formula:

(i) $\int_{-\infty}^{\infty} f(z)\, dz = 2\int_0^{\infty} f(z)\, dz$

(ii) $\int_0^{\infty} e^{-ax^2} x^{2n-1}\, dx = \frac{a^{-n}\Gamma n}{2}$

(iii) $\int_0^{\infty} e^{-ax} x^{n-1}\, dx = a^{-n}\Gamma n$

(iv) $\int_0^{\infty} e^{-x} x^{n-1}\, dx = \Gamma n$

(v) $f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\, dx \quad [-\infty < x < \infty]$

(vi) $f(x^2) = \frac{e^{-\frac{x^2}{2}} (x^2)^{\frac{n}{2}-1}}{2^{n/2}\, \Gamma\frac{n}{2}}$

(vii) $\Gamma(\frac{1}{2}) = \sqrt{\pi}$

(viii) $\beta(m,n) = \frac{\Gamma m \times \Gamma n}{\Gamma m+n}$

Chi-square variate: The square of a standard normal variate is known as chi-square variate with 1 d.f.

Thus, if $x \sim N(\mu, \sigma^2)$ then $z = \frac{x-\mu}{\sigma} \sim N(0,1)$ and $z^2 = \left(\frac{x-\mu}{\sigma}\right)^2$ is a chi-square variate with 1 d.f.

In general, if x_i = $(i = 1, 2, \ldots, n)$ are n independent normal variate with mean μ_i and variance σ_i^2 $(i = 1, 2, \ldots$ then, $\chi^2 = \sum_{i=1}^{n} \left(\frac{x_i - \mu_i}{\sigma_i}\right)^2$ is chi-square with n d.f.

Derivation of chi-square (χ^2) distribution

If x_i $(i = 1, 2, \ldots, n)$ are independent $N(\mu_i, \sigma_i^2)$, we want the distribution of

$$\chi^2 = \sum_{i=1}^{n} \left(\frac{x_i - \mu_i}{\sigma_i}\right)^2 ; \text{ Let } z_i = \frac{x_i - \mu_i}{\sigma_i}$$

$$\therefore \chi^2 = \sum_{i=1}^{n} z_i^2$$

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Since x_i are independent, z_i are also independent. Therefore

$$M_{\chi^2}(t) = M_{\sum_{i=1}^{n} z_i^2}(t)$$

$$\boxed{\begin{array}{l}\underline{\underline{\text{Law:}}} \\ M_x(t) = E(e^{tx}) = \int e^{tx} f(x)\,dx\end{array}}$$

$$\therefore M_{z_i^2}(t) = E\left(e^{tz_i^2}\right)$$

$$= \int_{-\infty}^{\infty} e^{tz_i^2} f(z_i)\,dz_i$$

$$= \int_{-\infty}^{\infty} e^{tz_i^2} \frac{1}{\sqrt{2\pi}} e^{-\frac{z_i^2}{2}}\,dz_i$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\left(\frac{1}{2}-t\right)z_i^2}\,dz_i$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{(1-2t)}{2} z_i^2} (z_i)^{2\left(\frac{1}{2}\right)-1}\,dz_i$$

$$= \frac{1}{\sqrt{2\pi}}\, 2\int_{0}^{\infty} e^{-\left(\frac{1-2t}{2}\right) z_i^2} (z_i)^{2\left(\frac{1}{2}\right)-1}\,dz_i$$

$$= \frac{1}{\sqrt{2\pi}} \cdot \frac{\cancel{2} \cdot \left(\frac{1-2t}{2}\right)^{-\frac{1}{2}} \Gamma\frac{1}{2}}{\cancel{2}}$$

$$= \frac{1}{\sqrt{2\pi}} \cdot \sqrt{\pi} \cdot \left(\frac{2}{1-2t}\right)^{1/2}$$

$$= (1-2t)^{-1/2}$$

$$\therefore M_{z_i^2}(t) = (1-2t)^{-1/2}$$

$$\therefore M_{\chi^2}(t) = \prod_{i=1}^{n} (1-2t)^{-1/2}$$

$$= (1-2t)^{-n/2}$$

which is the moment generating function of gamma variate with parameter $\frac{1}{2}$ and $\frac{n}{2}$.

$\boxed{\cancel{X^2}}$ Law: $$f(\chi^2) = \frac{e^{-\frac{\chi^2}{2}} (\chi^2)^{\frac{n}{2}-1}}{2^{n/2} \Gamma\frac{n}{2}}$$

Hence, by uniqueness theorem of m.g.f is

$$\chi^2 = \sum_{i=1}^{n} \left(\frac{x_i - \mu_i}{\sigma_i} \right)^2$$ is a gamma variate

with parameters $\frac{1}{2}$ and $n/2$

which is the required pdf of chi-square distribution ~~wh~~ with n degrees of freedom.

$\boxed{XX}$ M.G.F of chi-square distribution:

$$M_{\chi^2}(t) = E\left(e^{t\chi^2}\right)$$

$$= \int_0^\infty e^{t\chi^2} f(\chi^2)\, d\chi^2$$

$$\cancel{= \int_0^\infty e^{t\chi^2} \frac{1}{\sqrt{2\pi}} e^{-\frac{\chi^2}{2}}}$$

$$\int_0^\infty \frac{e^{t\chi^2} e^{-\frac{\chi^2}{2}} (\chi^2)^{\frac{n}{2}-1}}{2^{\frac{n}{2}} \Gamma\frac{n}{2}} d\chi^2$$

$$\int_0^\infty e^{-\left(\frac{1-2t}{2}\right)\chi^2} (\chi^2)^{\frac{n}{2}-1} d\chi^2$$

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$$= \frac{1}{2^{n/2}\,\Gamma\frac{n}{2}} \left(\frac{1-2t}{2}\right)^{-\frac{n}{2}} \Gamma\frac{n}{2}$$

$$= (1-2t)^{-n/2}$$

which is the m.g.f of χ^2 distribution with n d.f.

$\boxed{\cancel{Ax}}$ First four raw and central moments are as follows:

$$\mu_1' = \frac{\delta M_{\chi^2}(t)}{\delta t} \Big|_{t=0}$$

$$= \frac{\delta}{\delta t} (1-2t)^{-n/2} \Big|_{t=0}$$

$$= -\frac{n}{2} (1-2t)^{-\left(\frac{n}{2}+1\right)} \cdot (-2) \Big|_{t=0}$$

$$= n (1-2t)^{-\left(\frac{n}{2}+1\right)} \Big|_{t=0}$$

$$= n$$

$$\mu_2' = \frac{\delta^2 m_{(x^2)}(t)}{\delta t^2}\Big|_{t=0}$$

$$= \frac{\delta}{\delta t}\left[\frac{\delta m_{(x^2)}(t)}{\delta t}\right]\Big|_{t=0}$$

$$= \frac{\delta}{\delta t}\left[n(1-2t)^{-\left(\frac{n}{2}+1\right)}\right]$$

$$= -n\left(\frac{n}{2}+1\right)(1-2t)^{-\left(\frac{n}{2}+2\right)}\cdot(-2)$$

$$= n\left(\frac{n+2}{2}\right)\cdot 2\,(1-2t)^{-\left(\frac{n}{2}+2\right)}$$

$$= n(n+2)(1-2t)^{-\left(\frac{n}{2}+2\right)}\Big|_{t=0}$$

$$= n(n+2)$$

$$\therefore \mu_2 = \mu_2' - (\mu_1')^2$$

$$= n(n+2) - n^2$$

$$= 2n$$

$$\mu_3 = \frac{\delta^3 M(x^2)t}{\delta t^3}\Big|_{t=0}$$

$$= \frac{\delta}{\delta t}\left[\frac{\delta^2 M(x^2)t}{\delta t^2}\right]$$

$$= \frac{\delta}{\delta t}\left[n(n+2)(1-2t)^{-\left(\frac{n}{2}+2\right)}\right]$$

