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Section A

Theory of Equations:

* $a^2+b^2+c^2 = (a+b+c)^2 - 2(ab+bc+ca)$
* $a^3+b^3+c^3 = (a+b+c)^3 - 3(a+b+c)(ab+bc+ca) + 3abc$
* $(a+b)(b+c)(c+a) = (a+b+c)(ab+bc+ca) - abc$
* $a^2b + ab^2 + b^2c + bc^2 + ca^2 + c^2a = (a+b+c)(ab+bc+ca) - 3abc$
* $a^2b^2+b^2c^2+c^2a^2 = (ab+bc+ca)^2 - 2abc(a+b+c)$
* $\sum a = a+b+c$
* $\sum ab = ab+bc+ca$
* $\sum a^2 = a^2+b^2+c^2$
* $\sum a^2b^2 = a^2b^2+b^2c^2+c^2a^2$
* $\sum a^2b = a^2b+ab^2+b^2c+bc^2+c^2a+ca^2$
* $\sum a^3b = a^3b+ab^3+b^3c+bc^3+c^3a+ca^3$
* $\sum a^2bc = a^2bc+ab^2c+abc^2$

Theory 1: Every equation of nth degree has exactly n roots and no more.

Solution: Let $f(x) = x^n + P_1 x^{n-1} + P_2 x^{n-2} + \dots\dots + P_{n-1}x + P_n = 0$ — (1)

be an equation of n degree.

By the fundamental theorem of algebra $f(x)=0$ has a root say α.

That is say $f(\alpha)=0$ that is $(x-\alpha)$ is a factor of $f(x)$.

So, we can write $f(x) = (x-\alpha)\,\varphi_1(x)$, where $\varphi_1(x)$ is of $(n-1)$ degree.

Again $\varphi_1(x)=0$ has a root say β that is $\varphi_1(x) = (x-\beta)\varphi_2(x)$

$\therefore f(x) = (x-\alpha)(x-\beta)\varphi_2(x)$

Proceeding in this way, we can write $f(x) = (x-\alpha)(x-\beta)(x-\gamma)\dots\dots(x-K)\,\varphi_n(x)$

where, there are n linear factors on the right.

Hence $f(x)=0$ has exactly n roots say $\alpha, \beta, \gamma \dots K$ and no more.

(Proved)

ii) $\Sigma(b-c)^2 = (b-c)^2+(c-a)^2+(a-b)^2$

$= b^2-2bc+c^2+c^2-2ca+a^2+a^2-2ab+b^2$

$= 2(a^2+b^2+c^2) - 2(ab+bc+ca)$

$= 2\{(a+b+c)^2-2(ab+bc+ca)\} - 2(ab+bc+ca)$

$= 2(0^2-2p)-2p$

$= -4p-2p$

$= -6p$. Ans.

Problem 4: Find the roots of the equation $4x^3+20x^2-23x+6=0$ where two of the roots being equal.

Solution: Lets the roots are α, α and β

Given that $4x^3+20x^2-23x+6=0$

$\therefore \alpha+\alpha+\beta = -5 \Rightarrow 2\alpha+\beta = -5 \Rightarrow \beta = -(5+2\alpha)$ —— (i)

$\alpha\alpha+\alpha\beta+\beta\alpha = -23/4 \Rightarrow \alpha^2+2\alpha\beta = -23/4$ —— (ii)

and $\alpha\alpha\beta = -3/2 \Rightarrow \alpha^2\beta = -3/2$ —— (iii)

From equation (ii), we get

$$\alpha^2+2\alpha\beta = -23/4$$

$\Rightarrow \alpha^2-2\alpha(5+2\alpha) = -23/4$

$\Rightarrow \alpha^2-10\alpha-4\alpha^2 = -23/4$

$\Rightarrow 3\alpha^2+10\alpha = 23/4$

$\Rightarrow 12\alpha^2+40\alpha-23=0$

$\Rightarrow 12\alpha^2+46\alpha-6\alpha-23=0$

$\Rightarrow 2\alpha(6\alpha+23)-1(6\alpha+23)=0$

$\Rightarrow (6\alpha+23)(2\alpha-1)=0$

$\Rightarrow \alpha = -23/6$ and $\alpha = 1/2$

Now from (i), when $\alpha = -23/6$ then $\beta = -(5+2\times -23/6)$

$= -(5-\frac{23}{3}) = 8/3$ } Ans

and when $\alpha = 1/2$ then $\beta = -(5+2\times\frac{1}{2}) = -6$

Problem 5: Find the roots of the equation $8x^4-2x^3-27x^2+6x+9=0$ where the of the roots being equal but opposite in sign.

