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## Algorithms of Numerical Linear Algebra Assignment 4

Please submit your solution to StudOn until Wednesday, December 14, 2022, 23:59.

### Exercise 1 (Uniqueness of QR Factorization)

5P.

Let  $A \in \mathbb{C}^{m \times m}$  be nonsingular. Assume that there exists a QR factorization  $A = QR$  where  $Q$  is unitary and  $R$  is upper triangular. Prove that the QR factorization is unique when demanding  $r_{jj} > 0$  for all  $j$ .

Note: This means that both  $R$  and  $Q$  are uniquely defined.

Hint: You may use Theorem 23.1. from [1]

### Exercise 2 (LU decomposition of Banded Matrices)

6P.

Prove the following proposition: Let  $A \in \mathbb{C}^{m \times m}$  be a non-singular banded matrix with bandwidth  $2p + 1$ , i.e.,  $a_{ij} = 0$  for  $|i - j| > p$ . Furthermore, let  $A = LU$  be a LU-factorization without pivoting. Then, in addition to being triangular,  $L$  and  $U$  also have bandwidth  $2p + 1$ .

Hint: Show that the statement holds for the first row of  $U$  and use induction.

### Exercise 3 (Gaussian Elimination with Complete Pivoting)

9P.

**Make sure to follow the requirements for programming tasks stated on the information sheet!**

- Implement a Python function `L, U = lu(A)`, which implements gaussian elimination without pivoting, i.e.,  $LU = A$ . The input value is a matrix  $A \in \mathbb{R}^{m \times m}$ , the output values are triangular matrices  $L, U \in \mathbb{R}^{m \times m}$ .
- Implement a Python function `(i, j) = maxabs_idx(A)`, which returns the row and column indices of the maximum absolute element of  $A \in \mathbb{R}^{m \times n}$ , i.e.,  $i, j = \arg \max_{k, \ell} |a_{k, \ell}|$ .
- Implement a Python function `P, Q, L, U = lu_complete(A)`, which implements gaussian elimination with complete pivoting, i.e.  $LU = PAQ$ . The output values are  $\mathbb{R}^{m \times m}$  matrices  $L, U$  triangular and  $P, Q$  unitary.
- (optional) Test both functions with random input matrices `np.random.rand(100, 100)` and compare the error in the max-norm, e.g., `np.max(np.abs(L@U - A))`.

## References

- [1] L.N. Trefethen and D. Bau. *Numerical Linear Algebra*. Society for Industrial and Applied Mathematics, 1997.