Chapter — VII

De Moivre's Theorem

For all real values of n, $\cos n\theta + i\sin n\theta$ is a value of $(\cos\theta + i\sin\theta)^n$

<u>Case 1:</u> When n is a positive integer

Here,

$$(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2) = \cos\theta_1\cos\theta_2 + \cos\theta_1 \cdot i\sin\theta_2 + i\sin\theta_1 \cdot \cos\theta_2 + i^2\sin\theta_1\sin\theta_2$$

$$= \cos\theta_1\cos\theta_2 - \sin\theta_1 \cdot \sin\theta_2 + i(\sin\theta_1\cos\theta_2 + \sin\theta_2\cos\theta_1) \quad [\because i^2 = -1]$$

$$= \cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)$$

$$\left[\because \cos(A+B) = \cos A\cos B - \sin A\sin B,\ \sin(A+B) = \sin A\cos B + \cos A\sin B\right]$$

Similarly,

$$(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)(\cos\theta_3 + i\sin\theta_3) = \cos(\theta_1+\theta_2+\theta_3) + i\sin(\theta_1+\theta_2+\theta_3)$$

Proceeding in this way, the product of the n-factors

$$(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)\cdots(\cos\theta_n + i\sin\theta_n) = \cos(\theta_1+\theta_2+\cdots+\theta_n) + i\sin(\theta_1+\theta_2+\cdots+\theta_n)$$

Put in this identity,

$$\theta_1 = \theta_2 = \cdots = \theta_n = \theta$$

Then we get,

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$$

(proved)

Case 2: When n is a negative integer

Let

$$n = -m$$
$$= -(+m)$$

$\therefore$ m is a positive integer

Hence,

$$(\cos\theta + i\sin\theta)^n = (\cos\theta + i\sin\theta)^{-m}$$

$$= \frac{1}{(\cos\theta + i\sin\theta)^m}$$

$$= \frac{1 \cdot \cancel{(\cos\theta + i\sin}}{\cos m\theta + i\sin m\theta}$$

$$= \frac{1 \cdot (\cos m\theta - i\sin m\theta)}{(\cos m\theta + i\sin m\theta)(\cos m\theta - i\sin m\theta)}$$

[Multiple both numerator and divisor by $(\cos m\theta - i\sin m\theta$]

$$= \frac{\cos m\theta - i\sin m\theta}{(\cos m\theta)^2 - (i\sin m\theta)^2}$$

$$= \frac{\cos m\theta - i\sin m\theta}{\cos^2 m\theta - i^2\sin^2 m\theta}$$

$$= \frac{\cos m\theta - i\sin m\theta}{\cos^2 m\theta + \sin^2 m\theta} \quad [\because i^2 = -1]$$

$$= \frac{\cos m\theta - i\sin m\theta}{1}$$

$$= \cos m\theta - i\sin m\theta$$

$$\Rightarrow (\cos\theta + i\sin\theta)^n = \cos(-m)\theta + i\sin(-m)\theta \quad \left[\because \begin{matrix}\cos(-\theta) = \cos\theta \\ \sin(-\theta) = -\sin\theta\end{matrix}\right]$$

$$= \cos n\theta + \sin n\theta \quad [\because n = -m]$$

$$\therefore (\cos\theta + i\sin\theta)^n = \cos n\theta + \sin n\theta$$

(proved)

<u>Case 3:</u> When n is a fraction, positive or negative

Let, $n = \frac{p}{q}$ $\left[\begin{matrix}\text{where } q \text{ is a positive integer} \\ p \text{ is positive or negative integer}\end{matrix}\right]$

Now,

$$\left(\cos\frac{\theta}{q} + i\sin\frac{\theta}{q}\right)^q = \cos q\cdot\frac{\theta}{q} + i\sin\left(q\cdot\frac{\theta}{q}\right)$$

$$\Rightarrow \left(\cos\frac{\theta}{q} + i\sin\frac{\theta}{q}\right)^q = \cos\theta + i\sin\theta$$

$$\Rightarrow \left\{\left(\cos\frac{\theta}{q} + i\sin\frac{\theta}{q}\right)^q\right\}^{\frac{1}{q}} = (\cos\theta + i\sin\theta)^{\frac{1}{q}}$$

$$\Rightarrow \cos\frac{\theta}{q} + i\sin\frac{\theta}{q} = (\cos\theta + i\sin\theta)^{\frac{1}{q}}$$

$$\Rightarrow \left(\cos\frac{\theta}{q} + i\sin\frac{\theta}{q}\right)^p = \left\{(\cos\theta + i\sin\theta)^{\frac{1}{q}}\right\}^p$$

$$\Rightarrow \cos\frac{p\theta}{q} + i\sin\frac{p\theta}{q} = (\cos\theta + i\sin\theta)^{\frac{p}{q}}$$

$$\Rightarrow \cos n\theta + i\sin n\theta = (\cos\theta + i\sin\theta)^n \quad \left[\because n = \frac{p}{q}\right]$$

$$\therefore (\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$$

(proved)

Ex 3: iii

Let, $1+i = \overset{r\cos\theta}{\cancel{r\sin\theta}} + r i\sin\theta$

$\therefore \quad 1 = r\sin\theta$ —— (i)

$1 = r\cos\theta$ —— (ii)

$(i)^2 + (ii)^2 \Rightarrow$

$1+1 = r^2\sin^2\theta + r^2\cos^2\theta$

$\Rightarrow 2 = r^2(\sin^2\theta + \cos^2\theta)$

$\Rightarrow 2 = r^2 \cdot 1$

$\Rightarrow \sqrt{2} = r$

putting $r = \sqrt{2}$ in (i) we get

$1 = \sqrt{2}\sin\theta$

$\Rightarrow \frac{1}{\sqrt{2}} = \sin\theta$

$\Rightarrow \theta = \sin^{-1}\left(\frac{1}{\sqrt{2}}\right)$

$\Rightarrow \theta = \frac{\pi}{4}$

Therefore,

$1+i = r\cos\theta + ri\sin\theta$

$\Rightarrow 1+i = \sqrt{2}\cos\frac{\pi}{4} + \sqrt{2}\, i\sin\frac{\pi}{4}$

$\Rightarrow 1+i = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$

$\Rightarrow (1+i)^{\frac{1}{3}} = \left\{2^{\frac{1}{2}}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)\right\}^{\frac{1}{3}}$

$\Rightarrow (1+i)^{\frac{1}{3}} = 2^{\frac{1}{6}}\left\{\cos\left(2n\pi + \frac{\pi}{4}\right) + i\sin\left(2n\pi + \frac{\pi}{4}\right)\right\}^{\frac{1}{3}}$

$\Rightarrow (1+i)^{\frac{1}{3}} = 2^{\frac{1}{6}}\left\{\cos\left(\frac{8n\pi + \pi}{4}\right) + i\sin\left(\frac{8n\pi + \pi}{4}\right)\right\}^{\frac{1}{3}}$

$\Rightarrow (1+i)^{\frac{1}{3}} = 2^{\frac{1}{6}}\left\{\cos\frac{(8n+1)\pi}{12} + i\sin\frac{(8n+1)\pi}{12}\right\}$

when,

$n=0$

$$(1+i)^{\frac{1}{3}} = 2^{\frac{1}{6}}\left\{\cos\frac{(8.0+1)\pi}{12} + i\sin\frac{(8.0+1)\pi}{12}\right\}$$

$$= 2^{\frac{1}{6}}\left(\cos\frac{\pi}{12} + i\sin\frac{\pi}{12}\right)$$

when,

$n=1$

$$(1+i)^{\frac{1}{3}} = 2^{\frac{1}{6}}\left\{\cos\frac{(8.1+1)\pi}{12} + i\sin\frac{(8.1+1)\pi}{12}\right\}$$

$$= 2^{\frac{1}{6}}\left(\cos\frac{9\pi}{12} + i\sin\frac{9\pi}{12}\right)$$

$$= 2^{\frac{1}{6}}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$$

when,

$n=2$

$$(1+i)^{\frac{1}{3}} = 2^{\frac{1}{6}}\left\{\cos\frac{(8.2+1)\pi}{12} + i\sin\frac{(8.2+1)\pi}{12}\right\}$$

$$= 2^{\frac{1}{6}}\left(\cos\frac{17\pi}{12} + i\sin\frac{17\pi}{12}\right)$$

iii) Here,

$$1^{\frac{1}{3}} = (\cos 0 + i\sin 0)^{\frac{1}{3}}$$

$$= (\cos 2n\pi + i\sin 2n\pi)^{\frac{1}{3}}$$

$$= \cos\frac{2n\pi}{3} + i\sin\frac{2n\pi}{3}$$

Putting $n = 0, 1, 2$, the required values are

$$\cos 0 + i\sin 0,\ \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3},\ \cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}$$

$$= 1 + 0,\ -\frac{1}{2} + \sqrt{-1}\cdot\frac{\sqrt{3}}{2},\ -\frac{1}{2} + \sqrt{-1}\ \frac{-\sqrt{3}}{2}.$$

$$= 1,\ \frac{1}{2}\left(-1 + \sqrt{-3}\right),\ \frac{1}{2}\left(-1 - \sqrt{-3}\right)$$ (Ans.)

