Synthetic Aperture Radar

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Abstract

The general theory of side-looking synthetic aperture radar systems is developed. A simple circuit-theory model is developed; the geometry of the system determines the nature of the prefilter and the receiver (or processor) is the postfilter. The complex distributed reflectivity density appears as the input, and receiver noise is first considered as the interference which limits performance. Analysis and optimization are carried out for three performance criteria (resolution, signal-to-noise ratio, and least squares estimation of the target field). The optimum synthetic aperture length is derived in terms of the noise level and average transmitted power. Range-Doppler ambiguity limitations and optical processing are discussed briefly.

The synthetic aperture concept for rotating target fields is described. It is observed that, for a physical aperture, a side-looking radar, and a rotating target field, the azimuth resolution is λ/α where α is the change in aspect angle over which the target field is viewed.

The effects of phase errors on azimuth resolution are derived in terms of the power density spectrum of the derivative of the phase errors and the performance in the absence of phase errors.

Key Words—Aperture, radar, resolution, signal-to-noise, synthetic, system.

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I. Introduction

Coherent side-looking imaging radar [1], [2] is one of the most elegant electronic systems to be developed during the last decade. This particular system development is a part of the rather extensive activities in range-Doppler radars, which in turn are a part of a broad scope of activities in the processing of analog data (these activities include various applications of pulse compression to radar and communications, coherent optical data processing with recent emphasis on holography, laser radar and communications, work in automatic shape recognition, and the like).

The major concern of this (primarily tutorial) paper is coherent side-looking radar; however, in the interest of obtaining a unified picture, comments are made on physical aperture systems, and on the range-Doppler imaging of rigid rotating target fields. In all of these systems the obtainable cross-range resolution on the target field is determined by the change in aspect angle over which the target field is viewed; specifically, if α is the change in aspect angle in radians over which the target is viewed, then the obtainable resolution is approximately

$$\frac{\lambda}{2 \sin \frac{\alpha}{2}} \cdot$$

Imperfect phasing in a physical aperture and/or in Doppler-processing systems (such as in a side-looking radar) may well preclude obtaining the phase-error-free theoretical resolution; hence, the resolution limits due to phase errors will be described for the various systems under consideration. Also, optical processing and some elementary ambiguity limitations will be described briefly.

Resolution as a Performance Criterion [3]

Suppose the complex illumination on a linear aperture of length d is $H_0(x)$, then in the far field the amplitude and phase of the electric field at an angle θ from broadside (for small θ) is essentially

$$h(\theta) = \int_{-d/2}^{d/2} H_0(x) \exp\left(j \frac{2\pi x}{\lambda} \theta\right) dx.$$

Or if $2\pi x/\lambda = \omega$ and $\lambda H_0(\lambda\omega/2\pi) = H(\omega)$, then

$$h(\theta) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{j\omega\theta} H(\omega) d\omega \qquad (1)$$

where the illumination, $H(\omega)$, is subject to the constraint

$$H(\omega) = 0 \qquad \text{for } |\omega| > B = \frac{\pi d}{\lambda} \cdot$$

Typically, we desire that the beamwidth associated with h be narrow; i.e., the spread of $h(\theta)$ in θ is the resolution corresponding to the illumination $H(\omega)$, and the first-order problem is to select H so as to optimize the resolution. The illumination function H is constrained by the physical extent of the antenna (or optical aperture, depending on the value of λ). Of course, we have arrived at a rather classic mathematics problem; namely, given that $H(\omega)$ is constrained in bandwidth, how narrow can the Fourier transform of H be. This is something of a subject in itself [3, Ch. 5]—due partly to the fact that the time (or θ) duration of h and the ω spread of H can be quantitatively defined in several interesting ways. However, for our discussion, it is enough to recall one lower bound on the time-bandwidth product of a signal.

Let the time spread (or resolution) of $h(t)$ be defined as the width of an equivalent rectangle; specifically, if the maximum value of $|h(t)|^2$ occurs at $t=0$, then the resolution is defined quantitatively as

$$T = \frac{\int_{-\infty}^{\infty} |h(t)|^2 dt}{|h(0)|^2}.$$

Now if

$$h(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{j\omega t} H(\omega) d\omega$$

and $H(\omega)=0$ for $|\omega|>B$, then[1]

$$TB \geq \pi \tag{2}$$

and $T=\pi/B$ provided $H(\omega)$ is constant in $|\omega|<B$.

In terms of antenna illumination, the minimum obtainable beamwidth is $T=\Delta\theta=\pi/B=\pi/(\pi d/\lambda)=\lambda/d$ radians. which of course is the usual result; the simple observation here is that (within the definition of resolution used) the bandwidth limit (aperture limit) imposes an absolute lower bound on the angular resolution where this lower bound is trivially tied to the relation between time and frequency spreads of Fourier transform pairs.

Now at a range R, the position resolution provided by the angular resolution λ/d is $\rho=R\ \lambda/d$; however, d/R is (approximately) the angle subtended by the physical aperture d at range R. That is, if $\alpha=d/R$, then the positional resolution of the beam at any range (with these simplifying assumptions) is $\rho=\lambda/\alpha$.

The above simple remarks will be recast in a more complete imaging context in Section II; however, some additional simple observations on resolution as a performance criterion are in order. Specifically, consider a more or less arbitrary linear time-invariant system

$$y(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau$$

where h is the impulse response of the system. If we think of the output $|y|^2$ as the image of x, then $|h|^2$ is the image produced by a point source. In optical systems, antenna (illumination) design, and the synthesis of pulse compression networks, it is common practice to use resolution (the spread of the point "target" response $|h|^2$) as the (primary) performance criterion.

The idea is simply that if $|h|^2$ is concentrated on a short time interval, then y should "resemble" x. (Said in another way, if we strive to make h approach the δ-function, this should make y approach x.) It is somewhat interesting to compare resolution as a performance criterion with other criteria such as mean-square error, point target detection probability, etc. Suffice it to say here that if the system is constrained in bandwidth, the choice of system impulse response for small mean-square error is consistent with the choice of h for fine resolution. Specifically, if $H(\omega)=0$ for $|\omega|>B$, then

$$\begin{aligned} E(|y-x|^2) &= \int_{-\infty}^{\infty} |1-H(\omega)|^2 S_x(\omega)d\omega \\ &= \int_{|\omega|<B} |1-H|^2 S_x(\omega)d\omega \\ &\quad + \int_{|\omega|>B} S_x(\omega)d\omega \end{aligned}$$

where S_x is the power density spectrum of the input signal. Of course the H which minimizes the mean-square error is $H(\omega)=1$ for $|\omega|<B$, and this transfer function also minimizes the equivalent rectangle resolution under the same bandwidth constraint.

Actually work in pulse compression, radar waveform selection, and antenna illumination has resulted in something of a folklore on the desirable properties of the point target response h (required sidelobes etc.). For example, an $h(t)=1$ for $|t|<T$ and $h(t)=0$ for $|t|>T$ has most of the charms praised in the lore; however, if viewed in terms of the transfer function, $H(\omega)$, such a system contains rather fantastic phase distortion[2] (as well as amplitude distortion; however, compensation for the amplitude of a transfer function typically involves noise enhancement while phase compensation typically involves no noise penalty). Such phase distortion degrades equivalent rectangle resolution, mean-square error, etc., and would be intuitively viewed as offensive in an audio system. Comparable remarks can be made concerning the tapering of the amplitude frequency response of pulse

[1] As for the proof, we use $\int|h|^2=1/2\pi\int|H|^2$, and $h(0)=1/2\pi\int H$; then

$$T = 2\pi\frac{\int_{-B}^{B}|H|^2}{\left|\int_{-B}^{B}H\right|^2}.$$

The integrand in the denominator is viewed as $1\cdot H$, then the Schwarz inequality gives

$$\left|\int_{-B}^{-B} 1\cdot H\right|^2 \leq \int_{-B}^{B} d\omega \cdot \int_{-B}^{B}|H|^2 d\omega\cdot$$

Hence $T\geq(\pi/B)$ with equality when $H(\omega)$ is constant on $|\omega|<B$.

