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Higher Algebra Suggestion of RU

1. prove that $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix} = (b-c)(c-a)(a-b)(a+b+c)$

Soln: $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix}$

$$= \begin{vmatrix} 1 & 0 & 1 \\ a & b-a & c \\ a^3 & b^3-a^3 & c^3 \end{vmatrix} \quad c_2 \rightarrow c_2 - c_1$$

$$= \begin{vmatrix} 1 & 0 & 0 \\ a & b-a & c-a \\ a^3 & b^3-a^3 & c^3-a^3 \end{vmatrix} \quad c_3 \rightarrow c_3 - c_1$$

$$= \begin{vmatrix} b-a & c-a \\ b^3-a^3 & c^3-a^3 \end{vmatrix}$$

$$= \begin{vmatrix} b-a & c-a \\ (b-a)(b^2+ab+a^2) & (c-a)(c^2+ac+a^2) \end{vmatrix}$$

$$= (b-a)(c-a)\begin{vmatrix} 1 & 1 \\ b^2+ab+a^2 & c^2+ac+a^2 \end{vmatrix}$$

$$= (b-a)(c-a)(c^2+a^2+ac-b^2-ab-a^2)$$

$$= (b-a)(c-a)(c^2-b^2+ac-ab)$$

$$= (b-a)(c-a)\{c^2-b^2+a(c-b)\}$$

$$= (b-a)(c-a)\{(c+b)(c-b)+a(c-b)\}$$

$$= (b-a)(c-a)(c-b)(c+b+a)$$

$$= -(a-b)\cdot -(b-c)\cdot (c-a)(a+b+c)$$

$$= (a-b)(b-c)(c-a)(a+b+c)$$

Ans

2) Solve the equation $\begin{vmatrix} x & -6 & -1 \\ 2 & -3x & x-3 \\ -3 & 2x & x+2 \end{vmatrix} = 0$

Soln:

$$\begin{vmatrix} x-2 & -6+3x & -1-x+3 \\ 2 & -3x & x-3 \\ -3 & 2x & x+2 \end{vmatrix} = 0 \quad R_1 \to R_1 - R_2$$

$$\Rightarrow \begin{vmatrix} x-2 & 3(x-2) & -(x-2) \\ 2 & -3x & x-3 \\ -3 & 2x & x+2 \end{vmatrix} = 0$$

$$\Rightarrow (x-2) \begin{vmatrix} 1 & 3 & -1 \\ 2 & -3x & x-3 \\ -3 & 2x & x+2 \end{vmatrix} = 0$$

$$\Rightarrow (x-2) \begin{vmatrix} 1 & 3 & -1 \\ 0 & -3x-6 & x-1 \\ 0 & 2x+9 & x-1 \end{vmatrix} = 0 \quad \begin{matrix} R_2 = R_2 - 2R_1 \\ R_3 = R_3 + 3R_1 \end{matrix}$$

$$\Rightarrow (x-2)\cdot 1 \begin{vmatrix} -3x-6 & x-1 \\ 2x+9 & x-1 \end{vmatrix} = 0$$

$$\Rightarrow (x-2)\{-3x^2-6x+3x+6-(x-1)(2x+9)\}=0$$

$$\Rightarrow (x-2)\{-3x^2-3x+6-2x^2-9x+2x+9\}=0$$

$$\Rightarrow (x-2)\{-5x^2-10x+15\}=0$$

$$\Rightarrow (x-2)(-5)(x^2+2x-3)=0$$

$$\Rightarrow (x-2)(x+3)(x-2)=0$$

$$x = 2, -3, 1$$

3. prove that, $\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = (b-c)(c-a)(a-b)$

Soln: L.H.S =

$$\begin{vmatrix} 1 & a & a^2 \\ 0 & b-c & b^2-c^2 \\ 0 & c-a & c^2-a^2 \end{vmatrix} \quad \begin{matrix} R_2 \to R_2 - R_3 \\ R_3 \to R_3 - R_1 \end{matrix}$$

$$= \begin{vmatrix} b-c & b^2-c^2 \\ c-a & c^2-a^2 \end{vmatrix}$$

$$= (b-c)(c-a) \begin{vmatrix} 1 & b+c \\ 1 & c+a \end{vmatrix}$$

$$= (b-c)(c-a)(c+a-b-c)$$

$$= (b-c)(c-a)(a-b)$$

$$= (a-b)(b-c)(c-a)$$

= R.H.S

(proved)

