Recap:

(Various forms of Cauchy)

1. Recall: If $\gamma : [a,b] \longrightarrow \mathbb{C}$ is sufficiently nice and f is continuous on γ, then

$$\int_\gamma f(z)\,dz = \int_a^b f(\gamma(t))\,\gamma'(t)\,dt$$

So, if $\gamma(t) = x(t) + i y(t)$ [x & y are $\mathbb{R}$ functions]

$\gamma'(t) = x'(t) + i y'(t)$

(1) "FUNDAMENTAL THM"

If $\Omega \in \mathbb{C}$ is open and $\gamma : [a,b] \longrightarrow \Omega$ is a curve and $f : \Omega \longrightarrow \mathbb{C}$ admits a primitive

'nice'

$\exists\ F\ \Omega \longrightarrow \mathbb{C}$ s.t $F' = f$

$$\int_\gamma f(z)\,dz = F(\gamma(b)) - F(\gamma(a))$$

Also, if γ is closed loop, then

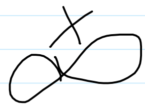
$$\int_\gamma f(z)\,dz = 0$$

[Note the conditions on Ω and γ]

Thm:

Theorem

Let C be a simple closed contour and let f be a holomorphic function defined on an open set containing C as well as its interior. Then $\int_C f(z)dz = 0$.



γ is simple if γ is 1-1 on $[a,b)$

γ is closed if $\gamma(a) = \gamma(b)$

Theorem

(More general form of Cauchy's theorem) *Let Ω be a simply connected domain in $\mathbb{C}$. Let $f(z)$ be a holomorphic function*

simply connected domain

Theorem

(More general form of Cauchy's theorem) *Let Ω be a simply connected domain in $\mathbb{C}$. Let $f(z)$ be a holomorphic function defined on Ω. Let C be a simple closed contour in Ω. Then* $\int_C f(z)dz = 0$

Simply connected domain

If $\gamma \in \Omega$ the $\text{Int}(\gamma) \in \Omega$

Theorem (Cauchy Integral Formula)

Let f be holomorphic everywhere on an open set Ω. Let γ a simple closed curve in Ω (oriented positively). If z_0 is interior to γ, then,

$$f(z_0) = \frac{1}{2\pi i}\int_\gamma \frac{f(z)dz}{z - z_0}.$$

oriented positively $\Rightarrow$ ↺

Tutorial 3 :-

Q) 1. Show that Cauchy Riemann equation take the form:

$$u_r = \frac{1}{r}v_\theta \text{ and } v_r = -\frac{1}{r}u_\theta$$

in polar coordinates.

$u_x = v_y$ and $u_y = -v_x$

$e^{i\theta} f'(re^{i\theta}) = e^{i\theta} f'(z_0)$

$$f(r,\theta) = \left[f(re^{i\theta}) = u(r,\theta) + i v(r,\theta) \right]$$

Mention: $z_0 = re^{i\theta} \neq 0$ (No need to mention because in Q $1/r$ term exists)

$ire^{i\theta}$

$\lim\limits_{z \to z_0} \dfrac{f(z) - f(z_0)}{z - z_0}$ should exist for differentiability at z_0

1 fix $\theta = \theta_0$ let $r \to r_0$ $[z_0 = r_0 e^{i\theta_0}]$

$$f'(z_0) = \lim_{r \to r_0} \left\{ \frac{u(r,\theta_0) - u(r_0,\theta_0)}{e^{i\theta_0}(r - r_0)} + i\,\frac{v(r,\theta_0) - v(r_0,\theta_0)}{e^{i\theta_0}(r - r_0)} \right\}$$