[2] The Fourier transform of a rectangular h is alternately positive and negative and negative H is the ultimate in phase distortion.

compression systems[3]—trading resolution for sidelobes. Again what is attractive in the lore can almost be viewed with horror from a transfer function distortion viewpoint.

It might be noted that resolution as a performance criterion (like concern for amplitude and phase distortion in an amplifier) can be (is) employed without specifying properties of the input (such as the power spectra of the signal and noise); this feature of resolution as a criterion has practical appeal. A somewhat more compelling appeal for resolution as a performance criterion occurs when system performance is limited by phase errors (or stochastic delays). Specifically, under such a disturbance resolution remains indicative of image quality while other popular criteria (mean-square error, etc.) tend to be inappropriate. For example, phase errors may not degrade resolution significantly, yet introduce enough distortion (misplacement of the output) to cause extravagant mean-square error. In such a case the quality of the output image may be excellent, but this fact would not be seen from, say, a mean-square error analysis.

II. Side-Looking Radar

With the major exception of ambiguity limitations, the range channel and azimuth channel of a side-looking radar can be analyzed independently. Hence, it is helpful to start with only the azimuth channel. To this end consider targets at just one range and a CW radar (it is simple enough later to pulse the radar to obtain range resolution in which case the reflecting material in a specific range resolution cell corresponds to the line of reflecting material to be considered now).

Mathematical Model

Let $A(\theta)$ be the two-way electric field pattern of the side-looking antenna in Fig. 1, and for simplicity let this pattern be real. The function $f(u)$ denotes the complex reflectivity density at position u. This reflectivity density is defined so that if $\cos \omega_0 t$ is transmitted from position x on the line of flight, then the signal returned at time t from the increment $(u, u+du)$ is

$$A(\theta)\,|f(u)|\ \cos\left[\omega_0\left(t-\frac{2s}{c}\right)-\gamma(u)\right]du$$

where $f=|f|e^{j\gamma}$. The total signal received at time t is given by summing over all incremental returns, thus

$$\left(\text{where } \theta \cong \frac{x-u}{R}\right):$$

$$g_b(t) = \int A\left(\frac{x-u}{R}\right)|f(u)| \cdot\cos\left[\omega_0\left(t-\frac{2s}{c}\right)-\gamma(u)\right]du.$$

[3] A procedure borrowed from the design of antenna illumination functions.

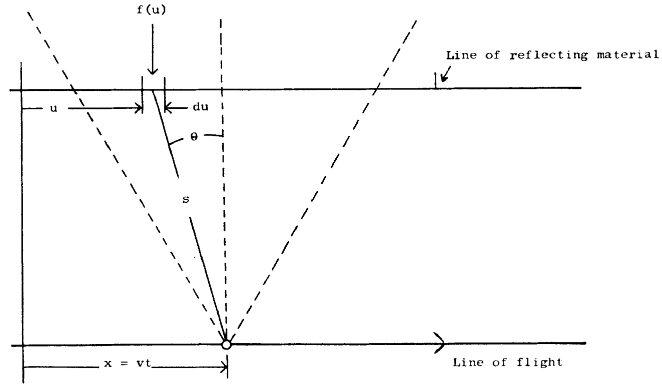

Fig. 1. Geometry of side-looking radar.

If g_b is mixed[4] with $\cos \omega_0 t$ and only the difference frequency is retained, we obtain

$$g_r(x) = \int A\left(\frac{x-u}{R}\right)|f(u)|\ \cos\left[\frac{2\omega_0}{c}s+\gamma(u)\right]du.$$

This mixer output is a function of time in that if the vehicle speed is v, then $x=vt$; however, since the basic objective of the system is to estimate (or image) a function of position (f), the data will be considered as a function of x.

If g_b is mixed in a quadrature mixer ($\sin \omega_0 t$), the signal at the difference frequency is

$$g_i(x) = \int A\left(\frac{x-u}{R}\right)|f(u)|\ \sin\left[\frac{4\pi}{\lambda}s+\gamma(u)\right]du.$$

Now the ordered pair of mixer outputs is a complex signal $g=g_r+jg_i$. It is convenient to approximate

$$s = \sqrt{R^2+(x-u)^2} \cong R + \frac{(x-u)^2}{2R}$$

and let f absorb the constant phase factor $\exp j(4\pi/\lambda)R$. Then if

$$W(x) = A\left(\frac{x}{R}\right)\exp j\,\frac{2\pi}{\lambda R}\,x^2$$

$$g(x) = W(x) * f(x). \tag{3}$$

Intrinsic Resolution

Of course the resolution of the basic data (without processing) is the spread of W; e.g., if the beamwidth of the illuminating antenna is α, then the unprocessed data provides a ground resolution of αR. On the other hand, the bandwidth of W offers the potential for much finer resolution. Specifically since W is a chirp pulse having an "instantaneous frequency" of $(4\pi/\lambda R)x$ for a point target located at $u=0$, the bandwidth of the data collected over

[4] For this analysis the carrier frequency, ω_0, will be removed, and the system analyzed in terms of complex lowpass signals; however, we could just as well treat the system in terms of real bandpass signals [3, chps. 6 and 9].

the beamwidth of the illuminating antenna is

$$2B = \frac{4\pi}{\lambda R}(\alpha R) \text{ radians/ft.}$$

In turn (2) suggests that the obtainable resolution is

$$\rho = \pi/B = \frac{\lambda}{2\alpha}\cdot \tag{4}$$

If d is the physical aperture, it is common to take $\alpha \cong \lambda/d$; then the "theoretical resolution" is $\rho = d/2$.

Before proceeding, consider the point target response without approximating s:

$$W(x) = A\left(\frac{x}{R}\right)\exp\left(j\frac{4\pi}{\lambda}\sqrt{R^2+x^2}\right).$$

The instantaneous frequency is

$$\frac{4\pi}{\lambda}\frac{x}{\sqrt{R^2+x^2}}\cdot$$

Thus, data considered over a path[5] of $\pm X$, gives rise to a bandwidth of $B = 4\pi/\lambda \sin \alpha/2$ where α is the change in aspect angle corresponding to this path length. Thus the anticipated resolution is

$$\rho = \frac{\lambda}{4\sin\alpha/2}\cdot \tag{5}$$

If one-way propagation is considered (as occurs, say, in an ordinary coherent optical system), then $(2\pi/\lambda)s$ appears in place of $(4\pi/\lambda)s$. In this case a coherent aperture which subtends an angle of α leads to a (diffraction-limited) resolution twice that given in (5).

Basic Optimization

The above remarks on resolution were based on the bandwidth of the data received from a point target; however, it was implicit that the Doppler histories were "compressed." That is, suppose we process the received data, g, with a filter having an impulse response $K(x)$. If noise is present at the output of the quadrature mixers ($n = n_r + jn_i$), then the final system output is

$$h = K * W * f + K * n.$$

The mathematical model is depicted in Fig. 2. For a point target f is a δ-function and the overall point target response is $h_0 = K * W$.

The first issue is that of selecting an appropriate K; here we suffer from too many available attractive theories. Specifically, K might be selected on the basis of resolution; alternatively, having introduced additive noise, a matched filter suggests itself as does the selection of K so as to minimize the mean-square error, $E(|f-h|^2)$. However, since W tends to be a pulse having a large time-

[5] Here we consider a broad illuminating beam so as to accommodate a Doppler history of any desired length.

