Q4.) a.) Now, the Laplacian Matrix with -4 in the centre is given as:

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & -4 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

Assume this matrix A can be written as $vw^T = A$ where $v, w \in \mathbb{R}^3$

Then for a $u \in \mathbb{R}^3$ $Au = vw^T u = (\langle u, w \rangle) v$

ie A maps vectors in $\mathbb{R}^3$ to a scalar multiple of $v \Rightarrow \text{Rank}(A) = 1$

However

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & -4 & 1 \\ 0 & 1 & 0 \end{pmatrix} \xrightarrow{R_3 \to R_3 - R_1} \begin{pmatrix} 0 & 1 & 0 \\ 1 & -4 & 1 \\ 0 & 0 & 0 \end{pmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{pmatrix} 1 & -4 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \xrightarrow{R_1 \to R_1 + 4R_2} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Is a rank $= 2$ matric $\Rightarrow\!\Leftarrow$

$\therefore A$ cannot be expressed as $vw^T \Rightarrow$ Not separable $\square$

b.) Now, observe that

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & -4 & 1 \\ 0 & 1 & 0 \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 0 & 0 \\ 1 & -2 & 1 \\ 0 & 0 & 0 \end{pmatrix}}_{A_1} + \underbrace{\begin{pmatrix} 0 & 1 & 0 \\ 0 & -2 & 0 \\ 0 & 1 & 0 \end{pmatrix}}_{A_2}$$

Here, A_1 and A_2 being rank = 1 matrices can be expresed as vw^T each $v, w \in \mathbb{R}^3$

$$\begin{bmatrix} 0 & 0 & 0 \\ 1 & -2 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} 1 & -2 & 1 \end{bmatrix}$$

and

$v := [0 \; 1 \; 0]^T$

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & -2 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}$$

$w := [1 \; -2 \; 1]^T$

So $Img = I$;
$$I * A = I * (A_1 + A_2)$$
$$= I * A_1 + I * A_2 \quad \text{(distributive)}$$
$$= I * (vw^T) + I * (wv^T)$$
$$= (I * v) * w^T + (I * w) * v^T \qquad \square$$

Therefore, we can implement with 1D conv.