Sheet 4.) Q2.)

16 December 2020 14:56

2. (a) Let $f : [a,b] \to \mathbb{R}$ be Riemann integrable and $f(x) \geq 0$ for all $x \in [a,b]$. Show that $\int_a^b f(x)dx \geq 0$. Further, if f is continuous and $\int_a^b f(x)dx = 0$, show that $f(x) = 0$ for all $x \in [a,b]$.

(b) Give an example of a Riemann integrable function on $[a,b]$ such that $f(x) \geq 0$ for all $x \in [a,b]$ and $\int_a^b f(x)dx = 0$, but $f(x) \neq 0$ for some $x \in [a,b]$.

Let $P := \{x_0, x_1, \ldots, x_n\}$ arbitrary of $[a,b]$

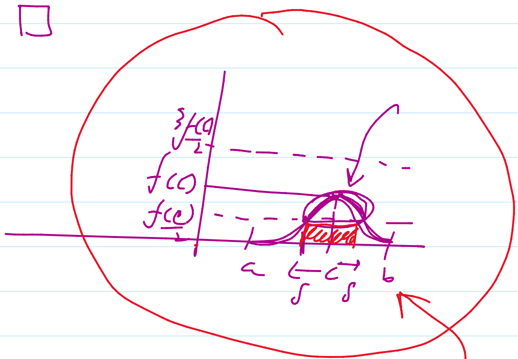
We know, $f(x) \geq 0$ on $[a,b] \Rightarrow f(x) \geq 0$ on $[x_{i-1}, x_i]$

$U(P) = M_i(x_i - x_{i-1})$ where M_i is $:= \sup\{f(x) : [x_{i-1}, x_i]\}$

$M_i \geq 0 \quad \forall [x_{i-1}, x_i]$

$\because f(x)$ is Reimann integrable on $[a,b]$

$$\Rightarrow \int_a^b f(x)\,dx = \overline{\int_a^b} f(x)\,dx \geq 0 \qquad \square$$



ii) Assume $\exists c \in [a,b]$ st $f(c) > 0$

Let $\epsilon = \frac{f(c)}{2}$ $\exists \delta > 0$ st

$$|x - c| < \delta \Rightarrow |f(x) - f(c)| < \frac{f(c)}{2}$$

$\boxed{2\delta \frac{f(c)}{2} > 0}$

$\therefore f(x) > \frac{f(c)}{2}$

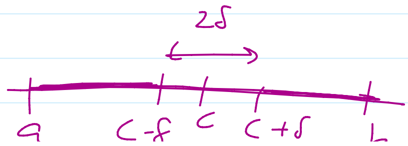
$f(x) \in \left(\frac{f(c)}{2}, \frac{3f(c)}{2}\right)$

