15.3 Double Integrals Over General Regions

1. Introduction on double integrals on any general region.
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CO-4 Apply the concept of double and triple integration to evaluate the integral, to find the moment of inertia of lamina and surface area.

Introduction on double integrals over regions D of general shape—

For double integrals, our objective is to evaluate it over general regions, not just the integration over rectangles.

We suppose that D is a bounded region, which means that D can be enclosed in rectangular region R as illustrated in fig:-1

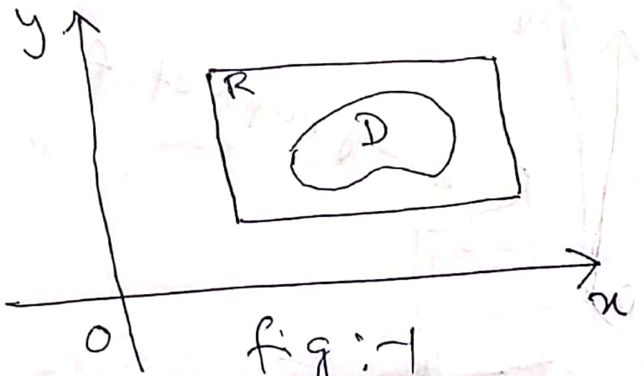


fig:-1

Now, we define a new function $F(x,y)$ with domain R by

$$F(x,y) = \begin{cases} f(x,y), & (x,y) \in D \\ 0, & (x,y) \in R \text{ but not in } D \end{cases} \quad \text{---(1)}$$

If F is integrable over R, then we define the double integral of f over D by

$$\iint_D f(x,y)\, dA = \iint_R F(x,y)\, dA \,,$$ where

$F(x,y)$ is given in eq(1). This means that it does not matter what rectangle R we use as long as it contains the domain D. Again, where $f(x,y) \geq 0$, $\iint_D f(x,y)\, dA$ is the volume of the solid that lies above D and under the surface $z = f(x,y)$ as shown in the following figure.