4.

prove that, $\begin{vmatrix} 1 & a & b+c \\ 1 & b & c+a \\ 1 & c & a+b \end{vmatrix} = 0$

Soln:

$$\text{L.H.S} = \begin{vmatrix} 1 & a & b+c \\ 1 & b & c+a \\ 1 & c & a+b \end{vmatrix}$$

$$= \begin{vmatrix} 1 & a & b+c \\ 0 & b-c & c+a-a-b \\ 0 & c-a & a+b-b-c \end{vmatrix} \quad \begin{matrix} R_2 = R_2 - R_1 \\ R_3 = R_3 - R_1 \end{matrix}$$

$$= \begin{vmatrix} b-c & c-b \\ c-a & a-c \end{vmatrix}$$

$$= (b-c)(c-a)(c-b)(a-b) \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix}$$

$$= (b-c)(c-a)(c-b)(a-b)(1-1)$$

$$= (b-c)(c-a)(c-b)(a-b) \cdot 0$$

$$= 0$$

$$= \text{R.H.S} \quad \text{(proved)}$$

5. Solve the equations $3x+5y-7z=13$
$4x+y-12z=6$
$2x+9y-3z=20$

Sol^n:

Given that, $3x+5y-7z=13$
$4x+y-12z=6$
$2x+9y-3z=20$

$$D = \begin{vmatrix} 3 & 5 & -7 \\ 4 & 1 & -12 \\ 2 & 9 & -3 \end{vmatrix}$$

Expanding along R_1

$= 3(-3+108)-5(-12+24)-7(36-2)$

$= 3(105)-5(12)-7(34)$

$= 315-60-238$

$= 315-298$

$= 17$

$$D_x = \begin{vmatrix} 13 & 5 & -7 \\ 6 & 1 & -12 \\ 20 & 9 & -3 \end{vmatrix}$$

Expanding along c_1

$= 13(-3+108) - 5(-18+240) - 7(54-20)$

$= 13(105) - 5(222) - 7(34)$

$= 1365 - 1110 - 238$

$= 17$

$$D_y = \begin{vmatrix} 3 & 13 & -7 \\ 4 & 6 & -12 \\ 2 & 20 & -3 \end{vmatrix}$$

Expanding along c_2

$= 3(-18+240) - 13(-12+24) - 7(80-12)$

$= 3(222) - 13(12) - 7(68)$

$= 666 - 156 - 476$

$= 34$

$$D_z = \begin{vmatrix} 3 & 5 & 13 \\ 4 & 1 & 6 \\ 2 & 7 & 20 \end{vmatrix}$$

Expanding along C_3

$= 3(20-54) - 5(80-12) + 13(36-2)$

$= 3(-34) - 5(68) + 13(34)$

$= -102 - 340 + 442$

$= -442 + 442$

$= 0$

By Cramer's Rule

$x = \frac{D_x}{D}$, $y = \frac{D_y}{D}$, $z = \frac{D_z}{D}$

$x = \frac{17}{17}$, $y = \frac{34}{17}$, $z = \frac{0}{17}$

$x = 1$, $y = 2$, $z = 0$

Hence the required value are $x, y, z = 1, 2, 0$

Ans.

