ME516S18 Equation Sheet

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1.15.18 - 5.4.18

A summary of select chapters of *Principles of Dynamics (2 Ed.)* by Donald T. Greenwood.

Module 1 (1.15-1.19): Chapter 1

Auxiliary:

Derivative of a vector

Considering a *vector function* of a scalar variable (magnitude and direction are dependent upon the value of scalar); the derivative of the vector $\mathbf{A}$ with respect to scalar u is defined by limit,

$$\frac{d\mathbf{A}}{du} = \lim_{\Delta u \to 0} \frac{\Delta \mathbf{A}}{\Delta u} \tag{1}$$

Derivative of vector sum $\mathbf{A} + \mathbf{B}$,

$$\Delta(\mathbf{A} + \mathbf{B}) = \Delta\mathbf{A} + \Delta\mathbf{B} \tag{2}$$

Vector differentiation is *distributive,*

$$\frac{d(\mathbf{A} + \mathbf{B})}{du} = \frac{d\mathbf{A}}{du} + \frac{d\mathbf{B}}{du} \tag{3}$$

And the following can be shown,

$$\frac{d}{du}(g\mathbf{A}) = \frac{dg}{du}\mathbf{A} + g\frac{d\mathbf{A}}{du} \tag{4}$$

$$\frac{d}{du}(\mathbf{A} \cdot \mathbf{B}) = \frac{d\mathbf{A}}{du} \cdot \mathbf{B} + \mathbf{A} \cdot \frac{d\mathbf{B}}{du} \tag{5}$$

$$\frac{d}{du}(\mathbf{A} \times \mathbf{B}) = \frac{d\mathbf{A}}{du} \times \mathbf{B} + \mathbf{A} \times \frac{d\mathbf{B}}{du} \tag{6}$$

Newton's Law of Motion:

1. Every body continues in its state of rest, or of uniform motion in a straight line, unless compelled to change that state by forces acting upon it.
2. The time rate of change of linear momentum of a bod is proportional to the force acting upon it and occurs in the direction in which the force acts.
3. To every action there is an equal and opposite reaction; that is, the mutual forces of two bodies acting upon each other are equal in magnitude and opposite in direction.

Laws of Motion for a Particle:

Stating **three** basic laws applying to the motion of a particle, the first being the *law of motion* which summarizes Newton's first two laws as,

$$\mathbf{F} = k\frac{d}{dt}(m\mathbf{v}) = km\mathbf{a} \tag{7}$$

where $m\mathbf{v}$ is linear momentum, $\mathbf{p} = m\mathbf{v}$. Choosing units such that $k = 1$,

$$\mathbf{F} = m\mathbf{a} \tag{8}$$

The second basic law expresses *collinearity* of the interaction of forces as the *law of action and reaction:*

> Two particles exert interaction forces on one another that are equal in magnitude, opposite in sense, and directed along straight line joining particles.

Finally the third basic law, *law of addition of forces*,

$$\mathbf{F} = \mathbf{P} + \mathbf{Q} \tag{9}$$

Frames of Reference:

Frames of reference are **always** defined as *inertial (Newtonian)* reference frames consisting of rigid coordinate axes such that particle motion is described by Newton's laws of motion, unless otherwise mentioned.
A relative reference frame, B, can be established that is *only translating* with respect to the original reference,A, at a constant velocity $\mathbf{v}_{B/A}$. Denoting particle velocity by observers on A and B as $\mathbf{v}_A$ and $\mathbf{v}_B$ respectively,

$$\mathbf{v}_A = \mathbf{v}_B + \mathbf{v}_{B/A} \tag{10}$$

Where upon differentiating,

$$\mathbf{a}_A = \mathbf{a}_B \tag{11}$$

It is seen that from the Newtonian point of view, forces, masses, and accelerations are seen as equal by both observes. This is referred to as the *Newtonian principle of relativity.*

D'Alemberts's Principle:

Writing Newton's law of motion for a particle in the form,

$$\mathbf{F} - m\mathbf{a} = 0 \tag{12}$$

effectively transforms the dynamically stated problem to a statically defined problem. Therefore, d'Alembert's principle simply states,

> The laws of static equilibrium apply to a dynamical system if the inertia forces, as well as the actual external forces, are considered as the forces acting on the system.

