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Module-04

linear Algebra - I
Vector Arithematic
Norm
Euclidean Inner product
Unit vector
Angle between two vectors
Cauchy-Schwarz Inequality
Orthogonality
orthogonal set
orthonormal set.
Gram-Schmidt process

2 marks Questions

Q1). How to find angle between two vectors

Q2) State cauchy schwartz inequallity (V, R^n)

Q3) when two vectors orthogonal ?

Q4) what is unit vectors ?

Q5) Define norm of a vector.

Q6) Define Dot product

Q7) Define Euclidean Inner product

Q8) find the angle between $u = (6,2,2)$ and $v = (3,0,9)$

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Q9) find angle between $u = (2,-1,4)$
and $v = (1,-2,3)$

Q10) $u = (1,-2,4)$ and $v = (-2,4,6)$
find $u \cdot v$.

Q11) $u = (-2,4,6)$ and $w = (-3,4,-5)$
find $u \cdot w$

Q12) If $u = (2,3,1)$ and $v = (3,0,4)$ then
find angle between u and v

1 mark

Q1) How to find Angle between two vectors.

Solution: Let u and v are two vectors
By using following formula, we can find Angle between two vectors.

$$\theta = \cos^{-1}\left(\frac{u \cdot v}{|u||v|}\right).$$

e.g. $u = (3,-2)$
$v = (1,7)$

$$u \cdot v = (3)\cdot(1) + (-2)(7)$$
$$= 3 - 14$$
$$u \cdot v = -11$$

$$|u| = \sqrt{(3)^2 + (-2)^2} = \sqrt{13},$$

$$|v| = \sqrt{1^2 + 7^2} = \sqrt{50}$$

$$\cos\theta = \frac{u \cdot v}{|u||v|} \Rightarrow \frac{-11}{\sqrt{13}\cdot\sqrt{50}}$$

$$\theta = \cos^{-1}\left(\frac{-11}{\sqrt{13}\cdot\sqrt{50}}\right)$$

Q2) when two vectors are orthogonal ?

Solution:

If the dot product of that two vectors is zero then that two vectors are orthogonal

e.g ① $u = (2, -3)$
$v = (-3, 2)$

$$u \cdot v = (2)(-3) + (-3)(2)$$
$$= -6 - 6$$
$$= -12$$
$$u \cdot v \neq 0$$

$\therefore$ u and v are not orthogonal.

② $u = (2, -3)$
$v = (3, 2)$

$$u \cdot v = (2)(3) + (-3)(2)$$
$$= 6 - 6$$
$$= 0$$
$$u \cdot v = 0$$

dot product of two vectors u and v are zero.

Therefore u and v are orthogonal.

Q3) what is unit vector ?

Solution. A vector which magnitude is one is called unit vector.

Formula = $\hat{v} = \dfrac{v}{|v|}$

unit vector

e.g. $v = (2,3)$ $\quad |v| = \sqrt{2^2+3^2}$
$= 5$

$\hat{v} = (2/5, 3/5)$

unit vector

Q4) find Angle between
$u = (6,2,2)$ and $v = (3,0,9)$

Solution:

$$u.v = (6).(3) + (2).(0) + (9).(2)$$
$$= 18 + 18$$
$$= 36$$

$$|u| = \sqrt{6^2+2^2+2^2} = \sqrt{36+4+4}$$
$$|u| = \sqrt{44}$$
$$|v| = \sqrt{3^2+0^2+9^2} = \sqrt{9+81} = \sqrt{90}$$

$$\theta = \cos^{-1}\left(\frac{u.v}{|u|.|v|}\right)$$
$$= \cos^{-1}\left(\frac{36}{\sqrt{44}\sqrt{90}}\right)$$
$$= \cos^{-1}\left(\frac{36}{\sqrt{11\times4\times9\times10}}\right)$$

$$= \cos^{-1}\left(\frac{36}{6\sqrt{110}}\right)$$

$$\boxed{\theta = \cos^{-1}\left(\frac{6}{\sqrt{110}}\right)}$$

Q5) Find Angle between $u = (2,-1,-4)$ and $v = (1,-2,3)$

solution: $u = (2,-1,-4)$
$v = (1,-2,3)$

$$u \cdot v = (2)(1) + (-1)(-2) + (-4)(3)$$
$$= 2 + 2 - 12$$
$$u \cdot v = -8$$

$$|u| = \sqrt{(2)^2 + (-1)^2 + (-4)^2}$$
$$= \sqrt{4+1+16}$$
$$|u| = \sqrt{21}$$

$$|v| = \sqrt{(1)^2 + (-2)^2 + (3)^2}$$
$$= \sqrt{1+4+9}$$
$$|v| = \sqrt{14}$$

$$\theta = \cos^{-1}\left(\frac{u \cdot v}{|u||v|}\right)$$

$$\boxed{\theta = \cos^{-1}\left(\frac{-8}{\sqrt{21}\cdot\sqrt{14}}\right)}$$

Q6) $u = (1,-2,4)$ and $v = (-2,4,6)$
find $u \cdot v$

Solution:
$$u = (1,-2,4)$$
$$v = (-2,4,6)$$

$$u \cdot v = (1)(-2) + (-2)(4) + (4)(6)$$
$$= -2 - 8 + 24$$
$$= -10 + 24$$
$$= 14$$

$$\boxed{u \cdot v = 14}$$

Q7) $u = (-2,4,6)$ and $w = (-3,4,-6)$
find $u \cdot w$

Solution:
$$u = (-2,4,6)$$
$$w = (-3,4,-6)$$

$$u \cdot w = (-2)(-3) + (4)(4) + (6)(-6)$$
$$= 6 + 16 - 36$$
$$= 22 - 36$$
$$= -14$$

$$\boxed{u \cdot w = -14}$$

Q8) If $u = (2,3,1)$ and $v = (3,0,4)$ then find angle between u and v.

Solution:

$$u = (2,3,1)$$
$$v = (3,0,4)$$

$$u \cdot v = (2)(3) + (3)\cdot(0) + (1)\cdot(4)$$
$$= 6 + 0 + 4$$
$$u \cdot v = 10$$

$$|u| = \sqrt{(2)^2 + (3)^2 + (1)^2}$$
$$= \sqrt{4+9+1}$$
$$|u| = \sqrt{14}$$

$$|v| = \sqrt{(3)^2 + (0)^2 + (4)^2}$$
$$= \sqrt{9+0+16}$$
$$= \sqrt{25}$$
$$|v| = 5$$

$$\theta = \cos^{-1}\left(\frac{u \cdot v}{|u||v|}\right)$$
$$= \cos^{-1}\left(\frac{10}{(\sqrt{14})(5)}\right)$$
$$\boxed{\theta = \cos^{-1}\left(\frac{2}{\sqrt{14}}\right)}$$

Q9) Define dot product.

Solution: Definition:

The Dot product (also known as the scalar product or inner product) of two vectors

$a = (a_1, a_2, a_3, \cdots, a_n)$ and
$b = (b_1, b_2, b_3, \cdots, b_n)$

is define as

$$a \cdot b = a_1 b_1 + a_2 b_2 + a_3 b_3 + \cdots + a_n b_n$$

e.g. ① $u = (2, 3)$, $v = (4, 5)$

$$u \cdot v = (2)(4) + (3)(5)$$
$$= 8 + 15$$
$$u \cdot v = 23$$

Q10) Define norm of a vector.

Solution:

Norm of a vector (magnitude):

The norm of a vector $v = (v_1, v_2, v_3, \cdots, v_n)$ is denoted by $\|v\|$ an defined as

$$\|v\| = \sqrt{v_1^2 + v_2^2 + v_3^2 + \cdots + v_n^2}$$

e.g. ① $v = (3, 4)$

$$\|v\| = \sqrt{3^2 + 4^2} = \sqrt{9 + 16} = \sqrt{25}$$

$$\|v\| = 5$$

Q11). Define Euclidean Inner product.

Solution:

Euclidean Inner product:

The Euclidean inner product (also known as the dot product or scalar product) of two vectors

$u = (u_1, u_2, \cdots, u_n)$ and

$v = (v_1, v_2, \cdots, v_n)$ in Euclidean space is defined as

$$u \cdot v = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n$$

Q12) State Cauchy Schwartz Inequality (V, R^n).