6) Solve the equations, $x+2y+3z=6$
$2x+4y+z=7$
$3x+2y+9z=14$

Soln:

Given that, $x+2y+3z=6$
$2x+4y+z=7$
$3x+2y+9z=14$

$$D=\begin{vmatrix} 1 & 2 & 3 \\ 2 & 4 & 1 \\ 3 & 2 & 9 \end{vmatrix}$$

Expanding along R_1

$=1(36-2)-2(18-3)+3(4-12)$

$=34-30-8$

$=-20$

$$D_x=\begin{vmatrix} 6 & 2 & 3 \\ 7 & 4 & 1 \\ 14 & 2 & 9 \end{vmatrix}$$

Expanding along c_1

$=6(36-2)-2(63-14)+3(14-56)$

$=204-98-126$

$=-20$

$$Dy = \begin{vmatrix} 1 & 6 & 3 \\ 2 & 7 & 1 \\ 3 & 14 & 9 \end{vmatrix}$$

Expanding along C_2

$= 1(63-14) - 6(18-3) + 3(28-21)$

$= 49 - 90 + 21$

$= 70 - 90$

$= -20$

$$Dz = \begin{vmatrix} 1 & 2 & 6 \\ 2 & 4 & 7 \\ 3 & 2 & 14 \end{vmatrix}$$

Expanding along C_3

$= 1(56-14) - 2(28-21) + 6(4-12)$

$= 42 - 14 - 48$

$= 42 - 62$

$= -20$

By using crameros Rule we get,

$$x = \frac{D_x}{D},\quad y = \frac{D_y}{D},\quad z = \frac{D_z}{D}$$

$$x = \frac{-20}{-20},\quad y = \frac{-20}{-20},\quad z = \frac{-20}{-20}$$

$$x = 1,\quad y = 1,\quad z = 1$$

Hence the required value are

$x, y, z = 1, 1, 1$ Ans

7.

Solve the equations.

$$3x + 2y - z = 20$$
$$2x + 3y + 6z = 70$$
$$x - y + 6z = 41$$

Soln:

Given that,

$$3x + 2y - z = 20$$
$$2x + 3y + 6z = 70$$
$$x - y + 6z = 41$$

$$D = \begin{vmatrix} 3 & 2 & -1 \\ 2 & 3 & 6 \\ 1 & -1 & 6 \end{vmatrix}$$

Expanding along R_1

$= 3(18+6) - 2(12-6) - 1(-2-3)$

$= 72 - 12 + 5$

$= 65$

$$Dx = \begin{vmatrix} 20 & 2 & -1 \\ 70 & 3 & 6 \\ 41 & -1 & 6 \end{vmatrix}$$

Expanding along C_1

$= 20(18+6) - 2(420-246) - 1(-70-123)$

$= 20(24) - 2(\cancel{-4}-174) - 1(-293)$

$= 480 - 348 + 193$

$= 325$

$$D_y = \begin{vmatrix} 3 & 20 & 1 \\ 2 & 70 & 6 \\ 1 & 41 & 6 \end{vmatrix}$$

Expanding along c_1

$= 3(420-246) - 20(12-6) - 1(82-70)$

$= 522 - 120 - 12$

$= 390$

$$D_z = \begin{vmatrix} 3 & 2 & 20 \\ 2 & 3 & 70 \\ 1 & -1 & 41 \end{vmatrix}$$

Expanding along c_3

$= 3(123+70) - 2(82-70) + 20(-2-3)$

$= 579 - 24 - 100$

$= 455$

using cramer's rule we get

$x = \frac{D_x}{D}$, $y = \frac{D_y}{D}$, $z = \frac{D_z}{D}$

$x = \frac{325}{65}$, $y = \frac{390}{65}$, $z = \frac{455}{65}$

$x = 5$, $y = 6$, $z = 7$

Hence the required value are $x, y, z = 5, 6, 7$

8) Solve the equation $x+2y+3z=14$

$2x+3y+4z=20$

$3x+4y+6z=33$

Sol:

Given that, $x+2y+3z=14$

$2x+3y+4z=20$

$3x+4y+6z=33$

$$D=\begin{vmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 6 \end{vmatrix}$$

