Tania

Successive Reduction

01. If $I_n = \int_0^{\pi/4} \tan^n\theta \, d\theta$, show that $I_n = \frac{1}{n-1} - I_{n-2}$, Hence find the value of $\int_0^{\pi/4} \tan^6 x \, dx$.

Solⁿ: Given that, $I_n = \int_0^{\pi/4} \tan^n\theta \, d\theta$

$$\text{or, } I_n = \int_0^{\pi/4} \tan^{n-2}\theta \tan^2\theta \, d\theta$$

$$= \int_0^{\pi/4} \tan^{n-2}\theta \, (\sec^2\theta - 1) \, d\theta$$

$$I_n = \int_0^{\pi/4} \tan^{n-2}\theta \sec^2\theta \, d\theta - \int_0^{\pi/4} \tan^{n-2}\theta \, d\theta$$

$$\text{or, } I_n = \int_0^{\pi/4} \tan^{n-2}\theta \, d(\tan\theta) - I_{n-2} \qquad \left[\because I_n = \int_0^{\pi/4} \tan^n\theta \, d\theta\right]$$

$$\text{or, } I_n = \left[\frac{\tan^{n-1}\theta}{n-1}\right]_0^{\pi/4} - I_{n-2}$$

$$\therefore I_n = \frac{1}{n-1} - I_{n-2}, \qquad I_n + I_{n-2} = \frac{1}{n-1}$$

$$I_n(n-1) + I_{n-2}(n-1) = 1$$

Replacing $(n-1)$ by n we get

$$n(I_{n+1} + I_{n-1}) = 1$$

2nd Part: Let $I_6 = \int_0^{\pi/4} \tan^6 x \, dx$

$$I_6 = \frac{1}{6-1} - \int_0^{\pi/4} \tan^4 x \, dx$$

$$= \frac{1}{5} - \left\{\frac{1}{4-1} - \int_0^{\pi/4} \tan^2 x \, dx\right\}$$

$$= \frac{1}{5} - \frac{1}{3} + \int_0^{\pi/4} (\sec^2 x - 1) \, dx$$

$$= \frac{3-5}{15} + \left[\tan x\right]_0^{\pi/4} - \left[\theta\right]_0^{\pi/4}$$

$$= \frac{-2}{15} + (1-0) - (\pi/4 - 0)$$

$$= \frac{-2+15}{15} - \frac{\pi}{4} = \left(\frac{13}{15} - \frac{\pi}{4}\right) \text{ Answer.}$$

02. Prove that if $U_n = \int_0^1 x^n \tan^{-1} x \, dx$, then $(n+1) U_n + (n-1) U_{n-2} = \frac{\pi}{2} - \frac{1}{n}$

Solution: Given that,

$$U_n = \int_0^1 x^n \tan^{-1} x \, dx$$

$$\therefore U_n = \int_0^{\pi/4} \tan^n\theta \; \theta \; \sec^2\theta \, d\theta$$

Put $\tan^{-1} x = \theta$

$x = \tan\theta$

$dx = \sec^2\theta \, d\theta$

x	0	1
θ	0	$\pi/4$

$$U_n = \int_0^{\pi/4} \theta \tan^n\theta \, d(\tan\theta)$$

$$= \left[\theta \int \tan^n\theta \, d(\tan\theta)\right]_0^{\pi/4} - \int_0^{\pi/4} \left\{\frac{d\theta}{d\theta} \int \tan^n\theta \, d(\tan\theta)\right\} d\theta$$

$$= \left[\theta \, \frac{\tan^{n+1}\theta}{n+1}\right]_0^{\pi/4} - \int_0^{\pi/4} \frac{\tan^{n+1}\theta}{n+1} \, d\theta$$

$$U_n = \frac{\pi}{4} \cdot \frac{1}{n+1} - \frac{1}{n+1} \int_0^{\pi/4} \tan^{n+1}\theta \, d\theta$$

$$(n+1) U_n = \frac{\pi}{4} - \int_0^{\pi/4} \tan^{n+1}\theta \, d\theta \longrightarrow (1)$$

