28 August 2021 15:02

RECAP :-

1 Singularity → "Where things go bad"

So, let $f : \Omega \to \mathbb{C}$ be a function

Let $z_0 \in \mathbb{C}$, z_0 is a singularity if :

→ (i) $z_0 \notin \Omega$

(ii) $z_0 \in \Omega$ but f is not holo. at z_0

For example, consider

1. $f : \mathbb{C} \to \mathbb{C}$ $f(z) = |z|$ all $z_0 \in \mathbb{C}$ are singularities
2. f $\mathbb{C}\setminus\{0\} \to \mathbb{C}$: $f(z) = 1/z$ $z_0 = 0$ is sing.

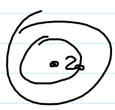
Try 3. $f :$ ____ $\to \mathbb{C}$ $f(z) = \dfrac{z}{\sin z}$

Types of Singularities :

ISOLATED Singularity is said to be ISOLATED

if f is holomorphic in some nbd of z_0

$\exists\, \delta > 0$ f is holo on $B_\delta(z_0) \setminus \{z_0\}$



Remark If set of singularities S is finite

then each sing. $z_0 \in S$ is ISOLATED

Class. of ISOLATED Singu :-

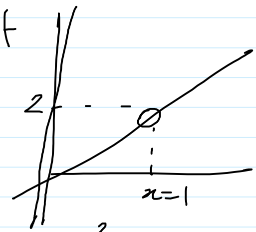
→ 1. Removable Sing.

$z_0 \in \mathbb{C}$ is said to be a removable sing. if

$\exists c \in \mathbb{C}$ s.t the f^n

$$g : \Omega \cup \{z_0\} \to \mathbb{C}$$

$$g(z) = \begin{cases} c & \text{if } z = z_0 \\ g(z_0) & \text{otherwise} \end{cases}$$



$\dfrac{x^2-1}{x-1}$

② Poles

z_0 is said to be a pole of f if

1. $\lim_{z \to z_0} f(z) = \infty$

2. $\lim_{z \to z_0} 1/f(z) = 0$

3. $\exists m \in \mathbb{N}$ s.t $\lim (z-z_0)^m f(z_0)$ exists.

$\frac{1}{(z-1)^3}$

$m = 3$ & $z_0 = 1$

③ Essential Singularity
Neither ① or ②

Tutorial - 4 :-

Q1.)

1. Show that there is a strict inequality

$$\left| \int_{|z|=R} \frac{z^n}{z^m - 1} dz \right| < \frac{2\pi R^{n+1}}{R^m - 1} \;; R > 1, m \geq 1, n \geq 0$$

Theorem : (Stronger ML inequality) :

let $f : \Omega \to \mathbb{C}$ be a continuous fn and $\gamma : [a,b] \to \Omega$ be a curve. Let $M > 0$ be such that

$$|f(\gamma(t))| \leq M \quad \forall t \in [a,b]$$

☆ and suppose $|f(\gamma(t))| < M$ for some $t_0 \in [a,b]$

Then ;

$$\left| \int_\gamma f(z)\, dz \right| < ML$$

Where L is the length of the curve.

[Note, ☆ holds at even one point ⇒ strict inequality]

Proof : Note

$$\int_a^b [M - |f(\gamma(t))|]\, |\gamma'(t)|\, dt \geq 0$$

By our assumptions (☆) the integral is strictly +ve

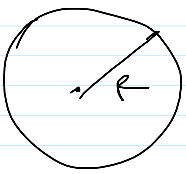
$$\int_a^b \left[M - |f(r(t))| \right] |r'(t)| \, dt > 0$$

Note. $$\int_a^b M |r'(t)| \, dt = ML$$

It follows. $$\int_a^b |f(r(t))| \, |r'(t)| \, dt < ML$$

$$\left| \int_a^b f(z) \, dz \right| \le \int_a^b |f(z)| \, dz \qquad \square$$

$$\left| \int_{|z|=R} \frac{z^n}{z^m - 1} dz \right| < \frac{2\pi R^{n+1}}{R^m - 1} \; ; R > 1, m \ge 1, n \ge 0$$



$$f(z) = \frac{z^n}{z^m - 1} \qquad |a-b| \ge \big| |a| - |b| \big|$$

$$\frac{|z^n|}{|z^m - 1|} = \frac{R^n}{|z^m - 1|} \le \frac{R^n}{\big| |z|^m - |1| \big|}$$

$$M := \left(\frac{R^n}{R^m - 1} \right)$$

find z_0 $\qquad z_0 = Re^{i\pi/m} \implies f(z_0) < M$

(point found)

$$\left| \int \underline{\quad} \right| \le M \, 2\pi R$$

$$= \frac{2\pi R^{n+1}}{R^m - 1}$$

Q 2.)