Solution:

Let $u = (u_1, u_2, u_3, \cdots, u_n)$ and $v = (v_1, v_2, v_3, \cdots, v_n)$ be two vectors of equal dimension and real component

The Cauchy-Schwarz inequality is stated as:

$$|u \cdot v| \leq |u| \cdot |v|$$

where

$$u \cdot v = \sum_{i=1}^{n} u_i v_i$$

$$|u| = \sqrt{\sum_{i=1}^{n} u_i^2}, \quad |v| = \sqrt{\sum_{i=1}^{n} v_i^2}$$

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5 marks

Q1) Find the cosine of the angle between u and v. If R^3 and R^4 have Euclidean Inner product.

i) $u = (-1, 5, 2)$, $v = (2, 4, 9)$

ii) $u = (1, 0, 1, 0)$, $v = (-3, -3, -3, -3)$

5 marks

Q2) Find all vectors in R^3 of Euclidean norm 1 that are orthogonal to $u_1 = (1, 1, 1)$, $u_2 = (1, 1, 0)$

5 marks

Q3) Using Cauchy Schwartz inequality prove that

$$\frac{b+c}{a} + \frac{c+a}{b} + \frac{a+b}{c} \geq 6$$

5 marks

Q4) Determine if there exists scalers K and l such that the vectors $u = (2, K, 6)$, $v = (l, 5, 3)$, and $w = (1, 2, 3)$ are mutually orthogonal w.r.t Euclidean Inner product.

Q5) Verify Cauchy Schwartz Inequality holds for

$u = (-2, 1)$, $v = (1, 0)$

where

$$|u \cdot v| = 3u_1v_1 + 2u_2v_2$$

5 marks

Q6) Find a unit vector orthogonal to both $(1,1,0)$ and $(0,1,1)$.

5 marks

Q7) Show that $u_1 = (2/3, -2/3, 1/3)$,
$u_2 = (2/3, 1/3, -2/3)$ and
$u_3 = (1/3, 2/3, 2/3)$
are orthonormal set.

5 marks

Q8) For $\begin{pmatrix} 2 & 6 \\ 1 & -3 \end{pmatrix}, \begin{pmatrix} 3 & 2 \\ 1 & 0 \end{pmatrix}$
Verify Cauchy Schwarz Inequality.

5 marks

Q9) $A = \begin{pmatrix} -1 & 2 \\ 6 & 1 \end{pmatrix}, B = \begin{pmatrix} 1 & 0 \\ 3 & 3 \end{pmatrix}$
using inner product
$\langle a \cdot b \rangle = a_1 b_1 + a_2 b_2 + a_3 b_3 + a_4 b_4$
where
$A = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}$ and $B = \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix}$.

5 marks

Q10) Find unit vector in R^3 orthogonal to both $(1,0,1)$, $(0,1,1)$.

marks

Q11) For what Values of a and b is the set $\{u, v\}$ orthonormal w.r.t Euclidean inner product for

$$u = \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right), v = \left(a, \frac{1}{\sqrt{2}}, -b\right).$$

5 marks

Q12) Show that $u_1 = \left(\frac{1}{5}, \frac{1}{5}, \frac{1}{5}\right), u_2 = \left(\frac{-1}{2}, \frac{1}{2}, 0\right)$ and

$$u_3 = \left(\frac{1}{3}, \frac{1}{3}, -\frac{2}{3}\right)$$

is orthogonal w.r.t Euclidean Inner product on R^3.

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20 KS

Q1) find the cosine of the angle between u and v. If R^3 and R^4 have Euclidean Inner product

i) $u=(-1,5,2)$, $v=(2,4,9)$

ii) $u=(1,0,1,0)$, $v=(-3,-3,-3,-3)$

Solution:

$|u| = \|u\|$

$|v| = \|v\|$

i) $u=(-1,5,2)$

$v=(2,4,9)$

We know that

$$\cos\theta = \frac{u.v}{|u||v|} \quad \text{———} (1)$$

$$u.v = (-1)(2)+(5)(4)+(2)(9)$$

$$= -2+20+18$$

$$u.v = 36$$

$$|u| = \sqrt{1+25+4} = \sqrt{30}$$

$$|v| = \sqrt{4+16+81} = \sqrt{101}$$

$$\therefore \cos\theta = \left(\frac{36}{\sqrt{30}.\sqrt{101}}\right)$$

$$\theta = \cos^{-1}\left(\frac{36}{\sqrt{30}.\sqrt{101}}\right)$$

$\therefore$ Angle between $u=(-1,5,2)$ and $v=(2,4,9)$ is

$$\boxed{\theta = \cos^{-1}\left(\frac{36}{\sqrt{30}.\sqrt{101}}\right)}$$

11) $u = (1,0,1,0)$, $v = (-3,-3,-3,-3)$

We know that

$$\cos\theta = \frac{u \cdot v}{|u| \cdot |v|} \qquad \left| \begin{array}{l} |u| = \|u\| \\ |v| = \|v\| \end{array} \right.$$

$$u \cdot v = (1)(-3) + (0)(-3) + (1)(-3) + (0)(-3)$$
$$= -3 + 0 - 3 + 0$$
$$u \cdot v = -6$$

$$|u| = \sqrt{2}$$

$$|v| = \sqrt{9+9+9+9} = \sqrt{36} = 6$$

$$\cos\theta = \frac{-\cancel{6}}{(\sqrt{2})\cancel{(6)}}$$

$$\cos\theta = \frac{-1}{\sqrt{2}}$$

$$\theta = \cos^{-1}\left(\frac{-1}{\sqrt{2}}\right)$$

$\therefore$ Angle between $u = (1,0,1,0)$ and $v = (-3,-3,-3,-3)$ is

$$\boxed{\theta = \cos^{-1}\left(\frac{-1}{\sqrt{2}}\right)}$$

marks

Q2) Find all vectors in R^3 of Euclidean norm 1 that are orthogonal to $u_1=(1,1,1)$, $u_2=(1,1,0)$.

Solution:

Step 1) Calculate the cross product of u_1 and u_2

Let $w = u_1 \times u_2$

$$w = \begin{vmatrix} i & j & k \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{vmatrix}$$

$$w = (-1)i - (-1)j + (0)k$$

$$w = -i + j$$

$$w = (-1,1,0) \quad \text{——— (1)}$$

Step 2) Calculate the magnitude of the resulting vector w.

$$\|w\| = \sqrt{(-1)^2+(1)^2+0^2}$$

$$= \sqrt{1+1+0}$$

$$\|w\| = \sqrt{2}$$

Step 3) Normalize the vector w to find the unit vectors

We know that

The unit vectors are $= \pm \dfrac{w}{\|w\|}$

$$w_1 = \frac{1}{\sqrt{2}}(-1,1,0) = \left(\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)$$

$$w_2 = \frac{-1}{\sqrt{2}}(-1,1,0) = \left(\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}, 0\right)$$

w_1 and w_2 are Normalize unit vectors

Step 4):

The unit vectors orthogonal to both $u_1 = (1,1,1)$ and $u_2 = (1,1,0)$ are $w_1 = \left(\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)$ and

$$w_2 = \left(\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}, 0\right).$$

marks

3) find a unit vector orthogonal to both $(1,1,0)$ and $(0,1,1)$.

solution:

Let $u = (1,1,0)$
$v = (0,1,1)$

step 1): Calculate the cross product of u and v.

Let $w = u \times v$

$$w = \begin{vmatrix} i & j & k \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{vmatrix}$$

$$w = i(1) - j(1) + k(1)$$

$$w = (1,-1,1) \quad \text{——} \quad ①$$

step 2) Calculate the magnitude of the resulting vector w.

$$||w|| = \sqrt{(1)^2+(-1)^2+(1)^2}$$

$$= \sqrt{1+1+1}$$

$$||w|| = \sqrt{3}$$

step 3). Normalize the vector w to find unit vectors

We know that

The unit vectors are $= \pm \dfrac{w}{||w||}$

$$\omega_1 = \frac{1}{\sqrt{3}}(1,-1,1) = \left(\frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$$

$$\omega_2 = \frac{-1}{\sqrt{3}}(1,-1,1) = \left(\frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}\right)$$

ω_1 and ω_2 are Normalize unit vectors.

