6th (and last :)) Tutorial .

Q1.)

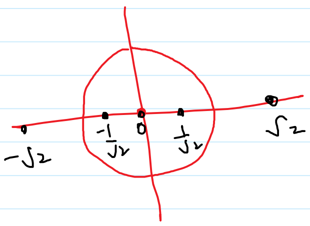
1. Evaluate $\int_0^{2\pi} \frac{cos^2(3x)dx}{5-4cos(2x)}$

$$1. \quad \cos(n\theta) = \frac{e^{in\theta} + e^{-in\theta}}{2} = \frac{z^n + z^{-n}}{2} \qquad [e^{i\theta} = z]$$

$$dz = e^{i\theta} i\, d\theta$$

$$z$$

$$\int_0^{2\pi} \frac{\cos^2(3x)}{5 - 4\cos(2x)}\, dx = \int_{|z|=1} \left(\frac{\left(\frac{z^3+z^{-3}}{2}\right)^2}{5 - 4\left(\frac{z^2+z^{-2}}{2}\right)} \right) \frac{1}{iz}\, dz$$



$$= -\frac{1}{8i} \int_{|z|=1} \underbrace{\frac{1}{z^5} \left(\frac{(z^6+1)^2}{(z^2-2)(z^2-\frac{1}{2})} \right)}_{f(z)} dz$$

set of poles $= \{ 0, \pm\sqrt{2}, \pm \frac{1}{\sqrt{2}} \}$

We calculate the residues for 0 & $\pm\frac{1}{\sqrt{2}}$

$$f(z) = \frac{(z^6+1)^2}{z^5 (z^2-2)(z^2-1/2)} = \frac{(z^{12} + 2z^6 + 1)}{z^5} \frac{1}{\left(1 - \left(\frac{5}{2}z^2 - z^4\right)\right)}$$

$$(z+1/\sqrt{2})(z-1/\sqrt{2})$$

$$= \left(\frac{z^{12} + 2z^6 + 1}{z^5} \right) \left[1 + (t) + t^2 \ldots \right]$$

$t \quad 1 \quad \frac{25}{4} \quad z^4 \quad \frac{1}{z^5}$

★ finding the coeff. of z^{-1} in the expansion of $f(z)$

The coefficient comes out to be $-1 + \frac{25}{4} = \frac{21}{4}$

Residue around $\pm \frac{1}{\sqrt{2}}$:- (calculate residue directly:)

$z = \frac{1}{\sqrt{2}}$:

$$\frac{\left(\left(\frac{1}{\sqrt{2}}\right)^6 + 1 \right)^2}{\left(\frac{1}{\sqrt{2}}\right)^5 \left(\left(\frac{1}{\sqrt{2}}\right)^2 - 2 \right) \left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} \right)} = 4\sqrt{2} \left(\frac{81/64}{-\frac{3}{2}\sqrt{2}} \right)$$

$$= \frac{-27}{8}$$

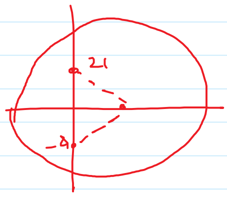
for $z = -1/\sqrt{2}$ also, the residue is $\frac{-27}{8}$

So $$-\frac{1}{8i} \int_{|z|=1} f(z)\, dz = -\frac{1}{8i} (2\pi i) \left[\frac{21}{4} - \frac{27}{8} - \frac{27}{8} \right]$$

$$= \boxed{\frac{3\pi}{8}}$$

Q2.)

