some other types of orderings

2. Non-examples: **($\mathbf{Z}$,≤)** , **($\mathbf{R}^+$, ≤)**
   - **($\mathbf{Z}$,≤)**: **$\mathbf{Z}$** itself has no least element.
   - **(Non-negative R , ≤)**:

     Nonnegative R itself does have a least element , but

     $\mathbf{R}^+$ $\subseteq$ **Nonnegative R** has no least element.

- Well-orderings are behind one of the most general proof techniques that exist: *mathematical induction*.

- The last 30 slides were a tiny crash course in the theory of *mathematical structures*
- Compare Rosen, chapter 7.6.