Step 4). The unit vectors orthogonal to both $u = (1,1,0)$ and $v = (0,1,1)$ are

$$\omega_1 = \left(\frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right) \text{ and}$$

$$\omega_2 = \left(\frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}\right)$$

Note! (In Question asking find one unit vector but we are finding two) if we are finding one vector it is Sufficient for our answer.

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Q4) Find unit vector in R^3 orthogonal to both $(1,0,1)$, $(0,1,1)$.

Solution: Let $u = (1,0,1)$
$v = (0,1,1)$

Step 1) Calculate the cross product of u and v.

Let $w = u \times v$

$$w = \begin{vmatrix} i & j & k \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{vmatrix}$$

$$w = i(-1) - j(1) + k(1)$$
$$w = (-1)i + j(-1) + k(1)$$
$$w = (-1,-1,1) \quad \text{——— } ①$$

Step 2) Calculate the Magnitude of the resulting vector w.

$$||w|| = \sqrt{(-1)^2 + (-1)^2 + (1)^2}$$
$$= \sqrt{1+1+1}$$
$$||w|| = \sqrt{3}$$

Step 3). Normalize the vector w to find unit vectors

We know that.

The unit vectors are $= \pm \dfrac{w}{||w||}$

$$w_1 = \frac{1}{\sqrt{3}}(-1,-1,1) = \left(\frac{-1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$$

$$w_2 = \frac{-1}{\sqrt{3}}(-1,-1,1) = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}\right)$$

w_1 and w_2 are Normalize unit vectors

Step 4)

The unit vectors orthogonal to both $u=(1,0,1)$ and $v=(0,1,1)$ are

$$w_1 = \left(\frac{-1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right) \text{ and}$$

$$w_2 = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}\right)$$

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Orthogonal & orthonormal set:

A set of vectors in a inner product space is called as orthogonal set if each vector in this set is orthogonal to every other vector in the set.

If the Norm of each vector is 1 then set is called orthonormal

5marks

Q5) show that $u_1 = \left(\frac{2}{3}, \frac{-2}{3}, \frac{1}{3}\right)$

$u_2 = \left(\frac{2}{3}, \frac{1}{3}, \frac{-2}{3}\right)$ and

$u_3 = \left(\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right)$

are orthonormal set.

Solution:

step 1): find dot product

$$u_1 \cdot u_2 = \left(\frac{2}{3}, \frac{-2}{3}, \frac{1}{3}\right) \cdot \left(\frac{2}{3}, \frac{1}{3}, \frac{-2}{3}\right)$$

$$= \left(\frac{2}{3}\right)\left(\frac{2}{3}\right) + \left(\frac{-2}{3}\right)\left(\frac{1}{3}\right) + \left(\frac{1}{3}\right)\left(\frac{-2}{3}\right)$$

$$= \frac{4}{9} - \frac{2}{9} - \frac{2}{9}$$

$$= \cancel{\frac{4}{9}} - \cancel{\frac{4}{9}}$$

$$= 0$$

$$u_1 \cdot u_2 = 0 \quad \text{——— } (1)$$

$$u_1 \cdot u_3 = \left(\tfrac{2}{3}, \tfrac{-2}{3}, \tfrac{1}{3}\right) \cdot \left(\tfrac{1}{3}, \tfrac{2}{3}, \tfrac{2}{3}\right)$$

$$= \left(\tfrac{2}{3}\right)\left(\tfrac{1}{3}\right) + \left(\tfrac{-2}{3}\right)\left(\tfrac{2}{3}\right) + \left(\tfrac{1}{3}\right)\left(\tfrac{2}{3}\right)$$

$$= \tfrac{2}{9} - \tfrac{4}{9} + \tfrac{2}{9}$$

$$= \cancel{\tfrac{4}{9}} - \cancel{\tfrac{4}{9}}$$

$$= 0$$

$$u_1 \cdot u_3 = 0 \quad \text{——— (2)}$$

$$u_2 \cdot u_3 = \left(\tfrac{2}{3}, \tfrac{1}{3}, \tfrac{-2}{3}\right) \cdot \left(\tfrac{1}{3}, \tfrac{2}{3}, \tfrac{2}{3}\right)$$

$$= \left(\tfrac{2}{3}\right)\left(\tfrac{1}{3}\right) + \left(\tfrac{1}{3}\right)\left(\tfrac{2}{3}\right) + \left(\tfrac{-2}{3}\right)\left(\tfrac{2}{3}\right)$$

$$= \tfrac{2}{9} + \tfrac{2}{9} - \tfrac{4}{9}$$

$$= \cancel{\tfrac{4}{9}} - \cancel{\tfrac{4}{9}}$$

$$= 0$$

$$u_2 \cdot u_3 = 0 \quad \text{——— (3)}$$

step 2) find Norm / Magnitude

$$\|u_1\| = \sqrt{\tfrac{4}{9} + \tfrac{4}{9} + \tfrac{1}{9}} = \sqrt{\tfrac{9}{9}} = 1$$

$$\|u_2\| = \sqrt{\tfrac{4}{9} + \tfrac{1}{9} + \tfrac{4}{9}} = \sqrt{\tfrac{9}{9}} = 1$$

$$\|u_3\| = \sqrt{\tfrac{1}{9} + \tfrac{4}{9} + \tfrac{4}{9}} = \sqrt{\tfrac{9}{9}} = 1$$

$\therefore$ from step ① and ②

$u_1 \cdot u_2 = 0$
$u_1 \cdot u_3 = 0$
$u_2 \cdot u_3 = 0$ } $\longrightarrow$ dot product of each pair is zero or// Each vector is orthogonal to other ~~vectors~~ vectors.

$\|u_1\| = 1$
$\|u_2\| = 1$
$\|u_3\| = 1$ } $\longrightarrow$ Norm of each vector is one.

$\therefore$ The Given vectors $u_1 = \left(\frac{2}{3}, \frac{-2}{3}, \frac{1}{3}\right)$

$u_2 = \left(\frac{2}{3}, \frac{1}{3}, \frac{-2}{3}\right)$

$u_3 = \left(\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right)$

are orthonormal set.

Hence proved.

8 marks

8a) show that

$u_1 = \left(\frac{1}{5}, \frac{1}{5}, \frac{1}{5}\right)$, $u_2 = \left(-\frac{1}{2}, \frac{1}{2}, 0\right)$

and $u_3 = \left(\frac{1}{3}, \frac{1}{3}, -\frac{2}{3}\right)$

is orthogonal with respect to Euclidean Inner product on R^3

solution

step 1) find dot product

$$\langle u_1, u_2 \rangle = u_1 \cdot u_2$$

$$u_1 \cdot u_2 = \left(\tfrac{1}{5}\right)\left(-\tfrac{1}{2}\right) + \left(\tfrac{1}{5}\right)\left(\tfrac{1}{2}\right) + \left(\tfrac{1}{5}\right)(0)$$

$$= -\frac{1}{10} + \frac{1}{10}$$

$$= 0$$

$$\Rightarrow \boxed{u_1 \cdot u_2 = 0}$$

$$u_1 \cdot u_3 = \left(\tfrac{1}{5}\right)\left(\tfrac{1}{3}\right) + \left(\tfrac{1}{5}\right)\left(\tfrac{1}{3}\right) + \left(\tfrac{1}{5}\right)\left(\tfrac{2}{3}\right)$$

$$= \frac{1}{15} + \frac{1}{15} - \frac{2}{15}$$

$$= \frac{2}{15} - \frac{2}{15}$$

$$\boxed{u_1 \cdot u_2 = 0}$$

$$u_2 \cdot u_3 = \left(\frac{-1}{2}\right)\left(\frac{1}{3}\right) + \left(\frac{1}{2}\right)\left(\frac{1}{3}\right) + (0)\left(\frac{-2}{3}\right)$$

$$= -\frac{1}{6} + \frac{1}{6}$$

$$\boxed{u_2 \cdot u_3 = 0}$$

Since Euclidean inner product of each pair of vector is zero

$\therefore$ The vectors

$$u_1 = \left(\frac{1}{5}, \frac{1}{5}, \frac{1}{5}\right)$$

$$u_2 = \left(\frac{-1}{2}, \frac{1}{2}, 0\right) \text{ and}$$

$$u_3 = \left(\frac{1}{3}, \frac{1}{3}, \frac{-2}{3}\right)$$

are orthogonal.