$$= e^{-i\theta_0}\left(u_r(r_0,\theta_0) + i v_r(r_0,\theta_0) \right)$$

2. fix $r = r_0$ and $\theta \to \theta_0$

$$f'(z_0) = \lim_{\theta \to \theta^0} \left\{ \frac{u(r_0,\theta) - u(r_0,\theta_0)(\theta-\theta_0)}{(\theta-\theta_0)r_0(e^{i\theta} - e^{i\theta_0})} + i\,\frac{v(r_0,\theta) - v(r_0,\theta_0)}{(\;)\, r_0(e^{i\theta} - e^{i\theta_0})} \right\}$$

$$= \frac{1}{r_0} \lim_{\theta \to \theta_0} \left(\frac{U_\theta(\theta_0, r_0)}{i e^{i\theta}} + \frac{i V_\theta(r_0, \theta_0)}{i e^{i\theta}} \right) e$$

$$= \frac{-i}{r_0 e^{i\theta}} \left(U_\theta(r_0, \theta_0) + i V_\theta(r_0, \theta_0) \right)$$

$$= \left(\frac{V_\theta(r_0, \theta_0)}{r_0} - i \frac{U_\theta(r_0, \theta_0)}{r_0} \right) e^{-i\theta_0}$$

$\Longrightarrow$

$z_0 = r_0 e^{i\theta_0}$

1 $\quad U_r = V_\theta / r, \quad V_r = - U_\theta / r \quad \square$

$z = r e^{i\theta}$

Q)

2. Prove Cauchy's theorem assuming Cauchy integral formula.

CIF: $\quad f(z_0) = \frac{1}{2\pi i} \int_\gamma \frac{f(z)\, dz}{z - z_0}$

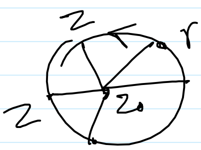
Cauchy's Thm: $\quad \int_\gamma f(z) = 0 \quad$ (NOTE: Condns on Domain or interior (γ))

f is holo. on an open set containing a simple contour γ and its interior (oriented positively)

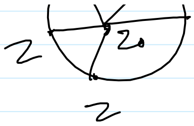
$g(z) = f(z)(z - z_0 - 1)$ $\quad$ g is holomorphic on the same open set

$$g(z_0) = \frac{1}{2\pi i} \int_\gamma \frac{g(z)\, dz}{z - z_0} = \frac{1}{2\pi i} \int_\gamma \frac{f(z)(z - z_0 - 1)\, dz}{(z - z_0)}$$

$$f(z_0) + g(z_0) = f(z_0) + f(z_0)(-1) = 0$$



$$= \frac{1}{2\pi i} \int_\gamma \frac{f(z) + f(z)(z - z_0 - 1)\, dz}{z - z_0}$$



$$2\pi i \int_\gamma \frac{}{z - z_0}$$

Since z_0 is in the interior of γ the den is $\neq 0$ non zero and we have a holo. f^n f on an open set containing γ & Int(γ)

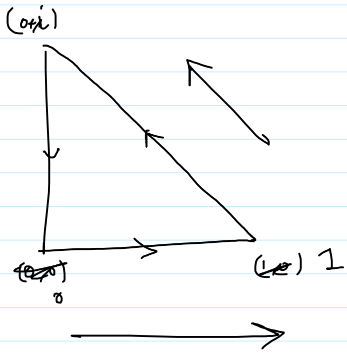
$$\frac{1}{2\pi i}\int_\gamma f(z) = 0$$

$$\Rightarrow \int_\gamma f(z) = 0 \qquad \square$$

Q:)

3. Let γ be the boundary of the triangle $\{0 < y < 1 - x;\ 0 \le x \le 1\}$ taken with the anticlockwise orientation. Evaluate:
a) $\int_\gamma Re(z)dz$ b) $\int_\gamma z^2 dz$

a)



