Partial Differention

1. **Problem**: If $v = x^2 + y^2 + z^2$, show that $xV_x + yV_y + zV_z = 2V$.

Solution: Given that

$$V = x^2 + y^2 + z^2 \longrightarrow ①$$

Partially differentiate equation ① with respect to x we get

$$V_x = 2x$$

$\therefore xV_x = 2x^2 \rightarrow ②$ [Multiply by x on both sides]

Similarly

$$V_y = 2y$$

$$yV_y = 2y^2 \longrightarrow ③$$

and $zV_z = 2z^2 \longrightarrow ④$

Adding eq^n ②, ③ and ④ we get

$$xV_x + yV_y + zV_z = 2(x^2 + y^2 + z^2)$$

or, $xV_x + yV_y + zV_z = 2V$ [using equation ①]

(Showed)

2. If $U = x^2y + y^2z + z^2x$, show that $U_x + U_y + U_z = (x+y+z)^2$.

Solution: Given that

$$U = x^2y + y^2z + z^2x \longrightarrow ①$$

Partially differentiate equation ① with respect to x, y and z respectively

$$U_x = 2xy + 0 + z^2 \longrightarrow ②$$

$$U_y = x^2 + 2yz + 0 \longrightarrow ③$$

$$U_z = 0 + y^2 + 2zx \longrightarrow ④$$

Adding eq^n ② ③ and ④ we get

$$U_x + U_y + U_z = 2xy + z^2 + x^2 + 2yz + y^2 + 2zx$$

or, $U_x + U_y + U_z = (x+y+z)^2$ showed

③ Problem: If $U = f(xyz)$, show that $xU_x = yU_y = zU_z$.

Solution: Given that $U = f(xyz) \longrightarrow ①$

Partially differentiate equation ① with respect to x we get

$$U_x = f'(xyz)\frac{\partial}{\partial x}(xyz)$$

$$U_x = f'(xyz)\, yz$$

$\therefore xU_x = f'(xyz)xyz \rightarrow ②$ (multiply by x on both sides)

Again, Differentiate eqⁿ ① w.r to y

$$U_y = f'(xyz)\frac{\partial}{\partial y}(xyz)$$

or, $U_y = f'(xyz)\, xz$

or $yU_y = f'(xyz)\, xyz \longrightarrow ③$

Similarly when partially differentiate equation ① w.r to z we get

$$U_z = f'(xyz)\frac{\partial}{\partial z}(xyz) = f'(xyz)\, xy$$

$$zU_z = f'(xyz)\, xyz \longrightarrow ④$$

From ②, ③ and ④ we conclude that

$$xU_x = yU_y = zU_z = f'(xyz)\, xyz$$

$\therefore xU_x = yU_y = zU_z$ (Showed)

④ Problem: If $U = \sin^{-1}\frac{x}{y} + \tan^{-1}\frac{y}{x}$, Prove that $x\frac{\partial U}{\partial x} + y\frac{\partial U}{\partial y} = 0$

Solⁿ: Given that, $U = \sin^{-1}\frac{x}{y} + \tan^{-1}\frac{y}{x} \longrightarrow ①$

Partially differentiate eqⁿ ① with respect to x we get

$$\frac{\partial U}{\partial x} = \frac{1}{\sqrt{1-\left(\frac{x}{y}\right)^2}}\frac{\partial}{\partial x}\left(\frac{x}{y}\right) + \frac{1}{1+\left(\frac{y}{x}\right)^2}\frac{\partial}{\partial x}\left(\frac{y}{x}\right)$$

$$= \frac{1}{\sqrt{\frac{y^2-x^2}{y^2}}} \cdot \frac{1}{y} + \frac{1}{\frac{x^2+y^2}{x^2}} \left(-\frac{y}{x^2}\right)$$

$$= \frac{y}{\sqrt{y^2-x^2}} \cdot \frac{1}{y} + \frac{x^2}{x^2+y^2} \left(-\frac{y}{x^2}\right)$$

$$\frac{\partial u}{\partial x} = \frac{1}{\sqrt{y^2-x^2}} - \frac{y}{x^2+y^2}$$

$$\therefore x\frac{\partial u}{\partial x} = \frac{x}{\sqrt{y^2-x^2}} - \frac{xy}{x^2+y^2} \longrightarrow (2)$$

