TUTORIAL 4

1. Show that there is a strict inequality

$$| \int_{|z|=R} \frac{z^n}{z^m - 1} dz |< \frac{2\pi R^{n+1}}{R^m - 1} \;; R > 1, m \geq 1, n \geq 0$$

2. A power series with center at the origin and positive radius of convergence, has a sum $f(z)$. If it known that $f(\bar{z}) = \overline{f(z)}$ for all values of z within the disc of convergence, what conclusions can you draw about the power series ?

3. This is called Taylor series with remainder :

$$f(z) = f(0) + zf'(0) + \tfrac{z^2}{2!}f''(0) + \cdots + \tfrac{z^N}{N!}f^N(0) + \tfrac{z^{N+1}}{(N+1)!}\int_0^1 (1-t)^N f^{N+1}(tz)dt$$

Use this to prove the following inequalities :

a) $| e^z - \sum_0^N \frac{z^n}{n!} |\leq \frac{|z|^{N+1}}{(N+1)!}; \quad Re(z) \leq 0.$

b) $| cos(z) - \sum_{i=0}^N \frac{(-1)^i z^{2i}}{2i!} |\leq \frac{|z|^{2N+2}\text{coshR}}{(2N+2)!}; \quad |Im(z)| \leq R$

4. By computing $\int_{|z|=1}(z + \frac{1}{z})^{2n}\frac{dz}{z}$, show that $\int_0^{2\pi} cos^{2n}\theta d\theta = \frac{2\pi}{4^n}\frac{2n!}{n!^2}$.

5. Let $f(z)$ be an entire function. Show that $f(z)$ is a polynomial of degree atmost n if and only if there exists a positive real constant C such that $|f(z)| < C|z^n|$ for all z with $|z|$ sufficiently large.

6. Let f and g be entire non-vanishing functions such that $(\frac{f'}{f})(\frac{1}{n}) = (\frac{g'}{g})(\frac{1}{n})$ for all $n \in \mathbb{N}$. Show that g is a non-zero scalar multiple of f.