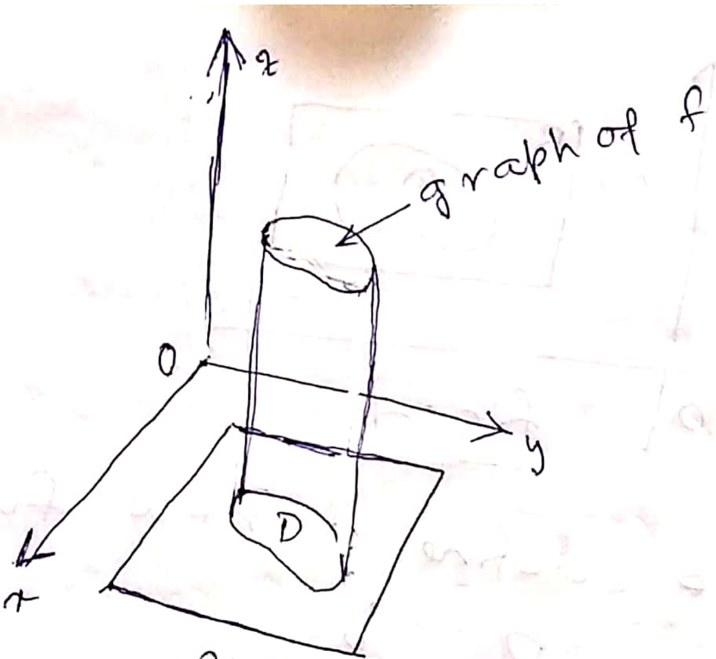


fig:-2

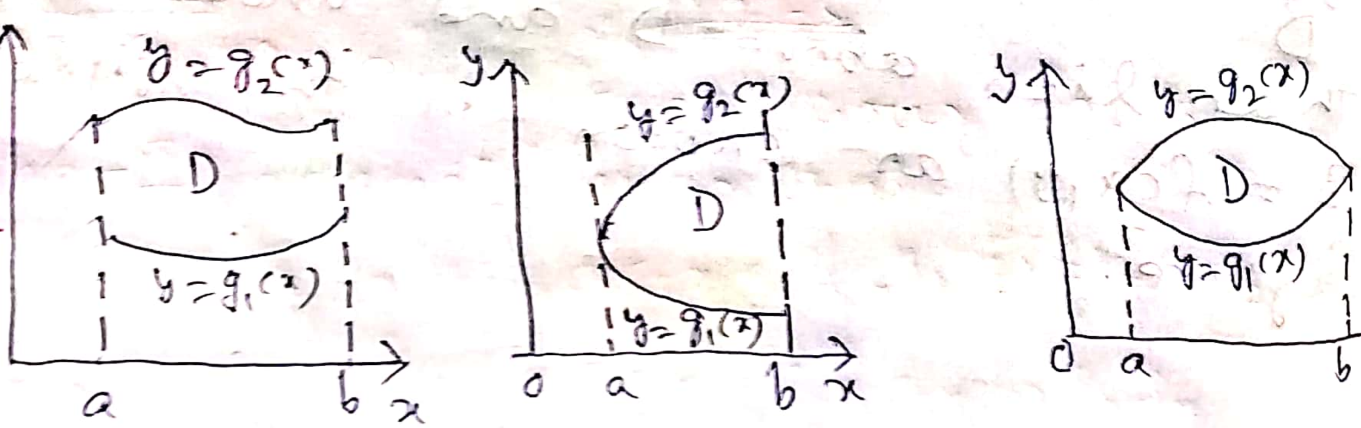
If f is continuous on D, then $\iint_R F(x,y)\,dA$ exists, therefore, $\iint_D f(x,y)\,dA$ exists. Now, this discussion leads to the following two types of regions.

Type:- I A plane region D is said to be of type-I if it lies between the graphs of two continuous fu^{n}s of x, that is,

$$D = \{ (x,y) \mid a \le x \le b,\ g_1(x) \le y \le g_2(x) \}$$

where g_1 and g_2 are continuous on $[a,b]$. Some examples of type-I regions are shown below.

fig-3



In order to evaluate $\iint_D f(x,y)\,dA$ when D is a region of type-I, we choose a rectangle $R=[a,b]\times[c,d]$ that contains D, as shown in fig-4. Here, under the assumption that F agrees with f on D and is 0 outside D. Then, by Fubini's theorem,

$$\iint_D f(x,y)\,dA = \iint_R F(x,y)\,dA = \int_a^b \int_c^d F(x,y)\,dy\,dx$$

Here, we observe that $F(x,y)=0$ if $y<g_1(x)$ or $y>g_2(x)$ because (x,y) then lies outside D. Therefore,

$$\int_c^d F(x,y)\,dy = \int_{g_1(x)}^{g_2(x)} F(x,y)\,dy = \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy$$

as $F(x,y)=f(x,y)$ when $g_1(x) \le y \le g_2(x)$.

This type-1 region is illustrated in fig:-4.

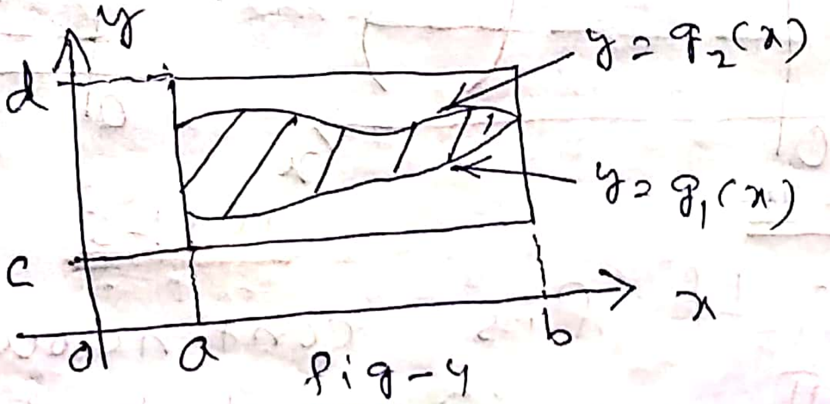


Fig-4

Def^n:- If f is continuous on a type-1 D such that

$$D = \{ (x,y) \mid a \le x \le b,\ g_1(x) \le y \le g_2(x) \}$$

then $$\iint_D f(x,y)\, dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\, dy\, dx$$

Note :- In the inner integral we regard x as being constant not only in $f(x,y)$ but also in the limits of integration, $g_1(x)$ and $g_2(x)$.

Similarly, the type-II region for the double integration is presented in fig-5.

