MA 205 Tutorial - 4

Autumn 2021

1 Question 1

Before starting the solution, we look at the stronger ML inequality.

Theorem 1 (The Stronger ML Inequality). *Let $f : \Omega \to \mathbb{C}$ is a continuous function, and $\gamma : [a, b] \to \Omega$ be a curve. Let $M > 0$ be such that*

$$|f(\gamma(t))| \le M \qquad \forall t \in [a, b]$$

And, suppose that $|f(\gamma(t))| < M$ for some $t \in [a, b]$. Then,

$$\left| \int_\gamma f(z)dz \right| < ML$$

where L is the length of the curve.

So, if $|f| < M$ even at one point, the ML inequality becomes strict.

Proof. Note that,

$$\int_a^b [M - |f(\gamma(t))|]|\gamma'(t)|dt \ge 0$$

as the integrand is nonnegative. But, by our assumption the integrand is not identically zero. Recall that integral is zero **iff** integrand is zero. Thus, it follows that

$$\int_a^b [M - |f(\gamma(t))|]|\gamma'(t)|dt > 0$$

As,

$$\int_a^b M|\gamma'(t)|dt = ML$$

It follows that

$$\int_a^b |f(\gamma(t))||\gamma'(t)|dt < ML$$

And since,

$$\left| \int_a^b f(z)dz \right| \leq \int_a^b |f(z)|dz$$

the theorem follows. □

Now, consider the function,

$$f(z) = \frac{z^n}{z^m - 1}$$

Note that,

$$\begin{aligned} \frac{|z^n|}{|z^{m-1}|} &= \frac{R^n}{|z^{m-1}|} \\ &\leq \frac{R^n}{||z|^m - 1|} \\ &= \frac{R^n}{R^m - 1} \end{aligned}$$

Take $M = \frac{R^n}{R^m-1}$. Note that $z_0 = Re^{\frac{i\pi}{m}}$ shows that $|f(z_0)| < M$. So, using The Stronger ML Inequality, we can conclude that

$$\begin{aligned} \left| \int_{|z|=R} \frac{z^n}{z^m - 1} dz \right| &\leq \int_{|z|=R} \left| \frac{z^n}{z^m - 1} \right| dz \\ &< M(2\pi R) \\ &= \frac{2\pi R^{n+1}}{R^m - 1} \end{aligned}$$

2 Question 2

Given

$$f(\overline{z}) = \sum_{n=0}^{\infty} a_n \overline{z}^n = \overline{f(z)} = \overline{\sum_{n=0}^{\infty} a_n z^n} = \sum_{n=0}^{\infty} \overline{a_n} \overline{z^n} \quad \forall\ |z| < r$$

i.e. the coefficients a_n can be given as $\frac{f^{(n)}(0)}{n!}$ and $\overline{a_n} = \overline{\frac{f^{(n)}(0)}{n!}}$. From which we directly have $a_0 = \overline{a_0}$. Also since the function is holomorphic. Calculating the derivative along the real line gives us (r is real)

$$f'(0) = \lim_{r \downarrow 0} \frac{f(r) - f(0)}{r}$$

Which is also real since r is real valued and we have $f(r) = f(\overline{r}) = \overline{f(r)}$. Thus inductively one may show that all the higher order derivatives are also real and therefore $a_k = \overline{a_k} \quad \forall k \in \mathbb{N}$. i.e. the coefficients of the power series are real.

Note one may also argue the same using term by term differentiability of a power series directly.

3 Question 3

Solution. 1. Note that the sum subtracted is simply the first $N+1$ terms of the Taylor expansion given. Thus, the quantity given within the modulus is simply

$$\frac{z^{N+1}}{(N+1)!}\int_0^1 (1-t)^N \exp(tz)\mathrm{d}t.$$

(We have used that $\exp^{(N+1)} = \exp$.)
Also, note that $|\exp(z)| = \exp(\Re z)$.