$$= -n(n+2)\left(\frac{n}{2}+2\right)(1-2t)^{-\left(\frac{n}{2}+3\right)}(-2)\Big|_{t=0}$$

$$= n(n+2)(n+4)(1-2t)^{-\left(\frac{n}{2}+3\right)}\Big|_{t=0}$$

$$= n(n+2)(n+4)$$

$$\mu_3 = \mu_3' - 3\mu_2'\mu_1' + 2(\mu_1')^3$$

$$= n(n+2)(n+4) - 3n(n+2)n + 2n^3$$

$$= n(n^2+6n+8) - 3(n^3+2n^2) + 2n^3$$

$$= n^3 + 6n^2 + 8n - 3n^3 - 6n^2 + 2n^3$$

$$= 8n$$

$$\mu_4' = \frac{\delta^4 M_{(x^2)}(t)}{\delta t^4}\Bigg|_{t=0}$$

$$= \frac{\delta}{\delta t}\left[\frac{\delta^3 M_{(x^2)}(t)}{\delta t^3}\right]$$

$$= \frac{\delta}{\delta t}\left[n(n+2)(n+4)(1-2t)^{-\left(\frac{n}{2}+3\right)}\right]_{t=0}$$

$$= -n(n+2)(n+4)\left(\frac{n}{2}+3\right)(1-2t)^{-\left(\frac{n}{2}+4\right)}(-2)\Big|_{t=0}$$

$$= n(n+2)(n+4)(n+6)(1-2t)^{-\left(\frac{n}{2}+4\right)}\Big|_{t=0}$$

$$= n(n+2)(n+4)(n+6)$$

$$\therefore \mu_4 = \mu_4' - 4\mu_3'\mu_1' + 6\mu_2'\mu_1'^2 - 3\mu_1'^4$$

$$= n(n+2)(n+4)(n+6) - 4n(n+2)(n+4)n + 6n(n+2)n^2 - 3n^4$$

$$= 12n^2 + 48n$$

∴ Now,

$$\beta_1 = \frac{\mu_3^2}{\mu_2^3} = \frac{(8n)^2}{(2n)^3} = \frac{64n^2}{8n^3} = \frac{8}{n}$$

which is positively skewed.

$$\beta_2 = \frac{\mu_4}{\mu_2^2} = \frac{12n^2+48n}{(2n)^2} = \frac{12n^2+48n}{4n^2}$$

$$= 3 + \frac{12}{n} > 3$$

So, the shape of the χ^2 distn is leptokurtic.

$\boxed{**}$ C.G.F of χ^2 distribution:

$$\phi_{\chi^2}(t) = \log M_{\chi^2}(t) = \log (1-2t)^{-n/2}$$

$$= -\frac{n}{2} \log(1-2t)$$

$$= -\frac{n}{2}\left(-2t - \frac{4t^2}{2} - \frac{8t^3}{3} - \frac{16t^4}{4} - \cdots\right)$$

$$= \frac{n}{2}\left(2t + \frac{4t^2}{2} + \frac{8t^3}{3} + \frac{16t^4}{4} + \cdots\right)$$

$= nt + nt^2 + \frac{4}{3}nt^3 + 2nt^4 + \cdots$

$\therefore K_1 =$ coefficient of $\frac{t}{1!} = n$

$K_2 =$ coefficient of $\frac{t^2}{2!} = 2n$

$K_3 =$ " " $\frac{t^3}{3!} = 8n$

$K_4 =$ coefficient of $\frac{t^4}{4!} = 48n$

$\therefore K_1 = \mu_1' = n$

$K_2 = \mu_2 = 2n$

$K_3 = \mu_3 = 8n$

$K_4 = \mu_4 - 3K_2^2 = 48n$

$\boxed{**}$ Properties of χ^2 distn :

$\boxed{1}$. If x has a density function $f(x) = e^{-x}$, $x > 0$; then $2x$ follows χ^2 distn with 2 df. The moment generating function of $2x$ is $M_{2x}(t) = E(e^{2xt})$

$$= \int_0^\infty e^{2xt} f(x)\, dx$$

$$= \int_0^\infty e^{2xt} e^{-x}\, dx$$

$$= \int_0^\infty e^{-x(1-2t)}\, dx$$

$$= \int_0^\infty e^{-(1-2t)x} \cdot x^{1-1}\, dx$$

$$= (1-2t)^{-1} \Gamma 1$$

$M_{2x}(t) = (1-2t)^{-1}$ which is the m.g.f of ~~a~~ χ^2 variate.

Hence $2x$ is distributed as χ^2 distn with 2 d.f.

$$f(\chi^2) = \frac{e^{-\frac{\chi^2}{2}} (\chi^2)^{\frac{2}{2}-1}}{2^{2/2} \gamma \frac{2}{2}} = \frac{e^{-\frac{\chi^2}{2}} (\chi^2 \ldots}{2}$$

$$= \frac{e^{-\frac{\chi^2}{2}}}{2} \quad ; (0, \infty) \chi^2$$

$\boxed{\cancel{Th}}$ If $x \to N(0,1)$ the $x^2 \to \chi^2$ distn with 1 d.f

Proof: Since $x \to N(0,1)$

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \quad -\infty < x < \infty$$

Now the m.g.f of x^2 is $M_{x^2}(t)$

$E(e^{tx^2})$

$$= \int_{-\infty}^{\infty} e^{tx^2} f(x)\, dx.$$

$$\therefore \int_{-\infty}^{\infty} e^{tx^2} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2\left(\frac{1-2t}{2}\right)} dx$$

$$= \frac{1}{\sqrt{2\pi}} \cdot 2 \int_{0}^{\infty} e^{-x^2\left(\frac{1-2t}{2}\right)} dx$$

$$= \frac{1}{\sqrt{2\pi}} \, 2 \int_{0}^{\infty} e^{-\left(\frac{1-2t}{2}\right)x^2} (x^2)^{\frac{1}{2}-1} dx$$

$$= \frac{1}{\sqrt{2\pi}} \frac{\cancel{2} \left(\frac{1-2t}{2}\right)^{-\frac{1}{2}} \Gamma\frac{1}{2}}{\cancel{2}}$$

$$= (1-2t)^{-1/2}$$

$$\therefore M_{(x^2)}(t) = (1-2t)^{-1/2}$$

Hence x^2 is distributed as χ^2 dist^n with 1 d.f.

$$\therefore f(\chi^2) = \frac{e^{-\frac{\chi^2}{2}} (\chi^2)^{\frac{1}{2}-1}}{2^{1/2}\,\Gamma\frac{1}{2}}$$

$$= \frac{e^{-\frac{x^2}{2}} (x^2)^{-1/2}}{\sqrt{2\pi}} \qquad 0 < x^2 < \infty$$

Additive property: If $\chi_i^2, i=1,2,3,\ldots,k$ are independent chi-square variates with n_i df respectively, then the $\sum_{i=1}^{k} \chi_i^2$ is also a chi-square variates with $N = \sum_{i=1}^{k} n_i$ df.

Proof: We get from the m.g.f of ~~ch~~ chi-square distribution,

$$M_{\chi_i^2}(t) = (1-2t)^{-n_i/2} \quad i=1,2,\ldots,k$$

Now, ~~the moment generating sum of the~~

Now, the m.g.f of the sum $\sum_{i=1}^{k} \chi_i^2$ is given by

$$M_{\sum_{i=1}^{k}\chi_i^2}(t) = M_{\chi_1^2}(t) \times M_{\chi_2^2}(t) \times M_{\chi_3^2}(t) \cdots \times M_{\chi_{nk}^2}(t)$$

$$= (1-2t)^{-n_1/2} \times (1-2t)^{-n_2/2} \times \cdots \times (1-2t)^{-n_k/2}$$

$$= (1-2t)^{-\frac{1}{2}(n_1+n_2+\cdots+n_k)}$$

$$= (1-2t)^{-\sum_{i=1}^{k} n_i/2}$$

$\Sigma \frac{(x_i - u_i)^2}{\sigma^2}$

Definition

A random variable χ^2 of the continuous type is said to have a chi-square distribution with n degrees of freedom (d.f.) if its p.d.f is defined as-

$$f(\chi^2)= e^{-\frac{\chi^2}{2}} (\chi^2)^{\frac{1}{2}} \Big/ 2^{\frac{n}{2}}\,\Gamma\frac{n}{2} \quad ; 0 < \chi^2 < \infty$$

Here n is the parameter of chi-square distribution and its known as degrees of freedom.