Solution: Given that $8x^4-2x^3-27x^2+6x+9=0$

Let the roots are $\alpha, -\alpha, \beta, \gamma$

$\therefore \alpha - \alpha + \beta + \gamma = \frac{2}{8} = \frac{1}{4} \Rightarrow \beta + \gamma = \frac{1}{4}$ —— (i)

$-\alpha^2 + \cancel{\alpha\beta} + \cancel{\alpha\gamma} - \cancel{\alpha\beta} - \cancel{\alpha\gamma} + \beta\gamma = -\frac{27}{8} \Rightarrow -\alpha^2 + \beta\gamma = -\frac{27}{8}$ —— (ii)

$-\alpha^2\beta - \alpha^2\gamma - \cancel{\alpha\beta\gamma} + \cancel{\alpha\beta\gamma} = -6/8 = -3/4 \Rightarrow -(\alpha^2\beta + \alpha^2\gamma) = -3/4$

$\Rightarrow \alpha^2(\beta+\gamma) = 3/4$ —— (iii)

and $-\alpha^2\beta\gamma = 9/8$ —— (iv)

Now, (iii) ÷ (i) $\Rightarrow \frac{\alpha^2(\beta+\gamma)}{(\beta+\gamma)} = \frac{3}{4} \times \frac{4}{1} \Rightarrow \alpha^2 = 3 \Rightarrow \alpha = \pm\sqrt{3}$

From (iv), we get $-\alpha^2\beta\gamma = \frac{9}{8} \Rightarrow -3\beta\gamma = \frac{9}{8} \Rightarrow \beta\gamma = -\frac{3}{8}$

$\Rightarrow \beta = \frac{-3}{8\gamma}$ —— (v)

Putting $\beta = -\frac{3}{8\gamma}$ in equation (i), we get

$$-\frac{3}{8\gamma} + \gamma = \frac{1}{4}$$

$$\Rightarrow \frac{-3+8\gamma^2}{8\gamma} = \frac{1}{4}$$

$$\Rightarrow \frac{-3+8\gamma^2}{2\gamma} = 1$$

$$\Rightarrow -3+8\gamma^2 = 2\gamma$$

$$\Rightarrow 8\gamma^2 - 2\gamma - 3 = 0$$

$$\Rightarrow 8\gamma^2 - 6\gamma + 4\gamma - 3 = 0$$

$$\Rightarrow 2\gamma(4\gamma-3) + 1(4\gamma-3) = 0$$

$$\Rightarrow (4\gamma-3)(2\gamma+1) = 0$$

$$\Rightarrow \gamma = 3/4 \text{ and } \gamma = -1/2$$

Putting in equation (v), when $\gamma = 3/4$ then $\beta = \frac{-3}{8\times\frac{3}{4}} = -1/2$

and when $\gamma = -1/2$ then $\beta = \frac{-3}{8\times -1/2} = 3/4$

$\therefore$ The roots are $\pm\sqrt{3}, 3/4, -1/2$ Ans.

Alternative / Previous Theorem 2: Find the relation between roots and coefficients.

Solution: Let $f(x) = p_0x^n + p_1x^{n-1} + p_2x^{n-2} + \ldots\ldots + p_n = 0$ be the equation of degree n and $\alpha_1, \alpha_2 \ldots \alpha_n$ be the roots.

Hence, we may write

$$p_0(x-\alpha_1)(x-\alpha_2)\ldots\ldots(x-\alpha_n) = p_0x^n + p_1x^{n-1} + p_2x^{n-2} + \ldots\ldots + p_n$$

$$\Rightarrow (x-\alpha_1)(x-\alpha_2)\ldots\ldots(x-\alpha_n) = x^n + \frac{p_1}{p_0}x^{n-1} + \frac{p_2}{p_0}x^{n-2} + \ldots\ldots + \frac{p_n}{p_0}$$

$$\Rightarrow x^n - \Sigma\alpha_1 x^{n-1} + \Sigma\alpha_1\alpha_2 x^{n-2} + \ldots\ldots + (-1)^n\alpha_1\alpha_2\ldots\alpha_n = x^n + \frac{p_1}{p_0}x^{n-1} + \frac{p_2}{p_0}x^{n-2} + \ldots\ldots + \frac{p_n}{p_0}$$

Equating the coefficient of like power of x, we get

$$\Sigma\alpha_1 = -\frac{p_1}{p_0}\ ;\ \Sigma\alpha_1\alpha_2 = \frac{p_2}{p_0}$$

$$\Sigma\alpha_1\alpha_2\alpha_3 = -\frac{p_3}{p_0}$$

$$\ldots\ldots\ldots\ldots$$

$$\alpha_1\alpha_2\ldots\alpha_n = (-1)^n\frac{p_n}{p_0}$$

which is the relation between roots and coefficients.

