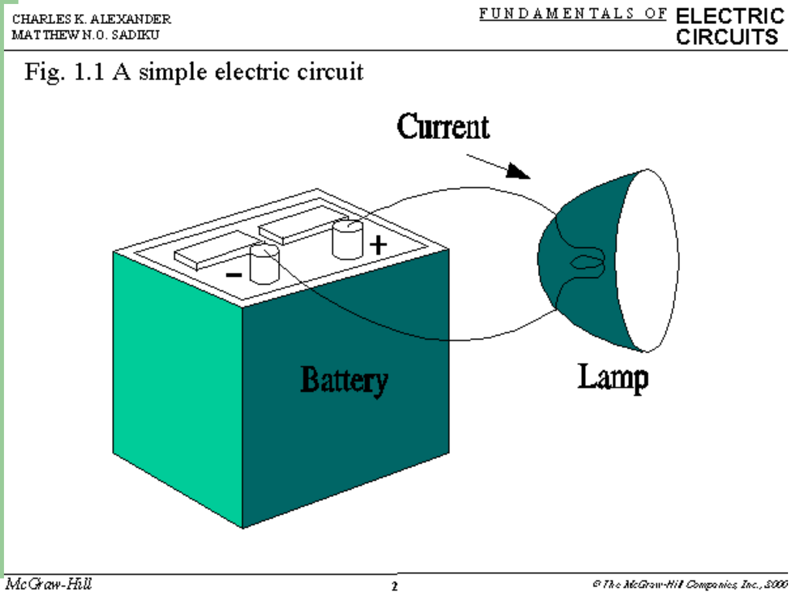
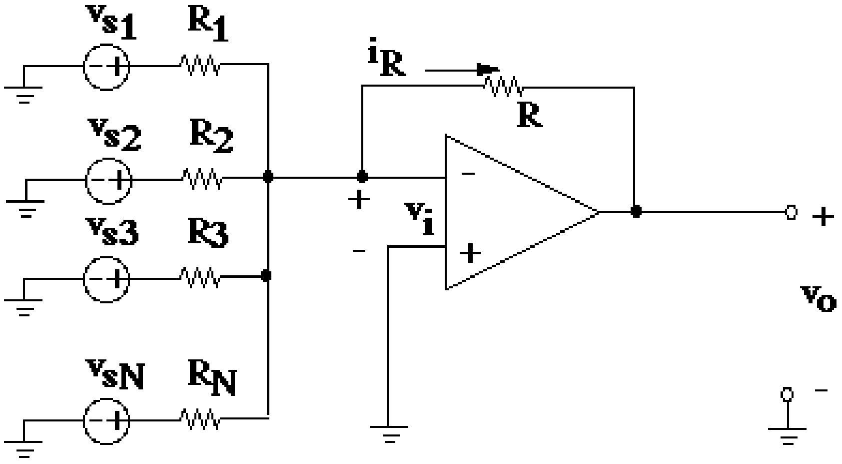
Electrical Circuit Analysis

Electric Circuit

An electric circuit is an interconnection of electrical elements.





Charge and Current

- Electric *charge* is a property of matter. Measured in coulombs (C).
- 1 C of charge requires 6.24×10^{18} electrons
- *Law of conservation of charge*: Charge can be moved, but it can not be created or destroyed.

Electric Current

- *Electric Current* is the flow of charge. It is measured in amperes (A)
- 1 A = 1 C/s
- Direct Current (DC): Current remains constant
- Alternating Current (AC): Current varies sinusoidally with time

$$i = \frac{dq}{dt}$$

where

$$\begin{aligned} i =& \quad \text{current in amperes} \\ q =& \quad \text{charge in coulombs} \\ t =& \quad \text{time in seconds} \end{aligned}$$

Electric Current

The charge transferred between time t_0 and t

$$q = \int_{t_o}^{t} i\,dt$$

Voltage

- The voltage across a circuit element is the energy absorbed or expended as a unit charge moves through the element
- Analogous to pressure in a hydraulic system
- Sometimes called potential difference
- Is a measure of the potential between *two points*
- Voltage pushes charge in one direction
- We use polarity ($+$ and $-$ on batteries) to indicate which direction the charge is being pushed

$$v = \frac{dw}{dq}$$

where

$v =$ voltage in volts

$w =$ energy in Joules

$q =$ charge in coulombs

Power

- Power: time rate of expending or absorbing energy
- Denoted by p
- Circuit elements that *absorb* power have a *positive* value of p
- Circuit elements that *produce* power have a *negative* value of p

$$p = \frac{dw}{dt} \qquad\qquad p = \pm vi$$

where

$$\begin{aligned}
p &= \quad \text{power in watts (W = J/s)} \\
w &= \quad \text{energy in joules (J)} \\
t &= \quad \text{time in seconds (s)} \\
v &= \quad \text{voltage in volts (V)} \\
i &= \quad \text{current in amperes (A)}
\end{aligned}$$

Energy

- *Law of Conservation of Energy*: the net power absorbed by a circuit is equal to 0.
- In other words, the total energy produced in a circuit is equal to the total energy absorbed
- *Energy*: capacity to do work, measured in joules (J)

$$w = \int_{t_0}^{t} p\,dt = \int_{t_0}^{t} (\pm vi)\,dt$$

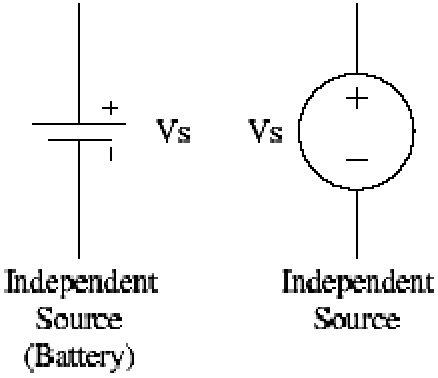
If current and voltage are constant (DC),

$$w = \int_{t_0}^{t} p\,dt = p(t - t_0)$$

Circuit Elements

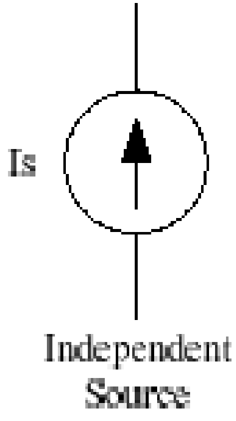
Ideal Independent Source: provides a specified voltage or current that is completely independent of other circuit elements

Ideal Independent Voltage Source:



Circuit Elements

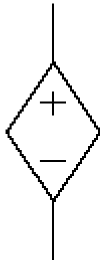
Ideal independent current source



Circuit Elements

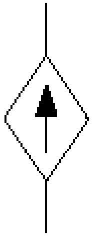
An ideal dependent (or controlled) source is an active element in which the source quantity is controlled by another voltage or current.

Ideal dependent voltage source



A voltage-controlled voltage source (VCVS).
A current-controlled voltage source (CCVS).
A voltage-controlled current source (VCCS).
A current-controlled current source (CCCS).

Ideal dependent current source



Resistance

- All materials resist the flow of current
- Resistance is usually represented by the variable R
- Depends on geometry and resistivity of the material
  - Copper 1.673e-8 ohm-meters
  - Lead 20.648e-8 ohm-meters
- Ohms per square for sheet.

A cylinder of length ℓ and cross-sectional area A has a resistance:

$$R = \rho \frac{\ell}{A}$$

where

$$\begin{aligned} R &= \quad \text{resistance of an element in ohms } (\Omega) \\ \rho &= \quad \text{resistivity of the material in ohm-meters} \\ \ell &= \quad \text{length of cylindrical material in meters} \\ A &= \quad \text{Cross sectional area of material in meters}^2 \end{aligned}$$

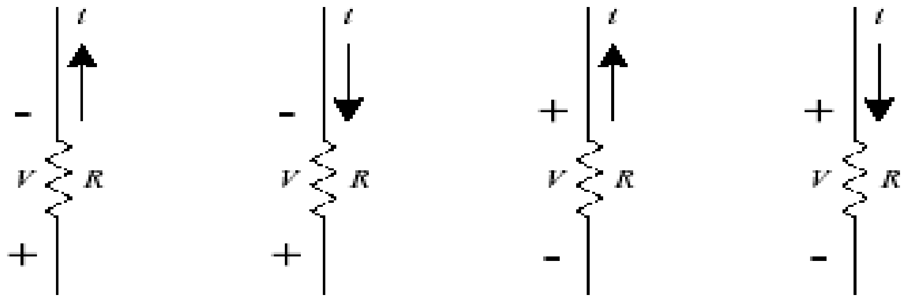
Resistance: Basic Concepts and Assumptions

- We will always measure resistance in Ohms
- Ohms are denoted by the Greek letter Omega: Ω
- Examples: 50 Ω, 1 kΩ, 2.5 MΩ
- Conductors (e.g. wires) have very low resistance ($< 0.1\ \Omega$) that can usually be neglected (i.e. we will assume wires have zero resistance)
- Insulators (e.g. air) have very large resistance (> 50 MΩ) that can usually be ignored (omitted from circuit for analysis)
- Resistors have a medium range of resistance and must be accounted for in the circuit analysis
- Conceptually, a light bulb is similar to a resistor
- Properties of the bulb control how much current flows and how much power is dissipated (absorbed & emitted as light and heat)

Passive Sign Convention

- Passive Sign Convention (PSC): Current enters the positive terminal of an element
- Most two-terminal circuit elements (e.g. batteries, light bulbs, resistors, switches) are characterized by a single equation that relates voltage to current: $v = \pm f(i)$ or $i = \pm g(v)$
- The PSC determines the sign of the relationship
  - If PSC is satisfied: $v = f(i)$ or $i = g(v)$
  - If PSC is not satisfied: $v = -f(i)$ or $i = -g(v)$
- This is also true of power
  - If PSC is satisfied: $p = vi$
  - If PSC is not satisfied: $p = -vi$