Thus, we get

$$\begin{aligned}\left|\int_0^1 (1-t)^N \exp(tz)\mathrm{d}t\right| &\le \int_0^1 \left|(1-t)^N \exp(tz)\right| \mathrm{d}t \\ &= \int_0^1 (1-t)^N \exp(t\Re z)\mathrm{d}t \\ &\le \int_0^1 (1-t)^N \mathrm{d}t = \frac{1}{N+1}\end{aligned}$$

Thus, we get the desired result as

$$\begin{aligned}\left|e^z - \sum_{n=0}^{N}\frac{z^n}{n!}\right| &= \left|\frac{z^{N+1}}{(N+1)!}\int_0^1 (1-t)^N \exp(tz)\mathrm{d}t\right| \\ &\le \frac{|z|^{N+1}}{(N+1)!}\frac{1}{N+1} \\ &\le \frac{|z|^{N+1}}{(N+1)!}.\end{aligned}$$

2. Note that the summation given can be seen as the first $2N+2$ terms. (All the coefficients from z^0 till z^{2n+1} since we know that the latter is 0.)

   Thus, the quantity given within the modulus is simply

$$\frac{z^{2N+2}}{(2N+2)!}\int_0^1 (1-t)^{2N+1}\cos^{(2N+2)}(tz)\mathrm{d}t.$$

Note that

$$\begin{aligned}|\cos(z)| &= \frac{1}{2}\left|e^{\iota z} + e^{-\iota z}\right| \\ &\le \frac{1}{2}\left(\left|e^{\iota z}\right| + \left|e^{-\iota z}\right|\right) \\ &= \frac{1}{2}(e^y + e^{-y}) \\ &= \cosh y.\end{aligned}$$

Now, note that $\cos^{(2N+2)}$ is either $\cos$ or $-\cos$. In either case, we have

$$\left|\cos^{(2N+2)}(tz)\right| \leq |\cosh ty|.$$

Note that $\cosh y$ is an increasing function of $|y|$. (For real y.) Thus, we see that

$$|\cosh ty| \leq |\cosh y|$$

for all $t \in [0, 1]$.

Moreover, if $|y| \leq R$, we get that

$$|\cosh ty| \leq |\cosh y| \leq \cosh R$$

for all $t \in [0, 1]$.

Thus, we get

$$\begin{aligned}\left|\int_0^1 (1-t)^{2N+1} \cos^{(2N+2)}(tz)\mathrm{d}t\right| &\leq \int_0^1 (1-t)^{2N+1} \left|\cos^{(2N+1)}(tz)\right| \mathrm{d}t \\ &\leq \int_0^1 (1-t)^{2N+1} \cosh R\mathrm{d}t \\ &= \frac{\cosh R}{2N+2}.\end{aligned}$$

As earlier, the desired result follows.

□

4 Question 4

By computing

$$\int_{|z|=1} \left(z + \frac{1}{z}\right)^{2n} \frac{1}{z}\mathrm{d}z,$$

show that

$$\int_0^{2\pi} \cos^{2n}\theta\mathrm{d}\theta = \frac{2\pi}{4^n} \cdot \frac{(2n)!}{(n!)^2}.$$

Solution. Recall the "generalised" Cauchy integral formula which tells us that

$$\int_{|w-z_0|=r} \frac{f(w)}{(w-z_0)^{n+1}}\mathrm{d}w = \frac{2\pi\iota}{n!} f^{(n)}(z_0)$$

where f is a function which is holomorphic on an open disc $D(z_0, R)$ and $r < R$. In this question, we take $z_0 = 0$, $r = 1$ and

$$f(z) = (z^2 + 1)^{2n}$$

which is defined and holomorphic on all of $\mathbb{C}$. (So we take $R = 2$, for example.)