Solution of cubic equation:: #ICE# // Gokul Chandra Roy // 01750-322800 //

Ex: Solve the cubic equation $x^3 - 18x - 35 = 0$ by cardan's method.

Solution: Given that $x^3 - 18x - 35 = 0$ ——— (i)

Put $x = m+n$ then $x^3 = (m+n)^3 = m^3 + n^3 + 3mn(m+n)$

$\Rightarrow x^3 - 3mn(m+n) - (m^3+n^3) = 0$

$\Rightarrow x^3 - 3mnx - (m^3+n^3) = 0$ ——— (ii)

Compairing equation (i) and (ii), we get

$-3mn = -18$ and $-(m^3+n^3) = -35$

$\Rightarrow mn = 6$ and $m^3+n^3 = 35$

$\Rightarrow (mn)^3 = 6^3$

$\Rightarrow m^3 n^3 = 216$

If m^3 and n^3 are the roots of the equation, then we can write $t^2 - (m^3+n^3)t + m^3 n^3 = 0$

$\Rightarrow t^2 - 35t + 216 = 0$

$\Rightarrow t = \dfrac{-(-35) \pm \sqrt{(-35)^2 - 4 \cdot 1 \cdot 216}}{2 \cdot 1} = \dfrac{35 \pm \sqrt{361}}{2} = \dfrac{35 \pm 19}{2}$

$\therefore t = 8$ and $t = 27$

$\therefore m^3 = 8$ and $n^3 = 27$

$\Rightarrow m = 2$ and $n = 3$

$\therefore$ ~~$x = m+n = 2+3 = 5$~~

Hence, the roots are x, $m\omega + n\omega^2$ and $m\omega^2 + n\omega$

$\therefore x = m+n = 2+3 = 5$

$m\omega + n\omega^2 = 2\left(\dfrac{-1+\sqrt{-3}}{2}\right) + 3\left(\dfrac{-1-\sqrt{-3}}{2}\right)$

$= \dfrac{-2+2\sqrt{-3}}{2} + \dfrac{-3-3\sqrt{-3}}{2}$

$= \dfrac{-2+2\sqrt{-3}-3-3\sqrt{-3}}{2} = \dfrac{-5-\sqrt{-3}}{2}$

and $m\omega^2 + n\omega = 2\left(\dfrac{-1-\sqrt{-3}}{2}\right) + 3\left(\dfrac{-1+\sqrt{-3}}{2}\right) = \dfrac{-2-2\sqrt{-3}}{2} + \dfrac{-3+3\sqrt{-3}}{2}$

$= \dfrac{-2-2\sqrt{-3}-3+3\sqrt{-3}}{2} = \dfrac{-5+\sqrt{-3}}{2}$

$\therefore$ The roots are $5, \dfrac{-5 \pm \sqrt{-3}}{2}$ (Ans.)

Ex: Solve the cubic equation $x^3-15x-126=0$

Solution: Given that $x^3-15x-126=0$ ——— (i)

Put $x=m+n$, then $x^3=(m+n)^3=m^3+n^3+3mn(m+n)$

$\Rightarrow x^3-3mn(m+n)-(m^3+n^3)=0$

$\Rightarrow x^3-3mnx-(m^3+n^3)=0$ ——— (ii)

Comparing the equation (i) and (ii), we get

$-3mn=-15$ and $-(m^3+n^3)=-126$

$\Rightarrow mn=5$ and $m^3+n^3=126$

$\Rightarrow m^3n^3=125$

If m^3 and n^3 are the roots of the equation, we can write

$t^2-(m^3+n^3)t+m^3n^3=0$

$\Rightarrow t^2-126t+125=0$

$\Rightarrow t^2-125t-t+125=0 \Rightarrow t(t-125)-1(t-125)=0$

$\Rightarrow (t-125)(t-1)=0 \Rightarrow t=125$ and $t=1$

$\therefore m^3=125$ and $n^3=1$

$\Rightarrow m=5$ and $n=1$

Hence the roots are x, $m\omega+n\omega^2$ and $m\omega^2+n\omega$

$\therefore x = m+n=5+1=6$

$$m\omega+n\omega^2=5\left(\frac{-1+\sqrt{-3}}{2}\right)+1\left(\frac{-1-\sqrt{-3}}{2}\right)=\frac{-5+5\sqrt{-3}}{2}+\frac{-1-\sqrt{-3}}{2}$$