Hence proved.

5 marks

Q7) For what values of a and b is the set $\{u, v\}$ orthonormal w.r.t. Euclidean inner product for

$$u = \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right), v = \left(a, \frac{1}{\sqrt{2}}, -b\right)$$

Solution:

Step 1) we know that

$$\langle u, v \rangle = u.v$$

$$u.v = 0 \quad \text{if } u \text{ and } v \text{ are orthogonal}$$

$$\frac{a}{\sqrt{2}} + 0 + \left(\frac{-b}{\sqrt{2}}\right) = 0$$

$$\frac{a}{\sqrt{2}} - \frac{b}{\sqrt{2}} = 0 \quad \text{--- (1)}$$

$$\Rightarrow a - b = 0 \Rightarrow \boxed{a = b}$$

Step 2) Find Norm / Maganitude

$$||u|| = \sqrt{\frac{1}{2} + 0 + \frac{1}{2}} = \sqrt{\frac{2}{2}} = 1$$

$$||v|| = \sqrt{a^2 + \frac{1}{2} + b^2}$$

Let $||v|| = 1 \Rightarrow \sqrt{a^2 + \frac{1}{2} + b^2} = 1$

$$\Rightarrow a^2 + \frac{1}{2} + b^2 = 1$$

$$\Rightarrow a^2 + b^2 = \frac{1}{2} \quad \text{--- (2)}$$

from eq (1) $a = b$

$$\Rightarrow a^2 + a^2 = \frac{1}{2}$$

$$\Rightarrow 2a^2 = \frac{1}{2}$$

$\Rightarrow a^2 = \frac{1}{4}$

taking squareroot

$\Rightarrow \boxed{a = \pm \frac{1}{2}}$

$\Rightarrow \boxed{a = b = \pm \frac{1}{2}}$ the set $\{u, v\}$ orthonormal w.r.t Euclidean Inner product.

Note: Northonormal = orthogonal + Norm one

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Q8) Verify Cauchy Schwartz inequality hold for $u=(-2,1)$, $v=(1,0)$ where
$$\langle u,v\rangle = 3u_1v_1 + 2u_2v_2$$

Solution: $u=(-2,1)$, $v=(1,0)$

We know that

Cauchy Schwartz inequality
$$|\langle u,v\rangle| \le \|u\|.\|v\| \quad \text{—— (1)}$$

Norm of vector with inner product
$$\|u\| = \sqrt{\langle u,u\rangle}$$

Step 1) Calculate the inner product of $\langle u,v\rangle$
$$\langle u,v\rangle = 3u_1v_1 + 2u_2v_2$$
$$= 3(-2)(1) + 2(1)(0)$$
$$= -6 + 0$$
$$\boxed{\langle u,v\rangle = -6} \quad \text{—— (*)}$$

Step 2) Calculate Norm of u and v

$$\|u\| = \sqrt{\langle u,u\rangle} \qquad u=(-2,1)$$
$$= \sqrt{3u_1u_1 + 2u_2u_2}$$
$$= \sqrt{3(-2)(-2) + 2(1)(1)}$$
$$= \sqrt{12+2}$$
$$\|u\| = \sqrt{14}$$

$$\|v\| = \sqrt{\langle v,v\rangle} = \sqrt{3v_1v_1 + 2v_2v_2}$$
$$= \sqrt{3(1)(1) + 2(0)(0)}$$
$$\|v\| = \sqrt{3}$$

step3) Verify the Cauchy-Schwarz inequality:

$$|\langle u, v\rangle| \leq \|u\| \cdot \|v\|$$

$$|-6| \leq \sqrt{14} \cdot \sqrt{3}$$

$$6 \leq \sqrt{42}$$

Squaring

$$36 \leq 42$$

This inequality is holds true

Conclusion:

The Cauchy-Schwarz inequality holds for the given vectors and inner product

Verified.

5 marks

Q9) For $\begin{pmatrix} 2 & 6 \\ 1 & -3 \end{pmatrix}, \begin{pmatrix} 3 & 2 \\ 1 & 0 \end{pmatrix}$ verify Cauchy-Schwarz inequality using inner product:

$$\langle A, B \rangle = a_1 b_1 + a_2 b_2 + a_3 b_3 + a_4 b_4$$

where

$$A = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \text{ and } B = \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix}.$$

Solution:

$$A = \begin{pmatrix} 2 & 6 \\ 1 & -3 \end{pmatrix}, \quad B = \begin{pmatrix} 3 & 2 \\ 1 & 0 \end{pmatrix}$$

We know that

Cauchy schwartz inequality

$$|\langle A, B \rangle| \leq \|A\| \cdot \|B\|$$

&

Norm of vector with Inner product

$$\|A\| = \sqrt{\langle A, A \rangle}$$

Step 1) Calculate the Inner product $\langle A, B \rangle$

$$\langle A, B \rangle = a_1 b_1 + a_2 b_2 + a_3 b_3 + a_4 b_4$$
$$= (2)(3) + (6)(2) + (1)(1) + (-3)(0)$$
$$= 6 + 12 + 1 + 0$$
$$= 19$$

$$\langle A, B \rangle = 19 \quad \text{———} (*)$$

Step 2) Calculate Norm / Maganitude of A and B

$$\|A\| = \sqrt{\langle A, A \rangle}$$
$$= \sqrt{\ }$$

$$= \sqrt{a_1 a_1 + a_2 a_2 + a_3 a_3 + a_4 a_4}$$

$$= \sqrt{a_1^2 + a_2^2 + a_3^2 + a_4^2}$$

$$||A|| = \sqrt{4+36+1+9}$$

$$= \sqrt{50} = \sqrt{25 \times 2}$$

$$||A|| = 5\sqrt{2}$$

$$||B|| = \sqrt{\langle B, B \rangle}$$

$$= \sqrt{b_1^2 + b_2^2 + b_3^2 + b_4^2}$$

$$= \sqrt{9+4+1+0}$$

$$= \sqrt{14}$$

$$||B|| = \sqrt{14}$$

step3) Verify the cauchy-schwarz inequality.

$$|\langle A, B \rangle| \leq ||A|| \cdot ||B||$$

$$|19| \leq (5\sqrt{2})(\sqrt{14})$$

$$19 \leq 5\sqrt{28}$$

$$361 \leq 25 \times 28 \qquad \text{squaring.}$$

$$361 \leq 700$$

$\therefore$ This inequality holds True.

Conclusion:

The Cauchy-Schwarz inequality holds for the given vectors and inner product.

Verified.

5 marks

Q10) For $A=\begin{pmatrix}-1 & 2\\ 6 & 1\end{pmatrix}$, $B=\begin{pmatrix}1 & 0\\ 3 & 3\end{pmatrix}$

verify Cauchy Schwartz inequality using inner product

$\langle A,B\rangle = a_1b_1 + a_2b_2 + a_3b_3 + a_4b_4$

where $A=\begin{pmatrix}a_1 & a_2\\ a_3 & a_4\end{pmatrix}$ and $B=\begin{pmatrix}b_1 & b_2\\ b_3 & b_4\end{pmatrix}$

Solution:

$$A=\begin{pmatrix}-1 & 2\\ 6 & 1\end{pmatrix},\quad B=\begin{pmatrix}1 & 0\\ 3 & 3\end{pmatrix}$$

