ME516S18 Equation Sheet

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1.15.18 - 5.4.18

A summary of select chapters of *Principles of Dynamics (2 Ed.)* by Donald T. Greenwood.

Module 1 (1.15-1.19): Chapter 1

Auxiliary:

Derivative of a vector

Considering a *vector function* of a scalar variable (magnitude and direction are dependent upon the value of scalar); the derivative of the vector $\mathbf{A}$ with respect to scalar u is defined by limit,

$$\frac{d\mathbf{A}}{du} = \lim_{\Delta u \to 0} \frac{\Delta \mathbf{A}}{\Delta u} \tag{1}$$

Derivative of vector sum $\mathbf{A} + \mathbf{B}$,

$$\Delta(\mathbf{A} + \mathbf{B}) = \Delta\mathbf{A} + \Delta\mathbf{B} \tag{2}$$

Vector differentiation is *distributive,*

$$\frac{d(\mathbf{A} + \mathbf{B})}{du} = \frac{d\mathbf{A}}{du} + \frac{d\mathbf{B}}{du} \tag{3}$$

And the following can be shown,

$$\frac{d}{du}(g\mathbf{A}) = \frac{dg}{du}\mathbf{A} + g\frac{d\mathbf{A}}{du} \tag{4}$$

$$\frac{d}{du}(\mathbf{A} \cdot \mathbf{B}) = \frac{d\mathbf{A}}{du} \cdot \mathbf{B} + \mathbf{A} \cdot \frac{d\mathbf{B}}{du} \tag{5}$$

$$\frac{d}{du}(\mathbf{A}\times\mathbf{B}) = \frac{d\mathbf{A}}{du}\times\mathbf{B} + \mathbf{A}\times\frac{d\mathbf{B}}{du} \tag{6}$$

Newton's Law of Motion:

1. Every body continues in its state of rest, or of uniform motion in a straight line, unless compelled to change that state by forces acting upon it.
2. The time rate of change of linear momentum of a bod is proportional to the force acting upon it and occurs in the direction in which the force acts.
3. To every action there is an equal and opposite reaction; that is, the mutual forces of two bodies acting upon each other are equal in magnitude and opposite in direction.

Laws of Motion for a Particle:

Stating **three** basic laws applying to the motion of a particle, the first being the *law of motion* which summarizes Newton's first two laws as,

$$\mathbf{F} = k\frac{d}{dt}(m\mathbf{v}) = km\mathbf{a} \tag{7}$$

where $m\mathbf{v}$ is *linear momentum*, $\mathbf{p} = m\mathbf{v}$. Choosing units such that $k = 1$,

$$\mathbf{F} = m\mathbf{a} \tag{8}$$

The second basic law expresses *collinearity* of the interaction of forces as the *law of action and reaction:*

> Two particles exert interaction forces on one another that are equal in magnitude, opposite in sense, and directed along straight line joining particles.

Finally the third basic law, *law of addition of forces*,

$$\mathbf{F} = \mathbf{P} + \mathbf{Q} \tag{9}$$

Frames of Reference:

Frames of reference are **always** defined as *inertial (Newtonian)* reference frames consisting of rigid coordinate axes such that particle motion is described by Newton's laws of motion, unless otherwise mentioned.
A relative reference frame, B, can be established that is *only translating* with respect to the original reference,A, at a constant velocity $\mathbf{v}_{B/A}$. Denoting particle velocity by observers on A and B as $\mathbf{v}_A$ and $\mathbf{v}_B$ respectively,

$$\mathbf{v}_A = \mathbf{v}_B + \mathbf{v}_{B/A} \tag{10}$$

Where upon differentiating,

$$\mathbf{a}_A = \mathbf{a}_B \tag{11}$$

It is seen that from the Newtonian point of view, forces, masses, and accelerations are seen as equal by both observes. This is referred to as the *Newtonian principle of relativity.*

D'Alemberts's Principle:

Writing Newton's law of motion for a particle in the form,

$$\mathbf{F} - m\mathbf{a} = 0 \tag{12}$$

effectively transforms the dynamically stated problem to a statically defined problem. Therefore, d'Alembert's principle simply states,

> The laws of static equilibrium apply to a dynamical system if the inertia forces, as well as the actual external forces, are considered as the forces acting on the system.

Considerable care must be taken in use of d'Alembert's principle, as the inertia forces *should not* be confused with external forces comprising total force, $\mathbf{F}$, that is applied to the particle. External forces are considered as contact and gravitational or other field forces applied to the particle.

Module 2 (1.22-2.2): Chapter 2

Kinematics:

Kinematics is the study of particles and rigid bodies, disregarding forces associated with motion. It is a purely mathematical study and does not consider physical laws. All motion is considered to be relative and no reference frame is more fundamental or absolute than another.

The position of particle P in the XYZ coordinate system of figure 1 is given by **r** drawn from the origin O to P. If P moves along curve C then the velocity **v**is in the direction tangent to the curve and has a magnitude equal to the speed. Considering velocity as a free vector, its origin can be imagined to be drawn from O at each instance of time, with the path C' describing the tip of the velocity **v**is known as the hodograph. So the velocity of the tip of **v**along the hodograph C' is the *acceleration* of P relative to the XYZ system. In summary,

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} \tag{13}$$

and the acceleration of P is given as,

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{d^2\mathbf{r}}{dt^2} \tag{14}$$

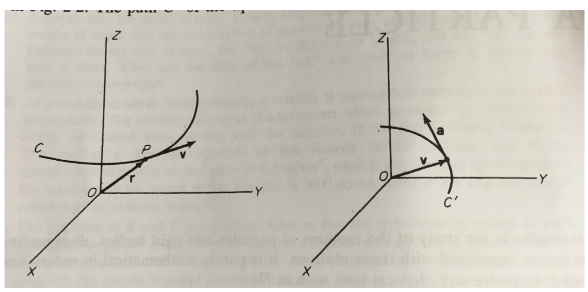

Figure 1: (Left) Position and velocity vectors of particle P as it moves along C. (Right) Hodograph showing velocity and acceleration.