$$=\frac{-5+5\sqrt{-3}-1-\sqrt{-3}}{2}=\frac{-6+4\sqrt{-3}}{2}=-3+2\sqrt{-3}$$

and $m\omega^2+n\omega=5\left(\frac{-1-\sqrt{-3}}{2}\right)+1\left(\frac{-1+\sqrt{-3}}{2}\right)$

$$=\frac{-5-5\sqrt{-3}}{2}+\frac{-1+\sqrt{-3}}{2}$$

$$=\frac{-5-5\sqrt{-3}-1+\sqrt{-3}}{2}=\frac{-6-4\sqrt{-3}}{2}=-3-2\sqrt{-3}$$

$\therefore$ The roots are 6, $-3\pm2\sqrt{-3}$. Ans.

5. $x^3 - 21x - 344 = 0$

Solution: Given that $x^3 - 21x - 344 = 0$ —— (i)

Put $x = m+n \Rightarrow x^3 = (m+n)^3 = m^3+n^3+3mn(m+n)$

$\Rightarrow x^3 - 3mn(m+n) - (m^3+n^3) = 0$

$\Rightarrow x^3 - 3mnx - (m^3+n^3) = 0$ —— (ii)

Comparing the equation (i) and (ii), we get

$-3mn = -21$ and $-(m^3+n^3) = -344$

$\Rightarrow mn = 7$ and $m^3+n^3 = 344$

$\Rightarrow m^3n^3 = 343$

If m^3 and n^3 are the roots of the equation, we can write

$t^2 - (m^3+n^3)t + m^3n^3 = 0$

$\Rightarrow t^2 - 344t + 343 = 0 \Rightarrow t^2 - 343t - t - 343 = 0$

$\Rightarrow t(t-343) - 1(t-343) = 0 \Rightarrow (t-343)(t-1) = 0$

$\Rightarrow t = 343$ and $t = 1$

$\therefore m^3 = 343$ and $n^3 = 1$

$\Rightarrow m = 7$ and $n = 1$

$\therefore$ Hence, the roots are x, $m\omega + n\omega^2$ and $m\omega^2 + n\omega$

$\therefore x = m+n = 7+1 = 8$

$m\omega + n\omega^2 = 7\left(\frac{-1+\sqrt{-3}}{2}\right) + 1\left(\frac{-1-\sqrt{-3}}{2}\right) = \frac{-7+7\sqrt{-3}}{2} + \frac{-1-\sqrt{-3}}{2}$

$= \frac{-7+7\sqrt{-3}-1-\sqrt{-3}}{2} = \frac{-8+6\sqrt{-3}}{2} = -4+3\sqrt{-3}$

and $m\omega^2 + n\omega = 7\left(\frac{-1-\sqrt{-3}}{2}\right) + 1\left(\frac{-1+\sqrt{-3}}{2}\right)$

$= \frac{-7-7\sqrt{-3}}{2} + \frac{-1+\sqrt{-3}}{2}$

$= \frac{-7-7\sqrt{-3}-1+\sqrt{-3}}{2} = \frac{-8-6\sqrt{-3}}{2}$

$= -4-3\sqrt{-3}$

$\therefore$ The roots are 8, $-4 \pm 3\sqrt{-3}$ Ans.

(i) $x^3+21x+342=0$

Solution: Given that $x^3+21x+342=0$ ——— (i)

put $x=m+n$ then $x^3=(m+n)^3=m^3+n^3+3mn(m+n)$

$$\Rightarrow x^3-3mn(m+n)-(m^3+n^3)=0$$

$$\Rightarrow x^3-3mnx-(m^3+n^3)=0 \text{ ——— (ii)}$$

Comparing the equation (i) and (ii), we get

$-3mn=21$ and $-(m^3+n^3)=342$

$\Rightarrow mn=-7$ and $m^3+n^3=-342$

$\Rightarrow m^3n^3=-343$

If m^3 and n^3 are the roots of equation, we can write

$t^2-(m^3+n^3)t+m^3n^3=0$

$\Rightarrow t^2+342t-343=0 \Rightarrow t^2+343t-t-343=0$

$\Rightarrow t(t+343)-1(t+343)=0 \Rightarrow (t+343)(t-1)=0$

$\Rightarrow t=-343$ and $t=1$

$\therefore m^3=-343$ and $n^3=1$

$\Rightarrow m=-7$ and $n=1$

$\therefore$ The roots of the equation are x, $m\omega+n\omega^2$ and $m\omega^2+n\omega$