Examples VII

Q1 Let

$a = r\cos\theta$ $\quad \therefore r^2\cos^2\theta + r^2\sin^2\theta = a^2+b^2$

$b = r\sin\theta$ $\quad \Rightarrow r^2(\sin^2\theta+\cos^2\theta) = a^2+b^2$

$\Rightarrow r^2 = a^2+b^2 \Rightarrow r = \sqrt{a^2+b^2}$

L.S

$$(a+ib)^{\frac{m}{n}} + (a-ib)^{\frac{m}{n}}$$

$$= (r\cos\theta + ir\sin\theta)^{\frac{m}{n}} + (r\cos\theta - ir\sin\theta)^{\frac{m}{n}}$$

$$= \{r(\cos\theta + i\sin\theta)\}^{\frac{m}{n}} + \{r(\cos\theta - i\sin\theta)\}^{\frac{m}{n}}$$

$$= r^{\frac{m}{n}}(\cos\theta + i\sin\theta)^{\frac{m}{n}} + r^{\frac{m}{n}}(\cos\theta - i\sin\theta)^{\frac{m}{n}}$$

$$= r^{\frac{m}{n}}\left(\cos\frac{m}{n}\theta + i\sin\frac{m}{n}\theta\right) + r^{\frac{m}{n}}\left(\cos\frac{m}{n}\theta - i\sin\frac{m}{n}\theta\right)$$

$$= r^{\frac{m}{n}}\left(\cos\frac{m}{n}\theta + \cancel{i\sin\frac{m}{n}\theta} + \cos\frac{m}{n}\theta - \cancel{i\sin\frac{m}{n}\theta}\right)$$

$$= \left(\sqrt{a^2+b^2}\right)^{\frac{m}{n}} \cdot 2\cos\frac{m}{n}\theta$$

$$= 2\cdot(a^2+b^2)^{\frac{m}{2n}} \cdot \cos\frac{m}{n}\cdot\tan^{-1}\frac{b}{a} \quad \left[\because \tan\theta = \frac{b}{a}\right]$$

$= R.S$

$\therefore$ L.S $=$ R.S (proved)

Q: Prove that,

$$(1+i)^{\frac{m}{n}} + (1-i)^{\frac{m}{n}} = 2^{\left(\frac{m}{2n}+1\right)} \cos\left(\frac{m}{n}\cdot\frac{\pi}{4}\right)$$

solution:

Let, $1 = r\cos\theta$ —— (i)

$i = r\, i\sin\theta$

$\Rightarrow 1 = r\sin\theta$ —— (ii)

$(i)^2 + (ii)^2 \Rightarrow$

$1^2 + 1^2 = r^2\cos^2\theta + r^2\sin^2\theta$

$\Rightarrow 2 = r^2(\sin^2\theta + \cos^2\theta)$

$\Rightarrow 2 = r^2 . 1$

$\Rightarrow \sqrt{2} = r$

Again,

$(ii \div i) \Rightarrow$

$\frac{r\sin\theta}{r\cos\theta} = \frac{1}{1}$

$\Rightarrow \tan\theta = 1$

$\Rightarrow \theta = \tan^{-1} 1$

$\Rightarrow \theta = 45^\circ$

$\Rightarrow \theta = \frac{\pi}{4}$

L.S

$(1+i)^{\frac{m}{n}} + (1-i)^{\frac{m}{n}}$

$= (r\cos\theta + ri\sin\theta)^{\frac{m}{n}} + (r\cos\theta - ri\sin\theta)^{\frac{m}{n}}$

$= \{r(\cos\theta + i\sin\theta)\}^{\frac{m}{n}} + \{r(\cos\theta - i\sin\theta)\}^{\frac{m}{n}}$

$= r^{\frac{m}{n}}\left(\cos\frac{m}{n}\theta + i\sin\frac{m}{n}\theta\right) + r^{\frac{m}{n}}\left(\cos\frac{m}{n}\theta - i\sin\frac{m}{n}\theta\right)$

$= r^{\frac{m}{n}}\left(\cos\frac{m}{n}\theta + \cancel{i\sin\frac{m}{n}\theta} + \cos\frac{m}{n}\theta - \cancel{\sin\frac{m}{n}\theta}\right)$

$= (\sqrt{2})^{\frac{m}{n}} . 2\cos\frac{m}{n}\theta$

$$= 2^{\frac{m}{2n}} \cdot 2 \cdot \left(\cos \frac{m}{n} \cdot \frac{\pi}{4}\right)$$

$$= 2^{\left(\frac{m}{2n}+1\right)} \cdot \left(\cos \frac{m}{n} \cdot \frac{\pi}{4}\right) = R.S$$

$\therefore$ L.S = R.S

(proved)

2 i) Given that,

$$x + \frac{1}{x} = 2\cos\theta, \quad y + \frac{1}{y} = 2\cos\phi$$

$$x^2 - 2\cos\theta\, x + 1 = 0$$

$\therefore$

$$x = \frac{2\cos\theta \pm \sqrt{4\cos^2\theta - 4}}{2}$$

$$= \frac{2(\cos\theta \pm i\sin\theta)}{2}$$

$$= \cos\theta \pm i\sin\theta$$

taking (+) sign,

$$x = \cos\theta + i\sin\theta$$

similarly,

$$y = \cos\phi + i\sin\phi$$

$\therefore$

$$xy = (\cos\theta + i\sin\theta)(\cos\phi + i\sin\phi)$$

$$= \cos(\theta+\phi) + i\sin(\theta+\phi)$$

$$\therefore \frac{1}{xy} = \left\{\cos(\theta+\phi) + i\sin(\theta+\phi)\right\}^{-1}$$

$$= \cos(\theta+\phi) - i\sin(\theta+\phi)$$

$$xy + \frac{1}{xy} = \cos(\theta+\phi) + i\sin(\theta+\phi) + \cos(\theta+\phi) - i\sin(\theta+\phi)$$

$$= 2\cos(\theta+\phi)$$

$$\therefore \frac{1}{2}\left(xy + \frac{1}{xy}\right) = \cos(\theta+\phi)$$

(proved)

$$\therefore x^m = \cos m\theta + i\sin m\theta$$

$$y^n = \cos n\phi + i\sin n\phi$$

$$\therefore \frac{x^m}{y^n} + \frac{y^n}{x^m} = \frac{\cos m\theta + i\sin m\theta}{\cos n\phi + i\sin n\phi} + \frac{\cos n\phi + i\sin n\phi}{\cos m\theta + i\sin m\theta}$$

$$= (\cos m\theta + i\sin m\theta)(\cos n\phi - i\sin n\phi)$$

$$+ (\cos n\phi + i\sin n\phi)(\cos m\theta - i\sin m\theta)$$

$$= \cos m\theta.\cos n\phi - \cancel{\cos m\theta.i\sin n\phi}$$

$$+ \cancel{i\sin m\theta.\cos n\phi} + \sin m\theta.\sin n\phi + \cos n\phi.\cos m\theta$$

$$- \cancel{\cos n\phi.i\sin m\theta} + \cancel{i\sin n\phi.\cos m\theta} + \sin n\phi\sin m\theta$$

$$= 2\sin m\theta.\sin n\phi + 2\cos m\theta.\cos n\phi$$

$$= 2\left(\sin m\theta \cdot \sin n\phi + \cos m\theta \cdot \cos n\phi\right)$$

$$= 2\cos\left(m\theta - n\phi\right)$$

(proved)

4| Given that,

$$x_r = \cos\frac{\pi}{2^r} + i\sin\frac{\pi}{2^r}$$

put $r = 1, 2, 3, \ldots$

$$x_1 = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2}$$

$$x_2 = \cos\frac{\pi}{2^2} + i\sin\frac{\pi}{2^2}$$

$$x_3 = \cos\frac{\pi}{2^3} + i\sin\frac{\pi}{2^3}$$

$$\ldots$$

Multiplying,

$$x_1 x_2 x_3 \ldots = \left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)\left(\cos\frac{\pi}{2^2} + i\sin\frac{\pi}{2^2}\right)\left(\cos\frac{\pi}{2^3} + i\sin\frac{\pi}{2^3}\right)\ldots$$

$$= \cos\left(\frac{\pi}{2} + \frac{\pi}{2^2} + \frac{\pi}{2^3} + \ldots\right) + i\sin\left(\frac{\pi}{2} + \frac{\pi}{2^2} + \frac{\pi}{2^3} + \ldots\right)$$

$$= \cos\left(\frac{\pi/2}{1-\frac{1}{2}}\right) + i\sin\left(\frac{\pi/2}{1-\frac{1}{2}}\right)$$

$$= \cos\left(\frac{\frac{\pi}{2}}{1/2}\right) + i\sin\left(\frac{\pi/2}{1/2}\right)$$

$$= \cos\left(\frac{\pi}{2}\cdot\frac{2}{1}\right) + i\sin\left(\frac{\pi}{2}\cdot\frac{2}{1}\right)$$

$$= \cos\pi + i\sin\pi$$

$$= -1 + i.0 = -1$$

(proved)

* $(1-x)^{-1} = 1 + x + x^2 + x^3 + \cdots$

H.W

Ex.5 Given that,

$$x + \frac{1}{x} = 2\cos\theta$$

$$\Rightarrow \frac{x^2+1}{x} = 2\cos\theta$$

$$\Rightarrow x^2+1 = 2x\cos\theta$$

$$\Rightarrow x^2 - 2x\cos\theta + 1 = 0$$

$$\therefore x = \frac{-(-2\cos\theta) \pm \sqrt{(-2\cos\theta)^2 - 4\cdot1\cdot1}}{2\cdot1}$$

$$= \frac{2\cos\theta \pm \sqrt{4\cos^2\theta - 4}}{2}$$

$$= \frac{2\cos\theta \pm \sqrt{4(\cos^2\theta - 1)}}{2}$$

$$= \frac{2\cos\theta \pm 2\sqrt{-\sin^2\theta}}{2}$$

$$= \frac{2(\cos\theta \pm \sqrt{-1\cdot\sin^2\theta})}{2}$$

$$= \cos\theta \pm i\sin\theta \quad [\because \sqrt{-1} = i]$$

Taking (+) sign

$$x = \cos\theta + i\sin\theta$$

Now $x^n + \frac{1}{x^n}$

$$= x^n + x^{-n}$$

$$= (\cos\theta + i\sin\theta)^n + (\cos\theta + i\sin\theta)^{-n}$$

$$= \cos n\theta + i\sin n\theta + \cos(-n)\theta + i\sin(-n)\theta$$

$$= \cos n\theta + \cancel{i\sin n\theta} + \cos n\theta - \cancel{i\sin n\theta}$$

$= 2\cos n\theta$

$\therefore \quad x^n + \frac{1}{x^n} = 2\cos n\theta$

(Showed)

5/1

Let, $a_1 + ib_1 = r_1(\cos\theta_1 + i\sin\theta_1)$

$\therefore \quad a_1 = r_1\cos\theta_1$ —(i) and $b_1 = r_1\sin\theta_1$ —(ii)

$(i)^2 + (ii)^2 \Rightarrow$

$$a_1^2 + b_1^2 = r_1^2\cos^2\theta_1 + r_1^2\sin^2\theta_1$$

$$\Rightarrow a_1^2 + b_1^2 = r_1^2(\cos^2\theta_1 + \sin^2\theta_1)$$

$$\Rightarrow r_1^2 = a_1^2 + b_1^2$$

$$\therefore r_1 = \sqrt{a_1^2 + b_1^2}$$

Again, $(ii) \div (i) \Rightarrow$

$$\frac{b_1}{a_1} = \frac{r_1\sin\theta_1}{r_1\cos\theta_1}$$

$$\Rightarrow \frac{b_1}{a_1} = \tan\theta_1$$

$$\Rightarrow \tan^{-1}\left(\frac{b_1}{a_1}\right) = \theta_1$$

Similarly, $r_2 = \sqrt{a_2^2 + b_2^2}$ and $\theta_2 = \tan^{-1}\left(\frac{b_2}{a_2}\right)$

