MA 205 Tutorial - 6

Autumn 2021

1 Question 1

$$\int_0^{2\pi} \frac{\cos^2 3x}{5 - 4\cos 2x}\,\mathrm{dx}$$

We would like to transform the integral to over the unit circle. As,

$$\begin{aligned}\cos(n\theta) &= \frac{e^{\iota n\theta} + e^{-\iota n\theta}}{2} \\ &= \frac{z^n + z^{-n}}{2}\end{aligned}$$

for $z = e^{\iota\theta}$. Then,

$$\begin{aligned}\int_0^{2\pi} \frac{\cos^2 3x}{5 - 4\cos 2x}\,\mathrm{dx} &= \int_{|z|=1} \left(\frac{(\frac{z^3+z^{-3}}{2})^2}{5 - 4\left(\frac{z^2+z^{-2}}{2}\right)} \right) \frac{1}{\iota z}\,\mathrm{dz} \\ &= \frac{1}{4\iota} \int_{|z|=1} \frac{1}{z^5} \left(\frac{(z^6+1)^2}{5z^2 - 2(z^4+1)} \right) \mathrm{dz} \\ &= -\frac{1}{8\iota} \int_{|z|=1} \frac{1}{z^5} \left(\frac{(z^6+1)^2}{(z^2-2)(z^2-1/2)} \right) \mathrm{dz}\end{aligned}$$

The integrand has the following poles

$$z = \left\{ 0, \pm\sqrt{2}, \pm\frac{1}{\sqrt{2}} \right\}$$

$\pm\sqrt{2}$ lie outside the unit circle, so we need not worry about it.
We will calculate residue for the remaining poles.
For $z = 0$, a pole of order 5, rather than differentiating the integrand 4 times, we can get the residue via Laurent Series.

$$\begin{aligned}\left(\frac{(z^6+1)^2}{z^5(z^2-2)(z^2-1/2)} \right) &= \left(\frac{z^{12} + 2z^6 + 1}{z^5} \right) \frac{1}{\left[1 - \left(\frac{5}{2}z^2 - z^4\right)\right]} \\ &= \left(\frac{z^{12} + 2z^6 + 1}{z^5} \right) \left[1 + \left(\frac{5}{2}z^2 - z^4 \right) + \left(\frac{5}{2}z^2 - z^4 \right)^2 + ... \right]\end{aligned}$$

We need to find coefficient of z^{-1} i.e. coefficient of z^4 in [...]. That comes out to be

$$-1 + \frac{25}{4} = \frac{21}{4}$$

For $z = \pm\frac{1}{\sqrt{2}}$, we can calculate residue directly.
$z = \frac{1}{\sqrt{2}}$:

$$\left(\frac{\left(\left(\frac{1}{\sqrt{2}}\right)^6 + 1\right)^2}{\left(\frac{1}{\sqrt{2}}\right)^5 \left(\left(\frac{1}{\sqrt{2}}\right)^2 - 2\right)\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right)} \right) = 4\sqrt{2}\left(\frac{\frac{81}{64}}{-\frac{3}{2}\sqrt{2}}\right)$$
$$= -\frac{27}{8}$$

For $z = -\frac{1}{\sqrt{2}}$ as well, residue comes out to be $-\frac{27}{8}$. So,

$$\begin{aligned} -\frac{1}{8\iota}\int_{|z|=1} \frac{1}{z^5}\left(\frac{(z^6+1)^2}{(z^2-2)(z^2-1/2)}\right) \mathrm{dz} &= -\frac{1}{8\iota}\cdot 2\pi\iota \cdot \left[\frac{21}{4} + \left(-\frac{27}{8}\right) + \left(-\frac{27}{8}\right)\right] \\ &= -\frac{1}{8\iota}\cdot 2\pi\iota\cdot\left(-\frac{3}{2}\right) \\ &= \frac{3}{8}\pi \end{aligned}$$

2 Question 2

Let

$$f(z) = \frac{2z^3 + z^2 + 4}{z^4 + 4z^2}$$

The set of singularities of the function are just poles at $\{0, \pm 2i\}$ all of which lie in the interior of the contour under consideration. Hence the integral can be given as

$$2\pi i \sum_{z_o \in \{0, \pm 2i\}} \mathrm{Res}(f; z_o)$$

Residue at $2i$ is given as

$$\lim_{z\to 2i}(z-2i)f(z) = \lim_{z\to 2i}(z-2i)\frac{2z^3+z^2+4}{z^4+4z^2} = \lim_{z\to 2i}\frac{2z^3+z^2+4}{(z+2i)z^2} = 1$$

Using symmetry arguments or otherwise, similarly as above one can see the residue at $-2i$ is also 1

We have a second order pole at 0. Thus the residue at 0 is given as $g'(0)$ where $g(z) = z^2 f(z) = \frac{1}{4}(2z^3 + z^2 + 4)(1 + \frac{z^2}{4})^{-1}$ (One can directly look for the coefficient of z in the expansion of g(z) to get the residue) which turns out to be 0