Using the formula gives us

$$\begin{aligned}\int_{|z|=1} \left(z + \frac{1}{z}\right)^{2n} \frac{1}{z} \mathrm{d}z &= \int_{|z|=1} \frac{(z^2+1)^{2n}}{z^{2n+1}} \mathrm{d}z \\ &= \frac{2\pi\iota}{(2n)!} f^{(2n)}(0)\end{aligned}$$

Thus, the task is now to compute $f^{(2n)}(0)$. Note that $f^{(2n)}(0)/(2n)!$ is precisely the coefficient of z^{2n} in the expansion of

$$(z^2+1)^{2n}.$$

Use binomial expansion, we see that

$$(z^2+1)^{2n} = \sum_{k=0}^{2n} \binom{2n}{k} (z^2)^k.$$

Thus, the desired coefficient is $\binom{2n}{n}$ and the integral is

$$\int_{|z|=1} \left(z + \frac{1}{z}\right)^{2n} \frac{1}{z} \mathrm{d}z = 2\pi\iota \binom{2n}{n}.$$

Now, we may compute the integral the menial way, i.e., by *parameterising and solving.*

Using the standard parameterisation of $z(t) = e^{\iota t}$ for $t \in [0, 2\pi]$, the integral becomes

$$\begin{aligned}\int_{|z|=1} \left(z + \frac{1}{z}\right)^{2n} \frac{1}{z} \mathrm{d}z &= \int_0^{2\pi} (2\cos t)^{2n} \frac{1}{e^{\iota t}} (\iota e^{\iota t}) \mathrm{d}t \\ &= 4^n \iota \int_0^{2\pi} \cos^{2n}(t) \mathrm{d}t.\end{aligned}$$

Equating it with the previous result gives us the desired answer. $\square$

5 Question 5

Let $f(z) = a_0 + a_1 z + \cdots + a_n z^n$ be a complex polynomial. Then clearly $f(z)$ is an entire function on $\mathbb{C}$.

Now $| f(z) |=| a_0 + a_1 z + \cdots + a_n z^n |$

$\leq \left\{| a_0 | + | a_1 || z | + \cdots | a_n || z^n |\right\} =| z^n | \left\{\frac{a_0}{|z^n|} + \cdots + | a_n |\right\}$

Therefore for $| z |\geq R > 0$, we have

$\left\{\frac{a_0}{|z^n|} + \cdots + | a_n |\right\} \leq \left\{\frac{a_0}{|R^n|} + \cdots + | a_n |\right\}$ for all $| z |\geq R > 0$

Thus $| f(z) |\leq C | z^n |$ for some positive constant C for all z with sufficiently large $| z |$.

Conversely, let f is an entire function such that there is a positive constant C such that $| f(z) |\leq C | z^n |$ for all z with $| z |$ sufficiently large.

Since f is entire , we have $f(z) = \sum\limits_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} z^k$.

Then applying Cauchy's estimates we get

$$| f^{(k)}(0) |\leq \tfrac{k!CR^n}{R^k} \text{ for all } | z |= R > 0.$$

Therefore for $k > n$ and letting $R \to \infty$ we see that $| f^{(k)}(0) |= 0$ for $k > n$. Thus f is a polynomial of degree $\leq n$.

6 Question 6

We define

$$h := \frac{g}{f}.$$

Since f is non-vanishing, h is defined on all of $\mathbb{C}$. We also note that h is non-vanishing since g is non-vanishing. Moreover, h is entire since f and g are entire. Now, we have

$$\frac{h'}{h} = \frac{g'f - gf'}{gf} = \frac{g'}{g} - \frac{f'}{f}.$$

Thus, we see that

$$\left(\frac{h'}{h}\right)\left(\frac{1}{n}\right) = 0 \text{ for all } n \in \mathbb{N}.$$

Since h is non-vanishing, we get

$$h'\left(\frac{1}{n}\right) = 0 \text{ for all } n \in \mathbb{N}.$$

Utilising the result from Question 7, Week 3, we conclude that $h' \equiv 0$. Since $\mathbb{C}$ is path-connected, h is a constant, say c. We thus have

$$\frac{g}{f} = c \implies g = c \cdot f.$$

Moreover, $c \neq 0$ since g is non-vanishing.