$u_x = 1 \quad v_y = 0$

$$\int_\gamma Re(z)$$

$\gamma_1(t) = t + 0i \qquad t \in [0,1]$

$\gamma_2(t) = 1 - t + ti \qquad t \in [0,1]$

$\gamma_3(t) = 0 + (1-t)i \qquad t \in [0,1]$

$$\int_{\gamma_1} Re(z)\,dz = \int_0^1 Re(\gamma_1(t))\,\gamma_1'(t)\,dt$$

$$= 1/2$$

$$\int_{\gamma_2} Re(z)\,dz = \int_0^1 (1-t)(-1+i)\,dt$$

$$= (-1+i)\int_0^1 1 - t\,dt$$

$$= (-1+i)(1/2)$$

$\int_{\gamma_3} Re(z)\, dz = \int \ldots = 0$

Ans) $I_1 + I_2 + I_3 = \boxed{i/2}$

ii) 1 closed simple loop

holomorphic fn. $f(z) = z^2$

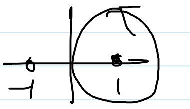
And domain is simply connected

$\Rightarrow$ Cauchy's Thm $\int_\gamma f(z)\,dz = 0$

Q4.)

4. Compute $\int_{|z-1|=1} \frac{2z-1}{z^2-1} dz$

$$\int_{|z-1|=1} \frac{2z-1}{(z+1)(z-1)}$$



$z_0 = 1$

Note: (If nothing is specified Assume γ ↺ oriented +vely)

Consider $f: \mathbb{C}\setminus\{-1\} \to \mathbb{C}$ $\quad f(z) = \frac{2z-1}{z+1}$ $\quad f(1) = 1/2$

$$f(1) = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z-1} dz$$

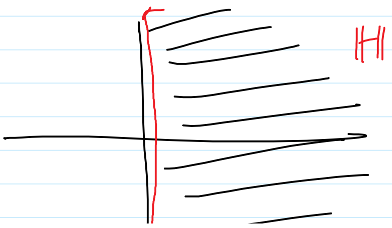
$$2\pi i \, \tfrac{1}{2} = \int_\gamma \frac{f(z)}{z-1} dz$$

What we want

Ans: $\boxed{\pi i}$

Q6.)

6. Let $\mathbb{H} = \{z \in \mathbb{C} | Re(z) > 0\}$ be the (strict) open right half plane. Construct a function f which is holomorphic on $\mathbb{H}$ and such that $f(\frac{1}{n}) = 0$ for $n \in \mathbb{N}$.



$f(n) = 0$

↳ $\sin(\pi z)$

$f: \mathbb{H} \to \mathbb{C}$ $\quad f(1/n) = 0$

$f(z) = \sin\left(\frac{\pi}{z}\right)$

$f(z) = \sin\left(\frac{\pi}{2}\right)$

$f(1/n) = \sin n\pi = 0$ ✓

f is non constant $f(2) = \sin(\pi/2) = 1$

f is holomorphic (why?) composition of $\sin(z)$ & (π/z)

Q 7.)

7. Let f be a holomorphic function on $\mathbb{C}$ such that $f(\frac{1}{n}) = 0$ for $n \in \mathbb{N}$. Show that f is a constant.

$f : \mathbb{C} \to \mathbb{C}$ st $f(1/n) = 0 \quad \forall\, n \in \mathbb{N}$

$\lim_{n\to\infty} f(1/n) = 0$ f is holomorphic & continuous on $\mathbb{C}$

$\Longrightarrow f(0) = 0$

Define $Z(f) := \{$ numbers such that $f(z) = 0\}$

$Z = \{1/n : n \in \mathbb{N}\} \cup \{0\}$

The zeros of this set are not isolated

(B_δ argument '0' for B_δ around 0 there is another zero of fn)

$\Longrightarrow$ Since the zeros are not discrete function is identically zero

Q 8)