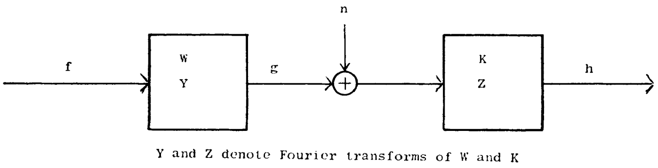

Fig. 2. Model for azimuth channel of side-looking radar.

bandwidth product ("time" duration of about αR; bandwidth of about $2\alpha/\lambda$; and a product, or available compression ratio, of about $2\alpha^2 R/\lambda \cong 2\lambda R/d^2$), the major obligation of the postfilter is that it phase compensate for the prefilter. That is, if h is to resemble f, it is intuitively clear that the phase of the frequency response Z should be the negative of the phase of Y (save possibly for a linear phase factor which would merely introduce time delay).

Actually, the use of the matched filter

$$Z = \frac{\overline{Y}}{S_n},$$

where S_n is the power density spectrum of the noise, generally involves some slight sacrifice of resolution in the interest of noise rejection. Conceptually the least squares receiver is splendid, it compromises between resolution and noise rejection. That is, the least squares receiver is

$$Z = \frac{\overline{Y}S_f}{|Y|^2 S_f + S_n}\cdot \tag{6}$$

As $S_n \to 0$, $Z \to 1/Y$ which provides a (theoretically) perfect image of f; and as $S_n \to \infty$, $Z \to (\overline{Y}/S_n)S_f$ which is about the same as the point target matched filter.

The question of optimizing the system in the presence of additive noise is not of overwhelming interest since imperfect coherence (phase errors) imposes, perhaps, more serious performance limitations. The effects of phase errors will be covered in the last Section. A rather basic question suggests itself at this stage; namely, granting that the least squares receiver (Z) is employed, how do we optimize the prefilter transfer function Y.

Let $e = h - f$, then from Fig. 2 we obtain the following, where $S = |Y|^2 S_f + S_n$:

$$S_e = \left|S^{1/2}Z - \frac{S_f\overline{Y}}{S^{1/2}}\right| + \frac{S_f S_n}{S}$$

which yields the optimum Z of (6). The corresponding mean-square error is

$$\overline{e^2} = \frac{1}{2\pi}\int \frac{S_f S_n}{S_f|Y|^2 + S_n}\,d\omega. \tag{7}$$

The problem is to minimize $\overline{e^2}$ over $|Y|^2$. Of course as $|Y|^2 \to \infty$, $\overline{e^2} \to 0$; thus for the minimization some constraint is required on the prefilter transfer function, Y. The most reasonable basic constraint is on the illumination power of the physical aperture. We now proceed to interpret this constraint in terms of Y.

Recall that

$$Y(\omega) = \int e^{-j\omega x} A\left(\frac{x}{R}\right) \exp\left(j \frac{2\pi}{\lambda R} x^2\right) dx.$$

To a good approximation:

$$|Y(\omega)|^2 = \frac{\lambda R}{2} A^2\left(\frac{\lambda}{4\pi}\omega\right). \tag{8}$$

As an approximation (8) simply states that in the direction θ from broadside we receive a Doppler of $4\pi\theta/\lambda$ radians per unit of distance traveled by the vehicle (or a Doppler frequency of $2v\theta/\lambda$ Hz), and that these Dopplers are weighted in amplitude by $A(\theta)$.

The point of (8) is that our selection of the prefilter $|Y|^2$ is equivalent to selecting the square of the two-way pattern A. Within the context of a real two-way pattern, $A(\theta)$ is the square of the one-way pattern $h(\theta)$ which is the Fourier transform of the physical illumination function $H(\xi)$ where ξ denotes position along the physical antenna. A transmitter power constraint amounts to a constraint on $\int|H|^2 \leq P_1$, or by Parseval's theorem $\int|h(\theta)|^2 d\theta \leq P_2$. Thus we wish to minimize (7) with $\int|Y|d\omega \leq P$ where P is directly proportional to transmitted power.

A radar scene rich in detail is well modeled by a random process f which has a broad spectral density; thus let $S_f(\omega)=F$ for $|\omega|<B$ and $S_f(\omega)=0$ otherwise and let the noise be white, $S_n=N$. To do the problem justice, a finite aperture constraint should be placed on the physical antenna; however, rather than treat this problem (which requires numerical analysis) the central issue can be dealt with in a simple way. The central issue is the following: for any F and N (signal-to-noise ratio), does the optimum Y pass the entire bandwidth associated with S_f. For this question it is enough to consider a rectangular illumination pattern. That is, we approximate h (and hence A and Y) as a constant over some angle so that (by the power constraint) $|Y|=P/b$ on a subinterval of length b in $|\omega|<B$, and proceed to find the optimum value of b. Equation (7) gives

$$2\pi E(|e|^2) = F(2B-b) + \frac{bF}{\frac{F}{N}\frac{P^2}{b^2}+1}.$$

The optimum b maximizes

$$b - \frac{b^3}{b^2+\frac{FP^2}{N}} \quad \text{or} \quad \frac{\frac{FP^2}{N}b}{b^2+\frac{FP^2}{N}}.$$

Thus with FP^2/N fixed, the appropriate Doppler bandwidth to be covered by the radar is

$$b = P\sqrt{F/N}. \tag{9}$$

A comparable problem occurs in the range channel of the radar; however, in this case [3, ch. 9] W is the complex modulation on the transmitted pulse and an energy constraint on the pulse gives $\int|W^2|^2 \leq D/2\pi$, so that $\int|Y|^2 \leq D$ and $|Y|^2 = D/b$. The quantity then to be maximized is

$$\frac{b\frac{FD}{N}}{b+\frac{FD}{N}}.$$

However, this is monotone in b (for $b>0$) so that for any assumed bandwidth B in the structure of the radar field, the (least squares) optimum is $b=B$; i.e., in the range dimension of a radar, limited energy in the transmitted pulses does not call for a compromise of the attempt to obtain arbitrarily fine resolution.

Of course illumination over a broad angle thins the power density over the Doppler bandwidth, and leads to a small (physical) reception aperture. Such compound payment for fine azimuth resolution accounts for the limited resolution ambitions called for in (9).

Two-Dimensional Imaging and Ambiguity Limitations

If the transmitter is pulsed, then for each range rather than having $g(t)$ of Fig. 2 available, a sampled version of $g(t)$ is available where the PRF of the radar determines the sampling rate. The detailed derivation of the equations and mathematical model for the range-azimuth system is not necessary here; however, a few remarks on the appropriate mathematical model are worthwhile. Let y denote range, $W_2(t)$ the complex modulation on each transmitted pulse, and $W_1(y)=W_2(2y/c)$. A two-dimensional reflectivity is introduced $f_0(x, y)$, and a receiver impulse response $K_2(t)$ is applied to each "range sweep" received by the radar with $K_1(y)=K_2(2y/c)$. There is a propagation phase shift which renders

$$f(x, y) = \exp\left(j\frac{4\pi}{\lambda}y\right) f_0(x, y)$$

the effective input to the system, and the observable data (to a good approximation) become

$$g(x, y) = f(x, y) * [W_y(x)W_1(y)]$$

where we now display the range dependence of the azimuth impulse response, $W_y(x)$. Finally, $g(x, y)$ is available only at $x=mX$ where X is the distance traveled between transmitted pulses and m is an integer variable (or index on the transmitted pulses); also receiver noise accompanies each range sweep. Thus, the model takes the form shown in Fig. 3.

The fact that the azimuth filters have a slow dependence on range presents no great conceptual or analytical problems so that the sampling stands out as the major additional feature to be considered. Over a broad class of

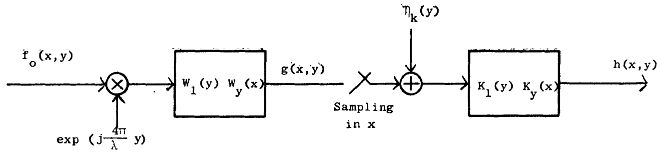

Fig. 3. Two-dimensional model.

system parameters, the sampling also introduces no appreciable effect; and in such cases it is seen that the elementary considerations discussed for the azimuth channel apply in the two-dimensional system and for that matter generalize in a rather obvious way to provide a theoretical basis for the entire system.