We know that

Cauchy Schwartz inequality

$$|\langle A,B\rangle| \leq \|A\|.\|B\|$$

& Norm of vector with Inner product

$$\|A\| = \sqrt{\langle A,A\rangle}$$

step 1) Calculate the Inner product $\langle A,B\rangle$

$$\begin{aligned}\langle A,B\rangle &= a_1b_1 + a_2b_2 + a_3b_3 + a_4b_4\\ &= (-1)(1) + (2)(0) + (6)(3) + (1)(3)\\ &= -1 + 0 + 18 + 3\\ &= 20\end{aligned}$$

$$\langle A,B\rangle = 20$$

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Step2) Calculate Norm/Maganitude of A and B

$$\|A\| = \sqrt{\langle A, A\rangle}$$

$$\left\{ \langle A, B\rangle = \sqrt{a_1b_1 + a_2b_2 + a_3b_3 + a_4b_4} \right\}$$

$$\|A\| = \sqrt{a_1^2 + a_2^2 + a_3^2 + a_4^2}$$

$$= \sqrt{(-1)^2 + (2)^2 + (6)^2 + (1)^2}$$

$$= \sqrt{1+4+36+1}$$

$$\|A\| = \sqrt{42}$$

$$\|B\| = \sqrt{\langle B, B\rangle}$$

$$= \sqrt{b_1^2 + b_2^2 + b_3^2 + b_4^2}$$

$$= \sqrt{1+9+9}$$

$$= \sqrt{19}$$

$$\|B\| = \sqrt{19}$$

Step3). Verify the Cauchy-Schwarz inequality

$$|\langle A, B\rangle| \leq \|A\| \cdot \|B\|$$

$$|20| \leq \sqrt{42} \cdot \sqrt{19}$$

$$20 \leq \sqrt{42} \cdot \sqrt{19}$$

$$400 \leq 42 \times 19$$

$$400 \leq 798$$

$\therefore$ This inequality holds true.

Conclusion:

The Cauchy-Schwarz inequality holds for the given vectors and inner product.

Verified.

5 marks

Q12) Determine if there exist scalar k and l such that the vector $u=(2,k,6)$, $v=(l,5,3)$, $w=(1,2,3)$ are mutually orthogonal w.r.t Euclidean inner product.

Solution: $u=(2,k,6)$, $v=(l,5,3)$, $w=(1,2,3)$

Step 1) find dot product

$$u\cdot w = (2,k,6)\cdot(1,2,3)$$
$$= (2)(1)+(k)(2)+(6)(3)$$
$$= 2+2k+18$$
$$u\cdot w = 2k+20$$

$u\cdot w=0$ orthogonal

$$2k+20=0$$
$$k+10=0$$
$$\Rightarrow \boxed{k=-10}$$

$$v\cdot w = (l,5,3)\cdot(1,2,3)$$
$$= (l)(1)+(5)(2)+(3)(3)$$
$$= l+10+9$$
$$v\cdot w = l+19$$

$v\cdot w=0$ orthogonal

$$l+19=0$$
$$\Rightarrow \boxed{l=-19}$$

Step 3) Verify orthogonality of u and v using the found values of k and l

$u = (2, -10, 6)$
$v = (-19, 5, 3)$

$u \cdot v = (2, -10, 6) \cdot (-19, 5, 3)$
$= (2)(-19) + (-10)(5) + (6)(3)$
$= -38 - 50 + 18$
$= -70$

$u \cdot v \neq 0$

∴ The vectors are not mutually orthogonal for these values of k and l.

Step 4)
Conclusion:
There do not exist scalars k and l such that the given vectors are mutually orthogonal.

*//
5marks

Q13). Verify Cauchy Schwartz inequality for the vectors $(-4,2,1)$, $(8,-4,-2)$. Also find an Angle between them.

Solution: $u = (-4,2,1)$
$v = (8,-4,-2)$

Step 1) find dot product of u and v

$$u \cdot v = (-4,2,1)\cdot(8,-4,-2)$$
$$= (-4)(8) + (2)(-4) + (1)(-2)$$
$$= -32 - 8 - 2$$
$$\boxed{u \cdot v = -42}$$

Step 2) find Norm/ Maganitude u and v

$u = (-4,2,1)$

$$\|u\| = \sqrt{(-4)^2 + (2)^2 + (1)^2}$$
$$= \sqrt{16+4+1}$$
$$\|u\| = \sqrt{21}$$

$v = (8,-4,-2)$

$$\|v\| = \sqrt{(8)^2 + (-4)^2 + (-2)^2}$$
$$= \sqrt{64+16+4}$$
$$\|v\| = \sqrt{84}$$

Step3) Verifying the Cauchy-Schwartz Inequality.

$$|u.v| \leq ||u||.||v||$$

$$|-42| \leq \sqrt{21}.\sqrt{84}$$

$$42 \leq \sqrt{21 \times 84}$$

$$42 \leq \sqrt{1764}$$

$$42 \leq 42$$

$$42 \times 42 = 1764$$

Hence Cauchy-Schwartz inequality is holds for given vectors and dot product. (Inner product)
Hence verified.

Step4) Find the Angle between u and v
We know that

$$\cos\theta = \frac{u.v}{||u||.||v||}$$

$$\cos\theta = \frac{-42}{\sqrt{21}.\sqrt{84}} \Rightarrow \frac{-42}{\sqrt{1764}}$$

$$\cos\theta = \frac{-\cancel{42}}{\cancel{42}}$$

$$\theta = \cos^{-1}(-1)$$

$$\boxed{\theta = 180^{\circ}}$$

5 marks

Q11) using Cauchy Schwartz inequality prove that, where $a,b,c \in R$

$$\frac{b+c}{a} + \frac{c+a}{b} + \frac{a+b}{c} \geq 6$$

Solution:

Step 1): The Cauchy-Schwartz inequality states that for real Numbers x_i, y_i

$$\left(\sum x_i y_i\right)^2 \leq \left(\sum x_i^2\right)\left(\sum y_i^2\right)$$

For positive x, $x + \frac{1}{x} \geq 2$

Rearrange the term & Group the terms

$$\frac{b+c}{a} + \frac{c+a}{b} + \frac{a+b}{c}$$

$$= \frac{b}{a} + \frac{c}{a} + \frac{c}{b} + \frac{a}{b} + \frac{a}{c} + \frac{b}{c}$$

$$= \left(\frac{b}{a} + \frac{a}{b}\right) + \left(\frac{c}{a} + \frac{a}{c}\right) + \left(\frac{c}{b} + \frac{b}{c}\right) \quad \text{--- } ①$$

Step 2) Apply the AM-GM inequality

For x positive, $x + \frac{1}{x} \geq 2$

$$\left.\begin{aligned} \frac{b}{a} + \frac{a}{b} &\geq 2 \\ \frac{c}{a} + \frac{a}{c} &\geq 2 \\ \frac{c}{b} + \frac{b}{c} &\geq 2 \end{aligned}\right\} \quad \text{--- } ②$$

step3) Form eq ① and ② we will get

$$\left(\frac{b}{a}+\frac{a}{b}\right)+\left(\frac{c}{a}+\frac{a}{c}\right)+\left(\frac{c}{b}+\frac{b}{c}\right) \geq 2+2+2$$

$$\left(\frac{b+c}{a}\right)+\left(\frac{c+a}{b}\right)+\left(\frac{a+b}{c}\right) \geq 6$$

Hence proved.

Cauchy Schwartz Inequality

OR using CSI for $x>0$ positive.

Let $u=\left(\sqrt{x}, \frac{1}{\sqrt{x}}\right)$, and $v=\left(\frac{1}{\sqrt{x}}, \sqrt{x}\right)$

$$(u \cdot v)^2 = \sqrt{x}\left(\frac{1}{\sqrt{x}}\right)+\frac{1}{\sqrt{x}}(\sqrt{x})$$

$$= 1+1$$

$$u \cdot v = 2$$

$$\|u\| = \sqrt{x+\frac{1}{x}}$$

$$\|v\| = \sqrt{\frac{1}{x}+x}$$

$$|u \cdot v| \leq \|u\| \cdot \|v\|$$

$$2 \leq \left(\sqrt{x+\frac{1}{x}}\right)\left(\sqrt{x+\frac{1}{x}}\right)$$

$$2 \leq x+\frac{1}{x}$$

$$\Rightarrow \boxed{x+\frac{1}{x} \leq 2}$$

*

Module-04

10 marks

Q1) Find orthonormal basis for subspaces of R^3 of $S = \{(1,1,1), (0,1,1), (0,0,1)\}$ by applying the Gram Schmidt process.

10 marks

Q2) Find orthonormal basis for subspaces of R^3 of $S = \{(1,1,1), (-1,1,0), (1,2,1)\}$ by applying the Gram Schmidt process.