Now

$$\frac{\partial u}{\partial y} = \frac{1}{\sqrt{1-\frac{x^2}{y^2}}} \frac{\partial}{\partial y}\left(\frac{x}{y}\right) + \frac{1}{1+\frac{y^2}{x^2}} \frac{\partial}{\partial y}\left(\frac{y}{x}\right)$$

$$= \frac{1}{\sqrt{\frac{y^2-x^2}{y^2}}} \left(-\frac{x}{y^2}\right) + \frac{1}{\frac{x^2+y^2}{x^2}} \cdot \frac{1}{x}$$

$$= - \frac{y}{\sqrt{y^2-x^2}} \cdot \frac{x}{y^2} + \frac{x^2}{x^2+y^2} \cdot \frac{1}{x}$$

$$\frac{\partial u}{\partial y} = - \frac{1}{\sqrt{y^2-x^2}} \cdot \frac{x}{y} + \frac{x}{x^2+y^2}$$

$$\therefore y\frac{\partial u}{\partial y} = - \frac{x}{\sqrt{y^2-x^2}} + \frac{xy}{x^2+y^2} \longrightarrow (3)$$

Adding (1) and (2), we get

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0$$

5. If $u = \log(x^3+y^3+z^3-3xyz)$, show that

(i) $\frac{\delta u}{\delta x} + \frac{\delta u}{\delta y} + \frac{\delta u}{\delta z} = \frac{3}{x+y+z}$

(ii) $\frac{\delta^2 u}{\delta x^2} + \frac{\delta^2 u}{\delta y^2} + \frac{\delta^2 u}{\delta z^2} = -\frac{3}{(x+y+z)^2}$.

Solution: Given that, $U=\log(x^3+y^3+z^3-3xyz) \longrightarrow (1)$

Partially differentiate (1) w.r to x, y and z respectively

$$\frac{\delta U}{\partial x}=\frac{1}{x^3+y^3+z^3-3xyz}\cdot(3x^2-3yz)=\frac{3(x^2-yz)}{x^3+y^3+z^3-3xyz} \longrightarrow (2)$$

$$\frac{\delta U}{\delta y}=\frac{3(y^2-xz)}{x^3+y^3+z^3-3xyz} \longrightarrow (3)$$

and $\frac{\delta y}{\delta y}$ $$\frac{\delta U}{\delta z}=\frac{3(z^2-xy)}{x^3+y^3+z^3-3xyz} \longrightarrow (4)$$

Adding above three equation we get

$$\frac{\delta U}{\delta x}+\frac{\delta U}{\delta y}+\frac{\delta U}{\delta z}=\frac{3(x^2+y^2+z^2-xy-yz-zx)}{x^3+y^3+z^3-3xyz}$$

$$=\frac{3(x^2+y^2+z^2-xy-yz-zx)}{(x+y+z)(x^2+y^2+z^2-xy-yz-zx)}$$

$$\therefore \frac{\delta U}{\delta x}+\frac{\delta U}{\delta y}+\frac{\delta U}{\delta z}=\frac{3}{x+y+z}.$$ [(11) নং নিজে কর]

Differentiate partially (2) we get

$$\frac{\delta^2 U}{\delta x^2}=\frac{(x^3+y^3+z^3-3xyz)\cdot 6x-3(x^2-yz)(3x^2-3yz)}{(x^3+y^3+z^3-3xyz)^2}$$

$$\frac{\delta^2 U}{\delta x^2}=\frac{6x^4+6xy^3+6z^3x-18x^2yz-9(x^4-x^2yz-x^2yz+y^2z^2)}{(x^3+y^3+z^3-3xyz)^2}$$

$$\frac{\delta^2 U}{\delta x^2}=\frac{-3(x^4-2xy^3-2xz^3+3y^2z^2)}{(x^3+y^3+z^3-3xyz)^2}$$

