Sheet 5 Q2

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2. Describe the level curves and the contour lines for the following functions corresponding to the values $c = -3, -2, -1, 0, 1, 2, 3, 4$:
(i) $f(x, y) = x - y$ (ii) $f(x, y) = x^2 + y^2$ (iii) $f(x, y) = xy$

ii) $c = -3, -2, -1, 0, 1, 2, 3, 4$

$f(x,y) = x^2+y^2$ Range of f (R) $=$ $[0,\infty)$

$\therefore$ Level curves for $c = -3, -2, -1$ do not exist

$\therefore$ Level " " $c = 0$ is a point $\{(0,0)\}$

Level curve $\| \; \{(x,y) \in \mathbb{R}^2 : x^2+y^2 = c\}$

$\therefore$ for $c = 1, 2, 3, 4$ L.C are circles centred at origin.

ii) Contour lines:

$\{(x,y,c) \in \mathbb{R}^3 : x^2+y^2 = c\}$

again for $c < 0$ D.N.E

$c = 0$ point at origin $\{(0,0,0)\}$

$c = 1, 2, 3, 4$ — circle in the $z = c$ plane.

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iii) $f(x,y) = xy$

Level curve $\{(x,y) \in \mathbb{R}^2 : xy = c\}$

$\therefore$ They will be rectangular hyperbolae 1, 3rd quad. for $c > 0$

2, 4th quad for $c < 0$ and $x = 0$ and $y = 0$ for $c = 0$

→ Contour lines

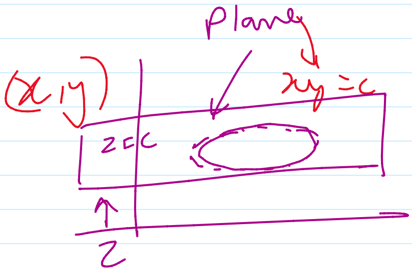
$\{(x,y,c) \in \mathbb{R}^3 : xy = c\}$

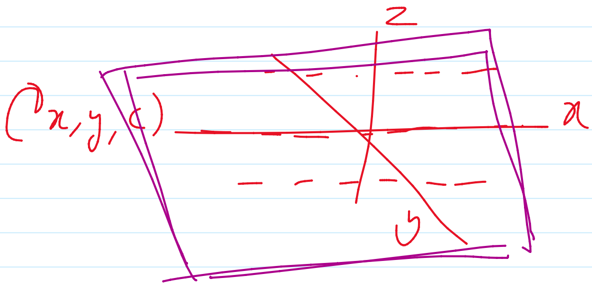
" $z = c$ " "

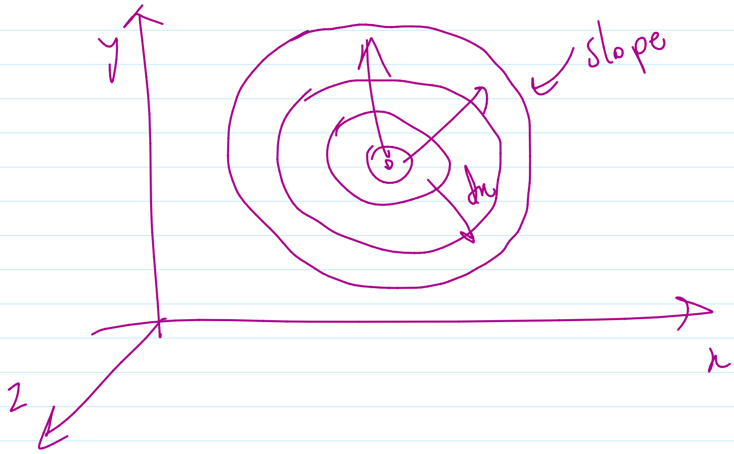
$\therefore$

→ But mention $z = c$ plane

Plane


Plane
(x,y)
$xy=c$
$z=c$
z


z
(x,y,c)
x
y


y
slope
x
z

Sheet 5 Q4

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4. Suppose $f,\ g : \mathbb{R} \to \mathbb{R}$ are continuous functions. Show that each of the following functions of $(x, y) \in \mathbb{R}^2$ are continuous:
(i) $f(x) \pm g(y)$ (ii) $f(x)g(y)$ (iii) $\max\{f(x),\ g(y)\}$ (iv) $\min\{f(x),\ g(y)\}$.