Angular Velocity:

In kinematics the primary concern is the translation of a point relative to a given reference frame, that is, with the position vector of the point and with its derivatives with respect to time. However, *general* motion also must consider changes in *orientation (rotation)* and it the rate of change of orientation that is expressed by means of the angular velocity.

Consider the motion of rigid body shown in figure 2 during infinitesimal time Δt. We note that the body has undergone both a translation of amount Δs and rotation of $\Delta\theta$, the order of translation and rotation is immaterial. The infinitesimal angular displacement $\Delta\theta$ is a vector whose magnitude is equal to the angle of rotation and whose direction is along an axis determined by those points not displaced by the infinitesimal rotation using the right-hand rule. The angular velocity, ω, therefore is defined as,

$$\omega = \lim_{\Delta t \to 0} \frac{\Delta\theta}{\Delta t} \tag{15}$$

The angular velocity is a free vector and will produce different results based upon the reference frame. It is dependent upon the body only and not of the base point.

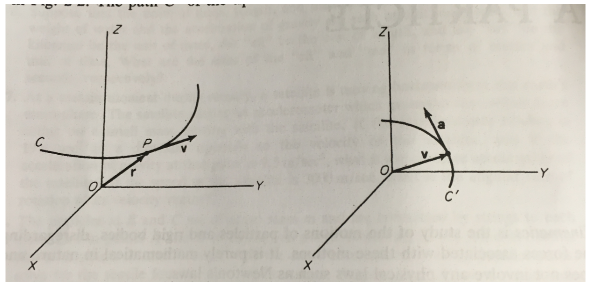

Figure 2: Motion of rigid body during infinitesimal time Δt

Rigid Body Motion About a Fixed Point:

In figure 3, the base point of the rigid body is assumed to be fixed at the origin O of the cartesian system XYZ. In calculation of the velocity of a point P that is fixed in the body using an inertial reference frame, $\mathbf{v}$is the absolute velocity of point P and $\omega$is the absolute angular velocity of the body. Rotation is occurring about the *instantaneous axis of rotation*, with the point P having speed,

$$\dot{s} = \omega r sin\theta \tag{16}$$

The velocity of P is of magnitude $\dot{s}$ and is directed along the tangent of the path. Therefore it is seen that for the general case of rigid-body rotation about a fixed point that,

$$\mathbf{v} = \omega \times \mathbf{r} \tag{17}$$

Differentiating with respect to time, the acceleration of P is given as,

$$\mathbf{a} = \omega \times \dot{\mathbf{r}} + \dot{\omega} \times \mathbf{r} \tag{18}$$

or noting that $\dot{\mathbf{r}} = \mathbf{v}$, the form,

$$\mathbf{a} = \omega \times (\omega \times \mathbf{r}) + \dot{\omega} \times \mathbf{r} \tag{19}$$

Where from the above, the term $\omega \times (\omega \times \mathbf{r})$ is the *centripetal acceleration* (directed towards the instantaneous axis of rotation) with the remaining terms being the *tangential acceleration.*

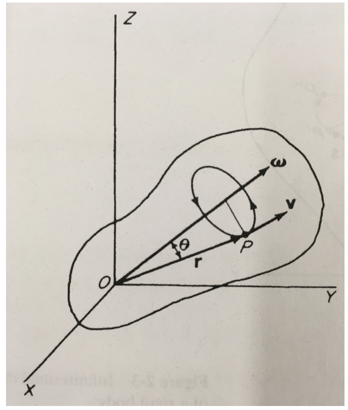

Figure 3: Rigid body motion about a fixed point.

Time Derivative of a Unit Vector:

As was the case of determining the derivative with respect to time of the position vector **r**in the previous section (magnitude is constant but direction changes), the time derivative of unit vectors are calculated similarly,

$$\dot{\mathbf{e_1}} = \omega \times \mathbf{e_1}\,; \dot{\mathbf{e_2}} = \omega \times \mathbf{e_2}\,; \dot{\mathbf{e_3}} = \omega \times \mathbf{e_3} \tag{20}$$

In the case where the unit vectors form an orthogonal system, vector multiplication is possible through the determinate expression, and writing the angular velocity in the form,

$$\omega = \omega_1 \mathbf{e_1} + \omega_2 \mathbf{e_2} + \omega_3 \mathbf{e_3} \tag{21}$$

obtains,

$$\dot{\mathbf{e_1}} = \begin{vmatrix} \mathbf{e_1} & \mathbf{e_2} & \mathbf{e_3} \\ \omega_1 & \omega_2 & \omega_3 \\ 1 & 0 & 0 \end{vmatrix} = \omega_3 \mathbf{e_2} - \omega_2 \mathbf{e_3} \tag{22}$$

Velocity and Acceleration of a Particle in Several Coordinate Systems:

