Tutorial 2.

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13.) ~~15.)~~ ~~18.)~~

Optional

~~7.)~~ 10.)

Sheet 2 :-

2, 5

Sheet 1.) 13.) ii)

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(ii) $f(x) = x \sin \frac{1}{x}$, if $x \neq 0$ and $f(0) = 0$

Discuss the continuity of this fn.

$\rightarrow$ for $x \neq 0$, $\frac{1}{x}$ - Continuous

$\sin(x)$ - continuous and x is conti.

$\therefore$ for $x \neq 0$ $f(x) = x \sin(\frac{1}{x})$ — product & composition of cont. fns

$f(x) \Rightarrow$ Continuous,

$\rightarrow$ for $x = 0$, $x \sin\left(\frac{1}{x}\right)$ we have to use $\epsilon - \delta$

By defn,

$f(x)$ is cont at $x = x_0$, if

Given $\epsilon > 0$, $\exists \delta > 0$ st

$$\underline{|x - x_0| < \delta} \implies |f(x) - f(x_0)| < \epsilon$$

$\delta = \epsilon$ (Rough - calculation δ — chosen)

$$|x| < \delta \qquad \left|x \sin \frac{1}{x}\right| \leq |x| \qquad \left(\because \left|\sin \frac{1}{x}\right| \leq 1\right)$$

$$\left|x \sin \frac{1}{x} - 0\right| \leq \epsilon$$
$$= \delta$$

$\therefore$ $\delta := \epsilon$ works so we are done. $\square$

Ideally :- Compute δ in rough beforehand, & then show why that δ works.

Rough work :-

$x_0 = 0$

$|f(x) - f(x_0)| \Rightarrow$ $|x| \longrightarrow f(\epsilon) = \delta$

$\boxed{|x| < \delta}$, $\left|x \sin \frac{1}{x} - 0\right| < \epsilon$ $f(\epsilon) = \epsilon$

$\boxed{|x| < \delta}$, $|x \sin \frac{1}{x} - 0| < \epsilon$ $f(\epsilon) = \epsilon$

$\Rightarrow |x \sin \frac{1}{x}| < \epsilon$ (we know $|\sin \frac{1}{x}| \leq 1$

$\leq$

$|x| < \epsilon =: \delta$

Tutorial 1. Q15.)

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all diff

15. Let $f(x) = x^2 \sin(1/x)$ for $x \neq 0$ and $f(0) = 0$. Show that f is differentiable on $\mathbb{R}$. Is f' a continuous function?

$\rightarrow$ for $x \neq 0$, x^2, $1/x$, $\sin(x)$ all diff

$\therefore$ $f(x)$ = diff.

Calculate derivative - chain rule.

$$f'(x) = x^2 \cos(1/x)(-1/x^2) + 2x\sin 1/x$$

$$= 2x\sin(1/x) - \cos(1/x)$$

$\rightarrow$ for $x = 0$,

$$\lim_{h\to 0} \frac{f(0+h) - f(0)}{h} = \text{exists \& is defined}$$

$$\frac{h^{\cancel{2}}\sin 1/h}{\cancel{h}} \Leftarrow 0 \quad \text{(from 13.) ii)}$$

$$\lim_{h\to 0} h\sin 1/h$$ $\boxed{\epsilon - \delta \text{ def}^n \text{ limits}}$

setting $\delta = \epsilon$, works

$\therefore$ $f(x)$ is diff $\forall x \in \mathbb{R}$.

Continuity of f'

We know, $\boxed{f'(0) = 0}$. (above)

but, $f'(x) = -\cos\frac{1}{x} + 2x\sin\frac{1}{x}$

setting $x_n = \frac{1}{2n\pi}$

$x_n \to 0$, $f'(x_n) = -1$

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$\boxed{\text{Sequential defn of continuity?}}$

Sequential definition of continuity :-

Cont. at x_0

$\lim_{n\to\infty} x_n \to x_0$ then, $f(x_n) \to f(x_0)$.

$\Longleftrightarrow$

$x_n = \frac{1}{2n\pi} \Big\downarrow n\to\infty$ $\quad \cos\left(\frac{1}{x_n}\right) \Rightarrow -1$

$x_0 = 0$ $\quad f'(x_n) \not\Rightarrow f'(x_0) = 0$

$\therefore$ Not Continuous

Case to Case Basis.

$x_n \leftarrow$ helpful | $\epsilon-\delta$ - helpful.

$x_n = \frac{1}{2n\pi}$ check $\lim_{n\to\infty} x_n \to x_0 = 0$ ✓

$\lim_{n\to\infty} f'(x_n) = f'(x_0) = f'(0)$

$f'(x) = -\cos\frac{1}{x} + 2x\sin\frac{1}{x}$

$\lim_{n\to\infty} f'(x_n) = -1 + 0$

$= -1 \neq 0$

Sheet 1. Q18.)

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18. Let $f : \mathbb{R} \to \mathbb{R}$ satisfy

$$f(x+y) = f(x)f(y) \text{ for all } x, y \in \mathbb{R}.$$

If f is differentiable at 0, then show that f is differentiable at every $c \in \mathbb{R}$ and $f'(c) = f'(0)f(c)$.