2. Evaluate $\int_{|z-2|=4} \frac{2z^3+z^2+4}{z^4+4z^2}\, dz$



Let $f(z) = \frac{2z^3+z^2+4}{z^4+4z^2}$ $\quad z^2(z^2+4)$

Singularities $S = \{0, \pm 2i\}$

$\therefore$ The integral evaluates to $2\pi i \sum_{z_0 \in \{0, \pm 2i\}} Res(f; z_0)$

Residue at $2i$:-

$$\frac{2(2i)^3 + (2i)^2 + 4}{(2i+2i)(2i)^2} = 1$$

Similarly, residue at $-2i$ is also $= 1$

Now for residue at 0 :- $g'(0)$ where

$$g(z) = z^2 f(z) = \frac{1}{4}(2z^3 + z^2 + 4)\left(1 + \frac{z^2}{4}\right)^{-1}$$

The residue turns out to be 0

$$\int_{|z-2|=4} f(z)\, dz = 2\pi i\,(0 + 1 + 1) = 4\pi i$$

[Important : in $|z - z_0| = r$ this is NOT r^2]

Q3.) 3. Show with and without using open mapping theorem that if $f(z)$ is a holomorphic function on a domain such that $|f(z)|$ is constant, then $f(z)$ is constant.

WITHOUT OMT :

$f = u + iv$ then we have $u^2 + v^2 = c$ (1)

If $c = 0$, we are done [why? $f(z) = 0$]

If $c \neq 0$, diff. (1) wrt x gives us.

$$u u_x + v v_x = 0 \quad \text{——— (i)}$$

" wrt y "

$$u u_y + v v_y = 0 \quad \text{——— (ii)}$$

Since the function is given to be holomorphic, it satisfies the CR equations, so we have :-

$$-u v_x + v u_x = 0 \quad \text{——— (iii)}$$

(i) & (iii) can be written as:

$$\underbrace{\begin{bmatrix} u & v \\ v & -u \end{bmatrix}}_{A} \begin{bmatrix} u_x \\ v_x \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$|A| = -u^2 - v^2 = -c \neq 0$ (By our assumption)

$\Longrightarrow$ A is of full rank

$\Longrightarrow$ Null space of A is $\{0\}$ $\quad u_x = 0 = v_x$

$\therefore$ $f' \equiv 0$ on Ω, again

Assuming Ω is path connected

$f = c$ on Ω.

WITH OMT:

Proof by contradiction, Suppose f is not constant

Then OMT, tells us that the image $f(\Omega)$ is an open subset of $\mathbb{C}$

However, $|f|$ being constant 'c' tells us that

$f(\Omega)$ is a subset of $\{z : |z| = c\}$

☆ No subset of $\{z : |z| = c\}$ is open (why?)

$\Rightarrow\!\Leftarrow$ Implies that f must be a constant.

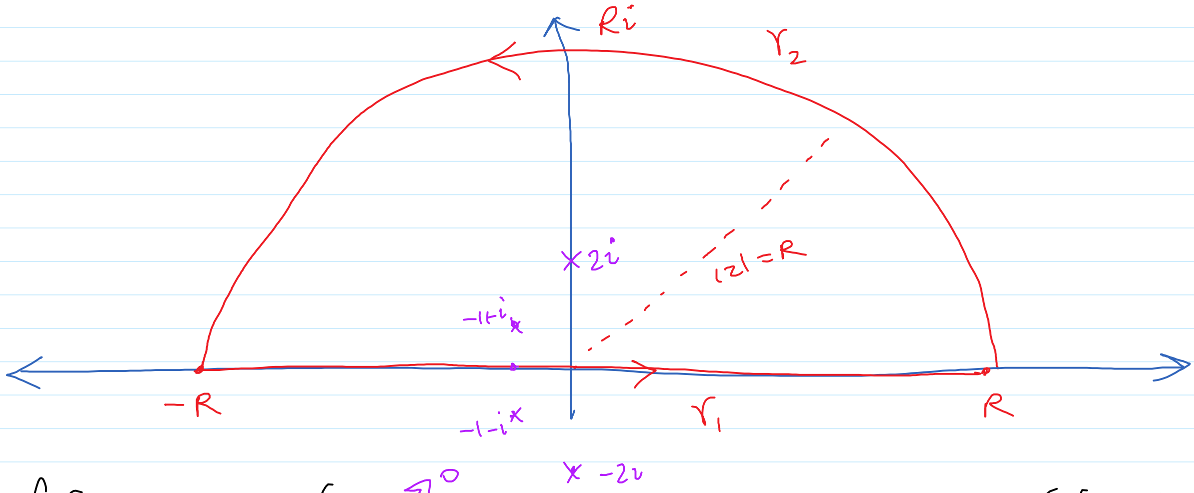
Q4.) 4. Show that $\int_{-\infty}^{\infty} \frac{x}{(x^2+2x+2)(x^2+4)} dx = -\pi/10$