Cartesian Coordinates

The position of particle P relative to the origin O of the system XYZ is shown in figure 4 is given by,

$$\mathbf{r}=\mathrm{x}\mathbf{i}+\mathrm{y}\mathbf{j}+\mathrm{z}\mathbf{k} \tag{23}$$

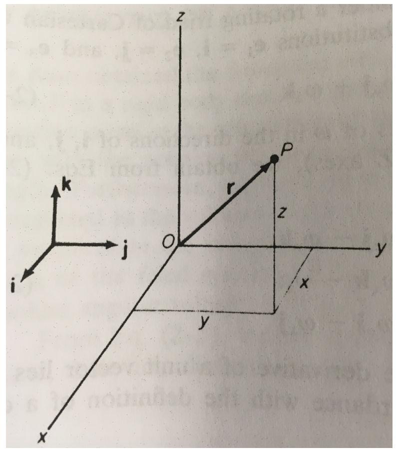

Figure 4: Cartesian coordinates and unit vectors.

The velocity is therefore given as,

$$\mathbf{v} = \dot{\mathbf{r}} = \dot{x}\mathbf{i} + \dot{y}\mathbf{j} + \dot{z}\mathbf{k} \tag{24}$$

and the corresponding acceleration,

$$\mathbf{a} = \ddot{x}\mathbf{i} + \ddot{y}\mathbf{j} + \ddot{z}\mathbf{k} \tag{25}$$

Considering the system XYZ to an inertial frame, the velocity and acceleration are *absolute*.

Cylindrical Coordinates

Now considering the case where cylindrical coordinates are used to express the XYZ system in terms of r, ϕ, z, as shown in figure 5, the position vector is defined as,

$$\text{r} = \text{r}\mathbf{e_r} + z\mathbf{e_z} \tag{26}$$

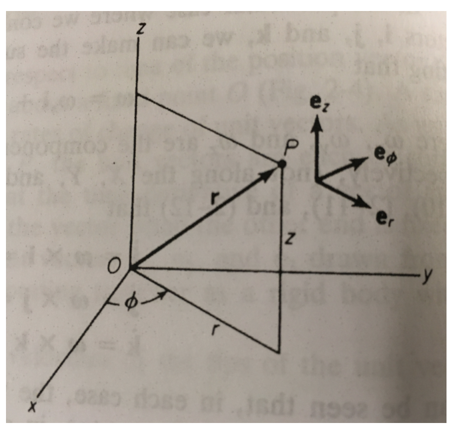

Figure 5: Cylindrical coordinates and unit vectors.

The velocity is found as,

$$\mathbf{v} = \dot{\mathbf{r}} = \dot{r}\mathbf{e_r} + \dot{z}\mathbf{e_z} + r\dot{\mathbf{e_r}} + z\dot{\mathbf{e_z}} \tag{27}$$

It is seen that $\mathbf{e_r}$ and $\mathbf{e}_\phi$ change direction as P moves through a general displacement, but $\mathbf{e_z}$ remains parallel to the z axis. Further, changes in direction of $\mathbf{e_r}$ and $\mathbf{e}_\phi$ are due solely to changes in ϕ, corresponding to rotations about the z axis. Therefore, the *absolute* rotation rate of the unit vector system is,

$$\omega = \dot{\phi}\mathbf{e_z} \tag{28}$$

and its shown that,

$$\begin{aligned} \dot{\mathbf{e_r}} &= \omega \times \mathbf{e_r} = \dot{\phi}\mathbf{e}_\phi \\ \dot{\mathbf{e}}_\phi &= \omega \times \mathbf{e}_\phi = -\dot{\phi}\mathbf{e}_r \\ \dot{\mathbf{e}}_z &= \omega \times \mathbf{e}_z = 0 \end{aligned} \tag{29}$$

In which the velocity is found as,

$$\mathbf{v} = \dot{r}\mathbf{e}_r + r\dot{\phi}\mathbf{e}_\phi + \dot{z}\mathbf{e}_z \tag{30}$$

Differentiating with respect to time provides the acceleration,

$$\mathbf{a} = \ddot{\mathbf{r}} = \ddot{r}\mathbf{e}_r + (\dot{r}\dot{\phi} + r\ddot{\phi})\mathbf{e}_\phi + \ddot{z}\mathbf{e}_z + \dot{r}\dot{\mathbf{e}}_r + r\dot{\phi}\dot{\mathbf{e}}_\phi + \dot{z}\dot{\mathbf{e}}_z \tag{31}$$

which can be simplified using the unit vector derivatives as,

$$\mathbf{a} = (\ddot{r} - r\dot{\phi}^2)\mathbf{e_r} + (r\ddot{\phi} + 2\dot{r}\dot{\phi})\mathbf{e}_\phi + \ddot{z}\mathbf{e}_z \tag{32}$$

Spherical Coordinates

Finally, considering the case where the XYZ system is cast into spherical coordinates, r, ϕ, θ, as shown in figure 6, such that $\mathbf{r} = r\mathbf{e_r}$. The velocity of $\mathbf{r}$is found as,

$$\mathbf{v} = \dot{\mathbf{r}} = \dot{r}\mathbf{e_r} + r\dot{\mathbf{e_r}} \tag{33}$$

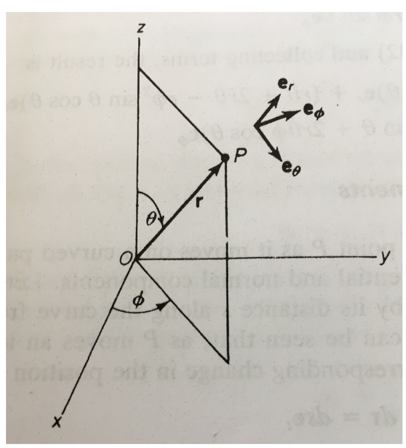

Figure 6: Spherical coordinates and unit vectors.