Considerable care must be taken in use of d'Alembert's principle, as the inertia forces *should not* be confused with external forces comprising total force, $\mathbf{F}$, that is applied to the particle. External forces are considered as contact and gravitational or other field forces applied to the particle.

Module 2 (1.22-2.2): Chapter 2

Kinematics:

Kinematics is the study of particles and rigid bodies, disregarding forces associated with motion. It is a purely mathematical study and does not consider physical laws. All motion is considered to be relative and no reference frame is more fundamental or absolute than another.
The position of particle P in the XYZ coordinate system of figure 1 is given by **r** drawn from the origin O to P. If P moves along curve C then the velocity **v**is in the direction tangent to the curve and has a magnitude equal to the speed. Considering velocity as a free vector, its origin can be imagined to be drawn from O at each instance of time, with the path C' describing the tip of the velocity **v**is known as the hodograph. So the velocity of the tip of **v**along the hodograph C' is the *acceleration* of P relative to the XYZ system. In summary,

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} \tag{13}$$

and the acceleration of P is given as,

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{d^2\mathbf{r}}{dt^2} \tag{14}$$

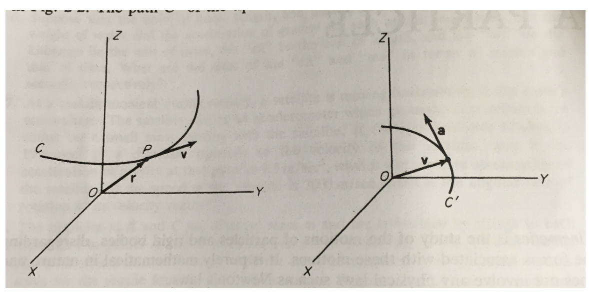

Figure 1: (Left) Position and velocity vectors of particle P as it moves along C. (Right) Hodograph showing velocity and acceleration.

Angular Velocity:

In kinematics the primary concern is the translation of a point relative to a given reference frame, that is, with the position vector of the point and with its derivatives with respect to time. However, *general* motion also must consider changes in *orientation (rotation)* and it the rate of change of orientation that is expressed by means of the angular velocity.
Consider the motion of rigid body shown in figure 2 during infinitesimal time Δt. We note that the body has undergone both a translation of amount Δs and rotation of $\Delta\theta$, the order of translation and rotation is immaterial. The infinitesimal angular displacement $\Delta\theta$ is a vector whose magnitude is equal to the angle of rotation and whose direction is along an axis determined by those points not displaced by the infinitesimal rotation using the right-hand rule. The angular velocity, ω, therefore is defined as,

$$\omega = \lim_{\Delta t \to 0} \frac{\Delta\theta}{\Delta t} \tag{15}$$

The angular velocity is a free vector and will produce different results based upon the reference frame. It is dependent upon the body only and not of the base point.

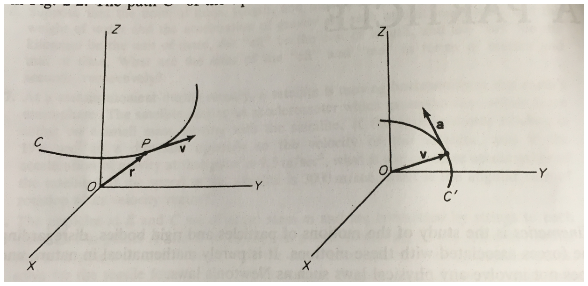

Figure 2: Motion of rigid body during infinitesimal time Δt

Rigid Body Motion About a Fixed Point:

