Tutorial 5

August 30, 2021

1. Locate and classify the type of singularities of :

a) $\frac{sin(1/z)}{(1+z^4)}$
b) $\frac{z^5 sin(1/z)}{(1+z^4)}$
c) $\frac{1}{sin(1/z)}$
d) $e^{\frac{1}{z}}$

2. Construct a meromorphic function on $\mathbb{C}$ with infinitely many poles.

3. Find Laurent expansions for the function $f(z) = \frac{2(z-1)}{z^2-2z-3}$ valid on the regions: (i) $0 \leq |z| < 1, (ii) 1 < |z| < 3, (iii) |z| > 3$.

4. Let D be a domain in $\mathbb{C}$ and let $z_0 \in D$. Suppose that z_0 is an isolated singularity of $f(z)$ and $f(z)$ is bounded in some punctured neighborhood of z_0 (that is, there exists $M > 0$ such that $|f(z)| \leq M$ for all $z \in U - z_0$ for neighborhood U). Show that $f(z)$ has a removable singularity at z_0.

5. A complex-valued function $f(z)$ on $\mathbb{C}$ is called doubly periodic if there exist linearly independent vectors $v, w \in \mathbb{C}$ over $\mathbb{R}$ such that $f(z + v) = f(z)$ and $f(z + w) = f(z)$ for all $z \in \mathbb{C}$. Show that any double periodic entire function is constant.

6. Show by transforming into an integral over the unit circle, that $\int_0^{2\pi} \frac{d\theta}{a^2+1-2acos\theta} = \frac{-2\pi}{1-a^2}$, where $a > 1$. Also compute the value when $a < 1$.

7. Show that if $a_1, a_2, ..., a_n$ are the distinct roots of a monic polynomial $P(z)$ of degree n, for each $1 \leq k \leq n$ we have the formula:

$$\prod_{j \neq k} (a_k - a_j) = P'(a_k)$$