Note: It is also called that the chi-square distribution is a particular case of two parameter gamma distribution with parameter $\lambda = \frac{1}{2}$ and $\mu = \frac{n}{2}$.

Definition of chi-square statistic

The square of a standard normal variate is known as a chi-square variate with 1 d.f. Thus if $X \sim N(\mu, \sigma^2)$, $Z = \frac{X-\mu}{\sigma}$ is N (0, 1) and $Z^2 = \left(\frac{X-\mu}{\sigma}\right)^2$ is a chi-square variate with 1 d.f.

In general if X_i (i = 1, 2, 3, … … … … n) are n independent normal variates with mean μ_i and variance σ_i^2, (I = 1, 2, 3, … … … … ,n) then $\chi^2 = \sum_{i=1}^{n}\left(\frac{X_i - \mu_i}{\sigma_i^2}\right)$ is a chi-square variate with n d.f.

Derivation

If X_i i = 1, 2, 3, …………., n are independent N (μ, σ^2). We want to find the distribution of $\chi^2 = \sum_{i=1}^{n}\left(\frac{X_i - \mu_i}{\sigma_i}\right)^2 = \sum_{i=1}^{n} u_i^2$; where, $u_i = \frac{X_i - \mu_i}{\sigma_i}$.

Since X_i's are independent u_i's are also independent. Therefore, the m.g.f. of χ^2 is-

$$M_{\chi^2}(t) = M_{\sum_{i=1}^{n} u_i^2}(t) = \prod_{i=1}^{n} M_{u_i^2}(t) = \left[M_{u_i^2}(t)\right]^n$$

Since u_i's N (0, 1) are identically distributed

$$\therefore f(u_i) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}u_i^2} \quad ; -\infty < u_i < \infty$$

$$Now,\ M_{u_i^2}(t) = E\left(e^{tu_i^2}\right) = \int_{-\infty}^{\infty} e^{tu_i^2} f(u_i)\,du_i = \int_{-\infty}^{\infty} e^{tu_i^2} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}u_i^2}\,du_i$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}u_i^2(1-2t)}\,du_i$$

Let,

$$y_i = u_i\sqrt{1-2t} \Rightarrow du_i = \frac{dy_i}{\sqrt{1-2t}}$$

$$\therefore M_{u_i^2}(t) = \frac{1}{\sqrt{1-2t}}\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}y_i^2}\,dy_i = \frac{1}{\sqrt{1-2t}}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-\frac{1}{2}y_i^2}\,dy_i$$

$$= \frac{1}{\sqrt{1-2t}}\frac{2}{\sqrt{2\pi}}\int_0^{\infty}e^{-\frac{1}{2}y_i^2}\,dy_i \qquad \left|\begin{array}{l} let\ z_i = \frac{1}{2}y_i^2 \Rightarrow y_i^2 = 2z_i \Rightarrow y_i = \sqrt{2z_i} \\ dy_i = \frac{du_i}{\sqrt{2z_i}} \end{array}\right.$$

$$= \frac{1}{\sqrt{1-2t}}\frac{2}{\sqrt{2\pi}}\int_0^{\infty}e^{-z_i}\frac{du_i}{\sqrt{2z_i}} = \frac{1}{\sqrt{1-2t}}\frac{1}{\sqrt{\pi}}\int_0^{\infty}e^{-z_i}z_i^{\frac{1}{2}-1}du_i$$

$$= \frac{1}{\sqrt{1-2t}}\frac{1}{\sqrt{\pi}}\Gamma\left(\tfrac{1}{2}\right) = \frac{1}{\sqrt{1-2t}}\frac{1}{\sqrt{\pi}}\sqrt{\pi} = (1-2t)^{-\frac{1}{2}} \qquad \left|\because \Gamma\left(\tfrac{1}{2}\right) = \int_0^{\infty}e^{-z_i}z_i^{\frac{1}{2}-1}du_i\right.$$

$$\therefore M_{\chi^2}(t) = \left[(1-2t)^{-\frac{1}{2}}\right]^n = (1-2t)^{-\frac{n}{2}}$$

This is the m.g.f. of a gamma variate with parameters $\mu = \frac{n}{2}$ and $\lambda = \frac{1}{2}$. Hence by uniqueness theorem of m.g.f.'s – $\chi^2 = \sum_{i=1}^{n}\left(\frac{X_i - \mu_i}{\sigma_i}\right)^2$ is a gamma variate with parameter n/2 and ½

$$\therefore dP(\chi^2) = \frac{\left(\frac{1}{2}\right)^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)}e^{-\frac{\chi^2}{2}}(\chi^2)^{\frac{n}{2}-1}\,d\chi^2 = \frac{1}{2^{\frac{n}{2}}\Gamma\left(\frac{n}{2}\right)}e^{-\frac{\chi^2}{2}}(\chi^2)^{\frac{n}{2}-1}\,d\chi^2;\ 0 < \chi^2 < \infty$$

This is the required probability differential function of chi-square distribution with n d.f.

Mean

The mean of chi-square distribution is-

$$E(\chi^2) = \mu_1' = \int_0^{\infty}\chi^2 f(\chi^2)\,d\chi^2 = \int_0^{\infty}\chi^2\frac{1}{2^{\frac{n}{2}}\Gamma\left(\frac{n}{2}\right)}e^{-\frac{\chi^2}{2}}(\chi^2)^{\frac{n}{2}-1}\,d\chi^2$$

$$= \frac{1}{2^{\frac{n}{2}}\Gamma\left(\frac{n}{2}\right)}\int_0^{\infty}e^{-\frac{\chi^2}{2}}(\chi^2)^{\frac{n}{2}+1-1}\,d\chi^2 = \frac{1}{2^{\frac{n}{2}}\Gamma\left(\frac{n}{2}\right)}\frac{\Gamma\left(\frac{n}{2}+1\right)}{\left(\frac{1}{2}\right)^{\frac{n}{2}+1}}$$

$$= \frac{1}{2^{\frac{n}{2}}\Gamma\left(\frac{n}{2}\right)}\frac{\frac{n}{2}\Gamma\left(\frac{n}{2}\right)}{(2)^{-\frac{n}{2}-1}} = \frac{n}{2}\times 2 = n \qquad \left|\because \Gamma(n+1) = n\Gamma n\right.$$

Therefore, mean of chi-square distribution is 'n'

Variance

$$V(\chi^2) = E\{(\chi^2)^2\} - [E(\chi^2)]^2; \quad Now,\ E\{(\chi^2)^2\} = \int_0^{\infty}(\chi^2)^2 f(\chi^2)\,d\chi^2$$

$$V(\chi^2) = \frac{1}{2^{n/2}\Gamma(n/2)}\int_0^\infty (\chi^2) e^{-\chi^2/2} (\chi^2)^{n/2-1}\, d\chi^2 = \frac{1}{2^{n/2}\Gamma(n/2)}\int_0^\infty e^{-\chi^2/2} (\chi^2)^{n/2+2-1}\, d\chi^2$$

$$= \frac{1}{2^{n/2}\Gamma(n/2)} \frac{\Gamma(n/2+2)}{(1/2)^{n/2+2}} = \frac{n/2\,(n/2+1)\,\Gamma(n/2)}{2^{n/2}\Gamma(n/2)}\, 2^{n/2+2} \quad \left|\; \because \int_0^\infty e^{-ax}(x)^{n-1}\,dx = \frac{\Gamma n}{a^{-n}};\right.$$

$$\text{and } \Gamma(n+1) = n\Gamma n$$

$$= \frac{n(n+2)}{4} \times 4 = n(n+2)$$

$$\therefore V(\chi^2) = n(n+2) - n^2 = n^2 + 2n - n^2 = 2n$$

Therefore, variance of chi-square distribution is '2n'

Moment Generating Function (m.g.f.)