Transmissions other than a periodic pulse train can be considered, and fundamentally we are simply dealing with a range-Doppler radar which suggests that most of the theory developed for range-Doppler radars (usually in an "air defense" radar context) applies to the side-looking coherent radar. Such is the case, and with some manipulation the two-dimensional point target response of the side-looking radar can be expressed as the ordinary range-Doppler ambiguity function [4] of the transmitted signal (tapered by the pattern of the illuminating antenna). However, this viewpoint does not particularly suggest much of a departure from a periodic pulse train.

With a periodic pulse train, the ambiguity limit of side-looking radar involves the need to have a PRF high enough to adequately sample the bandwidth of the Dopplers in the azimuth angle illuminated by the radar. This minimum PRF sets a maximum on the unambiguous range interval which can be imaged. In detail two frequencies ω_1 and ω_2 are ambiguous (alias) for sampling period T if $\exp j\omega_1 mT = \exp j\omega_2 mT$ for all m, and this is the case if

$$\omega_1 - \omega_2 = p\frac{2\pi}{T} \qquad (10)$$

where p is any integer. The Doppler frequencies are concentrated in a bandwidth (4)

$$\frac{4\pi}{\lambda R}(\alpha R)\ \text{rad/ft}$$

where α is the azimuth beamwidth. In this notation such a bandwidth will not involve ambiguous frequencies if the distance traveled between transmitted pulses (X) satisfies

$$\frac{2\pi}{X} \geq \frac{4\pi}{\lambda}\alpha.$$

That is,

$$X \leq \frac{\lambda}{2\alpha} = \rho$$

where ρ is the theoretical "resolution." Of course this is eminently reasonable; if we desire an along-track resolution of ρ we must transmit at least one pulse per ρ-distance traveled by the vehicle. With $\alpha \cong \lambda/d$, and a vehicle speed v this calls for the following PRF:

$$T = X/v \leq d/2v$$

$$\text{PRF} \geq \frac{2v}{d}\cdot$$

If the illuminating pattern had zero azimuth response outside the angle $\alpha = \lambda/2X$, and if the elevation pattern provided zero illumination outside a range interval of length $R_0 = (c/2)(X/v)$, then the system would contain no ambiguities. However, since a finite aperture is used some range ambiguities ("second time around" echoes) and Doppler ambiguities occur (possibly negligible in level).

As for the appearance of azimuth ambiguous responses, consider a point target under sidelobe illumination at an azimuth angle for which the Doppler is centered around the frequency $2\pi p/X$. After sampling the data collected at such an angle will have (very nearly) the same phase history as that appearing as the main beam of the antenna passes over the target. Such data will be compressed by the azimuth processor to yield a "false" response (with an amplitude determined by sidelobe levels). The positions of these ambiguous responses (relative to the main beam response) correspond to the condition that the instantaneous Doppler $((4\pi/\lambda R)x)$ is alias to zero Doppler:

$$\frac{4\pi}{\lambda R}x = p\frac{2\pi}{X}$$

$$x = p\frac{\lambda R}{2X} \qquad (p\text{—any integer}).$$

The situation is depicted[6] in Fig. 4.

The design problem posed by these ambiguity considerations is dominated by the question of determining the optimum choice of the illumination (azimuth and elevation) on the physical aperture (of specified dimensions). This problem has been effectively dealt with [5].

Optical Processing

To this point we have considered the data collected by the radar, what is to be done with the data to produce an image, and have analyzed performance in a general way. Some comment on how the data processing might be implemented is in order. The two-dimensional nature of the received data[7] suggests the possibility of employing optical data processing [6]–[8] which at least is conceptually appealing. Here we will only comment briefly on the possibility of such an implementation.

In the range dimension let us transmit a short pulse so that the range sweeps recorded on film have the desired

[6] Of course the indicated Doppler ambiguities are equivalent to the grating lobes which appear in a physical linear array which has directive widely spaced ($\gg\lambda/2$) elements.

[7] The received data is a function of time alone, but is two-dimensional as a sequence of range sweeps with the range dimension occurring in fast time along the sweep and the azimuth data occurring in slow time across the sweeps.

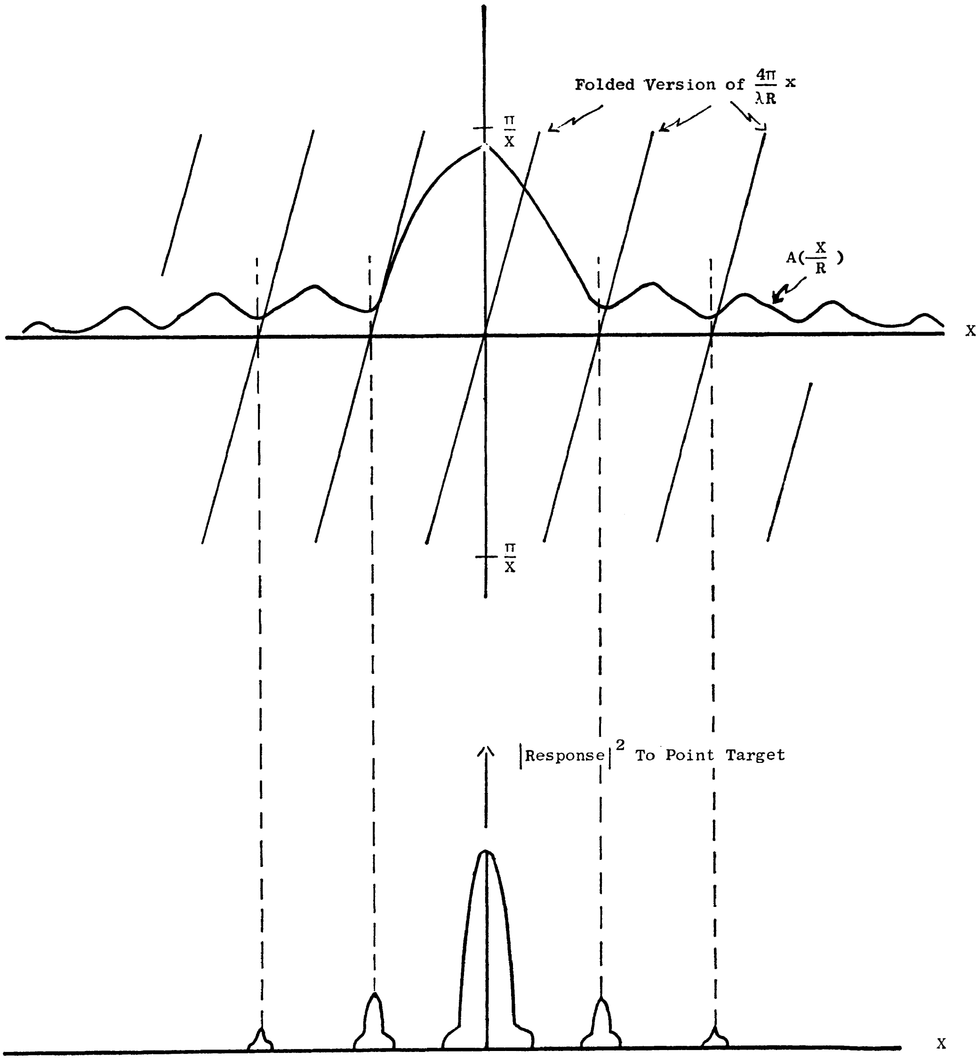

Fig. 4. Ambiguous sidelobes in azimuth response.

range resolution (i.e., the electronic receiver provides the processing depicted by K_1 in Fig. 3), then we optically synthesize only the azimuth processing. A suitable choice for the azimuth impulse response is the white noise matched filter; i.e.,

$$Z(\omega) = \overline{Y}(\omega) \quad \text{or} \quad K(x) = \overline{W}(-x).$$