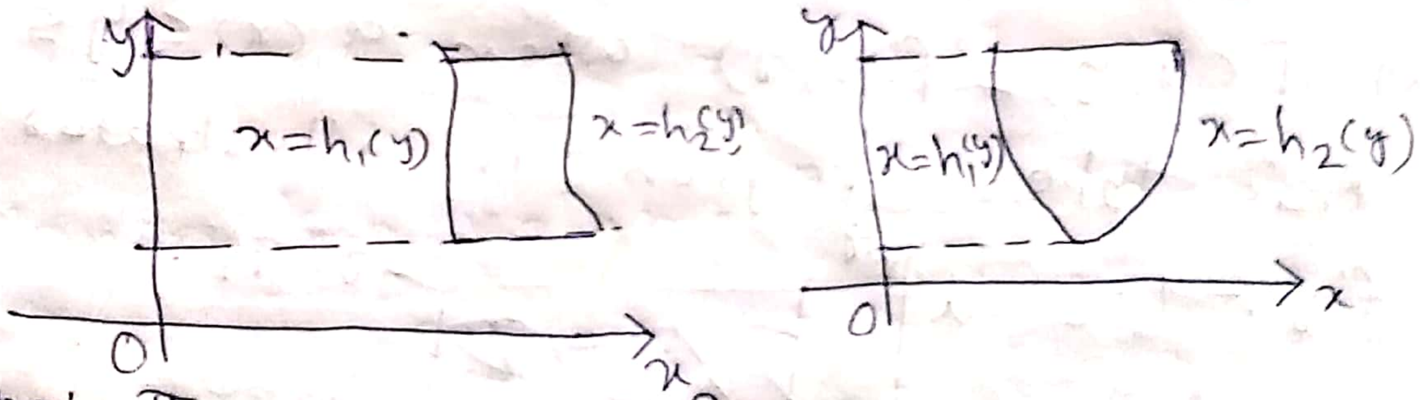


fig:-5

Type:- II

Def^n:- If f is continuous on a type-II region D such that

$$D=\{(x,y) \mid c\le y\le d,\ h_1(y)\le x\le h_2(y)\}$$

then $\iint\limits_D f(x,y)\,dA=\int_c^d\int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy$

where D is a type-II region.

<u>Example</u> Evaluate $\iint_D (x+2y)\,dA$, where D is the region bounded by the parabolas $y = 2x^2$ and $y = 1+x^2$.

Sol^n :- The two parabolas intersect when $2x^2 = 1+x^2 \Rightarrow x^2 = 1 \Rightarrow x = \pm 1$. It is a type-I region hence

$$D = \{(x,y) \mid -1 \le x \le 1,\ 2x^2 \le y \le 1+x^2\},$$

$y = 2x^2$ is the lower boundary and $y = 1+x^2$ is the upper boundary

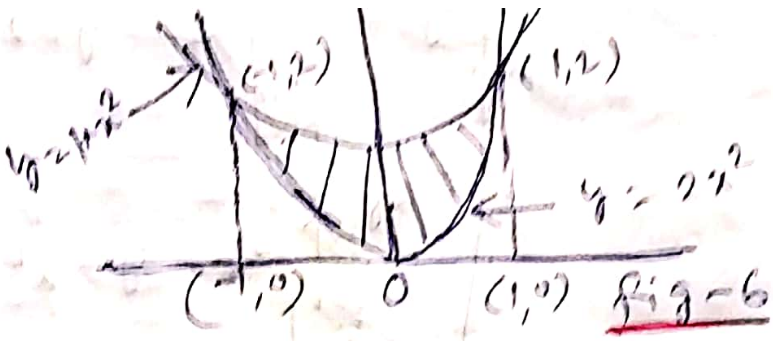


fig-6

as shown in fig-6.

$$\iint_D (x+2y)\, dA = \int_{-1}^{1}\int_{2x^2}^{1+x^2} (x+2y)\, dy\, dx$$

$$= \int_{-1}^{1} \Big[xy + y^2 \Big]_{y=2x^2}^{y=1+x^2} dx$$

$$= \int_{-1}^{1} \Big[x(1+x^2) + (1+x^2)^2 - x(2x^2) - (2x^2)^2 \Big] dx$$

$$= \int_{-1}^{1} (-3x^4 - x^3 + 2x^2 + x + 1)\, dx$$

$$= -\frac{3x^5}{5} - \frac{x^4}{4} + \frac{2x^3}{3} + \frac{x^2}{2} + x \Big]_{-1}^{1} = \frac{32}{15}$$

Example: Find the volume of the solid that lies under the paraboloid $z = x^2 + y^2$ and above the region D in the xy-plane bounded by the the line $y = 2x$ and the parabola $y = x^2$.