Expanding along R_1

$=1(18-16)-2(12-12)+3(8-9)$

$=2-0-3$

$=-1$

$$D_x=\begin{vmatrix} 14 & 2 & 3 \\ 20 & 3 & 4 \\ 33 & 4 & 6 \end{vmatrix}$$

Expanding along C_1

$=14(18-16)-2(120-132)+3(80-90)$

$=28+24-57$

$=-5$

$$Dy = \begin{vmatrix} 1 & 14 & 3 \\ 2 & 20 & 4 \\ 3 & 33 & 6 \end{vmatrix}$$

Expanding along c_2

$= 1(120-132) - 14(12-12) + 3(66-60)$

$= -12 - 0 + 18$

$= 6$

$$Dz = \begin{vmatrix} 1 & 2 & 14 \\ 2 & 3 & 20 \\ 3 & 4 & 33 \end{vmatrix}$$

Expanding along c_3

$= 1(99-80) - 2(66-60) + 14(8-9)$

$= 19 - 12 - 14$

$= 19 - 26$

$= -7$

using cramers rule we get,

$x = \frac{Dx}{D} = \frac{-5}{-1} = 5$

$y = \frac{Dy}{D} = \frac{6}{-1} = -6$

$z = \frac{Dz}{D} = \frac{-7}{-1} = 7$

Hence the require value are

$x, y, z = 5, -6, 7$ Ans.

2) Solve the equation, $3x-2y=5$
$4y-z=4$
$2z+3y=14$

Soln;

Given that, $3x-2y=5$, $4y-z=4$, $2z+3y=14$ or, $3x-2y+0.z=5$, $0.x+4y-z=4$, $0x+3y+2z=14$

$$D=\begin{vmatrix}3 & -2 & 0\\ 0 & 4 & -1\\ 0 & 3 & 2\end{vmatrix}$$

Expanding along R_1

$=3(8+3)+2(0-0)+0(0-0)$

$=33$

$$Dx=\begin{vmatrix}5 & -2 & 0\\ 4 & 4 & -1\\ 14 & 3 & 2\end{vmatrix}$$

Expanding along R_1

$=5(8+3)+2(8+14)+0(12-56)$

$=55+44$

$=99$

$$Dy = \begin{vmatrix} 3 & 5 & 0 \\ 0 & 4 & -1 \\ 0 & 14 & 2 \end{vmatrix}$$

Expanding along c_2

$= 3(8+14) - 5(0-0) + 0(0-0)$

$= 66$

$$Dz = \begin{vmatrix} 3 & -2 & 5 \\ 0 & 4 & 4 \\ 0 & 3 & 14 \end{vmatrix}$$

Expanding along c_3

$= 3(56-12) + 2(0-0) + 5(0-0)$

$= 132$

using crcamercs rule

$x = \frac{Dx}{D} = \frac{99}{33} = 3$

$y = \frac{Dy}{D} = \frac{66}{33} = 2$

$z = \frac{Dz}{D} = \frac{132}{33} = 4$

The required value arce $x=3, y=2, z=4$

$x, y, z = 3, 2, 4$ Ans

(9) Every equation of the nth degree has nth roots and no more.

prove: Suppose

$$p_0x^n + P_1x^{n-1} + P_2x^{n-2} + \cdots + P_n = 0$$ be

n th degree equation denote by $f(x)=0$

where

$$f(x) = P_0x^n + P_1x^{n-1} + P_2x^{n-2} + \cdots + P_n$$

The equation $f(x)=0$ has a roots real or imaginary.

Let this root be a_1 then $f(x)$ is divisible by $(x-a_1)$ so that,

$$f(x) = (x-a_1)\,\phi_1(x) \longrightarrow (1)$$

where $\phi_1(x)$ is a rotational function of degree $(n-1)$

Similarly,

$$\phi_1(x) = (x-a_2)\,\phi_2(x)$$

where $\phi_2(x)$ is a rotational function of degree $(n-2)$

putting the value of $\phi_1(x)$ in eq^n (1) we get

$f(x) = (x-a_1)(x-a_2)\,\phi_2(x)$

proceeding in the way we obtain

$f(x) = (x-a_1)(x-a_2)\cdots\cdots(x-a_n)$

Hence the equation $f(x)=0$ has n roots Since $f(x)$ vanishes when x has any of the values, $a_1, a_2, a_3 \cdots a_n$

(proved)