$\vdots \quad \vdots \quad \vdots \quad \vdots$

$r_n = \sqrt{a_n^2 + b_n^2}$ and $\theta_n = \tan^{-1}\left(\frac{b_n}{a_n}\right)$

Here,

$$a_1 + ib_1 = r_1\cos\theta_1 + i\, r_1 \sin\theta_1$$
$$= r_1(\cos\theta_1 + i\sin\theta_1)$$

$$a_2 + ib_2 = r_2\cos\theta_2 + i\, r_2\sin\theta_2$$
$$= r_2(\cos\theta_2 + i\sin\theta_2)$$

$$\cdots\cdots\cdots\cdots$$

$$a_n + ib_n = r_n(\cos\theta_n + i\sin\theta_n)$$

Given that,

$$(a_1+ib_1)(a_2+ib_2)\cdots\cdots(a_n+ib_n) = A+iB$$

$$\Rightarrow \{r_1(\cos\theta_1 + i\sin\theta_1)\}\{r_2(\cos\theta_2 + i\sin\theta_2)\}\cdots\{r_n(\cos\theta_n + i\sin\theta_n)\} = A+iB$$

$$\Rightarrow (r_1 r_2 \cdots r_n)\{\cos(\theta_1+\theta_2+\cdots+\theta_n) + i\sin(\theta_1+\theta_2+\cdots+\theta_n)\} = A+iB$$

Equating real and imaginary part, we get,

$$A = (r_1 r_2 \cdots r_n)\cos(\theta_1+\theta_2+\cdots+\theta_n) \quad \text{—— (iii)}$$

$$B = (r_1 r_2 \cdots r_n)\sin(\theta_1+\theta_2+\cdots+\theta_n) \quad \text{—— (iv)}$$

(iv) ÷ (iii) ⇒

$$\frac{B}{A} = \tan(\theta_1+\theta_2+\cdots+\theta_n)$$

$$\Rightarrow \tan^{-1}\frac{B}{A} = \theta_1+\theta_2+\cdots+\theta_n$$

$$\Rightarrow \tan^{-1}\frac{B}{A} = \tan^{-1}\frac{b_1}{a_1} + \tan^{-1}\frac{b_2}{a_2} + \cdots + \tan^{-1}\frac{b_n}{a_n}$$ (proved)

II

(iii)^2 + (iv)^2 $\Rightarrow$

$$A^2 + B^2 = (r_1 r_2 \cdots r_n)^2 \cdot \cos^2(\theta_1 + \theta_2 + \cdots + \theta_n) + (r_1 r_2 \cdots r_n)^2 \cdot \sin^2(\theta_1 + \theta_2 + \cdots + \theta_n)$$

$$\Rightarrow A^2 + B^2 = (r_1 r_2 \cdots r_n)^2 \left\{ \cos^2(\theta_1 + \theta_2 + \cdots + \theta_n) + \sin^2(\theta_1 + \theta_2 + \cdots + \theta_n) \right\}$$

$$\Rightarrow A^2 + B^2 = (r_1^2 \cdot r_2^2 \cdots r_n^2) \cdot 1$$

$$\Rightarrow A^2 + B^2 = (a_1^2 + b_1^2)(a_2^2 + b_2^2) \cdots (a_n^2 + b_n^2)$$

(proved)

9/ Given that

$$(1+x)^n = P_0 + P_1 x + P_2 x^2 + \cdots$$

putting $x = i$, then we get,

$$(1+i)^n = P_0 + P_1 i + P_2 i^2 + P_3 i^3 + P_4 i^4 \cdots$$

$$\Rightarrow (1+i)^n = P_0 + iP_1 - P_2 - iP_3 + P_4 \cdots \quad \text{—— (i)}$$

Let,

$$1 + i = r\cos\theta + ir\sin\theta$$

$$\therefore r\cos\theta = 1 \quad \text{—— (ii)}$$

$$r\sin\theta = 1 \quad \text{—— (iii)}$$

(ii)^2 + (iii)^2 $\Rightarrow$

$$r^2\cos^2\theta + r^2\sin^2\theta = 1^2 + 1^2$$

$$\Rightarrow r^2(\cos^2\theta + \sin^2\theta) = 2$$

$$\Rightarrow r^2 = 2 \Rightarrow r = \sqrt{2}$$

Again,

(iii) ÷ (ii) ⇒

$$\frac{r\sin\theta}{r\cos\theta} = \frac{1}{1}$$

$$\Rightarrow \tan\theta = 1$$

$$\Rightarrow \theta = \tan^{-1} 1$$

$$\Rightarrow \theta = \frac{\pi}{4}$$

Now,

$$(1+i)^n = (r\cos\theta + i r\sin\theta)^n$$

$$\Rightarrow (1+i)^n = \{r(\cos\theta + i\sin\theta)\}^n$$

$$\Rightarrow (1+i)^n = r^n\left(\cos n\cdot\frac{\pi}{4} + i\sin n\frac{\pi}{4}\right)$$

$$\Rightarrow (1+i)^n = (\sqrt{2})^n\left(\cos\frac{n\pi}{4} + i\sin\frac{n\pi}{4}\right)$$

$$\Rightarrow (1+i)^n = 2^{\frac{n}{2}}\left(\cos\frac{n\pi}{4} + i\sin\frac{n\pi}{4}\right)$$

$$\Rightarrow (P_0 - P_2 + P_4 - \cdots) + i(P_1 - P_3 + P_5 - \cdots) = 2^{\frac{n}{2}}\left(\cos\frac{n\pi}{4} + i\right.$$

Equating real and imaginary part we get,

$$P_0 - P_2 + P_4 - \cdots = 2^{\frac{n}{2}}\cos\frac{n\pi}{4}$$

$$P_1 - P_3 + P_5 - \cdots = 2^{\frac{n}{2}}\sin\frac{n\pi}{4}$$

(proved)

12| Given that,

$$x = \cos\theta + i\sin\theta \quad \text{——— (i)}$$

$$\therefore \frac{1}{x} = \frac{1}{\cos\theta + i\sin\theta}$$

$$\Rightarrow \frac{1}{x} = (\cos\theta + i\sin\theta)^{-1} \quad \text{——— (ii)}$$

$$(i+ii) \Rightarrow \Rightarrow \frac{1}{x} =$$

$$x + \frac{1}{x} = \cos\theta + i\sin\theta$$

12| Given that,

$$x = \cos\theta + i\sin\theta \quad \text{——— (i)}$$

$$\therefore x^{-1} = (\cos\theta + i\sin\theta)^{-1}$$

$$\Rightarrow \frac{1}{x} = \cos(-1)\theta + i\sin(-1)\theta$$

$$\Rightarrow \frac{1}{x} = \cos\theta - i\sin\theta \quad \text{——— (ii)}$$

$$(i+ii) \Rightarrow$$

$$x + \frac{1}{x} = \cos\theta + \cancel{i\sin\theta} + \cos\theta - \cancel{i\sin\theta}$$

$$\Rightarrow x + \frac{1}{x} = 2\cos\theta$$

$$\Rightarrow n\left(x + \frac{1}{x}\right) = 2n\cos\theta \quad \text{——— (iii)}$$

Again,

$$1 + \sqrt{1-a^2} = na$$

$$\Rightarrow \sqrt{1-a^2} = na - 1$$

$$\Rightarrow \left(\sqrt{1-a^2}\right)^2 = (na-1)^2$$

$$\Rightarrow \quad 1 - a^2 = n^2 a^2 - 2an + 1$$

$$\Rightarrow \quad 1 + 2an - 1 = a^2 n^2 + a^2$$

$$\Rightarrow \quad 2an = a^2 (n^2 + 1)$$

$$\Rightarrow \quad 2n = a (n^2 + 1)$$

$$\Rightarrow \quad \frac{1}{n^2+1} = \frac{a}{2n} \quad \text{——— (iv)}$$

Now,

$$\frac{a}{2n}(1+nx)\left\{1+\frac{n}{x}\right\} = \frac{a}{2n}\left(1+\frac{n}{x}+nx+n^2\right)$$

$$= \frac{a}{2n}\left\{(1+n^2)+n\left(x+\frac{1}{x}\right)\right\}$$

$$= \frac{1}{n^2+1}\left\{(1+n^2)+2n\cos\theta\right\} \quad \text{by (iii)}$$

$$= 1 + \frac{2n\cos\theta}{n^2+1}$$

$$= 1 + a\cos\theta$$

$$\therefore \quad 1 + a\cos\theta = \frac{a}{2n}(1+nx)\left\{1+\frac{n}{x}\right\}$$

(proved)

(5) + (6) $\Rightarrow$

$$(1+i)^n + (1-i)^n = (a_0 + ia_1 - a_2 + ia_3 + a_4 + \cdots) + (a_0 - ia_1 - a_2 + a_4 + \cdots)$$

$$\Rightarrow (1+i)^n + (1-i)^n = 2(a_0 - a_2 + a_4 - \cdots) \quad \text{——} (7)$$

(4) + (7) $\Rightarrow$

$$2^n + (1+i)^n + (1-i)^n = 2(a_0 + a_2 + a_4 + \cdots) + 2(a_0 - a_2 + a_4 - \cdots)$$

$$\Rightarrow 2^n + (1+i)^n + (1-i)^n = 4(a_0 + a_4 + a_8 + \cdots)$$

$$\Rightarrow \frac{2^n + (1+i)^n + (1-i)^n}{4} = a_0 + a_4 + a_8 + \cdots$$

$$\Rightarrow \frac{2^n}{2^2} + \frac{(1+i)^n + (1-i)^n}{4} = a_0 + a_4 + a_8 + \cdots$$

$$\Rightarrow 2^{n-2} + \frac{(1+i)^n + (1-i)^n}{4} = a_0 + a_4 + a_8 + \cdots$$

$$\Rightarrow 2^{n-2} + \frac{2^{\frac{n}{2}+1} \cdot \cos\frac{n\pi}{4}}{4} = a_0 + a_4 + a_8 + \cdots$$

(proved)

$$\Rightarrow 2^{n-2} + \left(2^{\frac{n}{2}+1-2} \cdot \cos\frac{n\pi}{4}\right) = a_0 + a_4 + a_8 + \cdots$$

$$\Rightarrow 2^{n-2} + 2^{\frac{n}{2}-1} \cos\frac{n\pi}{4} = a_0 + a_4 + a_8 + \cdots$$

(proved)

