NO RECAP TODAY

— I'll attach screenshots of thms wherever necessary

Tutorial 5:

Q1)

1. Locate and classify the type of singularities of :

a) $\frac{sin(1/z)}{(1+z^4)}$
b) $\frac{z^5 sin(1/z)}{(1+z^4)}$
c) $\frac{1}{sin(1/z)}$
d) $e^{\frac{1}{z}}$

Definition 15 (Singularities)

Let $f : \Omega \to \mathbb{C}$ be a function. A point $z_0 \in \mathbb{C}$ is said to be a singularity of f if

1. $z_0 \notin \Omega$, i.e., f is not defined at z_0, or
2. $z_0 \in \Omega$ and f is not holomorphic at z_0.

Definition 16 (Isolated singularity)

A singularity $z_0 \in \mathbb{C}$ is said to be *isolated* if there exists *some* $\delta > 0$ such that f is holomorphic on $B_\delta(z_0) \setminus \{z_0\}$.

$\frac{1}{z}$ $z_0 = 0$

Definition 18 (Removable singularity)

If an isolated singularity can be removed by defining the function by assigning a certain value at that point, we say that the singularity is removable.

$\exists c \in \mathbb{C}$ define st $f(z_0) = c$

$f(z) = \frac{\sin z}{z}$ $Re^{i\theta}$

$\sin z = \frac{e^{iz} - e^{-iz}}{2i}$

$= z - \frac{z^3}{3!} \ldots$

Definition 19 (Pole)

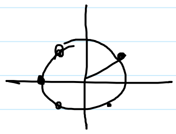
An isolated singularity z_0 is said to be a pole if $|f(z)| \to \infty$ as $z \to z_0$.

Definition 20 (Essential singularity)

An isolated singularity is called an essential singularity if it is neither a removable singularity nor a pole.

a.) $f(z) = \dfrac{\sin(1/z)}{1+z^4}$ $\qquad$ $z^4+1=0$ roots

and $z=0$



By first glance we see that set $S = \left\{ 0, \pm \dfrac{1 \pm i}{\sqrt{2}} \right\}$

Claim: Singularities of the form $z_0 \in \left(\dfrac{\pm 1 \pm i}{\sqrt{2}} \right)$ are poles

$$\lim_{z \to z_0} \frac{1}{f(z)} = \lim_{z \to z_0} \frac{1+z^4}{\sin(1/z)} = \lim_{z \to z_0} \frac{0}{\sin(1/z)} = 0 \qquad \Box$$

Non zero

$$\sin(1/z) = \frac{e^{i/z} - e^{-i/z}}{2i}$$

$$\lim_{y \to 0} f(z) = \lim_{y \to 0} \frac{\sin(1/iy)}{1+(iy)^4}$$

$$= \frac{1}{1} \lim_{y \to 0} \left(\frac{\overset{\infty}{e^{1/y}} - \overset{0}{e^{-1/y}}}{2i} \right)$$

$$= \text{not defined}$$

$$\lim_{x \to 0} f(z) = \lim_{x \to 0} \frac{\sin(1/x)}{1}$$ finite (limit does not exist)

$$|\sin(1/x)| \leq 1 \text{ for } x \in \mathbb{R}$$

$\therefore$ $z_0 = 0$ is neither removable nor a pole

$\Longrightarrow$ $z_0 = 0$ is an essential singularity b.) $f(z) = \dfrac{z^5 \sin(1/z)}{1+z^4}$

Set of sing. $S = \{ 0, \dfrac{\pm 1 \pm i}{\sqrt{2}} \}$

all the nonzero sing. are . poles

Regarding $z_0 = 0$ we approach $\mathbb{R}$ clearly get limit as 0.

"But" when we approach from imaginary axis

our limit $\longrightarrow \infty$ (Why? exp grows faster than polynomial decays)

$\lim_{y \to 0} y^5 e^{1/y}$ $h := 1/y$

$\left(\lim_{h \to \infty} \dfrac{e^h}{h^5} \overset{\text{Lopital}}{=} \dfrac{e^h}{5h^4} \dots \dfrac{e^h}{5!} \infty \right)$

c.) $f(z) = \dfrac{1}{\sin(1/z)}$ we need $\sin(1/z) \neq 0$

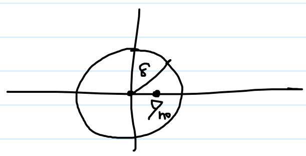
$z = 0$ is a singularity

Set of singularities $S = \{ 0 \cup \frac{1}{n\pi} \quad n \in \mathbb{Z} / \{0\} \}$

i) $z_0 = 0$ (Claim NOT ISOLATED)