10 marks

Q3) Find orthonormal basis for subspaces of R^3 of $S = \{(1,0,0), (3,7,-2), (0,4,1)\}$ by applying the Gram Schmidt process.

10 marks

Q4) Find orthonormal basis for subspace of R^3 of $S = \{(3,0,4), (-1,0,7), (2,9,11)\}$ by applying the Gram Schmidt process.

→ Gram schmidt orthogonalization process :-

it has Three steps :

① Begin with a basis for the Inner product space. it need not be orthogonal nor consist of unit vectors.

② Convert the basis to an orthogonal basis.

③ Normalize each vector in orthogonal basis to form an orthonormal basis

$$\begin{cases} w_1 = v_1 \\ w_2 = v_2 - \dfrac{\langle v_2, w_1 \rangle}{\langle w_1, w_1 \rangle} w_1 \\ w_3 = v_3 - \dfrac{\langle v_3, w_1 \rangle}{\langle w_1, w_1 \rangle} w_1 - \dfrac{\langle v_3, w_2 \rangle}{\langle w_2, w_2 \rangle} w_2 . \end{cases}$$

→ w_1, w_2, w_3 — orthogonal basis

→ $\dfrac{w_1}{\|w_1\|}, \dfrac{w_2}{\|w_2\|}, \dfrac{w_3}{\|w_3\|}$ — orthonormal basis.

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10 marks

Q1) find orthonormal basis for subspaces of R^3 of $S = \{(1,1,1), (0,1,1), (0,0,1)\}$ by applying the Gram Schmidt process.

Solution: $S = \{(1,1,1), (0,1,1), (0,0,1)\}$.

Step 1) initial vectors

$v_1 = (1,1,1)$

$v_2 = (0,1,1)$

$v_3 = (0,0,1)$

Step 2) find orthogonal vectors

① first orthogonal vector

$$\boxed{w_1 = v_1 = (1,1,1)}$$

② second orthogonal vector

$$w_2 = v_2 - \frac{\langle v_2, w_1 \rangle}{\langle w_1, w_1 \rangle} w_1$$

$$\langle v_2, w_1 \rangle = v_2 \cdot w_1 = (0)(1) + (1)(1) + (1)(1)$$
$$= 2$$

$$\langle w_1, w_1 \rangle = w_1 \cdot w_1 = (1)(1) + (1)(1) + (1)(1)$$
$$= 3$$

$$w_2 = (0,1,1) - \frac{2}{3}(1,1,1)$$
$$= (0,1,1) - \left(\frac{2}{3}, \frac{2}{3}, \frac{2}{3}\right)$$

$$\boxed{w_2 = \left(-\frac{2}{3}, \frac{1}{3}, \frac{1}{3}\right)}$$

③ Find 3rd orthogonal vector

$$w_3 = v_3 - \frac{\langle v_3, w_1 \rangle}{\langle w_1, w_1 \rangle} w_1 - \frac{\langle v_3, w_2 \rangle}{\langle w_2, w_2 \rangle} w_2$$

$$\langle v_3, w_1 \rangle = v_3 \cdot w_1 = (0)(1) + (0)(1) + (1)(1)$$
$$= 1$$

$$\langle w_1, w_1 \rangle = w_1 \cdot w_1 = (1)(1) + (1)(1) + (1)(1)$$
$$= 3$$

$$\langle v_3, w_2 \rangle = v_3 \cdot w_2 = (0)\left(-\frac{2}{3}\right) + (0)\left(\frac{1}{3}\right) + (1)\left(\frac{1}{3}\right)$$
$$= \frac{1}{3}$$

$$\langle w_2, w_2 \rangle = w_2 \cdot w_2 = \left(-\frac{2}{3}\right)^2 + \left(\frac{1}{3}\right)^2 + \left(\frac{1}{3}\right)^2$$
$$= \frac{4}{9} + \frac{1}{9} + \frac{1}{9}$$
$$= \frac{6}{9}$$
$$= \frac{2}{3}$$

$$w_3 = (0,0,1) - \frac{1}{3}(1,1,1) - \frac{\left(\frac{1}{3}\right)}{\left(\frac{2}{3}\right)}\left(-\frac{2}{3}, \frac{1}{3}, \frac{1}{3}\right)$$

$$= (0,0,1) - \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right) - \frac{1}{2}\left(-\frac{2}{3}, \frac{1}{3}, \frac{1}{3}\right)$$

$$= (0,0,1) - \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right) - \left(-\frac{2}{6}, \frac{1}{6}, \frac{1}{6}\right)$$

$$= (0,0,1) - \left(\frac{1}{3} - \frac{2}{6}, \frac{1}{3} + \frac{1}{6}, \frac{1}{3} + \frac{1}{6}\right)$$

$$= (0,0,1) - \left(0, \frac{3}{6}, \frac{3}{6}\right)$$

$$= (0,0,1) - \left(0, \frac{1}{2}, \frac{1}{2}\right)$$

$$\boxed{w_3 = \left(0, -\frac{1}{2}, \frac{1}{2}\right)}$$

$$\left.\begin{array}{l} w_1 = (1,1,1) \\ w_2 = \left(-\frac{2}{3}, \frac{1}{3}, \frac{1}{3}\right) \\ w_3 = \left(0, -\frac{1}{2}, \frac{1}{2}\right) \end{array}\right\} \text{are orthogonal vectors.}$$

Step3) Normalize the orthogonal Vectors

$$\alpha_1 = \frac{w_1}{\|w_1\|} = \frac{(1,1,1)}{\sqrt{1^2+1^2+1^2}}$$

$$= \frac{(1,1,1)}{\sqrt{3}}$$

$$\boxed{\alpha_1 = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)}$$

$$\alpha_2 = \frac{w_2}{\|w_2\|} = \frac{(-\frac{2}{3}, \frac{1}{3}, \frac{1}{3})}{\sqrt{(-\frac{2}{3})^2 + (\frac{1}{3})^2 + (\frac{1}{3})^2}}$$

$$= \frac{(-\frac{2}{3}, \frac{1}{3}, \frac{1}{3})}{\sqrt{\frac{4}{9} + \frac{1}{9} + \frac{1}{9}}}$$

$$= \frac{(-\frac{2}{3}, \frac{1}{3}, \frac{1}{3})}{\sqrt{\frac{6}{9}}} \Rightarrow \frac{3.(-\frac{2}{3}, \frac{1}{3}, \frac{1}{3})}{\sqrt{6}}$$

$$= \frac{(-2, 1, 1)}{\sqrt{6}}$$

$$\boxed{\alpha_2 = \left(\frac{-2}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right)}$$

$$\alpha_3 = \frac{w_3}{\|w_3\|} = \frac{(0, -\frac{1}{2}, \frac{1}{2})}{\sqrt{(0)^2 + (-\frac{1}{2})^2 + (\frac{1}{2})^2}}$$

$$= \frac{(0, -\frac{1}{2}, \frac{1}{2})}{\sqrt{\frac{1}{4} + \frac{1}{4}}} \Rightarrow \frac{(0, -\frac{1}{2}, \frac{1}{2})}{\sqrt{\frac{2}{4}}}$$

$$= \frac{(0, -\frac{1}{2}, \frac{1}{2})}{\frac{1}{\sqrt{2}}}$$

$$= \sqrt{2}\left(0, \frac{-1}{(\sqrt{2})^2}, \frac{1}{(\sqrt{2})^2}\right)$$

$$\boxed{\alpha_3 = \left(0, \frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)}$$

$$\left.\begin{aligned} \alpha_1 &= \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right) \\ \alpha_2 &= \left(\frac{-2}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right) \\ \alpha_3 &= \left(0, \frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) \end{aligned}\right\} \text{are orthonormal vectors}$$

Conclusion: The orthonormal basis is

$$\left\{\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right), \left(\frac{-2}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right), \left(0, \frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)\right\}.$$

10marks

Q2) Find orthonormal basis for subspaces of R^3 of $S = \{(1,1,1), (-1,1,0), (1,2,1)\}$ by applying the Gram Schmidt process.