4| Given that,

$$x_r = \cos\frac{\pi}{2^r} + i\sin\frac{\pi}{2^r}$$

Putting $r = 1, 2, 3, \ldots$ we get

$$x_1 = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} = e^{i\frac{\pi}{2}}$$

$$x_2 = \cos\frac{\pi}{2^2} + i\sin\frac{\pi}{2^2} = e^{i\frac{\pi}{2^2}}$$

$$x_3 = \cos\frac{\pi}{2^3} + i\sin\frac{\pi}{2^3} = e^{i\frac{\pi}{2^3}}$$

$\ldots\ldots\ldots\ldots$

$$\therefore x_1 x_2 x_3 \ldots\ldots \infty = e^{i\frac{\pi}{2}} \cdot e^{i\frac{\pi}{2^2}} \cdot e^{i\frac{\pi}{2^3}} \ldots\ldots$$

$$= e^{i\frac{\pi}{2}\left(1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots\right)}$$

$$= e^{i\frac{\pi}{2}\left(1 - \frac{1}{2}\right)^{-1}}$$

$$= e^{i\frac{\pi}{2}} \cdot \left(\frac{1}{2}\right)^{-1}$$

$$= e^{\left(i\frac{\pi}{2} \cdot 2\right)}$$

$$= e^{i\pi}$$

$$= \cos\pi + i\sin\pi$$

$$= -1 + i.0$$

$$= -1$$

(Proved)

Ex → VIII

Formula :

1. $\cos\alpha = 1 - \frac{\alpha^2}{2!} + \frac{\alpha^4}{4!} - \frac{\alpha^6}{6!} + \cdots$
2. $\sin\alpha = \alpha - \frac{\alpha^3}{3!} + \frac{\alpha^5}{5!} - \frac{\alpha^7}{7!} + \cdots$
3. $\sin(A+B) = \sin A \cos B + \cos A \sin B$
4. $\sin(A-B) = \sin A \cos B - \cos A \sin B$
5. $\cos(A+B) = \cos A \cos B - \sin A \sin B$
6. $\cos(A-B) = \cos A \cos B + \sin A \sin B$

17| Given that,

$$1 - \frac{2}{3!} + \frac{3}{5!} - \frac{4}{7!} + \cdots$$

$$= \frac{1}{2} \cdot 2\left(1 - \frac{2}{3!} + \frac{3}{5!} - \frac{4}{7!} + \cdots\right)$$

$$= \frac{1}{2}\left(2 - \frac{4}{3!} + \frac{6}{5!} - \frac{8}{7!} + \cdots\right)$$

$$= \frac{1}{2}\left(1 + 1 - \frac{3+1}{3!} + \frac{5+1}{5!} - \frac{7+1}{7!} + \cdots\right)$$

$$= \frac{1}{2}\left(1 + 1 - \frac{3}{3!} - \frac{1}{3!} + \frac{5}{5!} + \frac{1}{5!} - \frac{7}{7!} - \frac{1}{7!} + \cdots\right)$$

$$= \frac{1}{2}\left(1 + 1 - \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} - \frac{1}{6!} - \frac{1}{7!} + \cdots\right)$$

$$= \frac{1}{2}\left(1 - \frac{1}{2!} + \frac{1}{4!} - \frac{1}{6!} + \cdots\right) + \frac{1}{2}\left(1 - \frac{1}{3!} + \frac{1}{5!} - \frac{1}{7!} + \cdots\right)$$

$$= \frac{1}{2}\cos 1 + \frac{1}{2}\sin 1$$

$$= \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}\cos 1 + \frac{1}{\sqrt{2}}\sin 1\right)$$

$$= \frac{1}{\sqrt{2}}\left(\sin\frac{\pi}{4}\cos 1 + \cos\frac{\pi}{4}\cdot\sin 1\right)$$

$$= \frac{1}{\sqrt{2}}\sin\left(\frac{\pi}{4} + 1\right)$$

(Ans)

18| Given that,

$$x = \frac{2}{1!} - \frac{4}{3!} + \frac{6}{5!} - \frac{8}{7!} + \cdots \text{ to } \infty$$

and $$y = 1 + \frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots \text{ to } \infty$$

Now,

$$x = \frac{2}{1!} - \frac{4}{3!} + \frac{6}{5!} - \frac{8}{7!} + \cdots \infty$$

$$\Rightarrow x = \frac{1+1}{1!} - \frac{3+1}{3!} + \frac{5+1}{5!} - \frac{7+1}{7!} + \cdots \infty$$

$$\Rightarrow x = 1 + 1 - \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} - \frac{1}{6!} - \frac{1}{7!} + \cdots \infty$$

$$\Rightarrow x = \left(1 - \frac{1}{2!} + \frac{1}{4!} - \frac{1}{6!} + \cdots \infty\right) + \left(1 - \frac{1}{3!} + \frac{1}{5!} - \frac{1}{7!} + \cdots \infty\right)$$

$$\Rightarrow x = \cos 1 + \sin 1$$

$$\Rightarrow x^2 = (\cos 1 + \sin 1)^2$$

$$\Rightarrow x^2 = \cos^2 1 + \sin^2 1 + 2 \cos 1 \cdot \sin 1$$

$$\Rightarrow x^2 = 1 + \sin 2 \cdot 1$$

$$\Rightarrow x^2 = 1 + 2 - \frac{2^3}{3!} + \frac{2^5}{5!} - \frac{2^7}{7!} + \cdots$$

$$\Rightarrow x^2 = y$$

$$\therefore x^2 = y \text{ (proved)}$$

Chapter IX

Formulae

1. $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$
2. $e^x \cdot e^y = e^{x+y}$
3. $e^{ix} = \cos x + i \sin x$
4. $e^{-ix} = \cos x - i \sin x$
5. $\cos x = \frac{1}{2}\left(e^{ix} + e^{-ix}\right)$
6. $\sin x = \frac{1}{2i}\left(e^{ix} - e^{-ix}\right)$
7. $e^{2n\pi i} = \cos 2n\pi + i \sin 2n\pi = 1$
8. $a^x = e^{x \log a}$
9. $\sin\theta = 0 \Rightarrow \theta = n\pi$
10. $\cos\theta = 0 \Rightarrow \theta = (2n+1)\pi/2$
11. $\tan(A \pm B) = \dfrac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$
12. $\tan^{-1} x + \tan^{-1} y = \tan^{-1} \dfrac{x+y}{1-xy}$
13. $\tan 2A = \dfrac{2\tan A}{1 - (\tan A)^2}$

Ex.2 Let, $\alpha = r\cos\theta$ and $\beta = r\sin\theta$

$\therefore r = \sqrt{\alpha^2+\beta^2}$ and $\theta = \tan^{-1}\left(\frac{\beta}{\alpha}\right)$

$\therefore \alpha + i\beta = r\cos\theta + ir\sin\theta$

$\Rightarrow \alpha + i\beta = r(\cos\theta + i\sin\theta)$

$\Rightarrow \alpha + i\beta = re^{i\theta}$

$\Rightarrow \log(\alpha+i\beta) = \log(r.e^{i\theta})$

$\Rightarrow \log(\alpha+i\beta) = \log r + \log e^{i\theta}$

$\Rightarrow \log(\alpha+i\beta) = \log r + i\theta$ ——— (1)

Now,

$(\alpha+i\beta)^{x+iy} = e^{(x+iy).\log(\alpha+i\beta)}$

$= e^{(x+iy).(\log r + i\theta)}$ [using (1)]

$= e^{x\log r + ix\theta + iy\log r + i^2 y\theta}$

$= e^{(x\log r - y\theta) + i(x\theta + y\log r)}$

$= e^{x\log r - y\theta} . e^{i(x\theta + y\log r)}$

$= e^{x\log r - y\theta} . \{\cos(x\theta + y\log r) + i\sin(x\theta + y\log r)\}$

$= \{e^{x\log r - y\theta} \cos(x\theta + y\log r)\} + \{e^{x\log r - y\theta} . i\sin(x\theta + y\log r)\}$

$$= e^{x\log(\sqrt{\alpha^2+\beta^2}) - y\tan^{-1}\frac{\beta}{\alpha}} \cdot \cos\left\{x\left(\tan^{-1}\frac{\beta}{\alpha}\right) + y\log\sqrt{\alpha^2+\beta^2}\right\}$$

$$+ i\, e^{x\log\sqrt{\alpha^2+\beta^2} - y\tan^{-1}\frac{\beta}{\alpha}} \cdot \sin\left(x\tan^{-1}\frac{\beta}{\alpha} + y\log\sqrt{\alpha^2+\beta^2}\right)$$

(Ans.)

Ex. 1| Let,

$$\log(\alpha+i\beta) = x+iy \quad \text{—— (i)}$$

$$\therefore e^{x+iy} = \alpha+i\beta \quad \text{—— (ii)}$$

Again, Let,

$$\alpha = r\cos\theta \text{ ; and } \beta = r\sin\theta$$

$$\therefore r = \sqrt{\alpha^2+\beta^2} \text{ and } \theta = \tan^{-1}\left(\frac{\beta}{\alpha}\right)$$

Now,

$$\alpha+i\beta = r(\cos\theta + i\sin\theta)$$

$$= r.e^{i\theta}$$

$$= e^{\log r} \cdot e^{i\theta} \cdot e^{2n\pi i} \quad \left[\because e^{\log r} = r,\ e^{2n\pi i} = 1\right]$$

$$= e^{\log r + i(2n\pi+\theta)}$$

$$\therefore \alpha+i\beta = e^{\log r + i(2n\pi+\theta)} \quad \text{—— (iii)}$$

From (ii), we get

$$e^{x+iy} = e^{\log r + i(2n\pi+\theta)} \quad [\text{using (iii)}]$$

$$\Rightarrow x+iy = \log r + i(2n\pi+\theta)$$

Equating real and imaginary parts, we get

$x = \log r$ and $y = 2n\pi + \theta$

From (1). we get

$$\log(\alpha + i\beta) = \log r + i(2n\pi + \theta)$$

$$= \log\sqrt{\alpha^2+\beta^2} + i\left(2n\pi + \tan^{-1}\frac{\beta}{\alpha}\right)$$

or

Ex-1| Let,

$$\alpha = r\cos\theta$$
$$\beta = r\sin\theta$$

$\therefore r = \sqrt{\alpha^2+\beta^2}$ and $\theta = \tan^{-1}\frac{\beta}{\alpha}$

Here, $\alpha + i\beta = r\cos\theta + ir\sin\theta$

$$= r(\cos\theta + i\sin\theta)$$

$$= re^{i\theta}$$

$\therefore \alpha + i\beta = re^{i\theta}$

Now, $\log(\alpha + i\beta) = \log(re^{i\theta})$

$$= \log r + \log e^{i\theta}$$

$$= \log r + i\theta$$

$$= \log\sqrt{\alpha^2+\beta^2} + i\tan^{-1}\beta/\alpha$$

(Ans)