The m.g.f. of chi-square distribution with respect to origin is-

$$M_{\chi^2}(t) = E(e^{t\chi^2}) = \int_0^\infty e^{t\chi^2} f(\chi^2)\, d\chi^2 = \frac{1}{2^{n/2}\Gamma(n/2)}\int_0^\infty e^{t\chi^2} e^{-\frac{1}{2}\chi^2} (\chi^2)^{n/2-1}\, d\chi^2$$

$$= \frac{1}{2^{n/2}\Gamma(n/2)}\int_0^\infty e^{-\frac{1}{2}\chi^2(1-2t)} (\chi^2)^{n/2-1}\, d\chi^2 = \frac{1}{2^{n/2}\Gamma(n/2)} \frac{\Gamma(n/2)}{\left(\frac{1-2t}{2}\right)^{n/2}} \quad \left|\; \because \int_0^\infty e^{-ax} x^{n-1}\,dx = \Gamma n / a^n\right.$$

$$= (1-2t)^{-n/2} \quad |2t|<1$$

$$\therefore M_{\chi^2}(t) = (1-2t)^{-n/2} \quad |2t|<1 \quad \ldots\ldots\ldots (1)$$

Therefore, the m.g.f. of χ^2 -distribution about origin is $(1-2t)^{-n/2}$ $|2t| < 1$

Using binomial expansion for negative index we get from (1), if $|2t| < 1$

$$M_{\chi^2}(t) = 1 + \frac{n}{2}2t + \frac{\frac{n}{2}\left(\frac{n}{2}+1\right)}{2!}(2t)^2 + \frac{\frac{n}{2}\left(\frac{n}{2}+1\right)\left(\frac{n}{2}+2\right)}{3!}(2t)^3 + \ldots\ldots + \frac{\frac{n}{2}\left(\frac{n}{2}+1\right)\ldots\ldots\left(\frac{n}{2}+r-1\right)}{r!}(2t)^r + \ldots.$$

$\mu_r' = r$th raw moment $=$ coefficient of $t^r/r!$ in the expansion of $M_{\chi^2}(t)$

$$= 2^r\, \frac{n}{2}\left(\frac{n}{2}+1\right)\ldots\ldots\left(\frac{n}{2}+r-1\right) = n(n+2)(n+4)\ldots\ldots(n+2r-2)$$

$\therefore\ \mu_1' = n = \text{Mean}$ putting $r = 1$

$\mu_2' = n(n+2)$ putting $r = 2$

$\mu_3' = n(n+2)(n+4)$ putting $r = 3$

$\mu_4' = n(n+2)(n+4)(n+6)$ putting $r = 4$

Variance $= \mu_2 = \mu_2' - (\mu_1')^2 = n(n+2) - n^2 = 2n$

$$\mu_3 = \mu_3' - 3\mu_2'\mu_1' + 2(\mu_1')^3 = n(n+2)(n+4) - 3n(n+2) + 2n^3 = n^3 + 6n^2 + 8n - 3n^3 - 6n^2 + 2n^3$$

$$= 8n$$

$$\mu_4 = \mu_4' - 4\mu_3'\mu_1' + 6\mu_2'(\mu_1')^2 - 3(\mu_1')^4$$

$$= n(n+2)(n+4)(n+6) - 4n(n+2)(n+4)n + 6n(n+2)n^2 - 3n^4$$

$$= (n^2+2n)(n^2+10n+24) - 4n^2(n^2+6n+8) + 6n^3(n+2) - 3n^4$$

$$= n^4 + 12n^3 + 44n^2 + 48n - 4n^4 - 24n^3 - 32n^2 + 6n^4 + 12n^3 - 3n^4 = 48n + 12n^2$$

$$\beta_1 = \mu_3^2 / \mu_2^3\ \frac{(8n)^2}{(2n)^3} = 8/n \quad \therefore \text{Skewness} = \sqrt{\beta_1} = \sqrt{8/n}$$

$\frac{n}{2}\left(\frac{n}{2}+1\right)$

$= 2n\left(\frac{n}{2}+1\right)$

$= 2n \cdot \frac{n}{2} + 2n$

$= n(n+2)$

$(\chi^2)^{\frac{n}{2}-1}$

$\dfrac{e^{-\frac{\chi^2}{2}}}{2^{\frac{n}{2}}\Gamma\frac{n}{2}}$

$$\text{Kurtosis} = \beta_2 = \mu_4/\mu_2^2 = \frac{48n+12n^2}{(2n)^2} = 3+12/n$$

$\beta_1 = \frac{8}{n} > 0$; *so the curve is positively skewed*

$\beta_2 = 3 + 12/n > 3$; *so the curve is lepto kurtic*

If $n \to \infty$, $\beta_1 \to 0$ and $\beta_2 \to 3$, then chi-square distribution tends to normal distribution.

Cumulent Generating Function (c.g.f)

We know the m.g.f of chi-square distribution with n d.f. is-

$$M_{\chi^2}(t) = (1-2t)^{-n/2}\ ; \quad |2t|<1$$

Therefore, c.g.f of chi-square distribution is-

$$K_{\chi^2}(t) = \log\left(M_{\chi^2}(t)\right) = \log(1-2t)^{-n/2} = -\frac{n}{2}\log(1-2t)$$

$$= -\frac{n}{2}\left[-2t - \frac{(2t)^2}{2} - \frac{(2t)^3}{3} - \frac{(2t)^4}{4} - \dots\dots\dots\dots - \frac{(2t)^r}{r} - \dots\dots\right]$$

$$= \frac{n}{2}\left[2t + \frac{(2t)^2}{2} + \frac{(2t)^3}{3} + \frac{(2t)^4}{4} + \dots\dots\dots\dots + \frac{(2t)^r}{r} + \dots\dots\right]$$

K_r = rth cumulent = coefficient of $t^r/r!$ in the expansion of $K_{\chi^2}(t)$

$$= n/2 \cdot 2^r\,(r-1)! = n2^{r-1}\,(r-1)!$$

Putting r = 1, 2, 3, 4 then we get

$K_1 = n = \text{Mean} = \mu'_1$ putting r = 1

$K_2 = 2n = \text{variance} = \mu_2$ putting r = 2 1 Cx

$K_3 = 8n = \mu_3$ putting r = 3

$K_4 = 48n \quad \therefore \mu_4 = K_4 + 3K_2^2 = 48n+12n^2$ putting r = 4

Now, $\beta_1 = \mu_3^2/\mu_2^3 = (8n)^2/(2n)^3 = 8/n \quad \therefore$ *skewness,* $\sqrt{\beta_1} = \sqrt{8/n}$ Cq

Kurtosis, $\beta_2 = \mu_4/\mu_2^2 = \dfrac{48n+12n^2}{4n^2} = 3+12/n$ = (a×3×2

$\beta_1 = \frac{8}{n} > 0$, the curve of the distribution is positively skewed

$\beta_2 = 3 + \frac{12}{n} > 3$, the shape of the curve of the distribution is lepto kurtic

Additive Property

Statement: The sum of the two independent chi-square variates is also a chi-square variate with sum of their respective d.f.

i.e. if χ_1^2 and χ_2^2 are two independent chi-square variate with n_1 and n_2 d.f. respectively, then $\chi_1^2 + \chi_2^2$ is also a chi-square variatė with $n_1 + n_2$ d.f.

