Ramisha

Area of volume → 1 set

01. Find the area of the quadrant of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ between the major and minor axes. 2012

Sol^n: Given that,

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \longrightarrow (i)$$

or, $\frac{y^2}{b^2}=1-\frac{x^2}{a^2}$

or, $y^2=b^2\left(\frac{a^2-x^2}{a^2}\right)$

or, $y=\frac{b}{a}\sqrt{a^2-x^2}$

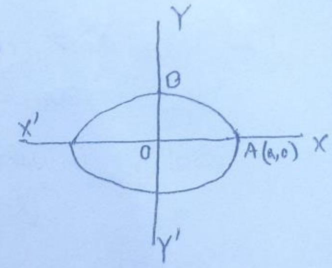


Figure: 01

Put $y=0$ in eq^n (i) we get

$x=\pm a$

Clearly, the area being bounded by the curve, the x-axis and the ordinates $x=0$ and $x=a$.

$\therefore$ The Required Area = Area of OABO

$$=\int_0^a \frac{b}{a}\sqrt{a^2-x^2}\,dx$$

$$=\frac{b}{a}\int_0^a \sqrt{a^2-x^2}\,dx$$

$$=\frac{b}{a}\int_0^{\pi/2} \sqrt{a^2(1-\sin^2\theta)}\;a\cos\theta\,d\theta$$

Put $x=a\sin\theta$

$dx=a\cos\theta\,d\theta$

x	0	a
θ	0	$\pi/2$

$$=\frac{b}{a}\int_0^{\pi/2} a^2\cos^2\theta\,d\theta$$

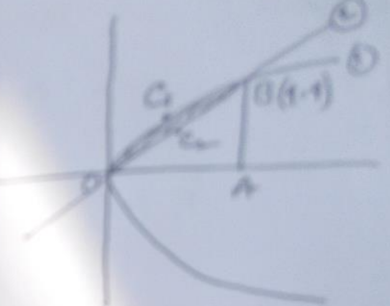
$$=ab\int_0^{\pi/2}\cos^2\theta\,d\theta$$

$$=ab\cdot\frac{1}{2}\frac{\pi}{2}=\frac{1}{4}\pi ab$$ (Answer)

* এটি Problem টি Find the area of $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ হয়, তাহলে

The Required Area = $4 \times$ area of OABO

02. Find the area of the segment cut off from $y^2=4x$ by the line $y=x$



Solⁿ: Given, $y^2=4x \longrightarrow ①$

& $y=x \longrightarrow ②$

Put $y=x$ in eqⁿ ① we get

$x^2=4x$

or, $x(x-4)=0$

$\therefore x=0$ or, $x=4$

The required Area = Area of $OABC_1O$ − Area of $OABC_2O$

$$= \int_0^4 \sqrt{4x}\,dx - \int_0^4 x\,dx$$

$$= 2\int_0^4 \sqrt{x}\,dx - \left[\frac{x^2}{2}\right]_0^4$$

$$= 2\left[\frac{x^{3/2}}{3/2}\right]_0^4 - \frac{16}{2}$$

$$= 2\,\frac{2}{3}\,4\sqrt{4} - 8$$

$$= \frac{32}{3} - 8$$

$$= \frac{8}{3} \text{ (Answer)}$$

03. 2016 W.S Show that the area bounded by the parabolas $y^2=4ax$ and $x^2=4ay$ is $\frac{16}{3}a^2$.