2. A power series with center at the origin and positive radius of convergence, has a sum $f(z)$. If it known that $f(\bar{z}) = \overline{f(z)}$ for all values of z within the disc of convergence, what conclusions can you draw about the power series ?

∞ $\qquad \overline{\quad \infty \quad}$

Given $f(\bar{z}) = \sum_{n=0} a_n \bar{z}^n = \overline{f(z)} = \sum_{n=0} \bar{a}_n z^n$

$= \sum_{n=0}^{\infty} \bar{a}_n \bar{z}^n$

(Important)

(Before we directly say $a_n = \bar{a}_n$) :-

We directly have $a_0 = \bar{a}_0$, also since the fn is holomorphic calculating derivative along $\mathbb{R}$ gives

$$f'(0) = \lim_{r \to 0} \frac{f(r) - f(0)}{r}$$

$$f(r) = f(\bar{r}) = \overline{f(r)}$$

Thus you may show inductively that all higher order derivatives are also real $\Longrightarrow a_k = \bar{a}_k \quad \forall k \in \mathbb{N}$

(Note, one may also argue the same using term by term differentiability of a power series.)

Q3.)

3. This is called Taylor series with remainder :

$f(z) = f(0) + zf'(0) + \frac{z^2}{2!}f''(0) + \cdots + \frac{z^N}{N!}f^N(0) + \frac{z^{N+1}}{(N+1)!}\int_0^1 (1-t)^N \underline{f^{N+1}(tz)}dt$

$f^{N+1}(z)$

$N+1$ th derivative of $f(tz)$

Use this to prove the following inequalities :

a) $| e^z - \sum_0^N \frac{z^n}{n!} | \leq \frac{|z|^{N+1}}{(N+1)!}; \quad Re(z) \leq 0.$

b) $| cos(z) - \sum_{i=0}^N \frac{(-1)^i z^{2i}}{2i!} | \leq \frac{|z|^{2N+2}\text{coshR}}{(2N+2)!}; \quad |Im(z)| \leq R$

a.) (i) $e^z - \sum_{n=0}^{N} \frac{z^n}{n!} = \frac{z^{N+1}}{N+1} \int_0^1 (1-t)^N \exp(tz)\, dt$

$$\left| \int_0^1 (1-t)^N \exp(tz)\, dt \right| \leq \int_0^1 \left| (1-t)^N \underline{\exp(tz)} \right| dt$$

Note $|\exp(z)| = |\exp(\mathrm{Real}(z))|$ $\quad = \int_0^1 (1-t)^N \exp(\mathrm{Real}(tz))\, dt$

$\Rightarrow e^x \leq 1 \quad \leq \int_0^1 (1-t)^N dt$

$= \frac{1}{N+1}$

$$= \frac{1}{N+1}$$

Thus we get :

$$\left| e^{z} - \sum_{n=0}^{N} \frac{z^n}{n!} \right| = \left| \frac{z^{N+1}}{(N+1)!} \int_0^1 (1-t)^N \exp(tz)\, dt \right|$$

$$\leq \frac{z^{N+1}}{(N+1)!} \left(\frac{1}{N+1} \right) (1)$$

$$\leq \frac{z^{N+1}}{(N+1)!} \qquad \square$$

b.) In the LHS modulus, we have

$$\frac{z^{2N+2}}{(2N+2)!} \int_0^1 (1-t)^{2N+1} \cos^{(2N+2)}(tz)\, dt$$

$$|\cos(z)| = \frac{1}{2} \left| e^{iz} + e^{-iz} \right|$$

$$\leq \frac{1}{2} \left(|e^{iz}| + |e^{-iz}| \right) \qquad y = iz$$

$$= \frac{1}{2} \left(e^{y} + e^{-y} \right)$$

$$= \cosh(y)$$

$\cos^{(2N+2)}(tz)$ are either $+\cos$ or $-\cos$

$$|\cos^{(2N+2)}(tz)| \leq |\cosh ty|$$

also, $\cosh ty$ is increasing for $|y|$

$$|\cosh ty| \leq |\cosh y| \qquad t \in (0,1)$$

If $|y| \leq R$

we have $|\cosh ty| \leq |\cosh y| \leq \cosh R$

$$\left| \int_0^1 (1-t)^{2N+1} \cos^{(2N+2)}(tz)\, dt \right| \leq \int_0^1 (1-t)^{2N+1} \left| \cos^{(2N+1)}(tz) \right| dt$$

$$\leq \int_0^1 (1-t)^{2N+1} \cosh R \, dt$$

$$= \frac{\cosh R}{2N+2} \quad \square$$

Q4)

$\int_C$ $\quad$ $\int_{\mathbb{R}}$

4. By computing $\int_{|z|=1}(z+\frac{1}{z})^{2n}\frac{dz}{z}$, show that $\int_0^{2\pi} cos^{2n}\theta d\theta = \frac{2\pi}{4^n}\frac{2n!}{n!^2}$.