$\Rightarrow f(x) > \frac{f(c)}{2}$

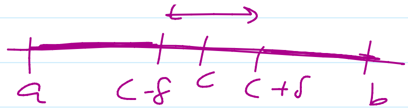
Consider partition :-

$P := \{x_0 = a,\ x_1 = c - \delta,\ x_2 = c + \delta,\ x_3 = b\}$

$L(P_0) = \sum m_i(x_i - x_{i-1})$



$$L(P_0) = \sum m_i (x_i - x_{i-1})$$

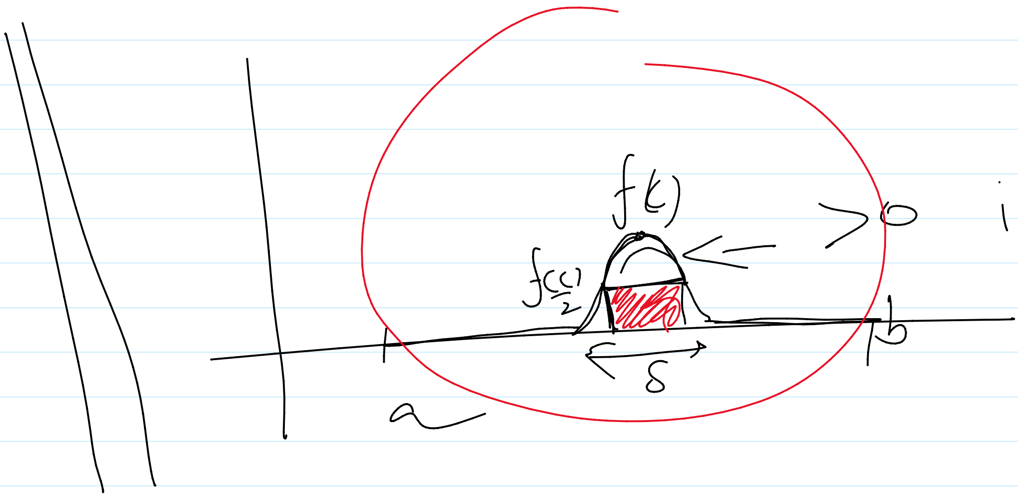


$$= 0 + \frac{f(c)}{\cancel{2}}(\cancel{2}\delta) \qquad 0$$

$$\int_a^b f(x)\,dx \geq \underline{\int_a^b} f(x)\,dx = \delta f(c) > 0$$

$\therefore$ $f(c) = 0 \quad \forall c$

& $f(x)$ is 0 in $[a,b]$



integral $\neq 0$

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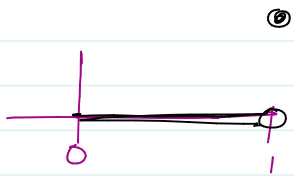
ii) consider $f(x) := \begin{cases} 0 & 0 \leq x < 1 \\ 1 & x = 1 \end{cases}$



interval $[a,b] \equiv [0,1]$

Consider any partition

$P := \{x_0, \dots, x_n\}$

$\sigma(P) = \sum_{i=1}^{n} f(c_i)(x_i - x_{i-1})$ $\qquad f(c_i) = 0 \quad \forall\ i < n$

Note $\quad 0 \le f(c_i) \le 1$

$$0 \le \sigma(P) \le \underline{\underline{(x_n - x_{n-1})}}$$

$\lim_{n\to\infty}$ by sand. theorem

$$\sigma(P) = 0$$

$$\& \quad f(x) \ne 0 \quad \forall x \; [0,1] \quad \blacksquare$$

Sheet 4) Q3.)

16 December 2020 14:57

3. Evaluate $\lim_{n\to\infty} S_n$ by showing that S_n is an approximate Riemann sum for a suitable function over a suitable interval:

(ii) $S_n = \sum_{i=1}^{n} \frac{n}{i^2+n^2}$

(iv) $S_n = \frac{1}{n}\sum_{i=1}^{n} \cos\frac{i\pi}{n}$

ii) Consider the interval $[0,1]$

$P_n := \{x_0, \dots, x_n\}$ $\quad \boxed{x_i = \frac{i}{n}} \Rightarrow x_0 = 0 \quad x_n = 1$

Moreover, $\|P_n\| = \frac{1}{n}$ $\quad P_n = \frac{i-(i-1)}{n} = \frac{1}{n}$

& $\lim_{n\to\infty} \|P_n\| = 0$

$$S_n = \sum_{i=1}^{n} \frac{n}{i^2+n^2}$$

$$= \sum_{i=1}^{n} \left(\frac{1}{1+\left(\frac{i}{n}\right)^2}\right)\left(\frac{1}{n}\right)$$

$$= \sum \frac{1}{1+x_i^2}(x_i - x_{i-1})$$

Consider $f : \mathbb{R} \mapsto \mathbb{R}$

$$f(x) = \frac{1}{1+x^2}$$

$[\ \ ]$

Note: $f(x)$ continuous on $[0,1]$

$\therefore f(x)$ is Reimann Integrable

$$\therefore \sigma(P_n) = \sum \frac{1}{1+x_i^2}(x_i - x_{i-1})$$

$$= S_n$$

$$\lim_{n\to\infty} \sigma(P_n) = \int_0^1 \frac{1}{1+x^2}\,dx$$

$$= \tan^{-1}(x) \Big|_0^1$$

$$= \frac{\pi}{4}$$

ii) $S_n = \frac{1}{n} \sum_{i=1}^{n} \cos\left(\frac{i\pi}{n}\right)$

Partition steps as before:

Consider $f(x) = \cos(\pi x)$

$f(x)$ cont on $[0,1]$ $\Rightarrow$ Reimann integrable

$$\sigma(P_n) = \sum_{i=1}^{n} \cos(\pi x_i)(x_i - x_{i-1})$$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \sigma(P_n)$$

$$= \int_0^1 \cos(\pi x)\,dx$$

$$= \frac{\sin \pi x}{\pi} \Big|_0^1$$

$$= 0 \;//$$

Sheet 4) Q4)

16 December 2020 14:59

4. Compute

(a) $\frac{d^2y}{dx^2}$, if $x = \int_0^y \frac{dt}{\sqrt{1+t^2}}$

(b) $\frac{dF}{dx}$, if for $x \in \mathbb{R}$ (i) $F(x) = \int_1^{2x} \cos(t^2)dt$ (ii) $F(x) = \int_0^{x^2} \cos(t)dt$.

