DE Moivre's Theorem

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3(i) Prove that $(a+ib)^{\frac{m}{n}} + (a-ib)^{\frac{m}{n}} = 2(a^2+b^2)^{\frac{m}{2n}} \cos\left(\frac{m}{n}\tan^{-1}\frac{b}{a}\right)$

Soln: let, $a = r\cos\theta$

$b = r\sin\theta$

$\therefore r = \sqrt{a^2+b^2}$, $\theta = \tan^{-1}\frac{b}{a}$

$\therefore a+ib = r(\cos\theta + i\sin\theta)$

$a-ib = r(\cos\theta - i\sin\theta)$

$$\begin{aligned}
L.H.S &= (a+ib)^{\frac{m}{n}} + (a-ib)^{m/n} \\
&= \{r(\cos\theta + i\sin\theta)\}^{\frac{m}{n}} + \{r(\cos\theta - i\sin\theta)\}^{\frac{m}{n}} \\
&= r^{\frac{m}{n}}(\cos\tfrac{m}{n}\theta + i\sin\tfrac{m}{n}\theta) + r^{\frac{m}{n}}(\cos\tfrac{m}{n}\theta - i\sin\tfrac{m}{n}\theta) \\
&= r^{\frac{m}{n}} . 2\cos\tfrac{m}{n}\theta \\
&= 2(\sqrt{a^2+b^2})^{\frac{m}{n}} \cos\tfrac{m}{n}(\tan^{-1}\tfrac{b}{a}) \\
&= 2(a^2+b^2)^{\frac{m}{2n}} \cos\left(\tfrac{m}{n}\tan^{-1}\tfrac{b}{a}\right) = R.H.S \text{ (Proved)}
\end{aligned}$$

④ If $x_r = \cos\frac{\pi}{2^r} + i\sin\frac{\pi}{2^r}$ then prove that

$x_1 x_2 x_3 \cdots$ to infinity $= -1$.

Soln: Given that

$$x_r = \cos\frac{\pi}{2^r} + i\sin\frac{\pi}{2^r}$$

put $r = 1, 2, 3, \cdots$

$x_1 = \cos\frac{\pi}{2^1} + i\sin\frac{\pi}{2^1} = e^{i\pi/2}$

$x_2 = \cos\frac{\pi}{2^2} + i\sin\frac{\pi}{2^2} = e^{i\pi/2^2}$

$x_3 = \cos\frac{\pi}{2^3} + i\sin\frac{\pi}{2^3} = e^{i\pi/2^3}$

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multiplying above equations

$\therefore x_1 x_2 x_3 \cdots\cdots$ infinity $= e^{i\pi/2} \cdot e^{i\pi/2^2} \cdot e^{i\pi/2^3} \cdots\cdots \infty$

$= e^{\frac{i\pi}{2} + \frac{i\pi}{2^2} + \frac{i\pi}{2^3} + \cdots \infty}$

$= e^{\frac{i\pi}{2}\left(1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots\cdots \infty\right)}$

$= e^{\frac{i\pi}{2} \cdot \left[\frac{1}{1 - 1/2}\right]}$

$= e^{i\pi}$

$= \cos\pi + i\sin\pi$

$= -1 + i.0$

$= -1$ (Proved)

⑤ If $(a_1+ib_1)(a_2+ib_2)\cdots\cdots(a_n+ib_n) = A+iB$, then show that

(i) $\tan^{-1}\frac{b_1}{a_1} + \tan^{-1}\frac{b_2}{a_2} + \cdots\cdots + \tan^{-1}\frac{b_n}{a_n} = \tan^{-1}\frac{B}{A}$.

