C-MA205

(Tut Batch 6)

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General Guidelines: (credits. $\int$(◠‿◠) dx)

1. Keep yourself muted unless you wish to say something
2. Remind me to Record! (Every tutorial)
3. When you wish to say something → Directly unmute and Speak
4. If you think I made a mistake unmute and tell

   case 1 → You correct me and save me the embarassment

   case 2 → I clear any misconception arised and we continue

   → 'Everyone Benefits'

5. Also, it is encouraged that after unmuting, you start with 'This is <your name>'.
   [Not necessary but gives an offline flair to this online Tut]

6. If you feel your doubt is a 'small clarification', put it in the chat, some benevolent soul may answer it there for you.
   [I will read the chat intermittently]

(Apologies in advance for bad handwriting / spelling miskates)

§. Tutorial 1
(03-08-2020)

Recap Topology

Open set : A set $U \subset \mathbb{C}$ is open if for every $z_0 \in U$ $(\exists \delta > 0)$ s.t $B_\delta(z_0) \subset U$

$$B_\delta(z_0) = \{ z \in \mathbb{C} : |z - z_0| < \delta \}$$

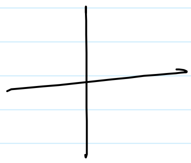
Closed set A set $U \subset \mathbb{C}$ is closed if $\mathbb{C} \setminus U$ is open

path connected : A subset $P \subset \mathbb{C}$ is said to be path connected if for every z_1, z_2 $\exists$ continuous fn $\sigma : [0,1] \to P$ st $\sigma(0) = z_1$ and $\sigma(1) = z_2$

Differentiation :

A function $f : \Omega \to \mathbb{C}$ is said to be (complex) differentiable at $z_0 \in \Omega$ if

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} \quad \text{exists}$$



Cauchy Reimann Eq^{n}s

$$U_x = V_y \quad \& \quad U_y = -V_x$$

$U_x = V_y$ & $U_y = -V_x$

$$f : \Omega \to \mathbb{R}^2$$

$$f(x,y) = (u(x,y), v(x,y))$$

Intuition from where U_x, V_x, u_y, V_y

$$z_0 = x_0 + iy_0$$

$$\left.\begin{aligned} u_x(x_0,y_0) &= V_y(x_0,y_0) \\ u_y(x_0,y_0) &= -V_x(x_0,y_0) \end{aligned}\right]$$

[(Existence of part. der continuous at point) + "] $\Rightarrow$ Complex diff.

$\Leftrightarrow$ $\mathbb{R}^2$ differentiability

Questions –

1. Show that a real polynomial that is irreducible has degree at most two. i.e., if

$$3 \to f(x) = a_0 + a_1x + \ldots + a_nx^n, \ a_i \in \mathbb{R},$$

then there are non-constant real polynomials g and h such that $f(x) = g(x)h(x)$ if $n \geq 3$.
(Assume FTA)

≤ 2

Solution: $\mathbb{C}[x] := \{$ polynomials in x with complex coefficients $\}$

$\mathbb{R}[x] := \{$ " real coeff $\}$

$\mathbb{R}[x] \subset \mathbb{C}[x]$

Case 1. $f(x)$ – has real root x_0

$h(x) = (x - x_0)$

$f(x) = (x-x_0)\, g(x) = h(x)\, g(x)$

$\deg(f) \geq 3 \quad \deg(h) = 1 \quad \deg(g) \geq 2 \quad \square$

Case 2: $f(x)$ – has a complex root z_0

$f(x) \in \mathbb{R}[x]$

$f(x) \in \mathbb{R}[x]$

$f(z_0) = 0$ — Given

$\overline{f(x)} = a_0 + a_1 x + \cdots a_n x^n$

$= \bar{a}_0 + \bar{a}_1 \bar{x} + \cdots + \bar{a}_n \bar{x}^n$

$a_i \in \mathbb{R} \implies \bar{a}_i = a_i$

$= a_0 + a_1 \bar{x} + \cdots + a_n \bar{x}^n$

$\overline{f(z_0)} = a_0 + a_1 \bar{z}_0 + \cdots a_n \bar{z}_0^n$

$\overline{f(z_0)}$ ← 0

$0 = f(\bar{z}_0)$

$\implies$ If z_0 is a root, so is $\bar{z}_0$

$g(x) = (x - z_0)(x - \bar{z}_0) \qquad g(x) \in \mathbb{C}[x]$

$= x^2 - \underline{(z_0 + \bar{z}_0)} x + \underline{z_0 \bar{z}_0}$

$= x^2 - 2\mathrm{Re}(z_0) x + |z_0|^2 \in \mathbb{R}[x]$

Again, $\deg(g) = 2$ and $\deg(f) \geq 3 \implies \deg(h) \geq 1$

$\Downarrow$

h - non constant

$\underline{f(x)} = \underbrace{(x - z_0)(x - \bar{z}_0)}_{\text{Real poly}} h(x)$

$f(x) = g(x) h(x)$

Q2.)