Similarly $$\frac{\delta^2 U}{\delta y^2}=\frac{-3(y^4-2yz^3-2yx^3+3z^2x^2)}{(x^3+y^3+z^3-3xyz)^2}$$

$$\frac{\delta^2 U}{\delta z^2}=\frac{-3(z^4-2zx^3-2zy^3+3x^2y^2)}{(x^3+y^3+z^3-3xyz)^2}$$

On Adding we get

$$\frac{\delta^2 v}{\delta x^2}+\frac{\delta^2 v}{\delta y^2}+\frac{\delta^2 v}{\delta z^2}=\frac{-3\left(x^2+y^2+z^2-yz-zx-xy\right)^2}{\left(x^3+y^3+z^3-3xyz\right)^2}$$

$$=\frac{-3\left(x^2+y^2+z^2-yz-xy-zx\right)^2}{(x+y+z)^2\left(x^2+y^2+z^2-xy-yz-zx\right)^2}$$

Hence $\frac{\delta^2 v}{\delta x^2}+\frac{\delta^2 v}{\delta y^2}+\frac{\delta^2 v}{\delta z^2}=\frac{-3}{(x+y+z)^2}$.

6. If $V=\sqrt{x^2+y^2+z^2}$, then Prove that $V_{xx}+V_{yy}+V_{zz}=\frac{2}{V}$

Solⁿ: Given that,

$$V=\sqrt{x^2+y^2+z^2} \longrightarrow (1)$$

$$\therefore V_x=\frac{1}{2\sqrt{x^2+y^2+z^2}}\cdot\frac{\partial}{\partial x}\left(x^2+y^2+z^2\right)$$

$$V_x=\frac{1}{2\sqrt{x^2+y^2+z^2}}\cdot 2x$$

$$\text{Or, } V_x=\frac{x}{\sqrt{x^2+y^2+z^2}}$$

Again

$$V_{xx}=\frac{\sqrt{x^2+y^2+z^2}\cdot 1-x\cdot\frac{1}{2\sqrt{x^2+y^2+z^2}}\cdot 2x}{\left(\sqrt{x^2+y^2+z^2}\right)^2}$$

$$=\frac{\sqrt{x^2+y^2+z^2}-\frac{x^2}{\sqrt{x^2+y^2+z^2}}}{x^2+y^2+z^2}$$

$$=\frac{x^2+y^2+z^2-x^2}{\left(x^2+y^2+z^2\right)\sqrt{x^2+y^2+z^2}}$$

$$V_{xx}=\frac{y^2+z^2}{\left(x^2+y^2+z^2\right)\sqrt{x^2+y^2+z^2}} \longrightarrow (2)$$

Similarly

$$V_{yy}=\frac{x^2+z^2}{\left(x^2+y^2+z^2\right)\sqrt{x^2+y^2+z^2}} \longrightarrow (3)$$

and $V_{zz} = \dfrac{x^2+y^2}{(x^2+y^2+z^2)\sqrt{x^2+y^2+z^2}} \longrightarrow (4)$

Adding equation (2), (3) and (4) we get

$$V_{xx} + V_{yy} + V_{zz} = \frac{2(x^2+y^2+z^2)}{(x^2+y^2+z^2)\sqrt{x^2+y^2+z^2}}$$

$$\therefore V_{xx} + V_{yy} + V_{zz} = \frac{2}{V} \text{ [using eq}^n \text{ (1)]}$$ Hence Proved.

7. If $U = e^{xyz}$, then Prove that $\dfrac{\delta^3 U}{\delta x \delta y \delta z} = (1+3xyz+x^2y^2z^2)e^{xyz}$

<u>Solution</u>: Given that, $U = e^{xyz} \longrightarrow (1)$

Partially diff. w. r. to x we get

$$\frac{\delta U}{\delta x} = e^{xyz}\frac{\partial}{\partial x}(xyz) = e^{xyz}yz \longrightarrow (2)$$

Partially diff. eq^n (2) w. r to y we get

$$\frac{\delta^2 U}{\delta x \delta y} = e^{xyz}\frac{\partial}{\partial y}(yz) + e^{xyz}\cdot\frac{\partial}{\partial y}(xyz)\cdot yz$$

$$= e^{xyz}z + e^{xyz}xz\cdot yz$$

$$\frac{\delta^2 U}{\delta x \delta y} = (z + xyz^2)e^{xyz} \longrightarrow (3)$$