Sol: Given that $y^2=4ax \longrightarrow ①$

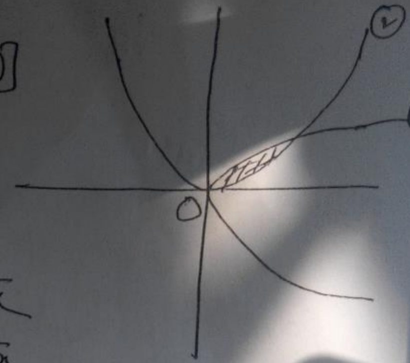
and $x^2=4ay \longrightarrow ②$

or, $(x^2)^2 = (4a)^2 y^2$

or, $x^4 = (4a)^2 \cdot 4a x$ [Using ①]

or, $x\{x^3 - (4a)^3\} = 0$

$\therefore x = 0$, or, $x = 4a$



Now from eqⁿ ① $y = \sqrt{4ax}$

$\therefore y = 2\sqrt{a}\sqrt{x}$

& from eqⁿ ② $y = \frac{x^2}{4a}$

The required area bounded by ① and ② is

$$= \int_0^{4a} 2\sqrt{a}\sqrt{x}\,dx - \int_0^{4a} \frac{x^2}{4a}\,dx$$

$$= 2\sqrt{a}\int_0^{4a} x^{\frac{1}{2}}\,dx - \frac{1}{4a}\int_0^{4a} x^2\,dx$$

$$= 2\sqrt{a}\left[\frac{x^{3/2}}{3/2}\right]_0^{4a} - \frac{1}{4a}\left[\frac{x^3}{3}\right]_0^{4a}$$

$$= 2\sqrt{a}\,\frac{2}{3}\cdot 4a\sqrt{4a} - \frac{1}{4a}\cdot\frac{1}{3}(4a)^3$$

$$= \frac{16a}{3}\sqrt{a}\cdot 2\sqrt{a} - \frac{1}{3}16a^2$$

$$= \frac{32}{3}a^2 - \frac{16}{3}a^2$$

$$= \frac{16}{3}a^2$$

Answer

2015

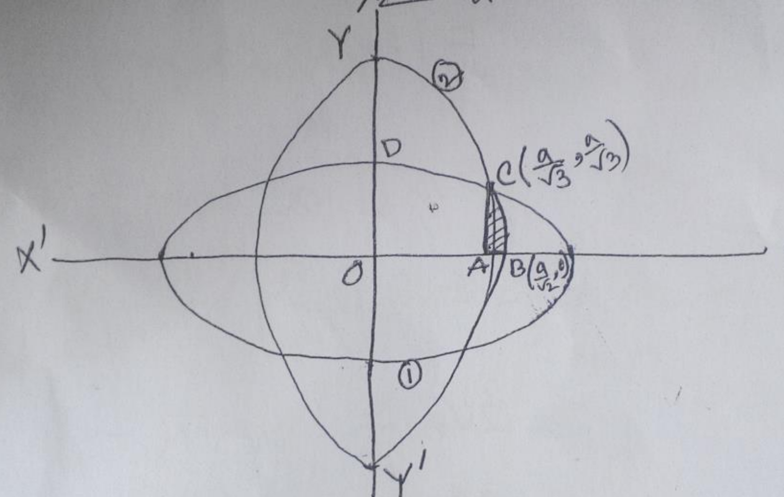
V.V.I 04. Find the area included between the ellipses $x^2+2y^2=a^2$ and $2x^2+y^2=a^2$.

Solution: Given, $x^2+2y^2=a^2 \longrightarrow (A)$

or, $\frac{x^2}{a^2}+\frac{y^2}{a^2/2}=1 \longrightarrow (1)$

and $2x^2+y^2=a^2 \longrightarrow (B)$

or; $\frac{x^2}{a^2/2}+\frac{y^2}{a^2}=1 \longrightarrow (2)$



From eqⁿ (A) & (B) $x^2+2y^2=2x^2+y^2$

or, $x^2+y^2-2y^2=0$

or, $x^2-y^2=0$

or, $y^2=x^2$

Put $y^2=x^2$ in (A) we get $3x^2=a^2$

$\therefore x=\frac{a}{\sqrt{3}}$

Therefore the abscissa (ভুজ) of C is $\frac{a}{\sqrt{3}}$

Now Put $y=0$ in eqⁿ (2) we get $x^2=\frac{a^2}{2}$

$\therefore x=\frac{a}{\sqrt{2}}$

Hence the point B is $\left(\frac{a}{\sqrt{2}}, 0\right)$.