2. Show that a non-constant polynomial $f(z_1, z_2)$ in complex variables z_1 and z_2 and with complex coefficients has infinitely many roots (in $\mathbb{C}^2$).
(Assume FTA)

(Lemma)

FTA $\implies$ non constant complex poly $f(x)$

FTA $\Rightarrow$ non constant complex poly $f(x)$

$\exists \quad f(x) = (x - x_0)\, g(x)$

Claim: Complex polynomial $f(x) \in \mathbb{C}[x]$ with degree $= n$ has exactly 'n' roots

Proof. By Induction

Base case $\quad \deg(f) = 1$

$$f(x) = a_0 + a_1 x \implies 1 \text{ root}$$

$$0 = a_0 + a_1 x$$

$$\iff x = -a_0/a_1 \quad [1 \text{ root}]$$

Inductive Hyp: $f(x)$ — deg $= n$ has n roots

$$\boxed{I(n) \implies I(n+1)}$$

$\deg(f) = n+1$

Then by FTA, $\quad f(x) = (x - x_0)\, g(x) \qquad \deg(g) = n$

$$x = x_0 \quad \text{or} \quad g(x) = 0$$

Counting roots with 'multiplicity'

$$1 + n \quad (\text{Inductive Hyp})$$

$$n+1 \text{ — roots} \qquad \square$$

$$f(z_1, z_2) = \boxed{\sum_{k=0}^{n} \underbrace{p_k(z_1)}_{\text{coeff.}}\, \underline{\underline{z_2}}^{\,k}} \qquad (\text{Assume } z_1 \text{ fixed})$$

$p_n(z_1) \neq 0$

choose $z_1 \in \mathbb{C} \mid p_n(z_1) \neq 0$

Also we know $\exists$ infinite $z_1 = \alpha$ such that $p_n(z_1) \neq 0$

such that $p_n(z_1) \neq 0$

$z_1 \in \mathbb{C}$ $\quad$ $\boxed{p_n(z_1) \ \deg = k}$

$\left|\mathbb{C} - \{\text{finite set}\}\right| = |\mathbb{C}|$

FTA, we know $\exists z_{2_0}$ such $z_2 = z_{2_0}$

$$f(\alpha, z_{2_0}) = 0$$

$\Longrightarrow$ Infinitely many roots

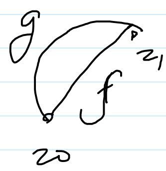
Q3)

3. Show that the complex plane minus a countable set is path-connected.

$\mathbb{C} \setminus \{\text{countable set}\}$ – path connected [Let S – countable set]

Fact: $\mathbb{C}$ – is path connected

Consider z_0 and $z_1 \in \mathbb{C} \setminus \{S\}$

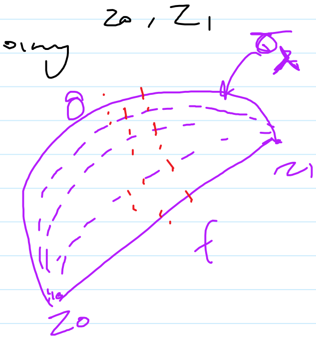


$f : [0,1] \longrightarrow$ straight line joining z_0, z_1

$g : [0,1] \longrightarrow$ semi, circular arc joining z_0, z_1

$$\sigma_\lambda(t) = \lambda f(t) + (1-\lambda) g(t)$$

$$\lambda \in (0,1)$$



Claim:

σ_λ : path in $\mathbb{C}$

proof: Cont. (why) f & g are cont.

$\sigma_\lambda(0) = z_0 \qquad \sigma_\lambda(1) = z_1$

If $\lambda_1 \neq \lambda_2 \Longrightarrow \sigma_{\lambda_1}(t) \neq \sigma_{\lambda_2}(t)$

Proof Assume $\sigma_{\lambda_1} = \sigma_{\lambda_2}$

$$\lambda_1 f(t) + (1-\lambda_1) g(t) = \lambda_2 f(t) + (1-\lambda_2) g(t)$$

$$(\lambda_1 - \lambda_2) f(t) = (\lambda_1 - \lambda_2) g(t)$$

$$f(t) \neq g(t) \implies \Longleftarrow$$

So, we have uncountable no of paths (why?)

Because $[0,1]$ — uncountable

$\{\mathbb{N}\}$ countable

$\implies \exists \lambda_0$ such that $\sigma_{\lambda_0}(t) \in S$

This works for all z_1 & z_2. We are done.

4.)