Again Partially differentiate eq^n (3) with respect to z we get

$$\frac{\delta^3 U}{\delta x \delta y \delta z} = (z + xyz^2)e^{xyz}\frac{\partial}{\partial z}(xyz) + \frac{\partial}{\partial z}(z+xyz^2)e^{xyz}$$

$$= (z + xyz^2)e^{xyz}\cdot xy + (1+2xyz)e^{xyz}$$

$$= (xyz + x^2y^2z^2 + 1 + 2xyz)e^{xyz}$$

Therefore $\dfrac{\partial^3 U}{\delta x \delta y \delta z} = (1+3xyz+x^2y^2z^2)e^{xyz}$

(Proved)

8. If $y = f(x+ct) + \phi(x-ct)$; then Prove $\frac{\delta^2 y}{\delta t^2} = c^2 \frac{\delta^2 y}{\delta x^2}$.

Solution: Given that

$$y = f(x+ct) + \phi(x-ct) \longrightarrow ①$$

Partially differentiate eqn ① with respect to x we get

$$\frac{\delta y}{\delta x} = f'(x+ct)\frac{\delta}{\delta x}(x+ct) + \phi'(x-ct)\frac{\partial}{\delta x}(x-ct)$$

$$= f'(x+ct)\cdot 1 + \phi'(x-ct)\cdot 1$$

Again diff. w.r to x we get

$$\frac{\delta^2 y}{\delta x^2} = f''(x+ct) + \phi''(x-ct) \longrightarrow ②$$

Partially differentiate equation ① w.r to t we get

$$\frac{\delta y}{\delta t} = f'(x+ct)\frac{\partial}{\partial t}(x+ct) + \phi'(x-ct)\frac{\partial}{\partial t}(x-ct)$$

$$= f'(x+ct)\,c + \phi'(x-ct)(-c)$$

$$\frac{\delta^2 y}{\delta t^2} = f'(x+ct)c^2 + c^2\phi'(x-ct)$$

$$\frac{\delta^2 y}{\delta t^2} = c^2\{f''(x+ct) + \phi''(x-ct)\} \longrightarrow ③$$

$$\frac{\delta^2 y}{\delta t^2} = c^2 \frac{\delta^2 y}{\delta x^2} \text{ [using eqn ②]}$$

Therefore $\frac{\delta^2 y}{\delta t^2} = c^2 \frac{\delta^2 y}{\delta x^2}$ (Proved)

9. If $x^2+y^2+z^2-2xyz = 1$, show that $\frac{dx}{\sqrt{1-x^2}} + \frac{dy}{\sqrt{1-y^2}} + \frac{dz}{\sqrt{1-}}$

Solution: Given that, $x^2+y^2+z^2-2xyz = 1 \longrightarrow ①$

Taking differentials

$$2xdx + 2ydy + 2zdz - 2xydz - 2xzdy - 2yzdx$$

or, $(x-yz)\,dx+(y-zx)\,dy+(z-xy)\,dz=0 \longrightarrow (2)$

Now From relation (1) we can write

$$x^2-2xyz=1-y^2-z^2$$

or, $x^2-2xyz+y^2z^2=1-y^2-z^2+y^2z^2$

or, $(x-yz)^2=(1-y^2)-z^2(1-y^2)$

or, $(x-yz)^2=(1-y^2)(1-z^2)$

Multiplying by $(1-x^2)$ on the both sides we get

$$(x-yz)^2(1-x^2)=(1-x^2)(1-y^2)(1-z^2)$$

or, $(x-yz)^2(1-x^2)=k^2$ let $k^2=(1-x^2)(1-y^2)(1-z^2)$

or $(x-yz)^2=\dfrac{k^2}{1-x^2}=\dfrac{k^2}{(\sqrt{1-x^2})^2}$

$\therefore\ x-yz=\dfrac{k}{\sqrt{1-x^2}}$

Similarly, $y-zx=\dfrac{k}{\sqrt{1-y^2}}$

$z-xy=\dfrac{k}{\sqrt{1-z^2}}$

Putting above these values in (2) we have

$$\frac{k\,dx}{\sqrt{1-x^2}}+\frac{k\,dy}{\sqrt{1-y^2}}+\frac{k\,dz}{\sqrt{1-z^2}}=0$$

or, $\dfrac{dx}{\sqrt{1-x^2}}+\dfrac{dy}{\sqrt{1-y^2}}+\dfrac{dz}{\sqrt{1-z^2}}=0$ (showed)