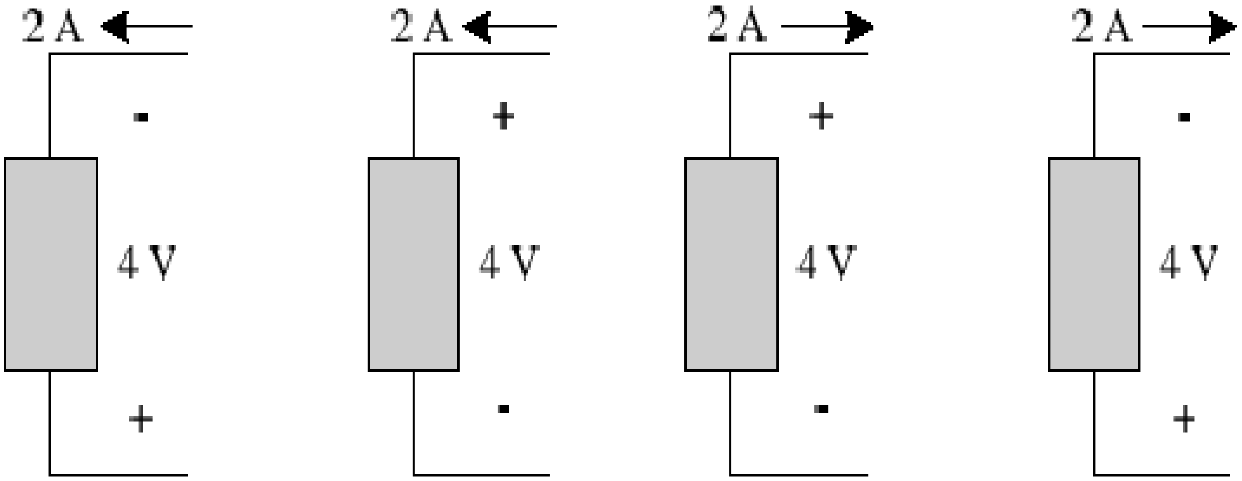
Resistors & Passive Sign Convention



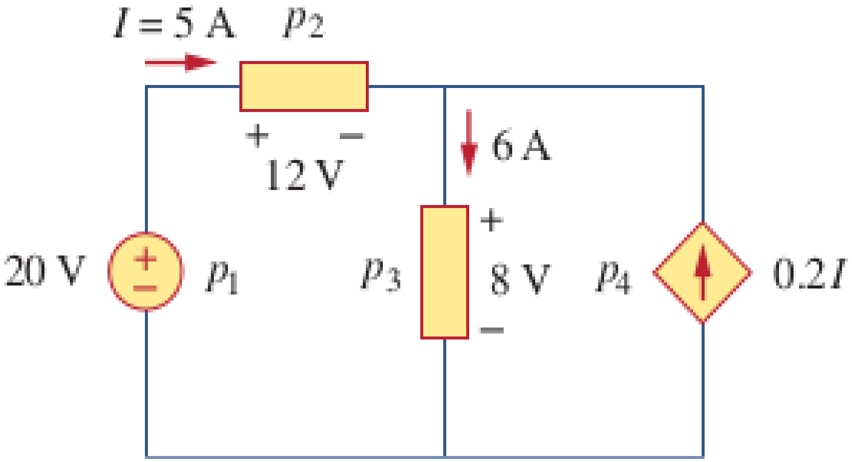
- Recall that relationships between current and voltage are sign sensitive
- Passive Sign Convention: Current enters the positive terminal of an element
  - If PSC satisfied: $v = iR$
  - If PSC not satisfied: $v = -iR$

PSC: Example

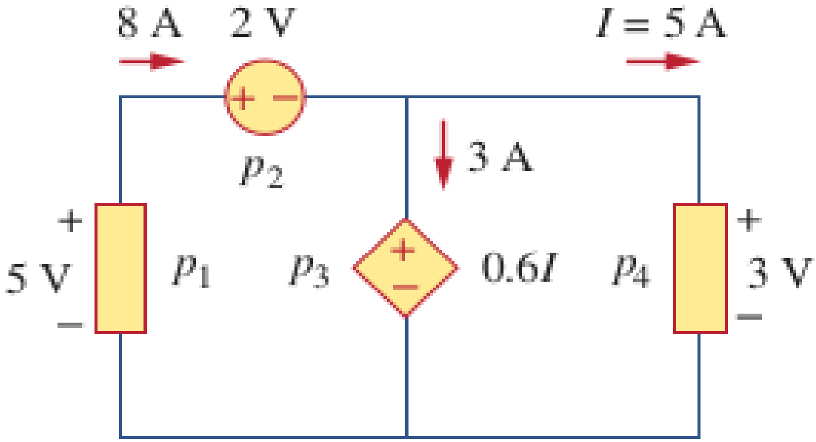
Find the power (absorbed) for each device.



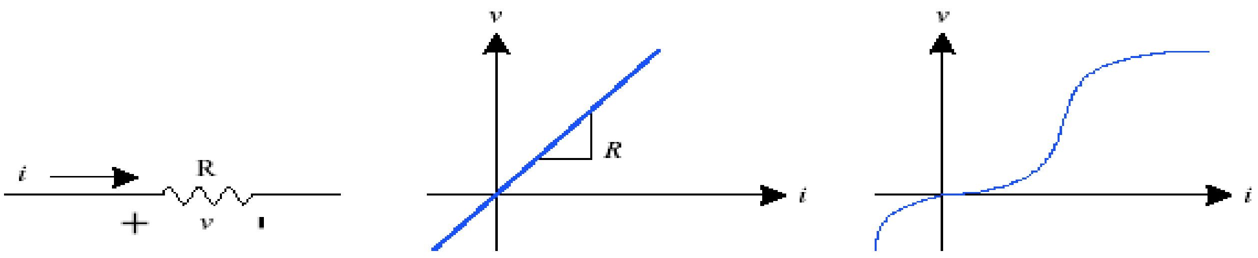
Calculate the power supplied or absorbed by each element in Fig.



Calculate the power supplied or absorbed by each element in Fig.



Ohm’s Law

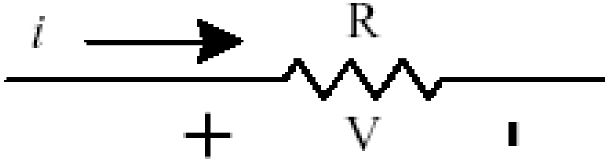


Linear
(Ohm's Law Applies)

Nonlinear
(Ohm's Law Does Not Apply)

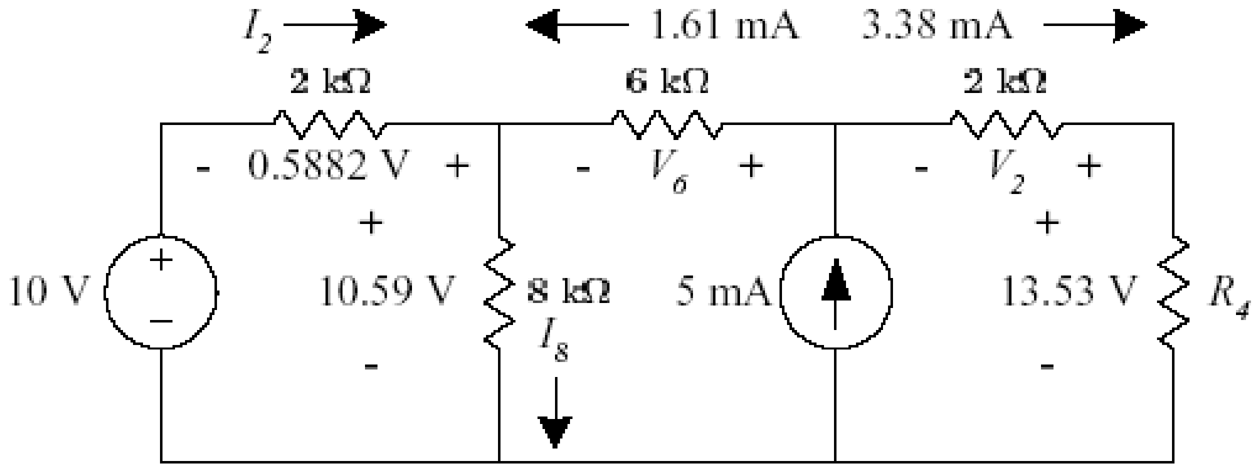
- As with all circuit elements, we need to know how the current through and voltage across the device are related
- Many materials have a complicated nonlinear relationship (including light bulbs): $v = \pm f(i)$
- Materials with a linear relationship satisfy Ohm’s law: $v = \pm mi$
- The slope, m, is equal to the resistance of the element
- Ohm’s Law: $v = \pm iR$
- Sign, $\pm$, is determined by the passive sign convention (PSC)

Other Eq. derived from Ohm's Law



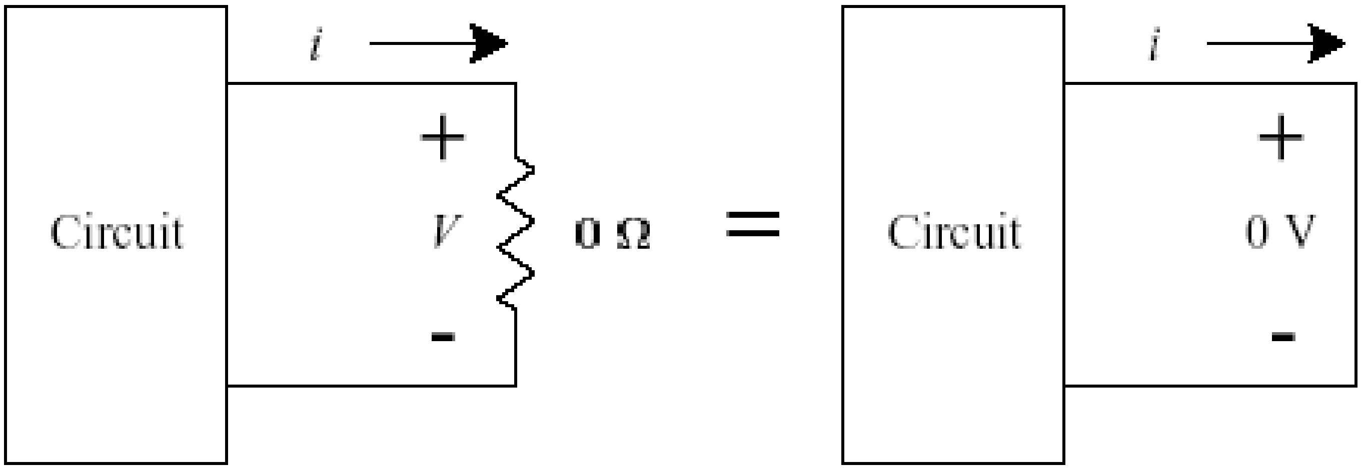
- Ohm's law implies: $i = \frac{v}{R}$
- Recall $p = \pm vi$. Therefore
  - $p = v\frac{v}{R} = \frac{v^2}{R}$
  - $p = (iR)i = i^2 R$
- Resistors cannot produce power so the power absorbed by a resistor will always be positive
- $1\ \Omega = 1\ \mathrm{V/A}$