Ex-3 Here,

$$i = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2}$$

$$= e^{i\frac{\pi}{2}}$$

$$= e^{i\frac{\pi}{2}} \cdot e^{2n\pi i} \quad \left[\because e^{2n\pi i} = 1\right]$$

$$= e^{i\frac{\pi}{2} + 2n\pi i}$$

$$= e^{i\frac{\pi}{2}(1+4n)}$$

Therefore,

$$\log i = \log e^{i\frac{\pi}{2}(4n+1)}$$

$$= i\frac{\pi}{2}(4n+1)$$

Therefore,

$$i^i = e^{i\log i}$$

$$= e^{i \cdot i\frac{\pi}{2}(4n+1)}$$

$$= e^{i^2\frac{\pi}{2}(4n+1)}$$

$$= e^{-\frac{\pi}{2}(4n+1)}$$

$$\therefore (i^i)_n = e^{-\frac{\pi}{2}(4n+1)}$$ (proved)

Ex-5] Given that

$$\tan \log (x+iy) = a+ib$$

$$\Rightarrow \log (x+iy) = \tan^{-1}(a+ib)$$

Therefore,

$$\log (x-iy) = \tan^{-1}(a-ib)$$

Now,

$$\log (x^2+y^2) = \log (x^2 - i^2 y^2)$$

$$= \log \{(x+iy)(x-iy)\}$$

$$= \log (x+iy) + \log (x-iy)$$

$$= \tan^{-1}(a+ib) + \tan^{-1}(a-ib)$$

$$= \tan^{-1} \frac{(a+ib)+(a-ib)}{1-(a+ib)(a-ib)}$$

$$= \tan^{-1} \frac{2a}{1-(a^2-\cancel{iab}+\cancel{iab}-i^2b^2)}$$

$$= \tan^{-1} \frac{2a}{1-(a^2+b^2)}$$

$$= \tan^{-1} \frac{2a}{1-a^2-b^2}$$

$$\therefore \log (x^2+y^2) = \tan^{-1} \frac{2a}{1-a^2-b^2}$$ (proved)

Ex-8 Here

$$\operatorname{Log}(\cos\theta + i\sin\theta)$$

$$= \log e^{i\theta}$$

$$= \log e^{i\theta} \cdot e^{2n\pi i}$$

$$= \log e^{i(2n\pi+\theta)}$$

$$= (2n\pi+\theta)i$$

$$= (2n\pi+\theta)\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$$

$$= (2n\pi+\theta)\left(e^{i\frac{\pi}{2}}\right)$$

Therefore,

$$\operatorname{Log}\left\{\operatorname{Log}(\cos\theta + i\sin\theta)\right\} = \log\left\{(2n\pi+\theta)\cdot e^{i\frac{\pi}{2}}\right\}$$

$$= \log(2n\pi+\theta) + \log e^{i\pi/2}$$

$$= \log(2n\pi+\theta) + \log e^{i\pi/2} \cdot e^{2k\pi i}$$

$$= \log(2n\pi+\theta) + \log e^{2k\pi i + i\pi/2}$$

$$= \log(2n\pi+\theta) + 2k\pi i + i\pi/2$$

$$= \log(2n\pi+\theta) + i\pi\left(2k + \frac{1}{2}\right)$$

$$\therefore \operatorname{Log}\left\{\log(\cos\theta + i\sin\theta)\right\} = \log(2n\pi+\theta) + i\pi\left(2k+\frac{1}{2}\right)$$

3vi Sol^n: Let, $a = r\cos\theta$ and $b = r\sin\theta$

$\therefore r = \sqrt{a^2+b^2}$ and $\theta = \tan^{-1}\frac{b}{a}$

Here,

$$\log(a+ib) = \log(r\cos\theta + ir\sin\theta)$$
$$= \log r(\cos\theta + i\sin\theta)$$
$$= \log r e^{i\theta}$$
$$= \log r + \log e^{i\theta}$$
$$= \log r + i\theta$$
$$= \log\sqrt{a^2+b^2} + i\tan^{-1}\frac{b}{a}$$

Similarly, $\log(a-ib) = \log\sqrt{a^2+b^2} - i\tan^{-1}\frac{b}{a}$

$$\therefore \log\frac{a-ib}{a+ib} = \log(a-ib) - \log(a+ib)$$
$$= \log\sqrt{a^2+b^2} - i\tan^{-1}\frac{b}{a} - \log\sqrt{a^2+b^2} - i\tan^{-1}\frac{b}{a}$$
$$= -2i\tan^{-1}\frac{b}{a}$$

Now,

$$\tan\left(i\log\frac{a-ib}{a+ib}\right) = \tan\left(-i\,2i\tan^{-1} b/a\right)$$
$$= \tan\left(2\tan^{-1} b/a\right)$$
$$= \frac{2\tan(\tan^{-1} b/a)}{1-\{\tan(\tan^{-1} b/a)\}^2}$$

$$= \frac{2\frac{b}{a}}{1-\frac{b^2}{a^2}}$$

$$= \frac{\frac{2b}{a}}{\frac{a^2-b^2}{a^2}}$$

$$= \frac{2b}{a} \times \frac{a^2}{a^2-b^2}$$

$$= \frac{2ab}{a^2-b^2}$$

$$\therefore \tan\left(i \log \frac{a-ib}{a+ib}\right) = \frac{2ab}{a^2-b^2}$$ (Showed)

4| Given that,

$$A+iB = \log(x+iy)$$

$$\Rightarrow e^{A+iB} = x+iy$$

$$\Rightarrow e^A . e^{iB} = x+iy$$

$$\Rightarrow e^A(\cos B + i\sin B) = x+iy$$

$$\Rightarrow e^A \cos B + i e^A \sin B = x+iy$$

Equating real and imaginary parts, we have

$$x = e^A \cos B, \quad y = e^A \sin B$$

Now, $\frac{y}{x} = \frac{e^A \sin B}{e^A \cos B}$

$$\Rightarrow \frac{y}{x} = \tan B \Rightarrow B = \tan^{-1}(y/x)$$ (proved)

Again,

$$x^2+y^2 = (e^A \cos B)^2 + (e^A \sin B)^2$$

$$\Rightarrow x^2+y^2 = e^{2A}(\cos^2 B + \sin^2 B)$$

$$\Rightarrow x^2+y^2 = e^{2A}$$

$$\Rightarrow \log(x^2+y^2) = \log e^{2A}$$

$$\Rightarrow \log(x^2+y^2) = 2A$$

$$\therefore A = \frac{1}{2}\log(x^2+y^2)$$ (proved)

8| Let, $x = r\cos\theta$ and $y = r\sin\theta$

$$\therefore r = \sqrt{x^2+y^2} \text{ and } \theta = \tan^{-1}\frac{y}{x}$$

Here,

$$x+iy = r\cos\theta + ir\sin\theta$$

$$\Rightarrow x+iy = r(\cos\theta + i\sin\theta)$$

$$\Rightarrow \log(x+iy) = \log(re^{i\theta})$$

$$\Rightarrow \log(x+iy) = \log r + \log e^{i\theta}$$

$$\Rightarrow \log(x+iy) = \log r + i\theta$$

$$\Rightarrow \log(x+iy) = \log\sqrt{x^2+y^2} + i\tan^{-1}\frac{y}{x}$$

$$\Rightarrow \log(x+iy) = p+iq$$

$$\Rightarrow \log\{\log(x+iy)\} = \log(p+iq) \quad \text{——— (i)}$$

Again Let $p = r\cos\theta$ and $q = r\sin\theta$

$\therefore r = \sqrt{p^2+q^2}$ and $\theta = \tan^{-1}\frac{q}{p}$

From equ^n (1), we get,

$$\begin{aligned}
\operatorname{Log}\{\log(x+iy)\} &= \operatorname{Log}(p+iq) \\
&= \log(r\cos\theta + i\, r\sin\theta) \\
&= \log\{r(\cos\theta + i\sin\theta)\} \\
&= \log(r e^{i\theta}) \\
&= \log r + \log e^{i\theta} \\
&= \log r + i\theta \\
&= \log\sqrt{p^2+q^2} + i\tan^{-1}\frac{q}{p} \\
&= \log(p^2+q^2)^{\frac{1}{2}} + i\tan^{-1}\frac{q}{p}
\end{aligned}$$

$$\therefore \operatorname{Log}\log(x+iy) = \frac{1}{2}\log(p^2+q^2) + i\tan^{-1}\frac{q}{p}$$

(proved)

If Given that,

$$\tan(x+iy) = u+iv$$

Similarly we get,

$$\tan(x-iy) = u-iv$$

Now

$$\tan 2x = \tan\{(x+iy)+(x-iy)\}$$

$$= \frac{\tan(x+iy)+\tan(x-iy)}{1-\tan(x+iy).\tan(x-iy)}$$

$$= \frac{u+iv+u-iv}{1-(u+iv)(u-iv)}$$

$$= \frac{2u}{1-(u^2-i^2v^2)}$$

$$= \frac{2u}{1-u^2-v^2}$$

$$\therefore \tan 2x = \frac{2u}{1-u^2-v^2}$$

$$\Rightarrow 1-u^2-v^2 = \frac{2u}{\tan 2x}$$

$$\Rightarrow 1-u^2-v^2 = 2u\cot 2x$$

$$\Rightarrow 1 = u^2+v^2+2u\cot 2x$$

$$\therefore u^2+v^2+2u\cot 2x = 1$$ (proved)

6) Given that,

$$i^{i^{i^{\cdots \text{ad inf}}}} = A + iB$$

$$\therefore i^{A+iB} = A + iB \quad \text{——— (i)}$$

Here,

$$i = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} = e^{i\pi/2}$$

$$\therefore i^{A+iB} = e^{i\pi/2(A+iB)}$$

$$= e^{\pi/2(iA - B)}$$

$$= e^{-B\cdot\pi/2 + i\,\pi/2\,A}$$

$$= e^{-\frac{B\pi}{2}} \cdot e^{i\frac{\pi A}{2}}$$

$$\therefore i^{A+B} = e^{-B\pi/2} \cdot e^{i\pi A/2} \quad \text{——— (ii)}$$

From, eq^n (i) and (ii) we get,

$$A + iB = e^{-B\pi/2} \cdot e^{i\pi A/2}$$

$$\Rightarrow A + iB = e^{-B\pi/2}\left[\cos\pi A/2 + i\sin\frac{\pi A}{2}\right]$$