10. Problem : If $ax^2+by^2+cz^2=1$ and $lx+my+nz=0$ then prove

$$\frac{dx}{bny-cmz}=\frac{dy}{clz-anx}=\frac{dz}{amx-bly}$$

Solⁿ: Given that,

$$ax^2+by^2+cz^2=1$$
$$lx+my+nz=0$$

Taking differentials above two equations, we get

$$2ax\,dx+2by\,dy+2cz\,dz=0$$
$$\text{or}\quad ax\,dx+by\,dy+cz\,dz=0 \longrightarrow (1)$$
$$\text{and}\quad l\,dx+m\,dy+n\,dz=0 \longrightarrow (2)$$

From eqⁿ (1) & (2) the rule of cross-multiplication

$$\frac{dx}{nby-cmz}=\frac{dy}{clz-anx}=\frac{dz}{amx-bly}\quad \text{(Proved)}.$$

(11). If $U=F(y-z,\,z-x,\,x-y)$, Prove that $\frac{\partial U}{\partial x}+\frac{\partial U}{\partial y}+\frac{\partial U}{\partial z}=$

Solⁿ: Given that, $U=F(y-z,\,z-x,\,x-y) \longrightarrow (1)$

Let us consider $x_1=y-z$, $x_2=z-x$, $x_3=x-y$

$$\therefore\ \frac{\partial x_1}{\partial x}=0,\quad \frac{\partial x_2}{\partial x}=-1,\quad \frac{\partial x_3}{\partial x}=1$$
$$\frac{\partial x_1}{\partial y}=1,\quad \frac{\partial x_2}{\partial y}=0,\quad \frac{\partial x_3}{\partial y}=-1$$
$$\frac{\partial x_1}{\partial z}=-1,\quad \frac{\partial x_2}{\partial z}=1,\quad \frac{\partial x_3}{\partial z}=0$$

The given function (1) becomes

$$U=F(x_1,x_2,x_3)$$
$$\therefore\ \frac{\partial U}{\partial x}=\frac{\partial U}{\partial x_1}\frac{\partial x_1}{\partial x}+\frac{\partial U}{\partial x_2}\frac{\partial x_2}{\partial x}+\frac{\partial U}{\partial x_3}\frac{\partial x_3}{\partial x}$$

$$\frac{\partial U}{\partial x} = \frac{\partial U}{\partial x_1}\cdot 0 + \frac{\partial U}{\partial x_2}(-1) + \frac{\partial U}{\partial x_3}(1)$$

$$\frac{\partial U}{\partial x} = -\frac{\partial U}{\partial x_2} + \frac{\partial U}{\partial x_3} \longrightarrow (1)$$

Similarly
$$\frac{\partial U}{\partial y} = \frac{\partial U}{\partial x_1}\frac{\partial x_1}{\partial y} + \frac{\partial U}{\partial x_2}\frac{\partial x_2}{\partial y} + \frac{\partial U}{\partial x_3}\frac{\partial x_3}{\partial y}$$

$$\frac{\partial U}{\partial y} = \frac{\partial U}{\partial x_1}\cdot 1 + \frac{\partial U}{\partial x_2}\cdot 0 + \frac{\partial U}{\partial x_3}\cdot(-1)$$

$$\therefore \frac{\partial U}{\partial y} = \frac{\partial U}{\partial x_1} - \frac{\partial U}{\partial x_3} \longrightarrow (2)$$

and
$$\frac{\partial U}{\partial z} = \frac{\partial U}{\partial x_1}\frac{\partial x_1}{\partial z} + \frac{\partial U}{\partial x_2}\frac{\partial x_2}{\partial z} + \frac{\partial U}{\partial x_3}\frac{\partial x_3}{\partial z}$$

$$= -\frac{\partial U}{\partial x_1} + \frac{\partial U}{\partial x_2} + 0 \longrightarrow (3)$$

