Q7 Consider a rotation of the coordinate system from (x,y) to $u = x\cos\theta - y\sin\theta$ and $v = x\sin\theta + y\cos\theta$

For proving that the laplatian operator (∇^2) is rotationally invariant we ~~can~~ have to show that

$$f_{xx} + f_{yy} = f_{uu} + f_{vv} \quad (\nabla^2 f(x,y) = \nabla^2 f(u,v))$$

$$f_x = \frac{\partial f}{\partial x} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial x} \quad \text{(Chain rules)}$$

$$f_x = \frac{\partial f}{\partial u}\cos\theta + \frac{\partial f}{\partial v}\sin\theta$$

$$f_{xx} = \frac{\partial}{\partial x} f_x = \frac{\partial f_x}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial f_x}{\partial v}\frac{\partial v}{\partial x}$$

$$= \frac{\partial}{\partial u}(\cos\theta f_u + \sin\theta f_v)\cos\theta + \frac{\partial}{\partial v}(\cos\theta f_u + \sin\theta f_v)$$

(Assuming $f_{uv} = f_{vu}$)

$$f_{xx} = f_{uu} \times \cos^2\theta + 2 f_{uv} \times \cos\theta \times \sin\theta + f_{vv} \times \sin^2\theta \quad - (1)$$

$$f_y = \frac{\partial f}{\partial y} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial y}$$

$$f_y = \frac{\partial f}{\partial u}(-\sin\theta) + \frac{\partial f}{\partial v}\cos\theta = -f_u \sin\theta + f_v\cos\theta$$

$$f_{yy} = \frac{\partial (f_y)}{\partial y} = \frac{\partial f_y}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial f_y}{\partial v}\frac{\partial v}{\partial y}$$

$$f_{yy} = \frac{\partial}{\partial u}(-f_u \sin\theta + f_v \cos\theta)(-\sin\theta) + \frac{\partial}{\partial v}(-f_u \sin\theta + f_v \cos\theta)\cos$$

$$f_{yy} = +f_{uu}\sin^2\theta - 2f_{uv}\sin\theta\cos\theta + f_{vv}\cos^2\theta \quad - (2)$$

(1) + (2) gives

$$f_{xx} + f_{yy} = f_{uu}(\sin^2\theta + \cos^2\theta) + f_{vv}(\sin^2\theta + \cos^2\theta)$$

$$f_{xx} + f_{yy} = f_{uu} + f_{vv}$$

Hence the laplacian operator is rotationally invariant