Example: Ohm's Law



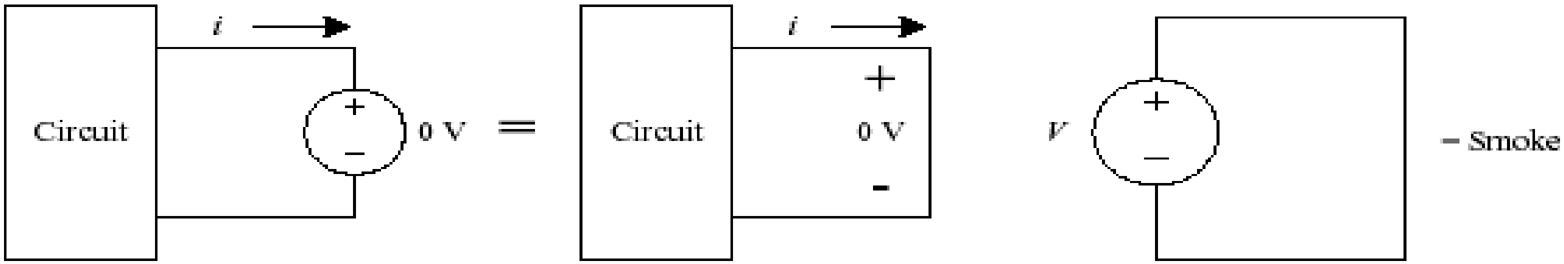
$$
\begin{aligned}
I_2 &= \\
V_6 &= \\
R_4 &= \\
V_2 &= \\
I_8 &=
\end{aligned}
$$

Short Circuit as Zero Resistance



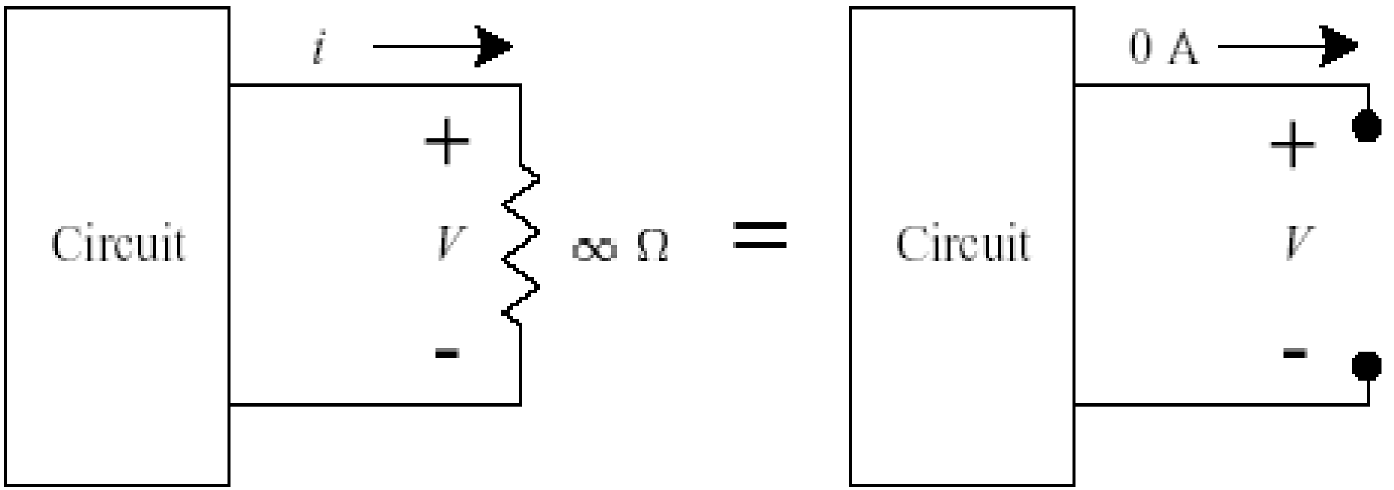
- An element (or wire) with $R = 0$ is called a short circuit
- Just drawn as a wire (line)

Short Circuit as Voltage Source (0V)



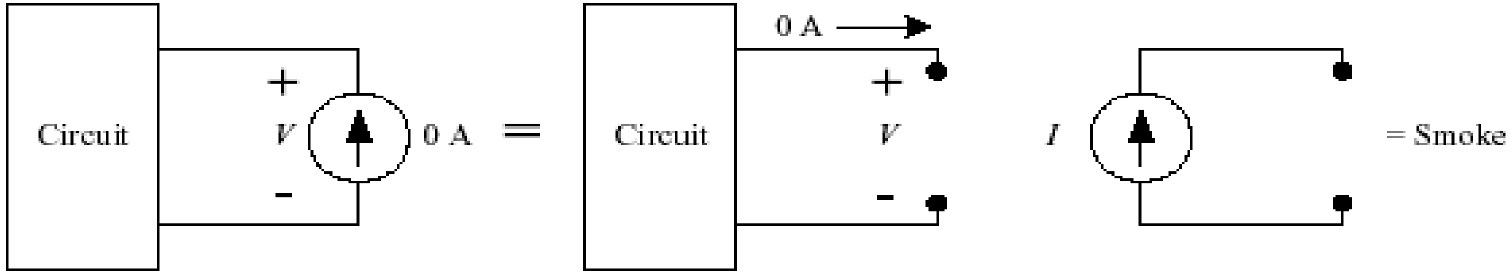
- An ideal voltage source $V_s = 0$ V is also equivalent to a short circuit
- Since $v = iR$ and $R = 0$, $v = 0$ regardless of i
- Could draw a source with $V_s = 0$ V, but is not done in practice
- Can not connect a voltage source to a short circuit
- Irresistible force meets immovable object
- In practice, the wire usually wins and the voltage source melts (if not protected)

Open Circuit



- An element (or wire) with $R = \infty$ is called a open circuit
- Just omitted

Open Circuit as Current Source (0 A)



- An ideal current source $I = 0$ A is also equivalent to an open circuit
- Could draw a source with $I = 0$ A, but is not done in practice
- Cannot connect a current source to an open circuit (spark coil)
- Irresistible force meets immovable object
- In practice, you blow the current source (if not protected)
- The insulator (air) usually wins. Else, sparks fly.

Conductance

- Sometimes conductance is specified instead of resistance
- Inverse of resistance
- $G = \frac{1}{R} = \frac{i}{v}$
- Units: siemens (S) or mhos ($\mho$)
- 1 S = 1 $\mho$ = 1 A/V
- In words: the ability of an element to conduct electric current

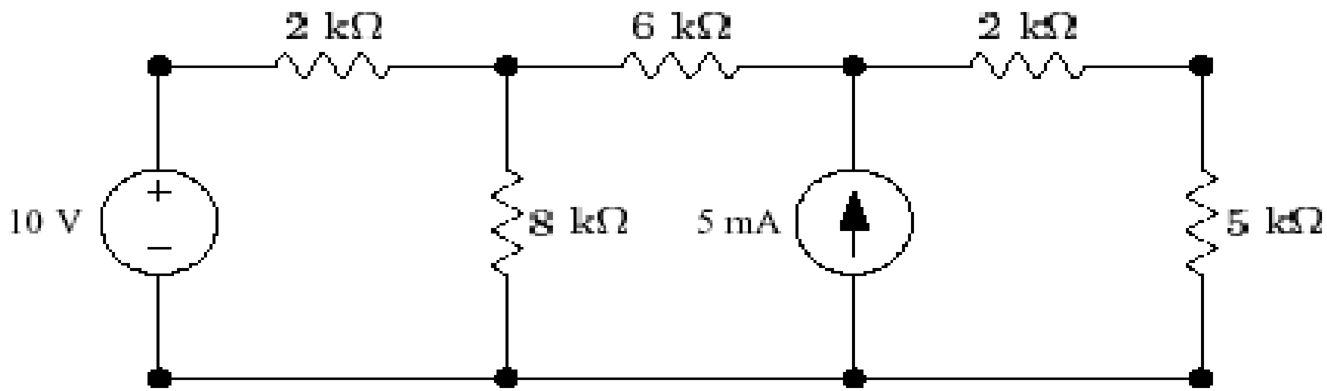
$$v = Ri \qquad p = vi = i^2 R = \frac{v^2}{R}$$

$$i = Gv \qquad p = vi = v^2 G = \frac{i^2}{G}$$

Circuit Building Blocks

- Before we can begin analysis, we need a common language and framework for describing circuits
- For this course, networks and circuits are the same
- Networks are composed of nodes, branches, and loops

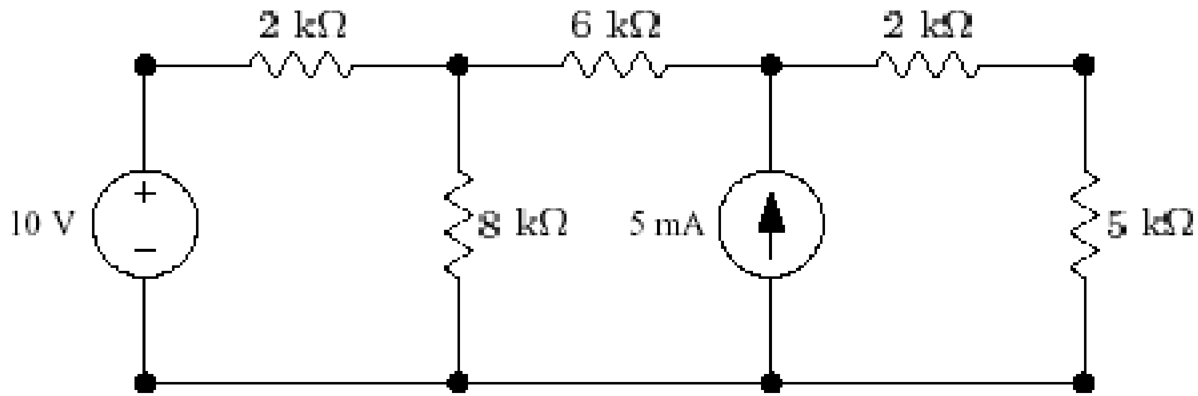
Branches



Example: How many branches?