Why $B_\delta(0)$

$\left| \frac{1}{n_0} \right| < \delta$



$\left|\frac{1}{n_0}\right| < \delta$

and we know $\implies |n_0| > \frac{1}{\delta} \quad n_0 \in \mathbb{Z}$

f not diff. at $z_0 = 1/n_0 \implies z_0 = 0$

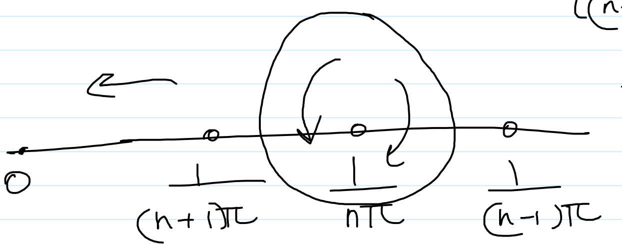
IS NOT ISOLATED

for $z_0 = \frac{1}{n\pi}$

$\delta : \min\left\{\frac{1}{(n-1)\pi} - \frac{1}{n\pi}, \frac{1}{n\pi} - \frac{1}{(n+1)\pi}\right\}$

Claim: (Poles)

$\longrightarrow$ Isolated



Pole proof: $\lim_{z \to \frac{1}{n\pi}} \frac{1}{f(z)} = \lim \sin(1/z) = 0$

d.) $f(z) = e^{1/z} \qquad z_0 = \{0\}$

Claim: Essential Singularity

$z_0 = x + iy$

$\lim_{y=0 \;\; x\to 0} f(x+iy) = \lim_{x \to 0} e^{1/x} = \infty$

(for real y)

$\lim_{y\to 0 \;\; x=0} f(x+iy) = \lim_{y \to 0} e^{-i/y}$ and $|e^{-i/y}| = 1$

$\implies$ Neither pole nor removable

$\Rightarrow$ Essential Sing

2. Construct a meromorphic function on $\mathbb{C}$ with infinitely many poles.

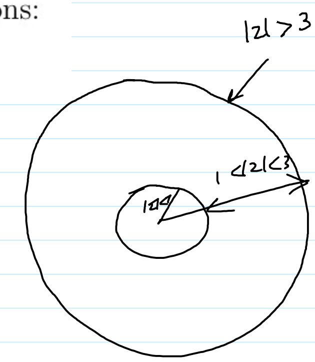
<u>Meromorphic function</u>: A function which is holomorphic everywhere except the set of isolated sing. of the fn s.t. all isolated sing are POLES

$g(z) = \sin(\pi z)$ is zero at all $z \in \mathbb{Z}$

$$f(z) := \frac{1}{g(z)} = \frac{1}{\sin(\pi z)}$$

$f : \mathbb{C} \setminus \mathbb{Z} \to \mathbb{C}$ is a meromorphic function with infinite poles

Q3) 3. Find Laurent expansions for the function $f(z) = \frac{2(z-1)}{z^2-2z-3}$ valid on the regions: (i) $0 \leq |z| < 1, (ii) 1 < |z| < 3, (iii) |z| > 3$.



$$f(z) = \frac{2(z-1)}{z^2-2z-3} = \frac{z+1 + z-3}{(z+1)(z-3)}$$

$$= \frac{1}{z+1} + \frac{1}{z-3}$$

(i) [We want to define power series s.t. power series <u>converges</u> to $f(z)$ within the given Annuli] (Aim)

$|z|<1 \Rightarrow \frac{1}{z+1} + \frac{1}{z-3}$

$$= \frac{1}{(1+z)} + \left(-\frac{1}{3}\right)\left(\frac{1}{1-\frac{z}{3}}\right)$$

[Write the overall coeff of z^n]

$$= \left(\sum_{n=0}^{\infty} (-1)^n z^n\right) - \frac{1}{3}\left(\sum_{n=0}^{\infty} \left(\frac{z}{3}\right)^n\right)$$

$= \left(\sum_{n=0} \ldots \right) \quad \frac{1}{3} \left(\sum_{n=0} \left(\frac{}{3}\right) \right)$