(ii) $(a_1^2+b_1^2)(a_2^2+b_2^2)\cdots\cdots(a_n^2+b_n^2) = A^2+B^2$

<u>Solⁿ</u> let $a_1 = r_1\cos\theta_1$

$b_1 = r_1\sin\theta_1$

$\therefore r_1 = \sqrt{a_1^2+b_1^2}$ and $\theta_1 = \tan^{-1}\frac{b_1}{a_1}$

similarly

$r_2 = \sqrt{a_2^2+b_2^2}$ and $\theta_2 = \tan^{-1}\frac{b_2}{a_2}$

$r_3 = \sqrt{a_3^2+b_3^2}$ and $\theta_3 = \tan^{-1}\frac{b_3}{a_3}$

$\cdots\cdots\cdots\cdots\cdots\cdots$

$\cdots\cdots\cdots\cdots\cdots\cdots$

$r_n = \sqrt{a_n^2+b_n^2}$ and $\theta_n = \tan^{-1}\frac{b_n}{a_n}$

Now $a_1+ib_1 = r_1(\cos\theta_1 + i\sin\theta_1)$

$a_2+ib_2 = r_2(\cos\theta_2 + i\sin\theta_2)$

$\cdots\cdots\cdots$

$a_n+ib_n = r_n(\cos\theta_n + i\sin\theta_n)$

$\therefore (a_1+ib_1)(a_2+ib_2)\cdots\cdots(a_n+ib_n)=(r_1r_2r_3\cdots r_n)\{\cos(\theta_1+\theta_2+\cdots+\theta_n)$
$+ i\sin(\theta_1+\theta_2+\theta_3+\cdots+\theta_n)\}$

$A+iB=(r_1r_2r_3\cdots r_n)\{\cos(\theta_1+\theta_2+\cdots+\theta_n)+i\sin(\theta_1+\theta_2+\cdots+\theta_n)\}$

Equating real and imaginary part on both sides

$\therefore r_1r_2\cdots r_n\cos(\theta_1+\theta_2+\cdots+\theta_n)=A$ ——— (i)

$r_1r_2\cdots r_n\sin(\theta_1+\theta_2+\cdots+\theta_n)=B$ ——— (ii)

From (ii) ÷ (i) we have

$\tan(\theta_1+\theta_2+\cdots+\theta_n)=\frac{B}{A}$

$\therefore \theta_1+\theta_2+\cdots+\theta_n=\tan^{-1}\frac{B}{A}$ ——— (iii)

Now putting the values of $\theta_1, \theta_2, \cdots, \theta_n$ in (iii) we have

$\tan^{-1}\frac{b_1}{a_1}+\tan^{-1}\frac{b_2}{a_2}+\cdots+\tan^{-1}\frac{b_n}{a_n}=\tan^{-1}\frac{B}{A}$ (showed)

(ii) Again, squaring and adding (i) & (ii), we have

$r_1^2r_2^2\cdots r_n^2\{\cos^2(\theta_1+\theta_2+\cdots+\theta_n)+\sin^2(\theta_1+\theta_2+\cdots+\theta_n)\}=A^2+B^2$

$\therefore r_1^2r_2^2\cdots r_n^2=A^2+B^2$

putting these value of $r_1, r_2, \cdots, r_n$ we have

$(a_1^2+b_1^2)(a_2^2+b_2^2)\cdots(a_n^2+b_n^2)=A^2+B^2$ (showed)

Find the equation whose roots are the 7th powers of the roots of the equation.

$$x^2-2x\cos\theta+1=0$$

Sol^n: Given that,

$$x^2-2x\cos\theta+1=0$$

$$\therefore x=\frac{2\cos\theta\pm\sqrt{4\cos^2\theta-4}}{2}$$

$$=\cos\theta\pm i\sin\theta$$

let $x_1=\cos\theta+i\sin\theta$ & $x_2=\cos\theta-i\sin\theta$

$\therefore x_1^7=(\cos\theta+i\sin\theta)^7$ & $x_2^7=(\cos\theta-i\sin\theta)^7$

$=\cos7\theta+i\sin7\theta$ $\qquad$ $=\cos7\theta-i\sin7\theta$

Now, we form an equation whose roots are x_1^7 & x_2^7

$$\therefore t^2-(x_1^7+x_2^7)t+x_1^7x_2^7=0$$

Now, $x_1^7+x_2^7=\cos7\theta+i\sin7\theta+\cos7\theta-i\sin7\theta$

$$=2\cos7\theta$$

and $x_1^7x_2^7=(\cos7\theta+i\sin7\theta)(\cos7\theta-i\sin7\theta)$

$$=\cos^27\theta-i\sin7\theta\cos7\theta+i\sin7\theta\cos7\theta+\sin^27\theta$$

$$=\cos^27\theta+\sin^27\theta$$

$$=1$$

The required equation is

$$t^2-2\cos7\theta\, t+1=0$$

9) If $(1+x)^n = P_0 + P_1 x + P_2 x^2 + \cdots$, then show that

$$P_0 - P_2 + P_4 - \cdots\cdots = 2^{n/2} \cos\frac{n\pi}{4}$$

$$P_1 - P_3 + P_5 - \cdots\cdots = 2^{n/2} \sin\frac{n\pi}{4}$$

<u>Soln</u> let, $1+i = r(\cos\theta + i\sin\theta)$, $(1+i)^n = r^n(\cos n\theta + i\sin n\theta)$