- Branch: a single two-terminal element in a circuit
- Segments of wire are not counted as elements
- Examples: voltage source, resistor, current source

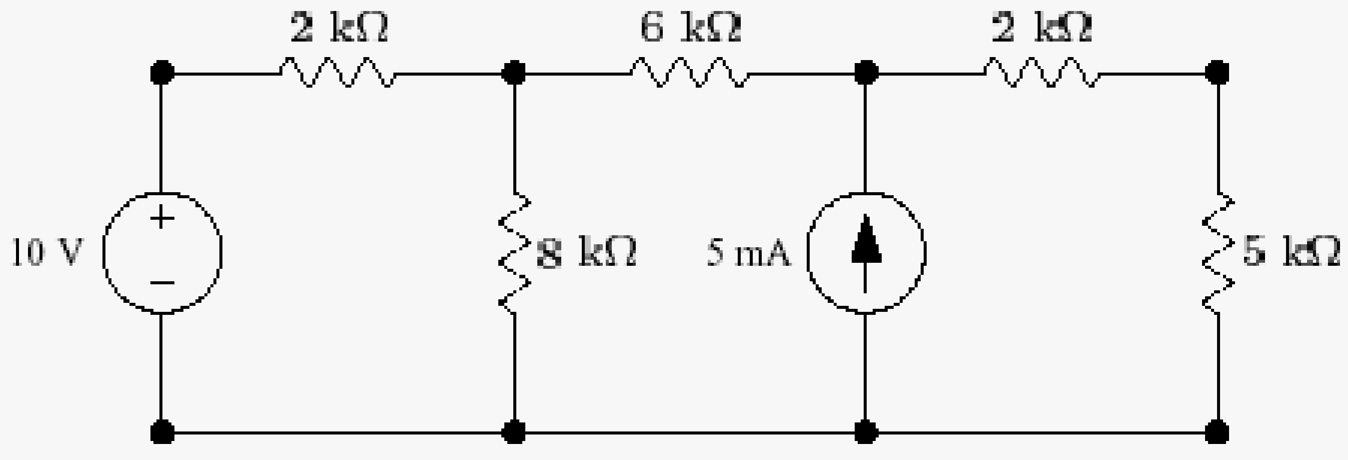
Nodes



Example: How many nodes? How many essential nodes?

- Node: the point of connection between two or more branches
- May include a portion of the circuit (more than a single point)
- Essential Node: the point of connection between three or more branches

Loops



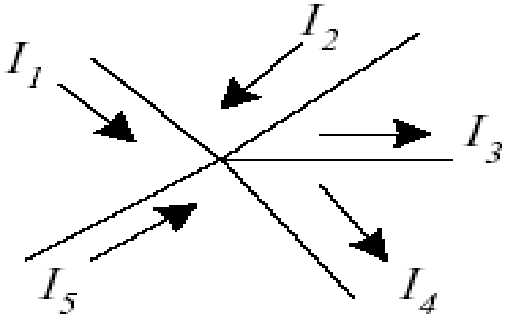
Example: How many loops?

- Loop: any closed path in a circuit

Overview of Kirchhoff's Laws

- The foundation of circuit analysis is
  - The defining equations for circuit elements (e.g. Ohm's law)
  - Kirchhoff's current law (KCL)
  - Kirchhoff's voltage law (KVL)
- The defining equations tell us how the voltage and current within a circuit element are related
- Kirchhoff's laws tell us how the voltages and currents in different branches are related

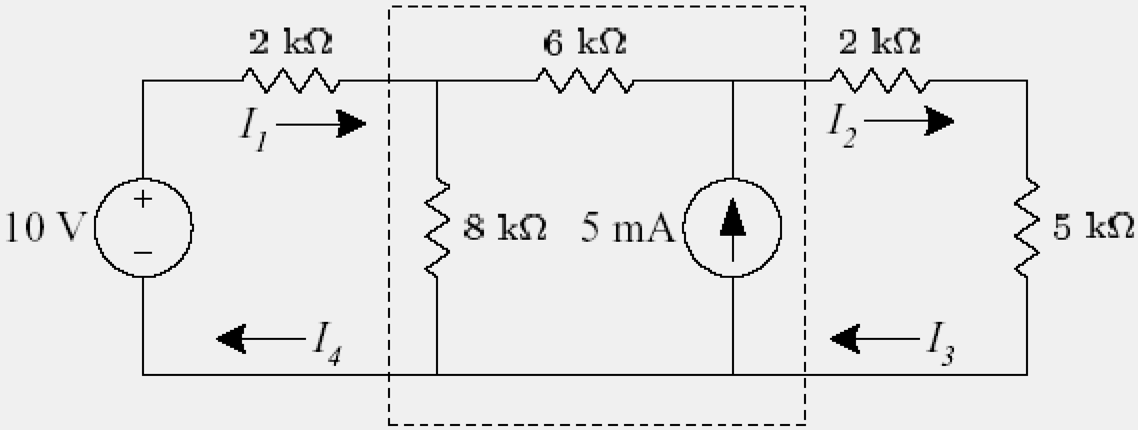
Kirchhoff's Current Law



$$
\begin{aligned}
I_1 + I_2 - I_3 - I_4 + I_5 &= 0 \\
I_1 + I_2 + I_5 &= I_3 + I_4
\end{aligned}
$$

- Kirchhoff's Current Law (KCL): the algebraic sum of currents entering a node (or a closed boundary) is zero
- The sum of currents entering a node is equal to the sum of the currents leaving a node
- Common sense:
  - All of the electrons have to go somewhere
  - The current that goes in, has to come out some place
- Based on law of conservation of charge and $\nabla j = \frac{\partial \rho}{\partial t} = 0$

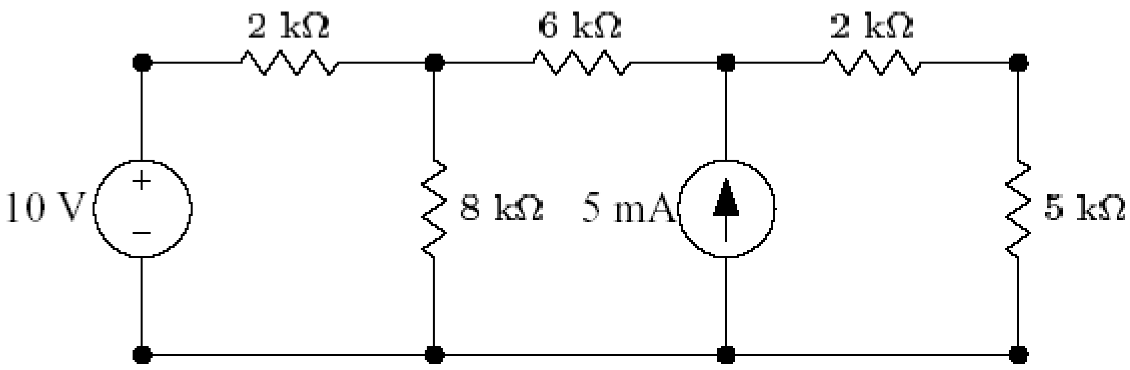
Kirchhoff's Current Law for Boundaries



$$
\begin{aligned}
I_1 - I_2 + I_3 - I_4 &= 0 \\
I_1 + I_3 &= I_2 + I_4
\end{aligned}
$$

- KCL also applies to closed boundaries for *all* circuits

KCL - Example



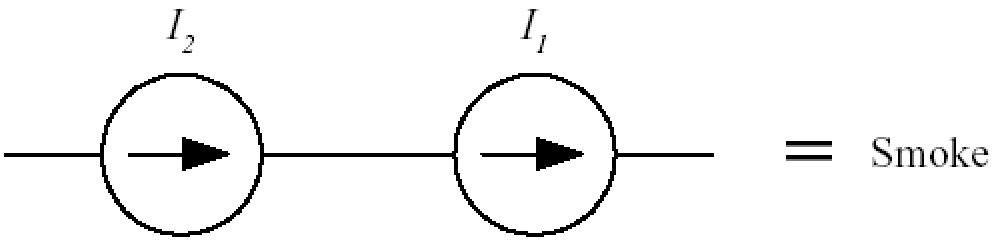
Apply KCL to each essential node in the circuit.

Essential Node 1:

Essential Node 2:

Essential Node 3:

Ideal Current Sources: Series



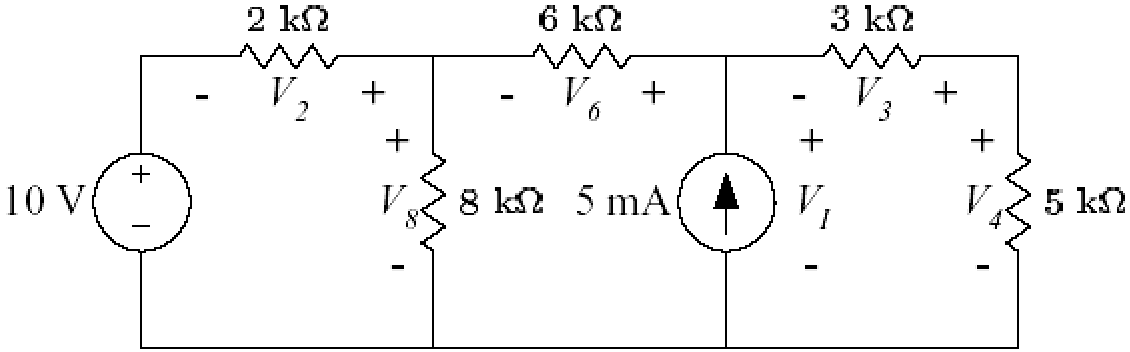
- Ideal current sources *cannot* be connected in series
- Recall: ideal current sources guarantee the current flowing through source is at specified value
- Recall: the current entering a circuit element must equal the current leaving a circuit element, $I_{in} = I_{out}$
- Could easily cause component failure (smoke)
- Ideal sources do not exist
- Technically allowed if $I_1 = I_2$, but is a bad idea