Adding eqn (1), (2) and (3) we get

$$\frac{\partial U}{\partial x} + \frac{\partial U}{\partial y} + \frac{\partial U}{\partial z} = 0 \quad \text{(Proved)}$$

12. Problem: If $U = x\phi(y/x) + \psi(y/x)$, show that

(i) $x\frac{\partial U}{\partial x} + y\frac{\partial U}{\partial y} = x\phi(y/x)$

(ii) $x^2 U_{xx} + 2xy U_{xy} + y^2 U_{yy} = 0$

Solution: Given that,
$$U = x\phi(y/x) + \psi(y/x) \longrightarrow (1)$$

Differentiate partially both sides with respect to x, we have

$$\frac{\partial U}{\partial x} = x\phi'(y/x)\frac{\partial}{\partial x}(y/x) + \frac{\partial}{\partial x}(x)\cdot\phi(y/x) + \psi'(y/x)\cdot\frac{\partial}{\partial x}(y/x)$$

$$\frac{\partial U}{\partial x} = x\phi'(y/x)\left(-\frac{y}{x^2}\right) + \phi(y/x) + \psi'(y/x)\left(-\frac{y}{x^2}\right)$$

$$\frac{\partial U}{\partial x} = -\frac{y}{x}\phi'(y/x) + \phi(y/x) + \left(-\frac{y}{x^2}\right)\psi'(y/x)$$

$$\therefore x\frac{\partial U}{\partial x} = -y\phi'(y/x) + \phi(y/x) - \frac{y}{x}\psi'(y/x) \longrightarrow (2)$$

Again :

$$\frac{\partial U}{\partial y} = x\phi'(y/x)\frac{\partial}{\partial y}(y/x) + \psi'(y/x)\frac{\partial}{\partial y}(y/x)$$

$$\text{or, } \frac{\partial U}{\partial y} = x\phi'(y/x)\frac{1}{x} + \psi'(y/x)\cdot\frac{1}{x}$$

$$\therefore y\frac{\partial U}{\partial y} = y\phi'(y/x) + \frac{y}{x}\psi'(y/x) \longrightarrow (3)$$

Adding equation (2) and (3) we get

$$x\frac{\partial U}{\partial x} + y\frac{\partial U}{\partial y} = 0 \quad \text{(Showed)}$$

(13.) Define Homogeneous Functions. State and Prove Euler's theorem on homogeneous functions.

Solution : Homogeneous Functions : A function $f(x,y)$ said to be homogeneous of degree n in the variables and y if it can be expressed in the form $x^n\phi(y/x)$ or in the form $y^n\phi\left(\frac{x}{y}\right)$.

Example let $f(x,y) = ax^2+2hxy+by^2$

$= x^2(a+2h\,y/x + b\,y^2/x^2) = x^2\phi(y/x$

therefore $ax^2+2hxy+by^2$ is homogeneous function of de[gree] two in x, y.

Statement : If $f(x,y)$ be a homogeneous function of x an[d y] of degree n, then $x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nf(x,y)$

Proof: Since $f(x,y)$ is a homogeneous function of degree n, let $f(x,y) = x^n \phi(y/x) \longrightarrow$ (a)

$$\therefore \cancel{\frac{\partial f}{\partial x} = x^n \phi'(y/x) + n x^{n-1} \phi}$$

$$\frac{\partial f}{\partial x} = x^n \phi'(y/x) \frac{\partial}{\partial x}(y/x) + n x^{n-1} \phi(y/x)$$

or, $$\frac{\partial f}{\partial x} = x^n \phi'(y/x)\left(\frac{-y}{x^2}\right) + n x^{n-1} \phi(y/x)$$

$$\therefore x\frac{\partial f}{\partial x} = -x \cdot \frac{y}{x^2} x^n \phi'(y/x) + n\, x \cdot x^{n-1} \phi(y/x)$$

$$x\frac{\partial f}{\partial x} = -\frac{y}{x} x^n \phi'(y/x) + n\, x^n \phi(y/x) \longrightarrow (1)$$