The required area $= 4 \times$ Area of OABC...

$= 4 \times$ [Area of OACDO + Area of ABCA]

$$= 4 \times \left[\int_0^{a/\sqrt{3}} \{y \text{ for } ①\} \, dx + \int_{a/\sqrt{3}}^{a/\sqrt{2}} \{y \text{ for } ②\} \, dx \right]$$

$$= 4 \times \left\{ \int_0^{a/\sqrt{3}} \frac{1}{\sqrt{2}} \sqrt{a^2 - x^2} \, dx + \int_{a/\sqrt{3}}^{a/\sqrt{2}} \sqrt{a^2 - 2x^2} \, dx \right\}$$

$$= \frac{4}{\sqrt{2}} \int_0^{a/\sqrt{3}} \sqrt{a^2 - x^2} \, dx + 4 \int_{a/\sqrt{3}}^{a/\sqrt{2}} \sqrt{a^2 - 2x^2} \, dx$$

$$= I_1 + I_2 \longrightarrow ③$$

Where, $I_1 = \frac{4}{\sqrt{2}} \int_0^{\frac{a}{\sqrt{3}}} \sqrt{a^2 - x^2} \, dx$, & $I_2 = 4 \int_{\frac{a}{\sqrt{3}}}^{a/\sqrt{2}} \sqrt{a^2 - 2x^2} \, dx$

$$\therefore I_1 = \frac{4}{\sqrt{2}} \left[\frac{x\sqrt{a^2 - x^2}}{2} + \frac{a^2}{2} \sin^{-1} \frac{x}{a} \right]_0^{\frac{a}{\sqrt{3}}}$$

$$= \frac{4}{\sqrt{2}} \left[\frac{a/\sqrt{3} \sqrt{a^2 - \frac{a^2}{3}}}{2} + \frac{a^2}{2} \sin^{-1} \frac{a/\sqrt{3}}{a} \right]$$

$$= \frac{2}{\sqrt{2}} \left[\frac{a}{\sqrt{3}} \cdot \sqrt{\frac{3a^2 - a^2}{3}} + a^2 \sin^{-1} \frac{1}{\sqrt{3}} \right]$$

$$= \frac{2}{\sqrt{2}} \left\{ \frac{a}{\sqrt{3}} \cdot \sqrt{\frac{2a^2}{3}} + a^2 \sin^{-1} \frac{1}{\sqrt{3}} \right\}$$

$$= \frac{\sqrt{2}}{} \left\{ \frac{a}{\sqrt{3}} \cdot \frac{\sqrt{2}a}{\sqrt{3}} + a^2 \sin^{-1} \frac{1}{\sqrt{3}} \right\}$$

$$I_1 = \frac{2}{3} a^2 + \sqrt{2} a^2 \sin^{-1} \frac{1}{\sqrt{3}}$$

Also, $I_2 = 4\int_{a/\sqrt{3}}^{a/\sqrt{2}} \sqrt{a^2-2x^2}\,dx$

$= 4\int_{a\sqrt{2/3}}^{a} \sqrt{a^2-t^2}\,\frac{1}{\sqrt{2}}\,dt$

$= \frac{4}{\sqrt{2}}\int_{a\sqrt{2/3}}^{a} \sqrt{a^2-t^2}\,dt$

$= \frac{4}{\sqrt{2}}\left[\frac{t\sqrt{a^2-t^2}}{2}+\frac{a^2}{2}\sin^{-1}\frac{t}{a}\right]_{a\sqrt{2/3}}^{a}$

$= \frac{4}{\sqrt{2}}\left[0+\frac{a^2}{2}\sin^{-1}\frac{a}{a}-\left\{\frac{a\sqrt{\frac{2}{3}}\sqrt{a^2-a^2\frac{2}{3}}}{2}+\frac{a^2}{2}\sin^{-1}\frac{a\sqrt{\frac{2}{3}}}{a}\right\}\right]$

