Q2

30 December 2020 15:02

2. Find the directions in which the directional derivative of $f(x,y) = x^2 + \sin xy$ at the point $(1,0)$ has the value 1.

[Directly from Slides]

lecture-17/18

$$f_x(x,y) = 2x + y\cos xy$$

$$f_y(x,y) = x\cos xy$$

$$\nabla f = (2x + y\cos xy,\ x\cos xy)$$

$$D_u f(1,0) = \nabla f(1,0) \cdot (u_1, u_2) = 1$$

$$(2(1)+0,\ 1\cos(0)) \cdot (u_1, u_2) = 1$$

$$2u_1 + u_2 = 1$$

$$u_1^2 + u_2^2 = 1 \qquad 5u_1^2 - 4u_1 = 0$$

$$\therefore u_1 = 0,\ \frac{4}{5}$$

$$\therefore (u_1, u_2) = \{ (0,1),\ (4/5,\ -3/5) \}$$

Q4

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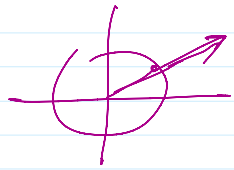
4. Find $D_{\underline{u}}F(2,2,1)$, where $F(x,y,z) = 3x - 5y + 2z$, and $\underline{u}$ is the unit vector in the direction of the outward normal to the sphere $\underline{x^2 + y^2 + z^2 = 9}$ at $(2,2,1)$.



$f: \mathbb{R}^3 \longrightarrow \mathbb{R}$ $\quad f(x,y,z) = x^2+y^2+z^2$

outward normal to sphere at $(2,2,1)$

$\nabla f(2,2,1)$

[Also in slides (maybe)]

$$\therefore \quad \vec{u} = \frac{\nabla f(2,2,1)}{\|\nabla f(2,2,1)\|}$$

$$\nabla f(x,y,z) = (2x, 2y, 2z)$$

$$\nabla f(2,2,1) = (2(2), 2(2), 2(1))$$

$$= (4,4,2)$$

$$u = \frac{4,4,2}{\sqrt{16 \times 2 + 4}} = \frac{(4,4,2)}{6} = \left(\frac{2}{3}, \frac{2}{3}, \frac{1}{3}\right)$$

$$D_u F(2,2,1) = \nabla F(2,2,1) \cdot \left(\frac{2}{3}, \frac{2}{3}, \frac{1}{3}\right)$$

$$= (3,-5,2) \cdot \left(\frac{2}{3}, \frac{2}{3}, \frac{1}{3}\right)$$

$$= -2/3 \quad //$$

Q5

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5. Given $\boxed{\sin(x+y)+\sin(y+z)=1,}$ find $\frac{\partial^2 z}{\partial x \partial y}$, provided $\cos(y+z) \neq 0$.

$\left(\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial y}\right)$

$\sin(x+y)+\sin(y+z)=1 \quad \left(\frac{\partial^2 z}{\partial x \partial y}\right)$ $\frac{\partial}{\partial x} \leftarrow$ 1st $\quad \frac{\partial}{\partial y} \leftarrow$ 2nd

We will assume that z is a sufficiently smooth fn of x, y $\quad \frac{\partial}{\partial x} \frac{\partial}{\partial y}$

Diff. wrt x first

$$\cos(x+y) + \cos(y+z)\left(\frac{\partial z}{\partial x}\right) = 0 \longrightarrow (1)$$

$$\boxed{\left(\frac{\partial z}{\partial x}\right)\left(\frac{\partial z}{\partial y}\right)} \text{ – final answer}$$

Diff. wrt y first

$$\frac{\partial (z)}{\partial y} \quad \cos(x+y) + \cos(y+z)\left(1+\frac{\partial z}{\partial y}\right) = 0 \quad — (2)$$

Diff (2) w.r.t x,

$$\frac{\partial^2 z}{\partial x \partial y} \rightarrow \quad -\sin(x+y) + \cos(y+z)\left(\frac{\partial^2 z}{\partial x \partial y}\right) + \left(\left[1+\frac{\partial z}{\partial y}\right]\right)\left(-\sin(y+z)\right)\left(\frac{\partial z}{\partial x}\right) = 0$$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\left(\sin(y+z)\left(\frac{\partial z}{\partial x}\right)\left[1+\frac{\partial z}{\partial y}\right] + \sin(x+y)\right)}{\cos(y+z)}$$

given $\cos(y+z) \neq 0$

$1+\frac{\partial z}{\partial y}$ and $\frac{\partial z}{\partial x}$

$$\frac{\partial^2 z}{\partial x \partial y} = \sin(y+z)\left(\frac{-\cos(x+y)}{\cos(y+z)}\right)\left(\frac{-\cos(x+y)}{\cos(y+z)}\right) + \sin(x+y)$$

$$\frac{\partial^2}{\partial x \partial y} = \frac{\ldots(y+z)\left(\frac{\ldots}{\cos(y+z)}\right)\left(\frac{\ldots}{\cos(y+z)}\right)\ldots}{\cos(y+z)}$$

hope:

$$\boxed{= \tan(y+z)\left(\frac{\cos^2(x+y)}{\cos^2(y+z)}\right) + \frac{\sin(x+y)}{\cos(y+z)}}$$