Equating real & imaginary we get,

$$A = e^{-\frac{B\pi}{2}}\cos\frac{A\pi}{2}$$

$$B = e^{-\frac{B\pi}{2}}\sin\frac{A\pi}{2}$$

Dividing, we get

$$\frac{B}{A} = \tan\left(\frac{A\pi}{2}\right)$$

$$\Rightarrow \tan\frac{\pi A}{2} = B/A \quad \text{(proved)}$$

Squaring and adding, we get

$$A^2 + B^2 = e^{\left(-\frac{B\pi}{2}\right)^2} \cdot \cos^2\frac{A\pi}{2} + e^{\left(-\frac{B\pi}{2}\right)^2} \cdot \sin^2\frac{A\pi}{2}$$

$$= e^{-B\pi}\left(\sin^2\frac{A\pi}{2} + \cos^2\frac{A\pi}{2}\right)$$

$$= e^{-B\pi}$$

(proved)

2) Let $\alpha = r\cos\theta$ and $\beta = r\sin\theta$

$\therefore r = \sqrt{\alpha^2+\beta^2}$ and $\theta = \tan^{-1}\frac{\beta}{\alpha}$

$$\therefore (\alpha+i\beta)^{x+iy} = e^{(x+iy)\log(\alpha+i\beta)}$$

$$= e^{(x+iy)\log(r\cos\theta + ir\sin\theta)}$$

$$= e^{(x+iy)\log re^{i\theta}}$$

$$= e^{(x+iy)(\log r + i\theta)}$$

$$= e^{x\log r + ix\theta + iy\log r - y\theta}$$

$$= e^{x\log r - y\theta + i(y\log r + x\theta)}$$

$$= e^{x\log r - y\theta} \cdot e^{i(y\log r + x\theta)}$$

$$= e^{x\log r - y\theta}\left[\cos(y\log r + x\theta) + i\sin(y\log r + x\theta)\right]$$

$$= e^{x\log\sqrt{\alpha^2+\beta^2} - y\tan^{-1}\frac{\beta}{\alpha}} \cdot \left[\cos\left(y\log\sqrt{\alpha^2+\beta^2} + x\tan^{-1}\frac{\beta}{\alpha}\right)\right.$$

$$\left. + i\sin\left(y\log\sqrt{\alpha^2+\beta^2} + x\tan^{-1}\frac{\beta}{\alpha}\right)\right]$$

The principal value of $(\alpha+i\beta)^{x+iy}$ is wholly real if imaginary part vanishes,

$$\therefore \sin\left(y \log\sqrt{\alpha^2+\beta^2} + x\tan^{-1}\frac{\beta}{\alpha}\right) = 0$$

$$\Rightarrow y \log\sqrt{\alpha^2+\beta^2} + x\tan^{-1}\frac{\beta}{\alpha} = n\pi \quad [n = 0, 1, 2, 3 \ldots]$$

$$\Rightarrow \frac{1}{2} y \log(\alpha^2+\beta^2) + x\tan^{-1}\frac{\beta}{\alpha} = 2n \cdot \frac{\pi}{2}$$

$$\therefore \frac{1}{2} y \log(\alpha^2+\beta^2) + x\tan^{-1}\frac{\beta}{\alpha} = \text{even multiple of } \pi/2$$

(proved)

Again,

The principle value of $(\alpha+i\beta)^{x+iy}$ is wholly imaginary if real part vanishes,

$$\cos\left(y \log\sqrt{\alpha^2+\beta^2} + x\tan^{-1}\frac{\beta}{\alpha}\right) = 0$$

$$\Rightarrow y \log\sqrt{\alpha^2+\beta^2} + x\tan^{-1}\frac{\beta}{\alpha} = (2n+1)\pi/2$$

$$\therefore \frac{1}{2} y \log(\alpha^2+\beta^2) + x\tan^{-1}\frac{\beta}{\alpha} = \text{odd multiple of } \pi/2$$

(proved)

Chapter - IX

1. $\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$

2. $\log(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \cdots$

3. $\tan^{-1}x = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots$

P: To expand θ in powers of $\tan\theta$ $\left(-\frac{\pi}{4} \le \theta < \frac{1}{4}\pi\right)$

Solution: We have,

$$i\tan\theta = \frac{i\sin\theta}{\cos\theta}$$

$$\Rightarrow i\tan\theta = \frac{2\, i\sin\theta}{2\cos\theta}$$

$$\Rightarrow i\tan\theta = \frac{e^{i\theta} - e^{-i\theta}}{e^{i\theta} + e^{-i\theta}}$$

$$\Rightarrow \frac{1}{i\tan\theta} = \frac{e^{i\theta} + e^{-i\theta}}{e^{i\theta} - e^{-i\theta}}$$

$$\Rightarrow \frac{1 + i\tan\theta}{1 - i\tan\theta} = \frac{e^{i\theta} + e^{-i\theta} + \cancel{e^{i\theta}} - \cancel{e^{-i\theta}}}{\cancel{e^{i\theta}} + e^{-i\theta} - \cancel{e^{i\theta}} + e^{-i\theta}}$$

$$\Rightarrow \frac{1 + i\tan\theta}{1 - i\tan\theta} = \frac{\cancel{2}e^{i\theta}}{\cancel{2}e^{-i\theta}}$$

$$\Rightarrow \frac{1 + i\tan\theta}{1 - i\tan\theta} = e^{i\theta + i\theta}$$

$$\Rightarrow \frac{1 + i\tan\theta}{1 - i\tan\theta} = e^{2i\theta}$$

$$\Rightarrow \log\left(\frac{1 + i\tan\theta}{1 - i\tan\theta}\right) = \log e^{2i\theta}$$

$$\Rightarrow \log(1 + i\tan\theta) - \log(1 - i\tan\theta) = 2i\theta$$

$$\Rightarrow \left(i\tan\theta - \frac{1}{2} i^2\tan^2\theta + \frac{1}{3} i^3\tan^3\theta - \dots\right) - \left(-i\tan\theta - \frac{1}{2} i^2\tan^2\theta - \frac{1}{3} i^3\tan^3\theta - \dots\right) = 2i\theta$$

$$\Rightarrow i\tan\theta - \frac{1}{2} i^2\tan^2\theta + \frac{1}{3} i^3\tan^3\theta - \dots + i\tan\theta + \frac{1}{2} i^2\tan^2\theta + \frac{1}{3} i^3\tan^3\theta + \dots = 2i\theta$$

$$\Rightarrow 2i\tan\theta + \frac{2}{3} i^3 \tan^3\theta + \frac{2}{5} i^5 \tan^5\theta + \cdots = 2i\theta$$

$$\Rightarrow 2i\tan\theta - \frac{2}{3} i\tan^3\theta + \frac{2}{5} i\tan^5\theta - \cdots = 2i\theta$$

$$\Rightarrow 2i\left(\tan\theta - \frac{1}{3}\tan^3\theta + \frac{1}{5}\tan^5\theta - \cdots\right) = 2i\theta$$

$$\Rightarrow \theta = \tan\theta - \frac{1}{3}\tan^3\theta + \frac{1}{5}\tan^5\theta - \cdots$$

This is called Gregory's series.

Putting $\tan\theta = x$ or $\theta = \tan^{-1}x$ we have

$$\tan^{-1}x = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \cdots$$

Again, putting $\theta = \frac{\pi}{4}$ in Gregory's series we have

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$$

Ex-2| We know, Gregory's series is

$$\theta = \tan\theta - \frac{1}{3}\tan^3\theta + \frac{1}{5}\tan^5\theta - \cdots \quad \text{--- (1)}$$

putting $\theta = \frac{\pi}{4}$ in eqn (1) we get

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots$$

$$= \left(1 - \frac{1}{3}\right) + \left(\frac{1}{5} - \frac{1}{7}\right) + \left(\frac{1}{9} - \frac{1}{11}\right) + \cdots$$

$$= \frac{3-1}{1\cdot 3} + \frac{7-5}{5\cdot 7} + \frac{11-9}{9\cdot 11} + \cdots$$

$$= \frac{2}{1\cdot 3} + \frac{2}{5\cdot 7} + \frac{2}{9\cdot 11} + \cdots$$

$$\therefore \frac{\pi}{4} = 2\left(\frac{1}{1.3} + \frac{1}{5.7} + \frac{1}{9.11} + \cdots\right)$$

$$\Rightarrow \frac{\pi}{8} = \frac{1}{1.3} + \frac{1}{5.7} + \frac{1}{9.11} + \cdots$$

(proved)

ex-2| Given that,

$$\tan x = n \tan y$$

$$\Rightarrow \frac{\sin x}{\cos x} = n \frac{\sin y}{\cos y}$$

$$\Rightarrow \frac{2i \sin x}{2 \cos x} = n \frac{2i \sin y}{\cos y}$$

$$\Rightarrow \frac{e^{ix} - e^{-ix}}{e^{ix} + e^{-ix}} = n \frac{e^{iy} - e^{-iy}}{e^{iy} + e^{-iy}}$$

$$\Rightarrow \frac{e^{ix}\left(e^{ix} - e^{-ix}\right)}{e^{ix}\left(e^{ix} + e^{-ix}\right)} = n \frac{e^{iy}\left(e^{iy} - e^{-iy}\right)}{e^{iy}\left(e^{iy} + e^{-iy}\right)}$$

$$\Rightarrow \frac{e^{2ix} - 1}{e^{2ix} + 1} = n \frac{e^{2iy} - 1}{e^{2iy} + 1}$$

$$\Rightarrow \frac{e^{2ix} + 1}{e^{2ix} - 1} = \frac{e^{2iy} + 1}{ne^{2iy} - n}$$

$$\Rightarrow \frac{e^{2ix} + 1 + e^{2ix} - 1}{e^{2ix} + 1 - e^{2ix} + 1} = \frac{e^{2iy} + 1 + ne^{2iy} - n}{e^{2iy} + 1 - ne^{2iy} + n}$$

$$\Rightarrow \frac{2e^{2ix}}{2} = \frac{e^{2iy}(1+n) + (1-n)}{e^{2iy}(1-n) + (1+n)}$$

$$\Rightarrow e^{2ix} = \frac{e^{2iy}\frac{1+n}{1+n} + \frac{1-n}{1+n}}{e^{2iy}\frac{1-n}{1+n} + \frac{1+n}{1+n}}$$

$$= \frac{e^{2iy} + \frac{1-n}{1+n}}{e^{2iy}\frac{1-n}{1+n} + 1}$$

$$= \frac{e^{2iy} + k}{ke^{2iy} + 1} \quad \left[\text{Let, } k = \frac{1-n}{1+n}\right.$$

$$= \frac{e^{2iy}(1 + ke^{-2iy})}{1 + ke^{2iy}}$$

Taking log on both sides, we have

$$\log e^{2ix} = \log\{e^{2iy}(1+ke^{-2iy})\} - \log(1+ke^{2iy})$$

$$\Rightarrow 2ix = \log e^{2iy} + \log(1+ke^{-2iy}) - \log(1+ke^{2iy})$$

$$= 2iy + \left(ke^{-2iy} - \frac{1}{2}k^2e^{-4iy} + \frac{1}{3}k^3e^{-6iy} - \dots\right)$$
$$- \left(ke^{2iy} - \frac{1}{2}k^2e^{4iy} + \frac{1}{3}k^3e^{6iy} - \dots\right)$$

$$= 2iy - k(e^{2iy} - e^{-2iy}) + \frac{k^2}{2}(e^{4iy} - e^{-4iy}) - \dots$$

$$= 2iy - k\,2i\sin 2y + \frac{k^2}{2}\cdot 2i\sin 4y$$
$$- \frac{k^3}{3}\,2i\sin 6y + \dots$$

$$\therefore 2ix = 2i\left(y - k\sin 2y + \frac{k^2}{2}\sin 4y - \frac{k^3}{3}\sin 6y + \cdots\right)$$

$$\Rightarrow x = y - k\sin 2y + \frac{k^2}{2}\sin 4y - \frac{k^3}{3}\sin 6y + \cdots$$

