RECAP

1. Given a sequence (a_n) of complex numbers, we have a sequence (S_n)

$$S_n := \sum_{k=1}^{n} a_k$$

We say that $\sum_{k=1}^{\infty} a_k$ converges if $\lim_{n\to\infty} S_n$ exists

Otherwise, we say it diverges

$$\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} S_n$$

Given a seq. (x_n) of reals, define

$$y_n := \sup \{ x_m : m \geq n \}$$

$$x_1, x_2 \ldots \underbrace{\{x_n, x_{n+1}, \ldots\}}_{\sup =: y_n})$$

Remarks: (About y_n)

$1, 2, 3, \ldots 4$

y_n is a monotonically decreasing sequence

$\lim_{n\to\infty} y_n$ exists (It can be ∞)

$$\limsup_{n\to\infty} x_n = \lim_{n\to\infty} y_n$$

Radius of convergence:-

Given a series of the form $\sum_{n=0}^{\infty} a_n (z-z_0)^n$ (PS)

$\exists R \in [0, \infty]$

PS converges for all $z: |z-z_0| < R$

" diverges for all $z: |z-z_0| > R$

Calculation -

Define $\alpha := \limsup (|a_n|)^{1/n}$

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Define $\alpha := \limsup_{n \to \infty} (|a_n|)^{1/n}$

$R := 1/\alpha$ (R - radius of conv as before)

Ratio test. Let (a_n) be defined as above

then (IF) $\alpha = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$ exists

then $R := 1/\alpha$

Tutorial 2 :-

Q1.)

1. If $u(X, Y)$ and $v(X, Y)$ are harmonic conjugates of each other, show that they are constant functions.

Proof. Since v is a harmonic conj of u

$u_x = v_y$ (i) and $u_y = -v_x$ (ii)

" u " v

$v_x = u_y$ (iii) and $v_y = -u_x$ (iv)

$\Longrightarrow$ $u_x = -u_x$ (i) (iv)

$u_y = -u_y$ (iii) (ii)

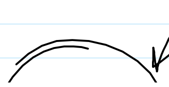
$\Longrightarrow$ $u_x = 0$ and $u_y = 0$

$\Longrightarrow$ $v_x = 0$ and $v_y = 0$

(Assumption)

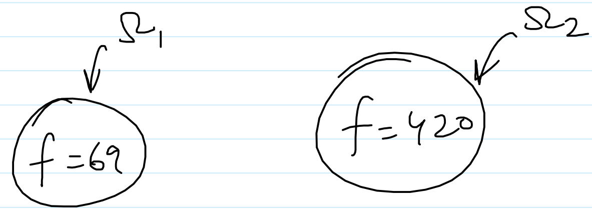
Since the domain is path - connected $\Longrightarrow$ u, v are constant $\square$

Ω_1 Ω_2



Ω := open subset

$\Omega :=$ open subset



Q2.)

2. Show that $u = XY + 3X^2Y - Y^3$ is harmonic and find its harmonic conjugate.

Given $\quad u(x,y) = xy + 3x^2y - y^3$

Show harmonic:

$$u_{xx} + u_{yy} = (0 + 6y + 0) + (0 + 0 - 6y) = 0$$

$\square$

To find harmonic conjugate:

$$u_x = v_y$$

$$v_y = y + 6xy$$

$$v(x,y) = \frac{y^2}{2} + 3xy^2 + f(x)$$

$$u_y = -v_x$$

$$3y^2 + f'(x) = -(x + 3x^2 - 3y^2)$$

$$f'(x) = -(x + 3x^2)$$

$$f(x) = -\frac{x^2}{2} - x^3 + C$$

$$v(x,y) = y^2/2 + 3xy^2 - x^2/2 - x^3$$

Alternative : Im part of $\left(z^3 + \frac{z^2}{2}\right)$

$z = x + iy$

$f(z) = z^3 + \frac{z^2}{2}$ is holomorphic on $\mathbb{C}$

$\Rightarrow$ $-$ve of Real part is harmonic conjugate

Q3.)