$$r\cos\theta = 1 \quad \& \quad r\sin\theta = 1$$

$$\therefore r = \sqrt{1^2+1^2} = \sqrt{2}$$

$$\therefore \tan\theta = 1$$

$$\therefore \theta = \pi/4$$

Now, put $x = i$, we get

$$(1+i)^n = P_0 + P_1 i - P_2 - iP_3 + P_4 + iP_5 - \cdots\cdots$$

$$= (P_0 - P_2 + P_4 - \cdots\cdots) + i(P_1 - P_3 + P_5 - \cdots\cdots)$$

$$r^n(\cos n\theta + i\sin n\theta) = (P_0 - P_2 + P_4 - \cdots\cdots) + i(P_1 - P_3 + P_5 - \cdots\cdots)$$

Equating real and imaginary parts on both sides, we get

$$P_0 - P_2 + P_4 - \cdots\cdots = r^n \cos n\theta$$

$$= (\sqrt{2})^n \cos\frac{n\pi}{4}$$

$$= 2^{n/2} \cos\frac{n\pi}{4}$$

and $P_1 - P_3 + P_5 - \cdots\cdots = r^n \sin n\theta$

$$= (\sqrt{2})^n \sin n\pi/4$$

$$= 2^{n/2} \sin\frac{n\pi}{4} \text{ (showed)}$$

(10) Find all the values of

(i) $(1+i)^{1/7}$ (ii) $(1+i)^{1/5}$ (iii) $(-i)^{1/6}$ (iv) $(-1)^{1/5}$

Solⁿ Here, $(1+i) = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$

$$(1+i)^{1/7} = (\sqrt{2})^{1/7}\left(\cos\frac{\pi}{4} + i\sin\pi/4\right)^{1/7}$$

$$= 2^{\frac{1}{14}}\left[\cos\frac{1}{7}(2n\pi+\pi/4) + i\sin\frac{1}{7}(2n\pi+\pi/4)\right]$$

Now putting $n=0,1,2,3,4,5,6$, then the required values are

$2^{1/14}\left(\cos\frac{\pi}{28} + i\sin\frac{\pi}{28}\right)$, $2^{\frac{1}{14}}\left(\cos\frac{9\pi}{28} + i\sin\frac{9\pi}{28}\right)$ etc.

(ii) Here $(1+i) = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\pi/4\right)$

$$(1+i)^{1/5} = (\sqrt{2})^{1/5}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)^{1/5}$$

$$= 2^{\frac{1}{10}}\left[\cos\frac{1}{5}(2n\pi+\pi/4) + i\sin(2n\pi+\pi/4)\right]$$

Now putting $n=0,1,2,3,4$ etc, then we have the values are,

$2^{\frac{1}{10}}\left(\cos\frac{\pi}{20} + i\sin\frac{\pi}{20}\right)$, $2^{\frac{1}{10}}\left(\cos\frac{9\pi}{20} + i\sin\frac{9\pi}{20}\right)$ etc.

(iii) $(-i)^{1/6} = \left(\cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2}\right)^{1/6}$

$$= \left\{\cos\left(2n\pi+\frac{3\pi}{2}\right) + i\sin\left(2n\pi+\frac{3\pi}{2}\right)\right\}^{1/6}$$

$$= \cos\frac{1}{6}\left(2n\pi+\frac{3\pi}{2}\right) + i\sin\frac{1}{6}\left(2n\pi+\frac{3\pi}{2}\right)$$

putting $n=0,1,2,3,4,5$, we get the required values

$\cos(4n+3)\frac{\pi}{12} + i\sin(4n+3)\frac{\pi}{12}$, $n=0,1,2,3,4,5$.