And $$\frac{\partial f}{\partial y} = x^n \phi'(y/x) \frac{\partial}{\partial y}(y/x)$$

$$= x^n \phi'(y/x)\, \frac{1}{x}$$

$$\therefore y\frac{\partial f}{\partial y} = \frac{y}{x} x^n \phi'(y/x) \longrightarrow (2)$$

Now Adding equation (1) and (2) we have

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = n\, x^n \phi(y/x)$$

or, $$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = n f(x,y)$$ [using eqn (a)]

Hence Proved

14. Problem: (a) If $U = \tan^{-1}\frac{x^3+y^3}{x-y}$, show that $x\frac{\partial U}{\partial x}+y\frac{\partial U}{\partial y}=s$

(b) If $U = \cos^{-1}\left\{\frac{x+y}{\sqrt{x}+\sqrt{y}}\right\}$, show that $x\frac{\partial U}{\partial x}+y\frac{\partial U}{\partial y}+\frac{1}{2}\cot U=0.$

(c) If $V=\sin^{-1}\left(\frac{x^2+y^2}{x+y}\right)$, then Prove that $xV_x+yV_y=\tan V.$

Solution: (a) Given that,

$$U=\tan^{-1}\frac{x^3+y^3}{x-y}$$

or, $\tan U = \frac{x^3+y^3}{x-y} = \frac{x^3\left\{1+\left(\frac{y}{x}\right)^3\right\}}{x\left\{1-\left(\frac{y}{x}\right)\right\}}$

or, $\tan U = x^2\phi(y/x)$ where $\phi(y/x)=\frac{1+(y/x)^3}{1-(y/x)}$

Therefore $\tan U$ is a homogeneous function of degree

$\therefore$ By Euler's theorem,

$$x\frac{\partial}{\partial x}(\tan U)+y\frac{\partial}{\partial y}(\tan U)=2\tan U$$

or, $x\sec^2 U\frac{\partial U}{\partial x}+y\sec^2 U\frac{\partial U}{\partial y}=2\tan U$

or, $x\frac{\partial U}{\partial x}+y\frac{\partial U}{\partial y}=2\tan U/\sec^2 U=2\tan U\cdot\frac{1}{\sec^2}$

or, $x\frac{\partial U}{\partial x}+y\frac{\partial U}{\partial y}=2\frac{\sin U}{\cos U}\cdot\cos^2 U$

or, $x\frac{\partial U}{\partial x}+y\frac{\partial U}{\partial y}=2\sin U\cos U=\sin 2U$

Hence $x\frac{\partial U}{\partial x}+y\frac{\partial U}{\partial y}=\sin 2U$ (showed)

(b) Given that, $U=\cos^{-1}\left\{\frac{x+y}{\sqrt{x}+\sqrt{y}}\right\}$

or, $\cos U = \frac{x+y}{\sqrt{x}+\sqrt{y}}$

or, $\cos U = \frac{x(1+y/x)}{\sqrt{x}(1+\sqrt{y/x})}$

or, $\cos u = x^{\frac{1}{2}}\left\{\frac{1+y/x}{1+\sqrt{(y/x)}}\right\}$

or, $\cos u = x^{\frac{1}{2}}\phi(y/x)$, let $\phi(y/x) = \frac{1+y/x}{1+\sqrt{y/x}}$.

$\therefore$ $\cos u$ is a homogeneous function of degree $\frac{1}{2}$.

By Euler's theorem we have

$$x\frac{\partial}{\partial x}(\cos u) + y\frac{\partial}{\partial y}(\cos u) = \frac{1}{2}\cos u$$

or, $-x\sin u\frac{\partial u}{\partial x} + y(-\sin u)\frac{\partial u}{\partial y} = \frac{1}{2}\cos u$

or, $-\sin u\left\{x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right\} = \frac{1}{2}\cos u$

or, $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{1}{2}\frac{\cos u}{-\sin u} = -\frac{1}{2}\cot u.$

Hence $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + \frac{1}{2}\cot u = 0$ (showed)

(c) Given that, $v = \sin^{-1}\frac{x^2+y^2}{x+y}$

or, $\sin v = \frac{x^2+y^2}{x+y}$

or, $\sin v = x^2\left\{1+(y/x)^2\right\}/x\left\{1+(y/x)\right\}$

or, $\sin v = x\left\{\frac{1+(y/x)^2}{1+(y/x)}\right\}$

or, $\sin v = x\,\phi(y/x)$, say $\phi(y/x) = \frac{1+(y/x)^2}{1+(y/x)}$

$\therefore$ $\sin v$ is a homogeneous function of degree 1 (one).