Replacing $(n+1)$ by (

$$\text{or, } (n-1) U_{n-2} = \frac{\pi}{4} - \int_0^{\pi/4} \tan^{n-1}\theta \, d\theta \longrightarrow (2)$$

Adding equation ① and ②

$$(n+1)U_n + (n-1)U_n = \frac{\pi}{4} + \frac{\pi}{4} - \int_0^{\pi/4} \tan^{n-1}\theta\,(\tan^2\theta + 1)\,d\theta$$

$$= \frac{\pi}{2} - \int_0^{\pi/4} \tan^{n-1}\theta \sec^2\theta\, d\theta$$

$$= \frac{\pi}{2} - \int_0^{\pi/4} \tan^{n-1}\theta\, d(\tan\theta)$$

$$= \frac{\pi}{2} - \left[\frac{\tan^n\theta}{n}\right]_0^{\pi/4}$$

$$= \frac{\pi}{2} - \frac{1}{n}$$

$$\therefore (n+1)U_n + (n-1)U_n = \frac{\pi}{2} - \frac{1}{n}.$$ (Proved)

03. If $U_n = \int_0^{\pi/2} \theta \sin^n\theta\, d\theta$ and $n > 1$ then Prove that $U_n = \frac{n-1}{n} U_{n-2} + \frac{1}{n^2}$.

Given $U_n = \int_0^{\pi/2} \theta \sin^n\theta\, d\theta$

$$= \int_0^{\pi/2} \theta \sin\theta \sin^{n-1}\theta\, d\theta = \int_0^{\pi/2} \theta \sin^{n-1}\theta \sin\theta\, d\theta$$

$$= \left[\theta \sin^{n-1}\theta(-\cos\theta)\right]_0^{\pi/2} - \int_0^{\pi/2} \left\{\frac{d}{d\theta}(\theta \sin^{n-1}\theta)\right\}(-\cos\theta)\, d\theta$$

$$= 0 + \int_0^{\pi/2} \left\{\sin^{n-1}\theta + (n-1)\theta \sin^{n-2}\theta \cos\theta\right\} \cos\theta\, d\theta$$

$$= \int_0^{\pi/2} \sin^{n-1}\theta \cos\theta\, d\theta + \int_0^{\pi/2} (n-1)\theta \sin^{n-2}\theta \cos^2\theta\, d\theta$$

$$U_n = \int_0^{\pi/2} \sin^{n-1}\theta\, d(\sin\theta) + (n-1)\int_0^{\pi/2} \theta \sin^{n-2}\theta\, \cancel{d(\sin\theta)}\, (1-\sin^2\theta)\, d\theta$$

$$U_n = \left[\frac{\sin^n\theta}{n}\right]_0^{\pi/2} + (n-1)\int_0^{\pi/2}\theta\sin^{n-2}\theta\,d\theta - (n-1)\int_0^{\pi/2}\theta\sin^n\theta\,d\theta$$

$$U_n = \frac{1}{n} + (n-1)U_{n-2} - (n-1)U_n$$

or, $U_n(1+n-1) = (n-1)U_{n-2} + \frac{1}{n}$

or, $nU_n = (n-1)U_{n-2} + \frac{1}{n}$

$\therefore U_n = \frac{n-1}{n}U_{n-2} + \frac{1}{n^2}$ Proved

04. If $U_n = \int_0^{\pi/2} x^n \sin x\,dx\ (n>0)$ then Prove that $U_n + n(n-1)U_{n-2} = n\left(\frac{1}{2}\pi\right)^{n-1}$