$f(x+y) = f(x)f(y)$ said ~~to~~ to be dif at $x=0$

$\Rightarrow$ $x=0, y=0$

$f(0) = f(0)^2 \Rightarrow f(0) = 0$ or $f(0) = 1$

Case 1 : $f(0) = 0$

$x=x, y=0,$ $\boxed{f(x) = 0}$ $\forall x \in \mathbb{R}$

diff. every where,

Case 2 : $f(0) = 1$,

derivative at x of $f(x)$ —

$f(0) = 1$

$$\lim_{h \to 0} \frac{\cancel{f(x+h)} - f(x)}{h}$$

$$\lim_{h \to 0} \frac{f(x)f(h) - f(x)}{h} \qquad \lim_{h \to 0} f(x)\left(\frac{f(h) - f(0)}{h}\right)$$

$f(x)f'(0)$

$$\boxed{\therefore f'(c) = f(c)f'(0)} \qquad \square.$$

Tutorial extra Q7.)

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7. Let $f:(a,b)\to\mathbb{R}$ and $c\in(a,b)$. Show that the following are equivalent:

(i) f is differentiable at c.

(ii) There exist $\delta>0$ and a function $\epsilon_1:(-\delta,\delta)\to\mathbb{R}$ such that $\boxed{\lim_{h\to0}\epsilon_1(h)=0}$ and

$$f(c+h)=f(c)+\alpha h+h\epsilon_1(h) \text{ for all } h\in(-\delta,\delta).$$

(iii) There exists $\alpha\in\mathbb{R}$ such that

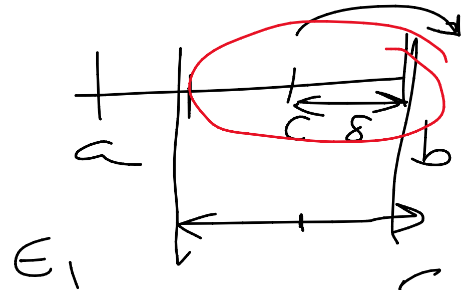
$$\lim_{h\to0}\left(\frac{|f(c+h)-f(c)-\alpha h|}{|h|}\right)=0.$$

$\alpha = f'(c)$

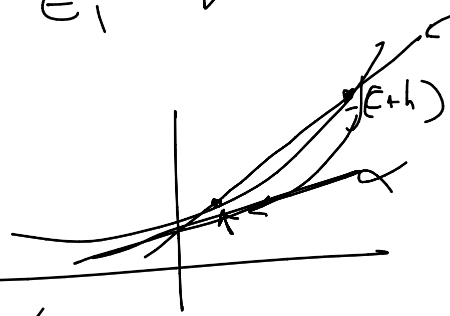
i $\iff$ ii $\iff$ iii $\iff$ i

i$\iff$ii) $\alpha := f'(c)$ (f is diff at c)

$\delta := \boxed{\min(c-a,\ b-c)}$



$$\epsilon_1(h) = \left(\frac{f(c+h)-f(c)}{h}\right) - \alpha$$



$$\lim_{h\to0}\epsilon_1(h) = \lim_{h\to0}\frac{f(c+h)-f(c)}{h} - \alpha$$

$\alpha := f'(c)$ $\qquad$ $\lim_{h\to0}\epsilon_1(h)=0$

$\lim f(n) = 0$ $\qquad \Longrightarrow$

$x \to 0$

Sheet 1 Q10 (optional.)

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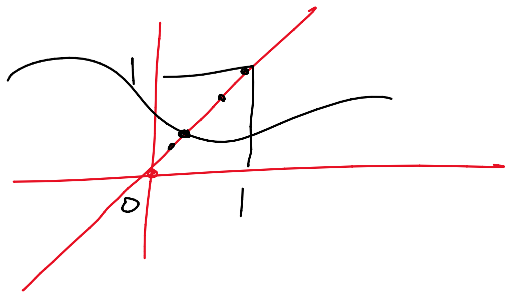
10. Show that any continuous function $f : [0,1] \to [0,1]$ has a fixed point.

fixed point $\implies$ $f(x) - x = 0$ for some $x_0 \in$ [domain]

One point.



Will discuss later.

Let $g(x) := f(x) - x$

$\therefore$ $g(x)$ cont. (sum of cont. fns)

$g(0) = f(0) - 0 \geq 0$

$g(1) = f(1) - 1 \leq 0$

If $g(0)$ or $g(1) = 0$ we are done

(We found fixed point)

Otherwise

$$\boxed{\begin{array}{l} g(0) > 0 \\ g(1) < 0 \end{array}} \longleftarrow \text{IVT}$$

Sheet 2 Q3

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Hint! LMVT

3. Let f be continuous on $[a, b]$ and differentiable on (a, b). If $f(a)$ and $f(b)$ are of different signs and $f'(x) \neq 0$ for all $x \in (a, b)$, show that there is a unique $x_0 \in (a, b)$ such that $f(x_0) = 0$.

f - Cont. $[a,b]$ diff (a,b)

$f(a)$ and $f(b)$ different signs

WLOG, $f(a) < 0$, $f(b) > 0$ (assume)

By intermediate value theorem (Why?) (continuous)

$\exists x_0 \in [a,b]$ st $f(x_0) = 0$ $\boxed{f(a) < 0 < f(b)}$

To show $\nexists$ $x_1 \in [a,b]$ st $f(x_1) = 0$

Proof by Contradiction:

Assume $\exists c_1$ $f(c_1) = 0$

Then, By LMVT on $[c_1, c]$ or $[c, c_1]$

$$\frac{f(c) - f(c_1)}{c - c_1} = f'(x) \quad \text{for some } x \in [c, c_1]$$

$$0 = f'(x)$$

Contradiction because $f'(x) \neq 0$ (Given)

$\therefore$ c is unique.

Suppose you assume $f'(x) > 0$ or $f'(x) < 0$

Aim:-

Use Given conditions ONLY

Use Given Conditions ONLY

(Minimum Assumptions)