In order to determine the derivatives of the unit vectors, i.e. $\dot{\mathbf{e_r}}$, $\dot{\mathbf{e}}_\theta$, $\dot{\mathbf{e}}_\phi$, its important to correlate changes in θ, ϕ. From figure 6 its seen that increases in θ causes rotation about an axis parallel to $\mathbf{e}_\phi$, while changes in ϕ expresses a rotation about the z axis. The total rotation therefore is,

$$\omega = \dot{\theta}\mathbf{e}_\phi + \dot{\phi}\mathbf{e}_z \tag{34}$$

From figure 6, it is seen that $\mathbf{e}_z$ can be expressed as a combination of the other unit vectors as,

$$\mathbf{e}_z = cos\theta\mathbf{e}_r - sin\theta\mathbf{e}_\theta \tag{35}$$

and along with previous equation, the angular velocity is expressed as,

$$\omega = \dot{\phi}cos\theta\mathbf{e}_r - \dot{\phi}sin\theta\mathbf{e}_\theta + \dot{\phi}\mathbf{e}_\phi \tag{36}$$

Taking advantage of orthogonality,

$$\dot{\mathbf{e}}_r = \omega \times \mathbf{e}_r = \begin{vmatrix} \mathbf{e}_r & \mathbf{e}_\theta & \mathbf{e}_\phi \\ \dot{\phi}cos\theta & -\dot{\phi}sin\theta & \dot{\theta} \\ 1 & 0 & 0 \end{vmatrix} = \dot{\theta}\mathbf{e}_\theta + \dot{\phi}sin\theta\mathbf{e}_\phi \tag{37}$$

Finally, the velocity and acceleration are found as,

$$\mathbf{v} = \dot{r}\mathbf{e}_r + r\dot{\theta}\mathbf{e}_\theta + r\dot{\phi}sin\theta\mathbf{e}_\phi \tag{38}$$

$$\mathbf{a} = (\ddot{r} - r\dot{\theta}^2 - r\dot{\phi}^2 sin^2\theta)\mathbf{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta} - r\dot{\phi}^2 sin\theta cos\theta)\mathbf{e}_\theta ...$$
$$... + (r\ddot{\phi}sin\theta + 2\dot{r}\dot{\phi}sin\theta + 2r\dot{\theta}\dot{\phi}cos\theta)\mathbf{e}_\phi \tag{39}$$

Tangential and Normal Components:

Both the velocity and acceleration of a point moving along a curve can be expressed in terms of *tangential* and *normal* components. From figures **??**, **??** and **??** its seen as P moves along s a distance ds, the position vector changes as,

$$d\mathbf{r} = ds\mathbf{e}_t; \;\therefore\; \mathbf{e}_t = \frac{d\mathbf{r}}{ds} \tag{40}$$

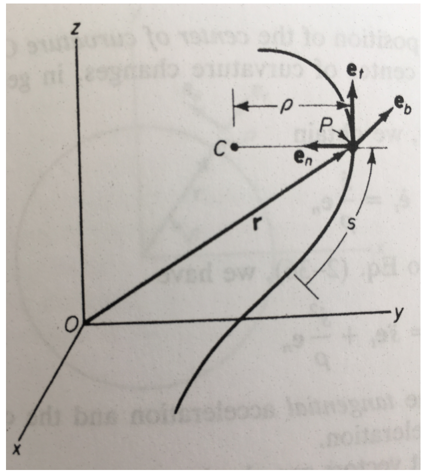

Figure 7: Tangential, normal, and binormal unit vectors along a curve.

Taking the derivative provides the velocity as,

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \dot{s}\frac{d\mathbf{r}}{ds} = \dot{s}\mathbf{e}_t \tag{41}$$

with the acceleration,

$$\mathbf{a} = \dot{\mathbf{v}} = \ddot{s}\mathbf{e}_t + \dot{s}\dot{\mathbf{e}}_t \tag{42}$$

Again from figures **??**, **??** and **??**, its noted that any changes in $\mathbf{e}_t$ are related to increasing s. Since the magnitude of $\mathbf{e}_t$ is constant, the direction of $d\mathbf{e}_t/ds$ is *normal* to $\mathbf{e}_t$, and lies in the plane of both $\mathbf{e}_t$ and $\mathbf{e}_n$ termed the *osculating plane*. The magnitude of $\dot{\mathbf{e}}_t$ is equal to its rotation rate, ω_b, about an axis perpendicular to the osculating plane, $\mathbf{e}_b$ (binormal axis) such as,

$$\dot{\mathbf{e}}_t = \dot{s}\frac{d\mathbf{e}_t}{ds} = \omega_b\mathbf{e}_n;\ \omega_b = \frac{\dot{s}}{\rho} \tag{43}$$

Substituting the expression for the rotation rate into the above equation for the definition of the acceleration,

$$\mathbf{a} = \ddot{s}\mathbf{e}_t + \frac{\dot{s}^2}{\rho}\mathbf{e}_n \tag{44}$$

where the first is component is the *tangential* acceleration and the second is the *centripetal* acceleration.