4b) $\frac{dF}{dx}$ i) $F(x) = \int_1^{2x} \cos(t^2)\,dt$

Consider $G : \mathbb{R} \mapsto \mathbb{R}$

$G(x) = \int_1^{x} \cos(t^2)\,dt$

extra.

If $x > 1$, $\exists b$ st $1 < x < b \ni x \in (1, b)$

and if $x < 1$, $\exists b$ st $b < x < 1$ $x \in (b, 1)$

$\int_1^x \cos(t^2)\,dt$ or $+\int_b^x \cos(t^2)\,dt - \int_b^1 \cos(t^2)\,dt$

$x \in (a, b)$ and $f(x)$ is continuous

$F'(x) = f(x)$

Irrespective of interval $f(x) = \cos(x^2)$ is continuous for all x.

$G'(x) = \cos(x^2)$

$F(x) = G(2x)$

$F'(x) = G'(2x)\ 2$

$$F'(x) = G'(2x)\ 2$$

$$= \underline{\underline{2\cos(4x^2)}}$$

b.) Let $G(x) = \int_0^x \cos(t)\, dx$

$\cos(x)$ – continuous

$\therefore$ By FTC part I,

$$G'(x) = \cos(x)$$

Let $F(x) = G(x^2)$

$\therefore$ $F'(x) = G'(x^2)\ 2x$ (chain rule)

$$= \underline{\underline{\cos(x^2)\ 2x}}$$

$$\int_1^x f(t)\, dt$$

FTC, $f(t)$ – cont. & $x \in [1, b]$

but what if $x < 1$?

$\exists b$ st $x \in [b, 1]$

Sheet 4.) Q6.)

16 December 2020 14:59

6. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous and $\lambda \in \mathbb{R}$, $\lambda \neq 0$. For $x \in \mathbb{R}$, let

$$g(x) = \frac{1}{\lambda}\int_0^x f(t)\sin\lambda(x-t)dt.$$

Show that $g''(x) + \lambda^2 g(x) = f(x)$ for all $x \in \mathbb{R}$ and $g(0) = 0 = g'(0)$.

Vacuously true.

$$g(x) = \frac{1}{\lambda}\int_0^x f(t)\sin(\lambda(x-t))\,dt \qquad \sin(A-B)$$

$$g(x) = \frac{1}{\lambda}\int_0^x f(t)\left(\sin(\lambda x)\cos(\lambda t) - \sin(\lambda t)\cos(\lambda x)\right)dt$$

$$= \frac{1}{\lambda}\left(\sin\lambda x\int_0^x f(t)\cos(\lambda t)\,dt - \cos(\lambda x)\int_0^x f(t)\sin(\lambda t)\,dt\right)$$

$\lambda\sin(\lambda x)$

Note: $f(t)$ is cont.

$\cos(t)$ / $\sin(t)$ cont

$\therefore$ Product is cont.

By FTC pt 1:

$$g'(x) = \frac{1}{\lambda}\cancel{\sin(\lambda x)}f(x)\cos(\lambda x)$$

$$-\frac{1}{\lambda}\cancel{\sin(\lambda x)}f(x)\cos(\lambda x)$$

$$+\frac{1}{\lambda}\left(\lambda\sin(\lambda x)\int_0^x f(t)\sin(\lambda t)\,dt + \lambda\cos(\lambda x)\int_0^x f(t)\cos(\lambda t)\,dt\right)$$

$g'(x) =$

$g'(0) = 0$

Continuous again

By FTC part 1.

$$g'(x) = \frac{1}{\cancel{\lambda}} \left(\cancel{\lambda} \sin(\lambda x) \int_0^x f(t) \sin(\lambda t)\, dt + \lambda \cos(\lambda x) \int_0^x f(t) \cos(\lambda t)\, dt \right)$$

$$g''(x) = \frac{1}{\cancel{\lambda}} \left(\cancel{\lambda} \sin^2(\lambda x) f(x) + \cancel{\lambda} \cos^2(\lambda x) f(x) \right)$$

$$+ \left(\int f(t) \sin(\lambda t)\, dt \right) \cos(\lambda x)(\lambda) + (-\sin(\lambda x)) \lambda \int_0^x f(t) \cos(\lambda t)\, dt$$

$$g''(x) = f(x) - \lambda^2 g(x)$$

$$f(x) = \lambda^2 g(x) + g''(x) \qquad \square$$