Solution:

$$S = \{(1,1,1), (-1,1,0), (1,2,1)\}$$

step1) initial vectors

$V_1 = (1,1,1)$

$V_2 = (-1,1,0)$

$V_3 = (1,2,1)$

step2) Find orthogonal vectors

① first orthogonal vector

$$\boxed{\omega_1 = V_1 = (1,1,1)}$$

② second orthogonal vector

$$\omega_2 = V_2 - \frac{\langle V_2, \omega_1 \rangle}{\langle \omega_1, \omega_1 \rangle} \omega_1$$

$$\langle V_2, \omega_1 \rangle = V_2 \cdot \omega_1 = (-1)(1) + (1)(1) + (0)(1)$$
$$= 0$$

$$\langle \omega_1, \omega_1 \rangle = \omega_1 \cdot \omega_1 = (1)^2 + (1)^2 + (1)^2$$
$$= 3$$

$$\omega_2 = (-1,1,0) - \frac{0}{3}(1,1,1)$$
$$= (-1,1,0) - (0,0,0)$$

$$\boxed{\omega_2 = (-1,1,0)}$$ //

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③ 3rd orthogonal vector

$$w_3 = v_3 - \frac{\langle v_3, w_1\rangle}{\langle w_1, w_1\rangle} w_1 - \frac{\langle v_3, w_2\rangle}{\langle w_2, w_2\rangle} w_2$$

$$\langle v_3, w_1\rangle = v_3 \cdot w_1 = (1)(1) + (2)(1) + (1)(1)$$
$$= 1 + 2 + 1$$
$$= 4$$

$$\langle w_1, w_1\rangle = w_1 \cdot w_1 = (1)^2 + (1)^2 + (1)^2$$
$$= 3$$

$$\langle v_3, w_2\rangle = v_3 \cdot w_2 = (1)(-1) + (2)(1) + (1)(0)$$
$$= -1 + 2$$
$$= 1$$

$$\langle w_2, w_2\rangle = w_2 \cdot w_2 = (-1)^2 + (1)^2 + (0)^2$$
$$= 2$$

$$w_3 = (1,2,1) - \frac{4}{3}(1,1,1) - \frac{1}{2}(-1,1,0)$$
$$= (1,2,1) - \left(\frac{4}{3}, \frac{4}{3}, \frac{4}{3}\right) - \left(\frac{-1}{2}, \frac{1}{2}, 0\right)$$
$$= (1,2,1) - \left(\frac{4}{3} - \frac{1}{2}, \frac{4}{3} + \frac{1}{2}, \frac{4}{3} + 0\right)$$

$$= (1,2,1) - \left(\frac{5}{6}, \frac{11}{6}, \frac{4}{3}\right)$$
$$= \left(1 - \frac{5}{6}, 2 - \frac{11}{6}, 1 - \frac{4}{3}\right)$$
$$\boxed{w_3 = \left(\frac{1}{6}, \frac{1}{6}, \frac{-1}{3}\right)}$$

$$\left.\begin{array}{l} w_1 = (1,1,1) \\ w_2 = (-1,1,0) \\ w_3 = \left(\frac{1}{6}, \frac{1}{6}, \frac{-1}{3}\right) \end{array}\right\}$$ are orthogonal vectors

step3). Normalize the orthogonal vectors

$$\alpha_1 = \frac{w_1}{\|w_1\|} = \frac{(1,1,1)}{\sqrt{1^2+1^2+1^2}} \Rightarrow \frac{(1,1,1)}{\sqrt{3}}$$

$$\boxed{\alpha_1 = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)}$$

$$\alpha_2 = \frac{w_2}{\|w_2\|} = \frac{(-1,1,0)}{\sqrt{(-1)^2+(1)^2+(0)^2}}$$

$$= \frac{(-1,1,0)}{\sqrt{2}}$$

$$\boxed{\alpha_2 = \left(\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)}$$

$$\alpha_3 = \frac{w_3}{\|w_3\|} = \frac{\left(\frac{1}{6}, \frac{1}{6}, \frac{-1}{3}\right)}{\sqrt{\left(\frac{1}{6}\right)^2+\left(\frac{1}{6}\right)^2+\left(\frac{1}{3}\right)^2}}$$

$$= \frac{\left(\frac{1}{6}, \frac{1}{6}, \frac{-1}{3}\right)}{\sqrt{\frac{1}{36}+\frac{1}{36}+\frac{1}{9}}}$$

$$= \frac{\left(\frac{1}{6}, \frac{1}{6}, \frac{-1}{3}\right)}{\sqrt{\cancel{\frac{6}{36}}_{6}}}$$

$$= \frac{\left(\frac{1}{6}, \frac{1}{6}, \frac{-1}{3}\right)}{\sqrt{\frac{1}{6}}}$$

$$= \sqrt{6}\cdot\left(\frac{1}{6}, \frac{1}{6}, \frac{-1}{3}\right)$$

$$\boxed{\alpha_3 = \left(\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{-\sqrt{6}}{3}\right)}$$ $\boxed{\text{or}}$ $\left(\frac{\sqrt{6}}{6}, \frac{\sqrt{6}}{6}, \frac{-\sqrt{6}}{3}\right)$

$$\left.\begin{aligned} \alpha_1 &= \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right) \\ \alpha_2 &= \left(\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right) \\ \alpha_3 &= \left(\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{-\sqrt{6}}{3}\right) \end{aligned}\right\} \text{are orthonormal vectors}$$

$\boxed{\text{OR}}$ $\left(\frac{\sqrt{6}}{6}, \frac{\sqrt{6}}{6}, \frac{-\sqrt{6}}{3}\right)$

Conclusion:

The orthonormal basis is

$$\left\{\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right), \left(\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right), \left(\frac{\sqrt{6}}{6}, \frac{\sqrt{6}}{6}, \frac{-\sqrt{6}}{3}\right)\right\}$$

10 marks

Q3) Find orthonormal basis for subspaces of R^3 of $S=\{(1,0,0),(3,7,-2),(0,4,1)\}$ by applying the Gram Schmidt process.

Solution: $S=\{(1,0,0),(3,7,-2),(0,4,1)\}$

Step 1) initial vectors

$v_1=(1,0,0)$

$v_2=(3,7,-2)$

$v_3=(0,4,1)$

Step 2) Find orthogonal vectors

① First orthogonal vector

$\boxed{w_1=v_1=(1,0,0)}$

② Second orthogonal vector

$$w_2=v_2-\frac{\langle v_2,w_1\rangle}{\langle w_1,w_1\rangle}\cdot w_1$$

$\langle v_2,w_1\rangle=v_2\cdot w_1=(1)(3)+(0)(7)+(0)(-2)$

$=3$

$\langle w_1,w_1\rangle=w_1\cdot w_1=(1)^2+(0)^2+(0)^2$

$=1$

$w_2=(3,7,-2)-\frac{3}{1}(1,0,0)$

$=(3,7,-2)-(3,0,0)$

$=(0,7,-2)$

$\boxed{w_2=(0,7,-2)}$

③ 3^{rd} orthogonal vector

$$w_3 = v_3 - \frac{\langle v_3, w_1 \rangle}{\langle w_1, w_1 \rangle} w_1 - \frac{\langle v_3, w_2 \rangle}{\langle w_2, w_2 \rangle} w_2$$

$$\langle v_3, w_1 \rangle = v_3 \cdot w_1 = (0,4,1) \cdot (1,0,0)$$
$$= (0)(1) + (4)(0) + (1)(0)$$
$$= 0$$

$$\langle w_1, w_1 \rangle = w_1 \cdot w_1 = (1,0,0) \cdot (1,0,0)$$
$$= (1)^2 + (0)^2 + (0)^2$$
$$= 1$$

$$\langle v_3, w_2 \rangle = v_3 \cdot w_2 = (0,4,1) \cdot (0,7,-2)$$
$$= (0)(0) + (4)(7) + (1)(-2)$$
$$= 28 - 2$$
$$= 26$$

$$\langle w_2, w_2 \rangle = w_2 \cdot w_2 = (0,7,-2) \cdot (0,7,-2)$$
$$= (0)^2 + (7)^2 + (-2)^2$$
$$= 49 + 4$$
$$= 53$$

$$w_3 = (0,4,1) - \frac{0}{1}(1,0,0) - \frac{26}{53}(0,7,-2)$$
$$= (0,4,1) - \left(0, \frac{182}{53}, \frac{-52}{53}\right)$$
$$= \left(0, 4 - \frac{182}{53}, 1 + \frac{52}{53}\right)$$
$$= \left(0, \frac{212-182}{53}, \frac{53+52}{53}\right)$$
$$\boxed{w_3 = \left(0, \frac{30}{53}, \frac{105}{53}\right)}$$

$$\left.\begin{aligned} w_1 &= (1,0,0) \\ w_2 &= (0,7,-2) \\ w_3 &= \left(0, \frac{30}{53}, \frac{105}{53}\right) \end{aligned}\right\}$$ are orthogonal vectors