Kirchhoff's Voltage Law - KVL

$$\sum_{m=1}^{M} V_m = 0$$

- Kirchhoff's Voltage Law (KVL): the algebraic sum of voltages around a closed path (ie loop) is zero

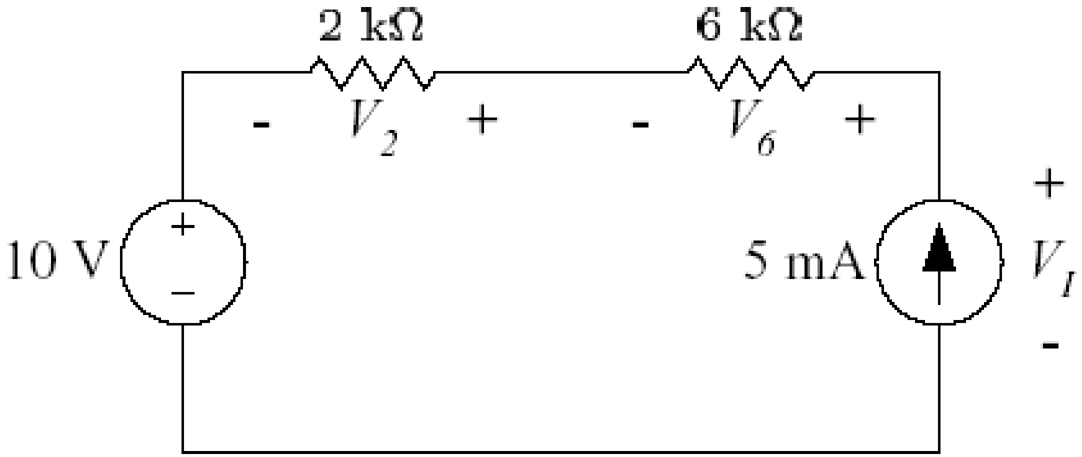
KVL - Example



Apply KVL to each loop in the circuit.

Loop 1:
Loop 2:
Loop 3:
Loop 4:
Loop 5:
Loop 6:

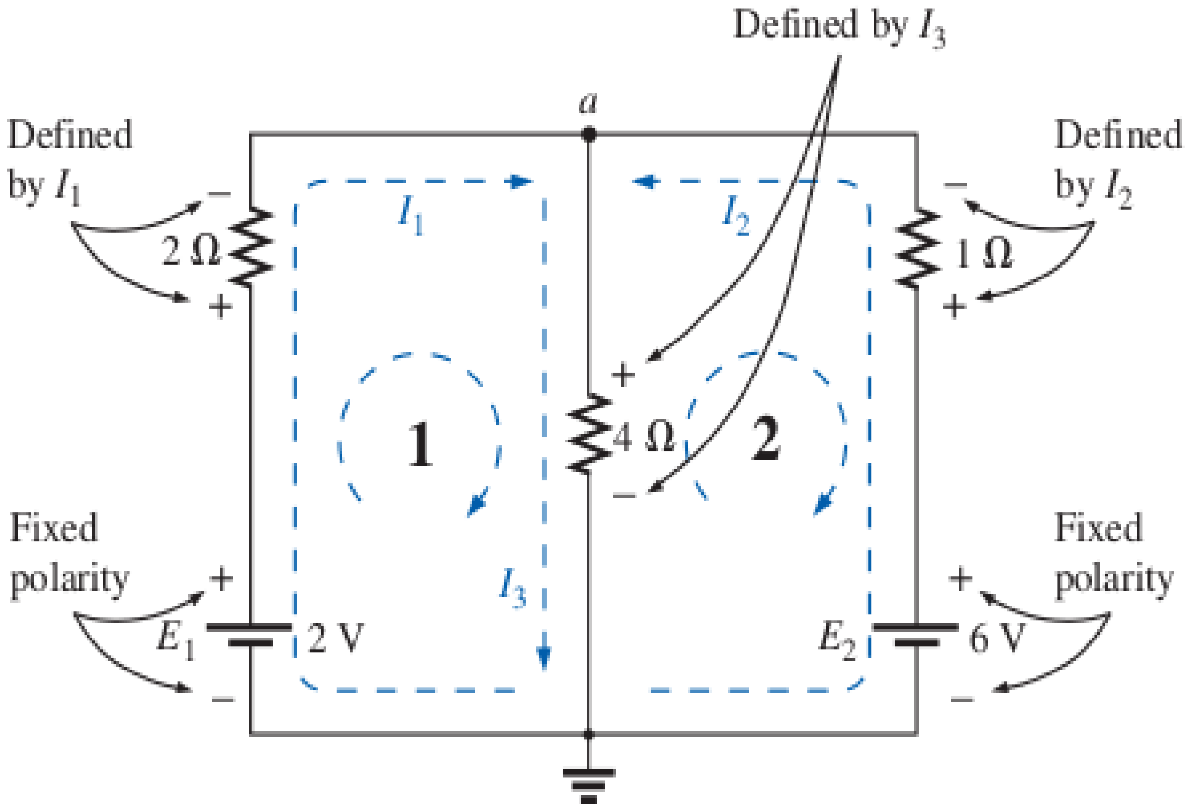
Example – Applying the Basic Laws



Find V_2, V_6 and V_I.

BRANCH-CURRENT ANALYSIS

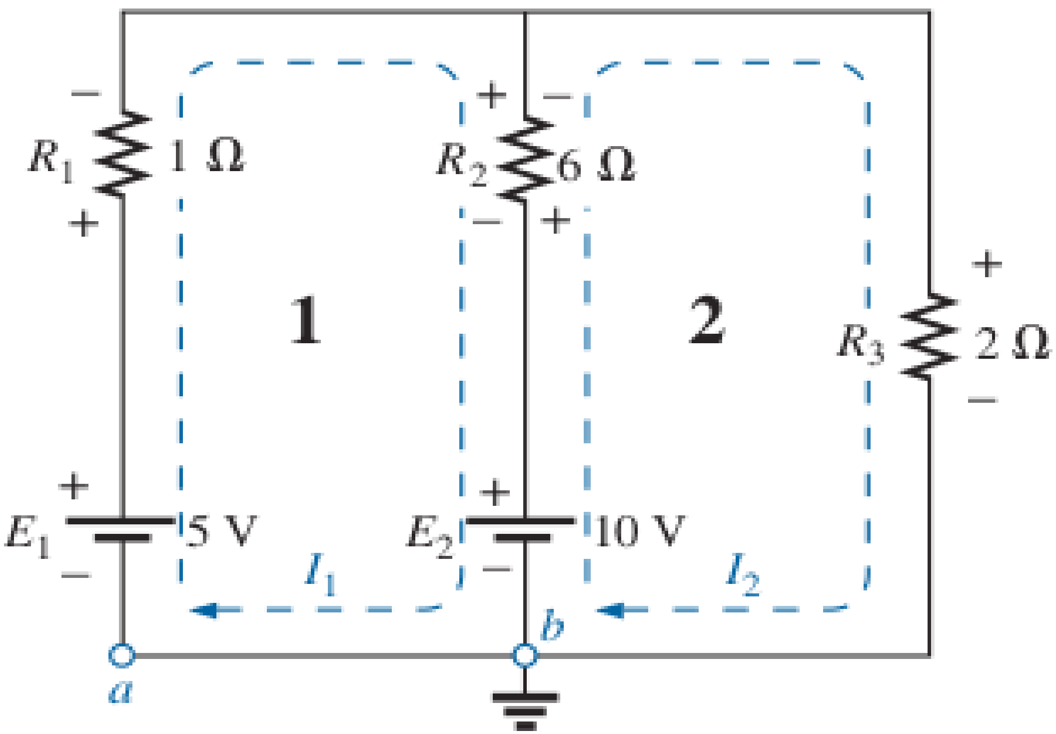
1. Assign a distinct current of arbitrary direction to each branch of the network.
2. Indicate the polarities for each resistor as determined by the assumed current direction.
3. Apply Kirchhoff's voltage law around each closed, independent loop of the network.
4. Apply Kirchhoff's current law at the minimum number of nodes that will include all the branch currents of the network.
5. Solve the resulting simultaneous linear equations for assumed branch currents.



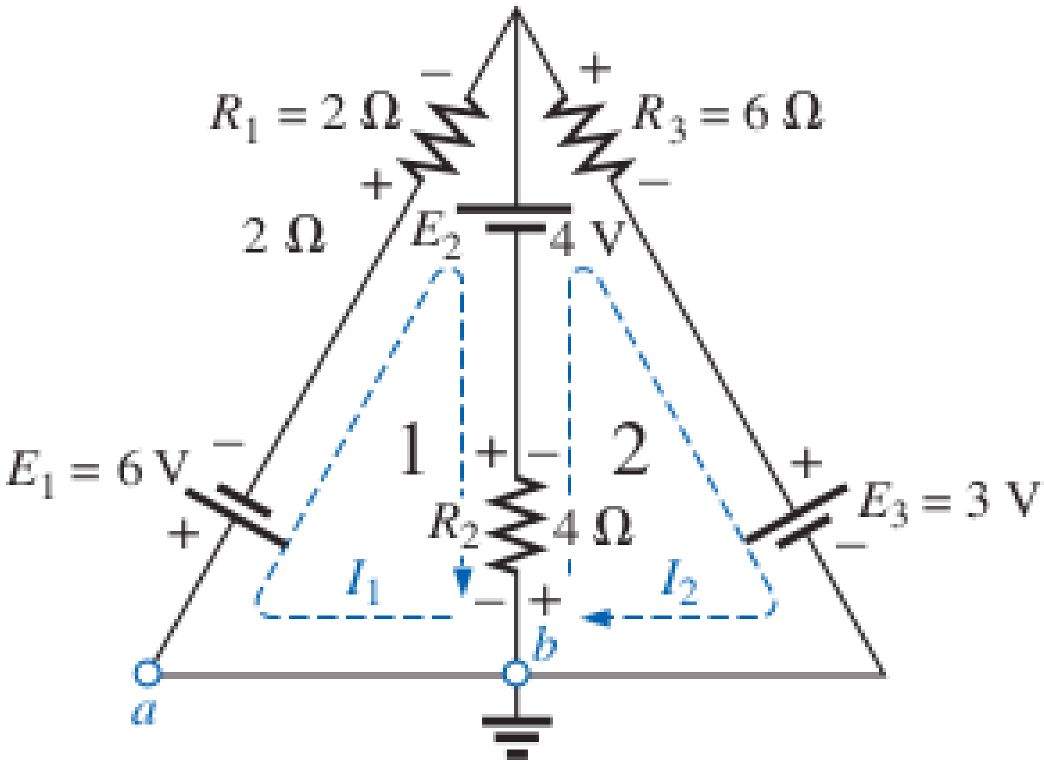
Defined by I_3
a
Defined
by I_1
Defined
by I_2
2 Ω
1 Ω
I_1
I_2
4 Ω
1
2
Fixed
polarity
Fixed
polarity
E_1
2 V
I_3
E_2
6 V