(Why convergent ?) $\quad \because |z|<1 \qquad (|z|<1<3)$

ii) $1 < |z| < 3$

$$\Rightarrow \frac{1}{1+z} + \frac{1}{z-3}$$

$$= \frac{1}{z}\left(\frac{1}{1+\frac{1}{z}}\right) - \frac{1}{3}\left(\frac{1}{1-\frac{z}{3}}\right)$$

$$= \frac{1}{z}\sum_{n=0}^{\infty}(-1)^n\left(\frac{1}{z}\right)^n$$

Same as above

Convergent because

$|z| > 1$

$\Leftrightarrow 1 > 1/|z|$

iii) $|z| > 3$

$$\Rightarrow \frac{1}{1+z} + \frac{1}{z-3}$$

$$= \frac{1}{z}\left(\frac{1}{1+\frac{1}{z}}\right) + \frac{1}{z}\left(\frac{1}{1-\frac{3}{z}}\right)$$

$$= \underbrace{\frac{1}{z}\sum_{n=0}^{\infty}(-1)^n(1/z)^n}_{*_1} + \underbrace{\frac{1}{z}\sum_{n=0}^{\infty}(3/z)^n}_{*_2}$$

Both convergent

$|z| > 3$

$\Rightarrow \frac{1}{|z|} < \frac{1}{3} < 1 \quad (*_1)$

$\Rightarrow \frac{1}{|z|} \times 3 < 1 \quad (*_2)$

Q4.)

4. Let D be a domain in $\mathbb{C}$ and let $z_0 \in D$. Suppose that z_0 is an isolated singularity

Q4.)

4. Let D be a domain in $\mathbb{C}$ and let $z_0 \in D$. Suppose that z_0 is an isolated singularity of $f(z)$ and $f(z)$ is bounded in some punctured neighborhood of z_0 (that is, there exists $M > 0$ such that $|f(z)| \leq M$ for all $z \in U - z_0$ for neighborhood U). Show that $f(z)$ has a removable singularity at z_0.

Solution) From the definition of isolated sing we have a $\delta > 0$: $B_\delta(z_0)$ is holomorphic for $B_\delta(z_0) \setminus \{z_0\}$

define $g(z) := f(z)(z - z_0)$ in this disc

$g : B_\delta(z_0) \setminus \{z_0\} \to \mathbb{C}$ so g is holomorphic on $B_\delta(z_0) \setminus \{z_0\}$

Moreover, $\lim_{z \to z_0} g(z) = 0$

(Why?) Because f is given to be bounded within $B_\delta(z_0)$

Theorem 23 (Riemann's Removable Singularity Theorem)

z_0 is a removable singularity of f iff $\lim_{z \to z_0} f(z)$ exists.

So, By RRST the sing. z_0 of $g(z)$ is a removable singularity

$h : B_\delta(z_0) \longrightarrow \mathbb{C}$ $\quad h(z) \begin{cases} g(z) & \text{for } z \neq z_0 \\ 0 & \text{for } z = z_0 \end{cases}$

$$h(z) = \cancelto{0}{a_0} + a_1(z - z_0) + a_2(z - z_0)^2 \ldots$$

$$\therefore f(z)\cancel{(z - z_0)} = a_1\cancel{(z - z_0)} + a_2(z - z_0)^2 \ldots$$

$$f(z) = a_1 + a_2(z - z_0) + \ldots$$

$$\lim_{z \to z_0} f(z) = a_1 \quad \text{(well defined)}$$

By RRST, f has a REMOVABLE sing at $z = z_0$

Q5.)

5. A complex-valued function $f(z)$ on $\mathbb{C}$ is called doubly periodic if there exist linearly independent vectors $v, w \in \mathbb{C}$ over $\mathbb{R}$ such that $f(z+v) = f(z)$ and $f(z+w) = f(z)$ for all $z \in \mathbb{C}$. Show that any double periodic entire function is constant.

$v, w \in \mathbb{C}$ are linearly independent over $\mathbb{R}$

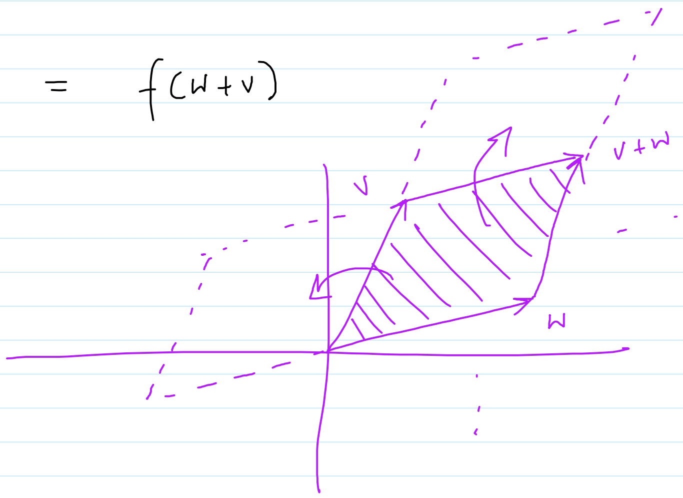
for $a_1, a_2 \in \mathbb{R}$ $\quad a_1 v + a_2 w = 0$

$$\implies a_1 = 0 \ \& \ a_2 = 0$$

$$f(z+w) = f(z) = f(z+w)$$

$$f(0) = f(w) = f(v) = f(w+v)$$

Just for some intuition:



Since, any $z \in \mathbb{C}$ can be written as

$z = av + bw \qquad a, b \in \mathbb{R}$

$$f(z) = f(av+bw) = f\big((\underbrace{\lfloor a \rfloor}_{\mathbb{Z}} + \{a\})v + (\underbrace{\lfloor b \rfloor}_{\mathbb{Z}} + \{b\})w\big)$$

define $\{a\} = a - \lfloor a \rfloor$ $\qquad = f(\{a\}v + \{b\}w)$

$\{b\} = b - \lfloor b \rfloor$

Proves that function takes values within shown parallelogram

$\Rightarrow$ Function is bounded let's say be an M

s.t $|f(z)| < M \quad \forall z \in \mathbb{C}$

By Liouville's Thm. a bounded entire fn is necessarily a constant.

[Result of Cauchy's Estimate]

Theorem 19 (Cauchy's estimate)

Suppose that f is holomorphic on $|z - z_0| < R$ and bounded by $M > 0$ on this disc. Then,

$$\left|f^{(n)}(z_0)\right| \leq \frac{n!M}{R^n}.$$

An easy application of this give us:

Theorem 20 (Liouville's Theorem)

Let $f : \mathbb{C} \to \mathbb{C}$ be holomorphic. If f is bounded, then f is constant!

Q6.)

6. Show by transforming into an integral over the unit circle, that $\int_0^{2\pi} \frac{d\theta}{a^2+1-2a\cos\theta} = \frac{-2\pi}{1-a^2}$, where $a > 1$. Also compute the value when $a < 1$.

$$\int_0^{2\pi} \frac{d\theta}{a^2+1-2a\cos\theta} = \int_0^{2\pi} \frac{d\theta}{a^2 - a(e^{i\theta}+e^{-i\theta}) + e^{i\theta}e^{-i\theta}}$$

$\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2}$

$1 = e^{i\theta} e^{-i\theta}$

$$= \int_0^{2\pi} \frac{d\theta}{(a-e^{i\theta})(a-e^{-i\theta})}$$

γ'

γ

$|z| = 1$

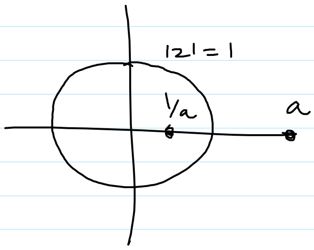
$$= \frac{1}{i} \int_0^{2\pi} \frac{ie^{i\theta}\, d\theta}{(a-e^{i\theta})(ae^{i\theta}-1)}$$

$a \in$

std CIF

$$= \frac{1}{i} \int_{|z|=1} \frac{dz}{(a-z)(az-1)}$$

$$= \frac{1}{i} \int_{|z|=1} \frac{\cdots}{(a-z)(az-1)}$$



$$= \frac{-1}{ia} \int_{|z|=1} \frac{dz}{(z-a)(z-1/a)}$$

$$= \int_{|z|=1} \frac{\frac{-1}{ia}\frac{1}{(z-a)}\, dz}{(z-1/a)}$$

$$2\pi i \left(f(1/a) \right) = 2\pi \cancel{i} \; \frac{-1}{\cancel{i}a} \frac{1}{(1/a - a)}$$

$$= \frac{-2\pi}{1-a^2} = \frac{2\pi}{a^2-1} \quad \square$$

Q7)

7. Show that if $a_1, a_2, ..., a_n$ are the distinct roots of a monic polynomial $P(z)$ of degree n, for each $1 \le k \le n$ we have the formula:
$\prod_{j \neq k} (a_k - a_j) = P'(a_k)$

Monic polynomial with roots a_i of degree n

$$P(z) = (z-a_1)(z-a_2) \cdots (z-a_n)$$

$$= \prod_{i=1}^{n} (z-a_i)$$

[Polynomial is well behaved]

By the product rule of diff (why?)

$$P'(z) = \sum_{i=1}^{n} \left(\prod_{j \neq i} (z-a_j) \right) \frac{d}{dz}(z-a_i)$$

$$= \sum_{i=1}^{n} \prod_{j \neq i} (z-a_j)$$

$$\therefore \; P'(a_k) = \sum_{i} \left(\prod_{j \neq i} (a_k - a_j) \right)$$

$i=1 \quad j \neq$

all terms are zero except $i=k$ term

$$f'(a_k) = \prod_{j \neq k} (a_k - a_j) \qquad \square$$