Soln :- type :-I region :-

$$D = \{ (x,y) \mid 0 \le x \le 2,\ x^2 \le y \le 2x \}$$

where the domain is illustrated in fig - 7.

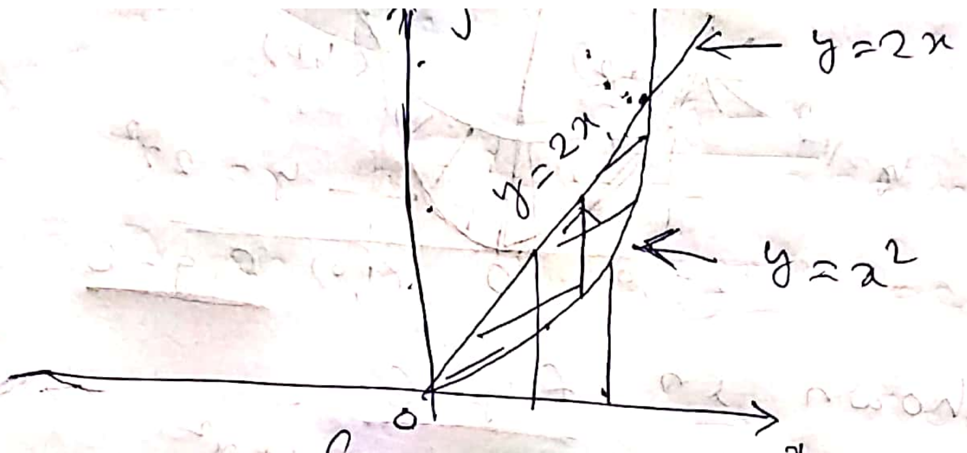


fig - 7

Hence the volume under $z=x^2+y^2$ and above D is

$$V=\iint_D (x^2+y^2)\,dA=\int_0^2\int_{x^2}^{2x}(x^2+y^2)\,dy\,dx$$

$$=\int_0^2\left[x^2y+\frac{y^3}{3}\right]_{y=x^2}^{y=2x}dx$$

$$= \int_0^2 \left[x^2 \times 2x + \frac{(2x)^3}{3} - x^4 - \frac{x^6}{3} \right] dx .$$

$$= \int_0^2 \left(-\frac{x^6}{3} - x^4 - \frac{14x^3}{3} \right) dx$$

$$= -\frac{x^7}{21} - \frac{x^5}{5} + \frac{7x^4}{6} \Big]_0^2 = \frac{216}{35} .$$

type-II region :

$$D = \{ (x, y) \mid 0 \leq y \leq 4, \ \frac{y}{2} \leq x \leq \sqrt{y} \}$$

where, the domain is illustrated in

fig - 8

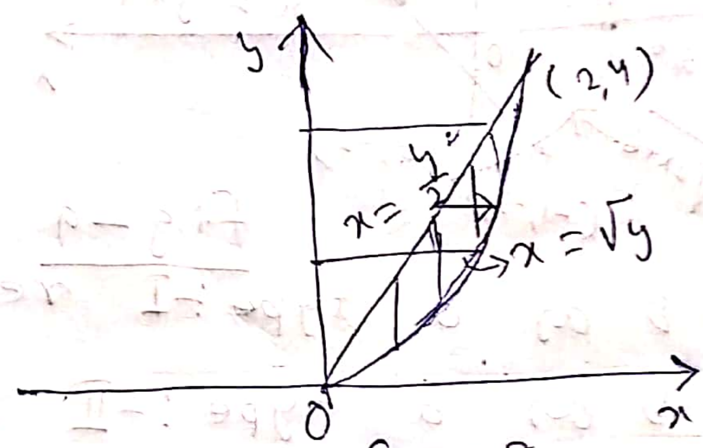


fig :- 8

Hence the volume under $z = x^2 + y^2$ and above D is

$$V = \iint_D (x^2 + y^2)\, dA = \int_0^4 \int_{\frac{y}{2}}^{\sqrt{y}} (x^2 + y^2)\, dx\, dy$$

$$= \int_0^4 \left[\frac{x^3}{3} + y^2 x \right]_{x=\frac{y}{2}}^{x=\sqrt{y}} dy$$

$$= \int_0^4 \left(\frac{y^{3/2}}{3} + y^{5/2} - \frac{y^3}{24} - \frac{y^3}{2} \right) dy$$

$$= \frac{216}{35}.$$