MA 205 Tutorial - 5

Autumn 2021

1 Question 1

(a) $\frac{\sin(1/z)}{z^4+1}$

Note that sin(1/z) is not defined at $z = 0$. And the denominator is not defined when $1 + z^4 = 0$ i.e. $z = \pm e^{\pm \iota \frac{\pi}{4}}$.

So, the set of singularities is $S = \left\{0, \pm\frac{1}{\sqrt{2}} \pm \iota\frac{1}{\sqrt{2}}\right\}$

All the singularities are isolated (finite in number). For any non-zero singularity, say z_0, it is easy to show that

$$\lim_{z \to z_0} \frac{1}{f(z)} = 0$$

Thus, the non-zero singularities satisfying $z^4 + 1 = 0$ are poles.

For $z = 0$, we show that 0 is an essential singularity i.e. it is neither a removable singularity nor a pole. Suppose that we approach 0 along positive imaginary axis, we have

$$\begin{aligned}\lim_{y \to 0} f(z) &= \lim_{y \to 0} \frac{\sin(1/\iota y)}{1 + (\iota y)^4} \\ &= \frac{1}{2} \lim_{y \to 0} (e^{\frac{1}{y}} - e^{-\frac{1}{y}})\end{aligned}$$

Now, the limit $e^{-\frac{1}{y}}$ exists, and $e^{\frac{1}{y}} = \infty$, so 0 is not a removable singularity. And when you approach 0 via real line, since sin is bounded on real line, we have that 0 is not a pole either.

So, 0 is a removable singularity.

(b) $\frac{z^5 \sin(1/z)}{z^4+1}$

The set of singularities is same as in part (a), and they are isolated. Similar argument can be used to show that each of the non zero singularity is a pole, and 0 is a removable singularity.

(c) $\frac{1}{\sin(1/z)}$

The set of singularities consist of the points where sin is zero, and where

$z = 0$. So the set of singularities is $S = \left\{0 \cup \frac{1}{n\pi} \qquad n \in \mathbb{Z} \setminus \{0\}\right\}$.
Here, 0 is not an isolated singularity, as for any neighbourhood around 0, one can find $\frac{1}{n\pi}$ in the neighborhood, for some integer n. Rest of the singularities are isolated. To see this for $\frac{1}{n\pi}$, where n is an integer, define

$$\epsilon = \max\left\{\left|\frac{1}{n\pi} - \frac{1}{(n+1)\pi}\right|, \left|\frac{1}{n\pi} - \frac{1}{(n-1)\pi}\right|\right\}$$

The punctured neighborhood $B_\epsilon(\frac{1}{n\pi})\setminus\{\frac{1}{n\pi}\}$ does not contain any other point of S.
The non zero singularities in S are isolated. Now, to see that these are poles, compute

$$\lim_{z\to n\pi} \frac{z - n\pi}{\sin\left(\frac{1}{z}\right)}$$

Note that this limit exists finitely (express it as limit in derivative), and is non zero when $n \neq 0$. This implies that for non zero n, $n\pi$ is a pole.
(As 0 is not an isolated singularity, we do not try to categorise it).

(d) $e^{\frac{1}{z}}$

The fraction $\frac{1}{z}$ is not defined at $z = 0$. We show that 0 is an essential singularity.
As $z \to 0$ along negative real line, $e^{\frac{1}{z}} \to 0$. And, when $z \to 0$ along negative imaginary axis, $e^{\frac{1}{z}}$ lies on the unit circle. Hence, 0 is neither a removable singularity nor a pole. So, 0 is an essential singularity.

2 Question 2

A function $f(z)$ is meromorphic in the neighbourhood of a point z_0 if either $f(z)$ or its reciprocal function is holomorphic in some neighbourhood of z_0. A pole of $f(z)$ is a zero of $1/f(z)$.

Consider the function
$$g(z) = sin(\pi z)$$
which is an entire function. Therefore we have $f(z) = 1/g(z)$ is a meromorphic function on $\mathbb{C}$
$$f(z) = \frac{1}{g(z)} = \frac{1}{sin(\pi z)}$$
which has a pole at every $z \in \mathbb{Z}$

3 Question 3

Solution. Note that
$$\frac{2(z-1)}{z^2 - 2z - 3} = \frac{1}{z-3} + \frac{1}{z+1}.$$

In each part, we expand each fraction as a Laurent series such that the series converges on that disc.

1. Here, we can write

$$\frac{1}{z-3} = -\frac{1}{3}\frac{1}{1-\frac{z}{3}} = -\frac{1}{3}\sum_{n=0}^{\infty}\left(\frac{z}{3}\right)^n$$

   and

$$\frac{1}{1+z} = \sum_{n=0}^{\infty}(-z)^n.$$

   Put them together to get the complete Laurent series.

2. Here, we can write

$$\frac{1}{z-3} = -\frac{1}{3}\frac{1}{1-\frac{z}{3}} = -\frac{1}{3}\sum_{n=0}^{\infty}\left(\frac{z}{3}\right)^n$$

   and

$$\frac{1}{1+z} = \frac{1}{z\left(1+\frac{1}{z}\right)} = \sum_{n=0}^{\infty}(-z)^{-n-1}.$$

   Put them together to get the complete Laurent series.