In figure 3, the base point of the rigid body is assumed to be fixed at the origin O of the cartesian system XYZ. In calculation of the velocity of a point P that is fixed in the body using an inertial reference frame, $\mathbf{v}$is the absolute velocity of point P and $\omega$is the absolute angular velocity of the body. Rotation is occurring about the *instantaneous axis of rotation*, with the point P having speed,

$$\dot{s} = \omega r sin\theta \tag{16}$$

The velocity of P is of magnitude $\dot{s}$ and is directed along the tangent of the path. Therefore it is seen that for the general case of rigid-body rotation about a fixed point that,

$$\mathbf{v} = \omega \times \mathbf{r} \tag{17}$$

Differentiating with respect to time, the acceleration of P is given as,

$$\mathbf{a} = \omega \times \dot{\mathbf{r}} + \dot{\omega} \times \mathbf{r} \tag{18}$$

or noting that $\dot{\mathbf{r}} = \mathbf{v}$, the form,

$$\mathbf{a} = \omega \times (\omega \times \mathbf{r}) + \dot{\omega} \times \mathbf{r} \tag{19}$$

Where from the above, the term $\omega \times (\omega \times \mathbf{r})$ is the *centripetal acceleration* (directed towards the instantaneous axis of rotation) with the remaining terms being the *tangential acceleration.*

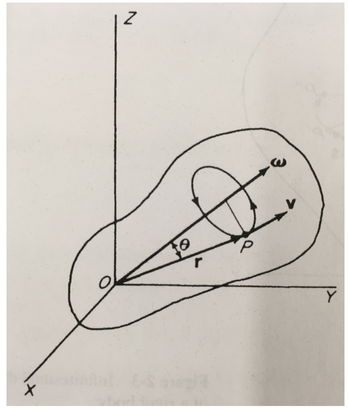

Figure 3: Rigid body motion about a fixed point.

Time Derivative of a Unit Vector:

As was the case of determining the derivative with respect to time of the position vector **r**in the previous section (magnitude is constant but direction changes), the time derivative of unit vectors are calculated similarly,

$$\dot{\mathbf{e_1}} = \omega \times \mathbf{e_1} \,; \dot{\mathbf{e_2}} = \omega \times \mathbf{e_2} \,; \dot{\mathbf{e_3}} = \omega \times \mathbf{e_3} \tag{20}$$

In the case where the unit vectors form an orthogonal system, vector multiplication is possible through the determinate expression, and writing the angular velocity in the form,

$$\omega = \omega_1 \mathbf{e_1} + \omega_2 \mathbf{e_2} + \omega_3 \mathbf{e_3} \tag{21}$$

obtains,

$$\dot{\mathbf{e_1}} = \begin{vmatrix} \mathbf{e_1} & \mathbf{e_2} & \mathbf{e_3} \\ \omega_1 & \omega_2 & \omega_3 \\ 1 & 0 & 0 \end{vmatrix} = \omega_3 \mathbf{e_2} - \omega_2 \mathbf{e_3} \tag{22}$$

Velocity and Acceleration of a Particle in Several Coordinate Systems:

Cartesian Coordinates

The position of particle P relative to the origin O of the system XYZ is shown in figure 4 is given by,

$$\mathbf{r} = \mathrm{x}\mathbf{i} + \mathrm{y}\mathbf{j} + \mathrm{z}\mathbf{k} \tag{23}$$

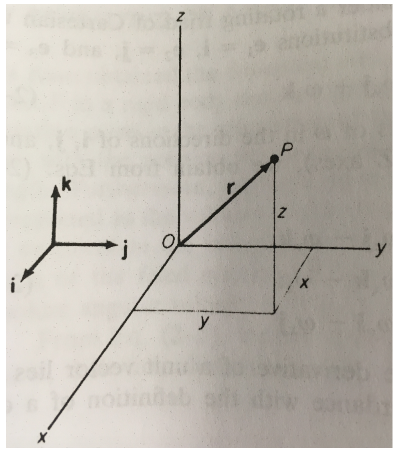

Figure 4: Cartesian coordinates and unit vectors.

The velocity is therefore given as,

$$\mathbf{v} = \dot{\mathbf{r}} = \dot{x}\mathbf{i} + \dot{y}\mathbf{j} + \dot{z}\mathbf{k} \tag{24}$$

and the corresponding acceleration,

$$\mathbf{a} = \ddot{x}\mathbf{i} + \ddot{y}\mathbf{j} + \ddot{z}\mathbf{k} \tag{25}$$

Considering the system XYZ to an inertial frame, the velocity and acceleration are *absolute*.