Proof:

Since, $\chi_1^2 \sim \chi^2$ distribution with n_1 d.f. then we get from m.g.f. of chi-squre distribution is- $M_{\chi_1^2}(t)=(1-2t)^{-n_1/2}$

Again, $\chi_2^2 \sim \chi^2$ distribution with n_1 d.f. then we get from m.g.f. of chi-squre distribution is- $M_{\chi_2^2}(t)=(1-2t)^{-n_2/2}$

Now, the m.g.f. of the sum of $(\chi_1^2 + \chi_2^2)$ is given by-

$$M_{\chi_1^2+\chi_2^2}(t) = M_{\chi_1^2}(t) \times M_{\chi_2^2}(t) \qquad \Big| \because \chi_1^2 \textit{ and } \chi_2^2 \textit{ are independent}$$

$$= (1-2t)^{-n_1/2} \times (1-2t)^{-n_2/2}$$

$$= (1-2t)^{-\left(n_1/2 + n_2/2\right)} = (1-2t)^{-\left(n_1+n_2/2\right)}$$

this is the m.g.f. of a chi-square variate with $(n_1 + n_2)$ d.f. Hence by uniqueness theorem of the m.g.f.'s $(\chi_1^2 + \chi_2^2)$ is also a chi-square variate with $(n_1 + n_2)$ d.f.

Generalisation of additive property

Statement

If χ_i^2 $i=1, 2, 3, \ldots \ldots \ldots, k$ are independent chi-square variates with n_i d.f. respectively, then the $\sum_{i=1}^{k} \chi_i^2$ is also a chi-square variate with $N=\sum_{i=1}^{k} n_i$ d.f.

Proof: We get from the m.g.f. of chi-square distribution,

$$M_{\chi_i^2}(t)=(1-2t)^{-n_i/2} \; ; \; i=1,2,\ldots \;\ldots.\;\ldots., k$$

Now, the m.g.f. of the sum $\sum_{i=1}^{k} \chi_i^2$ is given by

$$M_{\sum_{i=1}^{k}\chi_i^2}(t)=M_{\chi_1^2}(t)\times M_{\chi_2^2}(t)\times \ldots \ldots \times M_{\chi_{n_k}^2}(t) \;\Big|\because \chi_i^2\text{'}s \textit{ are independent}$$

$$= (1-2t)^{-n_1/2} \times (1-2t)^{-n_2/2} \times \ldots\ldots\ldots\ldots \times (1-2t)^{-n_k/2}$$

$$= (1-2t)^{-1/2(n_1+n_2+\ldots\ldots\ldots\ldots+n_k)} = (1-2t)^{-\sum^{k} n_i/2}$$

theorem of m.g.f.'s $\sum_{i=1}^{k}\chi_i^2$ is a chi-square variate with $N=\sum_{i=1}^{k} n_i$ d.f.

Chi-square distribution tends to normal distribution for large d.f. i.e. n $\rightarrow\infty$

We know for chi-square distribution with n d.f.- $E(\chi^2) = n$ and $V(\chi^2) = 2n$.

Then the standard variate is - $z=\dfrac{\chi^2-n}{\sqrt{2n}}=\dfrac{\chi^2}{\sqrt{2n}}-\sqrt{n/2}$

Now the m.g.f. of standard chi-square variate z is given by-

$$M_z(t)=M_{\chi^2/\sqrt{2n}-\sqrt{n/2}}(t)=e^{-t\sqrt{n/2}}M_{\chi^2/\sqrt{2n}}(t)=e^{-t\sqrt{n/2}}\left(1-2t/\sqrt{2n}\right)^{-n/2}$$

$$\because M_{\chi^2}(t)=(1-2t)^{-n/2}$$

$$\therefore K_z(t)=\log M_z(t)=\log\left[e^{-t\sqrt{n/2}}\left(1-2t/\sqrt{2n}\right)^{-n/2}\right]$$

$$=-t\sqrt{n/2}-n/2\log\left(1-t\sqrt{2/n}\right)$$

$$=-t\sqrt{n/2}-n/2\left[-t\sqrt{2/n}-t^2/2\,2/n-t^3/3\left(2/n\right)^{3/2}-\dots\dots\right]$$

$$=-t\sqrt{n/2}+t\sqrt{n/2}+t^2/2+0\left(1/\sqrt{n}\right)=t^2/2+0\left(1/\sqrt{n}\right)$$

Where $0\left(1/\sqrt{n}\right)$ are terms containing $\sqrt{n}$ and higher powers of n in the denominator.

Then,

$$\lim_{n\rightarrow\infty}K_z(t)=t^2/2\Rightarrow\lim_{n\rightarrow\infty}M_z(t)=e^{t^2/2}$$

$$i.e.\ M_z(t)=e^{t^2/2}\ as\ n\rightarrow\infty$$

This is the m.g.f of a standard normal variate. Hence by uniqueness theorem of m.g.f. standard hi-square variate tends to standard normal variate as n $\rightarrow\infty$. Therefore, chi-square distribution tends to normal distribution for large d.f.

Theorem: If χ^2 is a chi-square variate with n d.f. then prove that for large n, $\sqrt{2\chi^2}\sim N\left(\sqrt{2n},1\right)$

Proof: We know that for χ^2 distribution with n d.f. $E(\chi^2) = n$ and $V(\chi^2) = 2n$

Then the standard χ^2 variate, $z = \dfrac{\chi^2 - n}{\sqrt{2n}} \sim N(0, 1)$ *for large n*

Let us consider,

$$P\left(\frac{\chi^2 - n}{\sqrt{2n}} \le z\right) = P\left(\chi^2 \le n + z\sqrt{2n}\right) = P\left[2\chi^2 \le 2n + 2z\sqrt{2n}\right]$$

$$= P\left[\sqrt{2\chi^2} \le \sqrt{2n + 2z\sqrt{2n}}\right] = P\left[\sqrt{2\chi^2} \le \left(2n + 2z\sqrt{2n}\right)^{1/2}\right]$$

$$= P\left[\sqrt{2\chi^2} \le \left(2n\left\{1 + z\sqrt{2/n}\right\}\right)^{1/2}\right] = P\left[\sqrt{2\chi^2} \le \sqrt{2n}\left\{1 + z\sqrt{2/n}\right\}^{1/2}\right]$$

$$= P\left[\sqrt{2\chi^2} \le \sqrt{2n}\left\{1 + z/2\sqrt{2/n} + \frac{1/2\left(1/2 - 1\right)}{2!} z^2 \frac{2}{n} + \dots\dots\dots\right\}\right]$$

$$= P\left[\sqrt{2\chi^2} \le \sqrt{2n}\left\{1 + z/\sqrt{2n} - \frac{z^2}{2n} + \dots\dots\dots\right\}\right]$$

$$\approx P\left[\sqrt{2\chi^2} \le \sqrt{2n} + z\right];\ \textit{for large } n$$

$$= P\left[\sqrt{2\chi^2} - \sqrt{2n} \le z\right];\ \textit{for lrge } n$$

Since for large n, $\dfrac{\chi^2 - n}{\sqrt{2n}} \sim$ N (0, 1), we conclude that $\left(\sqrt{2\chi^2} - \sqrt{2n}\right) \sim$ N(0, 1),

which implies that $\sqrt{2\chi^2} \sim N(\sqrt{2n}, 1)$

Theorem: If χ^2 is a chi-square variate with n d.f. then $\sqrt{2\chi^2}$ is approximately distributed as $N\left(\sqrt{2n-1}, 1\right)$ variate as $n \to \infty$.

Proof: We know that for χ^2 distribution with n d.f. $E(\chi^2) = n$ and $V(\chi^2) = 2n$.