Art. 11 $f(x) = x^n + P_1 x^{n-1} P_2 x^{n-2} + \cdots + P_{n-1} x + P_n = 0$ is a polynomial and $a + \sqrt{2}b$ be a root of $f(x) = 0$ then $a - \sqrt{2}b$ will also be a root of $f(x) = 0$

Sol: Let, $a + \sqrt{2}b = x$

$f(x) = 0$

Then $f(a + \sqrt{2}b) = 0$

But, $f(a + \sqrt{2}b) = A + \sqrt{2}B$

or, $A + \sqrt{2}B = 0$

or, $\sqrt{2}B = -A$

$\therefore \sqrt{2} = -\frac{A}{B}$

Since $\sqrt{2}$ is an irrational number, so the ratio is impossible

Hence, $A = 0$ and $B = 0$

In the same way $A - \sqrt{2}B = f(a - \sqrt{2}b)$

Hence, $x = a - \sqrt{2}b$ is the other root of $f(x) = 0$

Art (12)

In an equation with real co-efficients, imaginary roots occur in pairs.

Soln: Let $f(x)=0$ be an equation with real co-efficien
Let $a+ib$ be an imaginary root of $f(x)=0$
It is now required to show that $a-ib$ is also a root of $f(x)=0$

The product factors of $f(x)$ corresponding to these roots

$$\{x-(a+ib)\}\{x-(a-ib)\}=(x-a)^2+b^2$$

Let $f(x)$ divided by $(x-a)^2+b^2$ then we have

$$f(x)=Q\{(x-a)^2+b^2\}+Rx+R' \text{ ———— (i)}$$

where Q is the quotient of degree $(n-2)$ in x and remainder, if any is $Rx+R'$

If we put $x=a+ib$ in eqn (i) then $f(x)=0$ by hypothesis, also,

$(x+a)^2+b^2=0$ hence $(a+ib)+R'=0$

~~or $Ra+R'=0$, $R.b=0$ ———— (ii)~~

or, $Ra+R'+iRb=0$ ———— (ii)

Equating to zero the real and imaginary parts of eqn (ii), we have.

$$Ra + R' = 0, \quad Rb = 0 \text{ ——— (iii)}$$

by hypothesis, $b \neq 0$

therefore, $R = 0$

Hence (iii) we have

$R' = 0$

Hence $f(x)$ exactly divisible by $(x-a)^2 + b^2$

i.e $\{x-(a-ib)\}\{x-(a-ib)\}$

Thus $x = a - ib$ is also a root.

13/ Descartes Rule of Sing:

The number of real positive roots on the equation $f(x)=0$ can not exeeed the number of changes in the sign of the co-efficients of the terms in $f(x)$ and the number of real negative roots roots can not exceed the number of changes in the sign of the co efficients of $f(-x)$.

14) Find the nature of the root of the equation
$3x^4+12x^2+5x-4=0$

Soln: Given that, $3x^4+12x^2+5x-4=0$ ——— (1)

Let $f(x)=3x^4+12x^2+5x-4=0$

Hence there are one change of sign in $f(x)$

So, $f(x)$ has one positive roots

Again, $f(-x)=3(-x)^4+12(-x)^2+5(-x)-4=0$

$\Rightarrow 3x^4+12x^2-5x-4=0$

Hence there are only one change of sign
Hence there is on negative roots in $f(x)=0$
As the given equation is of the fourth degree it must have four roots, Therefore there are 2 imaginary roots; one positive roots and one negative roots.

15) Show that, $x^6 - x^5 - 10x + 7 = 0$ has tow positive and four imaginary roots.

Soln: Give that,

$$f(x) = x^6 - x^5 - 10x + 7 = 0$$

There are tow changes of sing x^6 to $-x^5$ and from $-10x$ to 7 i.e from $+$ to $-$ and $-$ to $+$

Hence $f(x) = 0$ and tow positive roots

Again $f(-x) = x^6 + x^5 + 10x + 7$

There is no change of sign,

Hence, there is no negative root

As the equation $f(x) = 0$ is of 6th degree It has six roots of which tow roots are positive.