Cylindrical Coordinates

Now considering the case where cylindrical coordinates are used to express the XYZ system in terms of r, ϕ, z, as shown in figure 5, the position vector is defined as,

$$\mathrm{r} = \mathrm{r}\mathbf{e_r} + z\mathbf{e_z} \tag{26}$$

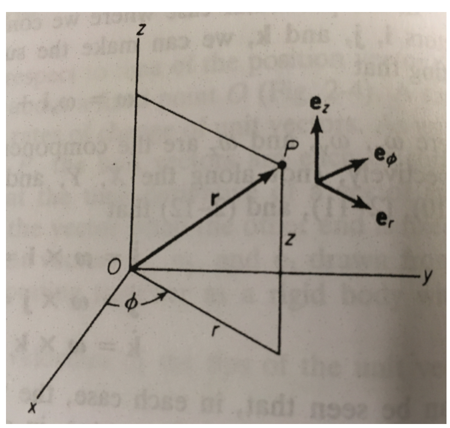

Figure 5: Cylindrical coordinates and unit vectors.

The velocity is found as,

$$\mathbf{v} = \dot{\mathbf{r}} = \dot{r}\mathbf{e_r} + \dot{z}\mathbf{e_z} + r\dot{\mathbf{e_r}} + z\dot{\mathbf{e_z}} \tag{27}$$

It is seen that $\mathbf{e_r}$ and $\mathbf{e}_\phi$ change direction as P moves through a general displacement, but $\mathbf{e_z}$ remains parallel to the z axis. Further, changes in direction of $\mathbf{e_r}$ and $\mathbf{e}_\phi$ are due solely to changes in ϕ, corresponding to rotations about the z axis. Therefore, the *absolute* rotation rate of the unit vector system is,

$$\omega = \dot{\phi}\mathbf{e_z} \tag{28}$$

and its shown that,

$$\begin{aligned}\dot{\mathbf{e_r}} &= \omega \times \mathbf{e_r} = \dot{\phi}\mathbf{e}_\phi \\ \dot{\mathbf{e}}_\phi &= \omega \times \mathbf{e}_\phi = -\dot{\phi}\mathbf{e}_r \\ \dot{\mathbf{e}}_z &= \omega \times \mathbf{e}_z = 0\end{aligned} \tag{29}$$

In which the velocity is found as,

$$\mathbf{v} = \dot{r}\mathbf{e}_r + r\dot{\phi}\mathbf{e}_\phi + \dot{z}\mathbf{e}_z \tag{30}$$

Differentiating with respect to time provides the acceleration,

$$\mathbf{a} = \ddot{\mathbf{r}} = \ddot{r}\mathbf{e}_r + (\dot{r}\dot{\phi} + r\ddot{\phi})\mathbf{e}_\phi + \ddot{z}\mathbf{e}_z + \dot{r}\dot{\mathbf{e}}_r + r\dot{\phi}\dot{\mathbf{e}}_\phi + \dot{z}\dot{\mathbf{e}}_z \tag{31}$$

which can be simplified using the unit vector derivatives as,

$$\mathbf{a} = (\ddot{r} - r\dot{\phi}^2)\mathbf{e_r} + (r\ddot{\phi} + 2\dot{r}\dot{\phi})\mathbf{e}_\phi + \ddot{z}\mathbf{e}_z \tag{32}$$

Spherical Coordinates

Finally, considering the case where the XYZ system is cast into spherical coordinates, r, ϕ, θ, as shown in figure 6, such that $\mathbf{r} = r\mathbf{e_r}$. The velocity of $\mathbf{r}$is found as,

$$\mathbf{v} = \dot{\mathbf{r}} = \dot{r}\mathbf{e_r} + r\dot{\mathbf{e_r}} \tag{33}$$

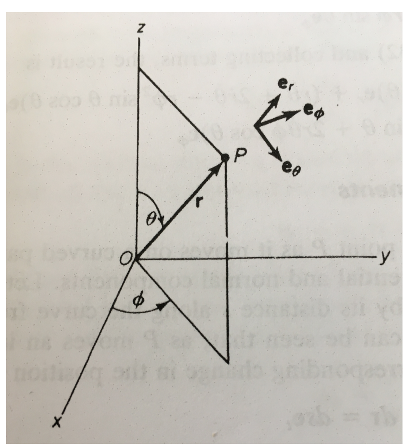

Figure 6: Spherical coordinates and unit vectors.