Then the standard χ^2 variate is defined as- $z = \dfrac{\chi^2 - n}{\sqrt{2n}} \sim$ N(0, 1) for large n

Now,

$$P\left[\sqrt{2\chi^2} - \sqrt{2n-1} \le z\right] = P\left[\sqrt{2\chi^2} \le z + \sqrt{2n-1}\right]$$

$$= P\left[2\chi^2 \le z^2 + 2z\sqrt{2n-1} + 2n - 1\right] = P\left[\chi^2 \le z^2/2 + z\sqrt{2n-1} + \left(n - 1/2\right)\right]$$

$$= P\left[\chi^2 - n \le z^2/2 + z\sqrt{2n-1} - 1/2\right] = P\left[\frac{\chi^2 - n}{\sqrt{2n}} \le z^2/2\sqrt{2n} + z\sqrt{2n-1/2n} - 1/2\sqrt{2n}\right]$$

$$= P\left[\frac{\chi^2 - n}{\sqrt{2n}} \le z^2/2\sqrt{2n} + z\sqrt{1 - 1/2n} - 1/2\sqrt{2n}\right] = P\left[\frac{\chi^2 - n}{\sqrt{2n}} \le z\right] \textit{as } n \to \infty$$

Since for large n, $\frac{\chi^2-n}{\sqrt{2n}} \sim N(0, 1)$.

Hence $\sqrt{2\chi^2}-\sqrt{2n-1}$ is approximately a standardized normal variate as n → ∞. In other words, $\sqrt{2\chi^2}$ is an $N\left(\sqrt{2n-1}, 1\right)$ variate as n → ∞.

Mode and Skewness of chi-square distribution

Mode of chi-square distribution is the solution of $f'\left(\chi^2\right)=0$ *and* $f''\left(\chi^2\right)<0$

Now,

$$f(\chi^2)=\frac{1}{2^{n/2}\Gamma\left(\frac{n}{2}\right)}e^{-\chi^2/2}\left(\chi^2\right)^{n/2-1}; \qquad 0<\chi^2<\infty$$

$$\therefore f'(\chi^2)=\frac{\partial}{\partial\chi}\left[\frac{1}{2^{n/2}\Gamma\left(\frac{n}{2}\right)}e^{-\chi^2/2}\left(\chi^2\right)^{n/2-1}\right]$$

$$=\frac{1}{2^{n/2}\Gamma\left(\frac{n}{2}\right)}\left[-\frac{1}{2}e^{-\chi^2/2}\left(\chi^2\right)^{n/2-1}+\left(\frac{n}{2}-1\right)\left(\chi^2\right)^{n/2-2}.e^{-\chi^2/2}\right]$$

$$=\frac{1}{2^{n/2}\Gamma\left(\frac{n}{2}\right)}e^{-\chi^2/2}\left(\chi^2\right)^{n/2-1}\left[\left(\frac{n}{2}-1\right)\frac{1}{\chi^2}-\frac{1}{2}\right]$$

But since $e^{-\chi^2/2}\left(\chi^2\right)^{n/2-1}\neq 0$, *then* $f'\left(\chi^2\right)=0$ *iff* $\left(n/2-1\right)\chi^2-1/2=0$

$$\Rightarrow \chi^2=n-2$$

It can be easily seen that at the point, $\chi^2 = n - 2$, $f''(\chi^2) < 0$.

Hence mode of chi-square distribution with n d.f. is (n – 2).

Also karl-Pearson's coefficient of skewness is –

$$S_k=\left(Mean-Mode\right)/S.D=\{n-(n-2)\}/\sqrt{2n}=\sqrt{2/n}$$

Comment: Since Karl-Pearson's coefficient of skewness is greater than zero for n ≥ 1, therefore, the chi-square distribution is positively skewed.

Properties of chi-square distribution

The properties of chi-square distribution are given below-

(i) Chi-square variate has a distribution of continuous type and its range is 0 to ∞,

(ii) It is an exact sampling distribution derived from normal distribution,

(iii) The p.d.f. of chi-square distribution with n d.f. is

$$f(\chi^2)=\frac{1}{2^{n/2}\Gamma\left(\frac{n}{2}\right)}e^{-\chi^2/2}(\chi^2)^{\frac{n}{2}-1}; \quad 0<\chi^2<\infty$$

(iv) Chi-square distribution with n d.f. has mean = n, variance = 2n and mode = n − 2. Thus mean of chi-square distribution is greater than its mode,

(v) Chi-square distribution possesses additive property i.e. sum of the two or more independent chi-square variates are also a chi-square variate,

(vi) For large d.f. chi-square distribution tends to normal distribution,

(vii) Pearson's coefficient of skewness for chi-square distribution is $\beta_1 = 8/n$ *and kurtosis*, $\beta_2 = 3+12/n$. So the shape of chi-square distribution is positively skewed and leptokurtic.

(viii) If χ^2_1 and χ^2_2 are independently distributed as chi-square distribution with n_1 and n_2 d.f. respectively, then the ratio-

$$\chi_1^2 \Big/ \chi_1^2+\chi_2^2 \sim \beta_1\left(\frac{n_1}{2},\frac{n_2}{2}\right) \text{ and } \chi_1^2 \Big/ \chi_1^2 \sim \beta_2\left(\frac{n_1}{2},\frac{n_2}{2}\right)$$

(ix) If χ^2 is a chi-square variate with n d.f. then $\sqrt{2\chi^2} \sim N\left(\sqrt{2N},1\right)$ for large n.

Application of chi-square distribution

It has a large number of applications in statistics, some of are given below-

(i) Chi-square distribution is used to test the hypothetical value of the population variance is $\sigma^2 = \sigma_0^2\ (say)$,

(ii) It is used to test the goodness of fit of any distribution,

(iii) It is used to test the independence of attributes,

(iv) It is used to test the homogeneity of the independent estimates of the population variances,

(v) It is used to test the homogeneity of the independent estimates of the population correlation coefficients and

(vi) It is used to combine various probabilities obtained from independent experiments to give a single test of significance.

Theorem: If χ_1^2 and χ_2^2 are two independent chi-square variate with n_1 and n_2 d.f. respectively, then $\chi_1^2 \Big/ \chi_2^2$ *is a* $\beta_2\left(n_1/2, n_2/2\right)$ variate.

Proof: Since χ_1^2 and χ_2^2 are two independent chi-square variate with n_1 and n_2 d.f. respectively, the their joint probability density function is given by-

$$f(\chi_1^2,\chi_2^2)=f(\chi_1^2)\times f(\chi_2^2)=\frac{1}{2^{n_1/2}\Gamma\left(n_1/2\right)}e^{-\chi_1^2/2}\left(\chi_1^2\right)^{n_1/2-1}\times\frac{1}{2^{n_2/2}\Gamma\left(n_2/2\right)}e^{-\chi_2^2/2}\left(\chi_2^2\right)^{n_2/2-1}$$

$$=\frac{1}{2^{n_1/2+n_2/2}\Gamma\left(n_1/2\right)\Gamma\left(n_2/2\right)}e^{-\left(\chi_1^2/2+\chi_2^2/2\right)}\left(\chi_1^2\right)^{n_1/2-1}\left(\chi_2^2\right)^{n_2/2-1};\ 0<(\chi_1^2,\chi_2^2)<\infty$$

Let us make the transformation- $u=\chi_1^2/\chi_2^2$ *and* $v=\chi_2^2$

So that $\chi_1^2=uv$ *and* $\chi_2^2=v$

Therefore, jacobian transformation J is given by-

$$J=\frac{\partial\left(\chi_1^2,\chi_2^2\right)}{\partial(u,v)}=\begin{vmatrix} v & u \\ 0 & 1\end{vmatrix}=v\ \therefore |J|=v;\qquad 0<u,v<\infty$$

Therefore, joint p.d.f. of u and v is-

$$f(u,v)=\frac{1}{2^{n_1/2+n_2/2}\Gamma\left(n_1/2\right)\Gamma\left(n_2/2\right)}e^{-\frac{1}{2}v(1+u)}(uv)^{n_1/2-1}(v)^{n_2/2-1}\times v;\ 0<u,v<\infty$$

$$=\frac{1}{2^{n_1/2+n_2/2}\Gamma\left(n_1/2\right)\Gamma\left(n_2/2\right)}e^{-\frac{1}{2}v(1+u)}u^{n_1/2-1}v^{\frac{n_1+n_2}{2}-1}$$