Simple Motions of a Point:

Circular Motion

Considering simple motion as shown in figure 8 with respect to polar coordinates (r, θ) the position vector is given as,

$$\mathbf{r} = r\mathbf{e}_r \tag{45}$$

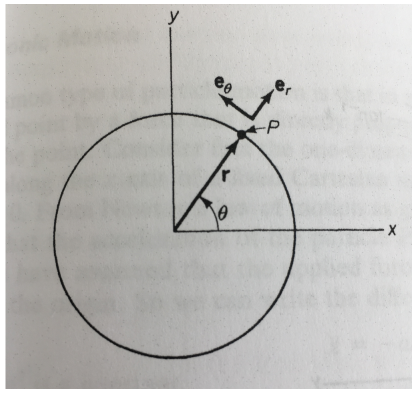

Figure 8: Circular motion of a point.

While the velocity and speed along the path are found to be,

$$\mathbf{v} = \dot{\mathbf{r}} = r\dot{\mathbf{e}}_r \tag{46}$$

$$v = r\dot{\theta} = r\omega \tag{47}$$

respectively. The acceleration is found by taking the derivative with respect to time,

$$\mathbf{a} = \dot{\mathbf{v}} = r\dot{\omega}\mathbf{e}_\theta + r\omega\dot{\mathbf{e}}_\theta \tag{48}$$

Since $\dot{\mathbf{e}}_\theta = -\,\omega\mathbf{e}_r$,

$$\mathbf{a} = -\frac{v^2}{r}\mathbf{e}_r + \dot{v}\mathbf{e}_\theta \tag{49}$$

Helical Motion

Not covered during this time.

Harmonic Motion

Not covered during this time.

Velocity and Acceleration of a Point in a Rigid Body:

Refer to sections "Frames of Reference", "Rigid Body Motion About a Fixed Point", and "Time Derivative of Unit Vector".

Vector Derivatives in a Rotating Systems:

Assume $\mathbf{A}$ to be a free vector observed in the system XYZ and in the rotating triad defined by $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$ as shown in figure 9. Due to the change in orientation, the time rate of change of $\mathbf{A}$ will not be the same from either reference frame even with a coordinate conversion. Instead, the absolute rate of change of $\mathbf{A}$ as expressed from a rotating reference frame is,

$$\dot{\mathbf{A}} = (\dot{\mathbf{A}})_r + \omega \times \mathbf{A} \tag{50}$$

and,

$$(\dot{\mathbf{A}})_A = (\dot{\mathbf{A}})_B + \omega_{B/A} \times \mathbf{A} \tag{51}$$

$$(\dot{\mathbf{A}})_B = (\dot{\mathbf{A}})_A + \omega_{A/B} \times \mathbf{A} \tag{52}$$

where $\omega_{B/A}$ is the rate of rotation of the rotating system as viewed from the fixed system and that $\omega_{A/B} = -\omega_{B/A}$.

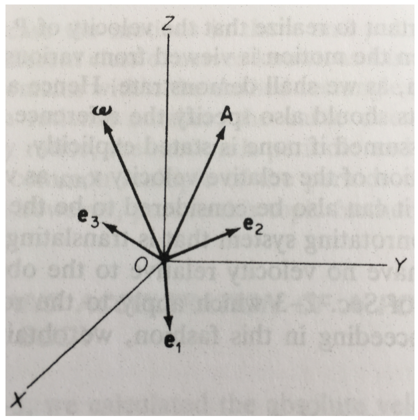

Figure 9: The vector **A** relative to fixed and rotating reference frames.

Motion of a Particle in a Moving Coordinate System:

Building upon previous material, the absolute velocity and acceleration of a particle P that is moving relative to *moving* coordinate system XYZ is found. From figure 10, the position relationship between **r**and **R** is,

$$\mathbf{r} = \mathbf{R} + \rho \tag{53}$$

and therefore,

$$\mathbf{v} = \dot{\mathbf{r}} = \dot{\mathbf{R}} + \dot{\rho} \tag{54}$$

In terms of the rotating system xyz the derivative of vector ρ is found from,

$$\dot{\rho} = (\dot{\rho})_{\mathbf{r}} + \omega \times \rho \tag{55}$$

along with the previous expression for velocity,

$$\mathbf{v} = \dot{\mathbf{R}} + (\dot{\rho})_r + \omega \times \rho \tag{56}$$

Where,

- $\dot{\mathbf{R}}$ is the absolute velocity of *O'*.
- $\omega \times \rho$ is the velocity of *P'* relative to *O'* as viewed from the fixed system.
- $\dot{\mathbf{R}} + \omega \times \rho$ is the absolute velocity of *P'*.

- $(\dot{\rho})_r$ is the velocity of P relative to O' as viewed from the rotating system, or the velocity of P relative to P' from the fixed system.

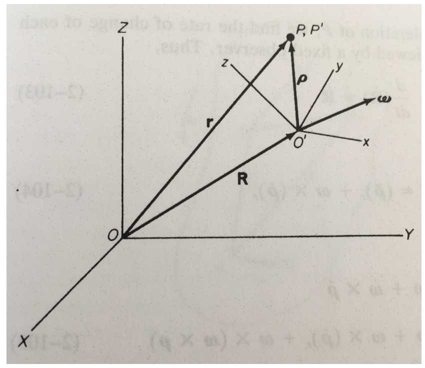

Figure 10: Position of a point P relative to a moving coordinate system.