Hence the equation has four imaginary roots.

16) If a, b, c are the roots of $x^3+px^2+qx+r=0$
Find the value of

(i) $(b+c)(c+a)(a+b)$ (ii) $\sum \frac{b^2+c^2}{bc}$

(iii) $\sum a^3$ (iv) $\sum \frac{b^2+c^2}{b+c}$

Soln: Given that, $x^3+px^2+qx+r=0$ —— (i)

Let, a, b, c are the roots of (i)

Sum of the roots

$$a+b+c=-p \longrightarrow \text{(ii)}$$

Sum of the products of the roots taken tow at a time

$$ab+bc+ca=q \longrightarrow \text{(iii)}$$

products of the roots

$$abc=-r$$

(i) $(b+c)(c+a)(a+b)$

$=(bc+ab+c^2+ac)(a+b)$

$=abc+a^2b+ac^2+a^2c+b^2c+ab^2+bc^2+abc$

$=ab(a+b+c)+bc(a+b+c)+ca(a+b+c)-abc$

$= (a+b+c)(ab+bc+ca) - abc$

$= -pq - (-r)$

$= r - pq$

(ii) $\Sigma \frac{b^2+c^2}{bc} = \frac{b^2+c^2}{bc} + \frac{a^2+b^2}{ab} + \frac{a^2+c^2}{ac}$

$= \frac{ab^2+ac^2+a^2c+b^2c+a^2b+bc^2}{abc}$

$= \frac{(a+b+c)(ab+bc+ca) - 3abc}{abc}$

$= \frac{(-p)(q) - 3(-r)}{-r}$

$= \frac{-pq+3r}{-r}$

$= \frac{3r-pq}{-r}$

(iii) $\sum a^3 = a^3+b^3+c^3$

$= (a+b+c)(a^2+b^2+c^2-ab-bc-ca)+3abc$

$= \{a^2+b^2+c^2-(ab+bc+ca)\}(a+b+c)+3abc$

$= \{(a+b+c)^2-2(ab+bc+ca)-(ab+bc+ca)\}(a+b+c)+3abc$

$= \{(-p)^2-2q-q\}\cdot(-p)+3(-r)$

$= (p^2-3q)(-p)+(3r)$

$= -p^3+3pq-3r$

$= 3pq-p^3-3r$ Ans.

(iv) $\sum \frac{b^2+c^2}{b+c} = \frac{b^2+c^2}{b+c}+\frac{c^2+a^2}{c+a}+\frac{a^2+b^2}{a+b}$

$= \frac{(b^2+c^2)(a+b)(c+a)+(c^2+a^2)(a+b)(b+c)+(a^2+b^2)(b+c)(c+a)}{(a+b)(b+c)(c+a)}$

$= \frac{(b^2+c^2)(ab+bc+ca+a^2)+(c^2+a^2)(ab+bc+ca+b^2)+(a^2+b^2)(\text{[illegible]})}{(a+b)(b+c)(c+a)}$

$$= \frac{(b^2+c^2)(ab+bc+ca)+a^2(b^2+c^2)+(c^2+a^2)(ab+bc+ca)+b^2(c^2+a^2)+(a^2+b^2}{(a+b)(b+c)(c+a)}$$

$$(ab+bc+ca)+c^2(a^2+b^2)$$

$$= \frac{(ab+bc+ca)(a^2+b^2+b^2+c^2+c^2+a^2)+a^2b^2+c^2a^2+b^2c^2+a^2b^2+c^2a^2+b^2c^2}{(a+b)(b+c)(c+a)}$$

$$= \cancel{2(ab+bc+ca)\{(a+b+c)^2-2(ab+bc+ca)\}+2\{}$$

$$= \frac{2(ab+bc+ca)(a^2+b^2+c^2)+2(a^2b^2+b^2c^2+c^2a^2)}{(a+b)(b+c)(c+a)}$$

$$= \frac{2(ab+bc+ca)\{(a+b+c)^2-2(ab+bc+ca)\}+2\{(ab+bc+ca)^2-2abc(a+b+c)}{(a+b+c)(ab+bc+ca)-abc}$$

$$= \frac{2q\{(-p)^2-2q\}+2\{q^2-2(-r)(-p)}{-p\cdot q-(-r)}$$

$$= \frac{2q(p^2-2q)+2(q^2-2pr)}{r-pq}$$

$$= \frac{2p^2q-4q^2+2q^2-4pr}{r-pq} = \frac{2p^2q-4pr-2q^2}{r-pq}$$ Ans

19) If a, b, c are the roots of $3x^3-2x^2+1=1$ Find the values of $\sum a^2 b$, $\sum a^3 b$