(Ans)

24) Given that,

$$\tan x = \frac{n\sin\alpha}{1 - n\cos\alpha}$$

$$\Rightarrow \frac{\sin x}{\cos x} = \frac{n\sin\alpha}{1 - n\cos\alpha}$$

$$\Rightarrow \frac{2i\sin x}{2\cos x} = \frac{ni\sin\alpha}{1 - n\cos\alpha}$$

$$\Rightarrow \frac{e^{ix} - e^{-ix}}{e^{ix} + e^{-ix}} = \frac{ni\sin\alpha}{1 - n\cos\alpha}$$

$$\Rightarrow \frac{e^{ix} - e^{-ix} + e^{ix} + e^{-ix}}{e^{ix} - e^{-ix} - e^{ix} - e^{-ix}} = \frac{ni\sin\alpha + 1 - n\cos\alpha}{ni\sin\alpha - 1 + n\cos\alpha}$$

$$\Rightarrow \frac{\cancel{2}e^{ix}}{-\cancel{2}e^{-ix}} = \frac{1 - n(\cos\alpha - i\sin\alpha)}{-(1 - n\cos\alpha - ni\sin\alpha)}$$

$$\Rightarrow \frac{e^{ix}}{e^{-ix}} = \frac{1 - n(\cos\alpha - i\sin\alpha)}{1 - n(\cos\alpha + i\sin\alpha)}$$

$$\Rightarrow e^{2ix} = \frac{1 - ne^{-i\alpha}}{1 - ne^{i\alpha}}$$

$$\Rightarrow \log e^{2ix} = \log\frac{1 - ne^{-i\alpha}}{1 - ne^{i\alpha}}$$

$$\Rightarrow 2ix = \log\left(1 - ne^{-i\alpha}\right) - \log\left(1 - ne^{i\alpha}\right)$$

$$\Rightarrow 2ix = \left\{ -ne^{-i\alpha} - \frac{n^2}{2} e^{-2i\alpha} - \frac{n^3}{3} e^{-3i\alpha} - \cdots \right\}$$

$$- \left\{ -ne^{i\alpha} - \frac{n^2}{2} e^{2i\alpha} - \frac{n^3}{3} e^{3i\alpha} - \cdots \right\}$$

$$= \left(-ne^{-i\alpha} - \frac{n^2}{2} e^{-2i\alpha} - \frac{n^3}{3} e^{-3i\alpha} - \cdots \right) + \left(ne^{i\alpha} + \frac{n^2}{2} e^{2i\alpha} + \frac{n^3}{3} e^{3i\alpha} + \cdots \right)$$

$$= n\left(e^{i\alpha} - e^{-i\alpha}\right) + \frac{n^2}{2}\left(e^{2i\alpha} - e^{-2i\alpha}\right) + \frac{n^3}{3}\left(e^{3i\alpha} - e^{-3i\alpha}\right) + \cdots$$

$$= n \cdot 2i \sin\alpha + \frac{n^2}{2} \cdot 2i \sin 2\alpha + \frac{n^3}{3} \cdot 2i \sin 3\alpha + \cdots$$

$$\therefore 2ix = 2i\left(n\sin\alpha + \frac{n^2}{2}\sin 2\alpha + \frac{n^3}{3}\sin 3\alpha + \cdots \right)$$

$$\Rightarrow x = n\sin\alpha + \frac{n^2}{2}\sin 2\alpha + \frac{n^3}{3}\sin 3\alpha + \cdots$$

(proved)

23| Given that,

$$\tan\theta = x + \tan\alpha$$

$$\Rightarrow \frac{\sin\theta}{\cos\theta} = x + \frac{\sin\alpha}{\cos\alpha}$$

$$\Rightarrow \frac{\sin\theta}{\cos\theta} = \frac{x\cos\alpha + \sin\alpha}{\cos\alpha}$$

$$\Rightarrow \frac{2i\sin\theta}{2\cos\theta} = \frac{i(x\cos\alpha + \sin\alpha)}{\cos\alpha}$$

$$\Rightarrow \frac{e^{i\theta} - e^{-i\theta}}{e^{i\theta} + e^{-i\theta}} = \frac{ix\cos\alpha + i\sin\alpha}{\cos\alpha}$$

$$\Rightarrow \frac{e^{i\theta} - e^{-i\theta} + e^{i\theta} + e^{-i\theta}}{e^{i\theta} - e^{-i\theta} - e^{i\theta} - e^{-i\theta}} = \frac{ix\cos\alpha + i\sin\alpha + \cos\alpha}{ix\cos\alpha + i\sin\alpha - \cos\alpha}$$

$$\Rightarrow \frac{2e^{i\theta}}{-2e^{-i\theta}} = \frac{ix\cos\alpha + (\cos\alpha + i\sin\alpha)}{-(-ix\cos\alpha + \cos\alpha - i\sin\alpha)}$$

$$\Rightarrow \frac{e^{i\theta}}{e^{-i\theta}} = \frac{ix\cos\alpha + e^{i\alpha}}{-ix\cos\alpha + e^{-i\alpha}}$$

$$\Rightarrow e^{2i\theta} = \frac{e^{i\alpha}(1 + ix\cos\alpha \cdot e^{-i\alpha})}{e^{-i\alpha}(1 - ix\cos\alpha \cdot e^{i\alpha})}$$

$$\Rightarrow e^{2i\theta} = e^{2i\alpha} \cdot \left(\frac{1 + ix\cos\alpha\, e^{-i\alpha}}{1 - ix\cos\alpha\, e^{i\alpha}}\right)$$

$$\Rightarrow \log e^{2i\theta} = \log\left\{e^{2i\alpha} \cdot \left(\frac{1 + ix\cos\alpha\, e^{-i\alpha}}{1 - ix\cos\alpha\, e^{i\alpha}}\right)\right\}$$

$$\Rightarrow 2i\theta = \log(e^{2i\alpha}) + \log\left(\frac{1 + ix\cos\alpha\, e^{-i\alpha}}{1 - ix\cos\alpha\, e^{i\alpha}}\right)$$

$$\Rightarrow 2i\theta = 2i\alpha + \left\{\log(1 + ix\cos\alpha\, e^{-i\alpha}) - \log(1 - ix\cos\alpha\, e^{i\alpha})\right\}$$

$$= 2i\alpha + \left\{\left(ix\cos\alpha\, e^{-i\alpha} - \frac{i^2x^2\cos^2\alpha}{2} \cdot e^{-2i\alpha} + \cdots\right) - \left(-ix\cos\alpha\, e^{i\alpha} - \frac{i^2x^2\cos^2\alpha}{2} \cdot e^{2i\alpha} - \cdots\right)\right\}$$

$$\Rightarrow 2i\theta = 2i\alpha + ix\cos\alpha\left(e^{i\alpha} + e^{-i\alpha}\right) + \frac{i^2 x^2 \cos^2\alpha}{2}\left(e^{2i\alpha} - e^{-2i\alpha}\right) + \cdots$$

$$\Rightarrow 2i\theta = 2i\alpha + ix\cos\alpha \cdot 2\cos\alpha - \frac{x^2\cos^2\alpha}{2} 2i\sin 2\alpha - \cdots$$

$$\Rightarrow 2i\theta = 2i\left(\alpha + x\cos^2\alpha - \frac{x^2}{2}\cos^2\alpha\sin 2\alpha - \frac{x^3}{3}\cos^3\alpha \cdot \cos 3\alpha + \cdots\right)$$

$$\therefore \theta = \alpha + x\cos^2\alpha - \frac{x^2}{2}\cos^2\alpha\sin 2\alpha - \frac{x^3}{3}\cos^3\alpha\cos 3\alpha + \cdots$$

(proved)

10| Given that,

$$\tan\theta = \frac{a\sin\alpha}{1 + a\cos\alpha}$$

$$\Rightarrow \frac{2i\sin\theta}{2\cos\theta} = \frac{ia\sin\alpha}{1 + a\cos\alpha}$$

$$\Rightarrow \frac{e^{i\theta} - e^{-i\theta}}{e^{i\theta} + e^{-i\theta}} = \frac{ia\sin\alpha}{1 + a\cos\alpha}$$

$$\Rightarrow \frac{e^{i\theta} - e^{-i\theta} + e^{i\theta} + e^{-i\theta}}{e^{i\theta} - e^{-i\theta} - e^{i\theta} - e^{-i\theta}} = \frac{a\,i\sin\alpha + 1 + a\cos\alpha}{a\,i\sin\alpha - 1 - a\cos\alpha}$$

$$\Rightarrow \frac{2e^{i\theta}}{-2e^{-i\theta}} = \frac{1 + a(\cos\alpha + i\sin\alpha)}{-\{1 + a\cos\alpha - a\, i\sin\alpha\}}$$

$$\Rightarrow \frac{e^{i\theta}}{e^{-i\theta}} = \frac{1 + a(\cos\alpha + i\sin\alpha)}{1 + a(\cos\alpha - i\sin\alpha)}$$

$$\Rightarrow e^{2i\theta} = \frac{1 + ae^{i\alpha}}{1 + ae^{-i\alpha}}$$

Taking log on both sides we get

$$\log e^{2i\theta} = \log(1 + ae^{i\alpha}) - \log(1 + ae^{-i\alpha})$$

$$\Rightarrow 2i\theta = \left\{ ae^{i\alpha} - \frac{a^2}{2}e^{2i\alpha} + \frac{a^3}{3}e^{3i\alpha} - \cdots \right\} - \left\{ ae^{-i\alpha} - \frac{a^2}{2}e^{-2i\alpha} + \frac{a^3}{3}e^{-3i\alpha} - \cdots \right\}$$

$$= a(e^{i\alpha} - e^{-i\alpha}) - \frac{a^2}{2}(e^{2i\alpha} - e^{-2i\alpha}) + \frac{a^3}{3}(e^{3i\alpha} - e^{-3i\alpha}) - \cdots$$

$$= a \cdot 2i\sin\alpha - \frac{a^2}{2} \cdot 2i\sin 2\alpha + \frac{a^3}{3} 2i\sin 3\alpha - \cdots$$

$$\therefore 2i\theta = 2i\left(a\sin\alpha - \frac{a^2}{2}\sin 2\alpha + \frac{a^3}{3}\sin 3\alpha - \cdots\right)$$

$$\Rightarrow \theta = a\sin\alpha - \frac{a^2}{2}\sin 2\alpha + \frac{a^3}{3}\sin 3\alpha \cdots \quad \text{(prove)}$$