Let the antenna pattern be even, then

$$K(x) = A\left(\frac{x}{y}\right) \exp - j\frac{2\pi}{\lambda y} x^2.$$

The desired output in azimuth at each range (y) is

$$\int A\left(\frac{u}{y}\right) \exp\left[-j\frac{2\pi}{\lambda y}u^2\right] g(x+u)du$$

where $g(x)$ is the data across the range sweeps. Rather than introduce the complex azimuth data in terms of real and imaginary parts, a carrier Doppler, ω_1, is inserted so that g appears as a complex modulation thus:

$$\begin{aligned} g(x) &= B(x)e^{j\beta(x)} \\ g_b(x) &= 2B(x)\cos\left[\omega_1 x + \beta(x)\right] + C \\ &= Be^{j\beta}e^{j\omega_1 x} + Be^{-j\beta}e^{-j\omega_1 x} + C \end{aligned}$$

where C is a bias which permits the recording of g_b as a positive function on ordinary film. As for the synthesis of K, a phase factor[8] $\exp(j(2\pi/\lambda y)u^2)$ can be introduced as a conical lens and the other part of the processing, $A(x/y)$, is not of much concern; however, it amounts to a wedged aperture so that we process only those Dopplers which are well illuminated. Now suppose g_b and the conical lens are placed essentially in contact in a plane wave of coherent light; let x denote the azimuth position on the film which is at the center of the conical lens, and let u denote distance from the center of the lens; then the complex amplitude of the emerging light is

[8] Since g_b contains $\bar{g}$ and g we need not use $\exp -j(2\pi/\lambda y)x^2$ which would correspond to a concave conical lens. Also, $2/\lambda y = 1/\lambda_0 f$ where λ_0 is the wavelength of the light and f is the variable focal length of the conical lens.

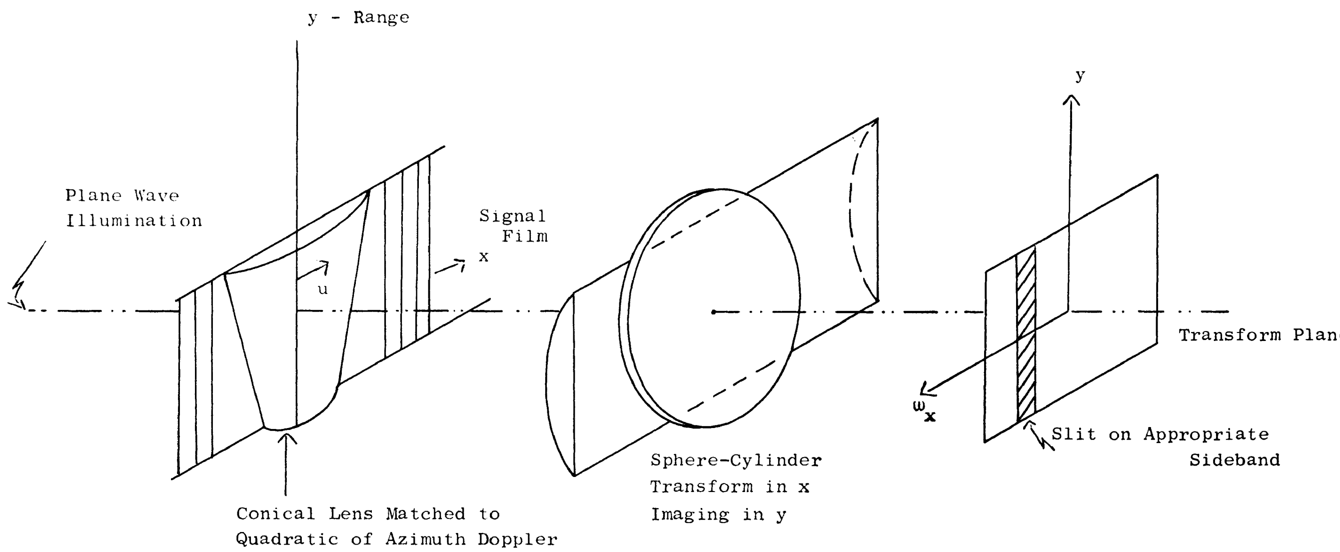

Fig. 5. Optical processing.

$$\begin{aligned} q(u) &= g(x+u)e^{j\omega_1(x+u)}A\left(\frac{u}{y}\right)\exp\left(j\frac{2\pi}{\lambda y}u^2\right) \\ &\quad + \bar{g}(x+u)e^{-j\omega_1(x+u)}A\left(\frac{u}{y}\right)\exp\left(j\frac{2\pi}{\lambda y}u^2\right) \\ &\quad + CA\left(\frac{u}{y}\right)\exp\left(j\frac{2\pi}{\lambda y}u^2\right) \end{aligned}$$

where u is the azimuth coordinate across the optical aperture.

If we take the Fourier transform of $q(u)$, the three terms produce outputs centered at ω_1, $-\omega_1$, and on axis (zero frequency). If ω_1 exceeds the highest frequency of gK, these three outputs are separated physically in the transform plane. Specifically, let us consider the transform

$$e^{-j\omega_1 x}\int \bar{g}(x+u)e^{-j\omega_1 u}A\left(\frac{u}{y}\right)\exp\left(j\frac{2\pi}{\lambda y}u^2\right)e^{-j\omega u}du.$$

At $\omega = -\omega_1$ (a slit in the frequency plane), the intensity of the light is

$$\left|\int \bar{g}(x+u)A\left(\frac{u}{y}\right)\exp\left(j\frac{2\pi}{\lambda y}u^2\right)du\right|^2$$

which is the desired processed radar image.

Thus with the variable focal length conical lens in place, the basic processing consists of recording the data on film with a bias and carrier offset frequency and then introducing the optical Fourier transform. The actual desired data appear at a position in the transform plane corresponding to the offset frequency. All of the above operations are in only the azimuth dimension and hence conceptually involve cylindrical lenses; however, a provision must be made to image the range dimension. This imaging involves another lens; either a cylinder acting in the range dimension, or an appropriate cylinder with a spherical two-dimensional Fourier transforming lens to provide the Fourier transform in azimuth and imaging in range. The final system takes the form shown in Fig. 5.

III. Imaging of a Rotating Target Field

The relative motion between the scene and the radar led to the ability to obtain fine azimuth resolution in a side-looking radar. Other forms of relative motion could hopefully lead to comparable results, and such is the case. A synthetic aperture effect, perhaps more basic than the side-looking radar, occurs if a radar scene simply rotates. Again, one can start by considering a line of reflecting material at some range; after the ability to obtain azimuth (or cross-range) resolution is displayed, the radar can be pulsed so as to lead to a two-dimensional imaging system. This additional synthetic aperture concept will not be developed in any detail; however, it is instructive to display another configuration in which Doppler processing provides (nearly) arbitrarily fine azimuth resolution at any range from a small aperture. The situation[9] is depicted in Fig. 6.

The amplitude and phase received from the increment at $(x, x+dx)$ is (in the CW case)

$$f(x)\exp\left(-j\frac{4\pi}{\lambda}s\right)dx.$$

The total received complex modulation as a function of time is

$$F_0(t) = \int_{-L/2}^{L/2}\exp\left(-j\frac{4\pi}{\lambda}s\right)f(x)dx.$$

[9] The range increment through the center of rotation is used; however, very little modification of the derivation is required for the other range increments.