(iv) $(-1)^{2/5} = (\cos\pi + i\sin\pi)^{2/5} = \{\cos(2n\pi+\pi) + i\sin(2n\pi+\pi)\}^{2/5}$

$$= \cos\frac{2}{5}(2n\pi+\pi) + i\sin\frac{2}{5}(\cancel{2n\pi+\pi}\ 2n\pi+\pi)$$

Now, putting $n=0,1,2,3,4$ we get the required values

(14) If n be a positive integer, then prove that.

$$(1+i)^n + (1-i)^n = 2^{\frac{n}{2}+1} \cos\frac{n\pi}{4}.$$

Solⁿ $1+i = \sqrt{2}.(\cos\frac{\pi}{4} + i\sin\pi/4)$ $[\because \cos\frac{\pi}{4} = \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}]$

$$(1+i)^n = (\sqrt{2})^n \left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)^n$$

$$= 2^{n/2}\left[\cos\frac{n\pi}{4} + i\sin\frac{n\pi}{4}\right]$$

similarly $(1-i)^n = 2^{n/2}\left[\cos\frac{n\pi}{4} - i\sin\frac{n\pi}{4}\right]$

$$\therefore (1+i)^n + (1-i)^n = 2^{n/2}\left\{\cos\frac{n\pi}{4} + i\sin\frac{n\pi}{4} + \cos\frac{n\pi}{4} - i\sin\frac{n\pi}{4}\right\}$$

$$= 2^{n/2}\left\{2\cos\frac{n\pi}{4}\right\}$$

$$= 2^{\frac{n}{2}+1}\cos\frac{n\pi}{4} \text{ (Proved)}$$

(19). If $(1+x)^n = a_0 + a_1 x + a_2 x^2 + \cdots\cdots$ (n being a positive integer), then prove that,

$$a_0 + a_4 + a_8 + \cdots\cdots = 2^{n-2} + 2^{\frac{1}{2}n-1}\cos\frac{1}{4}n\pi$$

Solⁿ Given that,

$$(1+x)^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots\cdots$$

putting $x = 1$ and -1 we have

$$(1+1)^n = a_0 + a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + a_7 + a_8 + \cdots$$

$$2^n = a_0 + a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + a_7 + a_8 + \cdots \quad \text{——— (1)}$$

and $(1-1)^n = a_0 - a_1 + a_2 - a_3 + a_4 - \cdots\cdots$

$$0 = a_0 - a_1 + a_2 - a_3 + a_4 - a_5 + a_6 - a_7 + a_8 - \cdots \quad \text{——— (2)}$$

Adding ① & ② we have,

$$2^n = 2(a_0 + a_2 + a_4 + a_6 + a_8 + \cdots\cdots) \text{ ——— (A)}$$

Again, putting $x = i$ and $-i$, we have

$$(1+i)^n = a_0 + a_1 i - a_2 - a_3 i + a_4 + a_5 i - \cdots\cdots \text{ ——— (3)}$$

$$(1-i)^n = a_0 - a_1 i - a_2 + a_3 i + a_4 - a_5 i - \cdots\cdots \text{ ——— (4)}$$

Adding (4) and (3)

$$(1+i)^n + (1-i)^n = 2(a_0 - a_2 + a_4 - a_6 + a_8 - \cdots\cdots) \text{ ——— (B)}$$

Again, Adding (A) and (B), we have

$$2^n + (1+i)^n + (1-i)^n = 2 \times 2(a_0 + a_4 + a_8 + \cdots\cdots)$$

$$\therefore 2^n + (1+i)^n + (1-i)^n = 2^2(a_0 + a_4 + a_8 + \cdots\cdots) \text{ ——— (C)}$$

Now, let $1 = r\cos\theta$ $1 = r\sin\theta$

$$r = \sqrt{2}, \text{ and } \theta = \pi/4$$

$$1+i = \sqrt{2}(\cos\pi/4 + i\sin\pi/4)$$

$$(1+i)^n = 2^{n/2}\left(\cos\frac{\pi}{4} + i\sin\pi/4\right) = 2^{n/2}\cos\left(\frac{n\pi}{4} + i\sin\frac{n\pi}{4}\right)$$

similarly $(1-i)^n = 2^{n/2}\left(\cos\frac{n\pi}{4} - i\sin\frac{n\pi}{4}\right)$

$$(1+i)^n + (1-i)^n = 2^{n/2} \cdot 2\cos\frac{n\pi}{4} = 2^{\frac{n}{2}+1}\cos\frac{n\pi}{4}$$

Hence from (C) we get

$$a_0 + a_4 + a_8 + \cdots\cdots = 2^{n-2} + 2^{\frac{n}{2}-1}\cos\frac{n\pi}{4}$$

(Proved)