To find the absolute acceleration of P,

$$\frac{d}{dt}(\dot{\mathbf{R}}) = \ddot{\mathbf{R}} \tag{57}$$

$$\frac{d}{dt}[(\dot{\rho})_r] = (\ddot{\rho})_r + \omega \times (\dot{\rho})_r \tag{58}$$

$$\frac{d}{dt}(\omega \times \rho) = \dot{\omega} \times \rho + \omega \times \dot{\boldsymbol{\rho}} ...$$

$$... = \dot{\omega} \times \rho + \omega \times (\dot{\boldsymbol{\rho}})_r + \omega \times (\omega \times \rho) \tag{59}$$

Therefore,

$$\mathbf{a} = \ddot{\mathbf{R}} + \dot{\omega} \times \rho + \omega \times (\omega \times \rho) + (\ddot{\rho})_{\mathbf{r}} + 2\omega \times (\dot{\boldsymbol{\rho}})_r \tag{60}$$

Where,

- $\ddot{\mathbf{R}}$ is the absolute acceleration of O'.
- $\dot{\omega} \times \rho$ and $\omega \times (\omega \times \rho)$ both represent the acceleration of P' relative to O' in the fixed system.
- $(\ddot{\rho})_r$ is the acceleration of the point P relative to the rotating system.
- $\mathbf{2}\omega \times (\dot{\boldsymbol{\rho}})_r$ is the *Coriolis acceleration.*

Module 3 (2.5-2.9): Chapter 2

Plane Motion:

This section contains lengthy paragraphs and many examples. It is best to review the examples in this section.

Module 4 (2.12-2.16): Chapter 3

Introduction:

The current chapter studies calculation of particle motion based with upon previous knowledge of acting forces. Consider the general case where an acting force is a function of position and velocity and time,

$$m\ddot{\mathbf{r}} = \mathbf{F}(\mathbf{r}, \dot{\mathbf{r}}, t) \tag{61}$$

This is in fact an ordinary differential equation composed of scalars *and* vectors - a rather complicated expression with little possibility of obtaining an analytical solution. However, from Newtonian mechanics the solution must be obtainable since complete knowledge of acting forces, along with initial conditions, should fully describe particle motion. Therefore, a variety of methods do in fact exist to obtain a solution to the above expression and are discussed in this chapter.

Direct Integration of the Equations of Motion:

Case 1 - Constant Acceleration

This is the simplest case in which acting force is constant in magnitude and direction. Therefore the general case equation of motion simplifies to,

$$\begin{aligned} m\ddot{x} &= F_x \\ m\ddot{y} &= F_y \\ m\ddot{z} &= F_z \end{aligned} \tag{62}$$

As directions of motion are orthogonal, acting forces are independent with respect to the XYZ system and can be studied individually. For the x case,

$$a = \ddot{x} = \frac{F_x}{m} \tag{63}$$

By direct integration with respect to time,

$$v = v_0 + at \tag{64}$$

and finally position by direct integration,

$$x = x_0 + v_0 t + \frac{1}{2}at^2 \tag{65}$$

Solving for time to achieve given speed or displacement respectively,

$$t = \frac{1}{a}(v - v_0) \tag{66}$$

$$t = \frac{1}{a}(\sqrt{2a(x - x_0) + v_0^2} - v_0) \tag{67}$$

and by cancelation of t in the above expressions,

$$v^2 = v_0^2 + 2a(x - x_0) \tag{68}$$

Motion of a Particle in a Uniform Gravitational Field

When the only acting force on a particle is a uniform gravitational field, particle motion is confined to the plane established from the initial velocity vector and the acting gravitational force.
From figure 11 assume the particle is at O at time t_0 with a velocity v_o directed at angle γ from the x-axis. The velocity components of the particle would be,

$$\dot{x}(0) = v_o cos\gamma$$

$$\dot{y}(0) = v_0 sin\gamma \tag{69}$$

with the acceleration components being,

$$a_x = 0$$

$$a_y = -grav. \tag{70}$$

From introductory courses, it is known that the velocity components of the particle will behave independently, such that,

$$v_x = v_0 cos\gamma$$

$$v_y = v_0 sin\gamma - grav.\,t \tag{71}$$

Noting constant accelerations and independence of vertical and horizontal motion, direct integration can be used to determine position,

$$x = v_0 t\, cos\gamma$$

$$y = v_0 t\, sin\gamma - \frac{1}{2} grav.\,t^2 \tag{72}$$

Where the time to reach a given distance can be found by solving for either x or y in the above.

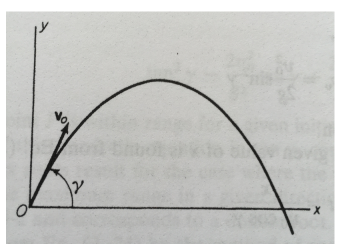

Figure 11: The trajectory of a particle in a uniform gravitational field.

Other useful expressions include determining the trajectory and vertex locations in terms of x and y. By solving the x-position equation for t and substituting into the y-position equation, the expression for the shape of the travel-parabola is given,

$$y = x\, tan\gamma - \frac{grav.\,x^2}{2v_0^2 cos^2\gamma} \tag{73}$$

and by differentiating with respect to x and setting the result equal to 0,

$$x_v = \frac{v_0^2}{grav.} sin\gamma cos\gamma = \frac{v_0^2}{2grav.} sin2\gamma \tag{74}$$

From the two above equations,

$$y_v = \frac{v_0^2}{2grav.} sin^2\gamma \tag{75}$$

Similarly, using the expressions for both arbitrary x-position and x-position of the vertex, the time to reach the vertex can be solved for,

$$t_v = \frac{v_0}{g} sin\gamma \tag{76}$$

where time of flight $t_f = 2t_v$ considering a uniform horizontal. Also considering a uniform horizontal, the range $R = 2x_v$.