$= \frac{4}{\sqrt{2}}\left\{+\frac{a^2}{2}\frac{\pi}{2}-\frac{a\sqrt{2/3}\sqrt{\frac{a^2}{3}}}{2}-\frac{a^2}{2}\sin^{-1}\sqrt{\frac{2}{3}}\right\}$

$= a^2\frac{\pi}{\sqrt{2}}-\frac{2}{\sqrt{2}}a^2\cdot\frac{\sqrt{2}}{3}-\frac{4}{\sqrt{2}}\frac{a^2}{2}\sin^{-1}\sqrt{\frac{2}{3}}$

$= \frac{a^2}{\sqrt{2}}\pi-\frac{2}{3}a^2-\sqrt{2}a^2\sin^{-1}\sqrt{\frac{2}{3}}$

$= -\frac{2}{3}a^2+\sqrt{2}a^2\left(\frac{\pi}{2}-\sin^{-1}\sqrt{\frac{2}{3}}\right)$

$= -\frac{2}{3}a^2+\sqrt{2}a^2\theta$

$\therefore I_2 = -\frac{2}{3}a^2+\sqrt{2}a^2\sin^{-1}\frac{1}{\sqrt{3}}$

Putting the value of I_1 & I_2 in eqn (3)

$\therefore$ The Area $= \frac{2}{3}a^2+\sqrt{2}a^2\sin^{-1}\frac{1}{\sqrt{3}}-\frac{2}{3}a^2+\sqrt{2}a^2\sin^{-1}\frac{1}{\sqrt{3}}$

$= 2\sqrt{2}a^2\sin^{-1}\frac{1}{\sqrt{3}}$. Answer

Put

$2x^2 = t^2$

$x^2 = \frac{t^2}{2}$

$x = \frac{t}{\sqrt{2}}$

$\therefore dx = \frac{1}{\sqrt{2}}dt$

x	$a/\sqrt{3}$	$a/\sqrt{2}$
t	$a\sqrt{2/3}$	a

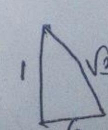
Again let

$\frac{\pi}{2}-\sin^{-1}\sqrt{2/3} = \theta$

$\therefore \frac{\pi}{2}-\theta = \sin^{-1}\sqrt{2/3}$

$\sin(\pi/2-\theta) = \sin\sin^{-1}\sqrt{2/3}$

$\cos\theta = \sqrt{2/3}$



$\sin\theta = \frac{1}{\sqrt{3}}$

$\theta = \sin^{-1}\frac{1}{\sqrt{3}}$

Rule: $\frac{ds}{dx} = \sqrt{1+\left(\frac{dy}{dx}\right)^2}$, S is the length of the arc (বৃত্ত)

$\int \frac{d}{dx}(s) = \int \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx \quad \therefore S = \int \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$

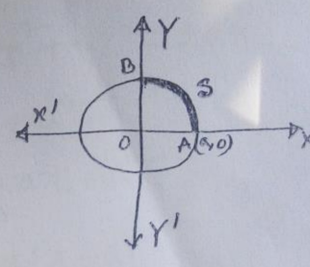
05. Find the Perimeter of the circle $x^2+y^2=a^2$. 2011

Solution: Given $x^2+y^2=a^2 \longrightarrow ①$

$\therefore y^2 = a^2-x^2 \Rightarrow y=\sqrt{a^2-x^2}$

Put $y=0$ we get $x=\pm a$

Diff. ① with respect to x



$2x + 2y\frac{dy}{dx} = 0$

$\therefore \frac{dy}{dx} = -\frac{x}{y}$

We know $\frac{ds}{dx} = \sqrt{1+\left(\frac{dy}{dx}\right)^2} = \sqrt{1+\frac{x^2}{y^2}} = \sqrt{\frac{y^2+x^2}{y^2}}$

$\Rightarrow \frac{ds}{dx} = \sqrt{\frac{a^2}{y^2}}$ [using ①]

