Sheet 2.) 8.) ii and iii

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8. In each case, find a function f which satisfies all the given conditions, or else show that no such function exists.

(i) $f''(x) > 0$ for all $x \in \mathbb{R}$, $f'(0) = 1$, $f'(1) = 1$

(ii) $f''(x) > 0$ for all $x \in \mathbb{R}$, $f'(0) = 1$, $f'(1) = 2$

(iii) $f''(x) \geq 0$ for all $x \in \mathbb{R}$, $f'(0) = 1$, $f(x) \leq 100$ for all $x > 0$

(iv) $f''(x) > 0$ for all $x \in \mathbb{R}$, $f'(0) = 1$, $f(x) \leq 1$ for all $x < 0$

8.) ii) $f''(x) > 0 \implies f'(x)$ strictly increasing.

Let $f'(x) = 1 + x$ $[\because f'(0) = 1 \quad f'(1) = 2 \quad f'(x)$ strictly increasing$]$

$\therefore f(x) = x + \frac{x^2}{2}$ works.

8.) iii) $f''(x) \geq 0 \quad f'(0) = 1 \quad \& \quad f(x) < 100 \quad \forall x > 0$

$f''(x) \geq 0 \Rightarrow f'(x)$ non-decreasing function.

$\therefore f'(x) \geq 1 \quad \forall x > 0$

By LMVT on $f(x)$ continuous in $[0, x]$ & diff in $(0, x)$

$\exists c \in (0, x) \mid$

$f'(c) = \frac{f(x) - f(0)}{x}$ and $f'(c) \geq 1$

$\therefore f(x) \geq x + f(0)$

let $x_0 = 200 + |f(0)|$ $(\because x_0 > 0)$

$\therefore f(x_0) \geq \underbrace{200 + |f(0)| + f(0)}_{\geq 200}$

$\therefore x_0 > 0 \implies f(x_0) \geq 200$ but $f(x) < 100 \quad \forall x > 0$

$\times$

$\therefore \nexists f$ satisfying the conditions

Sheet 2) Q10) i)
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10. Sketch the following curves after locating intervals of increase/decrease, intervals of concavity upward/downward, points of local maxima/minima, points of inflection and asymptotes. How many times and approximately where does the curve cross the x-axis?

(i) $y = 2x^3 + 2x^2 - 2x - 1$

(ii) $y = 1 + 12|x| - 3x^2$, $x \in [-2, 5]$

10) i) Polynomials are continuous and infinitely differentiable in $\mathbb{R}$.

$y' = 6x^2 + 4x - 2 = 2(3x-1)(x+1)$

Critical points: $x = 1/3$ and $x = -1$

$f'(x) > 0$ for $x \in (-\infty, -1) \cup (1/3, \infty)$

$f'(x) < 0$ for $x \in (-1, 1/3)$

$\therefore$ $x = -1$ — Maxima

$x = 1/3$ — Minima (By definition)

$f''(x) = 12x + 4$

$f''(x) < 0$ for $x < -1/3$

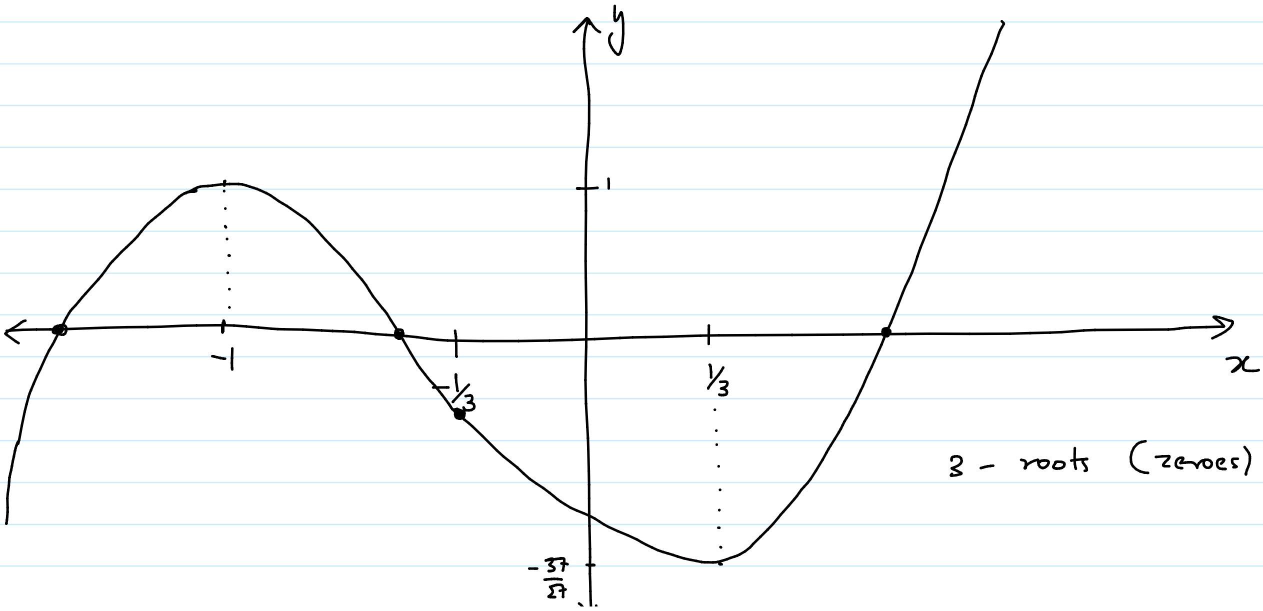
$f''(x) > 0$ for $x > -1/3$ $\therefore$ $x = -1/3$ is a point of inflection