less variables $\to$ more variables

$f(x) = x$

$f(x, y, z, w \dots) = f(x)$

i) $f^*(x,y) = f(x)$ $\quad f^* : \mathbb{R}^2 \mapsto \mathbb{R}$

We know $f(x)$ is continuous

$g^* : \mathbb{R}^2 \mapsto \mathbb{R}$ st $g^*(x,y) = g(y)$

f^* g^*

$\therefore f(x)$ is continuous

for given $\epsilon > 0$, $\exists\ \delta > 0$ st

$|x - x_0| < \delta \implies |f(x) - f(x_0)| < \epsilon$

$|f^*(x,y) - f^*(x_0,y_0)| < \epsilon$

$\therefore f^*(x,y)$ is continuous & sim. $g^*(x,y)$

i) $f^*(x,y) \pm g^*(x,y)$ — continuous ✓
(Thm in class)

$t = \ \ f(x) \pm g(y)$

$f : \mathbb{R}^2 \mapsto \mathbb{R}$ $\quad g : \mathbb{R}^2 \mapsto \mathbb{R}$

~~both cont~~

$f \pm g$

(x,y)

ii) Product of cont

$f^* g^*$ cont

fg cont

assume $f(x) > g(y)$

iii) $\max\{f(x), g(y)\} = \left(\dfrac{f(x) + g(y)}{2}\right) + \dfrac{|f(x) - g(y)|}{2} = f(x)$

from (i) $f(x) + g(y)$ is continuous

Consider $h(x) = |x|$

$h(t)$ cont cont $\therefore$ continuous by consequence

$$\boxed{g(f(x,y)) \qquad g : \mathbb{R} \longrightarrow \mathbb{R}}$$

cont

cont

f cont

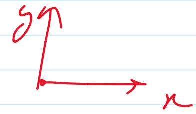
Check diff vs exam. partia

6. Examine the following functions for the existence of partial derivatives at $(0,0)$. The expressions below give the value at $(x,y) \neq (0,0)$. At $(0,0)$, the value should be taken as zero.

(i) $xy\dfrac{x^2 - y^2}{x^2 + y^2}$

(ii) $\dfrac{\sin^2(x+y)}{|x|+|y|}$

✓ X



6.) ii) $\dfrac{\sin^2(x+y)}{|x|+|y|}$ $\qquad f(0,0)=0$ $\qquad h\to 0^+$

f_x

$$\lim_{h\to 0} \frac{\left(\frac{\sin^2(h)}{|h|+0}\right)-(0)}{h} = \frac{\sin^2 h}{h|h|} \quad \begin{cases} 1 & h\to 0^+ \\ -1 & h\to 0^- \end{cases}$$

$\therefore$ RHL $\neq$ LHL $\qquad f_x(0,0)$ DNE.

f_y

$$\lim_{k\to 0} \frac{\sin^2(k)}{|k|k}$$

$\parallel$ $\qquad f_y(0,0)$ DNE

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8. Let $f(0,0) = 0$ and

$$f(x,y) = \begin{cases} x\sin(1/x) + y\sin(1/y), & \text{if } x \neq 0,\ y \neq 0 \\ x \sin 1/x, & \text{if } x \neq 0,\ y = 0 \\ y \sin 1/y, & \text{if } y \neq 0,\ x = 0. \end{cases}$$

Show that none of the partial derivatives of f exist at $(0,0)$ although f is continuous at $(0,0)$.

Note $f(0,0) = 0$

$$|x\sin 1/x + y \sin 1/y| \leq |x \sin 1/x| + |y \sin 1/y| \leq |x| + |y|$$

$\boxed{\delta = \epsilon/2}$ works

$|x - x_0| < \delta$, $|y - y_0| < \delta \implies |f(x,y) - f(x_0,y_0)| < \epsilon$

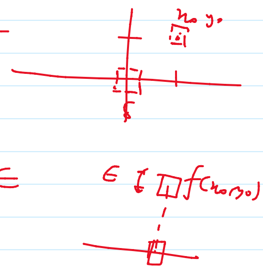
$x_0 = 0$, $y_0 = 0$, $\epsilon/2$, δ, δ, 0

$|x| + |y|$, δ δ $\boxed{2\delta = \epsilon}$

$\therefore$ Continuous at $(0,0)$

$$f_x(0,0) \lim_{h \to 0} \frac{h \sin 1/h - 0}{h} = \lim_{h\to 0} \sin \frac{1}{h} \quad \text{DNE}$$

Similarly $f_y(0,0) \lim_{k \to 0} \sin \frac{1}{k}$ DNE



By defn cont in 2 variables

$|x - x_0| < \delta$, $|y - y_0| < \delta \implies |f(x,y) - f(x_0,y_0)| < \epsilon$