Soln:

Given that, $3x^3-2x^2+1=1 \longrightarrow$ ①

Let, a, b, c are the roots of ①

sum of the roots

$$a+b+c=\frac{2}{3}$$

Sum of the product of the roots taken two at a time

$$ab+bc+ca=\frac{0}{3}=0$$

Sum of the products

$$abc=-\frac{1}{3}$$

① $\sum a^2 b = a^2 b + a^2 c + b^2 c + b^2 a + c^2 a + c^2 b$

$= (ab)(a+b) + ac(a+c) + bc(b+c)$

$= ab(a+b+c-c) + ac(a+b+c-b) + bc(a+b+c-a)$

$= (a+b+c)(ab+bc+ca) - 3abc$

$= \frac{2}{3}\cdot 0 - 3\cdot\left(-\frac{1}{3}\right)$

$= 1$

(ii) $\sum a^3b = a^3b + a^3c + b^3c + b^3a + c^3a + c^3b$

$= ab(a^2+b^2) + ac(a^2+c^2) + bc(b^2+c^2)$

$= ab(a^2+b^2+c^2-c^2) + ac(a^2+b^2+c^2-b^2) + bc(a^2+b^2+c^2-a^2)$

$= (a^2+b^2+c^2)(ab+bc+ca) - 3(abc)^2$

$= \{(a+b+c)^2 - 2(ab+bc+ca)\}(ab+bc+ca) - 3(abc)^2$

$= \{(\frac{2}{3})^2 - 2\cdot 0\}\cdot 0 - 3(-\frac{1}{3})^2$

$= 0 - \frac{3}{9}$

$= -\frac{1}{3}$ Ans.

(18) If α, β, γ are the roots of $3x^3-5x^2+2x+1=0$ Find the values of $\sum(\alpha-\beta)^2=0$ and $\sum \frac{\alpha^2}{\beta\gamma}$

Soln: Given that,

$$3x^3-5x^2+2x+1=0 \longrightarrow (i)$$

Let, α, β, γ are the roots of (i)

Some of the product roots

$$\alpha+\beta+\gamma=\frac{5}{3} \longrightarrow (ii)$$

Sum of the products of the roots taken tow at a time.

$$\alpha\beta+\beta\gamma+\gamma\alpha=\frac{2}{3} \longrightarrow (iii)$$

Sum of the products

$$\alpha\beta\gamma=-\frac{1}{3} \longrightarrow (iv)$$

(i) $\sum(\alpha-\beta)^2=(\alpha-\beta)^2+(\beta-\gamma)^2+(\gamma-\alpha)^2$

$$=\alpha^2-2\alpha\beta+\beta^2+\beta^2-2\beta\gamma+\gamma^2+\gamma^2-2\gamma\alpha+\alpha^2$$

$$=2(\alpha^2+\beta^2+\gamma^2)-2(\alpha\beta+\beta\gamma+\gamma\alpha)$$

$$=2\{(\alpha+\beta+\gamma)^2-2(\alpha\beta+\beta\gamma+\gamma\alpha)\}-2(\alpha\beta+\beta\gamma+\gamma\alpha)$$

$$=2\{(\tfrac{5}{3})^2-2\cdot\tfrac{2}{3}\}-2\cdot\tfrac{2}{3}$$

$= 2\left(\frac{25}{9} - \frac{4}{3}\right) - \frac{4}{3}$

$= 2\left(\frac{25-12}{9}\right) - \frac{4}{3}$

$= 2\left(\frac{13}{9}\right) - \frac{4}{3}$

$= \frac{26}{9} - \frac{4}{3}$

$= \frac{26-12}{9}$

$= \frac{14}{9}$

(ii) $\sum \frac{\alpha^2}{\beta\gamma} = \frac{\alpha^2}{\beta\gamma} + \frac{\beta^2}{\gamma\alpha} + \frac{\gamma^2}{\alpha\beta}$