28| Given that,

$$\cot y = \cot x + \operatorname{cosec}\alpha \operatorname{cosec} x$$

$$\Rightarrow \cot y = \frac{\cos x}{\sin x} + \frac{1}{\sin\alpha} \cdot \frac{1}{\sin x}$$

$$\Rightarrow \cot y = \frac{\cos x}{\sin x} + \frac{1}{a \sin x} \quad \left[\text{Let } \sin\alpha = a\right]$$

$$\Rightarrow \cot y = \frac{a\cos x + 1}{a \sin x}$$

$$\Rightarrow \frac{1}{\cot y} = \frac{a \sin x}{1 + a\cos x}$$

$$\Rightarrow \tan y = \frac{a \sin x}{1 + a\cos x}$$

$$\Rightarrow \frac{2i\sin y}{2\cos y} = \frac{i\, a \sin x}{1 + a\cos x}$$

$$\Rightarrow \frac{e^{iy} - e^{-iy}}{e^{iy} + e^{-iy}} = \frac{a\, i \sin x}{1 + a\cos x}$$

$$\Rightarrow \frac{e^{iy} + e^{-iy}}{e^{iy} - e^{-iy}} = \frac{1 + a\cos x}{a\, i \sin x}$$

$$\Rightarrow \frac{e^{iy} + \cancel{e^{-iy}} + e^{iy} - \cancel{e^{-iy}}}{\cancel{e^{iy}} + e^{-iy} - \cancel{e^{iy}} + e^{-iy}} = \frac{1 + a\cos x + a\, i \sin x}{1 + a\cos x - a\, i \sin x}$$

$$\Rightarrow \frac{2e^{iy}}{2e^{-iy}} = \frac{1 + a(\cos x + i\sin x)}{1 + a(\cos x - i\sin x)}$$

$$\Rightarrow e^{2iy} = \frac{1+ae^{ix}}{1+ae^{-ix}}$$

Taking log on both sides we get

$$\log e^{2iy} = \log\left(1+ae^{ix}\right) - \log\left(1+ae^{-ix}\right)$$

$$\Rightarrow 2iy = \left(ae^{ix} - \frac{a^2}{2}e^{2ix} + \frac{a^3}{3}e^{3ix} - \cdots\right)$$

$$- \left(ae^{-ix} - \frac{a^2}{2}e^{-2ix} + \frac{a^3}{3}e^{-3ix} - \cdots\right)$$

$$= a\left(e^{ix} - e^{-ix}\right) - \frac{a^2}{2}\left(e^{2ix} - e^{-2ix}\right)$$

$$+ \frac{a^3}{3}\left(e^{3ix} - e^{-3ix}\right) + \cdots$$

$$= a \cdot 2i \sin x - \frac{a^2}{2} \cdot 2i \sin 2x$$

$$+ \frac{a^3}{3} \cdot 2i \sin 3x + \cdots$$

$$\therefore 2iy = 2i\left(a \sin x - \frac{a^2}{2}\sin 2x + \frac{a^3}{3}\sin 3x + \cdots\right)$$

$$\Rightarrow y = a \sin x - \frac{a^2}{2}\sin 2x + \frac{a^3}{3}\sin 3x + \cdots$$

$$\Rightarrow y = \sin\alpha \sin x - \frac{1}{2}\sin^2\alpha \sin 2x + \frac{1}{3}\sin^3\alpha \sin 3x + \cdots$$

(proved)

21/ Here,

$$2\sqrt{3}\left[1-\frac{1}{3\cdot 3}+\frac{1}{5\cdot 3^2}+\frac{1}{7\cdot 5^3}+\cdots\right]$$

$$=2\sqrt{3}\times\sqrt{3}\left\{\frac{1}{\sqrt{3}}-\frac{1}{3\cdot(\sqrt{3})^2\cdot\sqrt{3}}+\frac{1}{5\cdot(\sqrt{3})^4\sqrt{3}}+\cdots\right\}$$

$$=6\left\{\frac{1}{\sqrt{3}}-\frac{1}{3}\frac{1}{(\sqrt{3})^3}+\frac{1}{5}\cdot\frac{1}{(\sqrt{3})^5}+\cdots\right\}$$

$$=6\left\{x-\frac{1}{3}x^3+\frac{1}{5}\cdot x^5-\cdots\right\}$$

[Putting $\frac{1}{\sqrt{3}}=x$]

$$=6\tan^{-1}x$$

$$=6\tan^{-1}\frac{1}{\sqrt{3}}$$

$$=6\cdot\frac{\pi}{6}$$

$$=\pi$$

$$\therefore \pi=2\sqrt{3}\left[1-\frac{1}{3\cdot 3}+\frac{1}{5\cdot 3^2}+\frac{1}{7\cdot 5^3}+\cdots\right]$$

(proved)

2 i) Here,

$$\left(\frac{2}{3}+\frac{1}{7}\right)-\frac{1}{3}\left(\frac{2}{3^3}+\frac{1}{7^3}\right)+\frac{1}{5}\left(\frac{2}{3^5}+\frac{1}{7^5}\right)-\cdots$$

$$=\frac{2}{3}+\frac{1}{7}-\frac{2}{3\cdot 3^3}-\frac{1}{3\cdot 7^3}+\frac{2}{5\cdot 3^5}+\frac{1}{5\cdot 7^5}-\cdots$$

$$=\left(\frac{2}{3}-\frac{2}{3\cdot 3^3}+\frac{2}{5\cdot 3^5}-\cdots\right)+\left(\frac{1}{7}-\frac{1}{3\cdot 7^3}+\frac{1}{5\cdot 7^5}-\cdots\right)$$

$$=2\left\{\frac{1}{3}-\frac{1}{3}\left(\frac{1}{3}\right)^3+\frac{1}{5}\left(\frac{1}{3}\right)^5-\cdots\right\}+\left\{\frac{1}{7}-\frac{1}{3}\cdot\left(\frac{1}{7}\right)^3+\frac{1}{5}\cdot\left(\frac{1}{7}\right)^5-\cdots\right\}$$

$$=2\tan^{-1}\frac{1}{3}+\tan^{-1}\frac{1}{7}$$

$$=\tan^{-1}\frac{1}{3}+\tan^{-1}\frac{1}{3}+\tan^{-1}\frac{1}{7}$$

$$=\tan^{-1}\frac{\frac{1}{3}+\frac{1}{3}}{1-\left(\frac{1}{3}\cdot\frac{1}{3}\right)}+\tan^{-1}\frac{1}{7}$$

$$=\tan^{-1}\frac{\frac{2+1}{3}}{\frac{9-1}{9}}+\tan^{-1}\frac{1}{7}$$

$$=\tan^{-1}\left(\frac{2}{3}\times\frac{9}{8}\right)+\tan^{-1}\frac{1}{7}$$

$$=\tan^{-1}\frac{3}{4}+\tan^{-1}\frac{1}{7}$$

$$=\tan^{-1}\frac{\frac{3}{4}+\frac{1}{7}}{1-\frac{3}{4}\cdot\frac{1}{7}}$$

$$= \tan^{-1}\left(\frac{\frac{21+4}{28}}{\frac{28-3}{28}}\right)$$

$$= \tan^{-1}\left(\frac{25}{28} \times \frac{28}{25}\right)$$

$$= \tan^{-1} 1$$

$$= \frac{\pi}{4}$$

$$\therefore \frac{\pi}{4} = \left(\frac{2}{3} + \frac{1}{7}\right) - \frac{1}{3}\left(\frac{2}{3^3} + \frac{1}{7^3}\right) + \frac{1}{5}\left(\frac{2}{3^5} + \frac{1}{7^5}\right)$$

(Showed)

2-iv | Here,

$$\left(1 - \frac{1}{3^{\frac{1}{2}}}\right) - \frac{1}{3}\left(1 - \frac{1}{3^{\frac{3}{2}}}\right) + \frac{1}{5}\left(1 - \frac{1}{3^{\frac{5}{2}}}\right) \cdots$$

$$= 1 - \frac{1}{3^{\frac{1}{2}}} - \frac{1}{3} + \frac{1}{3 \cdot 3^{3/2}} + \frac{1}{5} - \frac{1}{5 \cdot 3^{5/2}} \cdots$$

$$= \left(1 - \frac{1}{3} + \frac{1}{5} - \cdots\right) - \left\{\frac{1}{\sqrt{3}} - \frac{1}{3 \cdot 3^{3/2}} + \frac{1}{5 \cdot 3^{5/2}} \cdots\right\}$$

$$= \left\{1 - \frac{1}{3}(1)^3 + \frac{1}{5}(1)^5 \cdots\right\} - \left\{\frac{1}{\sqrt{3}} - \frac{1}{3(\sqrt{3})^3} + \frac{1}{5 \cdot (\sqrt{3})^5} \cdots\right\}$$

$$= \tan^{-1} 1 - \left\{\frac{1}{\sqrt{3}} - \frac{1}{3}\left(\frac{1}{\sqrt{3}}\right)^3 + \frac{1}{5}\left(\frac{1}{\sqrt{3}}\right)^5 \cdots\right\}$$

$$= \tan^{-1} 1 - \tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$$

$$= \frac{\pi}{4} - \frac{\pi}{6}$$

$$= \frac{3\pi - 2\pi}{12}$$

$$= \frac{\pi}{12}$$

$$\therefore \quad \frac{\pi}{12} = \left(1 - \frac{1}{3^{\frac{1}{2}}}\right) - \frac{1}{3}\left(1 - \frac{1}{3^{3/2}}\right) + \frac{1}{5}\left(1 - \frac{1}{3^{5/2}}\right) - \cdots\cdots$$

(Showed)