Department of Computer Science and Engineering
Indian Institute of Technology Palakkad
CS 2030: Data Structures and Algorithms
Problem Set 4

18 Sep 2018

1. Recall that the degree $d_G(v)$ of a vertex v in an undirected graph G is the number of vertices that are adjacent to v in G. Show that $\sum_{v \in V(G)} d_G(v) = 2|E(G)|$.

2. Suppose there is a football tournament contested by 15 teams. Is it possible to design the matches so that all teams have played exactly 5 matches at some point? Justify your answer.

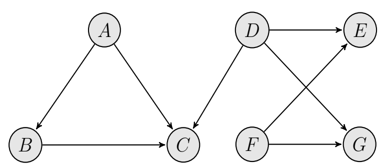
3. Show that in any group of six people, there are at least three of them who know each other mutually or who don't know each other mutually. Does this claim hold in any group of 5 people? Assume that "knowing" is a symmetric relation.

4. Show that in any group of n people, there must be at least two who know the same number of people among them. Do they also know the same set of people among them?

5. Given an adjacency-list representation of a directed graph, how long does it take to compute the out-degree of every vertex? How long does it take to compute the in-degrees?

6. A universal sink in a directed graph G is a vertex with in-degree $|V(G)| - 1$ and out-degree 0. Show that determining whether a directed graph G contains a universal sink can be determined in $\mathcal{O}(V)$ time, given an adjacency matrix for G.

7. Write the adjacency matrix for the below directed acyclic graph (DAG).



8. The multiplication of two adjacency matrices is similar to the normal matrix multiplication except that the multiplication is equivalent logical AND and addition is equivalent to logical OR. Using the adjacency matrix A found in the previous question, find A^2. What does an entry in the matrix A^2 signify? What does an entry in the matrix A^k signify?

9. Let B be the incidence matrix of a directed graph G. What does an entry in the matrix BB^T signify?

10. An undirected graph is a tree if it is connected and acyclic. Show that any tree on n vertices has exactly $n - 1$ edges.

11. Show that there is a unique path between any two vertices of a tree.

12. Design a linear-time algorithm which, given an undirected graph G and an edge $e \in E(G)$, determines whether G has a cycle containing e or not.

13. A triangle is a set of 3 vertices that are pairwise adjacent. Design a linear-time algorithm which, given an undirected graph G, determines whether G has a triangle or not.

14. Recall that the edges of a directed graph can be classified into four types: tree edge, back edge, forward edge, cross edge. Similarly, the edges of an undirected graph can be classified into two types: tree edge and non-tree edge (or back edge). Show that a graph has a cycle if and only if its DFS reveals a back edge.

15. Recall that a topological ordering is an ordering of the vertices of a directed graph G such that if (u, v) is an edge in G, then u appears before v in the ordering. Show that a directed graph has a topological ordering if and only if it is a DAG.

16. A sink in a directed graph is a vertex with out-degree 0 and a source in a directed graph is a vertex with in-degree 0. Show that every DAG has at least one source and at least one sink.

17. Let C_1 and C_2 be two strongly connected components of a directed graph G. Show that if there is an edge $(u, v) \in E(G)$ with $u \in C_1$ and $v \in C_2$, then there is no edge (x, y) in G with $x \in C_2$ and $y \in C_1$.

18. Give a linear-time algorithm for computing the reverse of a graph in adjacency list format.