Thus we obtain

$$\int_{|z-2|=4} \frac{2z^3+z^2+4}{z^4+4z^2} dz = 2\pi i \sum_{z_o \in \{0, \pm 2i\}} \operatorname{Res}(f; z_o) = 2\pi i(0+1+1) = 4\pi i$$

3 Question 3

Solution.
Without OMT: Writing $f = u + iv$ as usual, we see that

$$u^2 + v^2 \equiv c.$$

If $c = 0$, then we are done. Assume $c \neq 0$.
Differentiating the above w.r.t. x gives us

$$uu_x + vv_x = 0. \tag{$*$}$$

Similarly, differentiating w.r.t. y gives us

$$uu_y + vv_y = 0.$$

Using CR equations, the last equation can be re-written as

$$-uv_x + vu_x = 0. \tag{$**$}$$

($*$) and ($**$) together give us

$$\begin{bmatrix} u & v \\ v & -u \end{bmatrix} \begin{bmatrix} u_x \\ v_x \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Note that $\det \begin{bmatrix} u & v \\ v & -u \end{bmatrix} \equiv -c \neq 0$ and thus, $u_x = v_x \equiv 0$ on Ω.
This gives us that $f' \equiv 0$ on Ω and thus, f is constant, since Ω is connected.

With OMT: Suppose f is not constant. We show that this gives a contradiction.

By our assumption that f is not constant, OMT tells us that the image $f(\Omega)$ must be open in $\mathbb{C}$. However, $|f|$ being constant tells us that $f(\Omega)$ is a subset of the circle $\{z : |z| = c\}$, where c is the constant $|f|$ equals.

However, no subset of such a circle is open. (Why?) This shows that f must be constant. $\square$

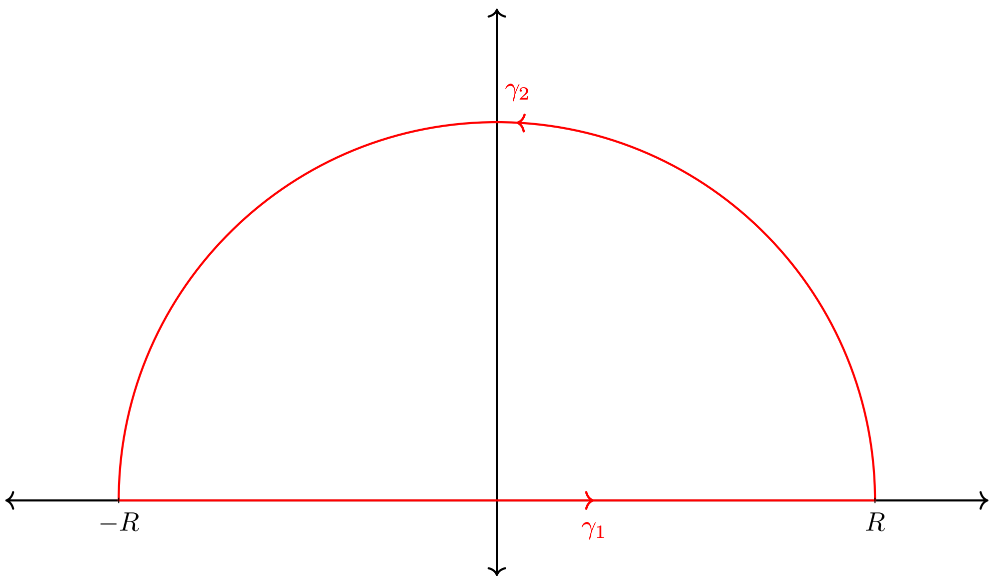
4 Question 4

Show that $\displaystyle\int_{-\infty}^{\infty} \frac{x}{(x^2+2x+2)(x^2+4)}\mathrm{d}x = -\frac{\pi}{10}$.

Solution. The usual technique. Define

$$f(z) := \frac{z}{(z^2+2z+2)(z^2+4)}.$$

Its poles are $-1 \pm \iota, \pm 2\iota$. Thus, if we take $R > 2$, then all the poles **in the upper half plane** are enclosed in the following contour.



Applying residue theorem to the above contour to get

$$\int_{\gamma_1} f(z)\mathrm{d}z + \int_{\gamma_2} f(z)\mathrm{d}z = 2\pi\iota \sum_{z\in\{2\iota,-1+\iota\}} \mathrm{Res}(f;z).$$

Along γ_1, the integral is what we want, in the limit $R \to \infty$. Along γ_2, a simple application of ML tells us that the integral is zero, in the limit $R \to \infty$ (can recall the result from class about rational functions where degree of numerator exceeds that of denominator by 2). Thus, we get the desired integral to be

$$2\pi\iota \sum_{z\in\{2\iota,-1+\iota\}} \mathrm{Res}(f;z).$$

Both the poles being considered above are simple and the residue calculation is simple.