$(-\infty, -1/3) \longrightarrow$ Concave $\quad (-1/3, \infty) \longrightarrow$ Convex

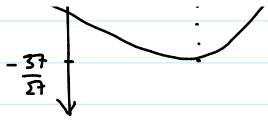
(Note that there are no asymptotes.)

$f(-1) = 1$ $\quad f(1/3) = \frac{2}{27} + \frac{2}{9} - \frac{2}{3} - 1 = \frac{2+6-18-27}{27} = -\frac{37}{27}$

$f(-1/3) = -\frac{2}{27} + \frac{2}{9} + \frac{2}{3} - 1 = \frac{-2+6+18-27}{27} = \frac{-5}{27}$



3 - roots (zeroes)


$-\frac{37}{27}$

Sheet 2) Q11)

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11. Sketch a continuous curve $y = f(x)$ having all the following properties:

$f(-2) = 8,\ f(0) = 4,\ f(2) = 0;\ \ f'(2) = f'(-2) = 0;$

$f'(x) > 0$ for $|x| > 2,\ f'(x) < 0$ for $|x| < 2;$

$f''(x) < 0$ for $x < 0$ and $f''(x) > 0$ for $x > 0.$

$f'(x) > 0 \qquad x \in (-\infty, -2) \cup (2, \infty)$

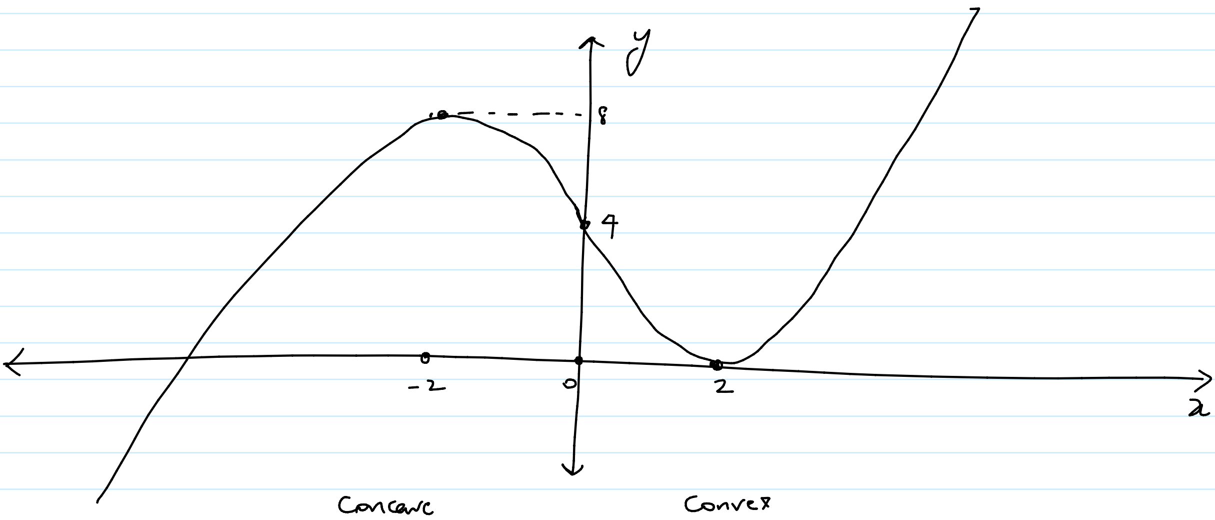
$f'(x) < 0 \qquad x \in (-2, 2)$

$\therefore\ x = -2$ – maxima $\qquad x = 2$ – minima

$f''(x) < 0 \qquad x < 0$ — concave

$f''(x) > 0 \qquad x > 0$ — convex

$\therefore\ x = 0$ – Point of Inflection



Sheet 3.) Q1.)