MESH ANALYSIS

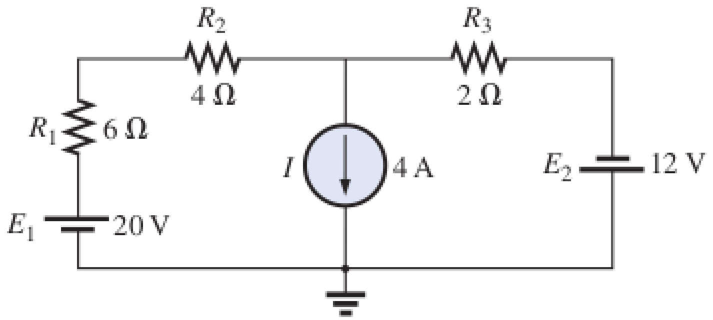
1. Assign a distinct current in the clockwise direction to each independent, closed loop of the network. It is not absolutely necessary to choose the clockwise direction for each loop current. In fact, any direction can be chosen for each loop current with no loss in accuracy, as long as the remaining steps are followed properly. However, by choosing the clockwise direction as a Standard.
2. Indicate the polarities within each loop for each resistor as determined by the assumed direction of loop current for that loop. Note the requirement that the polarities be placed within each loop.
3. Apply Kirchhoff's voltage law around each closed loop in the clockwise direction.
    a. If a resistor has two or more assumed currents through it, the total current through the resistor is the assumed current of the loop in which Kirchhoff's voltage law is being applied, plus the assumed currents of the other loops passing through in the same direction, minus the assumed currents through in the opposite direction.
    b. The polarity of a voltage source is unaffected by the direction of the assigned loop currents.
4. Solve the resulting simultaneous linear equations for the assumed loop currents.



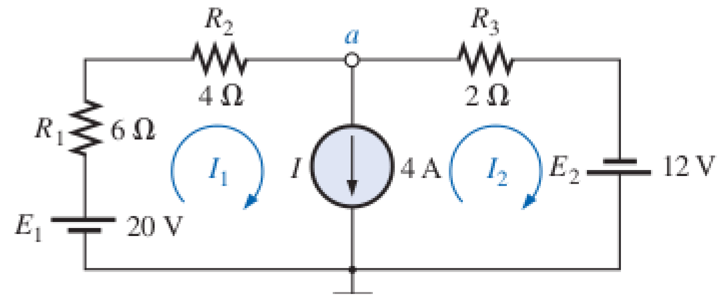
R1
1 Ω
R2
6 Ω
R3
2 Ω
E1
5 V
E2
10 V
1
2
I1
I2
a
b



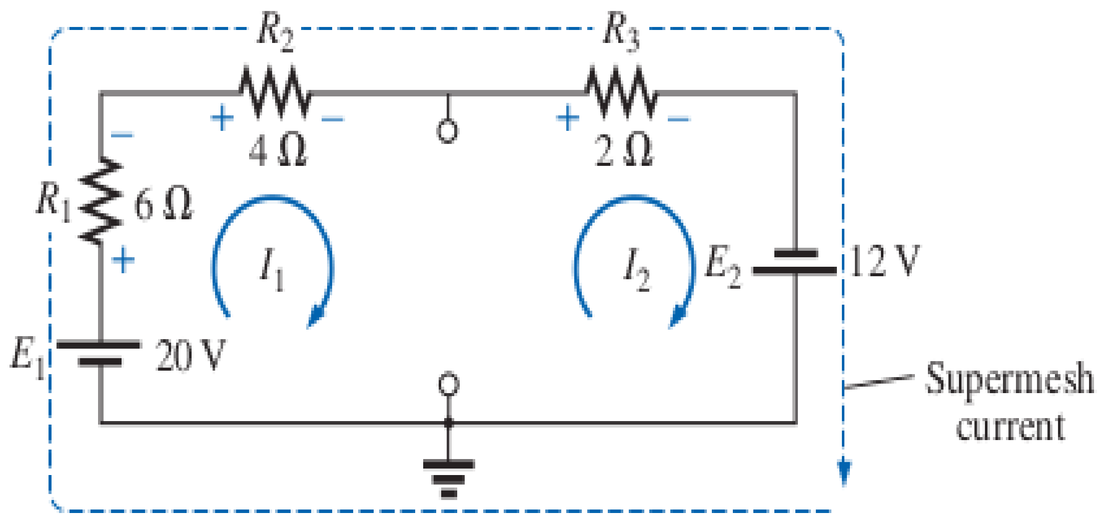
$R_1 = 2\ \Omega$
$R_3 = 6\ \Omega$
$2\ \Omega$
E_2
$4\ V$
$E_1 = 6\ V$
$E_3 = 3\ V$
1
2
R_2
$4\ \Omega$
I_1
I_2
a
b



R_2
R_3
4 Ω
2 Ω
R_1
6 Ω
I
4 A
E_2
12 V
E_1
20 V



R_2
a
R_3
4 Ω
2 Ω
R_1
6 Ω
I_1
I
4 A
I_2
E_2
12 V
E_1
20 V



R_2
R_3
4 Ω
2 Ω
R_1
6 Ω
I_1
I_2
E_2
12 V
E_1
20 V
Supermesh
current

Supermesh Currents

- On occasion there will be current sources in the network to which mesh analysis is to be applied. In such cases one can convert the current source to a voltage source (if a parallel resistor is present) and proceed as before or utilize a supermesh current and proceed as follows.

- Start as before and assign a mesh current to each independent loop, including the current sources, as if they were resistors or voltage sources. Then mentally (redraw the network if necessary) remove the current sources (replace with open-circuit equivalents), and apply Kirchhoff's voltage law to all the remaining independent paths of the network using the mesh currents just defined. Any resulting path, including two or more mesh currents, is said to be the path of a super-mesh current.
- Then relate the chosen mesh currents of the network to the independent current sources of the network, and solve for the mesh currents.

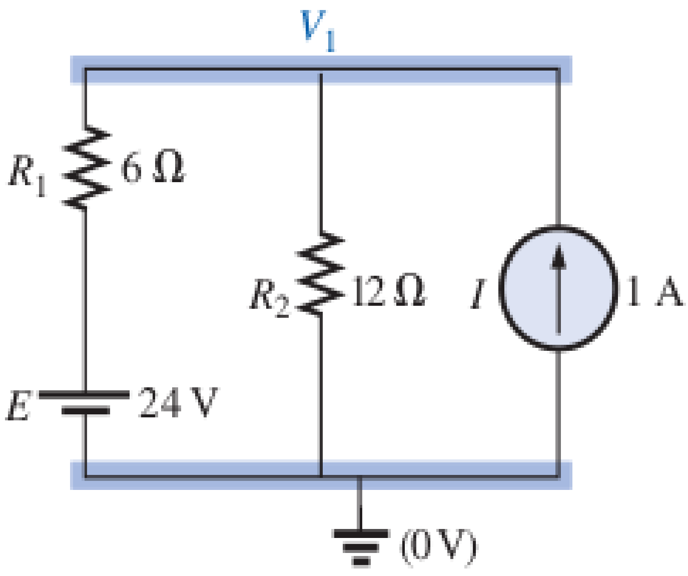
NODAL ANALYSIS

1. Determine the number of nodes within the network.

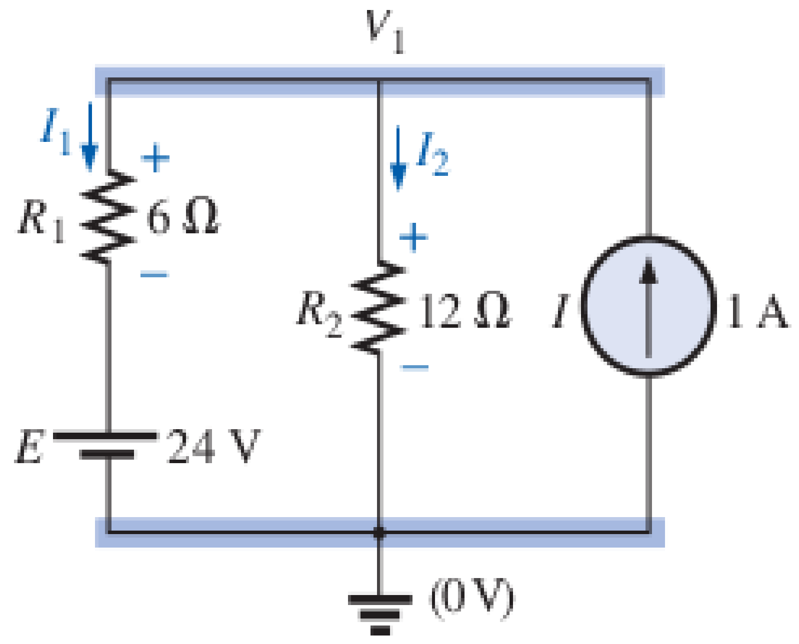
2. Pick a reference node, and label each remaining node with a subscripted value of voltage: V_1, V_2, and so on.

3. Apply Kirchhoff's current law at each node except the reference. Assume that all unknown currents leave the node for each application of Kirchhoff's current law. In other words, for each node, don't be influenced by the direction that an unknown current for another node may have had. Each node is to be treated as a separate entity, independent of the application of Kirchhoff's current law to the other nodes.