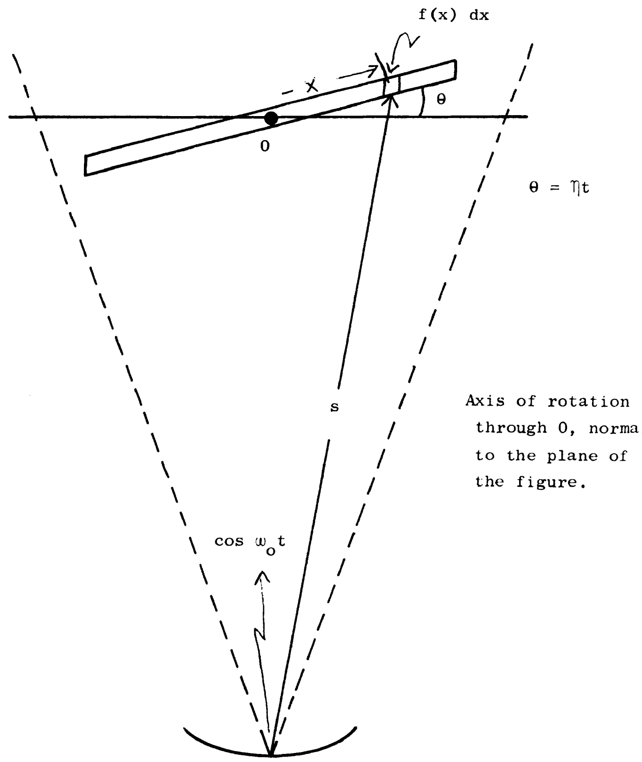

Fig. 6. Doppler processing for rotational motion.

Now $s=R+x \sin \theta=R+x \sin \eta t$ if R is large (where η is the angular velocity of the line of reflection); the constant phase associated with R is not of much concern, hence

$$F_0(t) = \int_{|x|<L/2} \exp\left(-j\frac{4\pi}{\lambda}x \sin \eta t\right) f(x)dx. \quad (11)$$

If we observe $F_0(t)$ for $|t|<T$, then we have the Fourier transform of the reflectivity over spatial frequencies

$$|\omega| < \frac{4\pi}{\lambda} \sin \eta T = B.$$

Such data on f provides a resolution capability of

$$\rho = \frac{\pi}{B} = \frac{\lambda}{4 \sin \eta T} = \frac{\lambda}{4 \sin \alpha/2}$$

where α is the change in aspect angle over which the target is viewed. Of course again we have the theoretical ability to achieve arbitrarily fine azimuth resolution at arbitrary range without the use of a large physical aperture on the radar. In this case the reflection at position x clearly gives rise to a Doppler frequency of $f_d=2\eta x/\lambda$ and a "bank of Doppler filters"[10] obviously resolves the target in the x-coordinate.

The side-looking radar can be viewed as a rotating target situation; specifically, (3) can be written thus:

$$g(vt) = \int_{|vt-u|<\alpha R/2} \exp\left[j\frac{2\pi}{\lambda R}(vt-u)^2\right] f(u)du$$
$$= \exp\left(j\frac{k}{2}t^2\right) \int \exp\left[-j\frac{4\pi}{\lambda}\left(\frac{v}{R}\right)tu\right] f_0(u)du \quad (12)$$

where $f_0(u)=\exp(j(2\pi/\lambda R)u^2)f(u)$, but f_0 and f provide the same image ($|f_0|^2=|f|^2$). Now consider a segment of the target field of length αR, and let $t=0$ correspond to when the center of this segment is closest to the radar vehicle; then if $g(vt)$ is modified by the known phase factor (which is the role of the conical lens in Fig. 5), the side-looking radar data is equivalent to that obtained by rotating the segment with an angular velocity of $\eta=v/R$. Also, the change in aspect angle in the side-looking radar case is the illumination beamwidth α.

In (11) let $\sin \eta t \cong \eta t$, then Fourier transform processing of the received data gives

$$q(\omega) = \frac{1}{2\pi} \int_{|t|<T} e^{j\omega \epsilon t} F_0(t)dt \quad (13)$$

and $h(x)=q(4\pi\eta x/\lambda)$ is the image of f over the spatial bandwidth $|\omega|<(2\pi\alpha/\lambda)$. With white noise included in the analysis, we process $F_0(t)+n(t)$. In this case transform processing is viewed as a "bank of matched filters," which for a point target at unknown position x_0 yields the maximum likelihood estimate of x_0.

IV. Effects of Phase Errors [9], [10]

Let us first consider the simple situation of the illumination of an aperture; specifically, let

$$F(\omega) = \int e^{-j\omega t} f(t)dt \quad (14)$$

where[11] $f(t)$ denotes the illumination function and $F(\omega)$ the resulting pattern. Of course (14) and (1) depict the same situation in different variables; in (14) let ω denote pattern angle in radians, then t is related to illumination position x by $t=2\pi x/\lambda$. If the intended illumination is distorted by a phase error $\alpha(t)$, the resulting modified pattern is

$$F_m(\omega) = \int e^{-j\omega t} e^{j\alpha(t)} f(t)dt. \quad (15)$$

As suggested by (11) and (12), the effects of $\alpha(t)$ on F_m carry over to the effects of phase errors on the point target response of side-looking radar and on the imaging of a rotating target field.

[10] The "bank of Doppler filters" amounts to taking the Fourier transform of $F_0(t)$ which is the appropriate processing on $F_0(t)$ to obtain an image of $f(x)$.

[11] The symbol f was previously used for reflectivity density; in this section our few references to reflectivity density will be denoted by $\hat{f}$. Also, α now denotes the phase error process rather than the change in aspect angle.

Elementary Observations

Our major concern is with the tendency for $|F_m|^2$ to be spread in ω compared to the phase error free response $|F|^2$; however, the first-order effect of the phase errors is to shift the pattern. Thus, we are obligated to consider both the shift in position of the beam and the loss in resolution due to phase errors (and, possibly higher-order effects as well). The basic situation is displayed[12] if we approximate the phase error by the first three terms of its Taylor series

$$\alpha(t) = \alpha_0 + vt + \tfrac{1}{2}at^2$$
$$v = \alpha'(0) \quad \text{and} \quad a = \alpha''(0).$$

Then

$$F_m(\omega) = \int e^{-j(\omega-v)t}e^{j1/2at^2}f(t)dt$$

where α_0 has been omitted since it has no effect on $|F_m|^2$.

If the phase distortion were known, we would modify f to eliminate the phase errors; hence, it is reasonable to consider α a random process and then optimize f against known (or estimated) statistics of α. Now within the Taylor series approximation the pattern shift is given by $\alpha'(0)$, and $\alpha''(0)$ remains as the factor tending to spoil resolution. The basic problem is to select the illumination so as to minimize resolution (on the average). Let us defer a careful treatment of this problem, and only optimize $f(t)$ over Gaussian[13] illuminations; i.e., let

$$f(t) = \exp - \tfrac{1}{2}ct^2.$$

Then $c^{1/2}$ is indicative of the size of aperture we should strive for in the presence of phase errors. Having noted the pattern shift, the modified pattern for resolution considerations is

$$\begin{aligned} F_m(\omega) &= \int e^{-j\omega t} \exp\left[-\tfrac{1}{2}(c + ja)t^2\right]dt \\ &= \sqrt{2\pi}(c + ja)^{-1/2} \exp\left[-\tfrac{1}{2}\omega^2(c + ja)^{-1}\right]. \end{aligned}$$

Thus

$$|F_m(\omega)|^2 = \frac{2\pi}{\sqrt{c^2 + a^2}} \exp\left[-\frac{\omega^2}{2}\left(\frac{2c}{c^2 + a^2}\right)\right].$$

The corresponding equivalent rectangle resolution is

$$\hat{\rho}_0 = \sqrt{\pi}\sqrt{\frac{c^2 + a^2}{c}}. \tag{16}$$

The radius of gyration resolution is

$$\rho = \sqrt{\frac{c^2 + a^2}{2c}}. \tag{17}$$

If we now treat α as a random process, $a = \alpha''(0)$ is a random variable, and a reasonable approach is to select c so as to minimize the mean-square resolution. For either measure of resolution, this calls for the c which minimizes $(c^2 + \overline{a^2})/c$ where now $\overline{a^2}$ denotes the mean-square value of α''. The optimum c is $c = \sqrt{\overline{a^2}} = \Delta a$. With this choice of antenna illumination the rms resolution is

$$\bar{\rho} = \sqrt{\Delta a}. \tag{18}$$

As a specific interpretation of the above simple observations let $\alpha(t)$ result from geometrical distortion of a physical aperture; in this context let $r(x)$ denote the departure of the aperture from the intended straight line. Then

$$\alpha(t) = \frac{2\pi}{\lambda} r\left(\frac{\lambda}{2\pi}t\right)$$
$$\alpha' = r'$$
$$\alpha'' = \frac{\lambda}{2\pi}r''.$$

In turn, if μ is the rms value of d^2r/dx^2, then the rms resolution is

$$\bar{\rho} = \sqrt{\frac{\lambda\mu}{2\pi}} \qquad \text{(radians)}$$
$$\bar{\rho}_0 = \sqrt{\lambda\mu}. \tag{19}$$

Also, the rms displacement of the beam is equal to the rms value of dr/dx.