Q8

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8. Analyze the following functions for local maxima, local minima and saddle points:
(i) $f(x,y) = (x^2 - y^2)e^{-(x^2+y^2)/2}$ (ii) $f(x,y) = x^3 - 3xy^2$

i) $f(x,y) = (x^2-y^2)\, e^{-(x^2+y^2/2)}$

Observe that f is a C^2 function

$f^* = (x^2-y^2)$ $g^* = e^{-(x^2+y^2/2)}$ both are differentiable and continuous

$f_x(x,y) = e^{\frac{-(x^2+y^2)}{2}} (2x - x^3 + xy^2)$

$f_y(x,y) = e^{\frac{-(x^2+y^2)}{2}} (-2y + y^3 - x^2y)$

(Check) Critical points $(0,0)$, $(\pm\sqrt{2},0)$, $(0,\pm\sqrt{2})$

$f_{xx}(x,y) = e^{\frac{-(x^2+y^2)}{2}} (2 - 5x^2 + x^2(x^2-y^2) + y^2)$

$f_{yy}(x,y) = e^{\frac{-(x^2+y^2)}{2}} (5y^2 - 2 + y^2(x^2-y^2) - x^2)$

$f_{xy}(x,y) = e^{\frac{-(x^2+y^2)}{2}} (xy(x^2-y^2))$

also, $f_{xy} = f_{yx}$ (why?)

(Because the function is double differentiable)

	f_{xx}	f_{xy}	f_{yy}	D
$(0,0)$	2	0	-2	-4
$(\sqrt{2},0)$	$-4/e$	0	$-4/e$	$16/e^2$
$(-\sqrt{2},0)$	$-4/e$	0	$-4/e$	$16/e^2$
$(0,\sqrt{2})$	$4/e$	0	$4/e$	$16/e^2$
$(0,-\sqrt{2})$	$4/e$	0	$4/e$	$16/e^2$

Very Important

(Refer lecture

$(0,\sqrt{2})$	$4/e$	0	$4/e$	$16/e^2$
$(0,-\sqrt{2})$	$4/e$	0	$4/e$	$16/e^2$

↑ (Refer lecture 19.)

(L-19)

$$\boxed{D = f_{xx}f_{yy} - (f_{xy})^2}$$

$f'' < 0 \Rightarrow$ maxima

$\therefore$ $(0,0)$ - saddle point

$(\pm\sqrt{2},0)$ - local maxima

$(0,\pm\sqrt{2})$ - local minima

→ If f is double differentiable we are $f_{xx}f_{yy} - (f_{xy})^2$

→ $D_\delta(x_0,y_0) \geq - \cdot$ $\frac{\delta}{x_0,y_0}$ or $\leq$ In all directions.

Confirming is a lot of work

General rule:

If 2nd der. exists, do $D = f_{xx}f_{yy} - f_{xy}^2$

If DNE D_δ

$D_\delta(x_0,y_0) \leftarrow$ Not very nice

Q9

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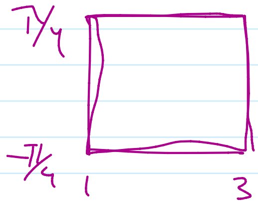


9. Find the absolute maximum and the absolute minimum of

$$f(x,y) = (x^2 - 4x)\cos y \text{ for } 1 \le x \le 3,\ -\pi/4 \le y \le \pi/4.$$

Observe that $f(x,y)$ is continuous (why?)

And, the function is bounded on the closed $D \subset \mathbb{R}^2$

So it attains its bounds.

Also,

f is differentiable in its domain

$f_x(x,y) = (2x-4)\cos y$ $\qquad$ $f_y(x,y) = -(x^2-4x)\sin y$

Interior points:- $1 < x < 3$ and $-\pi/4 < y < \pi/4$

The only critical point is $(2,0)$

$f(2,0) = -4$

Boundary points:-

1.) $x = 1$

$f(1,y) = -3\cos y$ $\qquad (-\pi/4 < y < \pi/4)$

$h(y) = -3\cos y$

$h'(y) = 3\sin y = 0$ at $y = 0$

$\therefore h(0) = -3$

2.) $x = 3$ $\quad f(3,y) = -3\cos y$ (again)

like (1.) optima is $f(3,0) = -3$

3) $f(x, \pm \pi/4) = \dfrac{x^2 - 4x}{\sqrt{2}}$

Consider $g : [1,3] \longrightarrow \mathbb{R}$

$$g(x) = \frac{x^2 - 4x}{\sqrt{2}}$$

$$g'(x) = \frac{2x - 4}{\sqrt{2}} \qquad g'(x) = 0 \text{ at } x = 2$$

$$\therefore\ g(2) = \frac{4 - 8}{\sqrt{2}} = -2\sqrt{2}$$

Finally

Corner points

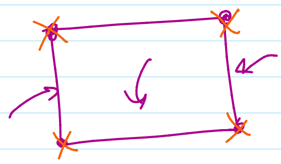
$$f(1, \pi/4) = -3/\sqrt{2}$$

$$f(1, -\pi/4) = -3/\sqrt{2}$$

$$f(3, \pi/4) = -3/\sqrt{2}$$

$$f(3, -\pi/4) = -3/\sqrt{2}$$

Results:- (Interior points & Critical, Boundary points, Corner points)



$$f(2,0) = -4$$

$$f(1,0) = f(3,0) = -3$$

$$f(2, \pm \pi/4) = -2\sqrt{2}$$

$$f(1, \pm \pi/4) = f(3, \pm \pi/4) = -3/\sqrt{2}$$