$\therefore x=m+n=-7+1=-6$

$$m\omega+n\omega^2=-7\left(\frac{-1+\sqrt{-3}}{2}\right)+1\left(\frac{-1-\sqrt{-3}}{2}\right)=\frac{7-7\sqrt{-3}}{2}+\frac{-1-\sqrt{-3}}{2}$$

$$=\frac{7-7\sqrt{-3}-1-\sqrt{-3}}{2}=\frac{6-8\sqrt{-3}}{2}=3-4\sqrt{-3}$$

$$m\omega^2+n\omega=-7\left(\frac{-1-\sqrt{-3}}{2}\right)+1\left(\frac{-1+\sqrt{-3}}{2}\right)=\frac{7+7\sqrt{-3}}{2}+\frac{-1+\sqrt{-3}}{2}$$

$$=\frac{7+7\sqrt{-3}-1+\sqrt{-3}}{2}=\frac{6+8\sqrt{-3}}{2}=3+4\sqrt{-3}$$

$\therefore$ The roots are -6, $3\pm4\sqrt{-3}$

(ii) $x^3+63x-316=0$

Solution: Given that $x^3+63x-316=0$ —— ①

Put $x=m+n$ then $x^3=(m+n)^3=m^3+n^3+3mn(m+n)$

$\Rightarrow x^3-3mn(m+n)-(m^3+n^3)=0$

$\Rightarrow x^3-3mnx-(m^3+n^3)=0$

comparing the equation ① and (ii) we get

$-3mn=63$ and $-(m^3+n^3)=-316$

$\Rightarrow mn=-21$ and $m^3+n^3=316$

$\Rightarrow m^3n^3=-9261$

If m^3 and n^3 are the roots of the equation, we can write

$t^2-(m^3+n^3)t+m^3n^3=0$

$\Rightarrow t^2-316t+9261=0$

$\Rightarrow t=\dfrac{-(-316)\pm\sqrt{(-316)^2-4.1.(-9261)}}{2.1}=\dfrac{316\pm370}{2}$

$\Rightarrow t=343$ and $t=-27$

$\therefore m^3=343$ and $n^3=-27$

$\Rightarrow m=7$ and $n=-3$

Here, The roots are x, $m\omega+n\omega^2$ and $m\omega^2+n\omega$

$\therefore x=m+n=7-3=4$

$m\omega+n\omega^2=7\left(\dfrac{-1+\sqrt{-3}}{2}\right)+(-3)\left(\dfrac{-1-\sqrt{-3}}{2}\right)$

$=\dfrac{-7+7\sqrt{-3}}{2}+\dfrac{3+3\sqrt{-3}}{2}=\dfrac{-7+7\sqrt{-3}+3+3\sqrt{-3}}{2}$

$=\dfrac{-4+10\sqrt{-3}}{2}=-2+5\sqrt{-3}$

and $m\omega^2+n\omega=7\left(\dfrac{-1-\sqrt{-3}}{2}\right)+(-3)\left(\dfrac{-1+\sqrt{-3}}{2}\right)$

$=\dfrac{-7-7\sqrt{-3}}{2}+\dfrac{3-3\sqrt{-3}}{2}=\dfrac{-7-7\sqrt{-3}+3-3\sqrt{-3}}{2}$

$=\dfrac{-4-10\sqrt{-3}}{2}=-2-5\sqrt{-3}$

The roots are $4, -2\pm5\sqrt{-3}$ Ans.

$x^3+72x-1720=0$

Solution: Given that $x^3+72x-1720=0$ —— (i)

put $x=m+n$ then $x^3=(m+n)^3=m^3+n^3+3mn(m+n)$

$\Rightarrow x^3-3mn(m+n)-(m^3+n^3)=0$

$\Rightarrow x^3-3mnx-(m^3+n^3)=0$ —— (ii)

Compairing the equation (i) and (ii) we get

$-3mn=72$ and $-(m^3+n^3)=-1720$

$\Rightarrow mn=-24$ and $m^3+n^3=1720$

$\Rightarrow m^3n^3=-13824$

if m^3 and n^3 are the roots of the equation, we can write

$t^2-(m^3+n^3)t+m^3n^3=0$

$\Rightarrow t^2-1720t-13824=0$

$\Rightarrow t=\frac{-(-1720)\pm\sqrt{(-1720)^2-4\cdot1(-13824)}}{2\cdot1}=\frac{1720\pm1736}{2}$