4. Solve the resulting equations for the nodal voltages.

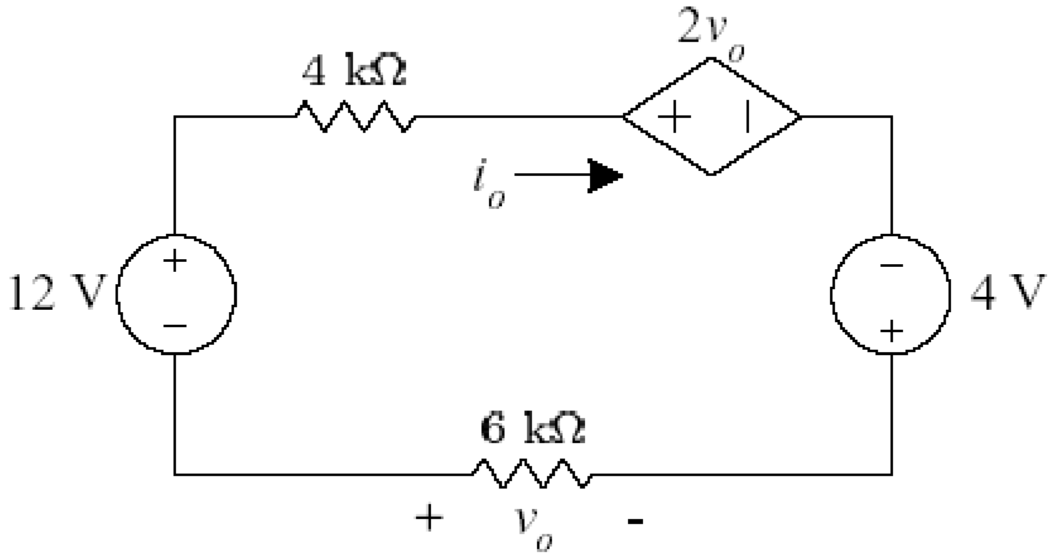


V1
R1
6 Ω
R2
12 Ω
I
1 A
E
24 V
(0 V)



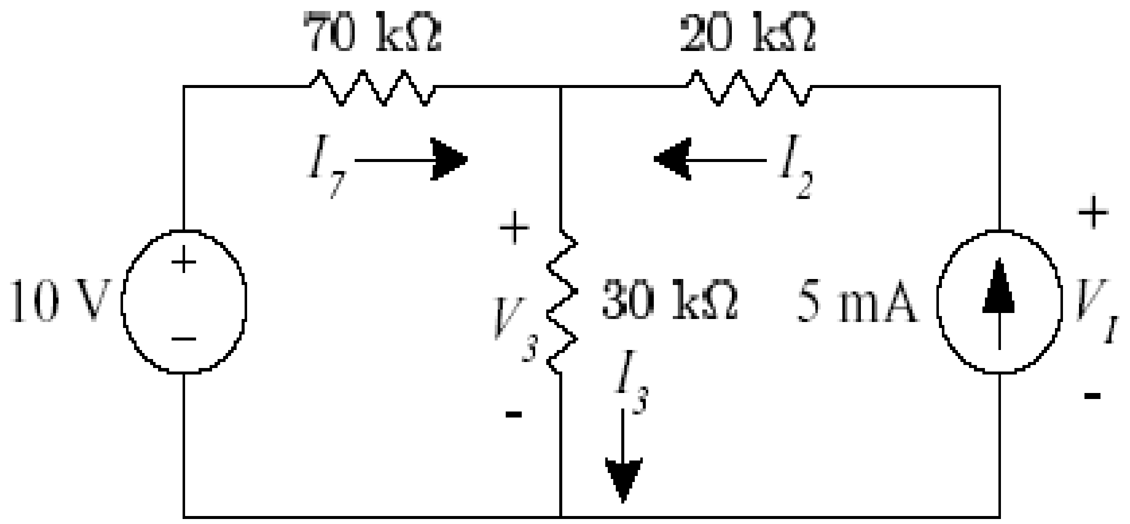
V1
I1
I2
+
−
R1
6 Ω
R2
12 Ω
I
1 A
E
24 V
(0 V)

Example – Applying the Basic Laws



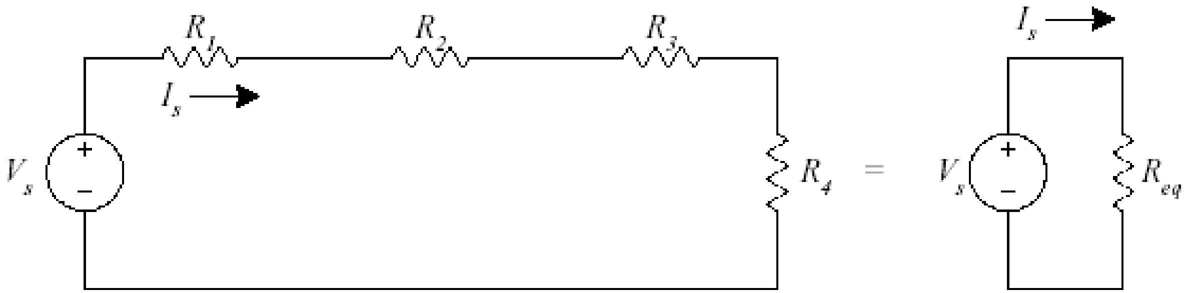
Find i_o and v_o.

Example – Applying the Basic Laws



Find I_7, I_3, I_2, V_3, and V_I.

Resistors in Series

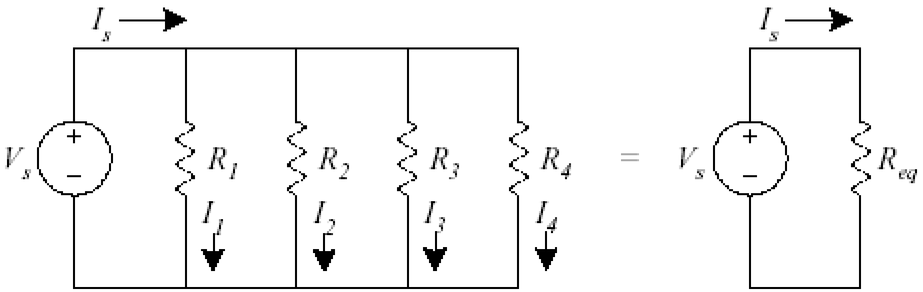


By KVL

$$
\begin{aligned}
V_s &= R_1 I_s + R_2 I_s + R_3 I_s + R_4 I_s \\
&= I_s (R_1 + R_2 + R_3 + R_4) \\
&= R_{eq} I_s \\
R_{eq} &= R_1 + R_2 + R_2 + R_4
\end{aligned}
$$

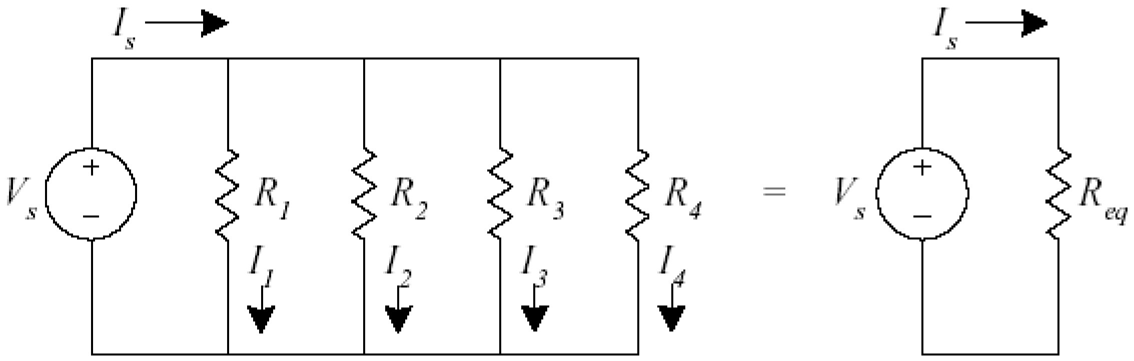
- Resistors in series add

Resistors in Parallel



$$
\begin{aligned}
I_s &= I_1 + I_2 + I_3 + I_4 \\
&= V_s/R_1 + V_s/R_2 + V_s/R_3 + V_s/R_4 \\
&= V_s\left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} + \frac{1}{R_4}\right) \\
&= \frac{V_s}{R_{eq}} \\
\frac{1}{R_{eq}} &= \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} + \frac{1}{R_4} \\
R_{eq} &= \frac{1}{\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} + \frac{1}{R_4}}
\end{aligned}
$$

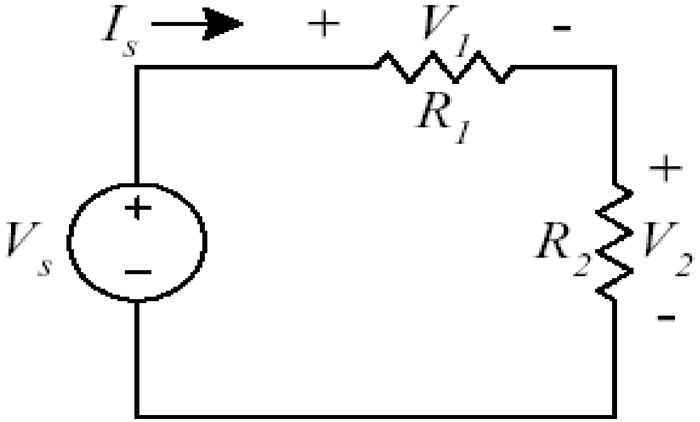
Resistors in Parallel



$$
\begin{aligned}
\frac{1}{R_{eq}} &= \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} + \frac{1}{R_4} \\
G_{eq} &= G_1 + G_2 + G_3 + G_4
\end{aligned}
$$

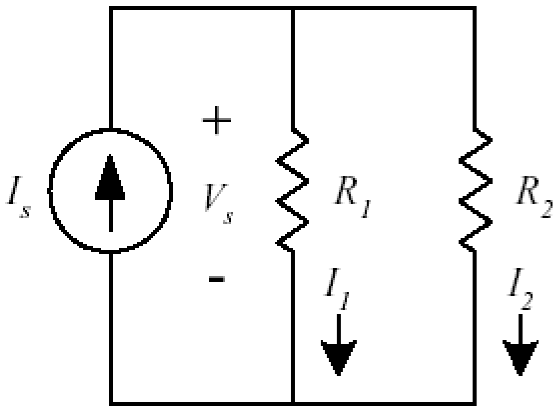
- Resistors in parallel have a more complicated relationship
- Easier to express in terms of conductance
- For two resistors: $R_{eq} = \frac{R_1 R_2}{R_1 + R_2}$