Case 2 - $\mathbf{F} = \mathbf{F}(t)$

If the acting force is only a function of time (not of position or velocity) the equation of motion is written as,

$$m\ddot{x} = F_x(t) \tag{77}$$

for the x-direction. Integration techniques to determine the velocity yield,

$$v = v_0 + \frac{1}{m}\int_0^t F_x(\tau)d\tau \tag{78}$$

While another integration results in,

$$x = x_0 + v_0 t + \frac{1}{m}\int_0^t [\int_0^{\tau_2} F_x(\tau_1)d\tau_1]d\tau_2 \tag{79}$$

SEE EXAMPLE 3-2

Case 3 - $\mathbf{F} = F_x(x)\mathbf{i} + F_y(y)\mathbf{j} + F_z(z)\mathbf{k}$

Again only considering the x component of the force,

$$m\ddot{x} = F_x(x) \tag{80}$$

By substituting $x \doteq v$,

$$\ddot{x} = \frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt} = v\frac{dv}{dx} \tag{81}$$

It is seen that,

$$mv\frac{dv}{dx} = F_x(x) \tag{82}$$

and with direct integration,

$$\frac{1}{2}m(v^2 - v_0^2) = \int_{x_0}^x F_x(x)dx \tag{83}$$

A final integration provides,

$$t = f(x_0, v_0, x) \tag{84}$$

where a solution is found in the form,

$$x = g(x_0, v_0, t) \tag{85}$$

SEE EXAMPLE 3-3

Case 4 - F $=F_x(\dot{x})\mathbf{i} + F_y(\dot{y})\mathbf{j} + F_z(\dot{z})\mathbf{k}$

Since each component of the acting force is still independent, the x-component is only considered,

$$m\frac{dv}{dt} = F_x(v) \tag{86}$$

Separation of variables provides,

$$t = m\int_{v_0}^{v} \frac{dv}{F_x(v)} \tag{87}$$

Where upon solving for the velocity,

$$v = \frac{dx}{dt} = g(v_o,\, t) \tag{88}$$

With a final integration,

$$x = f(x_0,\, v_0,\, t) \tag{89}$$

A similar approach is to use the expression $\ddot{x} = v\,dv/dx$ within in the equation of motion as,

$$mv\frac{dv}{dx} = F_x(v) \tag{90}$$

Integration provides,

$$m\int_{v_0}^{v} \frac{vdv}{F_x(v)} = x - x_0 \tag{91}$$

Subtracting v from the expression for t and the above expression gives the same result,

$$x = f(x_0,\, v_0,\, t) \tag{92}$$

SEE EXAMPLE 3-4

Module 5 (2.19-2.23): Chapter 3

Introduction:

In some cases of particle mechanics, general principles apply which are directly derivable from Newton's laws of motion. However, explicit study of these principles provides further insight into particle dynamics and may readily contribute to problem solution.

Work and Kinetic Energy:

The first principle to be discussed is the *principle of work and kinetic energy*, which states:

> The increase in kinetic energy of a particle in going from one arbitrary point to another is equal to the work done by the forces acting on the particle as it moves over the given interval.

To illustrate, consider a particle of mass m that moves along a path from location A to location B under the action of force **F** as shown in figure 12.

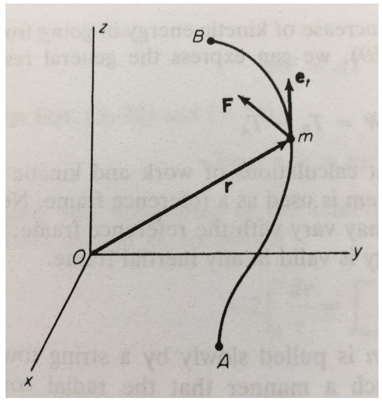

Figure 12: The path of a particle acted upon by an external force.

With respect to the position vector, the equation of motion is written as,

$$\mathbf{F} = m\ddot{\mathbf{r}} \tag{93}$$

Integrating with respect to position r,

$$\int_A^B \mathbf{F} \cdot d\mathbf{r} = \int_A^B m\ddot{\mathbf{r}} \cdot d\mathbf{r} \tag{94}$$

Where $d\mathbf{r}$ is taken to be in the $\mathbf{e}_t$ direction along each point of the curve such that,

$$\ddot{\mathbf{r}} \cdot d\mathbf{r} = \frac{1}{2}\frac{d}{dt}(\dot{\mathbf{r}} \cdot \dot{\mathbf{r}})dt = \frac{1}{2}d(v^2) \tag{95}$$

Upon substitution,

$$\int_A^B \mathbf{F} \cdot d\mathbf{r} = \frac{1}{2}m\,v_B^2 - \frac{1}{2}m\,v_A^2 \tag{96}$$

In the above expression, the left side represents the *work* done by force $\mathbf{F}$ on the particle as it moves from A to B. The right side expression represents the change in *kinetic energy* of the particle as it moves from A to B.
SEE EXAMPLE 3-5

Module 6 (2.26-3.2): Chapter 3

Conservative Forces:

Again from figure 12 suppose that force $\mathbf{F}$ has the following characteristics: (1) single-value function of only position; (2) the line integral

$$\int_A^B \mathbf{F} \cdot d\mathbf{r}$$

is a function of only the endpoints A and B such that it is independent of the path taken between. From calculus and (1),

$$\int_A^B \mathbf{F} \cdot d\mathbf{r} = -\int_B^A \mathbf{F} \cdot d\mathbf{r} \tag{97}$$

however from (2) the above is true for any and all paths between A and B such that,

$$\oint \mathbf{F} \cdot d\mathbf{r} = 0 \tag{98}$$

Any force which follows (1) and (2) is a *conservative force* in that it is not dissipative in nature and any mechanical process taking place under the influence of $\mathbf{F}$ is reversible. In connection with the previous section, the statement can be made that decrease in *potential energy* V of a particle in moving from A to B is equal to the work done on the particle by conservative force $\mathbf{F}$. Since the force has already been assumed to be a function of only its endpoints and with the above understanding,

$$\mathbf{F} \cdot d\mathbf{r} = -dV \tag{99}$$

In culmination,

$$W = \int_A^B \mathbf{F} \cdot d\mathbf{r} = -\int_A^B dV = V_A - V_B \tag{100}$$

Potential Energy:

If potential energy is assumed to be a single-value function of position only such that,

$$dV = \frac{\partial V}{\partial x}dx + \frac{\partial V}{\partial y}dy + \frac{\partial V}{\partial z}dz \tag{101}$$

then from the above and the previous relation between $\mathbf{F}$ and V,

$$\begin{aligned} F_x &= -\frac{\partial V}{\partial x} \\ F_y &= -\frac{\partial V}{\partial y} \\ F_z &= -\frac{\partial V}{\partial z} \end{aligned} \tag{102}$$

Where it is now seen that,

$$\mathbf{F} = -\nabla V \tag{103}$$

which states that $\mathbf{F}$ is in the direction of the largest spatial rate of decrease of V and is equal in magnitude to the rate of decrease.

Module 7 (3.5-3.16): Chapter 3

Linear Impulse and Momentum:

It was previously noted that *linear momentum* $\mathbf{p} = mv$. From Newton's laws of motion and assuming that m is constant,

$$\mathbf{F} = m\mathbf{a} = m\mathbf{v} = \dot{\mathbf{p}} \tag{104}$$

Integrating both sides,

$$\int_{t_1}^{t_2} \mathbf{F} dt = \int_{t_1}^{t_2} \dot{\mathbf{p}} dt = \mathbf{p}_2 - \mathbf{p}_1 \tag{105}$$

where $\mathbf{p}_2$ represents the linear momentum at time t_2. The integral of $\mathbf{F}$ is known as the *impulse* $\hat{\mathbf{F}}$. From the above the *principle of linear momentum* is given:

> The change in the linear momentum of a particle during a given interval is equal to the total impulse of the forces acting on the particle over the same interval.

The length of time between t_1 and t_2 does not alter the result - a force with duration approaching zero and very large magnitude is known as a *impulsive force.* Using the Dirac delta function, an impulsive force can be expressed as,

$$\mathbf{F} = \hat{\mathbf{F}}\delta(t - \tau) \tag{106}$$

where,

$$\delta(t) = \begin{cases} 0 & t \neq 0 \\ \infty & t = 0 \end{cases} ; \int_{-\infty}^{\infty} \delta(t) dt = 1 \tag{107}$$

From Newton's law of motion, its seen that if $\mathbf{F}$ =0 then linear momentum is constant which is the *principle of conservation of linear momentum.*

Angular Momentum and Angular Impulse:

Considering the case where the momentum vector is given as in figure 13, the *moment of momentum* or *angular momentum* above the point O is then given as,

$$\mathbf{H} = \mathbf{r}\times m\mathbf{v} \tag{108}$$

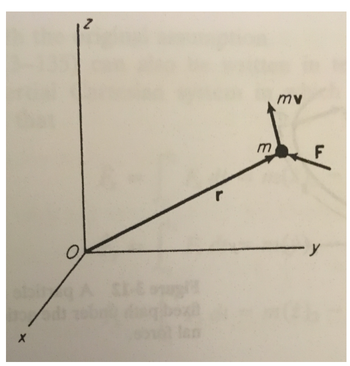

Figure 13: A particle moving relative to a fixed point O.

To make the relationship between force $\mathbf{F}$ and angular momentum $\mathbf{H}$, the above expression is differentiated,

$$\dot{\mathbf{H}} = \mathbf{r}\times m\ddot{\mathbf{r}} + \dot{\mathbf{r}} \times m\dot{\mathbf{r}} = \mathbf{r} \times m\ddot{\mathbf{r}} \tag{109}$$

From Newton's law of motion,

$$\mathbf{r} \times \mathbf{F} = \mathbf{r} \times m\ddot{\mathbf{r}} \tag{110}$$

with the left hand side of the above expression being the moment $\mathbf{M}$ of force $\mathbf{F}$ about point O,

$$\mathbf{M} = \mathbf{r} \times \mathbf{F} = \dot{\mathbf{H}} \tag{111}$$

The last expression explains that the moment about a fixed point of the total force applied to a particle is equal to the time rate of change of angular momentum of that particle about the same fixed point. Observation shows that if $\mathbf{M} = \mathbf{0}$ then the angular momentum is constant in magnitude and direction which is the *principle of conservation of angular momentum.*

Similarly as was seen between the relationship between linear impulse and linear momentum, the time integral of the moment $\mathbf{M}$ is known as *angular impulse* and is written as $\hat{\mathbf{M}}$.