At this stage we could proceed with the general analysis of the effects of phase errors; however, it seems appropriate to first relate the above type of analysis to synthetic aperture radar.

Application to Synthetic Aperture Radars

First consider the case of a rotating target field. In (11) let $\sin \eta t \cong \eta t$ and let $\hat{f}(x) = \delta(x - x_0)$ so that the point target response is under consideration. Also, let $F_0(t)$ be distorted by phase errors; then the available data are

$$e^{j\alpha(t)} \exp\left(-j\frac{4\pi}{\lambda}\eta x_0 t\right).$$

It is presumed that the image of this point target is formed via

$$F_m(\omega) = \int \exp\left[-j(\omega + \omega_0)t\right]e^{j\alpha(t)}f(t)dt \tag{20}$$

where $f(t)$ essentially depicts the amount of data to be processed and $\omega_0 = 4\pi\eta x_0/\lambda$. Now within the context of a Gaussian taper on the data (which limits the effects of the phase errors), the obtainable resolution in the ω-variable is $\sqrt{\Delta a}$ where Δa is the rms value of α''. In turn the positional (radius of gyration) resolution of the image is

$$\bar{\rho} = \frac{\lambda}{4\pi\eta}\sqrt{\Delta a}. \tag{21}$$

For example, suppose the phase errors are due to a time variation in the electrical-path length (either due to physical change in the distance between the target and the radar or due to time varying propagation conditions), and let $r(t)$ denote this change in propagation path; then

[12] Simple results are also obtained if we consider a periodic phase error, e.g., $\exp jA \cos \mu t = \Sigma J_k(A)e^{jk\mu t}$ which leads to a superposition of shifted versions of the phase error free response.

[13] This is the most simplifying choice mathematically and has some intuitive appeal; however, later it will be seen to have a deeper justification.

$\alpha(t)=(4\pi/\lambda)r(t)$ and if μ is the rms value of r'' we have

$$\bar{\rho} = \frac{1}{2\eta}\sqrt{\frac{\lambda\mu}{\pi}}. \tag{22}$$

It is worthwhile to observe that within the many simplifying assumptions of this analysis, the taper which provides the minimum resolution of (21) also maximizes the output signal-to-noise ratio. Specifically, if the observed data are accompanied by white additive noise, the noise at the output is

$$n_0(\omega) = \int e^{-j\omega t} n(t) f(t) dt$$

$$E(|n_0|^2) = \frac{N_0}{2\pi}\int |f(t)|^2 dt$$

where N_0 is the power density of the white additive noise. Since the mean-square noise out is independent of ω, let the position of the point target be $\omega_0=0$. Also, since $|f|^2=|e^{j\alpha}f|^2$, we have by Parseval's theorem that

$$E(|n_0|^2) = N_0 \int |F_m(\omega)|^2 d\omega.$$

Finally, with $\omega_0=0$ the peak signal out occurs at $\omega=0$; hence, the output noise-to-signal ratio is

$$N/S)_{\text{out}} = N_0 \frac{\int |F_m(\omega)|^2 d\omega}{|F_m(0)|^2}.$$

However, this is simply N_0 times the equivalent rectangle resolution. Thus within the Taylor series approximation (and Gaussian taper), processing for maximum signal-to-noise ratio and processing for minimum resolution are equivalent.

In the side-looking radar case phase errors (usually) enter as a multiplicative factor so as to modify the mathematical model of Fig. 2 thus:

$$h = [(W * \hat{f})e^{j\alpha} + n] * K.$$

One can easily optimize the receiver K with a least squares error criterion; however, the resulting mean-square error is not indicative of image quality. The reason for this is roughly that the phase error process, α, produces phase differences between h and $\hat{f}$ and shifts in position between h and $\hat{f}$ both of which introduce large values for $|h-\hat{f}|^2$. However, neither of these low-order effects render $|h|^2$ a poor quality image of $|\hat{f}|^2$. Thus as mentioned in the Introduction, resolution comes very much into its own as a performance criterion when we move from a concern for additive noise to a concern for phase errors (and related distortion).

Since the side-looking radar is a time[14] (space) invariant system, consider a point target at $x=0$; then if $W=A(x)\exp -j\frac{1}{2}kx^2$ and $K=B(x)\exp j\frac{1}{2}kx^2$ where $k=4\pi/\lambda R$, the point target response is

$$h(x) = \int A(u)B(x-u)\exp -j\frac{k}{2}[u^2-(x-u)^2] \cdot \exp j\alpha(u)du.$$

As for the image $|h|^2$ we can ignore the phase factor $\exp j\frac{1}{2}kx^2$; thus the response is

$$h(x) = \int \exp(-jkxu) \cdot \exp[j\alpha(u)]A(u)B(x-u)du. \tag{23}$$

A pesky detail appears because we are able to taper using A and/or B; except for additive noise considerations we can simply take $B=1$, and then (23) is in the exact form for which our resolution analysis applies. Alternatively, if the system achieves significant compression of the azimuth Doppler histories, then B is a broad function compared to h so that both sides of (23) are essentially zero for large x and $B(x-u)\cong B(u)$ for (the significant) small values of x. In any event the simple results mentioned at the beginning of Section IV apply with $\omega=kx$; specifically, (18) gives

$$\bar{\rho} = \frac{R\lambda}{4\pi}\sqrt{\Delta a}$$

where Δa is the rms value of α''. For example, if $\alpha(x)=(4\pi/\lambda)r(x)$ and μ is the rms value of r'', then

$$\bar{\rho} = \frac{R}{2}\sqrt{\frac{\lambda\mu}{\pi}}\ (ft) \tag{24}$$

which bears the expected relation to (19) and (22). Also, the previous observation about signal-to-noise ratio applies here by substantially the same derivation provided the tapering is obtained with $|K|$.

From the foregoing approximate analysis it should be clear that if the effects of phase errors on a Fourier transform are analyzed (which amounts to the physical aperture case), then by introducing appropriate scale factors the results apply to synthetic aperture systems (and other situations as well; e.g., pulse distortion due to dispersive propagation [9].

General Theory

It is analytically attractive to measure the shift in position due to phase errors by the first moment

$$\overline{\omega_\alpha} = \frac{\int \omega |F_m(\omega)|^2 d\omega}{\int |F_m(\omega)|^2 d\omega}.$$

If the second moment is

$$\overline{\omega_\alpha{}^2} = \frac{\int \omega^2 |F_m(\omega)|^2 d\omega}{\int |F_m(\omega)|^2 d\omega},$$

[14] It is assumed that α and n are stationary random processes. Also, the choices of W and K involve the assumption that the phase of K is matched to the phase of W in the absence of phase errors.

then the radius of gyration of the point target response (or beamwidth in the physical aperture case) is ρ_α where

$$\rho_\alpha{}^2 = \frac{\int (\omega - \overline{\omega_\alpha})^2 \,|\, F_m \,|^2 d\omega}{\int \,|\, F_m \,|^2 d\omega} = \overline{\omega_\alpha{}^2} - (\overline{\omega_\alpha})^2.$$

We could formulate higher-order central moments of $|F_m|^2$ and be led to formulas similar to those to be derived for resolution and displacement; however, the effort is greater and the rewards modest. The problem is to calculate the mean-square value of ρ_α and $\overline{\omega_\alpha}$; these quantities are depicted in Fig. 7.