$\therefore \frac{ds}{dx} = \frac{a}{y}$

Required Perimeter $= 4 \times$ The arc AB in the first Quadrant

$= 4 \times \int_0^a \frac{ds}{dx}\, dx$

$= 4\int_0^a \frac{a}{y}\, dx = 4a\int_0^a \frac{dx}{\sqrt{a^2-x^2}}$

$= 4a\left[\sin^{-1}\frac{x}{a}\right]_0^a$

$= 4a\left(\sin^{-1}1 - \sin^{-1}0\right) = 4a\frac{\pi}{2}$

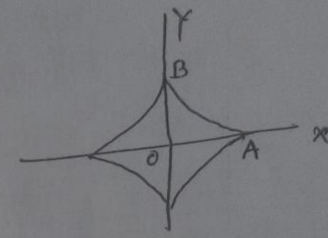
$= 2\pi a$ Answer.

06. Find the Perimeter of the astroid $x^{2/3}+y^{2/3}=a^{2/3}$

Sol^n: Given that $x^{2/3}+y^{2/3}=a^{2/3} \longrightarrow (1)$

Put $y=0$ we get $x=a$

Diff. (1) w. r. to x



$$\frac{2}{3}x^{-\frac{1}{3}}+\frac{2}{3}y^{-\frac{1}{3}}\frac{dy}{dx}=0$$

or, $$\frac{dy}{dx}=-\frac{y^{1/3}}{x^{1/3}}$$

We know $$\frac{ds}{dx}=\sqrt{1+\left(\frac{dy}{dx}\right)^2}=\sqrt{1+\frac{y^{2/3}}{x^{2/3}}}=\sqrt{\frac{x^{2/3}+y^{2/3}}{x^{2/3}}}$$

$$=\sqrt{\frac{a^{2/3}}{x^{2/3}}}=\frac{a^{\frac{1}{3}}}{x^{1/3}} \quad [\text{Using } (1)]$$

$\therefore$ The Required Perimeter $$=4\int_0^a \frac{ds}{dx}\,dx$$

$$=4\int_0^a \frac{a^{1/3}}{x^{1/3}}\,dx$$

$$=4a^{1/3}\int_0^a x^{-1/3}\,dx$$

$$=4a^{1/3}\left[\frac{x^{2/3}}{2/3}\right]_0^a \qquad \left|\ x^{-\frac{1}{3}+1}=x^{\frac{2}{3}}\right.$$

$$=4a^{\frac{1}{3}}\frac{3}{2}\cdot a^{2/3}$$

$$=6a^{\frac{1}{3}+\frac{2}{3}}=6a^{3/3}$$

$$=6a \text{ Answer}$$

2013, 2012, 2016

07. If S be the lenght of an arc of $3ay^2 = x(x-a)^2$ measured from the origin to the point (x,y) show that

vvF

$$3S^2 = 4x^2 + 3y^2$$

Solⁿ: Given that,

$$3ay^2 = x(x-a)^2 \longrightarrow (1)$$

Diff (1) w. r to x

$$3a \cdot 2y \frac{dy}{dx} = 1 \cdot (x-a)^2 + x \cdot 2(x-a) \cdot 1$$

$$\text{or, } 6ay \frac{dy}{dx} = x^2 - 2ax + a^2 + 2x^2 - 2ax$$

$$\text{or, } 6ay \frac{dy}{dx} = 3x^2 - 4ax + a^2$$

$$= 3x^2 - 3ax - ax + a^2$$

$$= 3x(x-a) - a(x-a)$$

$$= (3x-a)(x-a)$$

$$\therefore \frac{dy}{dx} = \frac{(3x-a)(x-a)}{6ay}$$

we know

$$\frac{ds}{dx} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \sqrt{1 + \frac{(3x-a)^2 (x-a)^2}{36a^2y^2}}$$

$$\cancel{\frac{ds}{dx} = \sqrt{\frac{36a^2y^2 + (9x^2 + a^2 - 6ax)(x^2 + a^2 - 2ax)}{36a^2y^2}}}$$

$$= \sqrt{1 + \frac{(3x-a)^2 (x-a)^2}{12a \cdot 3ay^2}} = \sqrt{1 + \frac{(3x-a)^2(x-a)^2}{12a \cdot x(x-a)^2}}$$