$\frac{n_1}{2}-1+\frac{n_2}{2}$

Now, the marginal p.d.f. of u is given by-

$$f(u)=\int_0^\infty f(u,v)dv=\frac{u^{n_1/2-1}}{2^{n_1/2+n_2/2}\Gamma\left(n_1/2\right)\Gamma\left(n_2/2\right)}\int_0^\infty e^{-\frac{1}{2}v(1+u)}v^{\frac{n_1+n_2}{2}-1}dv$$

$$=\frac{u^{n_1/2-1}}{2^{n_1/2+n_2/2}\Gamma\left(n_1/2\right)\Gamma\left(n_2/2\right)}\cdot\frac{\Gamma\left(\frac{n_1+n_2}{2}\right)}{\left(\frac{1+u}{2}\right)^{\frac{n_1+n_2}{2}}}=\frac{1}{\beta\left(\frac{n_1}{2},\frac{n_2}{2}\right)}\frac{u^{n_1/2-1}}{(1+u)^{\frac{n_1+n_2}{2}}}$$

$$\therefore f(u)=\frac{1}{\beta\left(\frac{n_1}{2},\frac{n_2}{2}\right)}\frac{u^{n_1/2-1}}{(1+u)^{\frac{n_1+n_2}{2}}};\ 0<u<\infty$$

This is the p.d.f. of beta variate of second kind.

Theorem: If χ_1^2 and χ_2^2 are two independent chi-square variate with n_1 and n_2 d.f. respectively, then $\chi_1^2/\left(\chi_1^2+\chi_2^2\right)$ *is a* $\beta_1\left(\frac{n_1}{2},\frac{n_2}{2}\right)$ variate.

Proof: Since χ_1^2 and χ_2^2 are two independent chi-square variate with n_1 and n_2 d.f. respectively, the their joint probability density function is given by-

$$f(\chi_1^2,\chi_2^2)=f(\chi_1^2)\times f(\chi_2^2)=\frac{1}{2^{n_1/2}\Gamma\left(n_1/2\right)}e^{-\chi_1^2/2}\left(\chi_1^2\right)^{n_1/2-1}\times\frac{1}{2^{n_2/2}\Gamma\left(n_2/2\right)}e^{-\chi_2^2/2}\left(\chi_2^2\right)^{n_2/2-1}$$

$$=\frac{1}{2^{n_1/2+n_2/2}\Gamma\left(n_1/2\right)\Gamma\left(n_2/2\right)}e^{-\left(\chi_1^2/2+\chi_2^2/2\right)}\left(\chi_1^2\right)^{n_1/2-1}\left(\chi_2^2\right)^{n_2/2-1};\ 0<\left(\chi_1^2,\chi_2^2\right)<\infty$$

Given that $u=\dfrac{\chi_1^2}{\chi_1^2+\chi_2^2}$ *and let* $v=\chi_1^2+\chi_2^2$

Then, $\chi_1^2=uv$ *and* $\chi_2^2=v-\chi_1^2=v-uv=v(1-u)$

Now the jacobian transformation J is –

$$J=\frac{\partial\left(\chi_1^2,\chi_2^2\right)}{\partial(u,v)}=\begin{vmatrix} v & u \\ -v & 1-u \end{vmatrix}=v-uv+uv=v;\ \therefore |J|=v$$

Therefore, the joint p.d.f. of u and v is given by-

$$f(u,v)=\frac{1}{2^{n_1/2+n_2/2}\Gamma\left(n_1/2\right)\Gamma\left(n_2/2\right)}e^{-\frac{1}{2}v}(uv)^{n_1/2-1}\{v(1-u)\}^{n_2/2-1}v;\ 0<v<\infty;0<v<1$$

$$=\frac{(u)^{n_1/2-1}(1-u)^{n_2/2-1}}{2^{n_1/2+n_2/2}\Gamma\left(n_1/2\right)\Gamma\left(n_2/2\right)}e^{-\frac{1}{2}v}\,v^{n_1/2+n_2/2-1}$$

Therefore, the marginal p.d.f. of u is given by-

$$f(u)=\int_0^\infty f(u,v)dv=\frac{(u)^{n_1/2-1}(1-u)^{n_2/2-1}}{2^{n_1/2+n_2/2}\Gamma\left(n_1/2\right)\Gamma\left(n_2/2\right)}\int_0^\infty e^{-\frac{1}{2}v}\,v^{n_1/2+n_2/2-1}dv$$

$$=\frac{(u)^{n_1/2-1}(1-u)^{n_2/2-1}}{2^{n_1/2+n_2/2}\Gamma\left(n_1/2\right)\Gamma\left(n_2/2\right)}\frac{\Gamma\left(n_1/2+n_2/2\right)}{\left(1/2\right)^{n_1/2+n_2/2}}$$

$$=\frac{\Gamma\left(n_1/2+n_2/2\right)}{\Gamma\left(n_1/2\right)\Gamma\left(n_2/2\right)}(u)^{n_1/2-1}(1-u)^{n_2/2-1}=\frac{1}{\beta\left(n_1/2+n_2/2\right)}(u)^{n_1/2-1}(1-u)^{n_2/2-1}$$

$$\therefore f(u)=\frac{1}{\beta\left(n_1/2+n_2/2\right)}(u)^{n_1/2-1}(1-u)^{n_2/2-1};\quad 0<u<1$$

This is the p.d.f. of beta variate of first kind. Thus $u = \frac{\chi_1^2}{\chi_1^2 + \chi_2^2}$ is a $\beta_1\left(\frac{n_1}{2}, \frac{n_2}{2}\right)$ variate.

Theorem: If χ_1^2 and χ_2^2 are two independent chi-square variate with n_1 and n_2 d.f. respectively, then $u = \chi_1^2 / \left(\chi_1^2 + \chi_2^2\right)$ and $v = \chi_1^2 + \chi_2^2$ are independently distributed as a $\beta_1\left(\frac{n_1}{2}, \frac{n_2}{2}\right)$ variate and v is χ^2 variate with (n_1+n_2) d.f.

Proof: Since χ_1^2 and χ_2^2 are two independent chi-square variate with n_1 and n_2 d.f. respectively, the their joint probability density function is given by-

$$f(\chi_1^2, \chi_2^2) = f(\chi_1^2) \times f(\chi_2^2) = \frac{1}{2^{n_1/2}\Gamma\left(\frac{n_1}{2}\right)} e^{-\chi_1^2/2} (\chi_1^2)^{n_1/2 - 1} \times \frac{1}{2^{n_2/2}\Gamma\left(\frac{n_2}{2}\right)} e^{-\chi_2^2/2} (\chi_2^2)^{n_2/2 - 1}$$

$$= \frac{1}{2^{n_1/2 + n_2/2}\Gamma\left(\frac{n_1}{2}\right)\Gamma\left(\frac{n_2}{2}\right)} e^{-\left(\chi_1^2/2 + \chi_2^2/2\right)} (\chi_1^2)^{n_1/2 - 1} (\chi_2^2)^{n_2/2 - 1} \; ; 0 < (\chi_1^2, \chi_2^2) < \infty$$

Given that $u = \frac{\chi_1^2}{\chi_1^2 + \chi_2^2}$ and let $v = \chi_1^2 + \chi_2^2$

Then, $\chi_1^2 = uv$ and $\chi_2^2 = v - \chi_1^2 = v - uv = v(1-u)$

Now the jacobian transformation J is –

$$J = \frac{\partial(\chi_1^2, \chi_2^2)}{\partial(u, v)} = \begin{vmatrix} v & u \\ -v & 1-u \end{vmatrix} = v - uv + uv = v; \therefore |J| = v$$