Let $f_m = e^{j\alpha} f$, then since F_m is the Fourier transform of f_m, $|F_m|^2$ is the transform of C_{f_m} where

$$C_{f_m}(t) = \int f_m(t + \tau)\bar{f}_m(\tau) d\tau.$$

It is worthwhile to note that

$$E(\,|\, F_m \,|^2) = E \int e^{-j\omega t} \left\{ \int \exp\,[j\alpha(t + \tau) - j\alpha(\tau)] \cdot f(t + \tau)\bar{f}(\tau) d\tau \right\} dt$$

$$= \int e^{-j\omega t} \{ R_\gamma(t) C_f(t) \} dt$$

$$= \frac{1}{2\pi} S_\gamma(\omega) * \,|\, F(\omega) \,|^2 \tag{25}$$

where $S_\gamma(\omega)$ is the power density spectrum of $\gamma(t) = e^{j\alpha(t)}$.

Thus as a first-order analysis of the effects of phase errors, the "expected image" is the phase error free image blurred through convolution with the power density of $e^{j\alpha}$; if phase errors are of serious concern, $|F|^2$ will behave like a δ-function and the expected response is $S_\gamma(\omega)$. The shortcoming of this observation is that the expected response provides a pessimistic estimate of performance in that the spread of $E(|F_m|^2)$ can well be due primarily to positional shift in the response (first moment). On the other hand, for large ω, $S_\gamma(\omega)$ is indicative of average sidelobe energy.

Since $|F_m|^2$ is the transform of C_{f_m},

$$C_{f_m}(t) = \frac{1}{2\pi} \int e^{j\omega t} \,|\, F_m \,|^2 d\omega \tag{26}$$

$$C_{f_m}(0) = \frac{1}{2\pi} \int \,|\, F_m \,|^2 d\omega \tag{27}$$

$$C_{f_m}''(0) = -\frac{1}{2\pi} \int \omega^2 \,|\, F_m \,|^2 d\omega. \tag{28}$$

Hence

$$\overline{\omega_\alpha{}^2} = -\frac{C_{f_m}''(0)}{C_{f_m}(0)}\,.$$

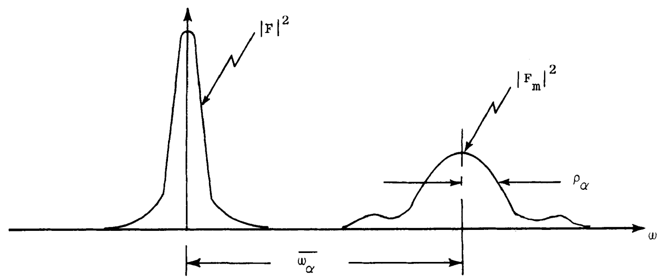

Fig. 7. Qualitative effects of phase errors.

It is easy to show that $-C_{f_m}''(t) = C_{f'_m}(t)$; also $C_{f_m}(0) = C_f(0)$. Thus we can take the expected value of $\overline{\omega_\alpha{}^2}$ (after expectation, the α subscript will be dropped):

$$C_f(0)\overline{\omega^2} = E[C_{f'_m}(0)]$$

$$f'_m = e^{j\alpha} f' + j\alpha' f e^{j\alpha}.$$

Let $E(\alpha') = 0$; then the cross terms in $E[C_{f'_m}]$ are zero and we get

$$C_f(0)\overline{\omega^2} = \int \,|\, f' \,|^2 + E[(\alpha')^2] \int \,|\, f \,|^2.$$

However, $\int |f'|^2 / C_f(0)$ is the second moment of the pattern in the absence of phase errors (which will be denoted $\overline{\omega_0{}^2}$), and $C_f(0) = \int |f|^2$; thus

$$\overline{\omega^2} = \overline{\omega_0{}^2} + E[(\alpha')^2]. \tag{29}$$

From (26) we also have

$$jC_f(0)\overline{\omega_\alpha} = C_{f_m}'(0)$$

$$C_{f_m}'(t) = \int f_m'(t + \tau)\bar{f}_m(\tau) d\tau$$

$$C_{f_m}'(0) = \int f'\bar{f} d\tau + j \int \alpha' \,|\, f \,|^2 d\tau.$$

Thus

$$C_f{}^2(0)(\overline{\omega_\alpha})^2 = \left[-jC_f'(0) + \int \alpha' \,|\, f \,|^2 d\tau \right]^2.$$

If $E(\alpha') = 0$, the cross terms again are zero; the first term squared is seen to give rise to the square of the first moment in the absence of phase errors, $(\overline{\omega_0})^2$. The term involving α' gives rise to

$$\iint R_{\alpha'}\,(\tau - \theta) \,|\, f(\tau) \,|^2 \,|\, f(\theta) \,|^2 d\tau d\theta = \frac{1}{8\pi^3} \int S_{\alpha'}\,(\omega) \,|\, C_F(\omega) \,|^2 d\omega$$

since $(1/2\pi)C_F$ is the transform of $|f|^2$, where F is the pattern in the absence of phase errors. The final result is

$$(\bar{\omega})^2 = E[(\omega_\alpha)^2]$$

$$= (\overline{\omega_0})^2 + \frac{1}{2\pi} \int S_{\alpha'}\,(\omega) \frac{|\, C_F(\omega) \,|^2}{|\, C_F(0) \,|^2} d\omega.$$

Finally, the mean-square resolution is

$$\rho^2 = \rho_0{}^2 + \frac{1}{2\pi}\int S_{\alpha'}(\omega)\left[1 - \frac{|C_F(\omega)|^2}{|C_F(0)|^2}\right]d\omega \qquad (30)$$

where

$$\rho_0{}^2 = \frac{\int \omega^2 |F|^2 d\omega}{\int |F|^2 d\omega}$$

if the phase error free pattern has zero first moment.

The problem of optimizing the system for resolution amounts to finding the phase error free pattern (F, or taper f) which minimizes ρ^2 of (30). This can be done (with numerical analysis) for any specific $S_{\alpha'}$, but seems to require considerable effort.

A worthwhile general result is available; namely, if we approximate $|C_F|^2$ for small ω by the first three terms of its Taylor series (which provides a good approximation if $S_{\alpha'}$ is concentrated at low frequencies), then the optimum $f(t)$ is found to be Gaussian

$$f(t) = \exp\left(-\tfrac{1}{2}ct^2\right)$$

and the optimum c is found to be $c^2 = E[(\alpha'')^2]$. In this case $\rho \cong \sqrt{\Delta a}$ where Δa is the rms value of α''. Of course this is in pleasant agreement with the simple analysis which resulted in (18).

What is more, for arbitrary $S_{\alpha'}$ (not necessarily concentrated at low frequencies), the said Gaussian taper renders $\rho \leq \sqrt{\Delta a}$. In addition since $\rho^2 \leq \overline{\omega^2}$, (29) easily indicates that ρ can be made less than Δv where Δv is the rms value of α'. Thus as a rather deep result we have that for arbitrary $S_{\alpha'}$, the obtainable rms resolution satisfies

$$\rho \leq \min\left(\Delta v, \sqrt{\Delta a}\right). \qquad (31)$$

In summary, (30) can be used to calculate the average resolution with any phase error spectrum and taper, f. More significantly, (30) permits the selection of the optimum taper in the presence of phase errors, and (31) provides a quick estimate (often sharp) of the minimum obtainable rms resolution. Finally, the power spectrum of $e^{j\alpha}$ provides a rough estimate of remote sidelobe energy levels.

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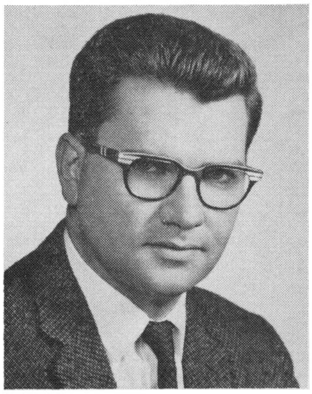

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