3. Find the radius of convergence of the following power series :

a) $\sum_{n=0}^{\infty} nz^n$
b) $\sum_{p \text{ prime}} z^p$
c) $\sum \frac{n! z^n}{n^n}$

$\alpha \implies R = \alpha^{-1}$

note we have

a.) $\alpha := \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{n+1}{n} \right| = 1$

$$R = \frac{1}{\alpha} = 1$$

b.) $\sum_{p \text{ is prime}} z^p$

$$a_n := \begin{cases} 1 & \text{when } n \text{ is prime} \\ 0 & \text{when } n \text{ is comp.} \end{cases}$$

$$\alpha := \limsup_{n\to\infty} (|a_n|)^{1/n}$$

$$= \limsup_{n\to\infty} (1)^{1/n}$$

$$= 1$$

$R := 1/\alpha = 1$

c.) $\sum_{n=0}^{\infty} \frac{n!}{n^n} z^n$

$a_n := n!/n^n$

$$\alpha := \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$$

$$= \lim_{n\to\infty} \left| \frac{\cancel{(n+1)!}}{(n+1)^{n+\cancel{1}}} \times \frac{n^n}{\cancel{n!}} \right|$$

$$= \lim_{n\to\infty} \left| \frac{n^n}{(n+1)^n} \right|$$

$$= \lim_{n\to\infty} \left| \frac{1}{(1+1/n)^n} \right|$$

$$= 1/e$$

$$R := 1/\alpha = e$$

Q4.) 4. Show that $L > 1$ in the ratio test (Lecture 3 slides) does not neccessarily imply that the series is divergent.

Solution :) Consider the sequence (a_n)

$$1/1^3,\ 1/1^2,\ 1/2^3,\ 1/2^2,\ \cdots\ 1/n^3,\ 1/n^2$$

$$a_{2n-1} = 1/n^3 \quad \text{and} \quad a_{2n} = 1/n^2$$

Why the series converges? :

Why the series converges? :

$$\left[\ 0 < \sum_{k=1}^{\infty} a_{2k} = 1 + \sum_{k=2}^{\infty} a_{2k}\right.$$

$$\left. < 1 + \sum_{k=2}^{\infty} \frac{1}{k(k-1)} = 2\ \right]$$

$$\left[\ 0 < \sum a_{2k} < 2\ \right]$$

$$\Longrightarrow \quad \sum a_{2k} \text{ is convergent}$$

And we also have $0 < \sum_{k=1}^{\infty} a_{2k-1} < \sum_{k=1}^{\infty} a_{2k}$

cubes in den   squares

$$\Longrightarrow \quad \sum a_{2k-1} \text{ is also convergent}$$

$\therefore$ Sum is convergent

Note:

$$L = \limsup_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$$

$$\geq \limsup_{n\to\infty} \left|\frac{a_{2n}}{a_{2n-1}}\right|$$

$$= \limsup_{n\to\infty} n = \infty$$

$$L = \infty \quad (\text{clearly} > 1)$$

(Not only does $L > 1$ not ensure divergence, but even $L = \infty$ cannot ensure divergence)

Q5 :)

5. Construct a infinitely differentiable function $f : \mathbb{R} \to \mathbb{R}$ which is non-zero but vanishes outside a bounded set. Show that there are no holomorphic functions which satisfy this property.

Ans) $f \; \mathbb{R} \longrightarrow \mathbb{R}$ such that

$$f(x) = \begin{cases} e^{\frac{-1}{1-x^2}} & x \in (-1,1) \\ 0 & \text{otherwise} \end{cases}$$

1. $f(0) = e^{\frac{-1}{1-0}} = e^{-1} = 1/e \neq 0$

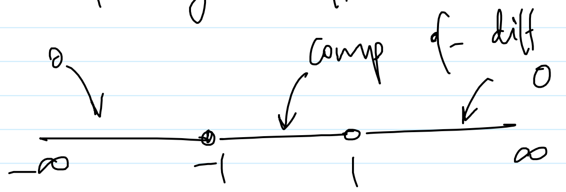
2: define $S := (-1,1)$

outside S $(\mathbb{R} \setminus S)$ $f = 0$

3. Infinitely differentiable

chain rule

$(-1,1)$



4: Any holomorphic function can be written as.