$\Rightarrow t=1728$ and $t=-8$

$\Rightarrow m^3=1728$ and $n^3=-8$

$\Rightarrow m=12$ and $n=-2$

Hence The roots are x, $m\omega+n\omega^2$ and $m\omega^2+n\omega$

$\therefore x=m+n=12-2=10$

$m\omega+n\omega^2=12\left(\frac{-1+\sqrt{-3}}{2}\right)+(-2)\left(\frac{-1-\sqrt{-3}}{2}\right)=-6+6\sqrt{-3}+1+\sqrt{-3}$

$=-5+7\sqrt{-3}$

And $m\omega^2+n\omega=12\left(\frac{-1-\sqrt{-3}}{2}\right)+(-2)\left(\frac{-1+\sqrt{-3}}{2}\right)$

$=-6-6\sqrt{-3}+1-\sqrt{-3}$

$=-5-7\sqrt{-3}$

$\therefore$ The roots are $10, -5\pm7\sqrt{-3}$. Ans.

$x^3 - 54x - 162 = 0$

Given that $x^3 - 54x - 162 = 0$ —— (i)

put $x = m+n$ then $x^3 = (m+n)^3 = m^3+n^3+3mn(m+n)$

$\Rightarrow x^3 - 3mn(m+n) - (m^3+n^3) = 0$

$\Rightarrow x^3 - 3mnx - (m^3+n^3) = 0$ —— (ii)

Comparing the equation (i) and (ii), we get

$-3mn = -54$ and $-(m^3+n^3) = -162$

$\Rightarrow mn = 18$ $\Rightarrow m^3+n^3 = 162$

$\Rightarrow m^3n^3 = 5832$

if m^3 and n^3 are the roots of the equation, we can write

$t^2 - (m^3+n^3)t + m^3n^3 = 0$

$\Rightarrow t^2 - 162t + 5832 = 0$

$\Rightarrow t = \frac{-(-162) \pm \sqrt{(-162)^2 - 4 \cdot 1 \cdot 5832}}{2 \cdot 1} = \frac{162 \pm 54}{2}$

$\Rightarrow t = 108$ and $t = 54$

$\therefore m^3 = 108$ and $n^3 = 54$

$\Rightarrow m = \sqrt[3]{27 \times 4}$ $\Rightarrow n = \sqrt[3]{54} = \sqrt[3]{27 \times 2}$

$= 3\sqrt[3]{4}$ $= 3\sqrt[3]{2}$

Hence the roots are x, $m\omega + n\omega^2$ and $m\omega^2 + n\omega$

$\therefore x = m+n = 3\sqrt[3]{4} + 3\sqrt[3]{2} = \sqrt[3]{108} + \sqrt[3]{54}$

$m\omega + n\omega^2 = 3\sqrt[3]{4}\left(\frac{-1+\sqrt{-3}}{2}\right) + 3\sqrt[3]{2}\left(\frac{-1-\sqrt{-3}}{2}\right)$

and $m\omega^2 + n\omega = 3\sqrt[3]{4}\left(\frac{-1-\sqrt{-3}}{2}\right) + 3\sqrt[3]{2}\left(\frac{-1+\sqrt{-3}}{2}\right)$

Ans.

* Solve the cubic equation $28x^3-9x^2+1=0$

Solution: Given that $28x^3-9x^2+1=0$ ——— (i)

Put $x=\frac{1}{y}$ in equation (i), we get

$$\frac{28}{y^3}-\frac{9}{y^2}+1=0 \Rightarrow \frac{28-9y+y^3}{y^3}=0$$

$$\Rightarrow y^3-9y+28=0 \text{ ——— (ii)}$$

Again put $y=(m+n)$ then $y^3=(m+n)^3=m^3+n^3+3mn(m+n)$

$$\Rightarrow y^3-3mn(m+n)-(m^3+n^3)=0$$

$$\Rightarrow y^3-3mny-(m^3+n^3)=0 \text{ ——— (iii)}$$

Compairing the equation (ii) and (iii) we get

$-3mn=-9$ and $-(m^3+n^3)=28$

$\Rightarrow mn=3$ and $m^3+n^3=-28$

$\Rightarrow m^3n^3=27$

if m^3 and n^3 are the roots of the equation, we can write

$$t^2-(m^3+n^3)t+m^3n^3=0$$

$$\Rightarrow t^2+28t+27=0 \Rightarrow t^2+27t+t+27=0$$

$$\Rightarrow t(t+27)+1(t+27)=0 \Rightarrow (t+27)(t+1)=0$$

$$\Rightarrow t=-27 \text{ and } t=-1$$

$\therefore m^3=-27$ and $n^3=-1$

$m=-3$ $\quad n=-1$

Now $y=m+n=-3-1=-4$

$$\therefore x=\frac{1}{y}=\frac{1}{-4}=-\frac{1}{4}$$

$$m\omega+n\omega^2=-3\left(\frac{-1+\sqrt{-3}}{2}\right)+(-1)\left(\frac{-1-\sqrt{-3}}{2}\right)$$