[Using (1)]

$$= \sqrt{\frac{12ax + (3x-a)^2}{12ax}}$$

$$= \frac{\sqrt{12ax + 9x^2 - 6ax + a^2}}{\sqrt{12ax}} = \frac{\sqrt{9x^2 + 6ax + a^2}}{\sqrt{12ax}}$$

$$\frac{ds}{dx} = \frac{\sqrt{(3x+a)^2}}{\sqrt{12ax}} = \frac{3x+a}{\sqrt{12ax}}$$

Since s be the arc length of ① measured fro(m) (0,0) to (x, y)

Hence $$S = \int_0^x \frac{3x+a}{\sqrt{12ax}}\,dx$$

$$= \frac{1}{\sqrt{12a}}\left\{\int_0^x \frac{3x}{\sqrt{x}}dx + \int_0^x \frac{a}{\sqrt{x}}dx\right\}$$

$$= \frac{1}{2\sqrt{3a}}\left\{\int_0^x 3x^{\frac{1}{2}}dx + a\int_0^x \frac{dx}{\sqrt{x}}\right\}$$

$$= \frac{1}{2\sqrt{3a}}\left\{3\frac{x^{3/2}}{3/2}\Big|_0^x + a\cdot 2\sqrt{x}\Big|_0^x\right\}$$

$$= \frac{1}{2\sqrt{3a}}\, 3\cdot\frac{2}{3}x^{3/2} + \frac{1}{2\sqrt{3a}}\cdot 2a\sqrt{x}$$

$$= \frac{1}{\sqrt{3a}}\, x^{3/2} + \frac{a}{\sqrt{3a}}\sqrt{x}$$

$$S = \frac{1}{\sqrt{3a}}\sqrt{x}(x+a)$$

$$\therefore \sqrt{3a}\,S = \sqrt{x}(x+a)$$

$$3aS^2 = x(x+a)^2 = x\{(x-a)^2 + 4ax\}$$

$$\Rightarrow 3aS^2 = x(x-a)^2 + 4ax^2$$

or, $3aS^2 = 3ay^2 + 4ax^2$ [Using ①]

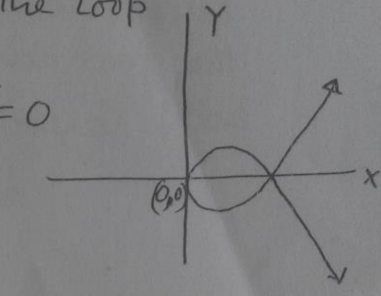
$$\therefore 3S^2 = 3y^2 + 4x^2$$

(Proved)

8. Find the whole length of the loop of the curve
$$3ay^2 = x(x-a)^2$$

Sol^n: Given that, $3ay^2 = x(x-a)^2 \longrightarrow (1)$

We see that Degree of y is even and highest degree of x is odd. Hence the Loop is symmetric about x-axis.



Put $y=0$ in (1) we get $x(x-a)^2=0$

$\therefore x=0$ and $x=a$

when $x=0$, then $y=0$

and when $x=a$, then $y=0$

Hence the loop is measured from $(0,0)$ to $(a,0)$

Now. Diff. (1) w.r.to x

$$6ay\frac{dy}{dx} = (x-a)(3x-a)$$

$$\frac{dy}{dx} = \frac{(x-a)(3x-a)}{6ay}$$

* See Calculation in the previous Problem (7)

$$\therefore \frac{ds}{dx} = \frac{3x+a}{\sqrt{12ax}} = \frac{3x+a}{2\sqrt{3a}\cdot\sqrt{x}}$$