Voltage Divider



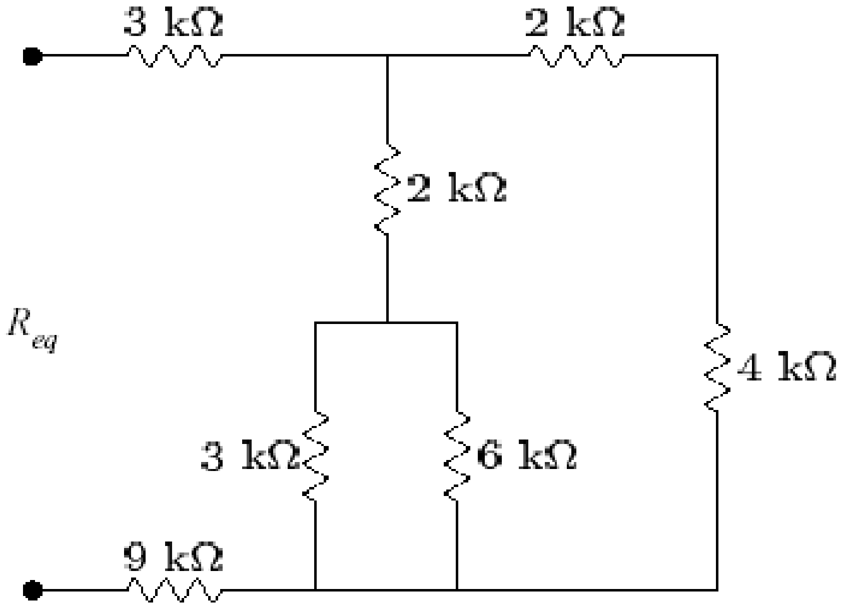
$$
\begin{aligned}
R_{eq} &= R_1 + R_2 \\
I_s &= \frac{V_s}{R_{eq}} \\
&= \frac{V_s}{R_1 + R_2} \\
V_2 &= I_s R_2 \\
&= V_s \frac{R_2}{R_1 + R_2} \\
V_1 &= I_s R_1 \\
&= V_s \frac{R_1}{R_1 + R_2}
\end{aligned}
$$

Current Divider



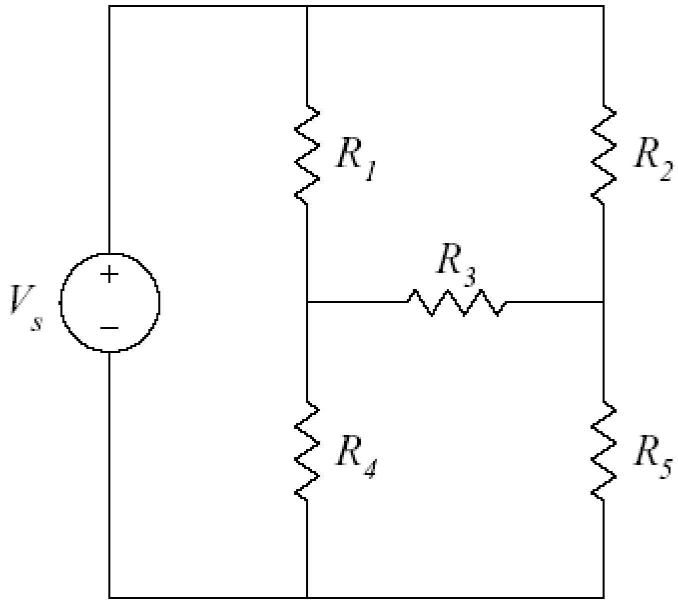
$$
\begin{aligned}
V_s &= I_s \frac{R_1 R_2}{R_1 + R_2} \\
I_2 &= \frac{V_s}{R_2} \\
&= I_s \frac{R_1}{R_1 + R_2} \\
I_1 &= \frac{V_s}{R_1} \\
&= I_s \frac{R_2}{R_1 + R_2}
\end{aligned}
$$

Resistor Network



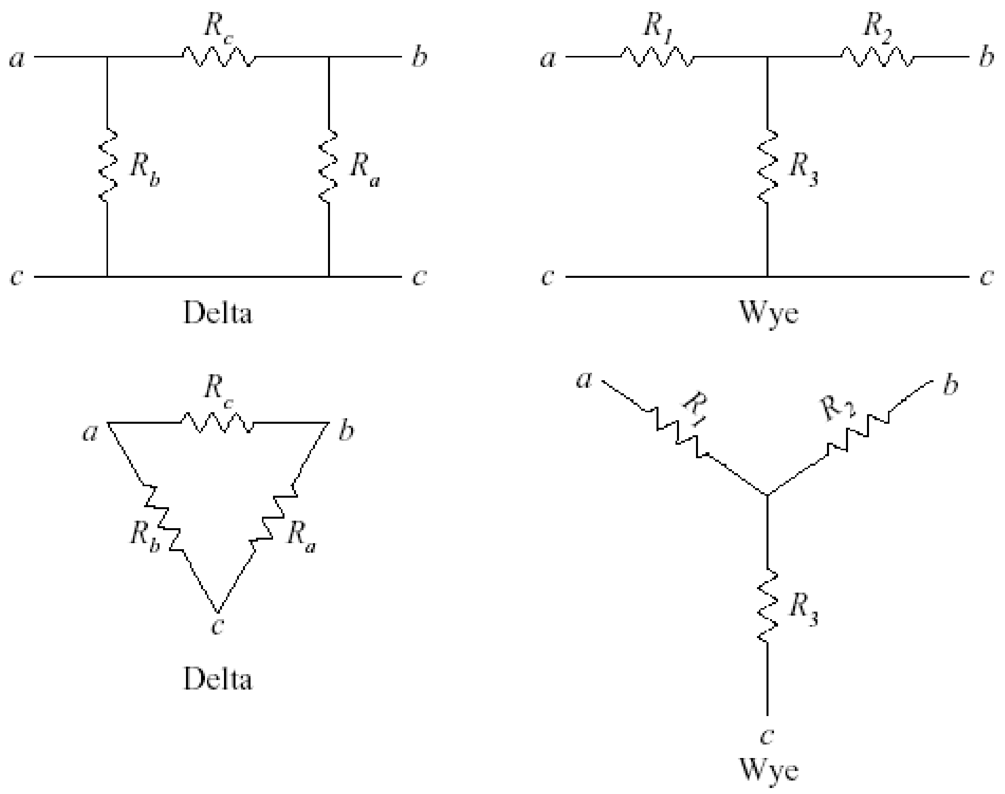
Find R_{eq}.

Resistor Network - Comments



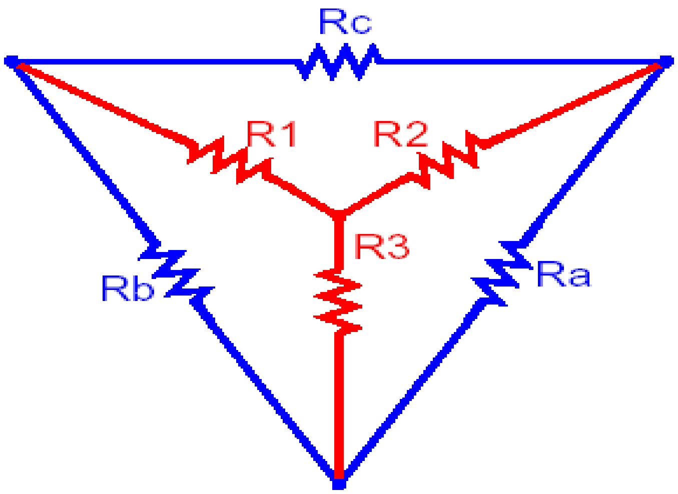
- Knowing the equivalent and parallel equivalents of resistors is not quite adequate
- There are some configurations that require one more tool

Delta ←→ Wye Transformations



- Every Delta network is functionally equivalent to a Wye network (and vice versa).

Delta ←→ Wye Transformations



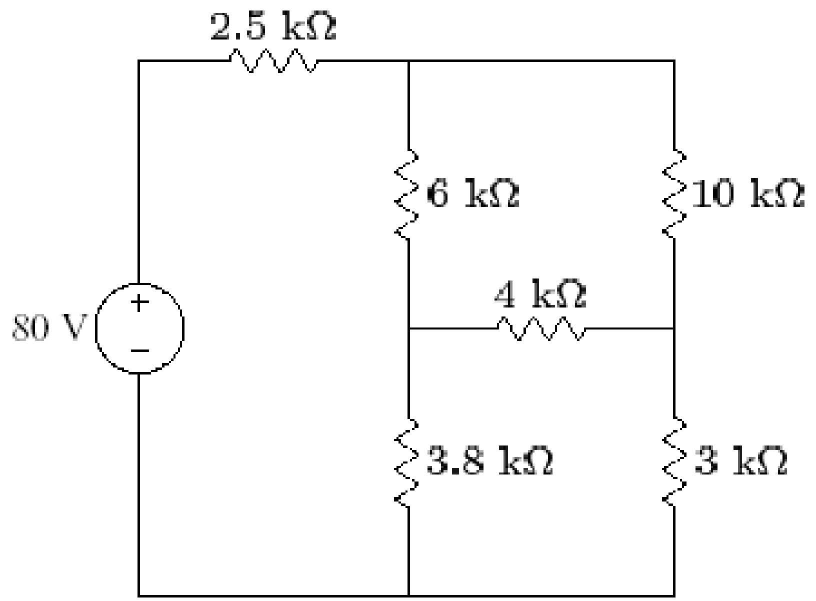
The following must be satisfied for equivalence

$$R_a = \frac{R_1R_2 + R_2R_3 + R_3R_1}{R_1} \qquad R_1 = \frac{R_bR_c}{R_a + R_b + R_c}$$

$$R_b = \frac{R_1R_2 + R_2R_3 + R_3R_1}{R_2} \qquad R_2 = \frac{R_cR_a}{R_a + R_b + R_c}$$

$$R_c = \frac{R_1R_2 + R_2R_3 + R_3R_1}{R_3} \qquad R_3 = \frac{R_aR_b}{R_a + R_b + R_c}$$

Example –
Delta ← → Wye Transformations



Find R_{eq} and Power delivered by source