Proof: $U_n = \int_0^{\pi/2} x^n \sin x\,dx$

Integrating by Parts

$$U_n = \left[x^n(-\cos x)\right]_0^{\pi/2} - \int_0^{\pi/2} nx^{n-1}(-\cos x)\,dx$$

$$= 0 + n\int_0^{\pi/2} x^{n-1}\cos x\,dx$$

Again Integrating by Parts

$$U_n = n\left[x^{n-1}\sin x\right]_0^{\pi/2} - n\int_0^{\pi/2}(n-1)x^{n-2}\sin x\,dx$$

$$U_n = n\left(\frac{\pi}{2}\right)^{n-1} - n(n-1)U_{n-2}$$

$\therefore U_n + n(n-1)U_{n-2} = n\left(\frac{\pi}{2}\right)^{n-1}$

(Proved)

05. Find the reduction formula for (i) $\int_0^{\pi/2} \sin^n x\,dx$ (ii) $\int_0^{\pi/2} \cos^n x\,dx$ and hence show that

$$\int_0^{\pi/2} \sin^n x\,dx = \int_0^{\pi/2} \cos^n x\,dx = \begin{cases} \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdots\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2} & \text{(when } n \text{ is even)} \\ \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdots\frac{4}{5}\cdot\frac{2}{3}\cdot 1 & \text{(when } n \text{ is odd)} \end{cases}$$

Solution: (i)

Let, $I_n = \int_0^{\pi/2} \sin^n x\,dx = \int_0^{\pi/2} \sin^{n-1}x\cdot\sin x\,dx$

$I_n = \left[-\sin^{n-1}x\cos x\right]_0^{\pi/2} - \int_0^{\pi/2}(n-1)\sin^{n-2}x\cos x\cdot(-\cos x)\,dx$

$= 0 + (n-1)\int_0^{\pi/2}\sin^{n-2}x\cos^2 x\,dx$

$= 0 + (n-1)\int_0^{\pi/2}\sin^{n-2}x(1-\sin^2 x)\,dx$

or, $I_n = (n-1)\int_0^{\pi/2}\sin^{n-2}x\,dx - (n-1)\int_0^{\pi/2}\sin^n x\,dx$

or, $I_n = (n-1)I_{n-2} - (n-1)I_n$

or, $I_n(1+n-1) = (n-1)I_{n-2}$

$\therefore I_n = \frac{n-1}{n}I_{n-2}$

which is the required reduction formula

[Similarly for Problem (ii) $I_n = \frac{n-1}{n} I_{n-2}$
(নিজে কর)]

$$\therefore I_n = \int_0^{\pi/2} \sin^n x\,dx = \int_0^{\pi/2} \cos^n x\,dx = \frac{n-1}{n} I_{n-2}$$

$$I_n = \frac{n-1}{n} I_{n-2} \longrightarrow (1)$$

Changing n into $(n-2)$, $(n-4)$, $(n-6)$ etc. successively we have from (1)

$$I_{n-2} = \frac{n-3}{n-2} I_{n-4}$$

$$I_{n-4} = \frac{n-5}{n-4} I_{n-6}$$

$$I_{n-6} = \frac{n-7}{n-6} I_{n-8} \text{ -------- etc}$$

$$\therefore I_n = \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-4} \text{ ----- } \frac{3}{4} \cdot \frac{1}{2} I_0$$

(when n is even)

$$\& \ I_n = \frac{n-1}{n} \frac{n-3}{n-2} \frac{n-5}{n-4} \text{ ------- } \frac{4}{5} \frac{2}{3} \cdot I_1$$

(when n is odd)

$$\text{But } I_0 = \int_0^{\pi/2} (\sin^0 x)^0 dx = \int_0^{\pi/2} dx = x\Big|_0^{\pi/2} = \frac{\pi}{2}$$

$$I_1 = \int_0^{\pi/2} \sin x\,dx = \left[-\cos x\right]_0^{\pi/2} = -0+1 = 1$$

Hence $I_n = \frac{n-1}{n} \frac{n-3}{n-2} \text{ ------- } \frac{3}{4} \cdot \frac{1}{2} \frac{\pi}{2}$ (when n is even)

$\& \ I_n = \frac{n-1}{n} \frac{n-3}{n-2} \text{ ------- } \frac{4}{5} \cdot \frac{2}{3} \cdot 1$ (when n odd)