The length of the loop is $= 2\int_0^a \frac{3x+a}{2\sqrt{3a}\sqrt{x}}\,dx$

$$= \frac{2}{2\sqrt{3a}}\int_0^a\left(\frac{3x}{\sqrt{x}} + \frac{a}{\sqrt{x}}\right)dx = \frac{1}{\sqrt{3a}}\int\left(3\sqrt{x}+a\frac{1}{\sqrt{x}}\right)dx$$

$$= \frac{1}{\sqrt{3a}}\left(3\cdot x^{3/2}\frac{2}{3} + a\,2\sqrt{x}\right)\Big|_0^a$$

$$= \frac{1}{\sqrt{3a}}\left(2a^{3/2} + 2a\sqrt{a}\right) = \frac{1}{\sqrt{3a}}\,4a\sqrt{a} = \frac{4}{\sqrt{3}}a$$

Ans

2015

9. Show that the length of the arc of the evolute $27ay^2 = 4(x-2a)^3$ of the parabola $y^2 = 4ax$, from the cusp to one of the points where the evolute meets the parabola is $2a(3\sqrt{3}-1)$.

Solⁿ: Given, $27ay^2 = 4(x-2a)^3 \longrightarrow (1)$

Differentiating w. r to x we get

$$27a \cdot 2y \frac{dy}{dx} = 12(x-2a)^2$$

$$\text{or, } \frac{dy}{dx} = \frac{6(x-2a)^2}{27ay} = \frac{2(x-2a)^2}{9ay}$$

$$\therefore 1 + \left(\frac{dy}{dx}\right)^2 = 1 + \frac{4(x-2a)^4}{81a^2y^2}$$

$$= 1 + \frac{4(x-2a)^4}{81 \cdot a^2 \cdot \frac{4(x-2a)^3}{27a}} \qquad \text{Put } y^2 = \frac{4(x-2a)^3}{27a}$$

$$= 1 + \frac{(x-2a)}{3 \cdot a}$$

$$= \frac{3a+x-2a}{3a} = \frac{x+a}{3a}$$

The Parabola $y^2 = 4ax$ meets eqⁿ (1)

$$\therefore 27a \cdot 4ax = 4(x-2a)^3$$

$$\text{or, } 27a^2x = x^3 - 3 \cdot 2a \cdot x^2 + 3x(2a)^2 - (2a)^3$$

$$\text{or, } x^3 - 6ax^2 + 12a^2x - 27a^2x - 8a^3 = 0$$

$$\text{or, } x^3 - 6ax^2 - 15a^2x - 8a^3 = 0$$

or, $x^3 + ax^2 - 7ax^2 - 7a^2x - 8a^2x - 8a^3 = 0$

or, $x^2(x+a) - 7ax(x+a) - 8a^2(x+a) = 0$

$\Rightarrow (x+a)(x^2 - 7ax - 8a^2) = 0$

$\Rightarrow (x+a)\{x^2 - 8ax + ax - 8a^2\} = 0$

$\Rightarrow (x+a)\{x(x-8a) + a(x-8a)\} = 0$

$\Rightarrow (x+a)(x+a)(x-8a) = 0$

or, $x = -a$, $x = 8a$

Put $y=0$ in ① we get $x = 2a$

Cusp of the curve is $x = 2a$

So the limits will be $2a$ to $8a$

The required arc length $= \displaystyle\int_{2a}^{8a} \sqrt{\left(\frac{x+a}{3a}\right)}\, dx$

$$= \frac{1}{\sqrt{3a}} \int_{2a}^{8a} \sqrt{x+a}\, dx$$

$$= \frac{1}{\sqrt{3a}} \left[(x+a)^{3/2} \cdot \frac{2}{3} \right]_{2a}^{8a}$$

$$= \frac{2}{3\sqrt{3a}} \left[(9a)^{3/2} - (3a)^{3/2} \right]$$

$$= \frac{2}{3\sqrt{3a}} (3a)^{3/2} \left\{ 3^{3/2} - 1 \right\} = \frac{2}{3\sqrt{3a}} 3a\sqrt{3a}(3\sqrt{3} - 1)$$

$$= 2a(3\sqrt{3} - 1)$$

(Proved)

2013

10. Find the volume and area of the curved surface of a paraboloid of revolution formed by revolving the parabola $y^2 = 4ax$ about the x-axis, and bounded by the section $x = x_1$.