$$h(z) = \sum_{i=0}^{\infty} \frac{h^{(i)}(z_0)}{i!} (z - z_0)^i \qquad z, z_0 \in \mathbb{C}$$

z_0 on the $\mathbb{R}$ > 1

$$h(z) = h(z_0) + \frac{h'(z_0)}{1}(z - z_0) \cdots$$

$= 0 + 0 + 0 \ldots = 0 \; // \; \Rightarrow\Leftarrow$

for some neighborhood around z_0

Proves that the function is not holomorphic

Q6.)

6. Show that $\exp : \mathbb{C} \to \mathbb{C}^{\times}$ is onto.

Let $z_0 \in \mathbb{C}^{x}$ we wish to show

$\exists \; z \in \mathbb{C}$ such that $\exp(z) = z_0$

Now $z_0 \neq 0$

$r_0 := |z_0| \neq 0$

and $w_0 = \dfrac{z_0}{r_0}$

Moreover $|w_0| = \dfrac{|z_0|}{|r_0|} = 1$ (well defined)

So, $w_0 = x_0 + iy_0$ for some $(x_0, y_0) \in \mathbb{R}^2$

and satisfies $x_0^2 + y_0^2 = 1$

$\therefore \; \exists \; \theta \in [0, 2\pi)$ such that

$x_0 = \cos\theta$ and $y_0 = \sin\theta$

define $z := \log(r_0) + i\theta$

define $z := \log(r_0) + i\theta$

we have $\exp(z) = \exp(\log r_0 + i\theta)$

$= \exp(\log r_0)\exp(i\theta)$

$= r_0(\cos\theta + i\sin\theta)$

$= z_0$ $\square$

Q7)

7. Show that $\sin, \cos : \mathbb{C} \to \mathbb{C}$ are surjective. (In particular, note the difference with real sine and cosine which were bounded by 1).

$$\sin(z) = \frac{e^{iz} - e^{-iz}}{2i} \qquad e^{iz} = t$$

Consider the quadratic

$$\frac{t - 1/t}{2i} = z_0$$

$$t^2 - 2iz_0 t - 1 = 0$$

[Thus always has 2 roots (why?) = FTA]

Let t_1 be a root

clearly $t_1 \neq 0$ (why?) $t_1 t_2 = -1$

$e^{iz} = t_1$ By Q6, $\exists z'$ such that

$e^{z'} = t_1$ $\boxed{z \overset{+}{=} i z'}$

$\sin(z) = z_0$ $\square$

$$\sin(z) = z_0 \qquad \square$$

[Do $\cos(z)$ yourself !]

Q8.) 8. Show that for any complex number z, $sin^2(z) + cos^2(z) = 1$.

$$\sin(z) = \frac{e^{iz} - e^{-iz}}{2i} \qquad \cos(z) = \frac{e^{iz} + e^{-iz}}{2}$$

$$\sin^2(z) = \frac{e^{2iz} + e^{-2iz} - 2}{-4}$$

$$\cos^2(z) = \frac{e^{2iz} + e^{-2iz} + 2}{4}$$

$$\frac{\cancel{-(e^{2iz} + e^{-2iz})} + 2}{4} + \frac{\cancel{(e^{2iz} + e^{-2iz})} + 2}{4}$$

$$= 1 \qquad \square$$

Nicer solution:

$f : \mathbb{C} \to \mathbb{C}$ [holomorphic]

Consider $f(z) = \sin^2(z) + \cos^2(z) - 1$

Observe that $f(z)$ $z \in \mathbb{R}$ $= 0$

So, $f(z) = 0$ on the $\mathbb{R}$ line

Since $\mathbb{R}$ is not a discrete set, then $f(z) = 0$ throughout $\mathbb{C}$