By Euler's theorem we have

$$x\frac{\partial}{\partial x}(\sin v) + y\frac{\partial}{\partial y}(\sin v) = 1\cdot\sin v$$

or, $x\cos v\frac{\partial v}{\partial x} + y\cos v\frac{\partial v}{\partial y} = \sin v$

or $x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} = \frac{\sin v}{\cos v}$

or, $xv_x + yv_y = \tan v$ $\quad\left|\ \frac{\partial v}{\partial x} = v_x, \frac{\partial v}{\partial y} = v_y\right.$

Hence Proved.

15. The radius of a right circular cone is measured as 5 i with a possible error of 0.01 inch, and altitude as 8 inches with a possible error of 0.024 inch. Find the possible relative & and percentage error in the volume as calculated from the measurements.

Solution: We know that the volume of a cone

$$V = \frac{1}{3}\pi r^2 h$$ where r is radius and h is the altitude of the co

Taking log on both sides we have

$$\log V = \log\frac{\pi}{3} + \{\log(r^2 h)\} = \log\left(\frac{\pi}{3}\right) + \{\log r^2 + \log h\}$$

$$\log V = \log\left(\frac{\pi}{3}\right) + (2\log r + \log h)$$

Taking Differentials we have

$$\frac{1}{V}dV = \cancel{\frac{\pi}{3}}\left(2\frac{1}{r}dr + \frac{1}{h}dh\right)$$

$$\text{or, } \frac{dV}{V} = \cancel{\frac{\pi}{3}}\left(2\cdot\frac{0.01}{5} + \frac{1}{8}\,0.024\right)$$

For our Problem
$r = 5$ inch
$dr = 0.01$ "
$h = 8$ "
and $dh = 0.024$ "

$$\text{or, } \frac{dV}{V} = \cancel{\frac{\pi}{3}}(0.004 + 0.003)$$

$$\text{or, } \frac{dV}{V} = \cancel{7.33303 \times 10^{-3}}\ 0.007$$

$\therefore$ Relative Error $= 7.0000 \times 10^{-3}$ inch or, 0.00700 inch

and Percentage Error $= 100\,\frac{dV}{V} = 100 \times 0.00700$ inch

$= 0.700\%$

Answer

16. If $V=(ax+by)^2-(x^2+y^2)$ where $a^2+b^2=2$ then show that $V_{xx}+V_{yy}=0$.

Solution: Given that $V=(ax+by)^2-(x^2+y^2) \longrightarrow (1)$

& $a^2+b^2=2 \longrightarrow (2)$

Partially Diff. (1) w. r. to x ~~at~~ two at a time

$V_x = 2(ax+by)\cdot\frac{\partial}{\partial x}(ax+by) - 2x + 0 = 2(ax+by)\cdot a - 2x$

$\therefore V_{xx} = 2a\cdot a - 2 = 2a^2-2 \longrightarrow (3)$

Similarly partially diff. (1) w.r to y we get

$V_y = 2(ax+by)\cdot\frac{\partial}{\partial y}(ax+by) - 0 - 2y$

$V_y = 2(ax+by)(0+b) = 2b(ax+by) - 2y$

$\therefore V_{yy} = 2b(0+b) - 2 = 2b^2-2 \longrightarrow (4)$

Now Adding equation (3) and (4) we get

$V_{xx}+V_{yy} = 2a^2-2+2b^2-2$

$= 2(a^2+b^2)-4$

$= 2\cdot2-4 = 0$ [Using eq^n (2)]

Hence $V_{xx}+V_{yy}=0$

(Showed)