$$=\frac{3-3\sqrt{-3}}{2}+\frac{1+\sqrt{-3}}{2}$$

$$=\frac{3-3\sqrt{-3}+1+\sqrt{-3}}{2}=\frac{4-2\sqrt{-3}}{2}=2-\sqrt{-3}$$

$$=2-i\sqrt{3}$$

P.T.O

$$\therefore \frac{1}{m\omega+n\omega^2} = \frac{1}{2-i\sqrt{3}} = \frac{2+i\sqrt{3}}{(2-i\sqrt{3})(2+i\sqrt{3})} = \frac{2+i\sqrt{3}}{2^2-i^2(\sqrt{3})^2}$$

$$= \frac{2+i\sqrt{3}}{4+3} = \frac{2+i\sqrt{3}}{7}$$

and $m\omega^2+n\omega = -3\left(\frac{-1-\sqrt{-3}}{2}\right)+(-1)\left(\frac{-1+\sqrt{-3}}{2}\right)$

$$= \frac{3+3\sqrt{-3}}{2} + \frac{1-\sqrt{-3}}{2} = \frac{3+3\sqrt{-3}+1-\sqrt{-3}}{2}$$

$$= \frac{4+2\sqrt{-3}}{2} = 2+\sqrt{-3} = 2+i\sqrt{3}$$

$$\therefore \frac{1}{m\omega^2+n\omega} = \frac{1}{2+i\sqrt{3}} = \frac{2-i\sqrt{3}}{(2+i\sqrt{3})(2-i\sqrt{3})} = \frac{2-i\sqrt{3}}{2^2-i^2(\sqrt{3})^2}$$

$$= \frac{2-i\sqrt{3}}{4+3} = \frac{2-i\sqrt{3}}{7}$$

$\therefore$ The roots are $~~x=~~ -\frac{1}{4}, \frac{2\pm i\sqrt{3}}{7}$. Ans.

Solve the cubic equation $x^3-15x^2-33x+847=0$

Solution: Let $f(x) = x^3-15x^2-33x+847$

$$\therefore f(x+h) = (x+h)^3-15(x+h)^2-33(x+h)+847$$

$$= x^3+3x^2h+\cdots-15x^2+\cdots$$

$$= x^3+3x^2(h-5)+\cdots\cdots$$

Put $h=5$ then the 2nd term is vanis.

Now dividing by $(x-5)$ that equation is:

1	−15	−33	847	5
	5	−50	−415	
1	−10	−83	432	
	5	−25		
1	−5	−108		
	5			
1	0			

Hence, the required equation $x^3+0.x^2+(-108)x+432=0$ ——— (i)

$$\Rightarrow x^3-108x+432=0$$

P.T.O

put $x = m+n$ then $x^3 = (m+n)^3 = m^3 + n^3 + 3mn(m+n)$

$$\Rightarrow x^3 - 3mn(m+n) - (m^3+n^3) = 0$$

$$\Rightarrow x^3 - 3mnx - (m^3+n^3) = 0 \quad \text{—— (ii)}$$

Compairing the equation (i) and (ii), we get

$$-3mn = -108 \quad \text{and} \quad -(m^3+n^3) = 432$$

$$\Rightarrow mn = 36 \quad \text{and} \quad m^3+n^3 = -432$$

$$\Rightarrow m^3n^3 = 46656$$

if m^3 and n^3 are the roots of the equation, we can write

$$t^2 - (m^3+n^3)t + m^3n^3 = 0$$

$$\Rightarrow t^2 + 432t + 46656 = 0$$

$$\Rightarrow t = \frac{-432 \pm \sqrt{(432)^2 - 4.1.46656}}{2.1} = \frac{-432 \pm 0}{2}$$