Recall. "Generalised" Cauchy Integral Formula

$$\boxed{\int_{|w-z_0|=r} \frac{f(w)}{(w-z_0)^{n+1}} dw = \frac{2\pi i}{n!} f^{(n)}(z_0)}$$

define, $\quad f(z) = (z^2+1)^{2n}$

Given formula gives.

$$\int_{|z|=1} \frac{(z^2+1)^{2n}}{z^{2n+1}} dz = \frac{2\pi i}{2n!} f^{(2n)}(0)$$

Evaluate $\quad f^{2n}(0) \quad \Big| \quad f(z) = (z^2+1)^{2n}$

$$f(z) = \sum_{k=0}^{2n} \binom{2n}{k} z^{2k}$$

$$f^{2n}(0) = \binom{2n}{n}(2n!) \Longrightarrow \int \left(\text{——}\right) = 2\pi i \binom{2n}{n}$$

Computing the integral using $|z|=1$ as γ. $[0,2\pi] \to r(t)e^{it}$

$$\int_{|z|=1} (z+1/z)^{2n} \, 1/z \, dz = \int_0^{2\pi} (2\cos t)^{2n} \frac{1}{\cancel{e^{it}}} i\cancel{e^{it}} \, dt$$

$z = \cos t + i\sin t$

$$2\pi i \binom{2n}{n} = 4^n i \int_0^{2\pi} (\cos t)^{2n} \, dt$$

$$2\pi i \binom{2n}{n} = 4 i \int_0 (\cos \ldots$$

$$\boxed{2\pi \binom{2n}{n} \frac{1}{4^n}}$$

$\square$

TRY

$$\int_0^{2\pi} \left[\cos(\cos t)\cosh(\sin t)\cos(nt) - \sin(\cos t)\sinh(\sin t)\sin(nt) \right] dt$$

Hint (Think of complex fn which param. gives this Real integral)

Q5.)

5. Let $f(z)$ be an entire function. Show that $f(z)$ is a polynomial of degree atmost n if and only if there exists a positive real constant C such that $|f(z)| < C|z^n|$ for all z with $|z|$ sufficiently large.

let $f(z) = a_0 + a_1 z + \cdot \; a_n z^n$ be a complex poly.

clearly, $f(z)$ is an entire fn on $\mathbb{C}$

$$|f(z)| = |a_0 + a_1 z + \ldots a_n z^n|$$

$$\leq |a_0| + |a_1 z| + \ldots |a_n z^n|$$

$$= |z|^n \left\{ \frac{|a_0|}{|z|^n} + \frac{|a_1|}{|z|^{n-1}} + \cdots + |a_n| \right\}$$

$\therefore$ for $|z| \geq R$, $R > 0$ we have

$$\left\{ \frac{a_0}{|z|^n} + \frac{a_1}{|z|^{n-1}} + \ldots + |a_n| \right\} \leq \left\{ \frac{a_0}{R^n} + \frac{a_1}{R^{n-1}} + \ldots + a_n \right\}$$

So, $f(z) \leq C|z|^n$ for $C = \frac{a_0}{R^n} + \frac{a_1}{R^{n-1}} + \ldots + a_n$

$\Leftarrow$ Proof: Conversely, let f be an entire function s.t $\exists\, C > 0$

$$|f(z)| \leq C|z|^n \quad \forall\, z \text{ with } |z| > R$$

(some R)

Since $f(z)$ is entire, we have $f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} z^k$

Since $f(z)$ is entire, we have $f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} z^k$

By Cauchy's Estimate

$$|f^{(k)}(0)| \leq \frac{k!\, C R^n}{R^k} \quad \forall\ |z| = R > 0$$

$\therefore$ for $k > n$ and letting $R \to \infty$ $\quad |f^{(k)}(0)| = 0$

$\implies$ f is polynomial of degree $\leq n$

Q6.)

6. Let f and g be entire non-vanishing functions such that $(\frac{f'}{f})(\frac{1}{n}) = (\frac{g'}{g})(\frac{1}{n})$ for all $n \in \mathbb{N}$. Show that g is a non-zero scalar multiple of f.

define $h := g/f$

$$h' = \frac{fg' - gf'}{f^2} \implies \frac{h'}{h} = \frac{fg' - gf'}{gf}$$

$$= \frac{g'}{g} - \frac{f'}{f}$$

So, $\underline{\underline{(h'/h)(1/n)}} = 0 \quad \forall\ n \in \mathbb{N}$

h'/h is a function and it vanishes on the set $Z := \{\frac{1}{n},\ n \in \mathbb{N}\}$

$\because$ Entire $\Rightarrow \Omega = \mathbb{C} \implies \{0\} \in \Omega$

[Same as Tut 3 Q7]

So, $\frac{h'}{h} \equiv 0$ identically $\implies h' = 0$

Since Ω is path connected,

$$g/f = c \implies \boxed{g = c \cdot f} \qquad \square$$