Therefore, the joint p.d.f. of u and v is given by-

$$f(u, v) = \frac{1}{2^{n_1/2 + n_2/2}\Gamma\left(\frac{n_1}{2}\right)\Gamma\left(\frac{n_2}{2}\right)} e^{-\frac{1}{2}v} (uv)^{n_1/2 - 1} \{v(1-u)\}^{n_2/2 - 1} v; 0 < v < \infty; 0 < v < 1$$

$$= \frac{(u)^{n_1/2 - 1} (1-u)^{n_2/2 - 1}}{2^{n_1/2 + n_2/2}\Gamma\left(\frac{n_1}{2}\right)\Gamma\left(\frac{n_2}{2}\right)} e^{-\frac{1}{2}v} v^{n_1/2 + n_2/2 - 1}$$

$$= \frac{\Gamma\left(\frac{n_1 + n_2}{2}\right) (u)^{n_1/2 - 1} (1-u)^{n_2/2 - 1}}{\Gamma\left(\frac{n_1}{2}\right)\Gamma\left(\frac{n_2}{2}\right)} \frac{e^{-\frac{1}{2}v} v^{n_1/2 + n_2/2 - 1}}{2^{n_1/2 + n_2/2}\Gamma\left(\frac{n_1 + n_2}{2}\right)}$$

$$f(u,v)=\frac{1}{\beta\left(\frac{n_1}{2},\frac{n_2}{2}\right)}(u)^{\frac{n_1}{2}-1}(1-u)^{\frac{n_2}{2}-1}\times\frac{e^{-\frac{1}{2}v}\,v^{\frac{n_1}{2}+\frac{n_2}{2}-1}}{2^{\frac{n_1}{2}+\frac{n_2}{2}}\Gamma\left(\frac{n_1+n_2}{2}\right)}=f(u)\times f(v)$$

Where,

$$f(u)=\frac{1}{\beta\left(\frac{n_1}{2},\frac{n_2}{2}\right)}(u)^{\frac{n_1}{2}-1}(1-u)^{\frac{n_2}{2}-1}\;;\;0<u<1 \text{ and}$$

$$f(v)=\frac{1}{2^{\frac{n_1}{2}+\frac{n_2}{2}}\Gamma\left(\frac{n_1+n_2}{2}\right)}e^{-\frac{1}{2}v}\,v^{\frac{n_1}{2}+\frac{n_2}{2}-1}\;;\;0<v<\infty$$

Since the joint p.d.f. of u and v is the product of their respective marginal p.d.f. u and v. Thus u and v are independent.

Theorem: If $X_1, X_2, \dots\dots\dots\dots, X_n$ be a random sample of size n from normal population with mean μ and variance σ^2, then $\sum_{i=1}^{n}\left(\frac{X_i-\mu}{\sigma}\right)^2$ has a chi-square distribution with n d.f.

Proof: Since $X_1, X_2, \dots\dots\dots\dots, X_n$ be a random sample of size n from normal population with mean μ and variance σ^2, then each sample X_i's will be distributed with mean μ and variance σ^2 as a normal distribution.

i.e. $$f(x_i)=\frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{x_i-\mu}{\sigma}\right)^2}\;;\quad -\infty<x_i<\infty; -\infty<\mu<\infty : i=1,2,\dots\dots n$$

Then $y_i=\dfrac{x_i-\mu}{\sigma}$ follows normal distribution with mean 0 and variance 1

Now, $\sum_{i=1}^{n}y_i^2=\sum_{i=1}^{n}\left(\frac{x_i-\mu}{\sigma}\right)^2$ is the sum of square of independent standardized normal variates. By the definition of chi-square distribution we know that the sum of squares of n independent standardized normal variates is a chi-square variate with n d.f.

Hence $\sum_{i=1}^{n}y_i^2=\sum_{i=1}^{n}\left(\frac{x_i-\mu}{\sigma}\right)^2$ follows chi-square distribution with n d.f.

Theorem: If $X_1, X_2, \ldots\ldots\ldots, X_n$ be a random sample of size n from normal population with mean μ and variance σ^2. Then the sample variance S^2 defined by

$s^2 = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})^2$. Show that $U = \frac{(n-1)s^2}{\sigma^2}$ i.e $U = \frac{\sum_{i=1}^{n}(x_i - \bar{x})^2}{\sigma^2}$ has a chi-square distribution with (n – 1) d.f.

Proof: We have $U = \frac{(n-1)s^2}{\sigma^2} = U = \frac{\sum_{i=1}^{n}(x_i - \bar{x})^2}{\sigma^2}$

Now,

$$\sum_{i=1}^{n}(x_i - \bar{x})^2 = \sum_{i=1}^{n}\{(x_i - \mu) - (\bar{x} - \mu)\}^2 = \sum_{i=1}^{n}(x_i - \mu)^2 - 2(\bar{x} - \mu)\sum_{i=1}^{n}(x_i - \mu) + n(\bar{x} - \mu)^2$$

$$= \sum_{i=1}^{n}(x_i - \mu)^2 - 2(\bar{x} - \mu)n(\bar{x} - \mu) + n(\bar{x} - \mu)^2 = \sum_{i=1}^{n}(x_i - \mu)^2 - 2n(\bar{x} - \mu)^2 + n(\bar{x} - \mu)^2$$

$$= \sum_{i=1}^{n}(x_i - \mu)^2 - n(\bar{x} - \mu)^2$$

$$\therefore U = \frac{\sum_{i=1}^{n}(x_i - \bar{x})^2}{\sigma^2} = \sum\left(\frac{x_i - \mu}{\sigma}\right)^2 - \left(\frac{\bar{x} - \mu}{\sigma/\sqrt{n}}\right)^2$$

But $\frac{\sum_{i=1}^{n}(x_i - \bar{x})^2}{\sigma^2}$ follows chi-square distribution with n d.f. and $\left(\frac{\bar{x} - \mu}{\sigma/\sqrt{n}}\right)^2$ follows also chi-square distribution with 1 d.f.

Hence, by the additive property of chi-square distribution $\frac{\sum_{i=1}^{n}(x_i - \bar{x})^2}{\sigma^2}$ follows chi-square distribution with (n – 1) d.f. Hence $U = \frac{(n-1)s^2}{\sigma^2}$ follows chi-square distribution with (n – 1) d.f.

Theorem: If the sum of two independent variates is distributed as χ^2 variate with $(n_1 + n_2)$ d.f. and one of them is distributed as χ^2 variate with n_1 d.f. then show that the other one is distributed as also χ^2 variate with n_2 d.f.

Proof: Let χ_1^2 *and* χ_2^2 are two independent variates, then according to the hypothesis $(\chi_1^2+\chi_2^2)$ is distributed as χ^2 variate with (n_1+n_2) d.f. Let us consider χ_1^2 is distributed as χ^2 variate with n_1 d.f. Then we have to show that χ_2^2 is a χ^2 variate with n_2 d.f.

Now,

$$M_{(x_1^2+x_2^2)}(t)=E\left\{e^{(x_1^2+x_2^2)t}\right\}=E\left\{e^{x_1^2 t}\times e^{x_2^2 t}\right\}=E\left(e^{x_1^2 t}\right)E\left(e^{x_2^2 t}\right)$$

$$\because \chi_1^2 \text{ and } \chi_2^2 \text{ are independent}$$

Therefore, $M_{(x_1^2+x_2^2)}(t)=M_{x_1^2}(t)\times M_{x_2^2}(t)$ .... (1)

Here, $M_{(x_1^2+x_2^2)}(t)=(1-2t)^{-\left(\frac{n_1+n_2}{2}\right)}$ *and* $M_{x_1^2}(t)=(1-2t)^{-n_1/2}$ using this value in (1)

we get,

$$(1-2t)^{-\left(\frac{n_1+n_2}{2}\right)}=(1-2t)^{-\left(\frac{n_1}{2}\right)}M_{x_2^2}(t)\Rightarrow(1-2t)^{-\left(\frac{n_1}{2}\right)}(1-2t)^{-\left(\frac{n_2}{2}\right)}=(1-2t)^{-\left(\frac{n_1}{2}\right)}M_{x_2^2}(t)$$

$$\therefore M_{x_2^2}(t)=(1-2t)^{\left(\frac{n_2}{2}\right)}$$

This is the m.g.f. of a χ^2 variate with n_2 d.f.