3. Here, we can write

$$\frac{1}{z-3} = \frac{1}{z}\frac{1}{1-\frac{3}{z}} = \frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{z}{3}\right)^{-n}$$

   and

$$\frac{1}{1+z} = \frac{1}{z\left(1+\frac{1}{z}\right)} = \sum_{n=0}^{\infty}(-z)^{-n-1}.$$

   Put them together to get the complete Laurent series.

Note that it has to be justified that each series that we wrote did converge on the given annulus. □

4 Question 4

Let Ω be a domain in $\mathbb{C}$, and let $z_0 \in \Omega$. Suppose that z_0 is an isolated singularity of f, and f is bounded in some punctured neighbourhood of z_0 (that is, there exists $M > 0$ and $\delta > 0$ such that $|f(z)| \leq M$ for all $z \in B_\delta(z_0) - \{z_0\}$). Show that f has a removable singularity at z_0.

Solution. Fix $\delta > 0$ such that f is bounded and holomorphic on the punctured disc of radius δ centered at z_0. (Why does such a δ exist?)

Define $g(z) := f(z)(z - z_0)$ on this punctured disc. Then, g is holomorphic on this punctured disc (why?). Moreover,

$$\lim_{z \to z_0} g(z) = 0.$$

(Why? Use the fact that f is bounded.)

Thus, by RRST, we see that z_0 is a removable singularity of g. Furthermore, defining $g(z_0) := 0$ makes it holomorphic on $B_\delta(z_0)$. (This is part of the conclusion of RRST.)
Thus, on $B_\delta(z_0)$, we can expand g as

$$g(z) = a_1(z - z_0) + a_2(z - z_0)^2 + \cdots .$$

(Why is there no constant coefficient above?)

Conclude that the following equality holds for $z \in B_\delta(z_0) - \{z_0\}$:

$$f(z) = a_1 + a_2(z - z_0) + a_3(z - z_0)^2 + \cdots .$$

Thus, z_0 is a removable singularity since defining $f(z_0) := a_1$ makes f holomorphic on $B_\delta(z_0)$. $\square$

5 Question 5

6 Question 6

Assuming $0 < a \neq 1$, we have

$$\begin{aligned}
\int_0^{2\pi} \frac{1}{a^2 - 2a\cos\theta + 1}\,\mathrm{d}\theta &= \int_0^{2\pi} \frac{1}{a^2 - a(e^{-\iota\theta} + e^{\iota\theta}) + e^{-\iota\theta} \cdot e^{\iota\theta}}\,\mathrm{d}\theta \\
&= \int_0^{2\pi} \frac{1}{(a - e^{\iota\theta})(a - e^{-\iota\theta})}\,\mathrm{d}\theta \\
&= \int_0^{2\pi} \frac{e^{\iota\theta}}{(a - e^{\iota\theta})(ae^{\iota\theta} - 1)}\,\mathrm{d}\theta \\
&= \frac{1}{\iota}\int_0^{2\pi} \frac{\iota e^{\iota\theta}}{(a - e^{\iota\theta})(ae^{\iota\theta} - 1)}\,\mathrm{d}\theta \\
&= \frac{1}{\iota}\int_{|z|=1} \frac{1}{(a - z)(az - 1)}\,\mathrm{d}z \\
&= -\frac{1}{a\iota}\int_{|z|=1} \frac{1}{(z - a)(z - 1/a)}\,\mathrm{d}z.
\end{aligned}$$

Note that for both cases $a > 1$ and $a < 1$, the integrand has exactly one pole within the unit circle. For $a > 1$, the pole is at $1/a$. Using Cauchy's Integral

Formula, we get

$$\begin{aligned}\int_0^{2\pi} \frac{1}{a^2 - 2a\cos\theta + 1}\,\mathrm{d}\theta &= -\frac{1}{a\iota} \cdot 2\pi\iota \frac{1}{1/a - a} \\ &= -\frac{2\pi}{1 - a^2}.\end{aligned}$$

For $a < 1$, the pole is at a, which gives us

$$\begin{aligned}\int_0^{2\pi} \frac{1}{a^2 - 2a\cos\theta + 1}\,\mathrm{d}\theta &= -\frac{1}{a\iota} \cdot 2\pi\iota \frac{1}{a - 1/a} \\ &= -\frac{2\pi}{a^2 - 1}\end{aligned}$$

7 Question 7

Since $P(z)$ is given to be a degree n monic polynomial over $\mathbb{C}$ with n distinct roots $a_1, \ldots, a_n$, we have

$$P(z) = (z - a_1) \cdots (z - a_n) = \prod_{i=1}^{n} (z - a_i).$$

Hence, applying the product rule for differentiation, we have

$$P'(z) = \sum_{i=1}^{n} \left(\prod_{j \neq i} (z - a_j) \right) \frac{d}{dz}(z - a_i) = \sum_{i=1}^{n} \prod_{j \neq i} (z - a_j).$$

For each $1 \leq k \leq n$, the term $\prod_{j \neq i}(z - a_j)$ does *not* contain the factor $z - a_k$ if and only if $i = k$. Hence, all but one of the terms in the RHS evaluate to zero at $z = a_k$, and we have $P'(a_k) = \prod_{j \neq k}(a_k - a_j)$.