In order to determine the derivatives of the unit vectors, i.e. $\dot{\mathbf{e_r}}$, $\dot{\mathbf{e}}_\theta$, $\dot{\mathbf{e}}_\phi$, its important to correlate changes in θ, ϕ. From figure 6 its seen that increases in θ causes rotation about an axis parallel to $\mathbf{e}_\phi$, while changes in ϕ expresses a rotation about the z axis. The total rotation therefore is,

$$\omega = \dot{\theta}\mathbf{e}_{\phi} + \dot{\phi}\mathbf{e}_z \tag{34}$$

From figure 6, it is seen that $\mathbf{e}_z$ can be expressed as a combination of the other unit vectors as,

$$\mathbf{e}_z = cos\theta\mathbf{e}_r - sin\theta\mathbf{e}_{\theta} \tag{35}$$

and along with previous equation, the angular velocity is expressed as,

$$\omega = \dot{\phi}cos\theta\mathbf{e}_r - \dot{\phi}sin\theta\mathbf{e}_{\theta} + \dot{\phi}\mathbf{e}_{\phi} \tag{36}$$

Taking advantage of orthogonality,

$$\dot{\mathbf{e}}_r = \omega \times \mathbf{e}_r = \begin{vmatrix} \mathbf{e}_r & \mathbf{e}_{\theta} & \mathbf{e}_{\phi} \\ \dot{\phi}cos\theta & -\dot{\phi}sin\theta & \dot{\theta} \\ 1 & 0 & 0 \end{vmatrix} = \dot{\theta}\mathbf{e}_{\theta} + \dot{\phi}sin\theta\mathbf{e}_{\phi} \tag{37}$$

Finally, the velocity and acceleration are found as,

$$\mathbf{v} = \dot{r}\mathbf{e}_r + r\dot{\theta}\mathbf{e}_{\theta} + r\dot{\phi}sin\theta\mathbf{e}_{\phi} \tag{38}$$

$$\mathbf{a} = (\ddot{r} - r\dot{\theta}^2 - r\dot{\phi}^2 sin^2\theta)\mathbf{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta} - r\dot{\phi}^2 sin\theta cos\theta)\mathbf{e}_{\theta}...$$
$$... + (r\ddot{\phi}sin\theta + 2\dot{r}\dot{\phi}sin\theta + 2r\dot{\theta}\dot{\phi}cos\theta)\mathbf{e}_{\phi} \tag{39}$$

Tangential and Normal Components

Both the velocity and acceleration of a point moving along a curve can be expressed in terms of *tangential* and *normal* components. From figures **??**, **??** and **??** its seen as P moves along s a distance ds, the position vector changes as,

$$d\mathbf{r} = ds\mathbf{e}_t; \;\therefore\; \mathbf{e}_t = \frac{d\mathbf{r}}{ds} \tag{40}$$

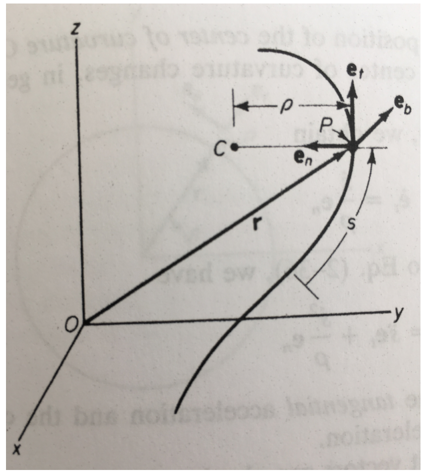

Figure 7: Tangential, normal, and binormal unit vectors along a curve.