4. Check for real differentiability and holomorphicity:

1. $f(z) = c$
2. $f(z) = z$
3. $f(z) = z^n, n \in \mathbb{Z}$
4. $f(z) = \mathrm{Re}(z)$
5. $f(z) = |z|$
6. $f(z) = |z|^2$
7. $f(z) = \bar{z}$
8. $f(z) = \begin{cases} \frac{z}{\bar{z}} & \text{if } z \neq 0 \\ 0 & \text{if } z = 0. \end{cases}$

Q1.) Q2) Real diff & holo (both)

3.) $f(z) = z^n$ $n \in \mathbb{Z}$

u classic defn

$n \geq 0$ $\boxed{\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}}$ exist

exists

for $z \neq z_0$, we know $\frac{z^n - z_0^n}{z - z_0} = \sum_{k=0}^{n-1} z^k z_0^{n-1-k}$

$\square$

$n < 0$

claim defined $\mathbb{C}\setminus\{0\}$ and $\frac{1}{f}$ is diff

(why?) $1/f$ is diff for $n \geq 0$ case

$\Longrightarrow$ $f(z)$ holomorphic on $\mathbb{C}\setminus\{0\}$

4.) $f(z) = \operatorname{Re}(z)$

$f(x+iy) = x + 0i$

$u(x,y) = x$ $\quad v(x,y) = 0$

$\star$ clearly f is real diff

$u_x = 1$ $\quad v_y = 0$ clearly $u_x \neq v_y$

$\Longrightarrow$ Not holomorphic

5) $f(z) = |z|$

$f(x+iy) = \sqrt{x^2+y^2} + 0i$

$u(x,y) = \sqrt{x^2+y^2}$ $\quad v(x,y) = 0$

On $\mathbb{R}^2 \setminus \{(0,0)\}$

The partial derivatives of u exist and are continuous

$\Longrightarrow$ Real diff everywhere except $(0,0)$

$\Longrightarrow$ f is real diff in set $\mathbb{C}\setminus\{0\}$

$u_x \neq 0$ $\quad v_y = 0$ $\quad u_x \neq v_y$

7.) $f(z) = \overline{z}$

$f(x+iy) = x - iy$

$u(x,y) = x$ $\quad v(x,y) = -y$

CR eqns

$u_x = 1$ $\quad v_y = -1$ $\quad u_x \neq v_y$

$\Rightarrow$ Not holomorphic anywhere

[Is it possible to have $f(z)$ such that f is holo nowhere but $f(f(z))$ is holo. everywhere]

8:) $f(z) = \begin{cases} \dfrac{z}{\bar{z}} & z \neq 0 \\ 0 & z = 0 \end{cases}$

$$f(x+iy) = \frac{x+iy}{x-iy} \frac{(x+iy)}{(x+iy)}$$

$$= \frac{x^2 - y^2 + 2xy\, i}{x^2+y^2}$$

$$= \left(\frac{x^2-y^2}{x^2+y^2}\right) + \left(\frac{2xy}{x^2+y^2}\right) i$$

$$u(x,y) = \frac{x^2-y^2}{x^2+y^2} \qquad v(x,y) = \frac{2xy}{x^2+y^2}$$

Real diff: (Recall MA 109)

(why?) $\Rightarrow$ Both u, v are not continuous at $(0,0)$

All other cases p. der. exist and are continuous

For everywhere except 0 ($\mathbb{C} \setminus \{0\}$)

Check CR equations

$$u_x = \frac{(x^2+y^2)2x - (2x)(x^2-y^2)}{(x^2+y^2)^2} = \frac{4xy^2}{(x^2+y^2)^2}$$

$$v_y = \frac{(x^2+y^2)2x - (2xy)(2y)}{} = 2x^3 - 2xy^2$$

$$(x^2+y^2)^2$$

$$2x^3 - 2xy^2 = 4xy^2$$

$$x^3 = 3xy^2 \implies (x=0)$$

$$\left[u_y = -v_x\right]$$

$$u_y = -\frac{4x^2y}{(x^2+y^2)^2} \qquad v_x = \frac{2y^3-2x^2y}{(x^2+y^2)^2}$$

$$\implies (y=0)$$

$$\implies x=0,\ y=0 \implies z=0 \implies \times$$

Edit: This was done after the tutorial ended.

$$U = \frac{x^2-y^2}{x^2+y^2} \qquad V = \frac{2xy}{x^2+y^2}$$

$$U_x = \frac{(x^2+y^2)2x - (x^2-y^2)(2x)}{(x^2+y^2)^2} = \frac{4xy^2}{(x^2+y^2)^2}$$

$$V_y = \frac{(x^2+y^2)2x - (2xy)(2y)}{(x^2+y^2)^2} = \frac{2x^3-2xy^2}{(x^2+y^2)^2}$$

$$u_y = \frac{(x^2+y^2)(-2y) - (x^2-y^2)(2y)}{(x^2+y^2)^2} = \frac{-4x^2y}{(x^2+y^2)^2}$$

$$V_x = \frac{(x^2+y^2)2y - (2xy)(2x)}{(x^2+y^2)^2} = \frac{2y^3-2x^2y}{(x^2+y^2)^2}$$

CR.1 $\implies u_x = v_y$

$\iff 4xy^2 = 2x^3 - 2xy^2$

$\iff 3xy^2 = x^3 \implies x=0$

CR · 2 $\Rightarrow$ $U_y = -V_x$

$\Rightarrow$ $y = 0$ (I was correct )

But at $z = 0$, Not Real diff

$\Rightarrow$ Holomorphic Nowhere