Solution: Given that $y^2 = 4ax \longrightarrow (1)$

$$y = 2\sqrt{a}\sqrt{x}$$

$$\therefore \frac{dy}{dx} = 2\sqrt{a}\,\frac{1}{2}x^{-\frac{1}{2}}$$

$$= \sqrt{\frac{a}{x}}$$

Now the required volume,

$$V = \pi \int_0^{x_1} y^2 dx \quad \text{(Rule)}$$

$$= \pi \int_0^{x_1} 4ax\, dx \quad [\text{Using eq}^n (1)]$$

$$= 4a\pi \left. \frac{x^2}{2} \right|_0^{x_1}$$

$$= 2a\pi\, x_1^2$$

Also, the required surface-area:

$$S = 2\pi \int_0^{x_1} y \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx \quad \text{(Rule)}$$

$$= 2\pi \int_0^{x_1} 2\sqrt{ax}\sqrt{1+\frac{a}{x}}\, dx$$

$$= 4\pi\sqrt{a} \int_0^{x_1} \sqrt{x}\, \frac{\sqrt{a+x}}{\sqrt{x}}\, dx = 4\pi\sqrt{a} \int_0^{x_1} (a+x)^{\frac{1}{2}} dx$$

$$= 4\pi\sqrt{a} \left[\frac{2}{3}(a+x)^{\frac{3}{2}}\right]_0^{x_1} = \frac{8}{3}\pi\sqrt{a}\left\{(a+x_1)^{3/2} - a^{3/2}\right\}$$

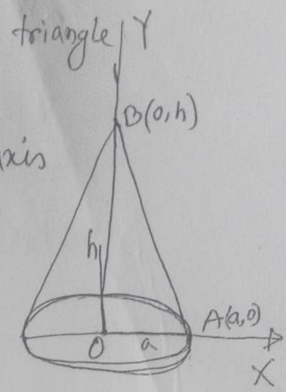
$$= \frac{8}{3}\pi\sqrt{a}\left\{(a+x_1)^{3/2} - a^{3/2}\right\}$$ Answer.

11. Show that the volume of a right circular cone of height h and base of radius "a" is $\frac{1}{3}\pi a^2 h$.

Solⁿ: Let OAB be the right angled triangle in which $OA = a$, OB = altitude = h,

Suppose O is the origin, OA the x-axis and OB, the y axis.

Then since $OA = a$, $OB = h$, then the equation of the line AB is



$$\frac{x}{a} + \frac{y}{h} = 1$$

$$\therefore \frac{x}{a} = 1 - \frac{y}{h}$$

$$x = a\left(1 - \frac{y}{h}\right)$$

Now the Required volume $= \int_0^h \pi x^2 \, dy$

$$= \pi \int_0^h a^2 \left(1 - \frac{y}{h}\right)^2 dy$$

$$= \pi a^2 \int_0^h \left(1 - 2\frac{y}{h} + \frac{y^2}{h^2}\right) dy$$

$$= \pi a^2 \left[y - \frac{2}{h}\frac{y^2}{2} + \frac{1}{h^2}\cdot\frac{y^3}{3}\right]_0^h$$

$$= \pi a^2 \left[h - h + \frac{1}{3}\cdot h\right]$$

$$= \frac{1}{3}\pi a^2 h$$

(Showed)

2015

12. Find the volume and the surface-area of the solid generated by revolving the cardioide $r = a(1-\cos\theta)$ or, $r = a(1+\cos\theta)$ about the initial line.