Taking the derivative provides the velocity as,

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \dot{s}\frac{d\mathbf{r}}{ds} = \dot{s}\mathbf{e}_t \tag{41}$$

with the acceleration,

$$\mathbf{a} = \dot{\mathbf{v}} = \ddot{s}\mathbf{e}_t + \dot{s}\dot{\mathbf{e}}_t \tag{42}$$

Again from figures **??**, **??** and **??**, its noted that any changes in $\mathbf{e}_t$ are related to increasing s. Since the magnitude of $\mathbf{e}_t$ is constant, the direction of $d\mathbf{e}_t/ds$ is *normal* to $\mathbf{e}_t$, and lies in the plane of both $\mathbf{e}_t$ and $\mathbf{e}_n$ termed the *osculating plane*. The magnitude of $\dot{\mathbf{e}}_t$ is equal to its rotation rate, ω_b, about an axis perpendicular to the osculating plane, $\mathbf{e}_b$ (binormal axis) such as,

$$\dot{\mathbf{e}}_t = \dot{s}\frac{d\mathbf{e}_t}{ds} = \omega_b\mathbf{e}_n;\ \omega_b = \frac{\dot{s}}{\rho} \tag{43}$$

Substituting the expression for the rotation rate into the above equation for the definition of the acceleration,

$$\mathbf{a} = \ddot{s}\mathbf{e}_t + \frac{\dot{s}^2}{\rho}\mathbf{e}_n \tag{44}$$

where the first is component is the *tangential* acceleration and the second is the *centripetal* acceleration.

Simple Motions of a Point

Circular Motion

Considering simple motion as shown in figure 8 with respect to polar coordinates (r, θ) the posistion vector is given as,

$$\mathbf{r} = r\mathbf{e}_r \tag{45}$$

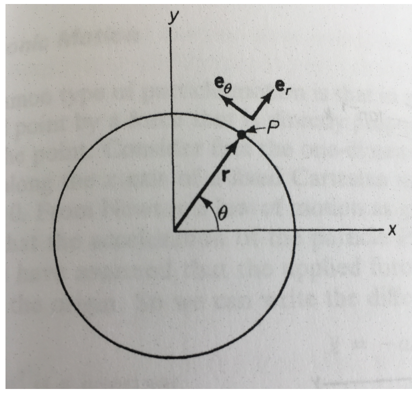

Figure 8: Circular motion of a point.

While the velocity and speed along the path are found to be,

$$\mathbf{v} = \dot{\mathbf{r}} = r\dot{\mathbf{e}}_r \tag{46}$$

$$v = r\dot{\theta} = r\omega \tag{47}$$

respectively. The acceleration is found by taking the derivative with respect to time,

$$\mathbf{a} = \dot{\mathbf{v}} = r\dot{\omega}\mathbf{e}_\theta + r\omega\dot{\mathbf{e}}_\theta \tag{48}$$

Since $\dot{\mathbf{e}}_\theta = -\,\omega\mathbf{e}_r$,

$$\mathbf{a} = -\frac{v^2}{r}\mathbf{e}_r + \dot{v}\mathbf{e}_\theta \tag{49}$$

Helical Motion

Not covered during this time.

Harmonic Motion

Not covered during this time.

Velocity and Acceleration of a Point in a Rigid Body

Refer to sections “Frames of Reference”, “Rigid Body Motion About a Fixed Point”, and “Time Derivative of Unit Vector”.

Vector Derivatives in a Rotating Systems

Assume $\mathbf{A}$ to be a free vector observed in the system XYZ and in the rotating triad defined by $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$ as shown in figure 9. Due to the change in orientation, the time rate of change of $\mathbf{A}$ will not be the same from either reference frame even with a coordinate conversion. Instead, the absolute rate of change of $\mathbf{A}$ as expressed from a rotating reference frame is,

$$\dot{\mathbf{A}} = (\dot{\mathbf{A}})_r + \omega \times \mathbf{A} \tag{50}$$

and,

$$(\dot{\mathbf{A}})_A = (\dot{\mathbf{A}})_B + \omega_{B/A} \times \mathbf{A} \tag{51}$$

$$(\dot{\mathbf{A}})_B = (\dot{\mathbf{A}})_A + \omega_{A/B} \times \mathbf{A} \tag{52}$$

where $\omega_{B/A}$ is the rate of rotation of the rotating system as viewed from the fixed system and that $\omega_{A/B} = -\omega_{B/A}$.