Step3) Normalize the orthogonal vectors

$$\alpha_1 = \frac{w_1}{\|w_1\|} = \frac{(1,0,0)}{\sqrt{1^2+0^2+0^2}} = (1,0,0)$$

$$\boxed{\alpha_1 = (1,0,0)}$$

$$\alpha_2 = \frac{w_2}{\|w_2\|} = \frac{(0,7,-2)}{\sqrt{0^2+7^2+(-2)^2}} = \frac{(0,7,-2)}{\sqrt{49+4}}$$

$$= \frac{(0,7,-2)}{\sqrt{53}}$$

$$\boxed{\alpha_2 = \left(0, \frac{7}{\sqrt{53}}, \frac{-2}{\sqrt{53}}\right)}$$

$$\alpha_3 = \frac{w_3}{\|w_3\|} = \frac{\left(0, \frac{30}{53}, \frac{105}{53}\right)}{\sqrt{0^2+\left(\frac{30}{53}\right)^2+\left(\frac{105}{53}\right)^2}}$$

$$= \frac{\left(0, \frac{30}{53}, \frac{105}{53}\right)}{\sqrt{\frac{900+11025}{(53)^2}}}$$

$$= \frac{53\left(0, \frac{30}{53}, \frac{105}{53}\right)}{\sqrt{11925}}$$

$$\alpha_3 = \left(0, \frac{30}{\sqrt{11925}}, \frac{105}{\sqrt{11925}}\right)$$

$$\sqrt{11925} = \sqrt{225 \times 53} = 15\sqrt{53}.$$

$$\alpha_3 = \left(0, \frac{30}{15\sqrt{53}}, \frac{105}{15\sqrt{53}}\right)$$

$$\boxed{\alpha_3 = \left(0, \frac{2}{\sqrt{53}}, \frac{7}{\sqrt{53}}\right)}$$

$$\left.\begin{aligned} \alpha_1 &= (1,0,0) \\ \alpha_2 &= \left(0, \frac{7}{\sqrt{53}}, \frac{-2}{\sqrt{53}}\right) \\ \alpha_3 &= \left(0, \frac{2}{\sqrt{53}}, \frac{7}{\sqrt{53}}\right) \end{aligned}\right\} \text{are orthonormal vectors}$$

Conclusion:

The orthonormal basis is

$$\left\{(1,0,0), \left(0, \frac{7}{\sqrt{53}}, \frac{-2}{\sqrt{53}}\right), \left(0, \frac{2}{\sqrt{53}}, \frac{7}{\sqrt{53}}\right)\right\}$$

10 marks

Q4) find orthonormal basis for subspace of R^3 of $S=\{(3,0,4),(-1,0,7),(2,9,11)\}$ by applying the Gram schmidt process.

Solution: $S=\{(3,0,4),(-1,0,7),(2,9,11)\}$

Step1) initial vectors

$V_1=(3,0,4)$

$V_2=(-1,0,7)$

$V_3=(2,9,11)$

Step2) ~~find orthonormal ba~~

find orthogonal vectors

① first orthogonal vector

$\boxed{w_1=V_1=(3,0,4)}$

② Second orthogonal vector

$$w_2=V_2-\frac{\langle V_2,w_1\rangle}{\langle w_1,w_1\rangle}\cdot w_1$$

$$\begin{aligned}\langle V_2,w_1\rangle = V_2\cdot w_1 &= (-1,0,7)\cdot(3,0,4)\\ &=(-1)(3)+(0)(0)+(7)(4)\\ &=-3+28\\ &=25\end{aligned}$$

$$\begin{aligned}\langle w_1,w_1\rangle = w_1\cdot w_1 &= (3,0,4)\cdot(3,0,4)\\ &=(3)(3)+(0)(0)+(4)(4)\\ &=9+16\\ &=25\end{aligned}$$

$$w_2 = (-1,0,7) - \frac{\cancel{25}^{\,1}}{25}(3,0,4)$$
$$= (-1,0,7) - (3,0,4)$$
$$= (-4,0,3)$$

$$\boxed{w_2 = (-4,0,3)}$$

③ 3rd orthogonal vector

$$w_3 = v_3 - \frac{\langle v_3, w_1 \rangle}{\langle w_1, w_1 \rangle} \cdot w_1 - \frac{\langle v_3, w_2 \rangle}{\langle w_2, w_2 \rangle} \cdot w_2$$

$$\langle v_3, w_1 \rangle = v_3 \cdot w_1 = (2,9,11) \cdot (3,0,4)$$
$$= (2)(3) + (9)(0) + (11)(4)$$
$$= 6 + 44$$
$$= 50$$

$$\langle w_1, w_1 \rangle = w_1 \cdot w_1 = (3,0,4) \cdot (3,0,4)$$
$$= (3)^2 + (0)^2 + (4)^2$$
$$= 9 + 16$$
$$= 25$$

$$\langle v_3, w_2 \rangle = v_3 \cdot w_2 = (2,9,11) \cdot (-4,0,3)$$
$$= (2)(-4) + (9)(0) + (11)(3)$$
$$= -8 + 0 + 33$$
$$= 25$$

$$\langle w_2, w_2 \rangle = w_2 \cdot w_2 = (-4,0,3) \cdot (-4,0,3)$$
$$= (-4)^2 + (0)^2 + (3)^2$$
$$= 16 + 9$$
$$= 25$$

$$w_3 = (2,9,11) - \frac{50}{25}(3,0,4) - \frac{25}{25}(-4,0,3)$$

$$= (2,9,11) - (6,0,8) - (-4,0,3)$$

$$= (2-6+4, 9-0-0, 11-8-3)$$

$$= (0,9,0)$$

$$\boxed{w_3 = (0,9,0)}$$

$$\left.\begin{aligned} w_1 &= (3,0,4) \\ w_2 &= (-4,0,3) \\ w_3 &= (0,9,0) \end{aligned}\right\} \text{ are orthogonal Vectors}$$

Step 3). Normalize the orthogonal vectors

$$\alpha_1 = \frac{w_1}{\|w_1\|} = \frac{(3,0,4)}{\sqrt{(3)^2+(0)^2+(4)^2}}$$

$$= \frac{(3,0,4)}{\sqrt{9+16}} \Rightarrow \frac{(3,0,4)}{\sqrt{25}}$$

$$= \frac{(3,0,4)}{5}$$

$$\boxed{\alpha_1 = \left(3/5, 0, 4/5\right)}$$

$$\alpha_2 = \frac{w_2}{\|w_2\|}$$

$$= \frac{(-4,0,3)}{\sqrt{(-4)^2+(0)^2+(3)^2}} \Rightarrow \frac{(-4,0,3)}{\sqrt{25}}$$

$$\alpha_2 = \frac{(-4,0,3)}{5}$$

$$\boxed{\alpha_2 = \left(\frac{-4}{5}, 0, \frac{3}{5}\right)}$$

$$\alpha_3 = \frac{w_3}{\|w_3\|} = \frac{(0,9,0)}{\sqrt{0^2+9^2+0^2}} \Rightarrow \frac{(0,9,0)}{\sqrt{81}}$$

$$= \frac{(0,9,0)}{9}$$

$$\boxed{\alpha_3 = (0,1,0)}$$

$$\left.\begin{aligned} \alpha_1 &= (3/5, 0, 4/5) \\ \alpha_2 &= (-4/5, 0, 3/5) \\ \alpha_3 &= (0,1,0) \end{aligned}\right\}$$ are orthonormal vectors

Conclusion:

The orthonormal basis is

$$\{(3/5, 0, 4/5), (-4/5, 0, 3/5), (0,1,0)\}$$