Indeed, to compute the pole at 2ι, we note that

$$(z - 2\iota)f(z) = \frac{z}{(z^2 + 2z + 2)(z + 2\iota)}.$$

Letting $z \to 2\iota$ above gives

$$\text{Res}(f; 2\iota) = \frac{2\iota}{(-4 + 4\iota + 2)(4\iota)} = \frac{1}{4(2\iota - 1)}.$$

Similarly compute the other residue. (Note that I haven't computed and checked that we actually get $-\pi/10$. Let me know if there's an error in the above calculation.) $\square$

5 Question 5

Rouche's Theorem : Let $f(z)$ and $g(z)$ be analytic functions on the bounded domain D that extend continuously to ∂D and satisfy

$$|f(z) + g(z)| < |f(z)| + |g(z)|$$

on ∂D. Then $f(z)$ and $g(z)$ have the same number of zeros in D counting multiplicity.

Solution to the problem : Let us consider that $g(z) = z^5 + z^2 - 6z + 3$ and $f(z) = 6z - 3$. Then $f(z) + g(z) = z^5 + z^2$.

Note that on the circle $|z| = 1$, we have

$$|f(z) + g(z)| \leq |z|^5 + |z|^2 = 2 < 3 = \big|6|z| - 3\big| \leq |f(z)| + |g(z)|.$$

Similar computation shows that on the circle $|z| = \frac{1}{3}$, we have

$$|f(z) + g(z)| < |f(z)| + |g(z)|.$$

Thus by Rouche's Theorem $f(z)$ and $g(z)$ have the same number of zeros in the annulus $\frac{1}{3} < |z| < 1$. So the number of zeros of the polynomial $g(z) = z^5 + z^2 - 6z + 3$ counted with multiplicity in the domain $\frac{1}{3} < |z| < 1$ is one.

6 Question 6

Let D be the annulus $1 < |z| < 2$. To show that $u(x, y) = \log(x^2 + y^2)$ is harmonic on D, we compute Δu at an arbitrary point (x, y) in the domain and show that it equals zero.

$$u_x(x,y) = \frac{2x}{x^2+y^2}, \quad u_y(x,y) = \frac{2y}{x^2+y^2}$$
$$\therefore u_{xx}(x,y) = \frac{-2x^2+2y^2}{(x^2+y^2)^2}, \quad u_{yy}(x,y) = \frac{2x^2-2y^2}{(x^2+y^2)^2}$$

Hence, $\Delta u = 0$ on D, so u is harmonic on D. However, there is no harmonic conjugate of u on D, as we show below.

Let $D_1 = D \setminus (-2,-1)$ and define $g\colon D_1 \to \mathbb{C}$ by $g(z) = 2\log z$ for all $z \in D_1$. Then, g is holomorphic on D_1, and $\Re(g(z)) = u(x,y)$ for every $z \in D_1$. Hence, $u(x,y)$ has a harmonic conjugate $v_1(x,y) = \Im(g(z)) = 2\operatorname{Arg}(z)$ on D_1.

Now, suppose (for the sake of contradiction) that $v(x,y)$ is a harmonic conjugate of $u(x,y)$ on D, so that $f(z) = u(x,y) + iv(x,y)$ is a holomorphic function on D. Then, in particular, v is a harmonic conjugate of u on D_1. Recall that any two harmonic conjugates of a harmonic function on a domain can only differ by an additive constant, so $v(x,y) = v_1(x,y) + c$ for some constant c, for all $(x,y) \in D_1$.

Therefore, $f_1(z) = f(z) - ic$ is a holomorphic function on D which agrees with $g(z)$ for all $z \in D_1$. Thus, we can extend $g(z)$ to a holomorphic function on all of D. In particular, we can extend $\operatorname{Arg}(z)$ to a holomorphic function on all of D.

But, this is a contradiction: for instance, recall that we have seen that it is not possible to define $\operatorname{Arg}(z)$ in a continuous fashion on any circle centered at the origin (a discontinuity is observed by approaching the negative real axis from above and from below).

Hence, there is no harmonic conjugate of $u(x,y)$ on all of D.

7 Question 7

Show that if $f(z)$ is a non-zero polynomial, then $g(z) = e^z f(z)$ has an essential singularity at ∞.

Solution. We will use the following definition to solve the said question.

Remark. *The nature of the singularity of f at ∞ is defined to be the nature of the singularity of $z \mapsto f(1/z)$ at 0.*

So we consider the function,

$$g : \mathbb{C}\backslash\{0\} \to \mathbb{C} \;\; g(z) := f(1/z) = e^{1/z} \sum_{k=0}^{n} a_k \left(\frac{1}{z}\right)^k$$

Since the limit: $\lim_{z \to 0} g(z)$ does not exist, 0 is not a removable singularity.
Also, the limit along the negative real axis:

$$\lim_{z \to 0^-} e^{1/z} \left(\frac{1}{z}\right)^k = 0 \neq \infty$$

By regular evaluation of real function limits. So a linear combination of the same with complex coefficients (to construct the polynomial) would also be 0. The singularity is neither removable, nor a pole $\implies$ It is an essential singularity

$\square$