Sol^n $\quad f(z) := \dfrac{z}{(z^2+2z+2)(z^2+4)}$ $\qquad M \in O(R^{-1-\delta})$

$\boxed{\delta > 0}$

poles $= \{-1 \pm i, \ \pm 2i\}$ If we take $R > 2$

then all poles are inside the contour defined by



$$\int_{\gamma_1} f(z)\,dz + \cancel{\int_{\gamma_2} f(z)\,dz}^{\,0} = 2\pi i \sum_{z_0 \in \{2i, -1+i\}} \operatorname{Res}(f, z_0)$$

But we want to find $\int_{\gamma_1} f(z)\,dz$ with $R \to \infty$

Along γ_2, by application of ML inequality,

the integral turns out to be 0

Residue at $2i$:-

$$\frac{\cancel{2i}}{\big((2i)^2 + 2(2i) + 2\big)(4\cancel{i})} = \frac{1}{4(2i-1)}$$

$$(-4 + 4i + 2)\,2$$

Residue at $-1+i$:-

$$\frac{(-1+i)}{(-1+i-(-1-i))\big((-1+i)^2+4\big)}$$

$$= \frac{-1+i}{(2i)(\cancel{1} - \cancel{1} - 2i + 4)}$$

$$= \frac{-1+i}{(2i)(4-2i)}$$

Answer :- $2\pi i \left(\dfrac{1}{4(2i-1)} + \dfrac{i+1}{4(2-i)} \right)$

$$= \frac{\pi}{2} i \left(\frac{1}{2i-1} + \frac{i+1}{2-i} \right)$$

$$\cancel{-2} \cancel{-i} + \cancel{2i} - 1$$

$$= \frac{\pi}{2} i \left(\frac{\cancel{2} - \cancel{i} + (i+1)(2i-1)}{(2i-1)(2-i)} \right)$$

$$= \frac{\pi}{2} i \left(\frac{-1}{4i + \cancel{2} - \cancel{2} + i} \right)$$

$$= \frac{\pi}{2} \cancel{i} \frac{(-1)}{5\cancel{i}} = -\frac{\pi}{10} \;//$$

Q5)

5. Compute the number of zeros of the polynomial $z^5 + z^2 - 6z + 3$ in the annulus $1/3 < |z| < 1$ using Rouche's theorem.

Rouche's Theorem : Let $f(z)$ and $g(z)$ be analytic functions on the bounded domain D that extend continuously to ∂D and satisfy

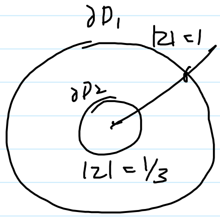
$$|f(z) + g(z)| < |f(z)| + |g(z)|$$

on ∂D. Then $f(z)$ and $g(z)$ have the same number of zeros in D counting multiplicity.

$g(z) = z^5 + z^2 - 6z + 3$ & $f(z) = 6z - 3$

$1/3 < 1/2 < 1$

$f(z) + g(z) = z^5 + z^2$



∂D_1 is $|z| = 1$

$$\underline{|f(z) + g(z)|} \leq |z|^5 + |z|^2 = 2 < 3 = |6|z| - 3|$$

$$\leq \underline{|f(z)| + |g(z)|}$$

Similarly for $|z| = 1/3$ (∂D_2)

$$|f(z) + g(z)| \leq |f(z)| + |g(z)|$$

So, by Rouche's thm, $f(z)$ & $g(z)$ have same no of zeros in $1/3 < |z| < 1$

$\Longrightarrow$ $g(z)$ has 1 zero in the domain.

Theorem 35 (Rouché's Theorem)

Let $f, g : \Omega \to \mathbb{C}$ be holomorphic. Let γ be closed curve in Ω. Suppose that

$$|f(z) - g(z)| < |f(z)|,$$

for all z on the image of γ.