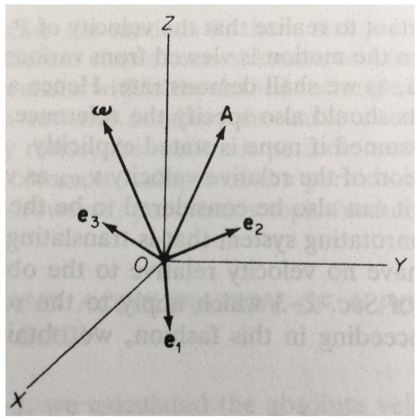

Figure 9: The vector **A** relative to fixed and rotating reference frames.

Motion of a Particle in a Moving Coordinate System

Building upon previous material, the absolute velocity and acceleration of a particle P that is moving relative to *moving* coordinate system XYZ is found. From figure 10, the position relationship between **r**and **R** is,

$$\mathbf{r} = \mathbf{R} + \rho \tag{53}$$

and therefore,

$$\mathbf{v} = \dot{\mathbf{r}} = \dot{\mathbf{R}} + \dot{\rho} \tag{54}$$

In terms of the rotating system xyz the derivative of vector ρ is found from,

$$\dot{\rho} = (\dot{\rho})_{\mathbf{r}} + \omega \times \rho \tag{55}$$

along with the previous expression for velocity,

$$\mathbf{v} = \dot{\mathbf{R}} + (\dot{\rho})_r + \omega \times \rho \tag{56}$$

Where,

- $\dot{\mathbf{R}}$ is the absolute velocity of *O'*.
- $\omega \times \rho$ is the velocity of *P'* relative to *O'* as viewed from the fixed system.
- $\dot{\mathbf{R}} + \omega \times \rho$ is the absolute velocity of *P'*.

- $(\dot{\rho})_r$ is the velocity of P relative to O' as viewed from the rotating system, or the velocity of P relative to P' from the fixed system.

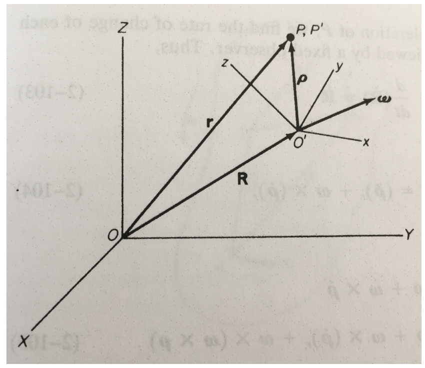

Figure 10: Position of a point P relative to a moving coordinate system.

To find the absolute acceleration of P,

$$\frac{d}{dt}(\dot{\mathbf{R}}) = \ddot{\mathbf{R}} \tag{57}$$

$$\frac{d}{dt}[(\dot{\rho})_r] = (\ddot{\rho})_r + \omega\times(\dot{\rho})_r \tag{58}$$

$$\frac{d}{dt}(\omega \times \rho) = \dot{\omega}\times\rho + \omega \times \dot{\boldsymbol{\rho}}...$$

$$... = \dot{\omega}\times\rho + \omega \times (\dot{\boldsymbol{\rho}})_r + \omega \times (\omega \times \rho) \tag{59}$$

Therefore,

$$\mathbf{a} = \ddot{\mathbf{R}} + \dot{\omega} \times \rho + \omega \times (\omega \times \rho) + (\ddot{\rho})_{\mathbf{r}} + 2\omega \times (\dot{\boldsymbol{\rho}})_r \tag{60}$$

Where,

- $\ddot{\mathbf{R}}$ is the absolute acceleration of O'.
- $\dot{\omega} \times \rho$ and $\omega \times (\omega \times \rho)$ both represent the acceleration of P' relative to O' in the fixed system.
- $(\ddot{\rho})_r$ is the acceleration of the point P relative to the rotating system.
- $\mathbf{2}\omega \times (\dot{\boldsymbol{\rho}})_r$ is the *Coriolis acceleration.*

Module 3 (2.5-2.9): Chapter 2

Plane Motion

This section contains lengthy paragraphs and many examples. It is best to review the examples in this section.

Module 4 (2.12-2.16): Chapter 3