$= \frac{\alpha^3 + \beta^3 + \gamma^3}{\alpha\beta\gamma}$

$= \frac{(\alpha+\beta+\gamma)^3 - 3(\alpha+\beta+\gamma)(\alpha\beta+\beta\gamma+\gamma\alpha) + 3\alpha\beta\gamma}{\alpha\beta\gamma}$

$= \frac{\frac{5^3}{3^3} - 3\cdot\frac{5}{3}\cdot\frac{2}{3} + \left(-\frac{1}{3}\right)}{-\frac{1}{3}}$

$= \frac{\frac{125}{27} - \frac{30}{9} - \frac{1}{3}}{-\frac{1}{3}}$

$$= \frac{\frac{125-90-9}{27}}{-\frac{1}{3}}$$

$$= \frac{26}{27} \times \frac{3}{-1}$$

$$= -\frac{26}{9} \text{ Ans.}$$

19) Solve the equation $x^3-12x^2-6x-10=0$ by the cardans method.

Soln:

Given that,

$$x^3-12x^2-6x-10=0 \longrightarrow (1)$$

Let, $x=(y+h)$ in eq (1)

$$y^3+3y^2(h-4)+3y(h^2-8h-2)+h^3-12h^2-6h-10=0$$

If, $h-4=0$, $\Rightarrow x=4$

$$\therefore y^3-54y-162=0 \longrightarrow (11)$$

Let, $y=(m+n)$

~~$y^3-3mny-(m^3+n^3)=0$~~

$$y^3=(m+n)^3$$
$$=m^3+n^3+3mn(m+n)$$

$= m^3+n^3+3mny$

$= y^3-3mny-(m^3+n^3)=0 \longrightarrow$ (iii)

Comparing (ii) and (iii), $mn=18$, $m^3+n^3=162$

Now $(m^3-n^3)^2=(m^3+n^3)^2-4m^3n^3$

$$= (162)^2-4(18)^3$$

$(m^3+n^3)^2 = 2916$

$\therefore m^3+n^3 = \pm 54$

$m^3-n^3=54$ [Taking positive sign]

Here $m^3+n^3=162$

$m^3-n^3=54$

$2m^3 = 216$

$\Rightarrow m^3 = 108$

$\Rightarrow m = 3\sqrt[3]{4},\ 3\sqrt[3]{4}\,\omega,\ 3\sqrt[3]{4}\,\omega^2$

Here $mn=18$

$n = \frac{18}{m}$

$\therefore n = \frac{6}{\sqrt[3]{4}},\ \frac{6}{\sqrt[3]{4}\,\omega},\ \frac{6}{\sqrt[3]{4}\,\omega^2}$

$\Rightarrow n = \frac{6}{\sqrt[3]{4}},\ \frac{6}{\sqrt[3]{4}}\omega^2,\ \frac{6}{\sqrt[3]{4}}\omega$

Since $y = m + n$

$$= 3\cdot\sqrt[3]{4} + \frac{6}{\sqrt[3]{4}},\ 3\cdot\sqrt[3]{4}\,\omega + \frac{6}{\sqrt[3]{4}},\ 3\sqrt[3]{4}\,\omega^2 + \frac{6}{\sqrt[3]{4}}\,\omega$$

Also,

$$x = y + h$$

$$= \frac{3\sqrt[3]{8} + 6}{\sqrt[3]{4}} + 4,\ \frac{3\sqrt[3]{8}\,\omega + 6\omega^2}{\sqrt[3]{4}} + 4,\ \frac{3\sqrt[3]{8}\,\omega^2 + 6\omega}{\sqrt[3]{4}} + 4$$

Ans.