Then,

$$N_\gamma(f) = N_\gamma(g).$$

Q6) 6. Show that the function $u(x, y) = log(x^2 + y^2)$ is harmonic on the annulus $1 < |z| < 2$. Does $u(x, y)$ have a harmonic conjugate ?

Let D be the annulus $1 < |z| < 2$

$u(x,y) = \log(x^2+y^2)$ $\qquad u_{xx} + u_{yy} = ?$

$u_x = \dfrac{2x}{x^2+y^2}$ $\qquad u_y = \dfrac{2y}{x^2+y^2}$

$u_{xx} = \dfrac{2(x^2+y^2) - 2x(2x)}{(x^2+y^2)^2}$ $\qquad u_{yy} = \dfrac{-2y^2+2x^2}{(x^2+y^2)^2}$

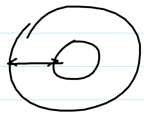
$= \dfrac{-2x^2 + 2y^2}{(x^2+y^2)^2}$

Laplacian Symbol

$\Longrightarrow \quad u_{xx} + u_{yy} = \Delta u(x,y) = 0$ on D

Proof that harmonic conjugate does not exist :-

Let $D_1 = D \setminus (-2,-1)$



define $g : D_1 \longrightarrow \mathbb{C}$ $\quad g(z) = 2\log(z)$

$g(z)$ is holo on D_1 & $Re(g(z)) = u(x,y)$

$\therefore$ harmonic conjugate $v(x,y) = Im(g(z)) = 2Arg(z)$ on D_1

For contradiction, suppose $v(x,y)$ is a harmonic conjugate of $u(x,y)$ s.t $f(z) = u + iv$ is a holomorphic function on D_1

Recall, any two harmonic conjugates of a harmonic fn can differ by a constant $\Longrightarrow v(x,y) = v_1(x,y) + c$ for some c_1 & $(x,y) \in D_1$

$\Longrightarrow$ $f_1(z) = f(z) - ic$ is a holo. fn on D_1 which agrees with $g(z)$ for $z \in D_1$

So we can extend $g(z)$ to a holo. fn over all of D

In particular, we can extend $Arg(z)$ to a holo. fn over D.

This is a contradiction, it is not possible to define $Arg(z)$ in a continuous fashion on any circle centred at origin.

Hence, there is no harmonic conjugate of $u(x,y)$ on all of D.

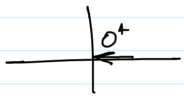
Q7) 7. Show that if $f(z)$ is a non-zero polynomial, then $g(z) = e^z f(z)$ has an essential singularity at ∞.

[fact $f(z)$ has essential sing at ∞ $\Longleftrightarrow$ $f(1/z)$ essential sing at 0]

$$g(1/z) = e^{1/z} f(1/z)$$

$$f(z) = \sum_{k=0}^{n} a_k z^k \implies f(1/z) = \sum_{k=0}^{n} a_k (1/z)^k$$

(Note $f(z)$ is non zero)



1.) Note that the $\lim_{z \to 0^+} g(1/z) = e^{1/z} f(1/z)$ does not exist

So, the singularity is <u>not</u> removable.

2.) $\lim_{z \to 0^-} e^{1/z} (1/z)^k = 0$ for all k
(along real axis)

$\implies$ linear combination over k to construct the polynomial is also $= 0$

The sing. is not a pole

$\implies$ Essential singularity.

The theorems of the course:

1. CIF (Cauchy integral form) 2. holo $\Leftrightarrow$ analytic 3. ODAD (Once diff is alway diff)

1. CIT 2. Holo $\Longleftrightarrow$ analytic 3. ODHD

(Cauchy's integral form) (Once diff is alway diff)

4 Liouville 5. CRT 6 MMT

7. Weierstrass product thm 8. Little Picard

9. Mittag-Lefler 10. OMT

Some very important ones:-

☆ CR equations ☆ ZAI (Zeros always Isolated)

☆ RRST ☆ Identity theorem

11. Proper + entire is non constant polynomial

12. Fund. Thm of Alg

13. Casorati-Weierstrass thm

14 Laurent Series

15. Schwarz Lemma