$$\Rightarrow t = -216 \quad \text{and} \quad t = -216$$

$$\therefore m^3 = -216 \quad \text{and} \quad n^3 = -216$$

$$\Rightarrow m = -6 \quad \text{and} \quad n = -6$$

Hence the roots are x, $m\omega + n\omega^2$ and $m\omega^2 + n\omega$

$$\therefore x = m+n = -6-6 = -12$$

$$m\omega + n\omega^2 = -6\left(\frac{-1+\sqrt{-3}}{2}\right) + (-6)\left(\frac{-1-\sqrt{-3}}{2}\right)$$

$$= 3 - 3\cancel{\sqrt{-3}} + 3 + 3\cancel{\sqrt{-3}}$$

$$= 6$$

and $m\omega^2 + n\omega = -6\left(\frac{-1-\sqrt{-3}}{2}\right) + (-6)\left(\frac{-1+\sqrt{-3}}{2}\right)$

$$= 3 + 3\cancel{\sqrt{-3}} + 3 - 3\cancel{\sqrt{-3}}$$

$$= 6$$

$\therefore$ The roots are -12, 6 and 6. Ans.

$\boxed{\text{Same question}}$ i) $x^3 - 12x^2 - 6x - 10 = 0$

ii).

Ex: $x^3 - 2x^2 - 6x - 10 = 0$

Given that $x^3 - 12x^2 - 6x - 10 = 0$

let $f(x) = x^3 - 12x^2 - 6x - 10 = 0$

$$f(x+h) = (x+h)^3 - 12(x+h)^2 - 6(x+h) - 10 = 0$$

$$= x^3 + 3x^2h + \cdots - 12x^2 + \cdots$$

$$= x^3 + 3x^2(h-4) + \cdots$$

put $h = 4$ thus the 2nd term is vanis.

Now dividing by $(x-4)$ that equation is

$$\begin{array}{rrrr|l}
1 & -12 & -6 & -10 & 4 \\
 & 4 & -32 & -152 & \\
\hline
1 & -8 & -38 & -162 & \\
 & 4 & -16 & & \\
\hline
1 & -4 & -54 & & \\
 & 4 & & & \\
\hline
1 & 0 & & &
\end{array}$$

Hence, the required equation

$$x^3 - 54x - 162 = 0 \quad \text{—— (i)}$$

Put $x = m+n$ then $x^3 = (m+n)^3 = m^3 + n^3 + 3mn(m+n)$

$$\Rightarrow x^3 - 3mn(m+n) - (m^3+n^3) = 0$$

$$\Rightarrow x^3 - 3mnx - (m^3+n^3) = 0 \quad \text{—— (ii)}$$

Compairing the equation (i) and (ii), we get

$-3mn = -54$ and $-(m^3+n^3) = -162$

$\Rightarrow mn = \frac{54}{3} = 18$ and $m^3 + n^3 = 162$

~~$\Rightarrow m^3n^3 = \left(\frac{\;\;}{3}\right)^3$~~

$\Rightarrow m^3n^3 = (18)^3 = 5832$

If m^3 and n^3 are the roots of the equation, we can write,

$t^2 - (m^3+n^3)t + m^3n^3 = 0$

$\Rightarrow \cancel{t^2 - 162t + 5832 = 0}$

$\Rightarrow t^2 - 162t + 5832 = 0$

$\Rightarrow t = \dfrac{-(-162) \pm \sqrt{(-162)^2 - 4.1.5832}}{2.1}$

$\Rightarrow t = \dfrac{162 \pm 54}{2}$

$\Rightarrow t = 108$ and $t = 54$

$\therefore m^3 = 108$ and $n^3 = 54$

$m = \sqrt[3]{27\times4}$ and $n = \sqrt[3]{27\times2}$

$= 3\sqrt[3]{4}$ $\quad = 3\sqrt[3]{2}$

Hence the roots are x, $m\omega + n\omega^2$ and $m\omega^2 + n\omega$

$$\left.\begin{aligned}
\therefore x &= m+n = 3\sqrt[3]{4} + 3\sqrt[3]{2} = \sqrt[3]{108} + \sqrt[3]{54} \\
m\omega + n\omega^2 &= 3\sqrt[3]{4}\left(\frac{-1+\sqrt{-3}}{2}\right) + 3\sqrt[3]{2}\left(\frac{-1-\sqrt{-3}}{2}\right) \\
\text{and } m\omega^2 + n\omega &= 3\sqrt[3]{4}\left(\frac{-1-\sqrt{-3}}{2}\right) + 3\sqrt[3]{2}\left(\frac{-1+\sqrt{-3}}{2}\right)
\end{aligned}\right\} \text{Ans.}$$