Here $(x_0, y_0) = (0,0)$

$|x| < \delta$, $|y| < \delta \implies |f(x,y)| < \epsilon$

Show why $\delta = \epsilon/2$ works

done. $|x| < \epsilon/2$ $|y| < \epsilon/2$

$$|f(x,y)| < |x| + |y| < \epsilon/2 + \epsilon/2 = \epsilon \quad \square$$

$$|f(x,y)| < \epsilon$$

Sheet 5 Q10

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$\frac{y}{|y|} = \begin{cases} 1 \Leftarrow 1 \\ -1 \end{cases}$ Not def

$2\epsilon \int \cdot f(x_0, y_0)$

10. Let ~~$f(x, y) = 0$ if $y = 0$ and~~

$$f(x, y) = \frac{y}{|y|}\sqrt{x^2 + y^2} \text{ if } y \neq 0.$$

Show that f is continuous at $(0, 0)$, $D_{\underline{u}}f(0, 0)$ exists for every vector $\underline{u}$, yet f is not differentiable at $(0, 0)$.

$\epsilon - \delta$

x_n

↳ consider

a) $|f(x,y)| = |\sqrt{x^2+y^2}| = \sqrt{x^2+y^2}$ $\quad D_\delta(x_0, y_0)$ $(0,0)$

$\boxed{\delta = \epsilon \quad \text{works}}$

$\sqrt{x^2+y^2} < \delta$ $\quad |f(x,y) - f(0,0)| = \sqrt{x^2+y^2} < \epsilon$

We showed continuity.

b) $D_u f(0,0)$ exists

$u = (u_1, u_2)$ $\quad u_1^2 + u_2^2 = 1$

$$D_u f(0,0) = \lim_{t \to 0} \frac{f(u_1 t, u_2 t) - f(0,0)}{t}$$

$$= \lim_{t \to 0} \frac{\frac{u_2 t}{|u_2 t|}\sqrt{u_1^2 t^2 + u_2^2 t^2}}{t}$$

$$= \lim_{t \to 0} \frac{\frac{u_2 \cancel{t}}{|u_2||\cancel{t}|} \cancel{|t|} \cancel{\sqrt{u_1^2 + u_2^2}}}{\cancel{t}} \to 1$$

$$= \frac{u_2}{|u_2|} \quad \text{if } u_2 \neq 0$$

derivative

If $u_2 = 0$, $f(x,y) = 0 \quad \therefore f_{u_2 = 0} = 0$

$$D_u(f) = \begin{cases} \frac{u_2}{|u_2|} & \text{if } u_2 \neq 0 \\ 0 & u_2 = 0 \end{cases}$$

$u_2 = 1 \Rightarrow f_y$

$u_2 = 0$

f_x

$f_x(x,y) = 0$

$f_y(x,y) = 1$

c) To show f is not diff at 0

$$\lim_{(h,k)\to(0,0)} \frac{|f(0+h,0+k) - f(0,0) - 0(h) - 1(k)|}{\|(h,k)\|}$$

$$\lim_{(h,k)\to(0,0)} \frac{\left|\frac{k}{|k|}\sqrt{h^2+k^2} - k\right|}{\sqrt{h^2+k^2}}$$

Hint: (Not diff so COUNTER EXAMPLE!)

$(x_n, y_n) \quad \left(\frac{1}{n}, \frac{m}{n}\right)$ $\quad$ m is finite

$\lim_{n\to\infty} (x_n, y_n) \longrightarrow (0,0)$

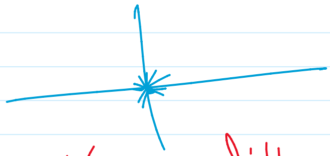
$$\frac{\cancel{\frac{m}{|m|}}\sqrt{\frac{1+m^2}{\cancel{n}}} - \frac{m}{\cancel{n}}}{\sqrt{\frac{1+m^2}{\cancel{n}}}}$$

$$\frac{\sqrt{1+m^2} - m}{\sqrt{m^2+1}} \quad \text{provided } m>0$$

$m = 1$, $\quad \frac{\sqrt{2}-1}{\sqrt{2}}$

$$\frac{\sqrt{2}-1}{\sqrt{2}} \neq \frac{\sqrt{5}-2}{\sqrt{5}}$$

$m = 2$ $\quad \frac{\sqrt{5}-2}{\sqrt{5}}$ $\quad \therefore$ Not diff. ✓



Consequently, f isn't diff at $(0,0)$