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Exercise 1. *Write down the Taylor series for (i)* $\cos x$*, (ii)* $\arctan x$ *about the point* 0*. Write down a precise remainder term* $R_n(x)$ *in each case.*

ii) $\arctan x = f(x)$

We use $\int \sum a_n x^n = \sum a_n \int x^n$

$\therefore$ Express $f'(x)$ as taylor polynomial & integrate.

$$f'(x) = \frac{1}{1+x^2}$$

$$= 1 - x^2 + x^4 \dots (-1)^n x^{2n} + (-1)^{n+1} \frac{x^{2n+2}}{1+x^2}$$

$$\therefore \int f'(x) = x - \frac{x^3}{3} + \frac{x^5}{5} \dots (-1)^n \frac{x^{2n+1}}{2n+1} + \int_0^x \frac{(-1)^n x^{2n+2}}{1+x^2}\,dx$$

$$\therefore P_{2n}(x) = \sum_{k=0}^{n} (-1)^k \frac{x^{2k+1}}{2k+1} \qquad R_n(x) = \int_0^x \frac{(-1)\,x^{2n+2}}{1+x^2}\,dx$$

(other methods available online, but this was easiest)

Sheet 3.) Q2.)

09 December 2020 17:00

Exercise 2. *Our examples of Taylor's series have usually been series about the point 0. Write down the Taylor series of the polynomial $x^3 - 3x^2 + 3x - 1$ about the point 1.*

Donkey Work method 1 -

$f(1) = 1 - 3 + 3 - 1 = 0$ $\quad$ $f'(1) = 3 - 6 + 3 = 0$

$f''(1) = 6 - 6 = 0$ $\quad$ $f'''(1) = 6$ $\quad$ $f^{(4)}(1) = f^{(5)}(1) = \dots = 0$

$$\therefore f(x) = f(1) + \frac{f'(1)}{1!}(x-1) + \frac{f''(1)}{2!}(x-1)^2 \dots$$

$$= 0 + 0 + 0 + \frac{6}{3.2.1}(x-1)^3 + 0 \dots$$

$$= (x-1)^3$$

Smart Method:

Taylor series of polynomial around point $\equiv$ Polynomial Expressed around the point

$$\therefore f(x) = x^3 - 3x^2 + 3x - 1$$
$$= (x-1)^3$$

$\therefore$ Taylor around $x_0 = 1$ $\Rightarrow$ $f(x) = (x-1)^3$

Sheet 3.) Q4.)

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Exercise 4. *Consider the series* $\sum_{k=0}^{\infty} \frac{x^k}{k!}$ *for a fixed* x. *Prove that it converges as follows. Choose* $N > 2|x|$. *We see that for all* $n > N$,

Correction

$$\frac{x^{n+1}}{(n+1)!} < \frac{1}{2} \cdot \frac{x^n}{n!}.$$

It should now be relatively easy to show that the given series is Cauchy, and hence (by the completeness of $\mathbb{R}$*), convergent.*

We wish to show the series converges by proving it is Cauchy.

$$S_n := \sum_{k=0}^{n} \frac{x^k}{k!}$$

Sequence S_n is cauchy if $|S_m(x) - S_n(x)| < \epsilon \quad \forall \; n, m > N$

choose N st. $N > 2|x| \longleftarrow (i)$

$$\therefore \frac{x^{N+1}}{(N+1)!} < \frac{x^N}{N!}\left(\frac{1}{2}\right) \quad \text{(follows from (i))}$$

WLOG, let $m > n$ and $m = n + r \quad r \in \mathbb{N}$

$$\therefore |S_m(x) - S_n(x)| = \left| \sum_{k=n+1}^{n+r} \frac{x^k}{k!} \right|$$

$$= \left| \frac{x^n}{n!}\left(\frac{1}{2} + \frac{1}{4} + \frac{1}{8} \cdots \frac{1}{2^r}\right) \right|$$

$$\leq \left| \frac{x^n}{n!} \right|$$

We wish to show $\left|\frac{x^n}{n!}\right| < \epsilon$ for $n > N$

choose K st. $K > |x|$ and $K < n$

$$\therefore |x|^{n-K} < K \cdot (K+1) \cdots (n-1)$$

$$\therefore \frac{|x|^n}{n!} < \frac{|x|^K}{(K-1)!\, n}$$

$$\therefore \frac{|x|^n}{n!} < \frac{|x|^k}{(K-1)!\, n} < \epsilon$$

$\therefore \quad \frac{|x|^n}{n!} < \frac{|x|^k}{(k-1)!\, n} < \epsilon$

Choose $N = \left\lfloor \frac{|x|^k}{(k-1)!\, \epsilon} \right\rfloor + 1$

Then $\frac{|x|^n}{n!} < \epsilon$ for $n > N$

$\therefore$ Sequence has been shown to be Cauchy,

$\because$ Cauchy sequences are convergent in $\mathbb{R}$, (Completeness of $\mathbb{R}$)

Green r. $\square$