8. Expand $\frac{1+z}{1+2z^2}$ into a power series around 0. Find the radius of convergence.

Consider $\dfrac{1}{1+2z^2} = 1 - (2z^2) + (2z^2)^2 - (2z^2)^3 \ldots$

$\ldots$? ($|2z^2| < 1 \iff |z| < 1/\sqrt{2}$)

consider

$1+2z^2$

when? $\left(|2z^2| < 1 \iff |z| < 1/\sqrt{2} \right)$

finite non zero poly. Power series

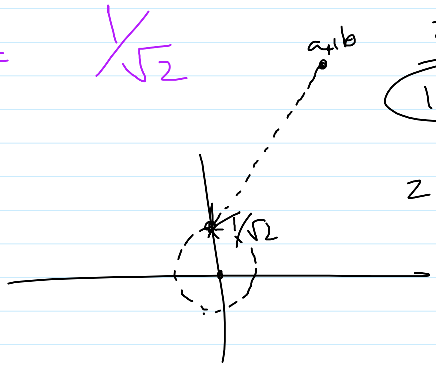
$$f(z) = \frac{1+z}{1+2z^2} = \underline{(1+z)}\left(1 - (2z^2) + (2z^2)^2 \dots\right)$$

$$= 1 + z - 2z^2 - 2z^3 + 4z^4 \dots$$

Radius of conv. of product

= Rad. of conv. of Power series

$\Rightarrow$ R of conv $f(z) = 1/\sqrt{2}$



$\frac{z+1}{1+2z^2}$

$z = 1/\sqrt{2}$

Q5.)

5. Show that if γ is a simple closed curve traced counter clockwise, the integral $\int_\gamma \bar{z}dz$ equals $2\imath Area(\gamma)$. Evaluate $\int_\gamma \bar{z}^m dz$ over a circle γ centered at the origin.

$$\gamma : [a,b] \to \mathbb{C}$$

$$\gamma(t) = x(t) + iy(t)$$

$$\gamma'(t) = x'(t) + iy'(t)$$

for Real value fns $x(t), y(t)$

$$\int_\gamma \bar{z}\, dz = \int_a^b \overline{\gamma(t)}\, \gamma'(t)\, dt$$

$$= \int_a^b (x(t) - iy(t))\,(x'(t) + iy'(t))\, dt$$

$$= \int_a^b xx' + yy' + i\int_a^b xy' - yx'\, dt$$

$$= \int_\gamma x\,dx + y\,dy + i\int_\gamma \underline{x\,dy - y\,dx}$$

Green's Thm: $\int_\gamma M dx + N dy = \iint_{Int(\gamma)} \partial N/\partial x - \partial M/\partial y \, d(x,y)$

$= \iint (0-0)\, d(x,y) + i \iint [1-(-1)]\, d(x,y)$

$= 0 + i(2) \iint_{Int(\gamma)} 1 \, d(x,y)$

$= 2i \, Area(Int(\gamma))$

ii) Around a circle $f(z) = \bar{z}^m$ (radius of $\bigcirc$ around origin is r)

$\int_\gamma \bar{z}^m dz$ let $\gamma(t) = r(\cos t + i\sin t) = re^{it}$

$\gamma'(t) = r(-\sin t + i\cos t)$

$= i\gamma(t)$

So, $\int_\gamma \bar{z}^m dz = \int_0^{2\pi} (\overline{\gamma(t)})^m \, i\gamma(t)\, dt$

$= i \int_0^{2\pi} (\overline{\gamma(t)})^{m-1} |\gamma(t)|^2 dt$

$= ir^2 \int_0^{2\pi} (\overline{\gamma(t)})^{m-1} dt$

$= ir^2 \int_0^{2\pi} r^{m-1} [\underbrace{\cos(m-1)t}_{K} - i\underbrace{\sin(m-1)t}_{K}]\, dt$

$m = 1$ or $m \neq 1$

If $m \neq 1 \implies$ Integral $= 0$

$m = 1$ Integral $= ir^2 \int_0^{2\pi} 1 \, dt$

$=$

$$\dot{i}\, 2\pi r^2$$

$$= 2\dot{i}\,(\text{Area}(r))$$